Zbornik radova 7(15)

THREE TOPICS FROM CONTEMPORARY MATHEMATICS

Matematički institut SANU

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Editor:
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PREFACE

The aim of *Zbornik radova* is to foster further growth of pure and applied mathematics, publishing papers which contain new ideas and scopes in the mathematics. The papers have to be prepared in such a manner that they can inform readers in a favourable way, introducing them in a narrower field of mathematical theories pointing at research possibilities. It can be for the individual use or for discussions in College or University seminars.

We are open for contacts and cooperations.

Bogoljub Stanković
Editor-in-Chief

Contents

Aleksandar T. Lipkovski

ALGEBRAIC GEOMETRY
(Selected Topics)

Contents

0. Introduction

First version of the present text appeared as one-semester course lecture notes in algebraic geometry. The course for graduate students of mathematics, of geometrical, topological and algebraic orientation, took place in the spring semester of 1994/95, and was organized on the initiative of Z. Marković, head of Mathematical Institute in Belgrade, with great support from my colleagues from the Belgrade GTA Seminar[1], especially R. Živaljević and S. Vrećica.

I had a difficult task. In a short course one should have reached some relevant topics of algebraic geometry. Basics of algebraic geometry require an ample preliminary material, mostly from commutative algebra, homological algebra and topology. I tried to avoid this and to include only a minimal amount of such material. Consequently, the style of writing is laconic, with many references to existing (excellent) textbooks in algebraic geometry, but on the other side, it is consistent, in order to be readable, with some effort of course. The scope of the course should include some of the interesting and important results in algebraic geometry. Two such results are included, both classical but very important: the 27 lines on a cubic surface and the Riemann–Roch theorem for curves. I leave to the reader to judge, whether my task has been solved, and to which extent.

The present text could serve different purposes. It could be used as an introduction for nonspecialists, who would like to understand what is going on in algebraic geometry, but are not willing to read long textbooks. It could also be used as a digest for students, who are preparing to take a serious course in algebraic geometry. Nowadays, algebraic geometry became an indispensable tool in many closely related or even far standing disciplines, such as theoretical physics, combinatorics and many others. Specialists in these fields may also find this text useful.

1. Rational algebraic curves

In the course of Calculus one evaluates indefinite integrals of the form

$$\int R(x, \sqrt{ax^2 + bx + c})dx \tag{1}$$

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[1]GTA stands for: Geometry, Topology, Algebra

where $R(x, y)$ is a rational function with two arguments. These are the simplest integrals with so called quadratic irrationalities. Some readers remember that these integrals are being calculated with the help of so-called Euler substitutions. There are three such substitutions (the types are not distinct):

Type I. If $a > 0$, one puts $\sqrt{ax^2 + bx + c} = t - \sqrt{a}x$

Type II. If $c > 0$, one puts $\sqrt{ax^2 + bx + c} = xt + \sqrt{c}$

Type III. If the polynomial has real roots λ and μ, $ax^2 + bx + c = a(x - \lambda)(x - \mu)$, and we use $\sqrt{ax^2 + bx + c} = t(x - \lambda)$.

In all three cases, the differential $R(x, y)dx$ is being rationalized and the integral evaluated in elementary functions.

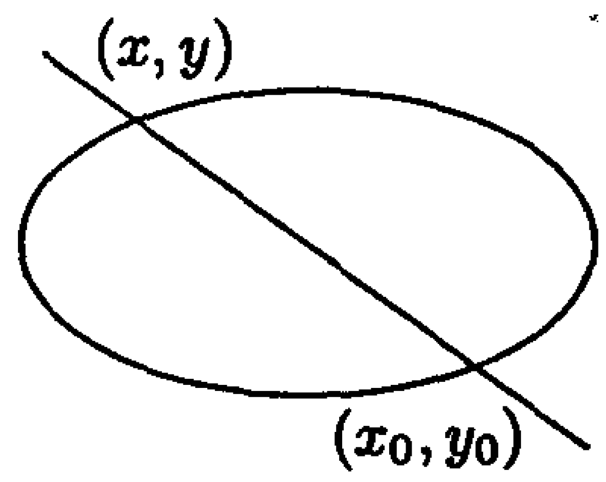
The Euler substitutions are described in traditional calculus textbooks, such as [30, p. 59]. A few students understand what is the real meaning of these substitutions. However, they have fine geometrical interpretation. Introduce the curve of second order

$$y^2 = ax^2 + bx + c \tag{2}$$

and its point (x_0, y_0). After the translation to that point, the equation of the curve is

$$(y - y_0)^2 + 2y_0(y - y_0) = a(x - x_0)^2 + 2ax_0(x - x_0) + b(x - x_0)$$

Let $y - y_0 = t(x - x_0)$ be the line through that point with variable slope t (see the figure)



The other, variable intersection point of the line and the curve (2) is obtained from the system

$$(t^2 - a)(x - x_0) = (2ax_0 + b) - 2y_0 t$$
$$y - y_0 = t(x - x_0)$$

whose solution (x, y) depends rationally on t (for almost all t):

$$x = x_0 + \frac{(2ax_0 + b) - 2y_0 t}{t^2 - a}$$
$$y = y_0 + t(x - x_0)$$

After substitution in the integral, one obtains rational function of argument t and the integral evaluates easily:

$$\int R(x, \sqrt{ax^2 + bx + c})dx = \int R(x(t), y(t))x'(t)dt = \cdots$$

Euler substitutions can be deduced from this general algorithm by different special choices of the point (x_0, y_0).

Type III. The point $(x_0, y_0) = (\lambda, 0)$ lies on the curve and we put $y = t(x - \lambda)$.

Type II. The point $(x_0, y_0) = (0, \sqrt{c})$ lies on the curve and we put $y - \sqrt{c} = xt$.

Type I. Here the situation is slightly more complicated. In the case $a > 0$ the curve (2) is a hyperbola with asymptotic directions $(\sqrt{a}, \pm 1)$. As starting point (x_0, y_0), one takes the point at the infinity of one of these two directions. Then the lines through that point are exactly the lines $y = \sqrt{a}x + t$ parallel to the asymptote. Each of these lines intersects the curve in one more point (x, y) and we use $y = \sqrt{a}x + t$.

From previous discussion one can see that the expressibility of the integral (1) in elementary functions is based on the following specific property of the conic (2). There exist rational functions $x = \xi(t)$; $y = \eta(t)$ of one argument t such that for different parameter values t, the corresponding point $(\xi(t), \eta(t))$ lies on the curve. In this way one obtains all (but one) points of the curve. More specifically, for each point $(x, y) \neq (x_0, y_0)$ on the curve (2) it is sufficient to draw a line through (x_0, y_0) with the slope $t = (y - y_0)/(x - x_0)$. We say that in such case curve (2) has rational parametrization. One can easily show that every plane curve of second order has a rational parametrization. Such curves have been called *unicursal.* Today one rather uses the term *rational curves.*

Let us now apply the above principle of evaluation of the integral (1) with irrationalities of the type $\sqrt{P(x)}$ where $P(x)$ is a polynomial of degree greater than 2. In this case, along with the integral

$$\int R(x, \sqrt{P(x)})dx \tag{3}$$

one should consider the curve

$$y^2 = P(x) \tag{4}$$

Here we have different behavior. Some of the curves (4) do admit rational parametrization, and some of them do not.

Examples. 1. It is obvious that the curve $y^2 = x^3$ has rational parametrization (which one?).

2. The curve $y^2 = x^3 + x^2$ also has rational parametrization $x = t^2 - 1$, $y = t(t^2 - 1)$, It is obtained when one finds the intersection points of the curve and the lines $y = tx$ through $(0, 0)$.

3. The curve $y^2 = x^3 + ax^2 + bx + c$ has rational parametrization if and only if the polynomial $x^3 + ax^2 + bx + c$ has multiple root.

When rational parametrization of the curve (4) exists, the integral (3) could be transformed into integral of rational function. How one evaluates the integral when there is no such parametrization? Interesting and complicated theory is obtained already when degree of the polynomial $P(x)$ equals 3 or 4. It is sufficient to consider only the latter case, since degree 3 could be transformed to degree 4 by rational transformations.

Example. For a given curve

$$y^2 = x^3 + ax^2 + bx + c \tag{5}$$

the right-hand side polynomial of degree 3 has at least one real root. Applying the translation along x axis one could make this root 0 i.e., one could put $c = 0$. After substitution $y = tx$ we get

$$x^2 + (a - t^2)x + b = 0$$
$$\left(x + \frac{a - t^2}{2}\right)^2 + b - \left(\frac{a - t^2}{2}\right)^2 = 0$$

The rational parametrization

$$x = u - \frac{a - t^2}{2} \quad y = t\left(u - \frac{a - t^2}{2}\right) \tag{6}$$

transforms the curve (5) in the curve of degree 4 with equation

$$u^2 = \frac{1}{4}t^4 - at^2 + \left(\frac{a^2}{4} - b\right)$$

The parametrization (6) is rationally invertible — it has a rational inverse

$$t = \frac{y}{x}, \quad u = x + \frac{a - (y/x)^2}{2}$$

This is a very important fact, as we will see later.

2. Plane algebraic curves. Polynomials in many variables

Now we should make the term "plane algebraic curve" more precise. Let K be field, so-called ground field, and $K[x, y]$ polynomial ring in two variables with coefficients in K.

Definition. Plane algebraic curve in the affine plane $K^2 = \mathbb{A}^2_K$ is the set of points in the plane defined by algebraic polynomial equation

$$X = \{(x, y) \in K^2 \mid f(x, y) = 0\}$$

where $f(x,y) \in K[x,y]$. This set is denoted $X = V(f)$.

Even such simple definition leads to several problems. Naturally, one would like to establish a one-to-one correspondence between sets and their equations. However, already in the real analytic geometry there exist examples where basically different equations define the same set, or equations define sets that seem unnatural to call "curves". So, in the real plane, equations $x = 0$ and $x^2 = 0$ define the x-axis, equations $x^2 + y^2 = 0$, $(x^2 + y^2)^2 = 0$ and $x^6 + y^4 = 0$ define the point $(0,0)$, and equation $x^2 + y^2 + 1 = 0$ defines the empty set. Immediately, two questions arise:

first, how to treat objects which are produced by this definition but do not agree with our intuitive notion of "curve";

second, in which extent is the set $X = V(f)$ determined by the polynomial f and how to modify the definition in order to get one-to-one correspondence.

These problems are present already in the course of analytical geometry for first-year undergraduates. Problem with curves which "are not curves" is being bypassed by calling them "degenerate", etc. The question, in which extent is equation determined by the set of points, is usually not treated at all. The first problem can be easily solved: one should consider complex numbers instead of real ones. This is known as the *complexification process*. In the general case, one should take the algebraic closure of the given field. The "empty" curves like $V(x^2 + y^2 + 1)$ then disappear. In the sequel the ground field K will always be algebraically closed, unless the opposite is explicitly stated. Usually, it will be the field of complex numbers $\mathbb{C}$.

As to the second problem, it is being answered by the following fact, known as Study's lemma[2]. Note that when polynomial f divides polynomial g, then $g = fh$ and every root of f is at the same time the root of g, that is $V(f) \subset V(g)$. In the case of algebraically closed field the converse is also true.

Lemma. *Let K be algebraically closed field and $f(x,y) \in K[x,y]$ irreducible polynomial. If the polynomial $g(x,y) \in K[x,y]$ has a zero in every point of the curve $X = V(f)$ (i.e., if $V(f) \subset V(g)$), then f divides g.*

Proof. Let $g \neq 0$ (in the opposite, f divides g). Then also $f \neq 0$. If f is constant, then f divides g. Suppose f is not constant, but an actual polynomial, say in y: $f(x,y) = a_0(x)y^n + \cdots \in (K[x])[y]$ with $a_0 \neq 0$, $n > 0$. Let us show that g is then also an actual polynomial in y. If $g = g(x) \in K[x]$, then $0 \neq a_0 g = a_0(x)g(x) \in K[x]$ and there should exist $\xi \in K$ such that $a_0(\xi)g(\xi) \neq 0$. Since K is algebraically closed, there exists $\eta \in K$ such that $f(\xi,\eta) = 0$, which is a contradiction to the choice of ξ. Therefore, $g(x,y) = b_0(x)y^m + \cdots \in (K[x])[y]$ with $b_0 \neq 0$, $m > 0$.

Let now $R = R(f,g) \in K[x]$ be the resultant of polynomials f and g with respect to y. Let $\xi \in K$ be such that $a_0(\xi) \neq 0$. Since K is algebraically closed, there exists $\eta \in K$ such that $f(\xi,\eta) = 0$. Then also $g(\xi,\eta) = 0$ and therefore $R(\xi) = 0$. In such way, $a_0 R = 0 \in K[x]$. Since $a_0 \neq 0$, it must be $R = 0$, which

[2]Eduard Study (1862–1930), German geometer (Fubini–Study metric in projective space)

means that f and g have a nontrivial common factor. However, f is irreducible and therefore f must divide g.

Corollary. *Irreducible factors of curve's equation are determined uniquely (up to ordering). If* $f = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ *is a factorization in irreducible factors, then* $V(f) = V(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}) = V(p_1 p_2 \dots p_k)$.

This proof can easily be generalized for more than two variables.

We see that in the case of algebraically closed field, one of all possible equations of a given curve is determined uniquely by the condition that it has no multiple factors i.e., it is reduced.

Study's lemma is a special case of a very important theorem, which could be considered also as a generalization of the main theorem of algebra. It is a famous Hilbert's[3] Nullstellensatz, which in its classical version states that "nontrivial" system of algebraic equations over an algebraically closed field always has a solution. Here "nontrivial" means that it is not possible to algebraically deduce a contradiction from the system. More precisely:

If the field $K$is algebraically closed and $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ are polynomials in n variables such that there are no polynomials $g_1, \dots, g_k \in K[x_1, \dots, x_n]$ for which it would be $g_1 f_1 + \dots + g_k f_k = 1$, then the system of algebraic equations

$$f_1(x_1, \dots, x_n) = 0$$
$$\dots$$
$$f_k(x_1, \dots, x_n) = 0$$

has a solution.

It is known that polynomial ring in one variable over a field is a PID (principal ideal domain): it is even Euclidean. This is a consequence of the existence of Euclidean gcd division algorithm. However, in polynomial rings in two variables this is no more true. For instance, ideal (x, y) cannot be generated by single polynomial. However, in 1868 Gordan[4] proved that it is possible to find a finite generating set of polynomials in every ideal. His proof was constructive – he described a construction of such basis. Many mathematicians tried to generalize Gordan's construction to the case of more than two variables, but nobody could overcome computing difficulties, and for twenty years this problem, known as Gordan's problem, remained open. In 1888 Hilbert proved in his famous basis theorem that every ideal in the polynomial ring with n variables has a finite basis. His proof was existential, not constructive. It has been said that Gordan, after he saw Hilbert's proof, said: "Das ist nicht Mathematik. Das ist Theologie!"[5]. Only when later Hilbert found a constructive proof, Gordan was satisfied and said that theology has its merits. Only after this, existential proofs in mathematics became legitimate.

[3]David Hilbert (1862–1943), German mathematician. Most famous for his list of problems for 20. century

[4]Paul Albert Gordan (1837–1912), German mathematician

[5]"This is not mathematics. This is theology!"

The notion of resultant was used in the proof of Study's lemma. Let us briefly describe it here.

Let A be a UFD (unique factorization domain, factorial ring), say polynomial ring over a field, and let $f, g \in A[x]$ be two polynomials with coefficients in A, $f = a_0x^m + \cdots + a_m$, $g = b_0x^n + \cdots + b_n$ (we allow also the possibility $a_0, b_0 = 0$). Since $A[x]$ is also a UFD, we are interested in their common divisors. The definition of a common divisor easily leads to the following lemma.

Lemma. *Polynomials f and g have nontrivial common divisor $\Leftrightarrow$ there exist polynomials $u, v \in A[x]$, $u, v \neq 0$ such that $\deg u < \deg f$, $\deg v < \deg g$ and $vf = ug$.*

If one writes this condition explicitly, one has $u = c_0x^{m-1} + \cdots + c_{m-1}$, $v = d_0x^{n-1} + \cdots + d_{n-1}$ and from equality $vf = ug$ one deduces

$$\sum_{i=0}^{m} a_i x^{m-i} \cdot \sum_{j=0}^{n-1} d_j x^{n-j-1} - \sum_{i=0}^{n} b_i x^{n-i} \cdot \sum_{j=0}^{m-1} c_j x^{m-j-1} = \cdots$$

$$= \sum_{k=0}^{m+n-1} \sum_{i+j=k} (a_i d_j - b_i c_j) x^{(m+n-1)-k} = 0$$

and comparing the coefficients for x one obtains system of $m+n$ linear equations

$$\sum_{i+j=k} (a_i d_j - b_i c_j) = 0 \quad (k = 0, \ldots, m+n-1)$$

or explicitly

$$\begin{array}{llllll}
a_0d_0 & & & -b_0c_0 & & = 0 \\
a_1d_0 & +a_0d_1 & & -b_1c_0 & -b_0c_1 & = 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
a_md_0 & & & & -b_1c_{m-1} & = 0 \\
 & & a_0d_{n-1} & -b_{n-1}c_0 & & = 0 \\
 & & a_1d_{n-1} & -b_nc_0 & & = 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
 & & a_md_{n-1} & & -b_nc_{m-1} & = 0
\end{array}$$

with $m+n$ indeterminates $c_0, \ldots c_{m-1}, d_0, \ldots, d_{n-1}$. This system has nontrivial solution if and only if its determinant equals 0.

Definition. Determinant of this system, i.e., the determinant

$$R(f,g) = \left| \begin{array}{cccccccc}
a_0 & a_1 & \ldots & a_m & & & & \\
 & a_0 & a_1 & \ldots & a_m & & & \\
 & & & & \ldots & & & \\
 & & & a_0 & a_1 & & \ldots & a_m \\
b_0 & b_1 & \ldots & b_n & & & & \\
 & b_0 & b_1 & \ldots & b_n & & & \\
 & & & & \ldots & & & \\
 & & & b_0 & b_1 & & \ldots & b_n
\end{array} \right| \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}b\\b\\b\\b\end{array}}\right\} m \end{array}$$

is called *resultant* of polynomials f, g with respect to x.

The resultant is a polynomial in coefficients a_i and b_j of degree $m+n$, homogeneous in each group of indeterminates. Preceding discussion proved the following theorem.

Theorem. *Polynomials f and g have nontrivial common divisor if anf only if $R(f,g)=0$.*

Applications. 1. Solution of systems with two polynomial equations. Let $f, g \in K[x,y] = (K[x])[y]$ be two polynomials and $R(f,g) = R(x) \in K[x]$. If (x_0, y_0) is a solution of the system

$$f(x,y) = 0, \quad g(x,y) = 0$$

then $R(x_0) = 0$. One consequence is, if the system has infinitely many solutions, then polynomials f and g have a nontrivial common divisor $h = \gcd(f,g)$ and the ideal $(f,g) = (h)$ is principal.

2. Parameter elimination. Suppose curve X is described by its rational parametrization

$$\begin{aligned} x &= P_1(t)/Q_1(t) \\ y &= P_2(t)/Q_2(t) \end{aligned}$$

Let $f(x,t) = P_1(t) - Q_1(t)x$, $g(y,t) = P_2(t) - Q_2(t)y$ and let $R = R(f,g) \in K[x,y]$ be the resultant of these polynomials. Then

$$(x_0, y_0) \in X \Leftrightarrow \exists t_0 : f(x_0, t_0) = g(y_0, t_0) = 0 \Leftrightarrow R(x_0, y_0) = 0$$

This means that the equation of X is $R(x,y) = 0$.

Example. Find the equation of the curve that has a parametrization $x = t^2$, $y = t^3 - t$. Here $f = t^2 - x$, $g = t^3 - t - y$ and

$$R(f,g) = \begin{vmatrix} 1 & 0 & -x & 0 & 0 \\ 0 & 1 & 0 & -x & 0 \\ 0 & 0 & 1 & 0 & -x \\ 1 & 0 & -1 & -y & 0 \\ 0 & 1 & 0 & -1 & -y \end{vmatrix} = y^2 - x^3 + 2x^2 - x$$

Therefore, the equation of X is $y^2 = x^3 - 2x^2 + x$. One could obtain it also without use of the resultant, but the present method is generally applicable.

3. Transcendence degree. Hilbert's Nullstellensatz

Let K be a field and L its extension. Subset $S \subset K$ is *algebraically independent* over K if there is no polynomial relation between elements in S, i.e., if there is no polynomial $f \in K[x_1, \ldots, x_n]$ such that $f(c_1, \ldots, c_n) = 0$ for some

$c_1, \dots, c_n \in S$. Family of algebraically independent sets is ordered by inclusion. Maximal elements of this family i.e., maximal algebraically independent sets are called *transcendence bases* of L over K. For example, in the field of rational functions $K(x_1, \dots, x_n)$ the set $\{x_1, \dots, x_n\}$ is one of its transcendence bases over K. Note that, if S is a transcendence basis of L over K, then L is algebraic over $K(S)$. Main statements about transcendence bases are analogous to corresponding statements about (linear) bases in vector spaces over fields.

Theorem. *A. Let L be the field generated over K by the set M and let $N \subset M$ be its algebraically independent subset. There exists a transcendence basis B between N and M. In other words, algebraically independent set can be extended to transcendence basis by adding elements from a given generating set.*

Theorem. *B. Every two transcendence bases of the field L over K have the same cardinality.*

Cardinality of any (and every) transcendence basis of L over K is called *transcendence degree* of that field extension.

Our next goal is to prove Hilbert's Nullstellensatz. That will be done in few steps.

Step 1. Let K be algebraically closed and L its finitely generated extension. There exist elements $z_1, \dots, z_{d+1}$ in L such that

1. they generate L over K,
2. $z_1, \dots, z_d$ are algebraically independent,
3. z_{d+1} is algebraic over $K(z_1, \dots, z_d)$.

Proof. Follows from the known theorem on the primitive element.

Step 2. Let K be algebraically closed field and $F_1, \dots, F_m \in K[t_1, \dots, t_n]$ polynomials. If the system of equations $F_1 = 0, \dots, F_m = 0$ has a solution in finitely generated extension L over K, then it has a solution in K also.

Proof. L is of the form $L = K(x_1, \dots, x_r, \eta)$ where $x_1, \dots, x_r$ are algebraically independent over K and η is algebraic over $K(x_1, \dots, x_r)$. Let $F(x_1, \dots, x_r, y) \in K(x_1, \dots, x_r)[y]$ be the minimal polynomial of η. Let now $(\xi_1, \dots, \xi_n)$ be the solution of the system in L^n. One has $\xi_i = C_i(x_1, \dots, x_r, \eta)$ for some polynomials $C_i(x_1, \dots, x_r, y) \in K(x_1, \dots, x_r)[y]$. Since F is minimal, there exist polynomials Q_i such that

$$F_i(C_1(x_1, \dots, x_r, y), \dots, C_n(x_1, \dots, x_r, y)) = F(x_1, \dots, x_r, y)Q_i(x_1, \dots, x_r, y)$$

identically with respect to $x_1, \dots, x_r, y$. Since K is infinite, there exist elements $\alpha_1, \dots, \alpha_n \in K$ such that all denominators in coefficients of polynomials $F, Q_1, \dots, Q_r, C_1, \dots, C_n \in K(x_1, \dots, x_r)[y]$ and also the highest order coefficient of polynomial F are different from 0 after substitution $x_i = \alpha_i$. Since K is algebraically closed, there exists $\beta \in K$ such that $F(\alpha_1, \dots, \alpha_r, \beta) = 0$. Then $\gamma_i = C_i(\alpha_1, \dots, \alpha_r, \beta)$ is the solution in K^n.

Step 3. If polynomials $F_1, \dots, F_m \in K[t_1, \dots, t_n]$ do not generate the unit ideal, then the system $F_1 = 0, \dots, F_m = 0$ has a solution in the field K.

Proof. Ideal $(F_1, \dots, F_m)$ is contained in some maximal ideal M. Therefore the quotient $L = K[t_1, \dots, t_n]/M$ is a field. Let the image of t_i in L be ξ_i. Obviously, $L = K(\xi_1, \dots, \xi_n)$ and $(\xi_1, \dots, \xi_n)$ is a solution of our system in the field L. According to Step 2, there exists a solution in K.

Step 4. If the polynomial G equals zero in all zero-points in K^n of polynomials $F_1, \dots, F_m$, then for some r, $G^r \in (F_1, \dots, F_m)$.

Proof. Introduce a new variable u and consider polynomials $F_1, \dots, F_m$, $uG - 1$ in the polynomial ring $K[t_1, \dots, t_n, u]$. According to assumption, they do not have common roots in K, and therefore (Step 3) generate the unit ideal: there exist polynomials $P_1, \dots, P_m, Q \in K[t_1, \dots, t_n, u]$ such that $P_1F_1 + \dots + P_mF_m + Q(uG-1) = 1$. This identity remains true after the substitution $u = 1/G$. Eliminating the denominator, one obtains the necessary statement. This proves the Hilbert's Nullstellensatz.

4. Algebraic sets and polynomial ideals

Definition of algebraic sets in higher dimensional space generalizes the notion of plane algebraic curves. Intuitively, algebraic set is a solution set of system of polynomial equations: If $f_1, \dots, f_m \in K[x_1, \dots, x_n]$, the set $V(f_1, \dots, f_m) = \{x \in K^n \mid f_1(x) = \dots = f_m(x) = 0\}$ of solutions of the system $f_1(x) = \dots = f_m(x) = 0$ is called *algebraic set* in the affine space K^n. For $m = 1$ (one equation), the corresponding set $V(f)$ is called *hypersurface*.

Even for plane algebraic curves it was not easy to establish a one-to-one correspondence between solution sets and equations: different equations could represent the same algebraic set. Instead of systems, let us consider their left-hand sides, that is, finite sets of polynomials. Instead of finite sets, it is useful to consider arbitrary sets of polynomials.

We shall use the notations $A = K[x_1, \dots, x_n]$ for ground polynomial ring and $X = K^n = \mathbb{A}^n_K$ for ambient affine point space in the whole section.

Definition. For any subset $S \subset K[x_1, \dots, x_n]$, algebraic set in X defined by S is the set $V(S) = \{\xi = (\xi_1, \dots, \xi_n) \in X \mid \forall f \in S, f(\xi) = 0\} \subset X$.

In this way one obtains the correspondence between subsets in A and subsets in X, that is, the mapping of partitive sets $V : \mathcal{P}(A) \to \mathcal{P}(X)$. Let us establish its elementary properties.

Lemma 1. *(a)* $S \subset T \Rightarrow V(T) \subset V(S)$ *(more equations, less solutions).*

(b) $V(\emptyset) = X$, $V(A) = \emptyset$. *(c)* $V(S_1 \cup S_2) = V(S_1) \cap V(S_2)$.

(d) $V(f_1, \dots, f_m) = V(f_1) \cap \dots \cap V(f_m)$ *(every algebraic set is the intersection of hypersurfaces).*

One can easily show that even for arbitrary unions $V(\bigcup_\alpha S_\alpha) = \bigcap_\alpha V(S_\alpha)$. Does the analogous statement hold for intersections, at least for finite ones?

Lemma 2. *If $I = (S)$ is the ideal generated by $S \subseteq A$, then $V(S) = V(I)$.*

According to the Hilbert basis theorem, the ring A is Noetherian, $I = (f_1, \dots, f_m)$ and $V(S) = V(I) = V(f_1, \dots, f_m)$. Therefore, every set $V(S)$ is algebraic set, described by finite set of equation.

One sees, that the mapping V defines (anti)epimorphism of partially ordered sets

$$V : \text{ideals in} A \to \text{algebraic sets in } X$$

When do the different ideals define the same algebraic set? Here the main role is played by the Hilbert's Nullstellensatz. It could be stated in the following manner:

$$\text{if } V(I) = \emptyset, \text{ then } I = A$$

and its generalized form:

$$\text{if } V(I) \subset V(f), \text{ then } f \in \sqrt{I}$$

Here $\sqrt{I} = \operatorname{Rad} I = \{a \in A \,|\, \exists r > 0 : a^r \in I\} \subset A$ is the *radical* of the ideal I. Construction of radical of a given ideal is possible in every commutative ring and has the following main properties, which could be easily proved.

Lemma 3. *(a) $\sqrt{I}$ is ideal in A; (b) $I \subset J \Rightarrow \sqrt{I} \subset \sqrt{J}$; (c) $I \subset \sqrt{I}$; (d) $\sqrt{\sqrt{I}} = \sqrt{I}$.*

Proposition. $V(I) = V(J) \Leftrightarrow \sqrt{I} = \sqrt{J}$

Proof. The direction $\Leftarrow$ follows from the easy fact that $V(I) = V(\sqrt{I})$. Let us prove the opposite direction $\Rightarrow$. If $V(I) \subset V(J)$ and $J = (f_1, \dots, f_m)$, one has $V(I) \subset V(f_1) \cap \dots \cap V(f_m) \Rightarrow f_1, \dots, f_m \in \sqrt{I} \Rightarrow J \subset \sqrt{I} \Rightarrow \sqrt{J} \subset \sqrt{\sqrt{I}} = \sqrt{I}$. $\sqrt{I}$ is the greatest element in the family of all ideals that define the algebraic set $V(f)$. It coincides with its own radical. The ideal I is a *radical ideal*, if it coincides with its radical: $I = \sqrt{I}$. In such way, the restriction of the mapping V

$$V : \text{radical ideals in } A \to \text{algebraic sets in } X$$

becomes a bijection, that is, (anti)isomorphism of ordered sets.

Example. In the case of hypersurface $V(f)$, if one factorizes the polynomial f into irreducible factors $f = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, it is easy to see that $\sqrt{(f)} = (f_{red})$ where $f_{red} = p_1 \dots p_k$. This agrees with earlier results on equations of plane algebraic curves, and motivates the notation of radical as a root.

If one considers the mapping V restricted to ideals only, its behavior with respect to unions and intersections becomes better.

Lemma 4. *(a) $\bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha) = V(\sum_\alpha I_\alpha)$; (b) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.*

Proof. (a) follows from Lemma 2. Let us now prove (b). Since for any two ideals in the ring A one has $I_1 I_2 \subset I_1 \cap I_2 \subset I_1, I_2$ and the mapping V is (anti)monotonic, $V(I_1) \cup V(I_2) \subset V(I_1 \cap I_2) \subset V(I_1 I_2)$. If $x \notin V(I_1) \cup V(I_2)$, then there exist $f_1 \in I_1$, $f_2 \in I_2$ such that $f_1(x) \neq 0$ and $f_2(x) \neq 0$. Therefore $(f_1 f_2)(x) \neq 0$ and $x \notin V(I_1 I_2)$.

The last property extends directly to finite unions of algebraic sets.

Lemmas 1, 2 and 4 show that the family AlgSets X of all algebraic sets in X is closed with respect to finite unions and arbitrary intersections, and it contains whole X and the empty set $\emptyset$. Therefore, this family is the family of closed sets of some topology on $X = K^n = \mathbb{A}_K^n$, called *Zariski topology*. Let us have a closer look on this topology.

It follows from Lemma 1(d), that the basis of this topology is the family of complements of hypersurfaces $D(f) = X \setminus V(f)$, so called *basic* or *principal open sets*. This indicates that the Zariski topology is very coarse: open sets are unions of complements of hypersurfaces.

For $n = 1$ the set $V(f)$ is finite, since it is the zero set of a polynomial in one variable. Therefore, in addition to the whole space and the empty set, closed sets in K^1 are only finite sets of points.

Finite sets are closed also for $n = 2$. Are there other closed sets? If the set $V(f_1, \dots, f_m)$ is not finite, it follows from Study's lemma that polynomials $f_1, \dots, f_m$ have nontrivial gcd h, therefore $f_i = hg_i$ and $V(f_1, \dots, f_m) = V(h) \cup V(g_1, \dots, g_m)$. The set $V(g_1, \dots, g_m)$ is finite, and $V(h)$ is a plane algebraic curve. In this way, closed sets are finite sets of points, plane algebraic curves and their unions.

The Zariski topology is compact, in the sense that every open covering contains a finite subcovering. If $\emptyset = \bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha) = V(\sum_\alpha I_\alpha)$, then applying the Nullstellensatz and the basis theorem one has $1 = f_1 g_1 + \dots + f_k g_k$ for some $f_i \in I_{\alpha_i}$ and therefore $\emptyset = V(I_{\alpha_1} + \dots + I_{\alpha_k}) = V(I_{\alpha_1}) \cap \dots \cap V(I_{\alpha_k})$.

One could introduce the inverse operation for V. If $Y \subset X$ is a subset, consider the set of all polynomials that are "annihilated" on this set:

$$I(Y) = \{f \in A \mid \forall x \in Y, f(x) = 0\}$$

It is easy to check the main properties of this operation.

Lemma 5. *(a) $I(Y)$ is ideal in A; (b) $Y \subset Z \Rightarrow I(Z) \subset I(Y)$; (c) $I(\emptyset) = A$, $I(X) = (0)$.*

Proposition. *(a) $J \subset I(V(J)) = \sqrt{J}$ for every ideal J in A;*
(b) $Y \subset V(I(Y)) = \overline{Y}$ for any subset Y in X (closure in the Zariski topology).

In this way, we obtained the mapping

$$I : \text{algebraic sets in } X \to \text{radical ideals in } A$$

as an inverse to the mapping V.

First step in classification of closed algebraic sets is the attempt to represent them as unions of simpler subsets. Lemma 4(b) shows that this is connected with product of ideals. It is known that, in a Noetherian ring, every ideal can be represented as a product of primary ideals, and every radical ideal as a product of prime ideals. This is a generalization of the factorization theorem for polynomials, and it is equivalent to it in the case of principal ideals:

$$\sqrt{(f)} = \sqrt{(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m})} = (p_1 p_2 \dots p_m) = (p_1)(p_2)\dots(p_m)$$

An arbitrary ideal I is *primary* if $ab \in I \wedge a \notin I \Rightarrow b \in \sqrt{I}$ and *prime* if $ab \in I \Rightarrow a \in I \vee b \in I$. Every prime ideal is radical. Decomposition of radical ideal in the product of primes is connected with decomposition of algebraic sets in intersection of irreducible ones:

Definition. Algebraic set $Y \subset X$ is *irreducible* if it can not be represented as union $Y = Y_1 \cup Y_2$ of two its proper algebraic subsets $Y_1, Y_2 \subset Y$, $Y_1, Y_2 \neq Y$.

Proposition. *Algebraic set Y is irreducible $\Leftrightarrow$ ideal $I(Y)$ is prime.*

Proof. If $Y = Y_1 \cup Y_2$ where $Y_i \neq Y$, then $\exists f_i \in I(Y_i) \setminus I(Y)$. However, $I(Y) = I(Y_1)I(Y_2)$ and $f_1 f_2 \in I(Y)$, so the ideal $I(Y)$ is not prime. Conversely, if this ideal is not prime, then $\exists f_i \notin I(Y)$ such that $f_1 f_2 \in I(Y)$. Let $I_i = I(Y) + (f_i)$ be ideals and $Y_i = V(I_i) = Y \cap V(f_i)$ corresponding closed sets ($i = 1, 2$). Then $Y = Y_1 \cup Y_2$ is a nontrivial decomposition, since

$$x \in Y \Rightarrow (f_1 f_2)(x) = 0 \Rightarrow f_1(x) = 0 \vee f_2(x) = 0 \Rightarrow x \in Y_1 \vee x \in Y_2$$

Proposition. *Every algebraic set can be decomposed in finite union of irreducible algebraic sets $Y = Y_1 \cup \dots \cup Y_k$, where $Y_i \not\subset Y_j$ for $i \neq j$. Such representation is determined uniquely (up to permutation).*

Mapping V defines a bijection between prime ideals and irreducible algebraic sets

$$V : \text{prime ideals in } A \to \text{irreducible algebraic sets in } X$$

Set of all prime ideals in commutative ring A is called (prime) *spectrum* of the ring A and denoted $\operatorname{Spec} A$.

