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THEORETICAL $\Sigma - D$ RELATION FOR SUPERNOVA REMNANTS

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Abstract. This lecture presents a summary of theoretically-derived relations between the radio surface brightness Σ and the diameter D of supernova remnants (SNRs): such relations are commonly known as $\Sigma - D$ relations. We argue that discrepancies between theoretically-derived relations and valid empirical relations derived, may be at least partially explained by taking into account thermal emission at radio frequencies from two particular types of SNRs.

1. INTRODUCTION

The relation between the surface brightness Σ and the diameter D of supernova remnants (SNRs) – the so-called $\Sigma - D$ relation – provides a convenient way to investigate the radio brightness evolution of SNRs. The $\Sigma - D$ relation relation was first presented and described by Shklovsky (1960ab) in the course of a theoretical analysis of synchrotron radiation from an expanding spherical nebula (that is, a theoretical construct to describe an SNR). Lequeux (1962) generalized the $\Sigma - D$ relation to the shell case to include the well-known shell-type SNR Cas A, and derived a relation which gave a better approximation to empirical relations than the relation derived by Shklovsky (1960ab). As inspired by the work of van der Laan (1962), Poveda and Woltjer (1968) described a modification to the original derivation presented by Shklovsky (1960ab), namely the magnetic field of the SNR was assumed to remain constant as the SNR expands. The $\Sigma - D$ relation derived by Poveda and Woltjer (1968) in this manner closely matched an empirical $\Sigma - D$ relation presented in the same paper. In addition, Kesteven (1968) derived the relation for a shell-type SNR assuming that the thickness of the shell of the SNR remains constant as the SNR expands. Despite the work of Poveda and Woltjer (1968) and Kesteven (1968), however, significant inconsistencies between empirical and theoretical $\Sigma - D$ relations remained. More recently, Duric and Seaquist (1986) derived a $\Sigma - D$ relation based on a theoretical interpretation that paralleled the work of Shklovsky (1960ab): specifically, Duric and Seaquist (1986) adopted both the version of Fermi's accelerating mechanism presented by Bell (1978ab) and the magnetic field model described by Gull (1973) and Fedorenko (1983). The most recent radio observations of SNRs indicate that the surface brightnesses of these sources decrease less rapidly than predicted by theory (e.g.

Case and Bhattacharya, 1998; Urošević, 2002; 2003). Shklovsky (1960b) described how the $\Sigma - D$ relation could be used to determine the distances to radio SNRs based on their surface brightnesses, assuming that this quantity is not distance-dependent. Therefore, the primary application of the $\Sigma - D$ relation is to provide an independent method to estimate the distances to radio SNRs.

Complementary radio observations of SNRs made during the development of theoretical $\Sigma - D$ relations have confirmed the existence of these relations in the form predicted by Shklovsky (1960ab). The first empirical $\Sigma - D$ relation was derived by Poveda and Woltjer (1968): shortly afterward, Milne (1970) derived an empirical $\Sigma - D$ relation and calculated distances to all 97 of the radio SNRs then known to exist in our galaxy. Many observational studies of the $\Sigma - D$ relation were conducted to determine precisely the distances to a specific set of calibrator sources and therefore improve the usefulness of the relation itself (Ilovaisky, 1972; Sakhibov and Smirnov, 1982; Huang and Thaddeus, 1985). Critical analyses of this relation have been conducted since the discoveries by Green (1984). Uncertainties in the distances to certain calibrators are the main weaknesses of the relations derived in this manner: in other words, there are not enough SNRs with precisely calculated distances for the derivation of a proper $\Sigma - D$ relation (Green, 1984). It has also been shown that the derivation of the $\Sigma - D$ relation is meaningful only for shell-type SNRs (Allakhverdiyev et al., 1983; 1986).

From the first studies of this relation, significant differences between theoretical and empirical results were established, with Green (1991) showing that the calibrators are too scattered on the $\Sigma - D$ diagram to derive a valid relation. However, Case and Bhattacharya (1998) derived an empirical $\Sigma - D$ relation – obtaining a much flatter slope than those seen in earlier works – and determined distances for all known shell-type Galactic SNRs. We believe that the discrepancies between theoretical and empirical $\Sigma - D$ relations may be at least partially explained by considering thermal bremsstrahlung emission from SNRs at radio frequencies. Here, we present an updated derivation of the $\Sigma - D$ relation which takes into account this thermal emission at radio frequencies, and we show that the inclusion of this emission helps decrease the discrepancy between theoretical and empirical $\Sigma - D$ relations.

2. THE THEORETICAL $\Sigma - D$ RELATION – A BRIEF REVIEW

2.1. BASIC THEORY (SHKLOVSKY 1960a)

We briefly present the original theory behind the $\Sigma - D$ relation as proposed by Shklovsky (1960a). The vast majority of radio emission detected from Galactic and extragalactic sources is synchrotron radiation produced by relativistic electrons gyrating in magnetic fields. We consider an ensemble of relativistic electrons with an energy distribution in the form of a power law,

$$n(E) = KE^{-\gamma}, \tag{1}$$

where $n(E)$ is the volume density of the relativistic electrons with energies between E and $E + \Delta E$, K is the coefficient of proportionality and γ is the synchrotron spectral index. The synchrotron emission power (emissivity) ϵ_ν of this ensemble from a unit volume at a given frequency ν may be expressed as

$$\varepsilon_\nu \propto KH^{1+\alpha}\nu^{-\alpha}, \tag{2}$$

where spectral index $\alpha = (\gamma - 1)/2$. The definition of the spectral index is through $S_\nu \propto \nu^{-\alpha}$, where S_ν is the flux density. The synchrotron surface brightness Σ_ν for an ensemble of relativistic electrons and positrons may be expressed as

$$\Sigma_\nu = \frac{S_\nu}{\Omega} = \frac{\varepsilon_\nu V}{D^2\pi^2} \propto DKH^{1+\alpha}\nu^{-\alpha}, \tag{3}$$

where Ω is the solid angle, D is the diameter of the spherical volume V of the ensemble with a constant volume emissivity ε_ν and ν is the frequency.

Assuming a constant value for γ, after short derivation (Shklovsky, 1960a), we obtain

$$K = K_0\Big(\frac{D_0}{D}\Big)^{\gamma-1}\Big(\frac{D_0}{D}\Big)^{3}. \tag{4}$$

Here, K represents the time-evolved value of K_0 as the SNR expands. A major assumption in the derivation presented by Shklovsky (1960a) was that as the spherical nebula (here, an SNR) expands, the structure of the magnetic field is approximately conserved. Therefore, the magnetic field flux must remain constant and H will have the following dependence on radius D:

$$H = H_0\Big(\frac{D_0}{D}\Big)^{2}. \tag{5}$$

Combining the relations given above, we may express the dependence of surface brightness Σ_ν on radius D as

$$\Sigma_\nu \propto D^{-4\alpha-4}. \tag{6}$$

Alternatively, we can express Σ_ν as a function of D as

$$\Sigma_\nu = AD^{-\beta}, \tag{7}$$

where A is a constant and $\beta = 4\alpha + 4$. Shklovsky (1960a) used Cas A to test this theory on the relationship between Σ_ν and D and predicted a 2% relative annual decrease in the observed flux density from this source. Radio observations from that era of Cas A indicated a somewhat lower rate of 1.5% per year. We note that even from the outset of research on the $\Sigma - D$ relation for Galactic SNRs, discrepancies between observation and theory were evident. Assuming an average spectral index of $\alpha = 0.5$ for radio SNRs (Clark and Caswell, 1976), the relation derived by Shklovsky (1960a) predicts a rather steep slope dependence for Σ_ν, namely

$$\Sigma_\nu \propto D^{-6}. \tag{8}$$

2.2. THE LEQUEUX (1962) MODIFICATION

Lequeux (1962) presented another derivation of the $\Sigma - D$ relation where the original relation derived by Shklovsky (1960a) was broadened to include shell-type SNRs. If we once again assume $H \propto D^{-2}$ and a mean spectral index α= 0.5 for shell-type SNRs, we obtain

$$\Sigma_\nu \propto D^{-5.8}. \tag{9}$$

The slope predicted by this relation is slightly shallower than the slope predicted by the relation presented by Shklovsky (1960a). When this model is applied to Cas A, an annual relative decrease of 1.7% in flux density is predicted, a value which is approximately 10% lower than the value predicted by Shklovsky (1960a). While this lower value is closer to the value measured by observations, it is still higher than those obtained from empirical relations.

If we assume a shell thickness ηD (where η is a constant such that $0 < \eta < 1$), we again obtain Eq. (3). We conclude that the result from Shklovsky (1960a) can be generalized directly for shell-type SNRs if we assume that the shell thickness remains a constant fraction of the radius of the SNR as the SNR expands.

2.3. THE POVEDA AND WOLTJER (1968) MODIFICATION

Poveda and Woltjer (1968) presented another derivation of the $\Sigma - D$ relation where the magnetic field H was assumed to be constant as the SNR expands, with the particular value of H depending on the amount of compression of the interstellar magnetic field by the SNR shock van der Laan (1962). If a constant H is included in the derivation presented by Shklovsky (1960a), a $\Sigma - D$ relation with a considerably flatter slope is derived as Eq. (3) becomes

$$\Sigma_\nu \propto D^{-2\alpha-2}. \tag{10}$$

If we again assume a value of 0.5 for α, we obtain the following expression for Σ_ν:

$$\Sigma_\nu \propto D^{-3}. \tag{11}$$

This theoretical relation was in good agreement with an empirical relation ($\beta \approx 8/3 \approx 2.67$) derived by Poveda and Woltjer (1968) in the same paper.