This (anti)isomorphism of ordered sets sends minimal irreducible algebraic sets in X (thus points) to maximal elements of the set $\operatorname{Spec} A$ (thus maximal ideals in the ring A). Recall that the ideal of the ring A is *maximal* if it is not contained in any proper ideal except itself, that is, if it is a maximal element in the set of all proper ideals in A. One has $V(I) = \{\xi\} = \{(\xi_1, \dots, \xi_n)\} \Leftrightarrow I = (x_1 - \xi_1, \dots, x_n - \xi_n) \Leftrightarrow I \subset A$ is maximal. We obtained a bijection

$$V : \text{maximal ideals in } A \to \text{points in } X$$

The set of all maximal ideals in the commutative ring A is called *maximal spectrum* and denoted $\operatorname{Max} A$ or $\operatorname{Specm} A$. One could identify $X \cong \operatorname{Max} A$, at least as sets of points. However, the meaning of this identification is much deeper than simple bijection of sets, which becomes visible in the general theory of schemes. On the base of maximal (or prime) spectrum of the ring one could recover the full geometrical structure of the corresponding algebraic set.

5. Regular functions and mappings. Rational functions. Dimension. Singularities

In the description of geometrical objects there are always natural functions on these objects[6]. For example, on vector spaces natural functions are linear functions, on topological spaces — continuous ones, on smooth manifolds — smooth functions, on complex varieties — holomorphic ones. What are the natural functions on algebraic sets? By analogy with other geometrical objects, it should be polynomial or rational functions. Such "naive" definition should be made more precise.

Let us start with polynomial functions. Let $V \subset X$ be algebraic set and $I \subset A$ corresponding radical ideal.

Definition. The function $f : V \to K$ is a polynomial or regular function on V if it is defined by a polynomial, that is, if there is a polynomial $F \in A = K[x_1, \dots, x_n]$ such that $\forall x \in V$, $f(x) = F(x)$.

All polynomial functions build a ring (and a K-algebra) with respect to usual operations of addition and multiplication of functions. Since two polynomials F and G define the same function $\Leftrightarrow \forall x \in V$, $F(x) - G(x) = 0 \Leftrightarrow F - G \in I$, this ring could be identified with quotient ring A/I of the polynomial ring by the defining ideal of the algebraic set V.

Definition. The ring A/I is called ring of regular functions or coordinate ring of the algebraic set V and denoted $K[V]$.

Examples. (a) If $V = \{x\}$ is a point, its corresponding ideal M is maximal and $A/M \cong K$ is the ground field: function in a point is uniquely determined by its value. More generally, for n points, $K[V] \cong \underbrace{K \oplus \cdots \oplus K}_{n}$.

(b) For $V = X$ one has $I = (0)$ and $K[V] = A$, which is natural.

(c) If $V = V(y - x^2)$ is a parabola in the plane, its coordinate ring is isomorphic to polynomial ring in one variable, that is, to coordinate ring of a straight line: $K[V] = K[x, y]/(y - x^2) \cong K[x]$.

(d) If $V = V(y^2 - x^3)$ is a semicubic parabola (a cusp curve), one has $K[V] = K[x, y]/(y^2 - x^3) \cong K[x] + K[x] \cdot y$. This is a K-algebra without zero-divisors, generated by two elements.

The ring $K[V]$ is always a finitely generated K-algebra. Could it be characterized by pure algebraic method? Since ideal I is radical, this algebra does not contain nontrivial nilpotent elements — it is *reduced*, as one says. The converse also holds: for any finitely generated reduced algebra B there exists an algebraic

[6]It can be said that the definition of functions describes the corresponding geometrical object. This is formalized via ringed spaces — spaces with structure sheaf of rings

set V such that $K[V] \cong B$. It is enough to chose generators of B over K, that is, represent the algebra in the form $B = K[b_1, \ldots, b_m]$ and consider the epimorphism from the corresponding polynomial ring $A = K[x_1, \ldots, x_m] \to K[b_1, \ldots, b_m] = B$, $x_i \mapsto b_i$. Its kernel I is the defining ideal for the algebraic set V. The choice of generators from the geometrical point of view corresponds to embedding of the algebraic set V into some affine space K^m and vice versa.

The properties of algebras $K[V]$ are analogous to the properties of polynomial ring. The main point is that in these rings two Hilbert's theorems hold — the basis theorem and the Nullstellensatz. These algebras are Noetherian, as quotients of Noetherian rings. The analogon of the Nullstellensatz is: if $g_1, \ldots, g_m \in K[V]$ and $f \in K[V]$ are such that $f(x) = 0$ for every $x \in V$ which satisfies the system $g_1(x) = \cdots = g_m(x) = 0$, then for some r, $f^r \in (g_1, \ldots, g_m) \subset K[V]$. Both theorems, as well as other properties, follow from the known properties of quotient rings, the main of which is that for any commutative ring B and its ideal I, natural epimorphism $h : B \to B/I = B'$ defines an order-preserving bijection between ideals in B/I and these ideals in B which contain I, in which radical ideals correspond to radical ideals, prime to prime ideals, maximal to maximal ideals. If J' is an ideal in B', the corresponding ideal in B is $J = h^{-1}(J') \supset I$ and one has $B/J \cong (B/I)/(J/I) = B'/J'$. Let $B = A$ be the polynomial ring, $B' = K[V]$ the coordinate ring and $I = I(V)$ the ideal of some algebraic set V. Ideal $J \supset I$ corresponds to the closed set $V(J) = W \subset V$. Ideal $J/I = I(W)/I(V)$ of algebraic subset W in algebraic set V is denoted by $I_V(W)$. Therefore, here we also have the corresponding bijections

$$\text{algebraic subsets in } V \Leftrightarrow \text{radical ideals in } K[V]$$
$$\text{irreducible algebraic subsets in } V \Leftrightarrow \text{prime ideals in } K[V]$$
$$\text{points in } V \Leftrightarrow \text{maximal ideals in } K[V]$$

The Zariski topology on V is induced from $X = K^n$. Its open base is also built by principal open sets $D(g) = V \setminus V(g), \quad g \in K[V]$.

Using regular functions one can define mappings which connect algebraic sets and play the role of morphisms in the corresponding category. Let $U \subset K^n$, $V \subset K^m$ be two algebraic sets and $\varphi : U \to V$ a mapping. Composition with φ defines a mapping

$$\varphi^* : \text{functions on } V \to \text{functions on } U$$

in the usual way, by the formula $\varphi^*(f) = f \circ \varphi$.

Definition. We say that φ is a *regular mapping*, if φ^* transforms regular functions into regular functions, that is, if $f \in K[V] \Rightarrow \varphi^*(f) \in K[U]$.

Proposition. *(a) φ is a regular mapping $\Leftrightarrow$ it is defined in coordinates with m regular functions, that is, there exist $f_1, \ldots, f_m \in K[U]$ such that $\varphi(x) = (f_1(x), \ldots, f_m(x)) \in V$ for all $x \in U$.*

(b) If φ is regular, $\varphi^* : K[V] \to K[U]$ is an algebra homomorphism.

(c) Conversely, for any algebra homomorphism $h : K[V] \to K[U]$ there is a regular mapping $\varphi : U \to V$ such that $\varphi^* = h$.

(d) The mapping

$$\begin{array}{ccc} U & & K[U] \\ \varphi\downarrow & \longmapsto & \uparrow\varphi^* \\ V & & K[V] \end{array}$$

is a contravariant functor. The category of algebraic sets and regular mappings is equivalent to the category of finitely generated K-algebras without nilpotents (so called affine K-algebras) and homomorphisms.

Proof. (a) If $y_i \in K[V]$ are coordinate functions (images of generators of polynomial ring in which the algebraic set V is defined) and $\varphi^*(y_i) = f_i \in K[U]$, then for $\forall x \in U$, i-th coordinate of the point $\varphi(x)$ is $y_i(\varphi(x)) = \varphi^*(y_i)(x) = f_i(x)$ and $\varphi(x) = (f_1(x), \dots, f_m(x))$. Conversely, if φ is a mapping of that form and $g \in K[V]$ a regular function on V, then for $\forall x \in U$, $\varphi^*(g)(x) = g(\varphi(x)) = g(f_1(x), \dots, f_m(x))$ is a polynomial function of coordinates x, that is, a regular function.

(b) is obvious.

(c) Let again $y_i \in K[V]$ be coordinate functions and $h(y_i) = f_i \in K[U]$. Define for $x \in U$, $\varphi(x) = (f_1(x), \dots, f_m(x))$ and prove that $\varphi(x) \in V$. Indeed, if $F \in I(V)$, then $F(y_1, \dots, y_m) = 0$. One has $0 = h(F(y_1, \dots, y_m)) = F(h(y_1), \dots, h(y_m)) = F(f_1, \dots, f_m)$ and $F(\varphi(x)) = h(F)(x) = 0$, and this means exactly that $\varphi(x) \in V$. For $x \in U$ and $g \in K[V]$ one has $\varphi^*(g)(x) = g(\varphi(x)) = g(f_1(x), \dots, f_m(x)) = g(h(y_1)(x), \dots, h(y_m)(x)) = h(g)(x)$ i.e., $\varphi^* = h$.

(d) This is also straightforward.

Definition. *Isomorphism of algebraic sets*[7] is isomorphism in the categorical sense, that is, a regular mapping which has inverse regular mapping. In this equivalence of categories, it corresponds to isomorphism of algebras, i.e., $U \cong V \Leftrightarrow K[U] \cong K[V]$.

Examples. 1. Projection $\varphi(x,y) = x$ is a regular mapping of the hyperbola $V = \{xy = 1\}$ in the line $\mathbb{A}^1$, but not an isomorphism (not even a set bijection). Corresponding algebras are $K[x,y]/(xy-1) \not\cong K[t]$.

2. Mapping $\varphi : \mathbb{A}^1 \to V = \{y^2 = x^3\}$, $t \mapsto (t^2, t^3)$ is regular and a set-theoretic bijection. However, it is not an isomorphism. The corresponding homomorphism of algebras $\varphi^* : K[V] = K[x,y]/(y^2 - x^3) \to K[t] = K[\mathbb{A}^1]$ is defined by $x \mapsto t^2$, $y \mapsto t^3$. Its image is $\operatorname{Im}\varphi^* = K[t^2, t^3] \subsetneq K[t]$. Since φ is a bijection, it has inverse mapping

$$\psi : V \to \mathbb{A}^1, \quad \psi(x,y) = \begin{cases} y/x, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

[7] *Biregular* isomorphism, as opposed to *birational* isomorphism which will be introduced later.

but it fails to be regular in the point $(0,0)$. One could give the following informal interpretation. Algebra $K[V]$ is smaller than $K[\mathbb{A}^1]$, since in the latter there is a polynomial function with derivative in 0 different from 0, and in the former there is no such function: mapping is "leveling" all tangent vectors in 0.

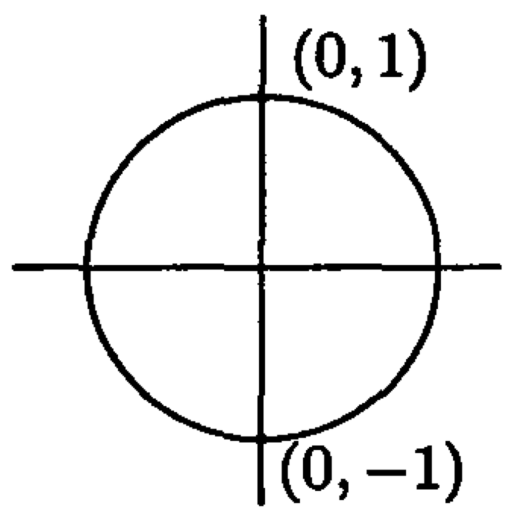
3. Parabola $y = x^k$ is isomorphic to the line $\mathbb{A}^1$. The corresponding isomorphisms are $\varphi(x,y) = x$, $\psi(t) = (t, t^k)$.

4. Let $V = \{y^2 = x^3 + x^2\}$ be the alpha-curve. As we already know, it has a rational parametrization $\varphi(t) = (t^2 - 1, t^3 - t)$. The parametrization defines a regular mapping $\varphi : \mathbb{A}^1 \to V$. Is this an isomorphism? More generally, is there an isomorphism of V and $\mathbb{A}^1$?

The examples show that, despite our wish to work only with polynomials, the involvement of rational functions is inevitable. Usual rational functions are not functions in a precise sense of the word - they have not to be defined everywhere. We are interested in rational functions on a given algebraic set, say curve C with equation $f(x,y) = 0$. Rational functions on the whole plane are the elements of the fraction field $K(x,y)$ of the polynomial ring $K[x,y]$. Two such rational functions may define the same function on C.

Example. On the circle $C : x^2 + y^2 = 1$ one has $x^2 = (1-y)(1+y)$, so the two rational functions

$$\varphi_1(x,y) = \frac{1-y}{x},$$
$$\varphi_2(x,y) = \frac{x}{1+y}$$



on C coincide in their common functional domain. Note that the domains of these two functions on C are different: the first is not defined in points $(0,1)$ and $(0,-1)$, the second only in $(0,-1)$. They coincide in the Zariski open subset $U = C \setminus \{(0,1),(0,-1)\}$ of the curve C (see fig.

Definition. Let $V \subset \mathbb{A}^n$ be irreducible closed set with coordinate ring $K[V]$. The fraction field $K(V)$ of the domain $K[V]$ is the *field of rational functions* on V (or simply the *function field* of V), and its elements *rational functions* on V.

Let $V \subset \mathbb{A}^n$ be a closed set, $x \in V$ a point, U its open neighborhood and $r : U \to K$ a function in the neighborhood. Function r is *regular at the point* x if there exist polynomial functions $f, g \in K[V]$ such that $g(x) \neq 0$ i.e., $x \in D(g)$ and $r = f/g$ on $U \cap D(g)$.

Proposition. *This local definition of regularity is consistent with the previous global one. In other words, if the function $r : V \to K$ is regular in every point $x \in V$, then $r \in K[V]$.*

Proof. From the definition, for every x there is a representation $r = f_x/g_x$ on $D(g_x)$. Due to compactness, $V = \bigcup_{x \in V} D(g_x) = D(g_1) \cup \cdots \cup D(g_m)$, or

$(g_x \,|x \in V) = (g_1, \dots, g_m)$ where $g_i = g_{x_i}$. Since $V(g_1, \dots, g_m) = \emptyset$, from the analogon of Nullstellensatz one has $(g_1, \dots, g_m) = 1$, or $g_1 h_1 + \cdots + g_m h_m = 1$. Consider the polynomial function $f = f_1 h_1 + \cdots + f_m h_m \in K[V]$. Since $g_i g_j = 0$ on $V(g_i g_j)$ and $f_i g_j = f_j g_i$ on $D(g_i g_j) = D(g_i) \cap D(g_j)$, then $g_i g_j (f_i g_j - f_j g_i) = 0$ on whole V. If we write $r = f_i/g_i = f_i g_i / g_i^2$, then we have $f_i g_j = f_j g_i$ on whole V. Therefore $f_j = 1 \cdot f_j = \sum_i f_j g_i h_i = \sum_i f_i g_j h_i = f \cdot g_j$ and $r = f \in K[V]$.

Definition. Let $V \subset \mathbb{A}^n$ be closed set, $K[V]$ its coordinate ring and $K(V)$ field of rational functions. If $x \in V$ is a point, all rational functions $r \in K(V)$ regular in x build a ring denoted by $\mathcal{O}_{x,V}$ or $\mathcal{O}_x$ and called the *local ring* of V at the point x. *Regular function* on whole V is a function, regular in every point of V. All regular functions on V also build a ring $\mathcal{O}(V)$. The preceding statement proves that $\mathcal{O}(V) = K[V]$. One has also $\mathcal{O}(V) = \bigcap_{x \in V} \mathcal{O}_{x,V} \subset \mathcal{O}_{x,V} \subset K(V)$.

The ring $\mathcal{O}_{x,V}$ consists of all rational functions from $K(V)$ which has a representation where x is not a zero of the nominator (a pole of the function). All regular functions which have a zero in x build a maximal ideal in $K[V]$ and the ring $\mathcal{O}_{x,V}$ is its minimal extension in which all elements of the complement of this ideal become invertible.

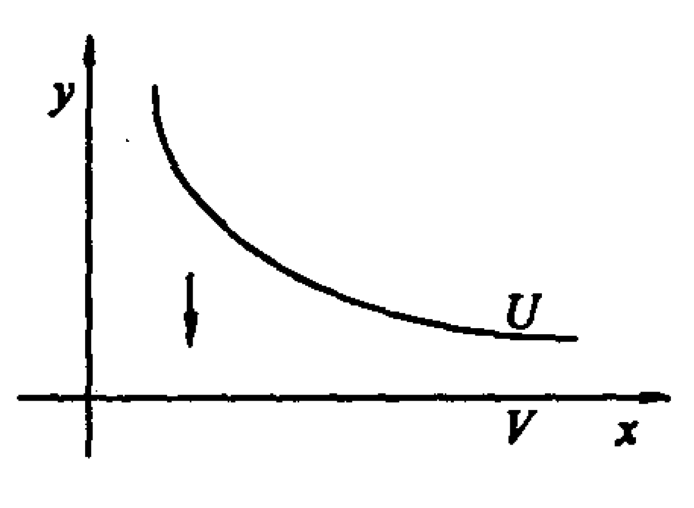
We could already note the importance of the principal open sets, which form the basis of the Zariski topology. The following result shall confirm this opinion.

Proposition. *Principal open sets are affine: they are isomorphic to affine closed sets.*

Proof. Let $V \subset \mathbb{A}^n$ be an affine closed set with coordinate ring $K[V]$, $f \in K[V]$ a regular function and $D(f) = V \setminus V(f) = \{x \in V \,|f(x) \neq 0\}$ principal open set. Let $J = I(V) \subset K[x_1, \dots, x_n]$ be the ideal of V and F defining polynomial for f. Introduce a new indeterminate y and consider the ideal $I = J + (yF - 1) \subset K[x_1, \dots, x_n, y]$. If $U = V(I) \subset \mathbb{A}^{n+1}$ is a closed set, then

$$K[U] = K[x_1, \dots, x_n, y]/\,(J + (yF - 1)) \cong (K[x_1, \dots, x_n]/J)\,[f^{-1}] = K[V][f^{-1}]$$

that is, $D(f) \cong U$. Geometrically, this is analogous to projection of the hyperbola on the axis (see the figure).



From the proof one can see that the principal open set $D(f)$ is the affine closed set corresponding to subalgebra $K[V][f^{-1}] \subset K(V)$. This subalgebra consists of all rational functions which in the denominator have only powers of f. In other words, it is a minimal extension of the algebra $K[V]$ in which the set $\{f, f^2, f^3, \dots\} \subset$

$K[V]$ becomes invertible. This construction, as well as the above construction of the local ring at the point, is called the *localization* of the ring with respect to multiplicative subset. It is very common in commutative algebra. Its oldest version is the construction of domain's fraction field, when all nonzero elèments become invertible.

Rational functions are used to define rational mappings, specific for algebraic geometry. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be two algebraic sets, and X irreducible.

Definition. Rational mapping $f : X \dashrightarrow Y$ is a mapping defined by m rational functions $f_1, \ldots, f_m \in K(X)$ with the formula $x \mapsto (f_1(x), \ldots, f_m(x))$ in every point $x \in X$ in which all functions are regular.

Rational mapping is not everywhere defined, only on an open set. The notation should also stress the fact, that we have a partial function here. It is however, uniquely defined by its values in the domain of definition. In other words, if one has two rational mappings (on different open sets), which coincide on some nonempty open set, then they are equal.

If the image of rational mapping $f : X \dashrightarrow Y$ is dense in Y, it defines a mapping of rational functions on Y to rational functions on X (by simple change of variables). In this way one has a monomorphism of fields $K(Y) \hookrightarrow K(X)$. Similarly to regular mappings and corresponding ring homomorphisms $K[Y] \to K[X]$, a functorial connection is defined between rational mappings and homomorphisms (i.e. inclusions) of function fields. That means that isomorphisms of fields correspond to "isomorphism" rational mappings.

Definition. Rational mapping is a *birational isomorphism*, if it has inverse rational mapping (inverse here means that the compositions are identities on nonempty open subsets!). Algebraic sets X and Y are *birationally isomorphic*, if there is a birational isomorphism between them, i.e., if $K(X) \cong K(Y)$. Algebraic set X is *rational*, if it is birationally isomorphic to affine space, that is, if $K(X) \cong K(x_1, \ldots, x_n)$.

As a result, there are two different equivalence relations and two classifications of algebraic sets. One is the finer classification up to isomorphism, or classification of coordinate rings, the other is the coarser birational classification, or classification of function fields.

Example. The alpha-curve $y^2 = x^3 + x^2$ is rational: it has a rational parametrization which defines isomorphism of its function field with the field of usual rational functions in one variable $K(x)$. The same holds for semicubic parabola $y^2 = x^3$. However, if in the plane cubic curve $y^2 = P_3(x)$ the right-hand-side polynomial has no multiple roots, it is not rational.

How should one properly define dimension of algebraic set? There are several characterizations of geometrical notion of dimension. The oldest description of dimension is probably one from the Euclid's "Elements": the point is the border of the line, the line is the border of the surface,... One says that the algebraic set V is of dimension d if d is a maximal length of strictly increasing chain $\{x\} = V_0 \subset V_1 \subset \cdots \subset V_d = V$ of irreducible subvarieties in V. Due to connection between irreducible subvarieties in V and prime ideals in $K[V]$, d is at the same time the maximal length of strictly increasing chain $(0) = I_0 \subset I_1 \subset \cdots \subset I_d$ of proper prime ideals in $K[V]$.

Definition. *Krull dimension* of commutative ring A is the maximal length of strictly increasing chain of proper prime ideals in A.

So, the dimension of algebraic set equals to Krull dimension of its coordinate ring. For example, $\operatorname{Krull}\dim K[x_1, \ldots, x_n] = n$, in accordance to our intuition.

In courses of commutative algebra it is shown that Krull dimension of the affine algebra (finitely generated reduced algebra over the ground field) is equal to the transcendence degree of the corresponding fraction filed i.e. function field: $\dim V = \operatorname{Krull}\dim K[V] = \operatorname{tr}\deg_K K(V)$ [1, p. 150]. What is the geometrical meaning of this equality? If $\operatorname{tr}\deg_K K(V) = d$, then one could chose d algebraically independent elements such that field extension $K(V) \supset K(x_1, \ldots, x_d) = K(\mathbb{A}^d)$ is algebraic. This extension defines a regular mapping $V \to \mathbb{A}^d$, so-called *normalization* of the algebraic set V. Normalization is a finite morphism, which means also that it is a finite covering, i.e., over each point of $\mathbb{A}^d$ there are at most d points of V. This gives us another geometrical explanation of dimension.

Every local ring $\mathcal{O}_{x,V}$ $(x \in V)$ has the same dimension $\dim V$. In local rings (rings with only one maximal ideal) there exists a connection between dimension and the maximal ideal itself: $\dim_K \mathcal{M}/\mathcal{M}^2 \geq \operatorname{Krulldim} \mathcal{O}$. The ring $\mathcal{O}_{x,V}$ (and the point x) is *regular*, if the exact equality holds. What is the meaning of the vector space $\mathcal{M}/\mathcal{M}^2$? It consists of linear parts (i.e., differentials) of all functions, regular and equal to zero in x. Therefore, its dual vector space $(\mathcal{M}/\mathcal{M}^2)^*$ plays the role of the tangent space to the algebraic set V at the point x. The above inequality means that there can exist points which are not regular in the sense that the dimension of the tangent space is strictly greater than the dimension of V itself. Such points are special points. If V is defined by its global equations, they can be characterized in the following way.

Definition. Point $x \in V$ is a *singular point* (*singularity*) of algebraic set $V = V(f_1, \ldots, f_k) \subset \mathbb{A}^n$ if it is a solution of the following system:

$$f_i(x) = \frac{\partial f_i}{\partial x_1}(x) = \cdots = \frac{\partial f_i}{\partial x_n}(x) = 0 \quad (i = 1, \ldots, k).$$

In other words, the singular points are the points where the rank of the Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{i=1,\ldots,k \\ j=1,\ldots n}}$ drops down. In regular points this rank is equal to codimension of V.

The algebraic definition of singular point, independent of the embedding of V in ambient affine space, was introduced by Zariski[8]. He also proved equivalence with above traditional analytical definition.

[8]Oscar Zariski (1899–1986), italian and american algebraic geometer.

Algebraic sets without singularities, nonsingular varieties, are the closest analogons of smooth manifolds or complex-analytic varieties. The presence of singular points complicates the structure of algebraic varieties and makes them interesting.

Example. There are two typical plane singular cubics, which correspond to simplest types of singularities. These are the alpha-curve $y^2 = x^3 + x^2$ with a nodal point (a "node"), and the semicubic parabola $y^2 = x^3$ with a cuspidal point (a "cusp").

The theory of singularities of algebraic varieties is a very deep theory, which itself requires a long introduction. There are many aspects of studying singularities, such as classification by discrete invariants, topological structure, resolution of singularities, etc. One fruitful method for investigation of singularities of hypersurfaces is given by a combinatorial-geometrical invariant called Newton polyhedron. It was introduced by Newton, but it attracted proper attention only recently, mainly in the work of Arnold's singularity group, in the 1970's (see [2]). Some simple, though interesting combinatorial connections between singularity and its Newton polyhedron were studied by the author [17].

6. Projectivization. Projective varieties

Besides the algebraic closure of the ground field, there is one more problem in correspondence between the curve as a set of points on one side, and its equation on the other. If the degree of the curve's equation is d, the number of intersection points with an arbitrary straight line is at most d, but it can also be less.

Let C be a plain curve of degree d, defined by equation $f(x, y) = 0$ where f is a polynomial of degree d. If $L: x = a + bt,\ y = c + dt$ is a straight line, the intersection of C with L is determined by the equation

$$f(a + bt, c + dt) = g(t) = a_0(a, b, c, d) \cdot t^d + \cdots$$

It can happen that some of the roots are multiple, that is, some intersections have higher order. The notion of intersection multiplicity resolves this problem (this, however, is not trivial). However, it can happen that the degree of the equation in t is strictly less than d, since the coefficient of the highest order term equals 0. In the case of hyperbola and its asymptotic lines, the intersection point has "gone to infinity". Therefore, the points at the infinity should be introduced. It is done with the process of projectivization.

There are many equivalent ways to define projective space. A common one is to define the n-dimensional projective space $\mathbb{P}^n$ over the field K as the set of all one-dimensional subspaces, that is, the set of all lines through the origin in the vector space K^{n+1}. If one considers a unit sphere in this space, each line intersects the sphere exactly in two antipodal points. For this reason, in topology, $\mathbb{P}^n$ is defined mostly as the sphere $S^n \subset K^{n+1}$ with its antipodal points identified. We are interested in analytical approach to this construction: ***n*-dimensional projective space** $\mathbb{P}^n = \mathbb{P}^n_K$ over K is the quotient of the set $K^{n+1} \setminus \{0\}$ by the equivalence relation, induced by homothety: $(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$

where $\lambda \neq 0$. Points in projective space are the corresponding equivalence classes, denoted $x = (x_0 : x_1 : \ldots : x_n)$. Therefore, $(x_0 : x_1 : \ldots : x_n) = (\lambda x_0 : \lambda x_1 : \ldots : \lambda x_n)$. Numbers x_i are ***homogeneous coordinates*** of the point x. Since there is always a coordinate different from 0, the space $\mathbb{P}^n$ is covered by sets $\mathbb{A}^n_i = \{x \in \mathbb{P}^n \mid x_i \neq 0\} = \{x = (x_0 : \ldots : \underset{i}{1} : \ldots : x_n)\} \cong K^n$, each of them isomorphic to the affine space $\mathbb{A}^n$. These are the *affine charts* of the projective space. The transition from homogeneous coordinates of the point $x = (x_0 : x_1 : \ldots : x_n)$ in $\mathbb{P}^n$ to coordinates $(1 : x_1/x_0 : \ldots : x_n/x_0) \cong (x_1/x_0, \ldots, x_n/x_0)$ in the affine chart $\mathbb{A}^n_0 \cong \mathbb{A}^n$ is called *dehomogenization* in x_0, and the converse transition from coordinates $(y_1, \ldots, y_n)$ in $\mathbb{A}^n$ to coordinates $(1 : y_1 : \ldots : y_n)$ in $\mathbb{P}^n$ homogenization.. The complements of affine charts $\mathbb{P}^n \setminus \mathbb{A}^n_i = \{x \in \mathbb{P}^n \mid x_i = 0\} = \left\{x = (x_0 : \ldots : \underset{i}{0} : \ldots : x_n)\right\} = \mathbb{P}^{n-1}_i \cong \mathbb{P}^{n-1}$ are isomorphic to the projective space of dimension $n-1$, that is $\mathbb{P}^n = \mathbb{A}^n_i \cup \mathbb{P}^{n-1}_i$. This decomposition is easily seen on the previous model also. If in the space K^{n+1} one considers the i-th coordinate hyperplane $X_i : x_i = 0$ and its parallel hyperplane $Y_i : x_i = 1$, then one could divide the lines through the origin into two types: the lines which intersect hyperplane Y_i and the lines which are parallel to it. Lines in the first family correspond to points of this hyperplane, and they form an affine space $Y_i \cong K^n = \mathbb{A}^n$. The other family of lines is the set of all one-dimensional subspaces of the vector space $X_i \cong K^n$. They form a projective space $\mathbb{P}^{n-1}$ of dimension $n-1$. Points in this projective space, that is, lines in X_i, represent the "points at infinity" of the corresponding parallel lines in the "finite" part Y_i. The whole projective space $\mathbb{P}^n$ is the (disjoint) union of its "finite" part $Y_i \cong \mathbb{A}^n$ and its "points at infinity" complement $\mathbb{P}^{n-1}$. Note that the distinction between finite points and points at infinity of the space $\mathbb{P}^n$ is only formal, since it depends on coordinates. Every point could be made finite or infinite by corresponding change of coordinates.

The next step is to define algebraic subsets in projective space. However, there is a small difference comparing to affine case. Polynomial equations in homogeneous coordinates in $\mathbb{P}^n$ can always be considered to be homogeneous. If $f \in K[s_0, s_1, \ldots, s_n]$ is a polynomial over K in $n+1$ indeterminates $s_0, s_1, \ldots, s_n$, then it is represented as a sum of its homogeneous components $f = f_0 + f_1 + \cdots + f_r$. If now $\xi = (\xi_0 : \xi_1 : \ldots : \xi_n)$ is a point in $\mathbb{P}^n$ for which $f(\xi) = 0$, then $f(\lambda\xi_0, \ldots, \lambda\xi_n) = f_0(\xi_0, \ldots, \xi_n) + \lambda f_1(\xi_0, \ldots, \xi_n) + \ldots + \lambda^r f_r(\xi_0, \ldots, \xi_n) = 0$ for all $\lambda \in K^*$. Since the field K is infinite, it follows that all $f_i(\xi_0, \ldots, \xi_n) = 0$.

The transition from homogeneous polynomial $f(s_0, \ldots, s_n) \in K[s_0, \ldots, s_n]$ to polynomial $f(1, t_1, \ldots, t_n) \in K[t_1, \ldots, t_n]$ is called *dehomogenization*, and the transition from polynomial $g(t_1, \ldots, t_n) \in K[t_1, \ldots, t_n]$ to homogeneous polynomial $s_0^{\deg g} \cdot g(s_1/s_0, \ldots, s_n/s_0) \in K[s_0, s_1, \ldots, s_n]$ (this is a homogeneous polynomial!) *homogenization* with respect to s_0.

Closed algebraic set $V \subset \mathbb{P}^n$ is the set of common zeros of the finite (or infinite) set of polynomials $f \in K[s_0, \ldots, s_n]$. The correspondence between closed sets V and ideals I is the same as in the affine case, only the ideals obtained are not arbitrary, but with every polynomial they contain also all its homogeneous components.

Definition. Ideal $I \subset K[s_0, \dots, s_n]$ is homogeneous if: $f \in I \Rightarrow$ all homogeneous components of f belong to I.

Every homogeneous ideal has a basis of homogeneous polynomials. Therefore, every closed set in projective space can be defined by homogeneous equations.

Another difference is the absence of Hilbert's Nullstellensatz: there are homogeneous ideals defining the empty set. They can be easily described.

Lemma. $V(I) = \emptyset \Leftrightarrow$ *for some* k, $I \supset I_k := (s_0, \dots, s_n)^k$.

Proof. The direction $\Leftarrow$ is obvious since $V(I_k) = \emptyset$. Conversely, let I be homogeneous and $V(I) = \emptyset$. Let $I = (f_1, \dots, f_r)$ be some homogeneous basis, $\deg f_i = m_i$. Dehomogenized system

$$\begin{aligned} f_1(1, t_1, \dots, t_n) &= 0 \\ \dots & \qquad\qquad (t_i = s_i/s_0) \\ f_r(1, t_1, \dots, t_n) &= 0 \end{aligned}$$

has no solutions, since an eventual solution would give a point in $V(I)$. From the Nullstellensatz, one has $1 = f_1(1,t)g_1(t) + \dots + f_r(1,t)g_r(t)$ in the ring $K[t_1, \dots, t_n]$. Homogenizing in s_0, that is multiplying by $s_0^{m_0}$, one has $s_0^{m_0} \in I$. Therefore, all $s_0^{k_0}, \dots, s_n^{k_n} \in I$. If now $m = \max(m_0, \dots, m_n)$ and $k = (m-1)(n+1)+1$, then in every monomial $s_0^{k_0} \cdot \ldots \cdot s_n^{k_n}$ with $k_0 + \dots + k_n \geq k$ at least one exponent $k_i \geq m \geq m_i$, and $I_k \subset I$.

Let $V \subset \mathbb{P}^n$ be a projective closed set. The process of dehomogenization in s_0 corresponds to intersection with affine chart $\mathbb{A}_0^n$. In other words, intersection $V \cap \mathbb{A}_0^n$ of the projective closed set V and an affine chart is an affine closed set. Its equations are obtained by dehomogenizing the equations of V with respect to s_0. It should be noted that $V \cap \mathbb{A}_0^n$ is closed as subset in $\mathbb{A}_0^n$ and open as subset in V. Conversely, let $W \subset \mathbb{A}_0^n \cong \mathbb{A}^n$ be an affine closed set. Homogenizing its equations with respect to s_0 one obtains equations of a projective closed set $V = \overline{W} \subset \mathbb{P}^n$ which represents the closure of the set W with respect to Zariski topology in $\mathbb{P}^n$, *projective closure* of W. It is obtained by adding the "points at infinity" to its "finite" part $W = V \cap \mathbb{A}_0^n$.

The coordinate ring of the projective closed set $V \subset \mathbb{P}^n$ is defined in the same way as in the affine case. It is a quotient ring $K[V] = K[s_0, \dots, s_n]/I(V)$. Since the ideal $I(V)$ is homogeneous, this ring is graded (in the affine case it may not be so). Its elements can not be interpreted as functions on V. Their value in points of V is not determined, since it depends on the choice of homogeneous coordinates. Also, the elements of its fraction field, "rational" functions, are not proper functions. Only those among them which originate from rational functions of degree 0 (that is, quotient of two polynomials of the same degree) define functions which have values in points of V, even then not all, but only the points in which the denominator is different from 0. Therefore, the definition of rational and regular function has to be changed, and one should use the local definition of regularity.

Lemma. (& definition). *Rational function in projective closed set $V \subset \mathbb{P}^n$ is the fraction of degree 0 in the field of fractions of the ring $K[V]$. The set $K(V)$ of all such fractions of degree 0 is a field, the field of rational functions of a projective closed set V. If $x \in V$, the rational function $r \in K(V)$ is regular in the point x if it has a representation $r = f/g$, $f, g \in K[V]$, $x \notin V(g) \subset V$. All functions regular in a given point x build a ring, denoted $\mathcal{O}_{x,V}$ or $\mathcal{O}_x$ and called local ring of V in x. Regular function on whole V is a function, regular in each point of V. All regular functions on V also build a ring $\mathcal{O}(V)$. One has $\mathcal{O}(V) = \bigcap_{x \in V} \mathcal{O}_{x,V} \subset \mathcal{O}_{x,V} \subset K(V)$.*

In the case of affine sets there are many regular functions, since $\mathcal{O}(V) = K[V]$. However, in the projective case, it is not so.

Proposition. *The only global regular functions on irreducible projective closed set V are constants: $\mathcal{O}(V) = K$.*

Proof. Let $K[V] = K[s_0, \dots, s_n]/I(V)$ and $x_i = s_i \bmod I(V)$ coordinate functions on V. Then $V = \bigcup_{i=1}^n D(x_i)$ since $\bigcap_{i=1}^n V(x_i) = V((x_0, \dots, x_n)) = \emptyset$. One could renumber coordinates in such way that $x_0, \dots, x_m \neq 0$, $x_{m+1}, \dots, x_n = 0$. Let $r \in \mathcal{O}(V)$ be global regular function. Then in every $D(x_i)$ function r has representation $r = f_i/x_i^{n_i}$ where $n_i = \deg f_i$. Let now $k = n_0 + \cdots + n_m$. If the sum of exponents is $k_0 + \cdots + k_m = k$, then the function $x_0^{k_0} \cdots x_m^{k_m} \cdot r \in K[V]$ (since at least one $k_i \geq n_i$). It follows that, if $K[V]_k$ is the subset of all homogeneous elements of degree k, then $r^l \cdot K[V]_k \subset K[V]_k$ for all $l \in \mathbb{N}$. Particularly, $r^l \cdot x_0^k \in K[V]_k$. This means that the ring $K[V][r]$ is finitely generated $K[V]$-submodule of a finitely generated (and Noetherian) $K[V]$-module $K[V] + 1/x_0^k \cdot K[V]$. Therefore r is integral over $K[V]$, that is, $r^p + a_1 r^{p-1} + \cdots + a_p = 0$ for some $a_i \in K[V]$. By taking homogeneous components of degree 0 in the ring of fractions of the ring $K[V]$, one sees by coefficient comparison that a_i could be replaced by their homogeneous components of degree 0 i.e., constants from K. Therefore, r is algebraic over an algebraically closed field K, and $r \in K$.

The definition of regular and rational mappings is the same as in the affine case. Regular mapping of projective varieties $f : X \to Y$ is a mapping which takes regular functions on Y in regular functions in X. Rational mapping $f : X \dashrightarrow Y$ can be defined in more ways. It is given by regular mapping $f : U \to V$ where $U \subset X$ and $V \subset Y$ are open sets, and two such mappings are identified if they agree on a common open set. Rational mapping is not a function in the proper sense. As a function it is defined on some maximal open subset, the domain of the rational mapping. If $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, rational mapping $f : X \dashrightarrow Y$ can be described by $m+1$ homogeneous forms $F_0, \dots, F_m$ of the same degree in $n+1$ indeterminates $x_0, \dots, x_n$, or by $m+1$ rational functions $f_0, \dots, f_m$ on X. Some important examples will be stated later.

Besides affine and projective closed sets, their open subsets also naturally appear. This is the most general type of variety we met by now.

Definition. Open subset of projective closed subset is called *quasiprojective variety.*

Affine and projective closed sets are both quasiprojective varieties. All previously defined notions are transferred to quasiprojective varieties: regular functions (local definition), rational functions, local ring of the functions, regular in a given point, field of rational functions. Also, the notions of regular and rational mapping are transferred, as well as the notions of isomorphism and birational isomorphism. Two irreducible varieties are birationally isomorphic if and only if they contain two isomorphic open subsets [33, p. 69].

Definition. Quasiprojective variety isomorphic to an affine (projective) closed set, is called *affine (projective) variety.*

These notions are introduced in order to study varieties independently of their embedding in the ambient space. As opposed to affine closed sets, the notion of affine variety is invariant with respect to isomorphism.

A property of a geometrical object is local, if every point of it has an open neighborhood in which this property holds. For example, being a closed set is a local property. In the study of local properties, we can always restrict ourselves to affine varieties.

Proposition. *If X is quasiprojective variety and $x \in X$, then x has a neighborhood isomorphic to an affine variety.*

Proof. Let $X \subset \mathbb{P}^n$, $X \cap \mathbb{A}^n = Y \setminus Z$ where $Y, Z \subset Y$ are closed in $\mathbb{A}^n$. Since $x \in Y \setminus Z$, there exists a polynomial $F \in K[\mathbb{A}^n]$ such that $F \in I(Z)$ and $F(x) \neq 0$. Then $V(F) \supset Z$ and $D(F) = Y \setminus V(F)$. Let the ideal $I(Y) = (F_1, \dots, F_m) \subset K[\mathbb{A}^n]$. Consider the closed set $W = V(F_1, \dots, F_m, y \cdot F - 1) \subset \mathbb{A}^{n+1}$. Then the projection $\mathbb{A}^{n+1} \to \mathbb{A}^n$ defines a mapping $\varphi : W \to D(F)$ and $\psi : D(F) \to W$ by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1/F(x_1, \dots, x_n))$.

Finally, let us state two important theorems which will be used in the sequel. In both cases, theorems are proved by local technique of reduction to affine case, and then by algebraic calculation in the polynomial ring.

The first theorem states that projective varieties behave better than affine with respect to regular mappings. Regular image of an affine variety need not be closed (example: a projection of hyperbola onto axis). This can not happen for projective varieties.

Theorem. (on closed image, [33, p. 76]). *The image of the projective variety X under a regular mapping $f : X \to Y$ is a closed set in Y.*

The second theorem is analogous to the corresponding theorem from differential geometry. A regular mapping foliates the domain into disjoint preimages of points—fibres over points. What is the dimension of each fibre? In differential geometry, it is equal to the difference between dimensions of the domain and its image. In algebraic situation this is the case "almost everywhere", that is, on an open subset.

Theorem. (on dimension of fibres, [33, p. 97]). *Let $f : X \to Y$ be regular mapping of an irreducible variety X of dimension n onto an irreducible variety Y of* *dimension m. Then $m \leq n$, fibres $f^{-1}(y)$ over $y \in Y$ have dimension* $\dim f^{-1}(y) \geq n-m$ *and the equality holds over a nonempty open set $U \subset Y$.*

7. Veronese and Grassmann varieties. Lines on surfaces

7.1. The Veronese variety. Homogeneous polynomials $F(x_0, \dots, x_n) = \sum_{i_0+\dots+i_n=m} a_{i_0\dots i_n} x_0^{i_0} \dots x_n^{i_n}$ of degree m in $n+1$ indeterminates form a vector space of dimension $d_m^n = \binom{n+m}{m}$. The form F defines a projective hypersurface $H = \{F=0\} \subset \mathbb{P}^n$ of degree m. Two forms define the same hypersurface if and only if they are proportional. Therefore, all projective hypersurfaces of degree m form a projective space $\mathbb{P}^{d_m^n-1}$ of dimension $d_m^n - 1 = \binom{n+m}{m} - 1$, with homogeneous coordinates $(v_{i_0\dots i_n} \mid i_0 + \dots + i_n = m)$.

Define a regular mapping $\mathcal{V}_m^n : \mathbb{P}^n \to \mathbb{P}^{d_m^n-1}$ by $(u_0 : \dots : u_n) \mapsto v_{i_0\dots i_n} = u^{i_0} \dots u^{i_n}$. Its image $\mathcal{V}_m^n(\mathbb{P}^n) = V_m^n \subset \mathbb{P}^{d_m^n-1}$ is called *Veronese variety*. It is defined by equations $v_{i_0\dots i_n} \cdot v_{j_0\dots j_n} = v_{k_0\dots k_n} \cdot v_{l_0\dots l_n}$ $(i_0+j_0 = k_0+l_0, \dots, i_n+j_n = k_n+l_n)$. Namely, if these equations define the variety X_m^n, then obviously $V_m^n \subset X_m^n$. Conversely, one deduces from these equations that on X_m^n at least one coordinate of the form $v_{0\dots m\dots 0}$ is different from zero, say $v_{m0\dots 0} \neq 0$. Then in the open set $\{v_{m0\dots 0} \neq 0\} \supset X_m^n$ the mapping

$$
\begin{aligned}
u_0 &= v_{m,0,\dots,0} \\
u_1 &= v_{m-1,1,\dots,0} \\
&\dots \\
u_n &= v_{m-1,0,\dots,1}
\end{aligned}
$$

is regular and inverse for $\mathcal{V}_m^n : \mathbb{P}^n \to \mathbb{P}^{d_m^n-1}$. So, $\mathcal{V}_m^n : \mathbb{P}^n \cong \mathcal{V}_m^n(\mathbb{P}^n) = V_m^n \subset \mathbb{P}^{d_m^n-1}$. The dimension of the Veronese variety V_m^n is n.

Example 1. For $n=1$, $m=3$, $\mathcal{V}_3^1 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. The equations of $V_3^1 \subset \mathbb{P}^3$ are

$$v_{03}v_{30} = v_{12}v_{21}, \quad v_{03}v_{21} = v_{12}^2, \quad v_{30}v_{12} = v_{21}^2$$

where $(v_{03} : v_{12} : v_{21} : v_{30})$ are the homogeneous coordinates in $\mathbb{P}^3$. Dehomogenization on $v_{03} \neq 0$, with notations $x = v_{12}/v_{03}$, $y = v_{21}/v_{03}$, $z = v_{30}/v_{03}$, gives

$$z = xy, \quad y = x^2, \quad xz = y^2$$

or $y = x^2$, $z = x^3$ since the ideal $(z - xy, y - x^2, xz - y^2) = (y - x^2, z - x^3)$. Therefore, the Veronese curve $V_3^1 \subset \mathbb{P}^3$ is exactly the *space cubic* (or the *norm-curve*) $t \mapsto (t, t^2, t^3)$.

Example 2. More generally, if $n=1$, the Veronese mapping $\mathcal{V}_m^1 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^m$ is $(x:y) \mapsto (x^m : x^{m-1}y : \dots : y^m)$. The system of equations for Veronese curve can be written as $V_m^1 = \{(x_0 : x_1 : \dots : x_m) \mid (x_0 : x_1) = (x_1 : x_2) = \dots = (x_{m-1} : x_m)\}$ or

$$\operatorname{rank} \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{m-1} \\ x_1 & x_2 & x_3 & \dots & x_m \end{pmatrix} \leq 1$$

In the affine chart the corresponding curve has rational parametrization $t \mapsto (t, t^2, \dots, t^m)$.

Example 3. If $F(x_0, \dots, x_n) = \sum\limits_{i_0+\dots+i_n=m} a_{i_0\dots i_n} x_0^{i_0} \dots x_n^{i_n}$ is a form of degree m and $H = \{F = 0\} \subset \mathbb{P}^n$ corresponding hypersurface, then $\mathcal{V}_m^n(H) = V_m^n \cap E$ where E is a hyperplane $\sum\limits_{i_0+\dots+i_n=m} a_{i_0\dots i_n} v_{i_0\dots i_n} = 0$ in $\mathbb{P}^{d_m^n-1}$. Using this fact it is easy to see that a complement of a hypersurface in $\mathbb{P}^n$ is an affine variety (that is, isomorphic to an affine closed set).

It is not difficult to prove that the Veronese variety $\mathcal{V}_m^n(\mathbb{P}^n) = V_m^n \subset \mathbb{P}^{d_m^n-1}$ is not contained in any linear subspace of $\mathbb{P}^{d_m^n-1}$.

7.2. The Grassmannian. Let V be the vector space of dimension n. The set $\mathrm{Gr}(r, V)$ of all r-dimensional subspaces of the space V is called the *Grassmannian* of V (of corresponding dimension). If $L \in \mathrm{Gr}(r, V)$ is one such subspace and $e_1, \dots, e_r$ its basis, it defines an element $e_1 \wedge \dots \wedge e_r \in \wedge^r V$ of the exterior power of the vector space V. If $e'_1, \dots, e'_r$ is another basis of L, then $e'_1 \wedge \dots \wedge e'_r = \alpha \cdot e_1 \wedge \dots \wedge e_r$ where $\alpha = \det C_{e \to e'} \neq 0$ is the determinant of the transition matrix. This means that the element $e_1 \wedge \dots \wedge e_r \in \wedge^r V$ defines a point in the projectivization. $\mathbb{P}(\wedge^r V)$, which does not depend on the choice of base, but only on the subspace L. In this way one obtains a mapping $P : \mathrm{Gr}(r, V) \to \mathbb{P}(\wedge^r V)$, $L \mapsto P(L)$. It is easily seen that this is an injection, that is, $\mathrm{Gr}(r, V) \hookrightarrow \mathbb{P}(\wedge^r V)$. If $e_1, \dots, e_n$ is a basis in V, $\{e_{i_1} \wedge \dots \wedge e_{i_r}\}$ is a basis in $\wedge^r V$, the dimension of this vector space equals $\binom{n}{r}$, and dimension of its projectivization equals $\binom{n}{r} - 1$. If $L \in \mathrm{Gr}(r, V)$, one has $P(L) = \sum\limits_{i_1<\dots<i_r} p_{i_1\dots i_r} \cdot e_{i_1} \wedge \dots \wedge e_{i_r}$. The homogeneous coordinates $\{p_{i_1\dots i_r}\}$ of the point $P(L) \in \mathbb{P}(\wedge^r V)$ are called the *Plücker*[9] *coordinates* of the subspace $L \in \mathrm{Gr}(r, V)$. However, the mapping P is not surjective. Let us determine the image $\mathrm{Im}\, P$. This reduces to a question, could one explicitly describe conditions that a vector $x \in \wedge^r V$ is decomposable, that is, has the form $x = f_1 \wedge \dots \wedge f_r$. In order to solve it, one introduces a new operation in the exterior algebra of a vector space, a mapping $V^* \times \wedge^r V \to \wedge^{r-1} V$ which "reduces" exterior degree, by the following inductive definition.

Let $u \in V^*$ be a linear function on V. For $x \in \wedge^0 V = K$ define $u \lrcorner x = 0$. For $x \in \wedge^1 V = V$ define $u \lrcorner x = (u, x) = u(x)$. In the general case, for vectors of the form $x \wedge y \in \wedge^r V$ $(r \geq 2)$ (which generate whole $\wedge^r V$) one defines $u \lrcorner (x \wedge y) = (u \lrcorner x) \wedge y + (-1)^r x \wedge (u \lrcorner y)$, and extends it linearly on arbitrary vectors. Finally, this mapping can be iteratively defined for vectors $u = u_1 \wedge \dots \wedge u_k \in \wedge^k V^*$ $(k \geq 2)$ and linearly extended on arbitrary vectors $u \in \wedge^k V^*$. One obtains a linear map $\wedge^k V^* \times \wedge^r V \to \wedge^{r-k} V$, $(u, x) \mapsto u \lrcorner x$, called *cancellation*[10].

Example. For $r = 1$ and $x, y \in V$ one has $u \lrcorner (x \wedge y) = u(x) \cdot y - u(y) \cdot x$. Particularly, for $x \neq 0$, $x^* \in V^*$ and $x^* \lrcorner (x \wedge y) = y$, which justifies the term.

[9]Julius Plücker (1801-1868), German geometer, who first introduced homogeneous coordinates and coordinate method in projective geometry. He was teacher of Felix Klein.

[10]in Russian "свертка"

The following lemma shows the connection between cancellation and the Plücker coordinates and can be proved straightforwardly.

Lemma. *Let $e_1, \dots, e_n$ be a basis in V and $e_1^*, \dots, e_n^*$ its dual basis in V^*. If $\{p_{i_1 \dots i_r}\}$ are Plücker coordinates of the vector x in e, then $p_{i_1 \dots i_r} = e_{i_r}^* \lrcorner (\dots (e_{i_1}^* \lrcorner x) \dots)$.*

Proposition. *A given vector $x \in \wedge^r V$ is of the form $x = f_1 \wedge \dots \wedge f_r$ for any $u \in \wedge^{r-1} V^*$ one has $(u \lrcorner x) \wedge x = 0$.*

Proof. Let us describe the proof in the case $n = 4$, $r = 2$. The direction $\Rightarrow$ is checked easily. Prove the opposite direction. Let $x = p_{12} e_1 \wedge e_2 + p_{13} e_1 \wedge e_3 + \dots \neq 0$ in some basis e_1, e_2, e_3, e_4 and, say, $p_{12} = 1$. Let u_1, u_2, u_3, u_4 be the dual basis in V^*. From previous properties one has $u_2 \lrcorner (u_1 \lrcorner x) = p_{12} = 1$, $(u_1 \lrcorner x) \wedge x = 0$, $u_2 \lrcorner ((u_1 \lrcorner x) \wedge x) = 0$, therefore $0 = (u_2 \lrcorner (u_1 \lrcorner x)) \wedge x - (u_1 \lrcorner x) \wedge (u_2 \lrcorner x) = x - (u_1 \lrcorner x) \wedge (u_2 \lrcorner x)$, and one has $x = (u_1 \lrcorner x) \wedge (u_2 \lrcorner x)$.

Vectors f_1 and f_2 in the decomposition $x = f_1 \wedge f_2$ can be described explicitly. Namely, if $x = e_1 \wedge e_2 + p_{13} e_1 \wedge e_3 + p_{14} e_1 \wedge e_4 + p_{23} e_2 \wedge e_3 + p_{24} e_2 \wedge e_4 + p_{34} e_3 \wedge e_4$ ($p_{12} = 1$) and if $f_1 = u_1 \lrcorner x = \dots = e_2 + p_{13} e_3 + p_{14} e_4$, $f_2 = u_2 \lrcorner x = \dots = -e_1 + p_{23} e_3 + p_{24} e_4$, then one sees that $x = f_1 \wedge f_2 \Leftrightarrow p_{34} = p_{13} p_{24} - p_{14} p_{23}$.

Note that it suffices to check the condition $(u \lrcorner x) \wedge x = 0$ only for basis vectors $u \in \wedge^{r-1} V^*$, which leads to a system of polynomial equations with respect to indeterminate Plücker coordinates. Therefore, the Grassmannian $\mathrm{Gr}(r, V) \cong \mathrm{Im}\, P \subset \mathbb{P}(\wedge^r V)$ has a natural structure of a projective algebraic variety, which parametrizes the set of all $(r-1)$-dimensional projective subspaces of a $(n-1)$-dimensional projective space. It is clear that this variety does not depend on the choice of the space V but only on its dimension n and therefore it is usually denoted $\mathrm{Gr}(r, n)$. In the sequel we shall be interested mostly in the case $n = 4$, $r = 2$, that is, the case of the Grassmannian $\mathrm{Gr}(2, 4)$ of all projective lines in a projective space. Here the defining system could be explicitly written down and it simplifies to single equation:

$$p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0$$

This equation defines a hypersurface $\Pi \subset \mathbb{P}^5$, *Plücker hypersurface.*

Lemma. *If vectors $f_1 = x_1 e_1 + \dots + x_4 e_4$, $f_2 = y_1 e_1 + \dots + y_4 e_4$ form a base of the plane L, then $f_1 \wedge f_2 = \sum (x_i y_j - x_j y_i) e_i \wedge e_j$ and Plücker coordinates of the corresponding line are $p_{ij} = x_i y_j - x_j y_i$. The plane $L = \mathrm{Span}\{f_1, f_2\} = \{u \lrcorner (f_1 \wedge f_2) \mid u \in V*\}$. If u has coordinates $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in the dual base u_1, u_2, u_3, u_4, that is, $u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4$, then $u \lrcorner (f_1 \wedge f_2) = u(f_1) f_2 - u(f_2) f_1 = \sum_i \alpha_i x_i \sum_j y_j e_j - \sum_i \alpha_i y_i \sum_j x_j e_j = \sum_i \left(\sum_j \alpha_j p_{ij} \right) e_i$ and projective coordinates of an arbitrary point of the corresponding projective line are $z_i = \sum_j p_{ij} \alpha_j \quad (i = 1, \dots, 4)$.*

7.3. Lines on surfaces in projective space. We have seen that surfaces of a given degree m in projective space $\mathbb{P}^3$ are parametrized by points of projective space $\mathbb{P}^k$ with $k = \binom{m+3}{3} - 1$. Lines in projective space $\mathbb{P}^3$ are parametrized by points of Plücker hypersurface $\Pi \subset \mathbb{P}^5$. We are interested in conditions when some surface contains some lines. Consider the product $\mathbb{P}^k \times \Pi$ and its subset $\Gamma = \{(\xi, \eta) \mid \eta \subset \xi\} \subset \mathbb{P}^k \times \Pi$ of all pairs (ξ, η) where ξ is a surface and η a line contained in it.

Proposition. *The set* $\Gamma = \{(\xi, \eta) \mid \eta \subset \xi\} \subset \mathbb{P}^k \times \Pi$ *is closed, i.e., it is a projective variety.*

This follows directly from the following lemma.

Lemma. *Let* $\eta \in \Pi$ *be a line in* $\mathbb{P}^3$ *with Plücker coordinates* p_{ij} $(1 \le i < j \le 4)$ *and* $\xi \in \mathbb{P}^k$ *a surface of degree* m *with coefficients* $q_{i_0 i_1 i_2 i_3}$ $(\sum i_k = m)$. *The condition* $\eta \subset \xi$ *is algebraic with respect to* p *and* q, *homogeneous on each group of indeterminates separately.*

Proof. Coordinates of arbitrary point on the line η are $z_i = \sum_j p_{ij}\alpha_j$ $(i = 1, \dots, 4)$ with indeterminate $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. If $F(z_1, z_2, z_3, z_4) = 0$ is a homogeneous equation of the surface ξ, then $\eta \subset \xi$ if and only if the equality $F\Big(\sum_j p_{1j}\alpha_j, \sum_j p_{2j}\alpha_j, \sum_j p_{3j}\alpha_j, \sum_j p_{4j}\alpha_j\Big) = 0$ holds for all α_i. This gives homogeneous equations for p and q.

Consider two projections $\varphi : \Gamma \to \mathbb{P}^k$, $(\xi, \eta) \mapsto \xi$ and $\psi : \Gamma \to \Pi$, $(\xi, \eta) \mapsto \eta$. These are regular mappings. The second projection ψ is surjective, since any line is contained in at least one (say, reducible) surface of degree m. Let us calculate the dimension of the fibre $\psi^{-1}(\eta) = \{(\xi, \eta) \mid \xi \supset \eta\}$. A coordinate transformation in $\mathbb{P}^3$ allows us to suppose that the equations for η are $z_0 = z_1 = 0$. Then the equation of the surface ξ which contains this line has to be of the form $F(z) = z_0 G(z) + z_1 H(z)$. The set of all such homogeneous forms in the space of all forms of degree m in 4 indeterminates forms a linear subspace of dimension l. The form $F(z) = \sum_{i_0 + \dots + i_3 = m} q_{i_0 \dots i_3} z_0^{i_0} \dots z_3^{i_3}$ is of the form $z_0 G(z) + z_1 H(z)$ $\Leftrightarrow$ in each summand $i_0 \ge 1$ or $i_1 \ge 1$, and this is a complement of the condition $i_0 = i_1 = 0$. Therefore $l =$ (the number of forms in 4 indeterminates) $-$ (the number of forms in 2 indeterminates) $= \binom{m+3}{3} - \binom{m+1}{1} = \frac{1}{6}m(m+1)(m+5)$. Now $\dim \psi^{-1}(\eta) = \frac{1}{6}m(m+1)(m+5) - 1$. All fibres have the same dimension and Γ is irreducible. According to the theorem of dimensions of fibres, $\dim \Gamma = \dim \psi(\Gamma) + \dim \psi^{-1}(\eta) = \frac{1}{6}m(m+1)(m+5) + 3$. Consider now the projection φ. According to the theorem on closed image, $\varphi(\Gamma) \subset \mathbb{P}^k$ is a closed subset of dimension $\dim \varphi(\Gamma) \le \dim \Gamma$. For a given surface ξ of degree m the fibre $\varphi^{-1}(\xi) \subset \Gamma$ consists of all pairs (ξ, η) for which the line η lays on the surface ξ. Obviously, if $\dim \Gamma < k = \binom{m+3}{3} - 1$, φ cannot be surjective, that is, there are surfaces of degree m which do not contain lines. For $\xi \in \varphi(\Gamma)$, from the theorem of dimension of fibres it follows that $\dim \varphi^{-1}(\xi) \ge \dim \Gamma - \dim \varphi(\Gamma)$. Compare the values of k and $\dim \Gamma$ for different m:

m	1	2	3	4
k	3	9	19	34
$\dim \Gamma$	5	10	19	33

If $m \geq 4$, then $\dim \Gamma < k$. This means that there is always a surface of degree $m \geq 4$ on which there are no lines at all. Consider more closely the cases $m = 1, 2, 3$. For $m = 1$, surfaces of degree 1 are planes, parametrized by points of $\mathbb{P}^3$ (principle of projective duality!). On any plane there are infinitely many lines—dimension of the fibre is ≥ 2. For $m = 2$, surfaces of degree 2 are quadrics, parametrized by points of $\mathbb{P}^9$. On any quadric there are infinitely many lines - dimension of the fibre is ≥ 1. The case $m = 3$ is most interesting. Cubic surfaces are parametrized by points of the space $\mathbb{P}^{19}$. Here the dimension of non-empty fibres is ≥ 0. Now prove that this lower bound is reached, that is, there exists a cubic surface with only finitely many lines. Consider the surface $xyz = 1$. In the finite part of the space, $\mathbb{A}^3$, it does not contain any line, whereas in the plane at infinity $\mathbb{P}^2$ it contains three lines $xyz = 0$. This means that $\dim \varphi(\Gamma) = \dim \Gamma = 19$ and that φ is surjective! We have just proved the following theorem.

Theorem. *Any cubic surface contains a line. The set of cubic surfaces that contain only finitely many lines is open in* $\mathbb{P}^{19}$.

This is a classical result, showing a very specific method of proof in classical algebraic geometry: one example has proved the theorem. Cubic surfaces have been extensively studied. One of the most complete monographs on the subject is [21].

8. Twenty-seven lines on a cubic surface

We have proved that on any cubic surface there is at least one line and that "almost all" cubic surfaces contain finitely many lines. How many? One of the most beautiful classical theorems of geometry says that any nonsingular cubic surface contains exactly 27 lines.

Let S be nonsingular cubic surface in $\mathbb{P}^3$. Note the following simple facts.

Lemma 1. *If* Π *is a plane, then* $S \cap \Pi$ *is a plane cubic curve.*

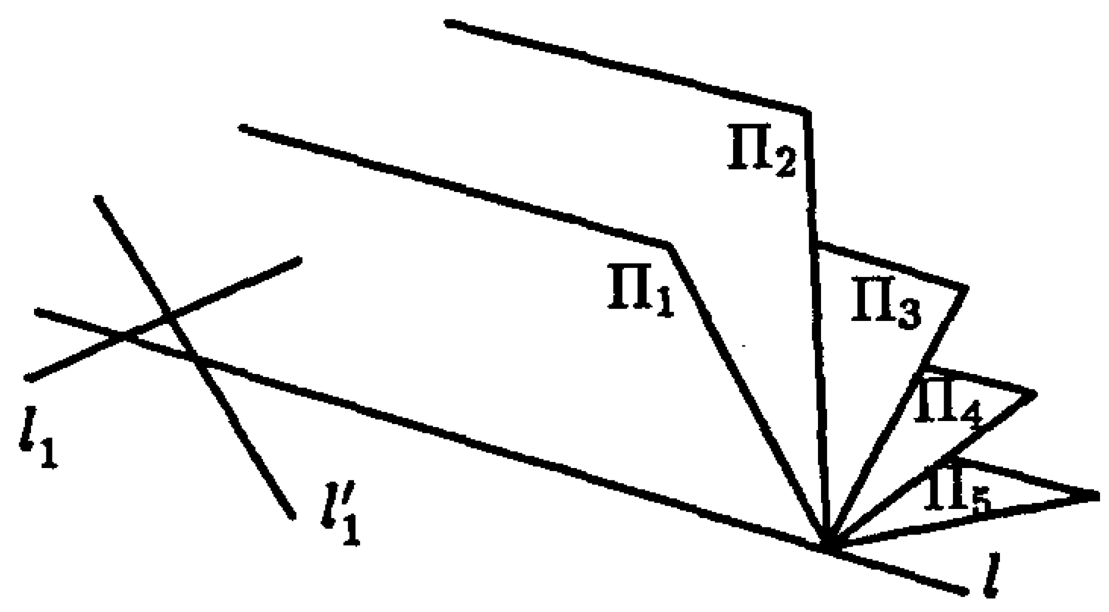
Lemma 2. *If this cubic contains a line* l*, then* $S \cap \Pi = l \cup \{conic\}$.

Lemma 3. *If this conic is reducible, then all three lines are different (there are no multiple lines) and their configuration belongs to one of the following two types*

Proof. Coordinates could be chosen so that $\Pi : \{t = 0\}$, $l : \{z = t = 0\}$ and $S : \{f(x, y, z, t) = 0\}$. If l is the multiple line of the intersection $S \cap \Pi$, then $f(x, y, z, t) = z^2 \cdot a(x, y, z, t) + t \cdot b(x, y, z, t)$ where a is a linear form and b a quadratic form. But then $\operatorname{Sing} S \supset \{z = t = b(x, y, z, t) = 0\} \neq \emptyset$ in $\mathbb{P}^3$.

Lemma 4. *If the point $P \in S$, then all lines l on S through P are coplanar (since all such $l \subset T_P(S)$) and there are at most three such lines.*

Lemma 5. *If the line $l \subset S$, then there exist exactly 5 planes $\Pi_1, \dots, \Pi_5$ for which the corresponding conic $Q = (S \cap \Pi) \setminus l$ is reducible, that is, $S \cap \Pi_i = l \cup (l_i \cup l_i')$ (see the figure)*



Proof. Let us choose the coordinates so that the line l has equation l : $\{z = t = 0\}$ and write the equation of the cubic surface in the form

$$S : f(x,y,z,t) \equiv a_1(z,t)x^2 + b_1(z,t)xy + c_1(z,t)y^2 + a_2(z,t)x + b_2(z,t)y + a_3(z,t) = 0$$

where a_1, b_1, c_1 are linear forms, a_2, b_2 quadratic forms and a_3 a cubic form. A bundle of planes through l has equation $\Pi : \mu z = \lambda t$, and one obtains the following equation of the conic in the intersection $S \cap \Pi$:

$$a_1(\lambda,1)x^2 + b_1(\lambda,1)xy + c_1(\lambda,1)y^2 + a_2(\lambda,1)xt + b_2(\lambda,1)yt + a_3(\lambda,1)t^2 = 0$$

This conic is reducible if and only if the corresponding determinant

$$\Delta = \begin{vmatrix} a_1 & b_1/2 & a_2/2 \\ b_1/2 & c_1 & b_2/2 \\ a_2/2 & b_2/2 & a_3 \end{vmatrix} = 0$$

equals zero. This is an equation of the fifth degree in λ. It has at most five roots, that is, at most five corresponding planes in which the conic is reducible. Let us prove that there are exactly five such planes, i.e., that all these roots are different. This will follow from nonsingularity of the cubic surface S. We could suppose that one of the roots is $\lambda = 0$ i.e., that $\Pi = \{z = 0\}$ is one of these planes. The intersection $S \cap \Pi$ consists of three lines with one of the above two configuration types.

Type 1. We can choose coordinates in such way that the three lines in the plane $z = 0$ are $t = 0$, $x = 0$ and $x = t$. The corresponding equation f is then $f = x(x-t)t + zg$ where g is quadratic form. Comparing the corresponding coefficients, we obtain $a_1 = t + \alpha z$, $a_2 = -t^2 + zd_1$ where d_1 is linear form, and $z | b_1, c_1, b_2, a_3$. Since S is nonsingular at the point $(0:1:0:0)$, one has $c_1 = \gamma z$, $\gamma \neq 0$.

Type 2. We can choose coordinates in such way that the three lines in the plane $z = 0$ are $t = 0$, $x = 0$ and $y = 0$. The corresponding equation f is then

$f = xyt + zg$ where g is quadratic form. Comparing the corresponding coefficients, we obtain $b_1 = t + \alpha z$ and $z \mid a_1, c_1, a_2, b_2, a_3$. Since S is nonsingular at the point $(0:0:0:1)$, one has $a_3 = \gamma z t^2 + \cdots$, $\gamma \neq 0$.

In both cases the determinant has the form $\Delta = z^2 h - \gamma z t^4$, and $z = 0$ is its single root.

Lemma 6. *Lines from different pairs l_i, l'_i $(i = 1, \ldots, 5)$ do not intersect. This follows from Lemma 4.*

Lemma 7. *If m is a line on S that does not intersect with l, then m intersects with exactly one line of each pair l_i, l'_i $(i = 1, \ldots, 5)$.*

Proof. Line m intersects with any plane in $\mathbb{P}^3$, therefore with Π_i. However, it can not be contained in Π_i since m does not intersect with l. Since $S \cap \Pi_i = l \cup (l_i \cup l'_i)$, m has to intersect with configuration $l_i \cup l'_i$, and due to Lemma 3 it can not intersect with both of these lines.

Let now l and m be two nonintersecting lines on S. From previous results it follows that such lines exist. The line l determines 10 lines l_i, l'_i $(i = 1, \ldots, 5)$, and exactly one of the each pair intersects with m. Change the notations in such way that l_i intersect with m. The line m also determines its 10 lines, 5 pairs of lines, and exactly one of the each pair is the line l_i. Let these be the lines l_i, l''_i $(i = 1, \ldots, 5)$. Each line l''_i does not intersect with any of lines l_j $(j \neq i)$ and therefore has to intersect all lines l'_j $(j \neq i)$. One has a configuration of $1 + 1 + 5 + 5 + 5 = 17$ lines.

Lemma 8. a) *Any 4 nonincident lines on S do not belong to a nonsingular quadric. (In such case the whole quadric would be contained in S, and the cubic S would be reducible.)*

b) Any 4 nonincident lines in $\mathbb{P}^3$ that do not belong to a nonsingular quadric, could have at most two common incident lines.

Lemma 9. *If n is a line on S different from the mentioned 17, then it intersects with exactly three of five lines l_i.*

Proof. If n intersects with at least four, then $n = l$ or $n = m$, which is a contradiction. If n intersects with at most two, then it has to intersect with at least three of five lines l'_i (since it intersects with exactly one line of each pair). Let these be, say, lines l_1, l'_2, l'_3, l'_4. These four nonincident lines on S already have two common incident lines l and l''_1. It follows then from Lemma 9 that $n = l$ or $n = l''_1$ which is again a contradiction.

Lemma 10. *For any choice of three indexes $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$ there is exactly one line $l_{ijk} \subset S$ that intersects with three lines l_i, l_j and l_k.*

Proof. Choose one of the indexes, say $i = 1$ and consider the line l_1. From Lemma 5, one has 10 lines intersecting with it. Four of them are l, l'_1, m, l''_1. There are six lines left. From Lemma 9, each of them intersects with exactly two of the lines l_2, l_3, l_4, l_5. Since $\binom{4}{2} = 6$, each possibility is being realized.

There are $\binom{5}{3} = 10$ new lines. These, with previous 17, add up to 27 lines on a cubic surface S.

Theorem. (Salmon, Cayley, 1849)[11] *On any nonsingular cubic surface there are exactly* 27 *lines.*

This remarkable theorem is among the most interesting results in geometry of the last century. The configuration of 27 lines has been extensively studied. In 1869 Wiener produced a model of a cubic surface with all its 27 lines real and visible in the model (see [35, p. 127]). The automorphism group of the configuration of 27 lines was first studied by Jordan[12][11]. The order of that group is $51840 = 2^7 3^4 5$, and it has been later classified as the Weyl group E_6. It has a simple subgroup of index 2 and of order 25920. There is a reach literature concerning 27 lines. In the 20th century it has been slowly (and unjustly) forgotten. With the renaissance of algebraic geometry in the fifties, the investigation of cubic surfaces had its culmination in papers and the book of Manin [21], and the 27 lines theorem became an inevitable part of many introductory courses of algebraic geometry. Our proof follows the book of Reid [25].

9. Number of equations. Multiple subvarieties. Weil divisors

In general, each additional equation in an affine or projective set's defining system decreases the dimension of the solution set by 1. However, it can happen that adding the equation does not change the dimension (if the equation is already contained in the ideal generated by previous ones). In other words, not all closed sets in $\mathbb{A}^n$ or $\mathbb{P}^n$ of codimension k could be defined by k equations. One has only $\operatorname{codim} V(f_1, \dots, f_k) \leq k$.