2.4. THE KESTEVEN (1968) MODIFICATION

Kesteven (1968) derived another theoretical $\Sigma - D$ relation where the shell thickness was assumed to remain constant as the SNR expands. This assumption produces a new dependence of H on D, namely

$$H = H_0 \frac{D_0}{D}. \tag{12}$$

Adopting this dependence of the magnetic field and repeating the derivation presented by Shklovsky (1960a), the expression for Σ_ν in Eq. (3) may be written as

$$\Sigma_\nu \propto D^{-3\alpha-3}. \tag{13}$$

Once again assuming that $\alpha = 0.5$, we may express Σ_ν as

$$\Sigma_\nu \propto D^{-4.5}. \tag{14}$$

This theoretical $\Sigma - D$ relation was in good agreement with empirically-determined relations of the early 1970s, such as those found by Milne (1970) ($\beta = 4.5$) and Ilovaisky and Lequeux (1972) ($\beta = 4$).

2.5. THE DURIC AND SEAQUIST (1986) MODIFICATION

Finally, we examine the derivation of a theoretical $\Sigma - D$ relation presented by Duric and Seaquist (1986), who considered shell SNRs in the adiabatic expansion phase. By incorporating the Sedov (1959) blast wave solution for SNR expansion, the generation and evolution of a magnetic field as described by Gull (1973) and lastly the acceleration of relativistic electrons by shocks as formulated by Bell (1978ab), Duric and Seaquist (1986) derived a model for the evolution with time of radio emission from an SNR.

Following Gull (1973), we assume that the ambient magnetic field H is amplified in the convection zone: it is the convection zone which provides the environment in which relativistic electrons can radiate efficiently. As the convection zone expands with the SNR, the dependence of H on D may be expressed as

$$H(D) = H_0\Big(\frac{D}{D_0}\Big)^{-\delta}, \tag{15}$$

where $1.5 \leq \delta \leq 2$ (Gull, 1973; Fedorenko, 1983).

Bell (1978b) gives an analytic expression for the synchrotron emissivity arising from such a distribution in a shocked gas. In terms of H and the velocity v (here, equivalent to the expansion velocity), we may express the emissivity ε at a given frequency ν as

$$\varepsilon(H,v) = g(\alpha)\varrho_0 H^{1+\alpha} v^{4\alpha}\Big(1 + \Big(\frac{7\times 10^9}{v}\Big)^2\Big)^\alpha \nu^{-\alpha}, \tag{16}$$

where ϱ_0 is the ambient density, $v \propto t^{-3/5}$ (Sedov, 1959), and

$$g(\alpha) = 0.217\times 10^{-37\alpha}\Big(\frac{\alpha}{0.75}\Big)(1.435)^{-\alpha}. \tag{17}$$

As in Sect. (2.1), using Sedov (1959) adiabatic solution $D \propto t^{2/5}$, we obtain

$$\Sigma(D) \propto D^{-(6\alpha+\delta\alpha+\delta-1)}\Big(1 + \Big(\frac{3.06\times 10^{18}\varrho_0}{x^5 E_0}\Big)D^3\Big)^\alpha, \tag{18}$$

where $x = 2.3$ and $\nu = 1$ GHz. The previous equation can be simplified for $D \ll 1$ pc and $D \gg 1$ pc, which represent two limits of interest to the present work. Applying these limits we finally obtain:

$$D \gg 1\text{pc} \rightarrow \Sigma(D) \propto D^{-(\delta+\delta\alpha+3\alpha-1)}, \quad (19)$$
$$D \ll 1\text{pc} \rightarrow \Sigma(D) \propto D^{-(\delta+\delta\alpha+6\alpha-1)}. \quad (20)$$

We can use this $\Sigma - D$ relation to help determine values for the coefficient A and the exponent β (see Eq. 7). We set $\delta = 2$, consistent with turbulent magnetic field amplification (Gull, 1973; Fedorenko, 1983) and adiabatic expansion of the convection zone: once more we adopt a value of 0.5 for α. Using these values, we obtain a $\Sigma - D$ relation of the following form:

$$D \gg 1\text{pc} \rightarrow \Sigma_{1\text{GHz}} \propto D^{-3.5}, \quad (21)$$
$$D \ll 1\text{pc} \rightarrow \Sigma_{1\text{GHz}} \propto D^{-5}. \quad (22)$$

From Eqs. (19) and (20) we conclude that the exponent β depends only on α and δ. Using $\alpha = 0.5$ and $1.5 \leq \delta \leq 2$ we obtain $2.75 \leq \beta \leq 3.5$. For large diameter SNRs ($D \gg 1$ pc), the $\Sigma - D$ relation may be expressed as

$$\Sigma_{1\text{GHz}} \propto D^{-(2.75 \leq \beta \leq 3.5)}. \quad (23)$$

Updated empirical $\Sigma - D$ relations (Urošević, 2002; 2003) indicate shallower slopes ($\beta \approx 2$) than the slope predicted by this relation.