Definition. The variety $X \subset \mathbb{A}^n$ of codimension k (and dimension $n-k$) is a *complete intersection* if $I(X) = (f_1, \dots, f_k) \subset K[x_1, \dots, x_n]$, and *set-theoretic complete intersection* if $I(X) = \sqrt{(f_1, \dots, f_k)}$. Clearly, each complete intersection is a set-theoretic one, but the converse does not hold. The variety X is a set-theoretic intersection if it can be represented as intersection of k hypersurfaces.

Examples. 1. [8, p. 242], [31, p. 290, ex. 4.9] In $\mathbb{A}^4$ the set $V(x_1, x_2) \cup V(x_3, x_4)$ has codimension 2, but can not be defined with two equations.

2. [31, p. 32, ex. 2.17] If $X \subset \mathbb{P}^3$ is the projective closure of the space cubic

$$x = t, \quad y = t^2, \quad z = t^3$$

the homogeneous ideal $I(X)$ can not be generated by 2 elements.

3. [31, p. 25, ex. 1.11] Affine space cubic can be defined with two equations, as intersection of two quadrics, a cylinder and a cone, since $I(X) = (y - x^2, y^2 - xz)$. However, if $X \subset \mathbb{A}^3$ is a space curve

$$x = t^3, \quad y = t^4, \quad z = t^5$$

[11] George Salmon (1819–1904), Irish mathematician. Arthur Cayley (1821–1895), English mathematician.

[12] Camille Jordan (1838–1922), French mathematician

then the corresponding ideal can not be generated by 2 elements. The intersection of any two of these three surfaces, besides the space curve, contains a coordinate axis. The general case of a space curve

$$x = t^{n_1}, \quad y = t^{n_2}, \quad z = t^{n_3}$$

has been treated only recently [32]. If $c_i \in \mathbb{N}$ are the least positive integers such that $n_i c_i \in n_j\mathbb{N}+n_k\mathbb{N}$ (here, the triple (i,j,k) goes through all three cyclic permutations of the triple $(1,2,3)$), then the ideal $I(X)$ has

(a) either two generators, $I(X) = (x^{c_1} - z^{c_3}, y^{c_2} - x^{r_{21}}z^{r_{23}})$ and the curve is a complete intersection. Example: $(n_1,n_2,n_3) = (4,5,6)$, $I(X) = (x^3 - z^2, y^2 - xz)$;

(b) or three generators, $I(X) = (x^{c_1} - y^{r_{12}}z^{r_{13}}, y^{c_2} - x^{r_{21}}z^{r_{23}}, z^{c_3} - x^{r_{31}}y^{r_{32}})$ and the curve is not a complete intersection. Example: $(n_1,n_2,n_3) = (3,4,5)$, $I(X) = (x^3 - yz, y^2 - xz, z^2 - x^2y)$.

Cases (a) and (b) could be distinguished algorithmically. The case (b) however, is a set-theoretic complete intersection, since there is always a polynomial $p(x,y,z)$ such that $I(X) = \sqrt{(p, z^{c_3} - x^{r_{31}}y^{r_{32}})}$.

In codimension 1 the corresponding equality holds. Any subvariety in $\mathbb{A}^n$ or $\mathbb{P}^n$ of codimension 1 can be given by one equation: it is a hypersurface. More generally, if X is a nonsingular projective or affine variety and $Y \subset X$ a subvariety of codimension 1, then near each of its points, it is defined by one equation, that is, $\forall x \in Y \ \exists U \ni x$ such that $I(Y \cap U) = (f) \subset K[U]$ is principal. In the proof, the factorial property of the local ring of regular point is used essentially [8, p. 241], [33, t. 1, pp. 90, 134].

Let us consider the multiplicity. Already in the case of curves we have seen that one has to take it into account. Radical ideals were introduced in order to remove the nilpotents from the coordinate ring. When one considers intersections of subvarieties, it becomes more complex. Let X be a variety and $Y, Z \subset X$ two its subvarieties with corresponding ideals $I(Y), I(Z)$. Then the intersection $Y \cap Z = V(I(Y) + I(Z))$. However, the sum of two radical ideals does not have to be radical, it can contain nilpotents.

Example 1. Let $X = \mathbb{A}^2$, $K[X] = K[x,y]$ and let $Y = V(y)$, $Z = V(y - x^2)$ be irreducible curves. The ideal $I(Y) + I(Z) = (y) + (y - x^2) = (x^2, y)$ and the coordinate ring $K[Y \cap Z] = K[x,y]/(x^2,y)$ has a nilpotent x. This corresponds to the intuitively clear fact that the parabola and the line are tangent to each other in their intersection point, the multiplicity of that point being 2.

Example 2. [14, p. 28] Consider the variety $V = \{xz = yz = 0\} \subset \mathbb{A}^3$ in the affine space. One has $V = V_1 \cup V_2$, where $V_1 = \{x = y = 0\}$ is the coordinate axis and $V_2 = \{z = 0\}$ the coordinate plane. For corresponding ideals one has $I(V) = (xz, yz) = (x,y) \cdot (z) = I_1 \cdot I_2$. The ideals $I_1 = I(V_1)$ and $I_2 = I(V_2)$ are prime, $I(V)$ is radical and $V = V_1 \cup V_2$ is the irreducible decomposition of V. The corresponding coordinate ring is $K[V] = K[x,y,z]/(xz,yz)$. Consider now the plane $W = \{x = z\} \subset \mathbb{A}^3$ with the ideal $I(W) = (x - z)$, and find the intersection $V \cap W$. The corresponding ideal is $I(V \cap W) = I(V) + I(W) = (xz, yz, x - z)$, the coordinate ring $K[V \cap W] = K[x,y,z]/(xz, yz, x-z) \cong K[x,y]/(x^2, xy)$. This is not an affine algebra: it contains a nilpotent x. This reflects the fact that in some way the origin is a double point of the intersection $V \cap W$, since it belongs to both components of the variety V.

Examples show that, although radical ideals helped to avoid varieties with multiple components, the multiplicity still appears when one considers intersections of varieties. The notion of multiplicity is one of the fundamental notions in algebraic geometry. To work with it, one must assign multiplicities to subvarieties and in this way introduce a new type of objects. The codimension 1 case is the simplest, since each such subvariety can be defined by a single equation. We will consider only this case.

Definition. Let X be an irreducible nonsingular variety. Consider the set of all its irreducible subvarieties of codimension 1, and call its elements *simple divisors*. The free abelian group generated by this set is denoted Div X and called the *divisor group* of the variety X, its elements are *divisors* on X. A divisor is, therefore, a formal linear combination $D = n_1C_1 + \cdots + n_kC_k$ of irreducible subvarieties $C_1, \ldots, C_k \subset X$ of codimension 1 with integer coefficients $n_1, \ldots, n_k \in \mathbb{Z}$. If all coefficients are nonnegative, we say that the divisor is *effective* and denote this by $D \geq 0$. Each divisor can be represented as a difference of two effective divisors. The number $d = n_1 + \cdots + n_k$ is called the *degree* of the divisor $D = n_1C_1 + \cdots + n_kC_k$ and defines an epimorphism $\deg : \mathrm{Div}(X) \to \mathbb{Z}$.

Divisors introduced by this definition are called sometimes *Weil divisors*[13], as opposed to more general *Cartier divisor*, which will be introduced later.

To a rational function $f(t) = \frac{(t-P_1)^{n_1}\ldots(t-P_k)^{n_k}}{(t-Q_1)^{m_1}\ldots(t-Q_l)^{m_l}} \in K(t)$ on the affine line $X = \mathbb{A}^1$ one could associate a divisor $D = n_1P_1 + \cdots + n_kP_k - m_1Q_1 - \ldots - m_lQ_l$, as a formal linear combination of zeros and poles with corresponding multiplicities as coefficients, the so-called *divisor of zeros and poles* of the rational function. The analogous construction is possible in the general case. Let X be an irreducible variety and $f \in K(X)$ a rational function on X, $f \neq 0$. If $C \subset X$ is an irreducible subvariety of codimension 1, then it is locally defined by one equation, that is, in some nonempty open set $U \subset X$ one has $C \cap U = V(p)$ where $p \in K[U]$.

Definition. 1) If the function $f \in K[U]$, that is, it is regular on U, then, since the intersection $\bigcap(p^k) = 0$, $\exists k \geq 0$ such that $p^k \mid f$, $p^{k+1} \nmid f$. This integer is uniquely determined and does not depend on the choice of local parameter p. The number k is called the *order* of (regular) function f along subvariety C and denoted $k = \mathrm{ord}_C f$.

2) If the function f is not regular, it has a representation $f = g/h$ where $g, h \in K[U]$. The *order* of (rational) function f along subvariety C is defined as the integer $\mathrm{ord}_C f = \mathrm{ord}_C g - \mathrm{ord}_C h$. This number does not depend on the choice of nonempty open set U. This follows from irreducibility of X, since two nonempty open sets always intersect and their intersection is nonempty and open.

[13]after André Weil (1906–), French mathematician, one of the founders of the Bourbaki group, and not after Hermann Weyl (1885–1955), German mathematician and physicist.

Lemma. *Order of function along subvariety has the following properties:*

1) $\operatorname{ord}_C(fg) = \operatorname{ord}_C f + \operatorname{ord}_C g$*;*

2) $\operatorname{ord}_C(f+g) \geq \min\{\operatorname{ord}_C f,\ \operatorname{ord}_C g\} \quad (f+g \neq 0)$.

Lemma. *For a given rational function f there are only finitely many irreducible subvarieties C of codimension 1 for which* $\operatorname{ord}_C f \neq 0$.

Definition. If $f \in K(X)^*$ is a rational function, the formal linear combination $(f) = \sum_{C \subset X} (\operatorname{ord}_C f) \cdot C$ has finitely many terms and represents a divisor on X. It is called the divisor of the function f. The sum of all terms with positive coefficient is the *divisor of zeros* $(f)_0$, and with negative coefficient the *divisor of poles* $(f)_\infty$ of the function f. One has $(f)_0 \geq 0$, $(f)_\infty \geq 0$ and $(f) = (f)_0 - (f)_\infty$.

The mapping div : $K(X)^* \to \operatorname{Div}(X)$, $f \mapsto (f)$ is a homomorphism of groups (the first group is multiplicative, the second one additive). Namely, one has $(fg) = (f) + (g)$.

The divisor of a regular function is effective. The converse also holds.

Lemma. *If $f \in K(X)^*$, then $(f) \geq 0 \Leftrightarrow f \in K[X]$.*

Proof. Let f be nonregular in $x \in X$. One has a representation $f = g/h \notin \mathcal{O}_x$, $g, h \in \mathcal{O}_x$. Since the ring $\mathcal{O}_x$ is factorial, one could consider g and h relatively prime. Let p be a prime factor of h, which does not divide g. The variety $V(p)$ has in some open neighborhood of x codimension 1, therefore $\overline{V(p)} = C \subset X$ is a subvariety of codimension 1 and $ord_C f < 0$.

Corollary. *On a nonsingular projective variety, rational function is determined uniquely up to constant factor by its divisor.*

Proof. If $(f) = (g)$, then $0 = (f) - (g) = (fg^{-1})$ and fg^{-1} is a global regular function on a projective variety, that is, constant.

Definition. Divisors of the form (f) where f is rational function on X, are called *principal divisors*. They form a subgroup $P(X) \subset \operatorname{Div} X$ of principal divisors in the group of all divisors. It is the image of the homomorphism div : $K(X)^* \to \operatorname{Div}(X)$.

Is every divisor principal? In other words, could one represent a given divisor as a divisor of zeros and poles of some rational function? The answer depends on variety X and it is not always affirmative. There could be also nonprincipal divisors. More precise answer is given by the factorgroup $\operatorname{Div}(X)/P(X) = \operatorname{Cl}(X)$, the *divisor class group* of the variety X. This quotient introduces a relation of linear equivalence of divisors: $D_1 \sim D_2 \Leftrightarrow D_1 - D_2 = (f)$ for some global rational function f.

Examples. 1. $\operatorname{Cl}(\mathbb{A}^n) = 0$. More generally, if the ring $K[X]$ is factorial, then $\operatorname{Cl}(X) = 0$.

2. $\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}$. [33, t. 1, p. 188], [31, p. 175]. Each irreducible subvariety $C \subset \mathbb{P}^n$ of codimension 1 is globally defined by a homogeneous equation, that is,

by irreducible homogeneous polynomial F, of degree m. Its affine parts are of the form $C \cap \mathbb{A}_0^n = V(F/x_0^m)$. Let $D = n_1C_1 + \cdots + n_kC_k$ be an effective divisor on $\mathbb{P}^n$ (with all $n_i > 0$), forms F_i define subvarieties C_i and let $F = F_1^{n_1} \cdots F_k^{n_k}$ be the corresponding product. Then $(F) = n_1(F_1) + \cdots + n_k(F_k) = D$ (more precisely, this holds in affine chart). In other words, each effective divisor on $\mathbb{P}^n$ is a divisor of a homogeneous polynomial form. Let now D be an arbitrary divisor, $D = D_1 - D_2$ its decomposition as difference of effective divisors, and $D_i = (G_i)$ $(i = 1, 2)$ their representation by homogeneous forms. Let $d_i = \deg D_i$ and $d = \deg D = d_1 - d_2$. Consider the rational function $f = G_1/x_0^d G_2$ and its principal divisor (f). This is a global rational function of degree 0, that is, an element of the field $K(\mathbb{P}^n)$. Its principal divisor is $(f) + dH = D_1 - D_2 = D$, where H is the divisor of the hyperplane $x_0 = 0$ (the *hyperplane section divisor*). This means that $D \sim dH$ and $\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z} \cdot H \cong \mathbb{Z}$.

Theorem. [31, p. 176] *Let X be a nonsingular variety, $Y \subset X$ its subvariety and $U = X \setminus Y$. Then*

(1) the mapping $\sum n_iC_i \mapsto \sum n_i(C_i \cap U)$ is an epimorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$;

(2) if $\mathrm{codim}_X Y \geq 2$, then it is an isomorphism $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$;

(3) if $\mathrm{codim}_X Y = 1$, then $1 \mapsto 1 \cdot Y$ defines a mapping $\mathbb{Z} \to \mathrm{Cl}(X)$ and the sequence $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$ is exact.

Example. If $Y \subset \mathbb{P}^2$ is an irreducible curve of degree d, then $\mathrm{Cl}(\mathbb{P}^2 \setminus Y) \cong \mathbb{Z}_d$. This can be easily proved by the previous theorem.

10. The divisor class group of nonsingular quadric and cone

Let us now determine the divisor class group of the nonsingular quadric Q. We shall represent the quadric by the so-called Segre embedding[14]. Stop for the moment to define the product of varieties. The set-theoretic (Cartesian) product of affine spaces is again an affine space: $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$, and similarly for corresponding closed subsets. However, for projective closed sets, the situation is more complex. How should one define a structure of a projective variety on the set-theoretic product of two projective lines? Define the Segre embedding $S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $(x_0 : x_1) \times (y_0 : y_1) \mapsto (z_{00} : z_{01} : z_{10} : z_{11})$ with $z_{ij} = x_iy_j$. The image $S(\mathbb{P}^1 \times \mathbb{P}^1)$ is on the quadric $Q = \{z_{00}z_{11} = z_{01}z_{10}\} \subset \mathbb{P}^3$. Conversely, if the point $(z_{00} : z_{01} : z_{10} : z_{11}) \in Q$, then at least one coordinate, say $z_{00} \neq 0$, and $S((z_{00} : z_{01}) \times (z_{00} : z_{10})) = (z_{00}z_{00} : z_{00}z_{01} : z_{00}z_{10} : z_{01}z_{10}) = (z_{00} : z_{01} : z_{10} : z_{11})$. So, the mapping S defines bijection $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q$, which makes it possible to transport the structure of algebraic variety, induced on the quadric Q from the ambient space $\mathbb{P}^3$, on the set $\mathbb{P}^1 \times \mathbb{P}^1$. Could one define this structure independently from the embedding? Homogeneous polynomial on Q has the form $F(z_{00}, z_{01}, z_{10}, z_{11}) = F(x_0y_0, x_0y_1, x_1y_0, x_1y_1) = G(x_0, x_1; y_0, y_1)$. This is a polynomial in two groups of variables, homogeneous in each group separately. The degree in each of the groups of variables need not to be equal: if $s = \deg_y G < \deg_x G = r$, then the equation

[14] Corrado Segre (1863–1924), Italian geometer, famous by his work in birational geometry.

$G = 0$ is equivalent to the system $y_0^{r-s}G = y_1^{r-s}G = 0$. Closed subsets in $\mathbb{P}^1 \times \mathbb{P}^1$ are defined by systems of polynomial equations of the form $G(x_0, x_1; y_0, y_1) = 0$, homogeneous in each group of variables separately. Subvarieties of codimension 1 in $\mathbb{P}^1 \times \mathbb{P}^1$ are defined by one such equation, as in standard projective space. Namely, if the polynomial $F(x_0, x_1; y_0, y_1)$ is homogeneous in each group of variables separately, and if F factorizes in product of two polynomials $F = G \cdot H$, then each of the factors must have the same homogeneity property. Let us determine the divisor class group $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$. The given divisor $D \in \mathrm{Div}(\mathbb{P}^1 \times \mathbb{P}^1)$, in addition to the usual degree $\deg D$, has two degrees $\deg_x D$ and $\deg_y D$ in each of the groups of variables, and one has $\deg D = \deg_x D + \deg_y D$. In this way, one obtains an epimorphism $\mathrm{Div}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{Z}^2$, $D \mapsto (\deg_x D, \deg_y D)$. It is straightforward to check that its kernel is exactly the principal divisor subgroup, so $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$. The pair $(\deg_x D, \deg_y D)$ is called type of the divisor D. Segre embedding $S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defines a homomorphism $\mathrm{Div}(\mathbb{P}^3) \to \mathrm{Div}(\mathbb{P}^1 \times \mathbb{P}^1)$ by intersection with Q. It extends to classes of divisors, and coincides with the diagonal embedding $\mathrm{Cl}(\mathbb{P}^3) = \mathbb{Z} \to \mathbb{Z}^2 = \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$, $1 \mapsto (1,1)$.

Example. Apply this to the case of the projective space cubic C. It has the following parametrization:

$$z_{00} = u^3, \quad z_{01} = u^2v, \quad z_{10} = uv^2, \quad z_{11} = v^3$$

Obviously, $C \subset Q$. Is it possible to represent C as intersection of the quadric and some surface? Consider the cone $K : z_{01}z_{11} = z_{10}^2$. The intersection is $K \cap Q = L \cup C$ where L is a line. The divisor class group is $\mathrm{Cl}(\mathbb{P}^3) = \mathbb{Z} \cdot H \cong \mathbb{Z}$ where H is the divisor of the (hyper)plane section, so $K \sim 2H$. The diagonal embedding gives $K = 2 \mapsto 2(1,1) = (2,2) = K \cap Q$. The type of this divisor is $K \cap Q = L + C = (2,2)$, and the type of the divisor $L = (1,0)$. Therefore, one has the type of $C = (2,2) - (1,0) = (1,2)$. Let now $Y \subset \mathbb{P}^3$ be the surface such that its intersection with the quadric is $Y \cap Q = C$. Then the type of the divisor $Y \cap Q$ is, on one side, $rC = r(1,2) = (r, 2r)$ and on the other, $dH = d(1,1) = (d,d)$. Since $(r, 2r) \neq (d,d)$, this means that the answer to the above question is negative: there is no surface which would intersect the quadric Q by the curve C!

Consider the cone $X = V(xy - z^2) \subset \mathbb{A}^3$, with ideal $I = I(X) = (xy - z^2) \subset K[x,y,z]$ and coordinate ring $K[X] = K[x,y,z]/(xy-z^2)$, and determine its divisor class group. The generators of the ring, the images of the indeterminates x, y, z we will denote also x, y, z. The directrisse of the cone is the line $Y = V(y,z) \subset X \subset \mathbb{A}^3$, and this is an irreducible subvariety in X of codimension 1 – a simple divisor in X. Corresponding chain of ideals in $K[x,y,z]$ is

$$(0) \subset (xy - z^2) \subset (y,z) \subset (x,y,z)$$

However, after the factorization by $I(X)$ one obtains the ideal $I_X(Y) = (y,z) \subset K[X]$, which is of height 1 but not principal! The corresponding chain of ideals is

$$(0) \subset (xy - z^2) \subset (y,z) \subset (x,y,z)$$

If $(y,z) = (f)$ in $K[X]$, then the corresponding originals in $K[x,y,z]$ would satisfy $y,z \in (f, xy - z^2)$, $f \in (y,z,xy-z^2) = (y,z)$, and this gives a contradiction when one considers f modulo ideal $(x,y,z)^2$ (that is, its linear components). So, $Y \neq 0$ in $\mathrm{Cl}(X)$! Let us prove that $2Y = 0$ in $\mathrm{Cl}(X)$, or that $2Y$ is principal. Consider the function $y \in K[X]$, and find a local equation of the set $V(y) \cap X = Y$ in the open subset $U = D(x) \cap X$. Then $K[U] = K[x,y,z,x^{-1}]/(xy - z^2) = K[x,x^{-1},z]$, $y = x^{-1}z^2$. One has $Y \cap U = V(y,z) \cap U$, the corresponding coordinate ring is $K[Y \cap U] = K[x,x^{-1},z]/(x^{-1}z^2, z) = K[x,x^{-1},z]/(z)$. The local equation of Y in U is $z = 0$, the local parameter z. The function $y \in K[U]$ has the form $y = x^{-1}z^2$. Therefore $\nu(y) = 2$ and the principal divisor is $(y) = 2Y$. Now, there is the exact sequence $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Y) \to 0$, where the first mapping is $1 \mapsto 1 \cdot Y$. The ring $K[X \setminus Y] = K[X \cap D(y)] = K[y, y^{-1}, z]$ is factorial, so $\mathrm{Cl}(X \setminus Y) = 0$. Therefore $\mathbb{Z} \to \mathrm{Cl}(X)$ is an epimorphism and $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

11. Group of points of nonsingular cubic. Elliptic curves

As we have seen, the degree of the divisor defines a natural homomorphism of the divisor group on the group of integers $\mathrm{Div}(X) \to \mathbb{Z}$. In some cases it factors through principal divisors (that is, the principal divisors have degree 0) and defines epimorphism $\mathrm{Cl}(X) \to \mathbb{Z}$. This was the case for projective space. This is also the case for nonsingular projective curves.

Theorem. [33, t. 1, pp. 205–209] *If X is a nonsingular projective curve, the degree of any principal divisor is* 0.

The kernel $\mathrm{Cl}^0(X)$ of $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is an important subgroup in $\mathrm{Cl}(X)$:

Theorem. *The following statements are equivalent:*

(1) the curve X is rational;

(2) the group $\mathrm{Cl}^0(X) = 0$, that is, $\mathrm{Cl}(X) \cong \mathbb{Z}$;

(3) there exist two different points $P, Q \in X$ such that $P \sim Q$.

Proof. If $\mathrm{Cl}^0(X) = 0$, each divisor of degree 0 is principal, and for any two different points $P, Q \in X$ there is a nonconstant rational function $f \in K(X)$ such that $P - Q = (f)$. This function defines a rational mapping $\varphi : X \to \mathbb{P}^1$ and $K(X) = K(f)$, P is a zero and Q is a pole of the function f.

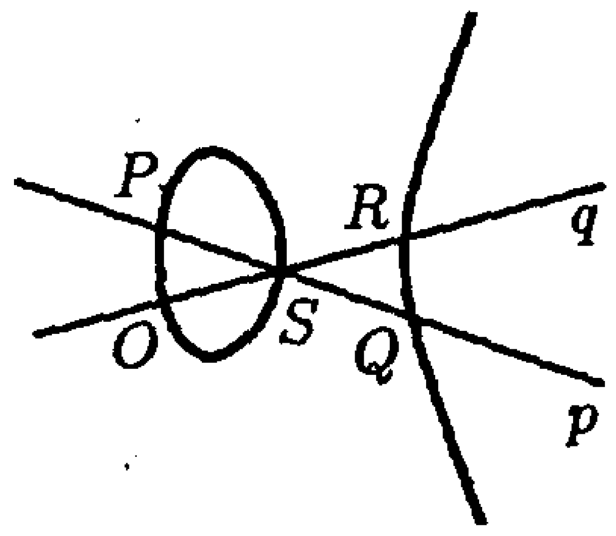
Theorem. *If X is a nonsingular cubic, then there exists a bijection $\mathrm{Cl}^0(X) = 0$, which induces the structure of Abelian group in the set of points of X.*

Proof. Let $O \in X$ be an arbitrary but fixed point. Define the mapping $X \to \mathrm{Cl}^0(X)$, $P \mapsto C_P = P - O$. It is injective since $C_P = C_Q \Rightarrow P - O \sim Q - O \Rightarrow P \sim Q$. In order to show that this mapping is also surjective, let us prove that any effective divisor $D \in \mathrm{Div}\, X$, $D > 0$, is equivalent to the divisor of the form $P + kO$, by induction on $\deg D$.

1. If $\deg D = 1$, then $D \sim P = P + 0 \cdot O$.

2. Let $\deg D > 1$. Then $D = D' + P$, $\deg D' = \deg D - 1$, $D' > 0$. By induction $D' \sim P + l \cdot O$. Then $D \sim P + Q + l \cdot O$. Find the point R such that

$P+Q \sim R+O$. We do this by a geometrical construction. Suppose that points P and Q are different, and let p be the line that they determine. Let S be the third point of intersection of that line and the cubic X. Let q be the line through O and S and let R be the third intersection point of the line q and cubic X. Then one has $P+Q+S \sim (p) \sim (q) \sim O+S+R$ and $P+Q \sim R+O$ (se the figure). In the case when some points coincide ($P=Q$ or $O=S$), one takes tangents instead of secants p and q.



We have proved that any effective divisor $D>0$ on X is equivalent to the divisor of the form $P+kO$, where O is a fixed point. Obviously, $k=\deg D-1$. If now D is a divisor of degree 0, then it is a difference of two effective divisors of the same degree $D=D_1-D_2 \sim (R+k\cdot O)-(Q+k\cdot O)=R-Q$. Using the same geometrical construction as above in reverse order, one finds the point S as intersection of X and the line through R and O, and then point P as intersection of the line through Q and S (see the same figure). One obtains $P+Q \sim R+O$ or $D \sim R-Q \sim P-O=C_P$. Therefore, the mapping $P \mapsto C_P$ is bijective.

This bijection introduces a structure of an Abelian group on the set of points of the curve X. This structure is defined purely geometrically, by the constructions described above. The point O is the neutral element. If P and Q are two points of our curve, the point R is their sum, and $S=-R$. One could prove directly that this is a group. The most complicated part is the proof of associative law, elementary but long.

How many nonisomorphic nonsingular cubics do exist? We shall give the answer to this question for complex ground field $\mathbb{C}$. The equation of the plane nonsingular cubic is $y^2=P_3(x)$ where the right-hand-side polynomial of the third degree does not have multiple roots. By translation and homothety in x one could obtain two of its three roots to be 0 and 1. In other words, $P_3(x)=x(x-1)(x-\lambda)$ where the third root $\lambda \neq 0,1$ parametrizes all such curves. However, for different parameter values one could get isomorphic curves: curves $y^2=x(x-1)(x+1)$ and $y^2=x(x-1)(x-2)$ are obviously isomorphic by isomorphism $x \mapsto 1-x$.

The three element permutation group S_3 (with six permutations) acts on the root triple $(0,1,\lambda)$. If we apply the linear transformation on x again after permutation, to obtain the root triple $(0,1,\bar{\lambda})$, it is easy to check that $\bar{\lambda}$ could take one of the following six values: λ, $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{\lambda-1}$, $\frac{\lambda-1}{\lambda}$. It follows directly that all six curves which correspond to these parameter values are isomorphic. Let us produce a function in λ, invariant with respect to this action and in this way remove the six-ply ambiguity. An obvious candidate would be

$$Q(\lambda) = (\lambda+1)\Big(\frac{1}{\lambda}+1\Big)((1-\lambda)+1)\Big(\frac{1}{1-\lambda}+1\Big)\Big(\frac{\lambda}{\lambda-1}+1\Big)\Big(\frac{\lambda-1}{\lambda}+1\Big)$$
$$= -\frac{(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2}{\lambda^2(\lambda-1)^2}$$

and another

$$J(\lambda) = 1 - \frac{1}{27}Q(\lambda) = 1 + \frac{(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2}{27\lambda^2(\lambda-1)^2} = \frac{4}{27}\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

Definition. If X is a nonsingular cubic with the equation $y^2 = x(x-1)(x-\lambda)$ $(\lambda \neq 0, 1)$, the complex number

$$j(X) = 2^6 3^3 J = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

is called the ***j-invariant*** of the curve X.

The function $\lambda \mapsto j(X)$ defines a six-to-one covering $j : \mathbf{A}^1 \to \mathbf{A}^1$, $\left\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\right\} \mapsto j(X)$, branched over points 0 and 1. The value of the j-invariant classifies nonsingular cubics, as the following theorem shows.

Theorem. [15, p. 249], [31],... *a) Two cubics X are Y isomorphic $\Leftrightarrow j(X) = j(Y)$.*

b) For any complex number a there is a curve X with $j(X) = a$.

Therefore, nonisomorphic nonsingular cubics are parametrized by points of complex line.

Nonsingular cubics are called also *elliptic curves*. This name originates from *elliptic integrals*. When the arc length of the ellipse (and other curves) is being calculated, the integrals of functions with radicals $\sqrt{P_4(x)}$ appear. In the spirit of lecture 1, these integrals are connected to curves $y^2 = P_4(x)$, which are birationally isomorphic to nonsingular cubics.

Classical theory of elliptic integrals culminated in the middle of the last century in the works of Legendre [15]. He reduced all elliptic integrals to the following three basic types: first type $F(\varphi) = \int_0^\varphi \frac{dx}{\sqrt{1-k^2\sin^2 x}}$, second type $E(\varphi) = \int_0^\varphi \sqrt{1-k^2\sin^2 x}\,dx$ and third type $G(\varphi) = \int_0^\varphi \frac{dx}{(\sin x - c)\sqrt{1-k^2\sin^2 x}}$. The arc length of the ellipse is expressed by the elliptic integral of the second type, and the first type appears in the arc length of the lemniscate. There is a family of curves with this property, discovered by Serret [28], which is connected to some interesting questions of the theory of elliptic curves and arithmetic. For more details see [23], [18].

As we have seen, the set of points of elliptic curve forms an Abelian group. What is this group?

Theorem. *As a group, the elliptic curve X is a two-dimensional torus:* $X \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.

Here $\Lambda \subset \mathbb{C}$ is a lattice, that is, a free additive subgroup $\Lambda = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ of rank 2 over $\mathbb{Z}$ ($\tau \notin \mathbb{R}$).

Proof. [31, pp. 414–416] We will sketch the proof. It requires some classical theory of complex functions.

Definition. The function $\wp(z) = \frac{1}{z^2} + \sum\limits_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$ of the complex argument z is called *Weierstrass function.*

The Weierstrass function and its derivative $\wp'(z) = -\sum_{\omega\in\Lambda} \frac{2}{(z-\omega)^3}$ are doubly periodic complex functions with periods 1 and τ, in other words Λ-periodic functions. They satisfy the equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ where coefficients $g_2 = 60 \sum\limits_{\omega\in\Lambda\setminus\{0\}} \omega^{-4}$ and $g_3 = 140 \sum\limits_{\omega\in\Lambda\setminus\{0\}} \omega^{-6}$, and the discriminant $\Delta = g_2^3 - 27g_3^2 \neq 0$.

It follows from these properties that if the lattice Λ is given, then the mapping $\mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ defined with $z \mapsto (\wp(z), \wp'(z))$ factors through homomorphism $\mathbb{C} \to \mathbb{C}/\Lambda$ and induces a bijection of the torus $\mathbb{C}/L$ and the cubic $y^2 = 4x^3 - g_2x - g_3$. Conversely, if the cubic is given, that is, two numbers g_2 and g_3 satisfying $\Delta = g_2^3 - 27g_3^2 \neq 0$, then one could show the existence of the lattice Λ such that its Weierstrass function satisfies the given equation.

One could obtain also the connection between j-invariant of the curve and its coefficients: $J = g_2^3/\Delta$ or $j(X) = 1728g_2^3/\Delta$.

Any lattice Λ or a complex number $\tau \notin \mathbb{R}$ defines an elliptic curve. Obviously, different values of τ could define isomorphic curves, exactly when their j-invariants coincide. The following theorem describes when this takes place.

Theorem. [31, p. 416] $J(\tau) = J(\tau') \Leftrightarrow \tau' = \frac{a\tau+b}{c\tau+d}$ *for some regular integer matrix* $\binom{a\ b}{c\ d} \in GL_2(\mathbb{Z})$

In the sequel we shall show the connection between elliptic curves and number theory.

Let X be an elliptic curve with fixed origin O and corresponding group structure. For any $n \in \mathbb{N}$ one has the mapping $\varphi_n : X \to X,\ P \mapsto nP$ with $\varphi_n(O) = O$. One could show that this is a regular (polynomial) morphism and homomorphism of the group structure.

Definition. *Endomorphism* of the elliptic curve with fixed origin (X, O) is algebraic morphism $f : X \to X$ which maps the point O again in O.

If f and g are two endomorphisms, define their sum in a usual way, pointwise $(f+g)(P) = f(P) + g(P)$, and their product as composition $(f \cdot g)(P) = f(g(P))$. The zero-element will be the constant function $O(P) = O$, the neutral element - the identity $1(P) = P$.

Lemma. *Endomorphism of elliptic curve is a homomorphism of its group structure. The set* $\mathrm{End}(X, O)$ *of all endomorphisms is a ring.*

Theorem. [31, p. 417], [15, p. 18] *There exists a bijection between endomorphisms of the elliptic curve* (X, O) *and complex numbers* $\alpha \in \mathbb{C}$ *which leave the lattice invariant* $\alpha\Lambda \subset \Lambda$. *In such way, an embedding is defined.*

Proof. From the Serre's theorem on isomorphism of algebraic and analytical structures on complex algebraic varieties [27], algebraic morphisms $f : X \to X$ correspond to holomorphic morphisms $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$. Each such mapping extends to the mapping $\bar{f} : \mathbb{C} \to \mathbb{C}$, $\bar{f}(\Lambda) \subset \Lambda$. It is a holomorphic homomorphism in the neighborhood of 0, so $\bar{f}(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $\bar{f}(z_1 + z_2) = \bar{f}(z_1) + \bar{f}(z_2)$. Comparing the coefficients in the equality $a_0 + a_1(z_1 + z_2) + a_2(z_1 + z_2)^2 + \cdots = (a_0 + a_1 z_1 + a_2 z_1^2 + \cdots) + (a_0 + a_1 z_2 + a_2 z_2^2 + \cdots)$ one obtains $a_0 = a_2 = a_3 = \cdots = 0$ and $\bar{f}(z) = a_1 z$.

So, $\mathrm{End}(X, O) = R = \{\alpha \in \mathbb{C} | \alpha\Lambda \subset \Lambda\}$ is a ring, $\mathbb{Z} \subset R \subset \mathbb{C}$.

Let us now analyze more closely rings of the form $R = \{\alpha \in \mathbb{C} | \alpha\Lambda \subset \Lambda\}$ where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ ($\tau \notin \mathbb{R}$) is a given lattice. Note that $R \subset \Lambda$, $R\Lambda \subset \Lambda$, that is, the lattice Λ is a R-module.

Lemma. *Each* $\alpha \in R$ *is integral algebraic number, and* R *is a subring of the ring of integral algebraic numbers* $\mathbb{O}$.

Proof. $\alpha\Lambda \subset \Lambda \Leftrightarrow \alpha \cdot 1 = a + b\tau$, $\alpha \cdot \tau = c + d\tau$ where a, b, c, d are integers. Therefore, $(a - \alpha)(d - \alpha) - bc = 0$ and $\alpha^2 - (a + d)\alpha + (ad - bc) = 0$.

Clearly, integers do leave any lattice invariant, or $\mathbb{Z} \subset R$. When does the lattice admit nontrivial endomorphisms?

Lemma. *The ring* R *is strictly greater than the ring of integers* $\mathbb{Z} \Leftrightarrow \tau$ *is algebraic number of degree 2 over* $\mathbb{Q}$.

Proof. One has: $\exists \alpha \in R, \alpha \notin \mathbb{Z}, \alpha\Lambda \subset \Lambda \Leftrightarrow \exists a, b, c, d \in \mathbb{Z}, b \neq 0$ such that $\alpha \cdot 1 = a + b\tau$, $\alpha \cdot \tau = c + d\tau$, and by elimination of α one obtains $b\tau^2 + (a - d)\tau - c = 0$. Conversely, if $\tau \in \mathbb{Q}(\sqrt{-D}) = \mathbb{Q}[\sqrt{-D}]$ for some $D \in \mathbb{Z}$, $D > 0$, $\tau = r + s\sqrt{-D}$, then

$$R = \{\alpha = a + b\tau \mid \alpha\tau = a\tau + b\tau^2 \in L\} = \{a + b\tau \in L \mid b\tau^2 \in L\}$$
$$= \{a + b\tau \mid a, b, 2br, b(r^2 + Ds^2) \in Z\}$$

since $b\tau^2 = -b(r^2 + Ds^2) + 2br\tau$. It is clear that R is strictly greater than $\mathbb{Z}$.

One has $R \subset O = \mathbb{O} \cap K$ where the field $K = \mathbb{Q}(\tau) = \mathbb{Q}[\tau] = \mathbb{Q}[\sqrt{-D}]$, and O is its ring of integers. Note that the lattice Λ is a projective R-module, since $R \otimes \mathbb{Q} = L \otimes \mathbb{Q} = \mathbb{Q}[\tau]$. One has $\mathrm{rank}_{\mathbb{Z}} R = \dim_{\mathbb{Q}} R \otimes \mathbb{Q} = \mathrm{rank}_{\mathbb{Z}} O = 2$. This means that for some $\rho \in O$, $O = \mathbb{Z} + \mathbb{Z}\rho$. Then $R \cap \mathbb{Z}\rho$ is a subgroup in $\mathbb{Z}\rho$, necessary of the form $R \cap \mathbb{Z}\rho = c \cdot \mathbb{Z}\rho$ for some positive integer $c \in \mathbb{N}$.

Lemma. *(& definition)* $R = \mathbb{Z} + c \cdot \mathbb{Z}\rho$. *The number* c *is called the conductor of the ring* R.

Proof. If $x = a + b\rho \in R$, then $b\rho = x - a \in R \cap \mathbb{Z}\rho = c \cdot \mathbb{Z}\rho$. In other words, $c \mid b$ and $x = a + c \cdot b'\rho$.

Example. Let $D = 1$, $K = \mathbb{Q}(i) = \mathbb{Q}[i]$ and $O = \mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}i$.

1. If $t = i$, then $R = O = \mathbb{Z}[i]$, $c = 1$ and $R = \mathbb{Z} + \mathbb{Z}i$.

2. If $t = 2i$, then $R = \mathbb{Z}[2i] \subsetneq O$, $c = 2$ and $R = \mathbb{Z} + 2\mathbb{Z}i = \{a + 2bi \mid a, b \in \mathbb{Z}\}$.

Let us return to elliptic curves. Integers correspond to "trivial" endomorphisms $\varphi_n : X \to X$, $P \mapsto nP$. Is it possible to describe all elliptic curves which allow also nontrivial endomorphisms? The answer is given by a wonderful theorem of number theory.

Definition. The elliptic curve $X = \mathbb{C}/\Lambda$ is a curve with *complex multiplication*, if $\mathrm{End}(X, O) \neq \mathbb{Z}$.

Theorem. *(Weber, Füter, Serre) If the curve X has complex multiplication, $R = \mathrm{End}(X, O)$ its ring of endomorphisms and $K = \mathbb{Q}[\sqrt{-D}]$ corresponding field of algebraic numbers, then*

(1) the invariant $j = j(X)$ is an integral algebraic number;

(2) Galois' group $\mathrm{Gal}(K(j)/K)$ is Abelian, of order $|\mathrm{Pic}(R)| = |\mathrm{Cl}(R)|$;

(3) number j is rational $\Leftrightarrow K(j) = K \Leftrightarrow |\mathrm{Cl}(R)| = 1 \Leftrightarrow$ ring O is factorial, and there are exactly 13 such values for j:

D (discriminant)	c (conductor)	j (invariant)
1	1	$2^6 3^3$
2		$2^6 5^3$
3		0
7		$-3^3 5^3$
11		-2^{15}
19		$-2^{15} 3^3$
43		$-2^{18} 3^3 5^3$
67		$-2^{15} 3^3 5^3 11^3$
163		$-2^{18} 3^3 5^3 23^3 29^3$
1	2	$2^3 3^3 11^3$
3		$2^4 3^3 5^3$
7		$3^3 5^3 17^3$
3	3	$-2^{15} 3 \cdot 5^3$

The groups $\mathrm{Cl}(R)$ and $Pic(R)$ which appear in the theorem are the class group of fractional ideals of the ring R, and the class group of projective R-modules of rank 1.

The question how many factorial rings O exist, has been answered only recently. From classical theoretical considerations it followed that there are at most ten, and the calculated tables for small values of D contained only nine such rings (the above table with conductor 1). It was a long-standing open problem whether there is tenth ring (the so called *problem of the tenth discriminant*). In 1967 Stark [29] answered negatively and showed that there is no such ring (see [12, p. 438], [3, p. 253]).

12. Cartier divisors and group of points of singular cubic

The notion of Weil divisor was introduced only for varieties that are nonsingular in codimension 1. In the case of curves, these are the nonsingular projective curves. But what about singular varieties? It would be possible to define divisors for arbitrary varieties as formal finite combinations $D = \sum n_i C_i$ of irreducible subvarieties of codimension 1. However, already the notion of principal divisor (and the divisor class group) does not work: the multiplicity of a rational function can not be always consistently defined along subvariety of codimension 1, since it may contain singular points. In this case one uses a different definition of divisor, suggested by connection between divisors and functions in projective space.

The notion of divisor occurs as the answer to a classical question: is there a rational function that has zeros ($n_i > 0$) and poles ($n_i < 0$) of given multiplicity n_i on given hypersurfaces C_i. If $D = \sum_i n_i C_i$ is a divisor on $\mathbb{P}^n$, each irreducible subvariety C_i of codimension 1 is globally defined by one polynomial homogeneous irreducible equation $g_i = 0$ and the solution of the problem is the rational function $f = \prod g_i^{n_i}$. This is a global rational function on $\mathbb{P}^n$ only if the degree of the divisor equals 0, that is, if $\sum_i n_i = 0$. However, in any affine chart $U_j = \{x_j \neq 0\}$ it defines a proper rational function $f_j = f/x_j^{(\Sigma n_i)}$. In addition, the family $\{U_j,\ f_j\}$ $(j = 0, \dots, n)$ has the property that functions f_j/f_k have neither zeros nor poles on intersections $U_j \cap U_k$, since corresponding factors cancellate.

For arbitrary nonsingular variety X, in an analogous way, each Weil divisor $D = \sum n_i C_i$ on X defines a family $\{U_j,\ f_j\}$ consisting of covering U_j of X and of rational functions $f_j \in K(U_j)^*$ on each element of the covering, such that function f_j on U_j cuts out the principal divisor $(f_j) = D \cap U_j$, and rational functions f_j/f_k have neither zeros nor poles on intersections $U_j \cap U_k$. One needs nonsingularity in order to describe each C_i locally by one equation $g_i = 0$. Such a family $\{U_j,\ f_j\}$ is called *coherent system of functions*. Conversely, coherent system of functions $\{U_j,\ f_j\}$ on X defines a divisor $D = \sum n_i C_i$ on X: note that $K(U_j)$ is the field of fractions of the factorial domain $K[U_j]$ and represent f_j in the form $f_j = \prod g_{ij}^{n_{ij}}$. The coherency conditions uniquely determine subvarieties C_i and multiplicities n_i. Two coherent systems of functions $\{U_j,\ f_j\}$ and $\{V_k,\ g_k\}$ define the same divisor if and only if corresponding principal divisors coincide: $(f_j) = (g_k)$ on intersections $U_j \cap V_k$, that is, if rational functions f_j/g_k have neither zeros nor poles on $U_j \cap V_k$. This defines equivalence on the family of coherent systems of functions. Corresponding equivalence classes are called *locally principal* (or *Cartier divisors*. A Weil divisor corresponds to each Cartier divisor and vice versa, and this is a bijection.

I63

The good property of Cartier divisors is that they can be defined for arbitrary variety X, even when Weil divisors can not. The product of two Cartier divisors $\{U_j, f_j\}$ and $\{V_k, g_k\}$ is a Cartier divisor $\{U_j \cap V_k, f_j g_k\}$, and this defines a group structure on the set $\operatorname{Ca Div}(X)$ of Cartier divisors. Analogously to Weil divisors, this operation is written additively and called the sum. Every global rational function f on X defines a principal Cartier divisor $\{X, f\}$. This defines a homomorphism $K(X)^* \to \operatorname{Ca Div}(X)$. The quotient group of $\operatorname{Ca Div}(X)$ by the subgroup of principal divisors is the group of Cartier divisor classes $\operatorname{Ca Cl}(X)$.

If the variety X is nonsingular in codimension 1, then there exist both Weil and Cartier divisors. The construction of Weil divisor, corresponding to Cartier one, defines a homomorphism $\operatorname{Ca Cl}(X) \to \operatorname{Cl}(X)$. The construction of Cartier divisor shows that this is an inclusion. However, it does not have to be a surjection, as the example of the simple cone shows. In this case, the group $\operatorname{Cl}(X) = \mathbb{Z}_2$ is generated by the class of the directrisse L of the cone. This directrisse can not be defined by one equation in any neighborhood of cone's vertex, since any function which should describe L as a set of points, cuts out the divisor $2L$. Therefore, L is not locally principal, every locally principal divisor is principal, and $\operatorname{Ca Cl}(X) = 0$.

Let us now calculate the Cartier divisor class group of the singular cubic $X = V(y^2z - x^3) \subset \mathbb{P}^2$ (the "cusp"-curve). That will introduce group structure on its set of nonsingular points, exactly as in the case of nonsingular cubic. Our construction follows that of [31, p. 187]. First prove an important lemma.

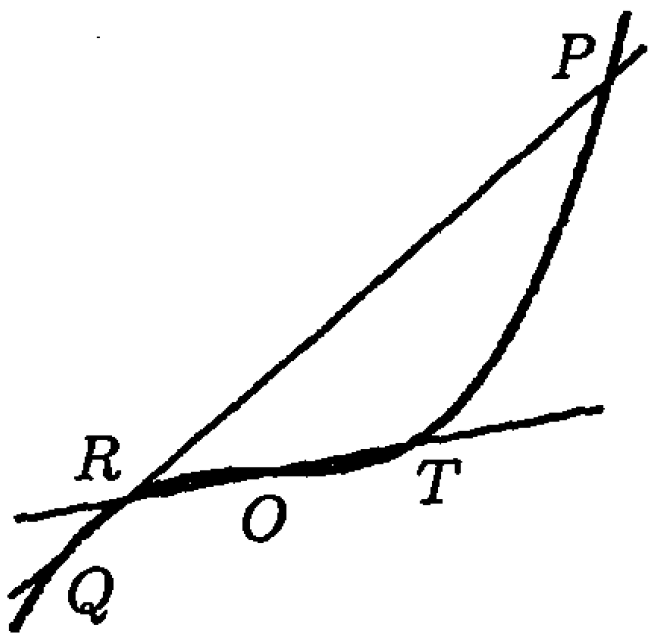
Lemma. ("removing the point from divisor's support"; (generalization see in [33, t. 1, p. 193]). *If X is a plane projective curve, $P \in X$ its point and $D \in \operatorname{Ca Div}(X)$ Cartier divisor on X, then there is a divisor $D' \sim D$ whose support does not contain the given point.*

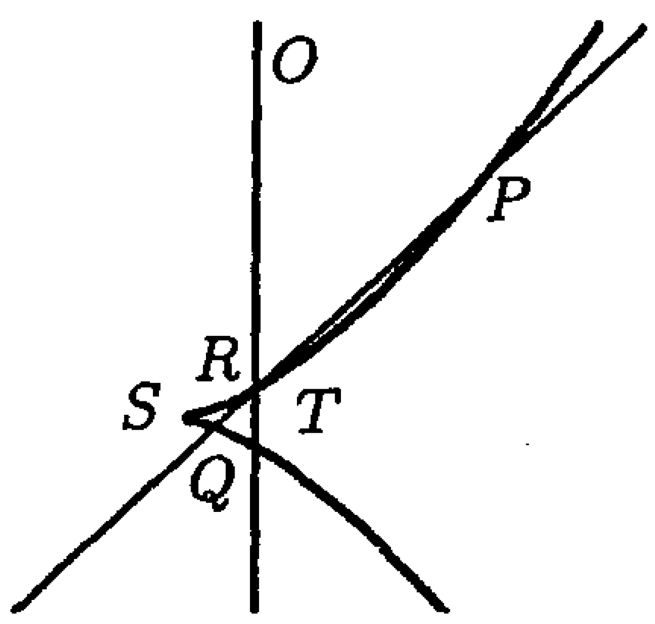
Proof. Let U be a neighborhood of P and f rational function defining locally principal divisor D in that neighborhood. Suppose that the support of the divisor contains the given point. This means that P is a zero or a pole of the function f, of some multiplicity n. Take a global rational function $g \in K(X)$ which in the point P has zero or pole of multiplicity n. The divisor $D' = D - (g)$ has the required property, since the function fg^{-1} is regular in some neighborhood of P.

Let now $X = V(y^2z - x^3) \subset \mathbb{P}^2$ with singular point $S = (0:0:1)$ and let $Y = X \setminus S$ be the nonsingular subvariety. Each Cartier divisor $D \in \operatorname{Ca Div}(X)$ is equivalent to divisor D' whose support does not contain the point S, that is, whose local equation does not have a pole in that point. Therefore, D' is a Weil divisor on Y. If D is principal, D' is such. The divisor D' has not to be uniquely determined, but its degree is. In such way, one defines the degree of divisor D, that is, a homomorphism $\operatorname{Ca Cl}(X) \to \mathbb{Z}$. Consider its subgroup $\operatorname{Ca Cl}(X)^0$ of divisor classes of degree 0 and, as in the case of nonsingular cubic, define a mapping $\varphi : Y \to \operatorname{Ca Cl}(X)^0$, $P \mapsto D_P = P - O$, where $O = (0:1:0)$ is a chosen point (point at infinity of the y-axis). One could also prove, by construction similar to nonsingular cubic case, that φ is a bijection. One should only note that if in the equality $P + Q = R + O$ points P and Q belong to Y, then the same holds for R.

The group operation on $\mathrm{Ca\,Cl}(X)^0$ is therefore carried to Y. The construction of the point $P+Q$ is as before: if R is the third intersection point of the line through P and Q with the curve Y and T the third intersection point of line through R and O with the curve, then $P+Q=T$.

Note that $Y = X \setminus S \cong \mathbb{A}^1$ since the curve X is rational. The corresponding isomorphism is given by the formula $(x:y:z) \mapsto x/y$ and its inverse $t \mapsto (t:1:t^3)$. However, $\mathbb{A}^1$ has its usual structure of (additive) group, which is carried by this isomorphism to Y, and this is exactly the described group structure: if $P=(u:1:u^3)$ and $Q=(v:1:v^3)$, then $T=((u+v):1:(u+v)^3)$. Namely, if one switches to the chart $y \neq 0$, intersection of the curve $Y: z = x^3$ and the line $z = \alpha x + \beta$ are the points P, Q and R whose x-coordinates are the roots of the equation $x^3 - \alpha x - \beta = 0$, that is, u, v and $-(u+v)$ respectively (Viet's rule!). The point T is symmetric to R with respect to the origin. Therefore, its x-coordinate equals $u+v$, which proves the assertion (see the figures).





13. Sheaves and Czech cohomology

In the past 40 years homology became an indispensable tool in algebraic geometry. In the context of algebraic varieties these concepts are easily introduced via sheaf theory. Sheaves represent one of the most important contemporary techniques in algebraic geometry, and also in other geometrical theories, everywhere where one has local constructions and needs global invariants. Sheaves are the most important tool for globalization in modern geometry. In this short review it is not possible to develop the sheaf theory in its full extent. However, we will try to give some motivation, main definitions and examples.

In the definition of fundamental geometrical objects such as topological and differential manifolds, complex analytical and algebraic varieties, the same general method is used. First, one introduces and studies objects which play the role of local models. For example, local models of differential manifolds are open domains in $\mathbb{R}^n$. Then one builds global object from local models by procedure of gluing (identification).

Example. [20, p. 47] Two copies of the real line $\mathbb{R}^1$ can be glued along its open subsets $U = \mathbb{R}^1 \setminus \{0\}$ in different ways, with two different identification functions $f : U \to U$. Using the function $f(x) = x$ one obtains the line with doubled origin 0, and using the function $f(x) = 1/x$ one obtains the circle—sphere S^1.

In the process of gluing one should take care of corresponding local structure. The local structure of a geometrical object is described by the set of permissible functions on that object. Continuous functions, differentiable functions, analytical functions, rational functions—all these are the classes of permissible functions for corresponding geometrical objects. By gluing of two local objects, identifying their parts, one must take care that on these common parts gluing takes permissible functions to permissible functions. One should know what permissible functions are, not only on the whole object, but also on its local parts. In such way one comes to a new type of structure, built by permissible functions. Let us explain it on the example of topological spaces, where the permissible functions are continuous functions. For any open subset $U \subset X$ one has a set $C(U)$ of all real continuous functions on U. It is a ring with respect to usual addition and multiplication of functions. If $V \subset U$, one has a homomorphism of rings $\rho_V^U : C(U) \to C(V)$ defined by restriction of functions $\rho_V^U(f) = f|_V$. Composition of restrictions is again a restriction: if $W \subset V \subset U$, then $\rho_W^V \circ \rho_V^U = \rho_W^U$. This provides us with the motivation for the following definition.

Definition. *Presheaf* of objects of a given category $\mathcal{C}$ (of sets, rings, Abelian groups,...) on a topological space X is a contravariant functor $\mathcal{F} : \operatorname{top} X \to \mathcal{C}$ from partially ordered structure of open sets in X (viewed as a category) to category $\mathcal{C}$. In other words, for each open subset $U \subset X$ there is an object $\mathcal{F}(U) \in \operatorname{Ob}\mathcal{C}$ of corresponding type (a set, a ring, an Abelian group,...), and for any inclusion of open sets $V \subset U$ there is a corresponding homomorphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$, $\rho_V^U \in \operatorname{Mor}\mathcal{C}$, with the following properties:

1) $\rho_U^U = id$; 2) if $W \subset V \subset U$, then $\rho_W^V \circ \rho_V^U = \rho_W^U$.

One uses the term "restriction" and the notation $\rho_V^U(f) = f|_V$ also in the general case, although objects $\mathcal{F}(U)$ are not necessarily sets of functions, and homomorphisms ρ_V^U-restrictions of functions. Elements of the set $\mathcal{F}(U)$ are called sections of the sheaf $\mathcal{F}$ over the open set U. Sections over whole X are called global sections. In the sequel, all objects $\mathcal{F}(U)$ will have at least the structure of Abelian group, and therefore one could speak about their subobjects, quotient objects, kernels and images of homomorphisms etc.

Let us return to the presheaf of real continuous functions on topological space X. It has one specific property, concerning families of functions on coverings of X. Namely, if $\{U_\alpha\}$ is an open covering of X, then each continuous function f on X is uniquely determined by its restrictions $f|_{U_\alpha}$ on U_α. Conversely, a given family of functions f_α on U_α determines a global function on whole X if and only if functions f_α are coherent on intersections i.e., if for any two indexes and, $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$. This property could be formalized in the following manner.

Definition. Presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if for any open subset U and its open cover $\{U_\alpha\}$ the following sequence of homomorphisms is exact:

$$0 \to \mathcal{F}(U) \xrightarrow{\varphi} \prod_{\alpha} \mathcal{F}(U_\alpha) \xrightarrow{\psi} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

where $\varphi : f \mapsto \{f|_{U_\alpha}\}$, $\psi : \{f_\alpha\} \mapsto \{f_\alpha|_{U_\alpha \cap U_\beta} - f_\beta|_{U_\alpha \cap U_\beta}\}$. In other words, $\operatorname{Ker}\varphi = 0$, that is, if the families of restrictions of two sections coincide then these two sections itself also coincide; and $\operatorname{Ker}\psi = \operatorname{Im}\varphi$, that is, any family of sections which coincide on intersections originates from some global section.

All important presheaves which we have already mentioned, are in fact sheaves. Such are: the sheaf of differentiable functions on a smooth manifold, the sheaf of analytical functions on a complex-analytic variety etc. In the case of sheaves, one could give the interpretation of sections and restrictions as proper functions and corresponding restrictions, by technique of étalé spaces. Even when a presheaf is not a sheaf, one could associate a sheaf to it, called *associated sheaf*, which is locally equal to given presheaf.

Example. If the space X has two connected components and $\mathcal{F} = \mathcal{Z}$ is a presheaf of constant functions with integer values (that is, for any open subset U, $\mathcal{F}(U) = \mathbb{Z}$) then it is not a sheaf: a family of two functions, one on each component, which take two different values (say 0 and 1), agrees on intersections (they are all empty), but does not originate from a global section.

Let $\mathcal{F}$ be a (pre)sheaf on X, $x \in X$ a point, U and V its open neighborhoods. One says that two sections $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ (over different neighborhoods) are equivalent if their restrictions coincide in some common neighborhood $W \subset U \cap V$. The quotient of the disjoint union $\coprod_{U:x\in U} \mathcal{F}(U)$ of all sections over all neighborhoods of a given point is called *germ* of a (pre)sheaf $\mathcal{F}$ at the point x and denoted $\mathcal{F}_x$. For example, an element of germ of sheaf of continuous functions at the point x is a function continuous in some neighborhood of that point, and two such functions are identified if they coincide in some (maybe smaller) neighborhood of x. The associated sheaf may be defined in the following way. Consider the disjoint union $\coprod_{x\in X} \mathcal{F}_x = E$ and the corresponding projection $\omega : E \to X$, $\mathcal{F}_x \to x$. Let E be topologized by the smallest (coarsest) topology in which ω is still continuous. One obtains the *étalé space* of the presheaf $\mathcal{F}$. For any open $U \subset X$, define $\mathcal{F}^+(U) = \Gamma(U, \mathcal{F})$ as a set of all continuous functions $s : U \to E$ such that $i = \omega \circ s : U \to E \to X$ is the identity on U. In this way one obtains a sheaf $\mathcal{F}^+$, the *associated sheaf* of the presheaf $\mathcal{F}$. What is the direct connection between and $\mathcal{F}(U)$? An element $s \in \mathcal{F}^+(U)$ can be interpreted as a family of sections $s_\alpha \in \mathcal{F}(U_\alpha)$, coherent on intersections $U_\alpha \cap U_{\alpha'}$, where $\mathcal{U} = \{U_\alpha\}$ is an open cover of U. Two such families of sections in two different coverings $\mathcal{U}$ and $\mathcal{V}$ are identified if they agree on cross-intersections $U_\alpha \cap V_\beta$. Presheaf $\mathcal{F}$ and associated sheaf $\mathcal{F}^+$ have equal germs $\mathcal{F}_x = \mathcal{F}^+_x$.

Example. Associated sheaf $\mathcal{F}^+$ of the presheaf $\mathcal{F}$ of constant functions on a topological space X is the sheaf of locally constant functions. Their germs coincide in each point (these are the functions, constant in some neighborhood of given point). Even their sections on each connected component of the space coincide. However, if the space has more than one component, then $\mathcal{F}^+ \neq \mathcal{F}$.

Morphism of sheaves is introduced in a standard categorical way: it is a natural transformation of functors. More precisely, morphism $\alpha : \mathcal{F} \to \mathcal{G}$ consists of a family of homomorphisms $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ commuting with restrictions: $\alpha_V \circ \rho^U_{V,F} = \rho^U_{V,G} \circ \alpha_U$. If all homomorphisms $\alpha_U : \mathcal{F}(U) \subset \mathcal{G}(U)$ are inclusions, we say that $\mathcal{F} \subset \mathcal{G}$ is a subsheaf. The definition of quotient sheaf is more complicated. Namely, if $\mathcal{F} \subset \mathcal{G}$ is a subsheaf, quotient groups $\mathcal{G}(U)/\mathcal{F}(U)$ form only a presheaf. By definition, a *quotient sheaf* $\mathcal{G}/\mathcal{F}$ is the corresponding associated sheaf. One could write $(\mathcal{G}/\mathcal{F})(U) = [\mathcal{G}(U)/\mathcal{F}(U)]^+$. Due to this construction, sections of the exact sequence of sheaves need not build an exact sequence. In other words, functor of sections is not right exact: if the sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0$ is exact, only the sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to (\mathcal{G}/\mathcal{F})(U)$ will be exact, and the last homomorphism need not be epimorphism. To the contrary, the functor of germs is exact: if the sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0$ is exact, then for all x sequence $0 \to \mathcal{F}_x \to \mathcal{G}_x \to (\mathcal{G}/\mathcal{F})_x \to 0$ is also exact (and vice versa).

Examples. 1. [20, p. 51] Let $X = S^1$. Consider the sheaf $\mathcal{C}$ of continuous functions on X, its subsheaf $\mathcal{Z}$ of all constant functions and presheaf $\mathcal{F}(U) = \mathcal{C}(U)/\mathcal{Z}(U)$. Cover X with two open sets: two half-circles overlapping at both ends, $X = U_1 \cup U_2$, $U_1 \cap U_2 = V_1 \cup V_2$. Let $f = 0$ be the zero-function on X, g a continuous function on X which equals 0 on V_1 and 1 on V_2 and let $f_1 = f|_{U_1}$, $f_2 = g|_{U_2}$. Then, obviously $f_1|_{V_1} - f_2|_{V_1} = 0$, $f_1|_{V_2} - f_2|_{V_2} = 1$. The pair $\{f_1, f_2\}$ defines a section of the sheaf $\mathcal{F}^+ = \mathcal{C}/\mathcal{Z}$ over U which does not originate from $\mathcal{F}(U) = \mathcal{C}(U)/\mathcal{Z}(U)$.

2. [7, p. 134] If $X = \mathbb{C}$, $\mathcal{O}$ the sheaf of holomorphic functions on X and $\mathcal{O}^*$ the sheaf of (multiplicative groups of) holomorphic functions which are everywhere different from 0, the morphism $\exp : \mathcal{O} \to \mathcal{O}^*$, locally defined by $f \mapsto \exp(f)$ is an epimorphism of sheaves, since it is epi on germs: any holomorphic function different from 0 at the point x may in some neighborhood of that point be written as $\exp(f)$ for some holomorphic function f. However, if U is the open ring around 0, then $\exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is not surjective.

Let us return to algebraic varieties. If X is an algebraic variety over algebraically closed field K, then generally there is no natural topology on the set X. The only topology which we could use is the Zariski topology. Which are the permissible functions? If $U \subset X$ is an open subset, let $\mathcal{O}(U)$ be the ring of regular functions on U. One obtains a sheaf $\mathcal{O}$ of rings on X, the *structure sheaf* of regular functions on X. If instead of regular, one takes rational functions and lets $\mathcal{K}(U)$ be the field of rational functions on U, one gets the sheaf $\mathcal{K}$ of fields of rational functions on X. This is a constant sheaf if X is irreducible.

Let us mention an important short exact sequence of sheaves (of multiplicative groups): $0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0$. If one compares the definition of Cartier divisor and the definition of quotient sheaf $\mathcal{K}^*/\mathcal{O}^*$, one sees that Cartier divisor on X is the same as global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$, that is, an element of the group $(\mathcal{K}^*/\mathcal{O}^*)(X) = \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. How to describe principal Cartier divisors? These are the classes of those coherent systems of functions $\{U_\alpha, f_\alpha\}$ for which there is a global function f such that $f_\alpha = f|_{U_\alpha}$. In other words, this is the image of the last morphism in the sequence of global sections $0 \to \mathcal{O}^*(X) \to \mathcal{K}^*(X) \to (\mathcal{K}^*/\mathcal{O}^*)(X)$ which needs not to be a surjection. Note one fact. Let $\{U_\alpha, f_\alpha\}$ be coherent system of (rational) functions. This means that on all $U_\alpha \cap U_\beta$, functions $g_{\alpha\beta} = f_\alpha/f_\beta$ are regular and different from 0, that is, $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. However, the system $\{g_{\alpha\beta}\}$ is not arbitrary - it satisfies some special coherency conditions. For any index triple (α, β, γ) one should have $g_{\alpha\beta}g_{\beta\gamma} = f_\alpha/f_\beta \cdot f_\beta/f_\gamma = f_\alpha/f_\gamma = g_{\alpha\gamma}$ on the intersection $U_\alpha \cap U_\beta \cap U_\gamma$. These conditions could be written in the form $g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$ and called the cocycle conditions. This homological terminology has its explanation, as we will shortly see.

The exact sequence in the definition of sheaf extends naturally into a complex, *Czech complex* of the sheaf, determined by the given covering. Let $\mathcal{F}$ be a sheaf on a topological space X and $\mathcal{U} = \{U_\alpha\}$ an open covering of X. Introduce the notation $U_{\alpha_0\alpha_1\ldots\alpha_k} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k}$ and define the cochain group $C^k(\mathcal{U}, \mathcal{F}) = \prod_{(\alpha_0,\alpha_1,\ldots,\alpha_k)} \mathcal{F}(U_{\alpha_0\alpha_1\ldots\alpha_k})$ for any $k \geq 0$ and also differentials $d = d^k : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$ with

$$d(\{s_{\alpha_0\alpha_1\ldots\alpha_{k+1}}\}) = \Big\{ \sum_{0 \leq i \leq k+1} (-1)^i s_{\alpha_0\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_{k+1}}|_{U_{\alpha_0\alpha_1\ldots\alpha_{k+1}}} \Big\}$$

Therefore, $d^0 : \{s_\alpha\} \mapsto \{(s_\beta - s_\alpha)|_{U_{\alpha\beta}}\}$, $d^1 : \{s_{\alpha\beta}\} \mapsto \{(s_{\beta\gamma} - s_{\alpha\gamma} + s_{\alpha\beta})|_{U_{\alpha\beta\gamma}}\}$ etc. Direct calculation shows that this is a proper differential, that is, $d^2 = 0$. One obtains Czech complex $C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U}, \mathcal{F}) \to \cdots$. Its cohomology groups $H^i(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$ $(i > 0)$, $H^0(U, \mathcal{F}) = \operatorname{Ker} d^0$ are called *Czech cohomology groups* of the sheaf $\mathcal{F}$ on the space X corresponding to covering $\mathcal{U}$. If one orders the set of indexes (for example, if the covering is finite) and leaves in the definition of C_k only all increasing k-tuples $\alpha_0 < \alpha_1 < \ldots < \alpha_k$, that is, if we eliminate all terms of the product which differ only by the sequence of open sets, one could check that cohomology will not change. In the same manner, if the covering has finite dimension, that is, if there exists an integer d such that the intersection of any $d+1$ elements of the covering is empty, then the cochains C_i for $i > d$ are trivial, Czech complex is finite and corresponding cohomologies are trivial starting from the position $d+1$. All this simplifies the explicit calculation.

The sheaf condition for $\mathcal{F}$ could be written also as $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. Let us note that this does not depend on the covering $\mathcal{U}$, thus justifying the notation $H^0(\mathcal{U}, \mathcal{F}) = H^0(X, \mathcal{F})$. This may not be so for higher cohomologies. In the general case, relation of refinement of coverings gives us the connection between cohomology groups of the same sheaf over two different coverings, and one takes the direct limit by all coverings. This theory has been developed by Cartan, Leray and Serre. Soon afterwards, Grothendieck has founded cohomological theory for sheaves in a more general context, using resolvents and derived functors. A very nice exposition of this theory may be found in [31]. We shall not discuss the general cohomological theories in this short report. For us it will do, that there exist cohomological groups $H^i(X, \mathcal{F})$ which do not depend on the covering and which satisfy all usual theorems of homology theory, and also that the calculation of Czech cohomology, described above, gives good results for some "well chosen" coverings. One of the most important results in homological algebra is the so-called long cohomological sequence: if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves, then there exists a long exact sequence of cohomology

$$
\begin{array}{l}
0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \\
\qquad\qquad\qquad\qquad\qquad\swarrow \\
\qquad\qquad\qquad H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H}) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\swarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad H^2(X,\mathcal{F}) \to H^2(X,\mathcal{G}) \to \cdots
\end{array}
$$

The beginning of the sequence is just the left exact sequence of global sections.

Examples. 1. [31, p. 284] Let $X = S^1$ be the one-dimensional sphere with the (already introduced) covering by two overlapping half-circles $\mathcal{U} = \{U_1, U_2\}$, $X = U_1 \cup U_2$, $U_1 \cap U_2 = V_1 \cup V_2$ and let $\mathcal{F} = \mathcal{Z}$ be the constant sheaf. One has $C^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(U) \times \mathcal{F}(V) = \mathbb{Z} \times \mathbb{Z}$, $C^1(U,\mathcal{F}) = \mathcal{F}(U \cap V) = \mathbb{Z} \times \mathbb{Z}$ and the corresponding differential in the Czech complex $0 \to C^0(\mathcal{U},\mathcal{F}) \xrightarrow{d} C^1(\mathcal{U},\mathcal{F}) \to 0 \to \cdots$ is $d : (a,b) \mapsto (b-a, b-a)$. Cohomology groups are $H^0(\mathcal{U},\mathcal{F}) = \operatorname{Ker} d = \mathbb{Z}$ and $H^1(\mathcal{U},\mathcal{F}) = \operatorname{Coker} d = C^1(\mathcal{U},\mathcal{F})/\operatorname{Im} d = \mathbb{Z}$.

2. [6, p. 61] Let $X = \mathbb{P}^1$ be the complex projective line with homogeneous coordinates u, v and usual affine covering $\mathcal{U} = \{U, V\}$, $U = \{v \neq 0\}$, $V = \{u \neq 0\}$, $U \cap V = \mathbb{C}^*$ and $\mathcal{F} = \mathcal{O}$ the sheaf of holomorphic functions. One has $C^0(\mathcal{U},\mathcal{O}) = \mathcal{O}(U) \times \mathcal{O}(V)$, $C^1(\mathcal{U},\mathcal{O}) = \mathcal{O}(U \cap V)$ and the differential $d : (f,g) \mapsto g - f$ where $f = \sum_{n\geq 0} a_n u^n \in \mathcal{O}(U)$, $g = \sum_{n\geq 0} b_n v^n \in \mathcal{O}(V)$. On the intersection $U \cap V$ one has $v = u^{-1}$ and

$$g - f = 0 \Leftrightarrow \sum_{n\geq 0} b_n u^{-n} - \sum_{n\geq 0} a_n u^n = 0 \Leftrightarrow a_0 = b_0,\ a_n = b_n = 0\ (n > 0)$$

Therefore, $H^0(\mathcal{U},\mathcal{O}) = \mathbb{C}$, that is, global holomorphic functions on X are only constants. For H^1 one gets

$$H^1(\mathcal{U},\mathcal{O}) = C^1(\mathcal{U},\mathcal{O})/\operatorname{Im} d = C[u,u^{-1}]\Big/\Big(\sum a_i u^i - \sum b_i u^{-i}\Big) = 0.$$

3. [9, p. 34] Let us calculate the cohomology groups of the structure sheaf on nonprojective quasiprojective algebraic variety. Let $X = \mathbb{A}^2 \setminus \{(0,0)\}$ be the plane without the origin, with coordinates u, v and covering $\mathcal{U} = \{U, V\}$, $U = \{v \neq 0\} = D(v)$, $V = \{u \neq 0\} = D(u)$ and let $\mathcal{O}$ be the sheaf of regular functions on X. One has $\mathcal{O}(U) = \mathcal{O}(D(v)) = K[u,v]_{(v)} = K[u,v,v^{-1}]$, $C^0(\mathcal{U},\mathcal{O}) = K[u,v,u^{-1}] \times K[u,v,v^{-1}]$ and $C^1(\mathcal{U},\mathcal{O}) = \mathcal{O}(U \cap V) = K[u,v,u^{-1},v^{-1}]$, and also $d : (f,g) \mapsto g-f$. One gets $H^0(\mathcal{U},\mathcal{O}) = \operatorname{Ker} d = K[u,v] = H^0(\mathbb{A}^2,\mathcal{O})$, that is, regular functions on X can be extended to the whole plane $\mathbb{A}^2$. Let us calculate now $H^1(\mathcal{U},\mathcal{O}) =$

$K[u,v,u^{-1},v^{-1}]/\operatorname{Im} d = K[\{u^{-m},v^{-n}|m,n>0\}]$, therefore $\dim_K H^1(\mathcal{U},\mathcal{O}) = \infty$. The dimension of cohomology groups is not necessarily finite.