3. POSSIBLE THERMAL EMISSION FROM SNRS

We present here two models which describe thermal bremsstrahlung emission from SNRs at radio frequencies. The efficiency of the radio emission through the thermal bremsstrahlung process increases as the density increases. There are two basic criteria for the production of a significant amount of radio emission through the thermal bremsstrahlung process from an evolved SNR in the adiabatic phase of evolution: the SNR must be evolving in a dense environment and its temperature must be low (but always greater than the recombination temperature).

3.1. THERMAL RADIATION FROM AN EVOLVED SNR IN THE ADIABATIC PHASE

We first consider an evolved SNR with a diameter D=200 pc, a surface brightness of $\Sigma = 10^{-22}$ W m^{-2} Hz^{-1} sr^{-1} at 1 GHz and a synchrotron shell with a thickness of 10 pc, representing 5% of the SNR diameter. Our adopted values for these properties correspond to those measured or indicated by observations of several evolved Galactic SNRs, such as the four radio loops observed by Spoelstra (1972), the source OA 184, as observed by Routledge et al. (1986) and SNR HB9 (observed by Leahy et al., 1998). For the evolved SNR considered here, the assumed surface brightness corresponds to an emissivity $\varepsilon_{1\text{GHz}} = 1.1 \times 10^{-38}$ ergs cm^{-3} sec^{-1} Hz^{-1} while the total luminosity emitted from the entire volume of the shell at 1 GHz is $L_{1\text{GHz}} = 3.8 \times 10^{23}$ ergs sec^{-1} Hz^{-1}. By integrating over the radio domain from 10^7 to 10^{11} Hz and assuming a spectral index $\alpha = 0.5$, we calculate a luminosity of L=7.5×10^{33} ergs sec^{-1}.

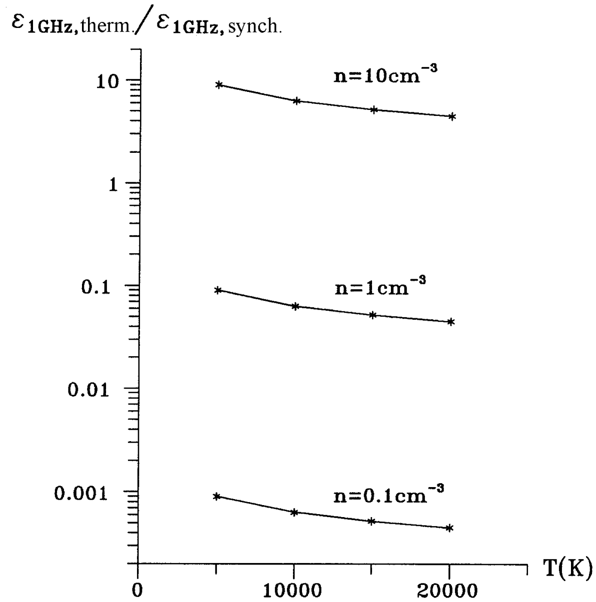

Figure 1: The ratios between the thermal and non-thermal (synchrotron) emissivities at 1 GHz as a function of frequency in the radio domain for the case of a warm ISM. The ratios are plotted for constant gas densities of $n = 0.1$, 1 and $10\,\mathrm{cm}^{-3}$.

We now estimate the amount of thermal bremsstrahlung radiation from the evolved SNR. For a density of $n \approx 1$ cm^{-3} and a temperature $T \approx 10^4$ K, thermal bremsstrahlung provides 10% of the energy ($\varepsilon_{1\mathrm{GHz}} \approx 10^{-39}$ ergs cm^{-3} sec^{-1} Hz^{-1}) at 1 GHz produced by the synchrotron mechanism at that frequency. We argue that thermal bremsstrahlung emission should represent a significant portion of the total radiation produced by SNRs. To illustrate this point, in Fig. 1 we have plotted the ratios of the thermal bremsstrahlung emissivity $\epsilon_{\mathrm{therm}}$ to the synchrotron emissivity $\epsilon_{\mathrm{synch}}$ (that is, the ratio of thermal to nonthermal emission) for the evolved SNR described here as a function of temperature at a frequency of 1 GHz for a range of values of gas density (0.1, 1.0 and 10 cm^{-3}). Notice that this ratio slowly decreases with increasing temperature for each value of the gas density. The thermal bremsstrahlung luminosity of the evolved SNR at 1 GHz is $L_{1\mathrm{GHz}} = 3.4 \times 10^{22}$ ergs sec^{-1} Hz^{-1}. Note that throughout the entire radio domain, the shell of the evolved SNR is optically thin: if we assume a gas density of $n \sim 1\,\mathrm{cm}^{-3}$ and a temperature of $T \sim 10^4$ K and a spectral index $\alpha = 0.5$ and integrate over the entire radio domain (from 10^7 to 10^{11}Hz), we calculate a luminosity of $L = 2.4 \times 10^{33}$ ergs sec^{-1}. The ratio of the thermal bremsstrahlung luminosity to the synchrotron luminosity of an evolved SNR expanding into a warm and dense interstellar medium is approximately one-third: clearly, thermal emission at radio frequencies from such sources should not be neglected.