4. [31, p. 284] Let us calculate cohomology groups of the sheaf of regular differential forms on the projective line. Let again $X = \mathbb{P}^1$ with usual coordinates and affine covering, and let Ω be the sheaf of regular differential forms [31, p. 224]. One has $C^0(\mathcal{U},\Omega) = \Omega(U)\times\Omega(V) = K[u]du \times K[v]dv$, $C^1(\mathcal{U},\Omega) = \Omega(U\cap V) = K[u,u^{-1}]du$ and $d: u \mapsto u,\ v\mapsto u^{-1},\ dv \mapsto -u^{-2}du$. Now, $\operatorname{Ker} d = \{(f(u)du, g(v)dv) \mid f(u)+u^{-2}g(u^{-1}) = 0\} = 0$ (where f and g are polynomials), and $H^0(\mathcal{U},\Omega) = 0$. Further,

$$\operatorname{Im} d = \{(f(u)+u^{-2}g(u^{-1}))\,du\} = \operatorname{Span}_K\{u^n du \mid n\in\mathbb{Z}\setminus\{-1\}\} \subset K[u,u^{-1}]du$$

so $H^1(\mathcal{U},\Omega) = K\cdot u^{-1}du$ and $\dim_K H^1(\mathcal{U},\Omega) = 1$.

14. Genus of algebraic variety

14.1. Topological genus of projective algebraic curve. Plane projective algebraic nonsingular curve (over the field of complex numbers) is a 2-dimensional compact smooth orientable manifold in $\mathbb{C}^2 = \mathbb{R}^4$. As it is known, such manifolds are uniquely classified by one integer parameter—topological genus g (this is the number of "handles" on X). This number is called *topological genus* of the corresponding nonsingular curve.

14.2. Arithmetical genus of projective variety. **Theorem.** (Hilbert's syzygy theorem) *Let $A = \mathbb{C}[x_0,\dots,x_n]$ and M finitely generated graded A-module ($M = \bigoplus_{k\geq 0} M_k$ as an Abelian group, and for any homogeneous polynomial f of degree d, $f\cdot M_k \subset M_{k+d}$). Then there exists a polynomial $P_M(t)\in\mathbb{Q}_n[t]$ with rational coefficients, of degree at most n, such that $\dim_{\mathbb{C}} M_k = p_M(k)$ for $k \gg 0$.*

Proof. Induction on n. 1. For $n=-1$, $A=\mathbb{C}$, M is a finite-dimensional $\mathbb{C}$-vector space, $M_k = 0$ for $k\gg 0$ and $P_M = 0$.

2. The inductive step. Multiplication homomorphism $\varphi: M_k \xrightarrow{\cdot x_n} M_{k+1}$ has kernel $\operatorname{Ker}\varphi = N' = \{m\in M : x_n m = 0\}$ and cokernel $\operatorname{Coker}\varphi = N'' = M/x_nM$, so one has an exact sequence of vector spaces

$$0 \to N'_k \to M_k \xrightarrow{\varphi} M_{k+1} \to N''_{k+1} \to 0$$

from which one has

$$\dim M_{k+1} - \dim M_k = \dim N''_{k+1} - \dim N'_k$$

Since multiplication with x_n annulates N' and N'', one can view it as finitely generated $\mathbb{C}[x_1,\dots,x_{n-1}]$-modules. By the induction hypothesis, $\dim N'_k = P'(k)$, $\dim N''_{k+1} = P''(k+1)$. We will use the following elementary lemma on polynomials.

Lemma. For any rational polynomial $f\in\mathbb{Q}[t]$ of degree d there exists a polynomial $g\in\mathbb{Q}[t]$ of degree $d+1$ such that $f(t) = g(t+1)-g(t)$.

Proof. Since $(t+1)^d - t^d = d \cdot t^{d-1} + \cdots$, the lemma follows by induction on degree of f.

From the lemma, $P''(k+1) - P'(k) = Q(k+1) - Q(k)$ for some polynomial Q, and

$$\dim M_{k+1} - \dim M_k = \dim N''_{k+1} - \dim N'_k = P''(k+1) - P'(k) = Q(k+1) - Q(k)$$

Therefore, $\dim M_k = Q(k)+$ const for $k \gg 0$. This proves the theorem.

Definition. If $X \subset \mathbb{P}^n$ is a projective algebraic variety and $M = \mathbb{C}[X] = \mathbb{C}[x_0, \ldots, x_n]/I(X)$ its homogeneous coordinate ring, viewed as $\mathbb{C}[x_0, \ldots, x_n]$-module, polynomial $P_X(t) := P_M(t)$ is called *Hilbert polynomial* of X.

Examples. 1) Projective n-dimensional space: $M = A = \mathbb{C}[x_0, \ldots, x_n]$, $M_k = \{$homogeneous forms of degree k with $n+1$ indeterminates$\}$, $\dim M_k = \binom{k+n}{n}$ and $P_M(t) = \binom{t+n}{n} = 1 \cdot t^n/n!$.

2) Projective hypersurface of degree d: $M = A/(f)$, where f is homogeneous of degree d. From the exact sequence $0 \to A_{k-d} \xrightarrow{\cdot f} A_k \to [A/(f)]_k \to 0$ one has $\dim[A/(f)]_k = \binom{k+n}{n} - \binom{k-d+n}{n}$ and $P_M(t) = \binom{t+n}{n} - \binom{t+n-d}{n} = d \cdot \frac{t^{n-1}}{(n-1)!} + \cdots$.

3) Particularly, for $n = 2$, that is, for plane projective algebraic curves of degree d one has $P_M(t) = \binom{t+2}{2} - \binom{t+2-d}{2} = d \cdot t + (1 - \frac{(d-1)(d-2)}{2})$.

Note that if $f \in \mathbb{Q}[t]$ is a rational polynomial of degree n such that in $n+1$ consequent integer points $k, k+1, \ldots, k+n \in \mathbb{Z}$ it has integer values, then it can be written in the form $P(t) = a_n \binom{t}{n} + a_{n-1} \binom{t}{n-1} + \cdots + a_0$ with integer coefficients. Therefore, the highest order coefficient of the Hilbert polynomial has the form $d/n!$ $(d \in \mathbb{Z})$, which can be guessed from previous examples. Examples also show that the degree of Hilbert polynomial equals the dimension of projective variety X. This is really so. One can show that not only the degree, but also the whole polynomial (all its coefficients) is an invariant of the variety, independent from the embedding $X \subset \mathbb{P}^n$. Some of the coefficients (the first and the last) have a special meaning and geometrical interpretation.

Definition. Let $P_X(t) = d \cdot \frac{t^r}{r!} + \cdots + P_X(0)$. The coefficient d is called *degree* of projective variety X. The integer $p_a(X) := (-1)^r[P_X(0) - 1]$ is called *arithmetical genus* of the variety X.

Example. Arithmetical genus of the plane algebraic nonsingular curve of degree d equals $(d-1)(d-2)/2$.

One can see that the definition of arithmetical genus really does not depend on the embedding $X \subset \mathbb{P}^n$ when it is expressed in terms of structure sheaf of the variety. Namely, if $\mathfrak{F}$ is a sheaf on X, its Euler characteristic is defined by $\chi(\mathfrak{F}) = \dim H^0(X, \mathfrak{F}) - \dim H^1(X, \mathfrak{F}) + \cdots$. One could prove that Euler characteristic of the structure sheaf $\mathcal{O}$ of regular functions on a variety X equals $\chi(\mathcal{O}) = P_X(0)$. Therefore, the arithmetic genus of nonsingular projective variety $X \subset \mathbb{P}^n$ equals $p_a(X) = (-1)^r[\chi(\mathcal{O}) - 1]$ where $r = \dim X$. For curves this is reduced to equality $p_a(X) = \dim H^1(X, \mathcal{O}) = h^1(X, \mathcal{O})$.

14.3. Geometrical genus of projective variety. The notion of geometrical genus appears for the first time in the works of Riemann, connected with maximal number of linearly independent global differential forms on a Riemann surface. On the language of sheaves this can be expressed in the following way. Let Ω be the sheaf of regular differential forms on projective nonsingular variety X of dimension r. The canonical sheaf of the variety X is the sheaf $\omega_X = \wedge^r \Omega$, and the dimension of the space of its global sections — geometrical genus.

Definition. $p_g(X) = \dim H^0(X, \omega_X)$.

Example. Let $X = \mathbb{P}^1$ be the complex projective line and $X = U \cup V$ its standard affine covering. Then $\omega = \Omega$ and its restriction on $U = \mathbb{A}^1$ is a free $\mathcal{O}$-module of rank 1, generated by the differential of the local coordinate du. Now,

$$
\begin{aligned}
&C^0(X,\omega) = \Omega(U) \times \Omega(V) = K[u]du \times K[v]dv,\\
&C^1(X,\omega) = \Omega(U \cap V) = K[u, 1/u]du,\\
&d : u \mapsto u,\ v \mapsto \tfrac{1}{u},\ dv \mapsto -\tfrac{1}{u^2}du,\\
&\operatorname{Ker} d = \Big\{(f(u)du, g(v)dv) \mid f(u) + \tfrac{1}{7}u^2 g(\tfrac{1}{u}) = 0\Big\} = 0,\\
&\operatorname{Coker} d = C^1 / \operatorname{Im} d,\ \operatorname{Im} d = \{[f(u) + u^{-2}g(1/u)]du\} = \operatorname{Span}\{u^n du, n \neq -1\},\\
&H^0(X,\omega) = 0,\ H^1(X,\omega) = K \cdot 1/u \cdot du \cong K,\ h^0 = 0,\ h^1 = 1.
\end{aligned}
$$

Therefore, the geometrical genus equals $p_g(\mathbb{P}^1) = 0$. We have also calculated $H^1(\mathbb{P}^1, \omega)$.

14.4. Equality of topological, algebraic and geometrical genus for nonsingular projective curves. The most important types of geometrical theorems are probably the duality theorems, connecting complementary homological objects (homology and cohomology, homology of complementary dimension etc.). These are the key theorems of geometry and topology. Such is the Serre's duality theorem for projective nonsingular varieties, which expresses sheaf cohomology in terms of higher derived functors of the functor $\operatorname{Hom}(-, \omega)$ of complementary dimension. Due to our space restrictions, we shall only state the theorem and the corollary, in which we are now interested.

Theorem. *$H^{r-i}(X, \mathcal{F})^* \cong \operatorname{Ext}^i(\mathcal{F}, \omega)$ for all $0 \le i \le r$ ($r = \dim X$).*

Corollary. Particularly, for $i = 0$ and $\mathcal{F} = \mathcal{O}$ (structure sheaf of regular functions) one obtains $H^0(X, \omega) = \operatorname{Hom}(\mathcal{O}, \omega) \cong H^r(X, \mathcal{O})^*$. Therefore, the geometrical genus equals $p_g(X) = h^0(X, \omega) = h^r(X, \mathcal{O})$. For curves, $r = 1$ and this leads to equality $p_g(X) = p_a(X)$. This equality is valid only for curves. For surfaces a new component appears, so-called *irregularity*. Its existence was known already in the Italian geometrical school. In this case $r = 2$ and

$$
\begin{aligned}
p_a(X) &= (-1)^2[\chi(\mathcal{O}) - 1] = h^0(X,\mathcal{O}) - h^1(X,\mathcal{O}) + h^2(X,\mathcal{O}) - 1\\
&= h^2(X,\mathcal{O}) - h^1(X,\mathcal{O}) = p_g(X) - \operatorname{irr}(X)
\end{aligned}
$$

The equality of arithmetical and topological genus for curves can be proved by complex-analytic means, naturally only when the main field is the field of complex numbers, that is, when the topological genus is defined. Note that both arithmetical and geometrical genera can be defined for varieties over any algebraically closed field, not only the field of complex numbers.

15. Vector space, associated to a divisor

Let $\infty \in \mathbb{P}^1$ be the point at infinity of the projective line. The polynomial $f \in K[\mathbb{P}^1 \setminus \infty]$ of degree n has in ∞ a pole of order n and does not have other poles. Vice versa, a rational function $f \in K(\mathbb{P}^1)$ which has only one pole at ∞, is a polynomial. One has

$$\deg f \leq n \Leftrightarrow (f) + n \cdot \infty \geq 0$$

The vector space of all polynomials of degree at most n (as a subset of the field of rational functions) could be described by this condition.

More generally, let X be a projective variety and $D \in \operatorname{Div} X$.

Definition. $L(D) = \{f \in K(X) \mid (f) + D \geq 0\} \subset K(X)$. This is a vector space over the ground field K, of dimension $\dim_K L(D) = l(D)$, the space of all rational functions whose zeros' and poles' divisor is bounded below by the divisor $-D$.

Remarks. 1. If $\deg D \leq 0$, then $l(D) = 0$. Global rational functions on a projective variety X have degree 0.

2. Spaces of equivalent divisors are isomorphic, that is, if $D_1 \sim D_2$, then $L(D_1) \cong L(D_2)$ and $l(D_1) = l(D_2)$. Namely, if $D_1 - D_2 = (g)$ where $g \in K(X)$ is a rational function, then the multiplication with g produces isomorphism of corresponding vector spaces, since

$$\begin{aligned} f \in L(D_1) \Rightarrow (f) + D_1 \geq 0 \Rightarrow (fg) + D_2 = (f) + (g) + D_2 = (f) + D_1 \geq 0 \\ \Rightarrow fg \in L(D_2) \end{aligned}$$

Therefore, $L(D)$ and $l(D)$ are well defined for classes of equivalent divisors.

A priori, the space $L(D)$ has not to be finitely dimensional. However, this is the case. We shall prove it for projective curves.

Theorem. *If X is a nonsingular projective curve and D divisor on X, then the number $l(D)$ is finite.*

Proof. Let $D = P_1 + \cdots + P_n - Q_1 - \cdots - Q_m$ $(n \geq m)$ with possible repetitions. Since $L(D) \subset L(P_1 + \cdots + P_n)$, one could consider $m = 0$. There is a sequence of vector subspaces $L(0) \subset L(P_1) \subset \cdots \subset L(P_1 + \cdots + P_n)$, and one sees that it suffices to prove $\dim L(D)/L(D-P) < \infty$. Let m be the multiplicity of P in the divisor D and let u be the local parameter in P. Now $f \in L(D) \Rightarrow (f) + D \geq 0$ $\Rightarrow \operatorname{ord}_P f \geq -m \Rightarrow (u^m f)(P) \in \mathbb{C}$. One has a linear mapping $L(D) \to \mathbb{C}$, $f \mapsto (u^m f)(P)$ whose kernel equals $\{f \in L(D) \mid (u^m f)(P) = 0\} = L(D-P)$. It follows that $\dim L(D)/L(D-P) \leq 1$.

The proof even provides an upper bound of the dimension $l(D) \leq \deg D + 1$. Attempting to calculate this dimension, Riemann obtained a lower bound, which was later named after him.

Theorem. (Riemann's inequality) *If g is the genus of the curve, then* $l(D) \geq \deg D + 1 - g$.

Riemann's student Roch made this inequality a precise equality, by calculating the additional term. In this way he arrived to very important theorem, named later after Riemann and Roch.

Theorem. (The Riemann–Roch theorem for curves) $l(D) - l(K - D) = \deg D + 1 - g$ *where K is the so called canonical divisor of the curve X.*

This theorem will be stated and proved later in a different context, using sheaves.

16. Linear systems

Vector space of rational functions associated to a given divisor is a very important object. In what follows, we will describe its connection to classical notion of linear system.

It is known that through each five points of the projective plane in the general position passes exactly one curve of second order. Less known is perhaps such fact: if the curve of third order passes through eight of nine intersection points built by three pairs of lines in the plane, then it passes also through the ninth point. These and similar geometrical theorems were very important in the classical geometry of the last century. They were often proved using linear systems. The simplest linear system is mentioned even today in courses of analytical geometry. This is the bundle of lines in the plane - a set of all lines in the plane passing through a given point. The condition of passing through point can be written as a linear condition on (general) coefficients of line's equation. When, instead of a line, one takes an arbitrary plane algebraic curve of a given order, besides the condition of passing through a point (which is a linear condition on coefficients of curve's equation), one prescribes also the highest multiplicity of this point on the curve (surprisingly, this is also a linear condition on the coefficients!) and finally if, instead of one point, one takes a finite set of points with prescribed multiplicities (that is, an effective divisor), then one obtains a linear system of equations on curve's coefficients, or linear system for short. This notion can be defined more precisely in a different way.

Let X be a projective nonsingular variety, $D \in \mathrm{Div}(X)$ a divisor on X and $L(D)$ the corresponding associated vector space.

Definition. *Complete linear system* on X, defined by the divisor D is a set of divisors $|D| = \{D' \in \mathrm{Div}^+ X \mid D' \sim D\} = \{(f) + D \mid f \in L(D)\}$.

Note that $(f) + D = (g) + D \Leftrightarrow (f) = (g) \Leftrightarrow f = \alpha g$, $\alpha \in K^*$ and therefore $|D| = \mathbb{P}(L(D)) = \mathrm{Gr}(L(D), 1)$ is a projective space of dimension $\dim |D| = l(D) - 1$, the projectivization of the vector space $L(D)$.

Definition. *Linear system* on X is a projective subspace of some complete linear system $|D|$.

Suppose that $L \subset |D|$ is a given linear system of dimension m. Since it is a projective subspace, it allows a coordinatization $L \cong \mathbb{P}^m$. Instead of identifying divisors of L with points of projective space, let us use the projective duality principle and identify them with linear forms on that space. In such way, one obtains coordinate isomorphism $\varphi : L \to (\mathbb{P}^m)^*$. Let $x_i \in (\mathbb{P}^m)^*$ be coordinate functions on $\mathbb{P}^m$ and $f_i = \varphi^{-1}(x_i) \in L(D)$ corresponding rational functions $(i = 0, \dots, m)$. Then a rational mapping $\Phi : X \to \mathbb{P}^m$, $\Phi(x) = (f_0(x), \dots, f_m(x))$ is defined. Show that conversely, any rational mapping of X in a projective space defines a linear system with chosen coordinatization. Let $\Phi : X \to \mathbb{P}^m$ be rational mapping and $- \circ \Phi : K(\mathbb{P}^m) \to K(X)$ the corresponding homomorphism of fields of rational functions. If $l \in (\mathbb{P}^m)^*$ is a linear form on $\mathbb{P}^m$ and $H \subset \mathbb{P}^m$ a hyperplane it defines, then $\Phi^{-1}(H) \subset X$ is the zeros' divisor of the regular function $l \circ \Phi \in K[X]$. Set of all such divisors when $l \in (\mathbb{P}^m)^*$ is a linear system L with coordinatization $(\mathbb{P}^m)^* \to L$, $H \mapsto \Phi^{-1}(H)$.

Examples. Consider the case $X = \mathbb{P}^n$ more closely. The class divisor group here is $\mathrm{Cl}(X) = \mathbb{Z}$, the given effective divisor $D \in \mathrm{Div}^+(X)$ is equivalent to a divisor (f) where f is a homogeneous polynomial of degree $d = \deg f = \deg D$. Vector space $L(D)$ is isomorphic to vector space V of all homogeneous polynomials of degree d and its dimension is $\binom{n+d}{d}$, and full linear system $|D| = L_d$ is its projectivization, of dimension $N = \binom{n+d}{d} - 1$. Linear system of dimension m is a projective subspace of that space, with basis consisting of $m + 1$ homogeneous polynomials of degree d. If these are $f_0(x), \dots, f_m(x) \in V \subset K[x_0, \dots, x_n]$, the corresponding rational mapping is $\Phi : \mathbb{P}^n \to \mathbb{P}^m$, $(x_0 : \dots : x_n) = x \mapsto (f_0(x) : \dots : f_m(x))$. Conversely, any rational mapping is defined by such polynomials, which for their part define linear system i.e., vector subspace of dimension $m + 1$ in vector space V of all homogeneous polynomials of corresponding degree.

1) $n = 1$, $d = 2$. Complete linear system

$$L_2 = \{(f) \mid f = a_{20}x_0^2 + a_{11}x_0x_1 + a_{02}x_1^2\}$$

defines a rational mapping $\Phi : \mathbb{P}^1 \to \mathbb{P}^2$, $\Phi(x_0 : x_1) = (x_0^2 : x_0x_1 : x_1^2)$ and $\Phi(\mathbb{P}^1)$ is a conic.

2) $n = 2$, $d = 2$. Complete linear system L_2 has projective dimension 5 and defines familiar Veronese rational mapping $\Phi : \mathbb{P}^2 \to \mathbb{P}^5$, $\Phi(x_0 : x_1 : x_2) = (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$ and $\Phi(\mathbb{P}^2) = \mathbb{P}^1 \times \mathbb{P}^1$.

3) Rational mapping $T : (x_0 : x_1 : x_2) \mapsto (\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}) = (x_1x_2 : x_2x_0 : x_0x_1)$ is known as Cremona transformation of projective plane $T : \mathbb{P}^2 \to \mathbb{P}^2$. What is the corresponding linear system? One has $L = \{(f) \mid f \in V'\}$ where $V' = \{a_0x_1x_2 + a_1x_2x_0 + a_2x_0x_1\} \subset V$ is a 3-dimensional subspace of 6-dimensional vector space of all homogeneous polynomials of degree 2 in 3 indeterminates. Each equation in V is the equation of a quadric passing through three points $P = (1:0:0)$, $Q = (0:1:0)$, $R = (0:0:1)$. Conversely, any quadric passing through

these three points should have equation of that form. Therefore, L is the linear system of all quadrics passing through points P, Q, R. Its projective dimension is 2 i.e., it is a two-parameter family of quadrics passing through three given points. Such points are called basic points of a linear system. What is their connection to starting rational mapping? The points in which the mapping Φ is not regular are the solutions of the system $x_1x_2 = x_2x_0 = x_0x_1 = 0$ and these are exactly the three basic points. This is a general fact, not simply a coincidence.

17. Sheaf, associated to a divisor

Let X be a nonsingular projective variety and D a divisor on X. If $\mathcal{O} \subset \mathcal{K}$ are sheaves of regular and rational functions on X, then the vector space $L(D)$ associated to divisor D appears on the level of global sections $L(D) \subset \mathcal{K}(X)$, and for $D = 0$, $L(0) = \mathcal{O}(X) = K$ (the only global regular functions on a projective variety are constants). We want to define a sheaf, whose sections over open U would play a role of the vector space $L(D \cap U)$. The given divisor D on X is locally principal, which means that any point has a neighborhood U such that $C_i \cap U = (f_i)$ or $D \cap U = \sum n_i(f_i) = (g)$, where $f_i \in K[U]$ are regular on U and $g = \prod f_i^{n_i}$. The condition $(f) + D \geq 0$ for $f \in L(D)$ locally on U is $(f) + \sum n_i(f_i) = (f) + (g) = (fg) \geq 0$ i.e., on each component $C_i \cap U$ of divisor D one has $\operatorname{ord} f \geq -n_i$, or $f \cdot g \in K[U] = \mathcal{O}(U)$ or equivalently $f \in 1/g \cdot \mathcal{O}(U)$. One sees that the role of the space $L(D \cap U)$ is played by the submodule of the field of rational functions in which the local equation g of the divisor D becomes invertible: $\mathcal{O}(U) \subset 1/g \cdot \mathcal{O}(U) \subset \mathcal{K}(U) = \mathcal{K}(X)$. This enables us to define the space associated to a divisor D in more general setting of Cartier divisors.

Definition. If $D \in \operatorname{CaDiv} X$ is a Cartier divisor on a variety X, $D = \{(U_i, f_i)\}$, the sheaf associated to D is the sheaf of submodules $\mathcal{O}(D) \subset \mathcal{K}$ of the sheaf of rational functions, generated by $1/f_i$ on U_i.

Obviously, $L(D) = H^0(X, \mathcal{O}(D))$ is the space of global sections of this sheaf, and $l(D) = \dim H^0(X, \mathcal{O}(D)) = h^0(\mathcal{O}(D))$ its dimension. Higher cohomology groups of the sheaf $\mathcal{O}(D)$ provide new integer invariants, and their alternating sum - Euler characteristic of the sheaf $\Im$: $\chi(\Im) = h^0(\Im) - h^1(\Im) + \cdots + (-1)^n h^n(\Im)$ where $h^i(\Im) = \dim H^i(X, \Im)$ and $n = \dim X$. The Riemann–Roch theorem could now be formulated in the following way. We will also give the sketch of its proof [31, p. 376]. Although it is not possible to explain all technical details in a short review, this proof illustrates the power of the technique of sheaves and their cohomology in modern geometry.

Theorem. (Riemann–Roch theorem for curves). *If X is a nonsingular projective curve, $\mathcal{O}$ its structure sheaf and D a divisor on X, then*

$$\chi(\mathcal{O}(D)) = \deg D + \chi(\mathcal{O})$$

Proof. Induction on degree of divisor. 1) If $D = 0$, then $\mathcal{O}(D) = \mathcal{O}$ and the statement is obvious.

2) Let the formula hold for divisor D and let $P \in X$. Consider P as a subvariety in X. Its structure sheaf is the "skyscraper"-sheaf $K = \mathcal{K}_P$ concentrated in the point P and zero outside it. Its sheaf of ideals is the sheaf $\mathcal{O}(-P)$. One has the short exact sequence of sheaves

$$0 \to \mathcal{O}(-P) \to \mathcal{O} \to \mathcal{K}_P \to 0$$

which, after tensoring with locally free sheaf $\mathcal{O}(D+P)$ of rank 1, gives the exact sequence

$$0 \to \mathcal{O}(-P) \otimes \mathcal{O}(D+P) \to \mathcal{O} \otimes \mathcal{O}(D+P) \to \mathcal{K}_P \otimes \mathcal{O}(D+P) \to 0$$

or

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to \mathcal{K}_P \to 0$$

Since the Euler characteristic is additive on exact sequences, and $\chi(\mathcal{K}_P) = 1$, one obtains $\chi(\mathcal{O}(D+P)) = \chi(\mathcal{O}(D)) + 1$ which proves the inductive step.

How to deduce the previous statement of the Riemann–Roch theorem from this one? Since it is a curve case, the Euler characteristic contains only two terms

$$\chi(\mathcal{O}) = h^0(\mathcal{O}) - h^1(\mathcal{O}) = \dim H^0(X,\mathcal{O}) - \dim H^1(X,\mathcal{O}) = 1 - p_a(X) = 1 - g$$
$$\chi(\mathcal{O}(D)) = h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) = l(D) - \dim H^1(\mathcal{O}(D))$$

The last term, introduced by Roch, could be interpreted in the following way. From Serre's duality theorem, one has

$$\begin{aligned} H^1(X,\mathcal{O}(D)) &\cong \mathrm{Hom}(\mathcal{O}(D),\omega) \cong \mathrm{Hom}(\mathcal{O},\omega \otimes \mathcal{O}(D)^*) \\ &= H^0(X,\omega \otimes \mathcal{O}(D)^*) = H^0(X,\mathcal{O}(K-D)) \end{aligned}$$

where K is the canonical divisor, which corresponds to the canonical sheaf of differential forms ω. Now $h^1(\mathcal{O}(D)) = h^0(\mathcal{O}(K-D)) = l(K-D)$ and the Riemann–Roch formula for curves takes its previous form:

$$l(D) - l(K-D) = \deg D + 1 - g$$

18. Applications of Riemann–Roch theorem for curves

From the Riemann–Roch theorem on nonsingular projective curves one could directly derive important corollaries on degree of canonical divisor, curves of genus 0 and 1, and other.

Application 1. Put $D = K$, then $l(K) - l(0) = \deg K - g + 1$. Since, according to definition of canonical divisor, $l(K) = g$, and since $l(0) = 1$ one obtains that the degree of the canonical divisor is $\deg K = 2g - 2$.

Application 2. If divisor D has a sufficiently high degree, or more precisely if $\deg D > 2g-2 = \deg K$, then $\deg(K-D) < 0$ and $l(K-D) = 0$. Therefore, $l(D) = \deg D + 1 - g$.

Application 3. Let X be a projective curve of genus 0 and $D = P \in X$ one point. Then

$$l(D) = \deg D + 1 - g + l(K-D) = 2 + l(K-D) \geq 2$$

This means that the vector space $L(D)$ contains a nonconstant rational function f with a pole of multiplicity 1 in the point P, that is, $(f)_\infty = P$. This function defines a rational mapping $f : X \dashrightarrow \mathbb{P}^1$ of degree 1. From this one could show that f is an isomorphism, that is, all curves of genus 0 are rational.

Application 4. Let X be a nonsingular projective curve of genus 1, $P \in X$ its point and $D = nP$. Then $\deg K = 0$ and for all $n > 0$ one has $\deg(K - nP) < 0$ and $h^0(K - nP) = 0$. Therefore $h^0(nP) = \deg nP - g + 1 = n$. There is a sequence of vector spaces $H^0(\mathcal{O}(P)) \subset H^0(\mathcal{O}(2P)) \subset \ldots \subset H^0(\mathcal{O}(nP)) \subset \ldots$ of strictly increasing dimension $1 < 2 < 3 < \cdots < n < \cdots$ Rational functions in $H^0(\mathcal{O}(nP))$ do have a pole in P of multiplicity at most n. Particularly, for $n = 2$ one has $\dim H^0(\mathcal{O}(2P)) = 2$ and this vector space has a basis $\{1, x\}$. Complete it to the basis $\{1, x, y\}$ of $H^0(\mathcal{O}(3P))$. The seven functions 1, x, y, x^2, xy, x^3, y^2 must be linearly dependent in the six-dimensional space $H^0(\mathcal{O}(6P))$. Each of them has only one pole in P, of multiplicity at most 0, 2, 3, 4, 5, 6, 6 respectively. One concludes that the coefficients in y^2 and x^3 should be different from zero. By homothety with respect to x and y one could transform these coefficients to get 1, so the linear combination has the form $y^2 + a_1xy + a_2y = x^3 + b_1x^2 + b_2x + b_3$. At last, by change $y + \frac{1}{2}(a_1x + a_2) \mapsto y$ (adding to a complete square) it could be transformed to the canonical form

$$y^2 = x^3 + c_1x^2 + c_2x + c_3$$

We obtained the canonical equation of nonsingular projective curve of genus 1. As we already know, X is nonsingular $\Leftrightarrow$ the right-hand-side polynomial has only simple roots. We conclude that every nonsingular projective algebraic curve of genus 1 is a plane curve defined by such equation in $\mathbb{P}^2$.

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B. Stanković

HYPERFUNCTIONS

Contents

Introduction

M. Sato ([27], [28]) introduced a new class of generalized functions, called hyperfunctions, as the n-th derived sheaf of the sheaf of holomorphic functions. He left without proof many details in these papers. To this day, subsequent papers of mathematicians, especially Japanese, completed these "gaps" ([3], [10], [13], [15], [18], [20], [30]).

Hyperfunctions have many important properties which are indispensable for an exquisite theory of partial differential equations, microfunctions, micro-local analysis, Fourier transform (cf. [13]). They became a major tool of several areas of analysis and applications.

The set of hyperfunctions forms a flabby sheaf on $\mathbf{R}^n$ [20]. Schwartz's space $\mathbf{D}'(\Omega)$ (Ω is an open set in $\mathbf{R}^n$) of distributions and the dual space of Gevrey class of functions on Ω are naturally contained in the space $\mathbf{B}(\Omega)$, of hyperfunctions on Ω (cf. [13]). For the relations between hyperfunctions and other generalized functions we refer to [4], [5], [19], [22] and [23].

Since Sato's theory utilizes the most advanced concept of sheaf cohomologies, it is not so popular as Schwartz distributions or Beurling and Roumieu ultradistributions. Also, there are a lot of introductory books on different types of generalized functions, but very few on Sato hyperfunctions. However there is a number of different approaches to hyperfunctions. Some of them are based on the same idea as Schwartz's distributions. Martineau [13] started with the space $\mathbf{A}'(\mathbf{R}^n)$ of analytic functionals carried by compact subsets of $\mathbf{R}^n$. For any open set $\Omega \subset \mathbf{R}^n$ the space of hyperfunctions on Ω is defined so that its elements are locally equal to those in $\mathbf{A}'(\mathbf{R}^n)$. A topology of hyperfunctions, has many exceptional features. (see also [1], [4], [13]). In the book [6] Imai introduced hyperfunctions from the viewpoint of applied mathematics.

In 1988 appeared Kaneko's book [7] (English edition) which is intended to be the first easily accessible introduction to Sato's hyperfunctions. Kaneko defines hyperfunctions using boundary value representation ("intuitive" method). Such an approach has been used from the very beginning only as an illustration. But after progress in the theory of Radon transform this approach has claimed its own place

AMS Subject Classification (1991): Primary 46F15; Secondary 32A45 in the foundation of hyperfunctions as a precise mathematical theory. The first rigorous proofs in this sense have been given by Morimoto [20]. There exist many papers on this subject. Kaneko's book is the first monograph with a systematic elaborated theory of hyperfunctions defined by boundary value representation.

Our aim is to draw attention, especially of young mathematicians, to hyperfunctions and to Kaneko's book which is the main reference in this text and can be the next step to make acquaintance of hyperfunction.

1. PRELIMINARIES

We repeat some standard part of the theory of sheaves and sheaf cohomology we need to introduce hyperfunctions. For this part one can consult any book on algebraic analysis and sheaves theory, for example [10].

1.1. Notation and notions

By $\mathbf{X}$ we denote a topological space and by S a locally closed set in $\mathbf{X}$. S is **locally closed set* in $\mathbf{X}$ if it can be written as the intersection of an open and an closed set in $\mathbf{X}$. Thus there exists an open set $U \subset \mathbf{X}$ containing S as relatively closed subset. In $\mathbf{R}$ every interval is locally closed.

A cone in $\mathbf{R}^n$ will be denoted by Γ or by Δ; $\operatorname{pr}\Gamma = \{x \in \Gamma;\ \|x\| = 1\}$; $\Gamma' \subset\subset \Gamma$ means that $\operatorname{pr}\overline{\Gamma'} \subset \operatorname{int}\Gamma$; $\Gamma^0 = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n; \xi x = \xi_1 x_1, \dots, \xi_n x_n \geq 0$ for every $x = (x_1, \dots, x_n) \in \Gamma\}$ is called the **dual cone* to Γ.

$\{F_\alpha;\ \alpha \in A\}$ is a **locally finite family of subset* of F if for every $x \in F$ and every neighbourhood $V(x)$ of x. $V(x) \cap F_\alpha \neq \emptyset$ only for a finite number $\alpha \in A$.

$E = \bigoplus_{\alpha \in A} E_\alpha$ is the **direct sum* of vector spaces E_α, $\alpha \in A$, if every $x \in E$ can be given in a unique way as the finite sum Σx_α, $x_\alpha \in E_\alpha$.

Let $\mathbf{U} = \{U \subset \mathbf{X};\ U \supset A\}$ be the set of open sets containing $A \subset \mathbf{X}$. To each $U \in \mathbf{U}$ there is associated a $\mathbf{C}$-vector space E_U and to each pair $U, V, \in \mathbf{U}$, $U \supset V$, there is associated a $\mathbf{C}$-linear mapping $\rho_{V,U} : E_U \to E_V$ (restriction) in such a way that: i) $\rho_{UU} = \mathrm{id}$; ii) $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$, whenever $U \supset V \supset W$. Then $\{E_U;\ U \in \mathbf{U}\}$ is an **inductive system of* $\mathbf{C}$*-vector spaces.* Let $E = \bigsqcup_{U \in \mathbf{U}} E_U$ ($\sqcup$ is formed by taking the union of E_U's regarding the E_U's as mutually unrelated). Introduce an equivalence relation $\sim$ in E as follows: $F \sim G$ ($F \in E_U$, $G \in E_V$) means that $\rho_{WU} F = \rho_{WV} G$ in E_W for some $W \subset U \cap V$. The **inductive limit* is

$$\varinjlim_{U \in \mathbf{U}} E_U = E/\sim .$$

Tere exists a natural mapping $\rho_U : E_U \to \varinjlim_{U \in \mathbf{U}} E_U$.