3.2. THERMAL RADIATION FROM A RELATIVELY YOUNG SNR IN THE ADIABATIC PHASE

We now consider relatively young SNRs in the adiabatic phase of evolution and estimate the amount of thermal bremsstrahlung emission expected from these sources at radio frequencies. Observations have already detected thermal bremsstrahlung absorption or emission at radio wavelengths from four relatively young SNRs: γ Cygni (Zhang et al., 1997), the Cygnus Loop (Leahy and Roger, 1998), HB21 (Zhang et al., 2002) and 3C 391 (Brogan et al., 2002). The typical diameters of these SNRs are 20 pc, the mean thicknesses of their synchrotron shells are 1 pc (that is, about 5% of the SNR diameter) and their average surface brightnesses at 1 GHz are $\sim 10^{-20}$ W m^{-2} Hz^{-1} sr^{-1}. If we consider a young SNR with these typical values for its diameter, synchrotron shell thickness and surface brightness at 1 GHz, respectively, we calculate a synchrotron emissivity of $\varepsilon_{1\mathrm{GHz}} = 1.1 \times 10^{-35}$ ergs cm^{-3} sec^{-1} Hz^{-1} . As in the case of the evolved SNR, we consider thermal bremsstrahlung emission from the young SNR in the adiabatic phase. For a density $n \approx 1\,\mathrm{cm}^{-3}$ and temperature $T \approx 10^6\ K$ (the electron temperature of a young SNR behind the shock wave) the emissivity of the thermal bremsstrahlung at 1 GHz is ($\varepsilon_{1\mathrm{GHz}} \approx 10^{-40}$ ergs cm^{-3} sec^{-1} Hz^{-1}). Therefore, we can neglect the thermal bremsstrahlung emissivity when compared to the synchrotron emissivity in the case of a young SNR expanding into an ambient medium with a density $n \approx 1\,\mathrm{cm}^{-3}$. However, if the young SNR evolves within a dense molecular cloud with a density $n \approx 300\,\mathrm{cm}^{-3}$ (again assuming $T \approx 10^6$K), the synchrotron emissivity and the thermal bremsstrahlung emissivity are approximately the same ($\varepsilon_{1\mathrm{GHz}} \approx 10^{-35}$ ergs cm^{-3} sec^{-1} Hz^{-1}). Therefore, a young SNR in the adiabatic phase of evolution which is expanding within a very dense medium will produce a significant amount of thermal bremsstrahlung radiation. At 1 GHz, the young SNR is optically thin for $n \sim 1000\,\mathrm{cm}^{-3}$ and $T \sim 10^6$ K and radio emission may be detected from the entire shell of the source. Significant amounts of radio and X-ray emission will not be detected from the interior of the SNR compared to the amounts of radio and X-ray emission from the relatively dense and luminous shell. Note that this medium (with $n \approx 100 - 1000\,\mathrm{cm}^{-3}$ and $T \approx 10^6$ K) is unstable, and this instability leads to a very rapid evolution by the SNR into the adiabatic phase. Furthermore, a young SNR located in a dense molecular cloud will evolve through the adiabatic phase more rapidly than an SNR expanding into a low density medium. The young SNR will also cool rapidly because the shock wave will quickly decelerate via its interaction with the very dense molecular environment. It is possible that we will detect a significant amount of thermal emission in the radio domain only in the case of young SNRs in the adiabatic stage of evolution which are located in regions of high ambient density, such as molecular clouds.

4. THERMAL RADIO EMISSION FROM SNRS AND A MODIFIED THEORETICAL $\Sigma - D$ RELATION

It is clear from the different derivations presented in Sect. 2 that values for the exponent β as determined by empirical $\Sigma - D$ relations are significantly less than values expected by theory. We argue that perhaps the empirical – theoretical inconsistency can be at least partially explained by the omission of thermal radio emission from SNRs in derivations of theoretical $\Sigma - D$ relations. As noted before, Shklovsky (1960a) derived the $\Sigma - D$ relation (Eq. 7) directly from synchrotron radiation theory: in that derivation, the thermal emission at radio frequencies was neglected even though it probably does influence the $\Sigma - D$ relation and in this Section we will show how thermal radio emission could influence this relation. If we again derive the $\Sigma - D$ relation, this time taking into account thermal radiation from the ionized gas cloud (that is, bremsstrahlung from free electrons moving through the fields established by positively charged ions) and associate it with relations previously derived for the synchrotron mechanism, we may obtain a $\Sigma - D$ relation with a significantly reduced value for β. Discussions on the effects of thermal radio emission on the $\Sigma - D$ relation have already been presented by Urošević et al. (2003ab).

The addition of the bremsstrahlung and synchrotron emissivities (that is, $\varepsilon_{\nu,\mathrm{therm.}} \propto R^{\theta_1}$ and $\varepsilon_{\nu,\mathrm{synch.}} \propto R^{-\theta_2}$, where θ_1 and θ_2 are constants which represent the thermal and synchrotron components of the exponent θ in the $\Sigma - D$ relation, respectively) is complicated because the term $\theta = \theta_1 - \theta_2$ cannot be calculated analytically. If the total emissivity ε_ν is defined to be the sum of the synchrotron emissivity and the thermal bremsstrahlung emissivity (that is, $\varepsilon_\nu = \varepsilon_{\nu,\mathrm{synch.}} + \varepsilon_{\nu,\mathrm{therm.}}$) and if we consider only large radii of the expanding SNR, the total emissivity as determined only by analytic methods is dominated by one of the two emissivities rather than a combination of the two emissivities and the $\Sigma - D$ relations derived in this manner all differ very significantly from the empirical relations. Because an analytic solution does not exist, we use here an approximate method – the convolution method – to combine the two emissivities from the different emission mechanisms. In this Section, the combined emissivity that we derive through the convolution method will yield a new $\Sigma - D$ relation where the slope will be reduced and more closely approximate the empirical relations.

4.1. THE $\Sigma - D$ RELATION FOR THERMAL RADIATION FROM AN IONIZED GAS CLOUD: THE CASE OF CONSTANT TEMPERATURE

For the derivation of the $\Sigma - D$ relation based on thermal emission from an ionized gas cloud, we will apply an algorithm applied by Shklovsky (1960a) for the derivation of the relation for synchrotron emission from SNRs. Based on the theory of the bremsstrahlung radiation applied to an ionized gas cloud, we adopt a volume emissivity of the following form (Rohlfs and Wilson, 1996):

$$\varepsilon_\nu = \frac{4}{3}\frac{Z^2 e^6}{c^3}\frac{N_i N_e}{m^2}\Big(\frac{2m}{\pi k T}\Big)^{\frac{1}{2}} \ln\frac{p_2}{p_1}, \tag{24}$$

where T is thermodynamic temperature of the medium and N_i and N_e are the volume concentrations of the ions and electrons, respectively. The mass and charge of the electron are denoted as m and e, respectively, while Z represents the atomic number. The collision parameter p represents the shortest distance between an ion and an electron in the course of the electron's accelerated motion in the ion field. The interval (p_1, p_2) spans all of the permitted values for the collision parameter: in this case, the upper limit for p_2 corresponds to the average distance between the ions (in other words, the Debye length) while limiting values for the parameter p_1 require quantum mechanical considerations, which are traditionally collected in the Gaunt factor. Eq. (24) is derived for interactions with large values of the collision parameter: therefore, considering the typical energy levels, interactions among the particles are weak. For this reason, bremsstrahlung theory has been developed for straight-line motion of the electron in the ion field where the parameter p represents the shortest distance between interacting particles. The $\Sigma - D$ relation derived under these circumstances is applicable for diffuse media where the particles are far away from each other and the energy change of the accelerated particles is small.

We assume that the temperature and density of the particles does not change with changing distance from the center of the object. This is consistent with the model for the hot interstellar medium (HIM) described by McKee and Ostriker (1977) and therefore the collision parameter is independent of the radius. From Eqs. (3) and (24), we obtain

$$\varepsilon_\nu = \mathrm{const} \quad \text{and} \quad \Sigma_\nu \propto D. \tag{25}$$

From inspection of this relation, we notice that as the size of the SNR increases, its surface brightness also increases: this result is consistent with our expectations for an optically thin medium. Urošević et al. (2003a) showed that the medium is transparent for the specific frequency used for the construction of the $\Sigma - D$ relation (that is, 1 GHz).