Example. Let Ω be an open set in $\mathbf{R}$ and U an open set in $\mathbf{C}$, a neighbourhood of Ω in $\mathbf{C}$. By $\mathbf{O}(U)$ is denoted the set of holomorphic functions on U. Then $A(\Omega) = \varinjlim_{U \supset \Omega} \mathbf{O}(U)$ is the set of **real analytic functions* on Ω.

1.2. Presheaves and sheaves

We say that a **presheaf* $\mathbf{F}$ of $\mathbf{C}$-vector spaces on $\mathbf{X}$ is given if: i) to each open set $V \subset \mathbf{X}$ there is associated a $\mathbf{C}$-vector space $\mathbf{F}(V)$ and ii) to each pair (V, W), $V \supset W$, there is associated a $\mathbf{C}$-linear mapping $\rho_{W,V} : \mathbf{F}(V) \to \mathbf{F}(W)$ such that: a) $\rho_{VV} = \text{id}$; b) $\rho_{ZW} \circ \rho_{WV} = \rho_{ZV}$, $Z \subset V \subset W$. Every element f of $\mathbf{F}(V)$ is called a **section of* $\mathbf{F}$ *on* U. We also write $\rho_{WV}(f) = f_{|W}$ (**restriction of* $f \in \mathbf{F}(V)$ on $W, W \subset V$).

A presheaf $\mathbf{F}$ is a **sheaf on* $\mathbf{X}$ if for any open covering $\{U_\lambda : \lambda \in \Lambda\}$ of an open set $V \subset \mathbf{X}$ we have the following properties: iii) if $f \in \mathbf{F}(V)$ and $f|_{U_\lambda} = 0$ for every U_λ, $\lambda \in \Lambda$, then $f \equiv 0$ (0 is the zero element of $\mathbf{F}(V)$); iv) for a family $\{f_\lambda; \lambda \in \Lambda\}$; $f_\lambda \in \mathbf{F}(U_\lambda)$, such that $f_\lambda|_{U_\lambda \cap U_\eta} = f_\eta|_{U_\lambda \cap U_\eta}$, $U_\lambda \cap U_\eta \neq \emptyset$, there exists $f \in \mathbf{F}(V)$ which has the property $f|_{U_\lambda} = f_\lambda$, $\lambda \in \Lambda$.

$A \subset V$ is the **support* of $f \in \mathbf{F}(V)$ if $V \setminus A$ is the largest open set contained in V on which f is zero.

Remark. Usually presheaves and sheaves are defined for Abelian groups with ρ_{WV} Abelian group homomorphism.

Examples 1. The sheaf $\mathbf{O}$ of holomorphic functions on $\mathbf{C}^n$; to each open set $V \subset \mathbf{C}^n$ there is associated $\mathbf{O}(V)$.

2. The presheaf $\mathbf{L}_1$ on $\mathbf{R}$ (Lebesgue integrable functions). $\mathbf{L}_1$ is not a sheaf because iii) is not satisfied. Let $U_\lambda = (-\lambda, \lambda)$ and $f_\lambda = 1$ for $\lambda \in \mathbf{R}_+$. We can not find an $f \in \mathbf{L}_1(\mathbf{R})$ such that $f|_{U_\lambda} = 1$ for every $\lambda \in \mathbf{R}_+$.

3. The sheaf $\mathbf{A}$ of real analytic functions on $\mathbf{R}^n$.

Let $\mathbf{F}$ and $\mathbf{G}$ be two (pre)sheaves on $\mathbf{X}$. A family $h = \{h_V\}$ of $\mathbf{C}$-linear mappings, $h_V : \mathbf{F}(V) \to \mathbf{G}(V)$ is **a (pre)sheaf homomorphism* if the following diagram commutes:

$$\begin{array}{ccc} \mathbf{F}(V) & \xrightarrow{h_V} & \mathbf{G}(V) \\ {\scriptstyle \rho^{\mathbf{F}}_{WV}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathbf{G}}_{WV}} \\ \mathbf{F}(W) & \xrightarrow[h_W]{} & \mathbf{G}(W) \end{array}$$

Sheaf homomorphisms do not enlarge the support of a section.

The linear differential operator with real analytic coefficients is a homomorphism of the sheaf $\mathbf{A}$ of real analytic functions.

$\mathbf{F}$ is said to be a **subsheaf* of the sheaf $\mathbf{G}$ if for every open set $V \subset \mathbf{X}$ there is associated the inclusion $i_V : \mathbf{F}(V) \to \mathbf{G}(V)$ such that $i = \{i_V\}$ constitutes a sheaf homomorphism. We write in short $\mathbf{F} \subset \mathbf{G}$.

The restriction of the sheaf $\mathbf{F}$ to the open set $V \subset \mathbf{X}$ is the sheaf defined by: $W \to \mathbf{F}(W)$ for every open set $W \subset V$; we denote it by $\mathbf{F}/_V$ (attention, $\mathbf{F}/_V$ is a sheaf and $\mathbf{F}(V)$ is a vector space).

A sheaf $\mathbf{F}$ on $\mathbf{X}$ is **flabby* if for every open set $V \subset \mathbf{X}$, $\rho_{V\mathbf{X}} : \mathbf{F}(\mathbf{X}) \to \mathbf{F}(V)$ is surjective.

Proposition 1.1. *If* $\mathbf{F}$ *is flabby, then for every pair of open sets* (U, V), $U \supset V$, *the restriction* $\rho_{VU} : \mathbf{F}(U) \to \mathbf{F}(V)$ *is surjective.*

Proof. For a given $v \in \mathbf{F}(V)$ there exists $x \in \mathbf{F}(\mathbf{X})$ such that $\rho_{V\mathbf{X}}(x) = v$; let $\rho_{U\mathbf{X}}(x) = u$, then $v = \rho_{V\mathbf{X}}(x) = \rho_{VU} \circ \rho_{U\mathbf{X}}(x) = \rho_{VU}(u)$, where $u \in \mathbf{F}(U)$. □

Let S be a locally closed set in $\mathbf{X}$ and U an open neighbourhood of it containing S as a relatively closed subset. Denote by $\Gamma_S(\mathbf{X}, \mathbf{F}) = \{s \in \mathbf{F}(U); \operatorname{supp} s \subset S\}$, where $\mathbf{F}$ is a sheaf on $\mathbf{X}$.

Proposition 1.2. *The definition of the* $\mathbf{C}$*-vector space* $\Gamma_S(\mathbf{X}, \mathbf{F})$ *does not depend on the choice of* U.

Proof. Let U_1 and U_2 be two such open neighbourhoods of S. Then $U_1 \cap U_2$ is again such an open neighbourhood of S. Hence, it suffices to show that the restriction

$$i : \{s \in \mathbf{F}(U_1);\ \operatorname{supp} s \subset S\} \to \{s \in \mathbf{F}(U_2); \operatorname{supp} s \subset S\}$$

is an isomorphism when $U_1 \supset U_2$. But this is obvious because if $s \in \mathbf{F}(U_2)$, $\operatorname{supp} s \subset S \subset U_2 \subset U_1$, then s can be extended to

$$s' \in \mathbf{F}(U_1),\ \ s'|_{U_2} = s,\ \ s'|_{U_1 \setminus U_2} = 0. \qquad \square$$

A direct consequence of Proposition 1.2 is

Proposition 1.3. $\Gamma_U(\mathbf{X}, \mathbf{F}) = \mathbf{F}(U)$; $\Gamma_S(\mathbf{X}, \mathbf{F}) = \Gamma_S(U, \mathbf{F}|_U)$, *where* S *is relatively closed subset of the open set* U; *if* S *is closed, then* $\Gamma_S(\mathbf{X}, \mathbf{F}) = \{s \in \mathbf{F}(\mathbf{X}), \operatorname{supp} s \subset S\}$.

Proposition 1.4. *Let* V *be an open set in* $\mathbf{X}$ *and* S *a locally closed set in* $\mathbf{X}$. *The correspondence* $V \to \Gamma_{S \cap V}(\mathbf{X}, \mathbf{F})$ *constitutes a sheaf on* $\mathbf{X}$ *denoted by* $\mathbf{T}_S(\mathbf{F})$. *It may also be regarded as a sheaf on* S.

Proof. It is obvious that $\mathbf{T}_S(\mathbf{F})$ is a presheaf. Also iii) and iv) follow from the fact that $\mathbf{F}$ is a sheaf.

Taking S as a topological space with the topology induced by $\mathbf{X}$, then $\mathbf{T}_S(\mathbf{F})(V) = \Gamma_{S \cap V}(\mathbf{X}, \mathbf{F})$ and $V \to \mathbf{T}_S(\mathbf{F})(V)$, $V \cap S \neq \emptyset$, where V is any open set in $\mathbf{X}$, defines a sheaf on S. □

Remark. If U is an open set in $\mathbf{X}$, then $\mathbf{T}_U(\mathbf{F}) = \mathbf{F}|_U$ and $\mathbf{T}_S(\mathbf{F})(\mathbf{X}) = \Gamma_S(\mathbf{X}, \mathbf{F})$.

Proposition 1.5. *If* $\mathbf{F}$ *is flabby, then* $\mathbf{T}_S(\mathbf{F})$ *is flabby, as well.*

Proof. Let U be an open set in $\mathbf{X}$ containing S as a relatively closed subset. We will prove that for any open set $V \subset \mathbf{X}$, $V \cap S$ is relatively closed subset of $V \cap U$. By definition of a locally closed set we have $S = O_S \cap Z_S$, where O_S is an open set in $\mathbf{X}$ and Z_S is a closed set in $\mathbf{X}$. Then $S \cap V = (Z_S \cap O_S) \cap V = Z_S \cap (O_S \cap V)$. Hence, $S \cap V$ is locally closed in $\mathbf{X}$.

To prove that $S \cap V$ is relatively closed in $V \cap U$ take an $x \in V \cap U$ and $x \notin S \cap V$. Since S is a relatively closed in U, there exists an open set $O \ni x$, $O \subset U$ such that $S \cap O = \emptyset$. The open set $O \cap V \ni x$ and $O \cap V \subset U \cap V$. Also,

$$(O \cap V) \cap (V \cap S) = (O \cap V) \cap S = (O \cap S) \cap V = \emptyset.$$

Consequently $S \cap V$ is relatively closed in $V \cap U$.

By Proposition 1.2, $\mathbf{T}_S(\mathbf{F})(V) = \Gamma_{S\cap V}(\mathbf{X}, \mathbf{F}) = \{s \in \mathbf{F}(V \cap U);\ \mathrm{supp}\, s \subset S \cap V\}$. Therefore, for $s \in \mathbf{T}_S(\mathbf{F})(V)$, $s|_{(U\setminus S)\cap(U\cap V)} = 0$. By Proposition 1.1 there exists an $s' \in \mathbf{F}((U \setminus S) \cup (U \cap V))$ such that $s'|_{U\setminus S} = 0$, $s'|_{V\cap U} = s$. By the same Proposition, s' can be extended to $\tilde{s} = \mathbf{F}(U)$, $\tilde{s}|_{U\setminus S} = 0$. Consequently $\tilde{s} \in \Gamma_S(\mathbf{X}, \mathbf{F}) = \mathbf{T}_S(\mathbf{F})(\mathbf{X})$. □

Let $\mathbf{F}$ be a (pre)sheaf on the topological space $\mathbf{X}$. For an $x \in \mathbf{X}$ and any open neighbourhood V of x,

$$\mathbf{F}_x = \varinjlim_{x \in V} \mathbf{F}(V),$$

is called the **stalk of* $\mathbf{F}$ *at* x. An element of $\mathbf{F}_x$ is called **a germ of sections* of $\mathbf{F}$ at x. A germ consists of local sections of $\mathbf{F}$, defined in a neighbourhood of x, which coincide on a smaller neighbourhood of x. A section $s \in \mathbf{F}(V)$ defines a germ $s_x \in \mathbf{F}_x$ at every point $x \in V$.

Proposition 1.6. *If* $\mathbf{F}$ *is a sheaf and* $s \in \mathbf{F}(V)$, *then* $s = 0$ *if and only if* $s_x = 0$ *for all* $x \in V$.

The proof is a direct consequence of the definition of a sheaf (see property iii)).

Attention. Make a distinction of s_x and $s(x)$; $s_x = 0$ means that $s(y) = 0$ for y belonging to a neighbourhood of x.

For a presheaf $\mathbf{F}$ on $\mathbf{X}$ and for every open set $V \subset \mathbf{X}$ we construct the vector space $\overline{\mathbf{F}}(V) = \{\overline{s} : V \to \bigsqcup_{x\in V} \mathbf{F}_x$, such that for each $x \in V$ there exists an open set $W \subset V$, $W \ni x$ and $t \in \mathbf{F}(W)$, with the property that $\overline{s}(y) = t(y)$ for every $y \in W\}$.

Proposition 1.7. *Let* V *be any open set in* $\mathbf{X}$. *The correspondence:* $V \to \overline{\mathbf{F}}(V)$ *with canonical restriction gives a sheaf on* $\mathbf{X}$ *and* $\overline{\mathbf{F}}_x = \mathbf{F}_x$.

Proof. It is obvious that $\overline{\mathbf{F}}$ is a presheaf. First the verification of iii). Let $\{U_\lambda\}$ be an open covering of the open set $V \subset \mathbf{X}$ and let $\overline{s} \in \overline{\mathbf{F}}(V)$, $\overline{s}|_{U_\lambda} = 0$. There exists an open set W, $x \in W \subset U_\lambda$, and $t \in \mathbf{F}(W)$ such that $\overline{s}(y) = t(y) = 0$ for every $y \in W$. It follows that $\overline{s}(x) = 0$ as an element of $\mathbf{F}_x$ for every $x \in U_\lambda$ and for every $U_\lambda \in \{U_\lambda\}$. By definition of $\overline{s}$, $\overline{s} = 0$.

Verification of iv). Given $\{\overline{s}_\lambda\}$, $\overline{s}_\lambda \in \overline{\mathbf{F}}(U_\lambda)$ with the property $\overline{s}_\lambda|_{U_\lambda\cap U_\eta} = \overline{s}_\eta|_{U_\lambda\cap U_\eta}$, where $U_\lambda \cap U_\eta \neq \emptyset$. We construct $\overline{s} \in \overline{\mathbf{F}}(V)$ such that $\overline{s}|_{U_\lambda} = \overline{s}_\lambda$ in the following way: if $x \in V$, then there exists U_λ, $x \in U_\lambda$; now $\overline{s}(x) = \overline{s}_\lambda(x)$.

At the end we prove that $\overline{\mathbf{F}}_x = \mathbf{F}_x$ (These two spaces are isomorphic). Let $\overline{s}_x \in \overline{\mathbf{F}}_x$, then $\overline{s}_x$ is given by an element $\overline{t} \in \overline{\mathbf{F}}(V)$, where V is an open set containing

x. We can take a smaller open set $W \ni x$ such that $\bar{t}|_W = t \in \mathbf{F}(W)$. Then t determines an element of $\mathbf{F}_x$. Hence we constructed a mapping $\overline{\mathbf{F}}_x \to \mathbf{F}_x$. By the construction, it is surjective and an isomorphism. □

The constructed sheaf $\overline{\mathbf{F}}$ is called **the sheaf associated with the presheaf* $\mathbf{F}$.

Example. Let $\mathbf{X} = \mathbf{R}$ and V be an open set in $\mathbf{R}$. By $V \to \mathbf{L}_1(V)$ is defined the presheaf of Lebesgue integrable functions. This is not a sheaf. The sheaf associated with this presheaf is the **sheaf of locally integrable functions* on $\mathbf{R} : V \to \mathbf{L}_{\mathrm{loc}}(V)$.

Let G be a fixed vector space associated to every open set $V \subset \mathbf{X}$, $V \to \mathbf{F}(V) = G$. Take the identity mapping of G as the restriction. Then $V \to G$ defines a presheaf $\mathbf{F}$ on $\mathbf{X}$. It is not a sheaf in general. The property iv) is not always satisfied. Suppose that V is not connected, namely that $V = U_1 \cup U_2$ where U_1 and U_2 are open set and $U_1 \cap U_2 = \emptyset$. Let $g_1 \in \mathbf{F}(U_1) = G$ and $g_2 \in \mathbf{F}(U_2) = G$, $g_1 \neq g_2$. We can not find a $g \in \mathbf{F}(V) = G$ such that $g|_{U_1} = g_1$ and $g|_{U_2} = g_2$.

The sheaf associated to this presheaf $\mathbf{F}$ is called the **constant sheaf* $\mathbf{G_X}$. The difference between $\mathbf{F}(V)$ and $\overline{\mathbf{F}}(V)$ appears when V is not connected.

If $G = \{0\}$, $\mathbf{G_X}$ is a sheaf for each $\mathbf{X}$; it is denoted by 0.

Given a sheaf $\mathbf{G}$ on $\mathbf{X}$ and its subsheaf $\mathbf{F}$. The correspondence: $V \to \mathbf{G}(V)/\mathbf{F}(V)$ (the quotient space) for open sets $V \subset \mathbf{X}$ gives a presheaf on $\mathbf{X}$ (property iv) is not always satisfied). The sheaf associated with this presheaf is called the **quotient sheaf of* $\mathbf{G}$ *by* $\mathbf{F}$ and denoted by $\mathbf{G}/\mathbf{F}$.

Let $h : \mathbf{F} \to \mathbf{G}$ be a sheaf homomorphism and V an open set in $\mathbf{X}$. $V \to \ker h_V$ determines a subsheaf of $\mathbf{F}$ denoted by $\mathbf{Ker}\, h$ (kernel of h). We shall prove that $\mathbf{Ker}\, h$ is a sheaf.

$\ker h_V = \{f \in \mathbf{F}(V);\ h_V(f) = 0\}$ is a vector space. With restrictions $\rho_{WV}, W \subset V$, $V \to \ker h_V$ is a presheaf. Property iii). Let $\{U_\lambda\}$ be an open covering of the open set V and $f \in \ker h_V$, $f|_{U_\lambda} = 0$, $U_\lambda \in \{U_\lambda\}$. Since $\mathbf{F}$ is a sheaf, $f = 0$ on V and $0 \in \ker h_V$. Property iv). With the same open covering $\{U_\lambda\}$ of V let $f_\lambda \in \ker h_{U_\lambda} = \{f \in \mathbf{F}(U_\lambda);\ h_{U_\lambda}(f) = 0\}$. If $U_\lambda \cap U_\eta \neq \emptyset$, then by supposition, $f_\lambda = f_\eta$ on $U_\lambda \cap U_\eta$. Since $\mathbf{F}$ is a sheaf, there exists $f \in \mathbf{F}(V)$ such that $f|_{U_\lambda} = f_\lambda$. By the property of the sheaf homomorphism we have

$$\rho^{\mathbf{G}}_{U_\lambda V} \circ h_V(f) = h_{U_\lambda} \circ \rho^{\mathbf{F}}_{U_\lambda, V}(f) = h_{U_\lambda}(f_\lambda) = 0.$$

Hence, $h_V(f)|_{U_\lambda} = 0$. Since $\mathbf{G}$ is also a sheaf, $h_V(f) = 0$ and $f \in \ker h_V$.

The correspondence: $V \to \operatorname{im} h_V$ for an open set $V \subset \mathbf{X}$ defines a presheaf. The sheaf associated with it is denoted by $\mathbf{Im}\, h$ (image of h).

Example. Consider the sheaf homomorphism $\dfrac{d}{dz} : \mathbf{O} \to \mathbf{O}$, where $\mathbf{O}$ is the sheaf of holomorphic functions on $\mathbf{C}$. $\mathbf{Ker}\, \dfrac{d}{dz}$ is the constant sheaf $\mathbf{C_C}$. The image of $\left(\dfrac{d}{dz}\right)_V : \mathbf{O}(V) \to \mathbf{O}(V)$, where V is an open set in $\mathbf{C}$, consists of all functions

f whose contour integrals around any "hole" in V, if such a "hole" exists in V, are all zero because in this case

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi \in \mathbf{O}(V) \text{ and } \frac{d}{dz}F(z) = f(z),$$

where $z, z_0 \in V$. The sheaf associated with the presheaf: $V \to \text{im} \left(\frac{d}{dz}\right)_V(V)$ is $\mathbf{O}$ $\left(\mathbf{Im}\ \frac{d}{dz} = \mathbf{O}\right)$.

The presheaf homomorphism $h : \mathbf{F} \to \mathbf{G}$ induces the C-linear mapping $h_x : \mathbf{F}_x \to \mathbf{G}_x$ in the following way: $\mathbf{F}_x \ni s_x \xrightarrow{h_x} (h_V(s))_x$, where $s \in s_x$, $s \in \mathbf{F}(V)$, $V \ni x$. We have to prove that this definition does not depend on the chosen representative of s_x and the open set $V \subset \mathbf{X}$. Let $t \in s_x$, $t \in \mathbf{F}(W), W \ni x$. By definition of s_x there exists $Z \subset V \cap W$ such that $t(y) = s(y)$, $y \in Z$, or

$$\rho^{\mathbf{F}}_{ZW}(t) = \rho^{\mathbf{F}}_{ZV}(s).$$

By the property of homomorphism h we have:

$$\rho^{\mathbf{G}}_{ZV} \circ h_V(s) = h_Z \circ \rho^{\mathbf{F}}_{ZV}(s) = h_Z \circ \rho^{\mathbf{F}}_{ZW}(t) = \rho^{\mathbf{G}}_{ZW} \circ h_W(t).$$

Hence, $h_V(s)(y) = h_W(t)(y)$, $y \in Z$ and $(h_V(s))_x = (h_W(t))_x$.

Proposition 1.8. $(\mathbf{Im}\, h)_x = \text{im}\, h_x$.

Proof. Denote by $\mathbf{H}$ the presheaf $V \to \text{im}\, h_V$, where V is any open set in $\mathbf{X}$. Then by Proposition 1.7, $(\mathbf{Im}\, h)_x = \mathbf{H}_x$ for every $x \in \mathbf{X}$. By definition of h_x, $\mathbf{H}_x = \text{im}\, h_x$ because of $\mathbf{H}_x = \varinjlim_{V \ni x} \text{im}\, h_V$. □

Proposition 1.9. *If* $\mathbf{F}$ *and* $\mathbf{G}$ *are two sheaves and* $\mathbf{F} \subset \mathbf{G}$, *then* $\mathbf{F} = \mathbf{G}$ *is equivalent to* $\mathbf{F}_x = \mathbf{G}_x$ *for all* $x \in \mathbf{X}$.

Proof. Denote by $i = (i_V)$ inclusion: $\mathbf{F} \to \mathbf{G}$. If $\mathbf{F}_x = \mathbf{G}_x$, then i_x is surjective. We have to prove that i_V is surjective for every open set $V \subset \mathbf{X}$. Suppose that $\xi \in \mathbf{G}(V)$, then $\xi \in \xi_x \in \mathbf{G}_x$, $x \in V$. There exists $s_x \in \mathbf{F}_x$ such that $s_x = \xi_x$. Consequently, there exists $s^x \in \mathbf{F}(W_x)$, $W_x \ni x$, $W_x \subset V$ sich that $\xi(y) = s^x(y)$, $y \in W_x$. The family of open sets $\{W_x; x \in V\}$ is an open covering of V. By property iv) there exists $f \in \mathbf{F}(V)$ such that $f|_{W_x} = s^x$ for every $x \in V$. Consequently, $f = \xi$ on V. If $\mathbf{F} = \mathbf{G}$ it is clear that $\mathbf{F}_x = \mathbf{G}_x$ for every $x \in \mathbf{X}$. □

1.3. Sheaf cohomology

Let $\mathbf{F} \xrightarrow{h} \mathbf{G} \xrightarrow{k} \mathbf{H}$ be a sequence of sheaf homomorphisms where $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are sheaves on $\mathbf{X}$. This **sequence* is said to be **exact at* $\mathbf{G}$ if $\mathbf{Im}\, h = \mathbf{Ker}\, k$. (For short, **exact sequence)*. In particular, $0 \to \mathbf{G} \xrightarrow{k} \mathbf{H}$ is exact at $\mathbf{G}$ if and only if k is injective; $\mathbf{F} \xrightarrow{h} \mathbf{G} \to 0$ is exact at $\mathbf{G}$ if and only if h is surjective.

The same definition is for the exact sequence of vector spaces.

Proposition 1.10. *The sequence* $\mathbf{F} \xrightarrow{h} \mathbf{G} \xrightarrow{k} \mathbf{H}$ *is exact at* $\mathbf{G}$ *if and only if the sequence of vector spaces.* $\mathbf{F}_x \xrightarrow{h_x} \mathbf{G}_x \xrightarrow{k_x} \mathbf{H}_x$ *is exact at* $\mathbf{G}_x$ *for every* $x \in \mathbf{X}$.

Proof. According to propositions 1.8 and 1.9 the following three assertions are equivalent: $\mathbf{Im}\, h = \mathbf{Ker}\, k$; $(\mathbf{Im}\, h)_x = (\mathbf{Ker}\, k)_x$; $\operatorname{im} h_x = \ker k_x$ for every $x \in \mathbf{X}$. □

If $\mathbf{F} \xrightarrow{h} \mathbf{G} \xrightarrow{k} \mathbf{H}$ is exact, then the sequence of vector spaces

$$\mathbf{F}(V) \xrightarrow{h_V} \mathbf{G}(V) \xrightarrow{k_V} \mathbf{H}(V)$$

is not necessarily exact ($(\mathbf{Im}\, h)_V$ has not to be equal to $\operatorname{im} h_V$). But if the above sequence of vector spaces is exact at $\mathbf{G}(V)$ for all open sets V which constitute a fundamental system of neighbourhood of x, then $\mathbf{F}_x \xrightarrow{h_x} \mathbf{G}_x \xrightarrow{k_x} \mathbf{H}_x$ is exact in $\mathbf{G}_x$.

Proposition 1.11. *Let* $\mathbf{F}$, $\mathbf{F}'$ *and* $\mathbf{F}''$ *be sheaves on* $\mathbf{X}$, S *be locally closed in* $\mathbf{X}$ *and* V *be an open set in* $\mathbf{X}$.

a) *If* $0 \longrightarrow \mathbf{F}' \xrightarrow{h'} \mathbf{F} \xrightarrow{h} \mathbf{F}''$ *is an exact sequence at* $\mathbf{F}'$ *and* $\mathbf{F}$, *then the following sequences of vector spaces are exact*

(1) $0 \to \mathbf{F}'(V) \xrightarrow{h'_V} \mathbf{F}(V) \xrightarrow{h_V} \mathbf{F}''(V)$;

(2) $0 \to \Gamma_S(\mathbf{X}, \mathbf{F}') \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{F}'')$.

b) *If* $0 \to \mathbf{F}' \xrightarrow{h'} \mathbf{F} \xrightarrow{h} \mathbf{F}'' \to 0$ *is an exact sequence and if further* $\mathbf{F}'$ *is flabby, then the following sequences of vector spaces are exact*

(3) $0 \to \mathbf{F}'(V) \xrightarrow{h'_V} \mathbf{F}(V) \xrightarrow{h_V} \mathbf{F}''(V) \to 0$;

(4) $0 \to \Gamma_S(\mathbf{X}, \mathbf{F}') \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{F}'') \to 0$.

Proof. a) (1) First we shall show that h'_V is injective. Suppose that $s' \in \mathbf{F}'(V)$ and $h'_V(s') = 0$. The injectivity of h' implies that $h'_x(s'_x) = 0$ (cf. Proposition 1.10) for every $x \in V$. Thus there exists a neighbourhood $W_x \subset V$ of x such that $s'|_{W_x} = 0$. In such a way we constructed an open covering $\{W_x; x \in V\}$ of V. By property iii), $s' = 0$. Therefore h'_V is injective.

Next we will prove that $\operatorname{im} h'_V \subset \ker h_V$. Since by Proposition 1.10, $(h_x \circ h'_x)(s'_x) = 0$ for $s' \in \mathbf{F}(V)$ and for each $x \in V$, one can find a neighbourhood $W(x)$ of x such that $(h_V \circ h'_V)(s')|_{W(x)} = 0$. Since $\mathbf{F}''$ is a sheaf, by property iii) it follows that $(h_V \circ h'_V)(s') = 0$. Consequently, $\operatorname{im} h'_V \subset \ker h_V$.

It remains to prove that $\operatorname{im} h'_V \supset \ker h_V$. Let $s \in \mathbf{F}(V)$ such that $h_V(s) = 0$. Then for each $x \in V$, $h_x(s_x) = 0$ holds. By the exactness of the sequence in $\mathbf{F}_x$ there exists $s'_x \in \mathbf{F}'_x$ such that $h'_x(s'_x) = s_x$. This implies that $h'_{W_x}(s^x)|_{W_x} = s|_{W_x}$ for an open set $W_x \ni x$, $W_x \subset V$, and $s^x \in \mathbf{F}'(W_x)$, $s^x \in s'_x$. Since h'_{W_x} is injective, s^x is unique. Therefore we have $s^x|_{W_x \cap W_y} = s^y|_{W_x \cap W_y}$. By property iv), there exists $s'' \in \mathbf{F}'(V)$ such that $s''_{W_x} = s^x|_{W_x}$ for every $x \in V$. Thus $h'_V s'' = s$ and $\ker h_V \subset \operatorname{im} h'_V$.

a)(2) Let S be relatively closed in the open set U. It is only to be shown that $\operatorname{supp} s'' \subset S$ provided that $\operatorname{supp} s \subset S$, where s and s'' are as in the above.

Note that $h'_{V\setminus S}(s''|_{V\setminus S}) = 0$ and that $h'_{V\setminus S}$ is injective. Therefore $s''_{V\setminus S} = 0$ and $\operatorname{supp} s'' \subset S$.

In b) it suffices to show that h_V is surjective. We omit this proof. One can find it in [10, Proposition 1.1.2]. □

Corollary 1.1. *Let* $0 \to \mathbf{F}' \xrightarrow{h'} \mathbf{F} \xrightarrow{h} \mathbf{F}'' \to 0$ *be an exact sequence of sheaves on a topological space* $\mathbf{X}$. *If* $\mathbf{F}'$, $\mathbf{F}$ *are flabby, then* $\mathbf{F}''$ *is also flabby.*

Proof. Since $\mathbf{F}'$ is flabby, by Proposition 1.11b) each row of the following commutative diagram is exact

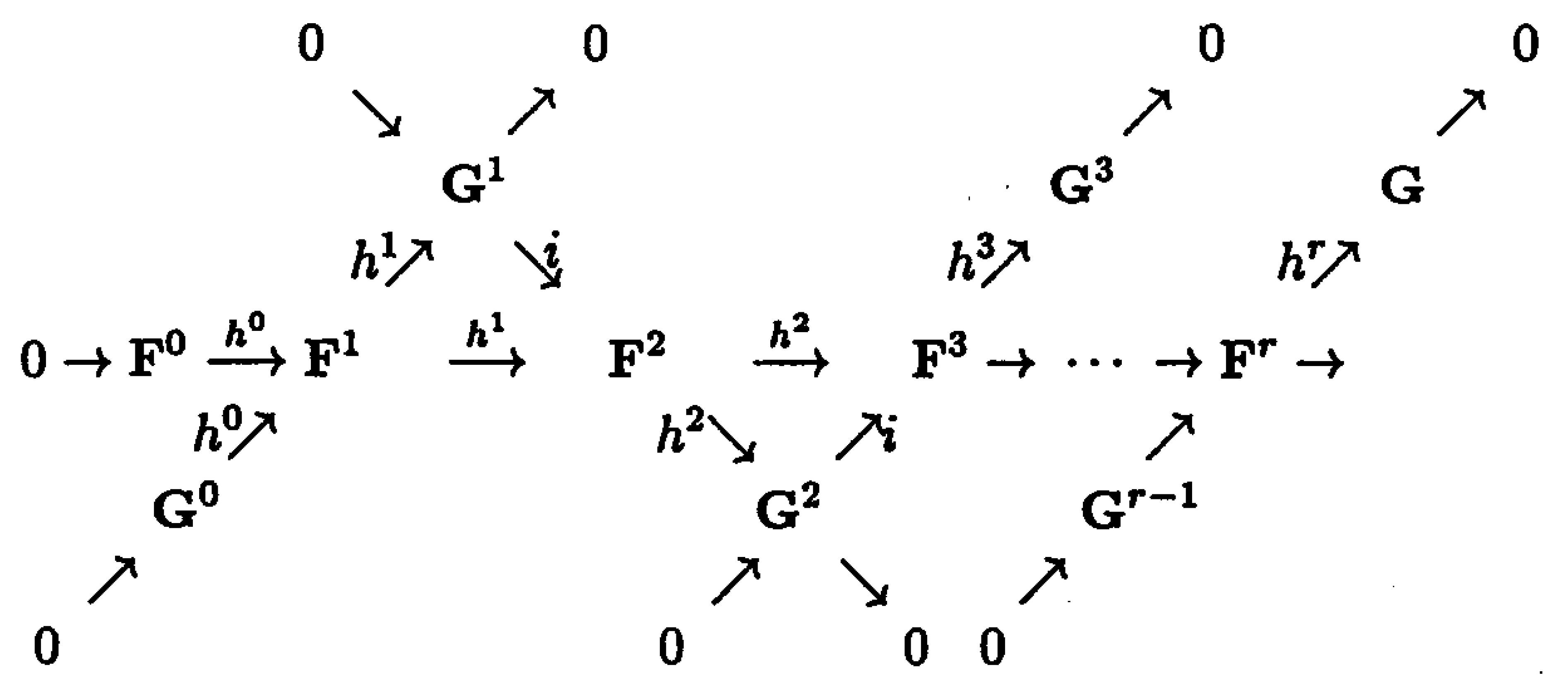
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{F}'(\mathbf{X}) & \xrightarrow{h'_{\mathbf{X}}} & \mathbf{F}(\mathbf{X}) & \xrightarrow{h_{\mathbf{X}}} & \mathbf{F}''(\mathbf{X}) & \longrightarrow & 0 \\ & & \rho'_{V\mathbf{X}}\downarrow & & \rho_{V\mathbf{X}}\downarrow & & \rho''_{V\mathbf{X}}\downarrow & & \\ 0 & \longrightarrow & \mathbf{F}'(V) & \xrightarrow{h'_V} & \mathbf{F}(V) & \xrightarrow{h_V} & \mathbf{F}''(V) & \longrightarrow & 0 \end{array}$$

Thus h_V is surjective. Because of the flabbiness of $\mathbf{F}$, $\rho_{V\mathbf{X}}$ is also surjective. Let $s'' \in \mathbf{F}''(V)$, then there exists an element $s \in \mathbf{F}(\mathbf{X})$ such that $h_V \circ \rho_{V\mathbf{X}}(s) = s''$. By the commutativity of the above diagram, $s'' = \rho''_{V\mathbf{X}} \circ h_{\mathbf{X}}(s)$; $h_{\mathbf{X}}(s) \in \mathbf{F}''(\mathbf{X})$ is the desired extension of s''. Hence $\mathbf{F}''$ is flabby. □

Corollary 1.2. *Let* $0 \to \mathbf{F}^0 \xrightarrow{h^0} \mathbf{F}^1 \xrightarrow{h^1} \cdots \to \mathbf{F}^r \xrightarrow{h^r} \mathbf{G} \to 0$ *be an exact sequence of sheaves on* $\mathbf{X}$. *If* $\mathbf{F}^j$, $0 \le j \le r$ *are all flabby, then* $\mathbf{G}$ *is also flabby. Furthermore, the following sequences are exact*

$$0 \to \mathbf{F}^0(V) \xrightarrow{h^0_V} \cdots \to \mathbf{F}^r(V) \xrightarrow{h^r_V} \mathbf{G}(V) \to 0,$$
$$0 \to \Gamma_S(\mathbf{X}, \mathbf{F}^0) \to \cdots \to \Gamma_S(\mathbf{X}, \mathbf{F}^r) \to \Gamma_S(\mathbf{X}, \mathbf{G}) \to 0.$$

Proof. The given long exact sequence can be decomposed into slanted short exact sequences as follows:



Corollary 1.1 to the slanted exact sequences successively from the left, we can see that every $\mathbf{G}^j$, $j = 0, 1, \ldots, r-1$, and $\mathbf{G}$ are flabby. Applying Proposition 1.11b) the corresponding short sequences of vector spaces

$$0 \to \mathbf{G}^{j-1}(V) \to \mathbf{F}^j(V) \to \mathbf{G}^j(V) \to 0, \ \ j = 1, \ldots, r,$$

are all exact. Combining these short sequences into one in the reversed procedure of that applied above, we obtain the first long exact sequence of vector spaces.