4.2. THE $\Sigma - D$ RELATION FOR SYNCHROTRON RADIATION AND THERMAL BREMSSTRAHLUNG RADIATION FROM AN IONIZED GAS CLOUD: THE CASE OF CONSTANT TEMPERATURE

The final result of the theory given by Shklovsky (1960a) is $\varepsilon_\nu \propto D^{-7}$. The relation presented by Shklovsky (1960a) is scaled by the maximum value $\varepsilon_{\max}$ of the emissivity of SNRs at the outset of their evolution: we can therefore express the normalized emissivity $\varepsilon_{\mathrm{norm}}$ as

$$\varepsilon_{\mathrm{norm}} = \frac{\zeta_1}{\varepsilon_{\max}} D^{-7}, \tag{26}$$

where ζ_1 is a constant which contains the portion of the synchrotron emissivity which does not depend on D. The maximum value of the emissivity $\varepsilon_{\max}$ corresponds to the minimum radius of the SNR ($D_{\min}$), while the minimum value of the emissivity corresponds to the maximum radius of the SNR ($D_{\max}$) which in turn corresponds to an SNR at the end of its evolution (that is, the dissipation phase). If the emissivity from Eq. (25) is convoluted with emissivity from Eq. (26), we obtain the following integral expression for ε as a function of time:

$$\varepsilon(t) = \frac{\zeta_1}{\varepsilon_{\max}} \int\limits_{D_{\min}}^{D_{\max}} \frac{\zeta_2}{(t-D)^7} dD. \tag{27}$$

Here, ζ_2 is another constant which contains the portion of the thermal bremsstrahlung emissivity which does not depend on D. For a qualitative analysis, this integral may be approximated as

$$\varepsilon(t) \approx \frac{\zeta_3}{\varepsilon_{\max}} \int_0^\infty \frac{1}{(t-D)^7} dD. \tag{28}$$

Here ζ_3 is the product of the constants ζ_1 and ζ_2 and the integral is evaluated over the range of D=0 through $D = +\infty$ to describe the expansion of the SNR from very small values (nearly zero) at the beginning of the explosion to very large values (limiting case is ∞) at the end of its lifetime. This integral has the following solution:

$$\varepsilon(t) \propto \int_0^\infty \frac{1}{(t-D)^7} dD \propto t^{-6}. \tag{29}$$

Combining this equation and Eq. (3) gives:

$$\Sigma_\nu \propto D^{-5}, \tag{30}$$

Therefore, the introduction of the thermal component to the relation derived by Shklovsky (1960a) leads to a form of the $\Sigma - D$ relation with a significantly flatter slope.

The theoretical interpretation presented by Duric and Seaquist (1986) yields a relation for evolved SNRs in the following form: $\Sigma \propto D^{-3.5}$ ($\varepsilon_\nu \propto D^{-4.5}$). If we assume a synchrotron shell model for the SNR that is consistent with the model of McKee and Ostriker (1977) (that is, for an SNR with low density interior), the thermal flux from the low density interior can be neglected in comparison with the flux from the denser cool X-ray gas (that is, the warm medium with $T \sim 10^4$ K, $n \sim 1$ cm^{-3} located in the inner rim of shell) because the particle concentration is greater in the shell, resulting in greater efficiency in thermal radiation from the shell (Eq. 24). The convolution integral in this case is:

$$\varepsilon(t) \propto \int_0^\infty \frac{1}{(t-D)^{4.5}} dD \propto t^{-3.5}. \tag{31}$$

Similar to the previous convolution example, this equation becomes:

$$\Sigma_\nu \propto D^{-2.5}. \tag{32}$$

This relation has a value for β which is closest to the Galactic empirical $\Sigma - D$ relation obtained by Case and Bhattacharya (1998) ($\beta = 2.4$), once again assuming an average value of 0.5 for the spectral index α.

4.3. THE $\Sigma - D$ RELATION FOR THERMAL BREMSSTRAHLUNG RADIATION FROM AN IONIZED GAS CLOUD: THE CASE OF VARIABLE TEMPERATURE

Since the SNR is assumed to be in the adiabatic phase (*i.e.*, the SNR is cooling adiabatically as it expands), we start with the adiabatic equation, expressed as

$$TV^{\gamma-1} = const. \tag{33}$$

In the case of a spherical cloud and assuming $\gamma = \frac{5}{3}$ (*i.e.*, assuming that the gas in the SNR interior behaves like an ideal gas), we obtain the following dependence of temperature with respect to cloud radius:

$$T \propto D^{-2}. \tag{34}$$

Assuming a constant density insures that the collision parameter is independent of the radius of the cloud (once again see Eq. 24). Substituting Eq. (34) into Eq. (24), we may therefore express the emissivity as

$$\varepsilon_\nu \propto D. \tag{35}$$

According to the Eq. (3), we then have:

$$\Sigma_\nu \propto D^2. \tag{36}$$

Since it is well-known that SNRs also possess relativistic electrons which emit synchrotron radiation, we may consider these relativistic particles to derive another constraint on the dependence of emissivity on radius. If the total energy of a particle is much greater than its rest mass, the rest mass may therefore be ignored when considering the particle's total energy. Similar to the case of an ideal gas, if we neglect relativistic corrections for temperatures $T \leq 10^6$K (Rybicki and Lightman, 1979) and set $\gamma = \frac{4}{3}$, we derive the following expression for emissivity with respect to cloud radius:

$$\varepsilon_\nu \propto D^{0.5}. \tag{37}$$

Therefore, following the model presented by McKee and Ostriker (1977), these relations yield a $\Sigma - D$ relation for thermal emission from SNRs of the following form:

$$\Sigma_\nu \propto D^{1.5 \leq -\beta \leq 2.0}. \tag{38}$$

4.4. THE $\Sigma - D$ RELATION FOR SYNCHROTRON RADIATION AND THERMAL BREMSSTRAHLUNG RADIATION FROM AN IONIZED GAS CLOUD: THE CASE OF VARIABLE TEMPERATURE

In this Sect. we derive a $\Sigma - D$ relation based on the combination of synchrotron radiation and thermal bremsstrahlung radiation from an ionized gas cloud (that is, a theoretical construct for an SNR) in the case where the temperature varies throughout the cloud.

The theoretical model described by Duric and Seaquist (1986) yields a $\Sigma - D$ relation in the following form: $\Sigma \propto D^{-3.5}$ (for evolved SNRs) and $\Sigma \propto D^{-5}$ ($\varepsilon \propto D^{-6}$, for young SNRs). As in the case considered in Sect. 4.2, if we assume a shell model for the SNR we can expect thermal flux from the shell. In this case, flux from the low density interior may be neglected because the particle concentration is higher in the shell, resulting in a greater efficiency of thermal radiation from the ionized gas cloud. Relativistic particles in the shell (and probably in the X-ray emitting region) will contribute, thereby introducing the thermal component to the total emissivity as is shown in Eq. (37). The convolution integrals (in the cases of both evolved and young SNRs) are

$$\text{Evolved SNRs} \rightarrow \varepsilon(t) \propto \int_0^\infty \frac{D^{0.5}}{(t-D)^{4.5}} dD \propto t^{-3} \tag{39}$$

$$\text{Young SNRs} \rightarrow \varepsilon(t) \propto \int_0^\infty \frac{D^{0.5}}{(t-D)^{6}} dD \propto t^{-4.5} \tag{40}$$

Similar to the previous convolution, we obtain

$$\text{Evolved SNRs} \rightarrow \Sigma_\nu \propto D^{-2} \tag{41}$$

$$\text{Young SNRs} \rightarrow \Sigma_\nu \propto D^{-3.5} \tag{42}$$

If we once again assume an average spectral index for SNRs of $\alpha = 0.5$, the first relation has a value for β which is closest to the latest "shallower master" empirical $\Sigma - D$ relations (Urošević, 2002; 2003). However, the second relation yields a value for β which is closer to that derived for the very rich young radio SNR population found in M82 (Huang et al., 1994), that is $\beta = 3.4$ (Urošević et al., 2004; Arbutina et al., 2004).

The surface brightness relation given in Eq. (41) for evolved SNRs decreases with an inverse square-law dependence as the radius of the SNR increases, giving a solution for the simple spherical expansion of the SNR as the luminosity remains constant. This effect – the independence of luminosity with respect to SNR diameter – has already been noted and described by Stanković et al. (2003) and Arbutina et al. (2004). The relation $\Sigma_\nu \propto S_\nu/\theta^2$ (where θ is the angular diameter), when combined with $D \propto \theta d$ (where d is distance to the remnant) and $L_\nu \propto S_\nu d^2$ (where L_ν is the radio luminosity of the remnant per unit frequency interval), yields the following relation:

$$\Sigma_\nu \propto L_\nu D^{-2}. \tag{43}$$

In the case where luminosity is independent of radius, this relation simplifies to the relation given in Eq. (37).

5. SUMMARY

(i) We have presented a brief review of theoretical $\Sigma - D$ relations previously published in the literature: this review helps to emphasize the inconsistencies between theoretical and empirical relations.

(ii) We have considered the thermal emission from SNRs at radio frequencies and included this emission in a model of the total radio emission from an SNR. We also developed two models describing young and evolved SNRs in the adiabatic phase of evolution, both of which emit significant amounts of thermal bremsstrahlung emission at radio frequencies.

(iii) By modifying the theory presented in Sect. (2) through the introduction of the thermal bremsstrahlung mechanism to describe SNR evolution in the adiabatic phase, we have derived $\Sigma - D$ relations which are in closer agreement to the empirical results than previous theoretical models.

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