For the second long sequence of vector spaces we have only to take care of the support of sections. □

Let $\mathbf{F}$ be a sheaf on $\mathbf{X}$. A **flabby resolution* of $\mathbf{F}$ is an exact sequence

$$0 \xrightarrow{i} \mathbf{F} \to \mathbf{L}^0 \xrightarrow{h^0} \mathbf{L}^1 \xrightarrow{h^1} \cdots$$

with flabby sheaves $\mathbf{L}^j$, $j = 0, 1, \ldots$. The smallest integer r such that $\mathbf{L}^j = 0$, $j > r$ (if it exists) is called **the length of this resolution.* The minimum of the lengths of all flabby resolutions of $\mathbf{F}$ is called **the flabby dimension of* $\mathbf{F}$, denoted by $\operatorname{fl\,dim}\mathbf{F}$. Flabby dimension measures, roughly speaking, how far the sheaf $\mathbf{F}$ is distant from flabbiness.

If $\mathbf{F}$ is flabby, then $r = 0$ since $0 \to \mathbf{F} \xrightarrow{i} \mathbf{F} \to 0$ is an exact sequence ($\mathbf{L}^0 = \mathbf{F}$).

Proposition 1.12. *Every sheaf possesses a flabby resolution.*

Proof. For a sheaf $\mathbf{F}$ on $\mathbf{X}$, we first construct a flabby sheaf $\mathbf{C}^0(\mathbf{F})$ such that $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F})$ is exact. Let $\mathbf{C}^0(\mathbf{F})$ be the sheaf constructed in the following way. Let V be an open set in $\mathbf{X}$. To V it corresponds the vector space

$$\mathbf{C}^0(\mathbf{F})(V) = \left\{ s^0 : V \to \bigsqcup_{x \in V} \mathbf{F}_x \text{ such that } s^0(x) \in \mathbf{F}_x \right\}.$$

If $s \in \mathbf{F}(V)$, then s defines an element $s^0 \in \mathbf{C}^0(\mathbf{F})$ where $s^0(x) = s_x \in \mathbf{F}_x$. Thus, inclusion $i : \mathbf{F} \to \mathbf{C}^0(\mathbf{F})$ is a sheaf homomorphism.

The property that $\mathbf{C}^0(\mathbf{F})$ is flabby is obvious. In this way we constructed $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F})$.

Next, for the quotient sheaf $\mathbf{C}^0(\mathbf{F})/\mathbf{F}$ we construct $\mathbf{C}^0(\mathbf{C}^0(\mathbf{F})/\mathbf{F})$ in the same way as above and denote it by $\mathbf{C}^1(\mathbf{F})$. Now we construct the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & & \searrow & \nearrow & & \\
 & & & & & \mathbf{C}^0(\mathbf{F})/\mathbf{F} & & & \\
 & & & & k\nearrow & & \searrow i_1 & & \\
0 \to \mathbf{F} & \xrightarrow{i} & \mathbf{C}^0(\mathbf{F}) & & & \xrightarrow{p} & & \mathbf{C}^1(\mathbf{F}) & \\
 & & \nearrow i & & & & & & \\
 & \mathbf{F} & & & & & & & \\
\nearrow & & & & & & & & \\
0 & & & & & & & &
\end{array}
$$

The sequence $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F})$ is exact because $\mathbf{Ker}\, i = \{0\}$. The two slanted sequences: $0 \to \mathbf{C}^0(\mathbf{F})/\mathbf{F} \to \mathbf{C}^1(\mathbf{F})$ and $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F}) \xrightarrow{k} \mathbf{C}^0(\mathbf{F})/\mathbf{F}$ are also exact. Hence, $\mathbf{Ker}\, p = \mathbf{Ker}\,(i_1 \circ k) = \mathbf{Im}\, i$ and $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F}) \xrightarrow{p} \mathbf{C}^1(\mathbf{F})$ is exact.

If we continue the same procedure, then we obtain a flabby resolution of $\mathbf{F}$ given by the flabby sheaves $\mathbf{C}^j(\mathbf{F})$, $j = 0, 1, ...$ The constructed resolution is called the **canonical flabby resolution.* □

Let $\{K^n\}$ be a sequence of C-vector spaces and $\{d_n\}$ be a sequence of C-linear mappings, $d_n : K^n \to K^{n+1}$ such that $d^n \circ d^{n-1} = 0$, $n \in \mathbf{N}$. Then the sequence of pairs $\{(K^n, d^n); n \in \mathbf{N}\}$ is called **a cochain complex of* C*-vector spaces* and is denoted by $K^\bullet$ or $(K^\bullet, d^\bullet)$. An element of K^n is called **an n-cochain.* By definition, $\operatorname{im} d^{n-1} \subset \ker d^n$, $n \in \mathbf{N}$.

An element of $\ker d^n$ is called **an n-cocycle;* an element of $\operatorname{im} d^{n-1}$ is called **an n-coboundary.* The quotient space $\ker d^n / \operatorname{im} d^{n-1}$ is said to be **the cohomology of degree n of the complex* $(K^\bullet, d^\bullet)$ which is denoted by $H^n(K^\bullet)$$\cdot$$H^n(K^\bullet)$ is a vector space, but according to the traditional terminology (which started with a sequence $\{K^n\}$ of Abelian groups it is called sometimes the n-th **cohomology group.*

If $H^n(K^\bullet) = 0$, then the sequence $\{K^n\}$ is exact at the term K^n. Hence, cohomologies provide the concept for measuring the non-exactness of a sequence of vector spaces.

Let $\mathbf{F}$ be a sheaf on $\mathbf{X}$ and $\{\mathbf{C}^j(\mathbf{F})\}$ be the sequence of flabby sheaves from the canonical flabby resolution. Denote by $\Gamma_S(\mathbf{X}, \mathbf{C}^\bullet(\mathbf{F}))$ the complex of spaces $\{\Gamma_S(\mathbf{X}, \mathbf{C}^j(\mathbf{F})),\ j = 0, 1, ...\}$.

The sequence of vector spaces

$$0 \to \Gamma_S(\mathbf{X}, \mathbf{C}^0(\mathbf{F})) \xrightarrow{d_0} \Gamma_S(\mathbf{X}, \mathbf{C}^1(\mathbf{F})) \xrightarrow{d_1} \cdots$$

is not necessarily exact. The cohomology of degree n of the complex $\Gamma_S(\mathbf{X}, \mathbf{C}^\bullet(\mathbf{F}))$ we denote by $H^n_S(\mathbf{X}, \mathbf{F}) = H^n(\Gamma_S(\mathbf{X}, \mathbf{C}^\bullet(\mathbf{F})))$ and call it **the n-th relative (local)* *cohomology of the pair* $(\mathbf{X}, \mathbf{X} \setminus S)$ *with coefficients in* $\mathbf{F}$ having support in S. If S is an open set U in $\mathbf{X}$, then we denote it by $H^n(U, \mathbf{F}) = H^n(\Gamma(U, \mathbf{C}^{\bullet}(\mathbf{F})))$ and call it **the n-th absolute (global) cohomology of the open set* U with coefficients in $\mathbf{F}$.

Note that $H^n_S(\mathbf{X}, \mathbf{F})$ can be defined by any flabby resolution of $\mathbf{F}$ (cf. [7, Theorem 1.1.1]).

Proposition 1.13. *For a sheaf* $\mathbf{F}$, $H^0_S(\mathbf{X}, \mathbf{F}) = \Gamma_S(\mathbf{X}, \mathbf{F})$. *If* $\mathbf{F}$ *is flabby, then* $H^n_S(\mathbf{X}, \mathbf{F}) = 0$, $n \geq 1$.

Proof. By Proposition 1.11a)(2) the sequence of vector spaces

$$0 \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{C}^0(\mathbf{F})) \xrightarrow{d_0} \Gamma_S(\mathbf{X}, \mathbf{C}^1(\mathbf{F}))$$

is exact. Hence, by the definition of the 0-th cohomology, $H^0_S(\mathbf{X}, \mathbf{F}) = \mathbf{Ker}\, d^0 = \Gamma_S(\mathbf{X}, \mathbf{F})$.

For the second part of the assertion, let us suppose that $\mathbf{F}$ is flabby. Cut the canonical flabby resolution to a bounded sequence

$$0 \to \mathbf{F} \to \mathbf{C}^0(\mathbf{F}) \xrightarrow{d_0} \mathbf{C}^1(\mathbf{F}) \to \cdots \xrightarrow{d_n} \mathbf{C}^{n+1}(\mathbf{F}) \xrightarrow{d^{n+1}} \mathbf{Im}\, d^{n+1} \to 0.$$

By Corollary 1.2 after Proposition 1.11 the last term $(\mathbf{Im}\, d^{n+1})$ is also flabby and

$$0 \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{C}^0(\mathbf{F})) \xrightarrow{d_0} \Gamma_S(\mathbf{X}, \mathbf{C}^1(\mathbf{F})) \xrightarrow{d_1} \cdots \xrightarrow{d_n} \Gamma_S(\mathbf{X}, \mathbf{C}^{n+1}(\mathbf{F}))$$

is exact. It follows that $H^n_S(\mathbf{X}, \mathbf{F}) = 0$, $n \geq 1$. □

The **n-th derived sheaf* $\mathbf{H}^n_S(\mathbf{F})$ of $\mathbf{F}$ is the sheaf associated with the following presheaf: $V \to H^n_{S\cap V}(\mathbf{X}, \mathbf{F})$. As we noted in Proposition 1.4 this presheaf can be regarded as the presheaf $S \cap V \to H^n_{S\cap V}(\mathbf{X}, \mathbf{F})$ and $\mathbf{H}^n_S(\mathbf{F})$ can be considered as a sheaf on S.

Since S is a locally closed set in $\mathbf{X}$, there exists an open set $U \subset \mathbf{X}$ containing S as relatively closed subset. Then $H^n_S(\mathbf{X}, \mathbf{F}) = H^n_S(U, \mathbf{F}|_U)$ and $H^n_S(\mathbf{X}, \mathbf{F}) = H^n_S(U, \mathbf{F})$ (cf. Proposition 1.2).

A closed set S in $\mathbf{X}$ is called **purely m-codimensional with respect to a sheaf* $\mathbf{F}$ if $\mathbf{H}^j_S(\mathbf{F}) = 0$ for all $j \neq m$.

Proposition 1.14. (Sato's theorem). $\mathbf{R}^n \subset \mathbf{C}^n$ *is purely n-codimensional relative to the sheaf* $\mathbf{O}$.

Sato's theorem gives a cohomological property of holomorphic functions. We omit the proof. A discussion of this theorem and its proof can be find in [7, Part II, Chapter 6, §5].

We have seen that: $V \to H^n_{S\cap V}(\mathbf{F})$ is only a presheaf. The next proposition gives a sufficient condition that such a presheaf is also a sheaf. First we shall discuss the case $n = 0$ and cite a lemma.

Since $H^0_S(\mathbf{X}, \mathbf{F}) = \Gamma_S(\mathbf{X}, \mathbf{F})$ (Proposition 1.13) and $V \to \Gamma_{S\cap V}(\mathbf{X}, \mathbf{F})$ is the sheaf $\mathbf{T}_S(\mathbf{F})$ (Proposition 1.4), $V \to H^0_{S\cap V}(\mathbf{X}, \mathbf{F})$ defines always a sheaf.

Lemma 1.1. *Let* $0 \to \mathbf{F} \to \mathbf{L}^0 \to \mathbf{L}^1 \to \cdots$ *be a flabby resolution of* $\mathbf{F}$ *and* $\mathbf{T}_S(\mathbf{L}^{\bullet})$ *the correspondent sequence of sheaves* $\mathbf{T}_S(\mathbf{L}^j)$, $j = 0, 1, \ldots$:

$$\mathbf{T}_S(\mathbf{L}^{\bullet}) : 0 \to \mathbf{T}_S(\mathbf{L}^0) \xrightarrow{d^0} \mathbf{T}_S(\mathbf{L}^1) \xrightarrow{d^1} \cdots$$

Then $\mathbf{H}^n_S(\mathbf{F}) = \mathbf{Ker}\, d^n / \mathbf{Im}\, d^{n-1}$.

The proof is based on the inductive limit of the family of complexes and we omit it. (cf. [7, Lemma 5.2.8 and the remark after Definition 5.3.4]).

Proposition 1.15. *If* $\mathbf{H}^j_S(\mathbf{F}) = 0$ *for* $0 \leq j \leq n-1$, *then the presheaf:* $V \to H^n_{S\cap V}(V, \mathbf{F})$ *is a sheaf and hence* $\mathbf{H}^n_S(\mathbf{F})(V) = H^n_{S\cap V}(V, \mathbf{F})$.

Proof. By Lemma 1.1, the complex of sheaves $\mathbf{T}_S(\mathbf{L}^{\bullet})$ given above

$$0 \to \mathbf{T}_S(\mathbf{L}^0) \xrightarrow{d^0} \mathbf{T}_S(\mathbf{L}^1) \xrightarrow{d^1} \cdots$$

is exact up to the $(n-1)$-st term. Then

$$0 \to \mathbf{T}_S(\mathbf{L}^0) \xrightarrow{d^0} \mathbf{T}_S(\mathbf{L}^1) \xrightarrow{d^1} \cdots \mathbf{T}_S(\mathbf{L}^{n-1}) \xrightarrow{d^{n-1}} \mathbf{Im}\, d^{n-1} \to 0$$

is an exact sequence. By Proposition 1.5 every $\mathbf{T}_S(\mathbf{L}^j)$, $j = 1, \ldots$ is flabby. By Corollary 1.2 the sheaf $\mathbf{Im}\, d^{n-1}$ is also flabby and for any open set $V \subset \mathbf{X}$

$$0 \to \Gamma_{S\cap V}(V, \mathbf{L}^0) \xrightarrow{d^0_V} \Gamma_{S\cap V}(V, \mathbf{L}^1) \xrightarrow{d^1_V} \cdots \to \Gamma_{S\cap V}(V, \mathbf{L}^{n-1}) \xrightarrow{d^{n-1}_V} (\mathbf{Im}\, d^{n-1})(V) \to 0$$

is exact. Now, we can construct the commutative diagram:

$$\begin{array}{ccc}
\Gamma_{S\cap V}(V, \mathbf{L}^{n-1}) & \xrightarrow{d^{n-1}_V} & \Gamma_{S\cap V}(V, \mathbf{L}^n) \\
 & \searrow^{d'^{n-1}} \quad \nearrow_{i} & \\
 & (\mathbf{Im}\, d^{n-1})(V) & \\
 & \nearrow \quad \searrow & \\
 & 0 \qquad\quad 0 &
\end{array}$$

From this diagram it follows that $(\mathbf{Im}\, d^{n-1})(V) = \operatorname{im} d^{n-1}_V$. The sequence

$$0 \to \mathbf{Im}\, d^{n-1} \to \mathbf{Ker}\, d^n \to \mathbf{H}^n_S(\mathbf{F}) \to 0$$

is exact. Since $\mathbf{Im}\, d^{n-1}$ is a flabby sheaf, by Proposition 1.11 b) (1),

$$0 \to (\mathbf{Im}\, d^{n-1})(V) \to (\mathbf{Ker}\, d^n)(V) \to \mathbf{H}^n_S(\mathbf{F})(V) \to 0$$

is exact. Consequently

$$\mathbf{H}^n_S(\mathbf{F})(V) = (\mathbf{Ker}\, d^n)/(\mathbf{Im}\, d^{n-1})(V) = \ker d^n_V / \operatorname{im} d^{n-1}_V = H^n_{S\cap V}(V, \mathbf{F}). \qquad \square$$

1.4. Čech cohomology

Let $\mathbf{F}$ be a sheaf on a topological space $\mathbf{X}$ and $\mathbf{U} = \{U_\lambda;\ \lambda \in \Lambda\}$ be an open covering of $\mathbf{X}$. Denote by $\sigma = (\sigma(0), \dots, \sigma(n))$ a permutation of the set $\{0, 1, \dots, n\}$. Denote by $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n}$ the equivalence class related to the intersection $U_{\lambda_0} \cap \dots \cap U_{\lambda_n}$ as follows: Classify all the symbols $b_{\lambda_0 \dots \lambda_n}$ into two sets by the relation: $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n} = \operatorname{sgn} \sigma\ \operatorname{sgn} b_{\lambda_{\sigma(0)} \dots \lambda_{\sigma(n)}}$. In particular, if in $(\lambda_0, \dots, \lambda_n)$ two elements are equal, then the expression $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \doteq 0$.

Consider the set of formal expressions

$$\sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n}, \quad \varphi_{\lambda_0 \dots \lambda_n} \in \mathbf{F}(U_{\lambda_0} \cap \dots \cap U_{\lambda_n}).$$

for a fixed $n \in \mathbf{N}_0$ and with the above convention on $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n}$. This set constitutes a $\mathbf{C}$-vector space with the $\mathbf{C}$-linear operations and it is denoted by $C^n(\mathbf{U}, \mathbf{F})$.

We also define a subspace of $C^n(\mathbf{U}, \mathbf{F})$. Let S be a closed set in $\mathbf{X}$ and $\mathbf{U}' = \{U_\lambda;\ \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$, be an open covering of $\mathbf{X} \setminus S$. Then by definition

$$C^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}) = \\ = \Big\{ \sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n} \in C^n(\mathbf{U}, \mathbf{F}); \varphi_{\lambda_0 \dots \lambda_n} = 0 \\ \text{if } (\lambda_0, \dots, \lambda_n) \in (\Lambda')^{n+1} \Big\}.$$

Furthermore, let $\{\delta^n\}$ be a sequence of $\mathbf{C}$-linear mappings which map $C^n(\mathbf{U}, \mathbf{F}) \to C^{n+1}(\mathbf{U}, \mathbf{F})$ as follows:

$$\delta^n \Big(\sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n} \Big) \\ = \sum_{(\lambda_0, \dots, \lambda_n, \lambda_{n+1}) \in \Lambda^{n+2}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n \lambda_{n+1}} \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}}},$$

where

$$\varphi_{\lambda_0 \dots \lambda_n} \in \mathbf{F}(U_{\lambda_0} \cap \dots \cap U_{\lambda_n}) \quad \text{and}$$
$$\varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}}} \equiv \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_0} \cap \dots \cap U_{\lambda_n} \cap U_{\lambda_{n+1}}} \in \mathbf{F}(U_{\lambda_0} \cap \dots \cap U_{\lambda_n} \cap U_{\lambda_{n+1}}).$$

We shall prove that $\delta^{n+1} \circ \delta^n = 0$, $n = 0, 1, \dots$

$$\delta^{n+1} \circ \delta^n \Big(\sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n} \Big) = \\ = \sum_{(\lambda_0, \dots, \lambda_n, \lambda_{n+1}, \lambda_{n+2}) \in \Lambda^{n+3}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n \lambda_{n+1} \lambda_{n+2}} \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}} \cap U_{\lambda_{n+2}}}.$$

Because of $\varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}} \cap U_{\lambda_{n+2}}} = \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+2}} \cap U_{\lambda_{n+1}}}$ and $\operatorname{sgn} b_{\lambda_0 \dots \lambda_{n+1} \lambda_{n+2}} = - \operatorname{sgn} b_{\lambda_0 \dots \lambda_{n+2} \lambda_{n+1}}$ the correspondent terms cancel each other in pairs. Consequently, $\delta^{n+1} \circ \delta^n = 0$, $n = 0, 1, \dots$

It is clear that δ^n maps $C^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F})$ into $C^{n+1}(\mathbf{U} \bmod \mathbf{U}', \mathbf{F})$. In such a way we have two cochain complexes of C-vector spaces, $C^{\bullet}(\mathbf{U}, \mathbf{F}) = \{C^n(\mathbf{U}, \mathbf{F}), \delta^n\}$ and $C^{\bullet}(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}) = \{C^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}), \delta^n\}$. Let us denote by $H^n(\mathbf{U}, \mathbf{F}) = H^n(C^{\bullet}(\mathbf{U}, \mathbf{F}))$ and by $H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}) = H^n(C^{\bullet}(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}))$ and call them the n-th **(absolute) cohomology group of the covering* $\mathbf{U}$ *with coefficients in* $\mathbf{F}$ and the n-th **relative cohomology group of the relative covering* $\mathbf{U} \bmod \mathbf{U}'$ *with coefficients in* $\mathbf{F}$, respectively.

We shall cite two theorems without proofs.

Proposition 1.16. (Leray's theorem). *Let* $\mathbf{X}$ *be a topological space and* $F \subset \mathbf{X}$ *be a closed set. Let* $\mathbf{V} = \{V_\lambda,\ \lambda \in \Lambda\}$ *be a covering of* $\mathbf{X}$ *and suppose that its part* $\mathbf{V}' = \{V_\lambda;\ \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$, *is a covering of* $\mathbf{X} \setminus F$. *Then, for a sheaf* $\mathbf{F}$ *on* $\mathbf{X}$, *there exist canonical mappings as follows:*

$$\tilde{c}^n_{\mathbf{V}} : H^n(\mathbf{V} \bmod \mathbf{V}', \mathbf{F}) \to H^n_F(\mathbf{X}, \mathbf{F}).$$

In addition, if $H^n(V_{\lambda_0} \cap \ldots \cap V_{\lambda_k}, \mathbf{F}) = 0$, $n \geq 1$, *holds for any family of indices, then the above mappings are isomorphisms. (The covering* $\{V_\lambda;\ \lambda \in \Lambda\}$ *satisfying this condition is called the Leray covering for the sheaf* $\mathbf{F}$*).*

For the proof see for example [7, p. 268].

Before we cite the next theorem we shall recall some notions of complex analysis of several variables.

A domain $U \subset \mathbf{C}^n$ (an open and connected set in $\mathbf{C}^n$) is said to be **a domain of holomorphy* if for every boundary point $z \in \partial U$ there exists a function $f \in \mathbf{O}(U)$ such that it cannot be analytically continued to any neighbourhood of z. An open set $V \subset \mathbf{C}^n$ is called a **Stein open set* if each connected component of it is a domain of holomorphy. The intersection of Stein open sets is also a Stein open set.

Proposition 1.17. (Oka–Cartan–Serre theorem). *Let* $V \subset \mathbf{C}^n$ *be a Stein open set. Then* $H^n(V, \mathbf{O}) = 0$, $n \geq 1$.

For the proof see for example [7, pp. 307–308].

2. HYPERFUNCTIONS OF SEVERAL VARIABLES

First we give a cohomological definition of the sheaf $\mathbf{B}$ of hyperfunctions following Sato's approach [28]. Secondly we pass to the "intuitive" definition and elaborate it following Kaneko's ideas and results [7].

2.1. Cohomological definitions of hyperfunctions

Definition 2.1. (Sato). $\mathbf{B} = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})$ (regarded as a sheaf on $\mathbf{R}^n$).

Proposition 2.1. *Let* Ω *be an open set in* $\mathbf{R}^n$ *and let* U *be an open set in* $\mathbf{C}^n$ *such that* $\Omega = \mathbf{R}^n \cap U$ *and that* Ω *is relatively closed in* U. *Then* $\mathbf{B}(\Omega) = H^n_\Omega(U, \mathbf{O}) = H^n_\Omega(\mathbf{C}^n, \mathbf{O})$.

Proof. By Proposition 1.2, $H^n_\Omega(U,\mathbf{O}) = H^n_\Omega(\mathbf{C}^n,\mathbf{O})$. Now by definition of $\mathbf{H}^n_{\mathbf{R}^n}$ and by propositions 1.14 and 1.15

$$\mathbf{B}(\Omega) = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})(\Omega) = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})(U) = H^n_{\mathbf{R}^n\cap U}(U,\mathbf{O}) = H^n_\Omega(U,\mathbf{O}). \qquad \square$$

We can relate $\mathbf{B}(\Omega)$ with the n-th relative cohomology group.

Let Ω be an open set in $\mathbf{R}^n$. By Grauert's theorem (cf. [7, p. 311]) there exists a Stein open set $U \subset \mathbf{C}^n$ such that $\Omega = \mathbf{R}^n \cap U$ and that Ω is relatively closed in U. Denote by:

$$\begin{aligned} U_j &= U \cap \{z = (z_1, \dots, z_n) \in \mathbf{C}^n;\ Im z_j \neq 0\},\ j = 1, \dots, n;\\ \mathbf{U} &= \{U, U_1, \dots, U_n\};\quad \mathbf{U}' = \{U_1, \dots, U_n\};\\ U\#\Omega &= U_1 \cap \dots \cap U_n = \{z \in U; Im z_j \neq 0,\ j = 1, \dots, n\};\\ U\#_j\Omega &= U_1 \cap \dots \cap U_{j-1} \cap U_{j+1} \cap \dots \cap U_n =\\ &= \{z \in U;\ \operatorname{Im} z_k \neq 0,\ \ k = 1, \dots, j-1,\ j+1, \dots, n\}. \end{aligned} \tag{2.1}$$

Proposition 2.2. *Let Ω be an open set in $\mathbf{R}^n$ and let U be a Stein open set in $\mathbf{C}^n$ such that $\Omega = \mathbf{R}^n \cap U$ and that Ω is relatively closed in U. Then $\mathbf{B}(\Omega) \cong H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O})(\mathbf{B}(\Omega)$ is isomorphic to $H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O}))$, where the families of covering $\mathbf{U}$ and $\mathbf{U}'$ are as above.*

Proof. Let U be taken as a topological space and Ω as the closed subset of U, then $\mathbf{U}$ is a covering of U and $\mathbf{U}'$ a covering of $U \setminus \Omega$. If U is a Stein open set, then $U_j = U \cap \{z \in \mathbf{C}^n; \operatorname{Im} z_j \neq 0\}$, $j = 1, \dots, n$, is also a Stein open set because $\{z \in \mathbf{C}^n;\ \operatorname{Im} z_j \neq 0\}$ is Stein. Also $U_{k_1} \cap \dots \cap U_{k_i}$ for any set of indices which belong to $\{1, \dots, n\}$ is Stein. By Proposition 1.17, $H^n(U_{k_1} \cap \dots \cap U_{k_i}, \mathbf{O}) = 0$, $n \geq 1$ for any set of indices which belong to $\{1, \dots, n\}$. By Proposition 1.16 and Proposition 2.1

$$H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O}) \cong H^n_\Omega(U, \mathbf{O}) = \mathbf{B}(\Omega). \tag{2.2}$$

Corollary 2.1. *Let $\Omega, U, \mathbf{U}$ and $\mathbf{U}'$ be as in Proposition 2.2; then*

$$\mathbf{B}(\Omega) \cong \mathbf{O}(U\#\Omega) \Big/ \sum_{j=1}^{n} \mathbf{O}(U\#j\Omega). \tag{2.3}$$

Proof. By Proposition 2.2, $\mathbf{B}(\Omega) = H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O})$. We have to construct $H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O})$ when $U, \mathbf{U}$ and $\mathbf{U}'$ are given as in Proposition 2.2.

A relative n-cochain is only of the form $\operatorname{sgn} b_{0\dots n}\varphi_{0\dots n}$, $\varphi_{0\dots n} \in \mathbf{O}(U_0 \cap \dots \cap U_n) = \mathbf{O}(U_1 \cap \dots \cap U_n)$, where $U_0 \equiv U$. This n-cochain is in the same time the n-cocycle.

A relative $(n-1)$-cochain has the form

$$\sum_{j=1}^{n} \operatorname{sgn} b_{0\dots j-1\, j+1\dots n}\varphi_{0\dots j-1\, j+1\dots n},\ \varphi_{0\dots j-1 j+1\dots n} \in \mathbf{O}(U\#_j\Omega).$$

Its boundary is

$$\sum_{j=1}^{n}(-1)^j \operatorname{sgn} b_{0\dots n}\varphi_{0\dots j-1j+1\dots n}.$$

Consequently (2.3) is true.

Corollary 2.2. *In one-dimensional case* (2.3) *has the form*

$$\mathbf{B}(\Omega) \cong \mathbf{O}(U \setminus \Omega)/\mathbf{O}(U). \tag{2.4}$$

Proof. In this case $\Omega = \mathbf{R} \cap U$ and $\mathbf{U}'$ consists of only one element $U_1 = \{z \in U, \operatorname{Im} z \neq 0\}$. Then $U\#\Omega = U_1 = U \setminus \Omega$ and $U\#_1\Omega = U$. With this notation (2.3) gives (2.4). □

Consequently, in one-dimensional case, $\mathbf{B}(\Omega)$ is given by the quotient space $\mathbf{O}(U \setminus \Omega)/\mathbf{O}(U)$. Every equivalence class $[F]$, where $F \in \mathbf{O}(U \setminus \Omega)$, is considered to be a hyperfunction f on $\Omega \subset \mathbf{R}$; the function F is called a **defining function of* f.

In many-dimensional case we have the same situation. Every equivalence class $[F]$ where $F \in \mathbf{O}(U\#\Omega)$ is considered to be a hyperfunction $f \in \mathbf{B}(\Omega)$, where Ω is an open set belonging to $\mathbf{R}^n$. F is called the **defining function* of f and we write $f = [F]$.

Proposition 2.3. *The sheaf* $\mathbf{B}$ *is flabby.*

For the proof see [7, pp. 350–351].

$f \in \mathbf{B}(\Omega)$ is said to be 0 on an open set $\Omega' \subset \Omega$ if $f|_{\Omega'} = 0$. **The support* of $f \in \mathbf{B}(\Omega)$ (for short $\operatorname{supp} f$) is the complement in Ω of the largest open subset of Ω on which f equals zero.

Between different operations on hyperfunctions we define some of them. Denote by Ω an open set in $\mathbf{R}^n$.

Let $f = [F]$ and $g = [G]$ be elements of $\mathbf{B}(\Omega)$ and λ, η be two complex numbers. Then $\lambda f + \eta g = [\lambda F + \eta G] \in \mathbf{B}(\Omega)$; thus $\mathbf{B}(\Omega)$ has a $\mathbf{C}$-vector space structure.

For a real analytic function $\varphi \in \mathbf{A}(\Omega)$ there exists an open set $U \subset \mathbf{C}^n$ such that $\Omega \subset U$ and $\varphi \in \mathbf{O}(U)$. Therefore we can define the multiplication by $\varphi \in \mathbf{A}(\Omega) : \varphi f = [\varphi F]$, where $f = [F] \in \mathbf{B}(\Omega)$.

Every $f = [F] \in \mathbf{B}(\Omega)$ has all derivatives. If we adopt the abbreviation: $D_x^\alpha = D_1^{\alpha}\dots D_n^{\alpha_n}$, $D_j = \partial/\partial x_j$, $j = 1, \dots, n$, then $D_x^\alpha f = [D_z^\alpha F]$. Moreover, the linear partial differential operator with real analytic coefficients $P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ acts as a sheaf homomorphism on the sheaf $\mathbf{B}$, $(|\alpha| = \alpha_1 + \dots + \alpha_n)$.

The sheaf $\mathbf{A}$ of real analytic functions: $\Omega \to \mathbf{A}(\Omega)$ is a subsheaf of $\mathbf{B}$. To define this natural mapping $\mathbf{A} \xrightarrow{i} \mathbf{B}$, let us start with an element $\varphi \in \mathbf{A}(\Omega)$ and let U be an open set in $\mathbf{C}^n$ such that φ is holomorphic on U. Introduce the function ϕ such that

$$\phi(z) = \varphi(z), \ z \in (\Omega + i\Gamma_\sigma); \ \phi(z) = 0, \ z \in (U\#\Omega) \setminus (\Omega + i\Gamma_\sigma)$$

where Γ_σ is any orthant in $\mathbf{R}^n$. Then the looked-for mapping i is: $\varphi \to [\phi]$. The defined mapping i does not depend on the chosen Γ_σ.

**The singular support* of $f \in \mathbf{B}(\Omega)$ (for short sing supp f) is the complement in Ω of the largest open set $\Omega' \subset \Omega$ such that $f|_{\Omega'}$ is real analytic.

The next proposition shows an important property of the sheaf $\mathbf{B}$ and also that many properties of this sheaf can be obtained from properties of the holomorphic functions.

Proposition 2.4. *Let Ω be an open set in $\mathbf{R}^n$. If $g \in \mathbf{B}(\Omega)$, then the equation $(\partial/\partial x_1)f(x) = g(x)$ admits a solution $f \in \mathbf{B}(\Omega)$ and every solution $(\partial/\partial x_1)f(x) = 0$ is a hyperfunction depending only on the variables $(x_2, \dots, x_n)$.*

Proof. Since $\mathbf{B}$ is flabby, g can be extended to an element belonging to $\mathbf{B}(\mathbf{R}^n)$. Thus we can take $\Omega = \mathbf{R}^n$ and $g \in \mathbf{B}(\mathbf{R}^n)$. Let G be a defining function of g, $G \in \mathbf{O}(\mathbf{C}^n \# \mathbf{R}^n)$. From the theory of holomorphic functions there exists a function $F \in \mathbf{O}(\mathbf{C}^n \# \mathbf{R}^n)$ such that $(\partial/\partial z_1)F(z) = G(z)$. Then the sought hyperfunction is $f = [F]$.

The second part of the proof is not so easy because the hyperfunction zero is defined by any element of the vector space $\sum_{j=1}^{n} \mathbf{O}(U \# j\Omega)$.

By the same reason as in the first part of the proof we can take $\Omega = \{x \in \mathbf{R}^n; |x_j| < q,\ j = 1, \dots, n\}$. Denote by U the convex open set in $\mathbf{C}^n$, $U = \Omega + i\mathbf{R}^n$, and by F the defining function of f which satisfies the equation $(\partial/\partial x_1)f(x) = 0$. Then F satisfies

$$(2.5) \qquad (\partial/\partial z_1)F(z) = \sum_{j=1}^{n} G_j(z)|_{U\#\Omega},\ G_j \in \mathbf{O}(U \# j\Omega),\ j = 1, \dots, n.$$

By the same property of holomorphic functions, we used in the first part of the proof, there exist $H_j \in \mathbf{O}(U \# j\Omega)$, $j = 1, \dots, n$, such that $(\partial/\partial z_1)H_j(z) = G_j(z)$, $j = 1, \dots, n$, because $U \# j\Omega$ is an open set in $\mathbf{C}^n$ consisting of convex components. Consequently (2.5) has now the form

$$\frac{\partial}{\partial z_1}(F(z) - \sum_{j=1}^{n} H_j(z)|_{U\#\Omega}) = 0.$$

It follows that $F(z) - \sum_{j=1}^{n} H_j(z)|_{U\#\Omega} \in \mathbf{O}(U\#\Omega)$ and depends on $(z_2, \dots, z_n)$ only.

Denote by Γ_σ^n the σ-th orthant in $\mathbf{R}^n$ and by $V_\sigma = (\Omega + i\Gamma_\sigma) \cap U$, then $U\#\Omega = \bigcup_\sigma V_\sigma$. If by Ω_1 is denoted the set $\Omega_1 = \{|x_j| < q;\ j = 2, \dots, n\}$, then the function $F(z) - \sum_{j=1}^{n} H_j(z)|_{U\#\Omega}$ can be continued to $(\{|x_1| < q\} + i\mathbf{R}) \times (\Omega_1 + i\Gamma_\sigma^{n-1})$, being constant in z_1.

This shows that f is a hyperfunction which depends on $(x_2, \dots, x_n)$ only. □

A more general assertion can be proved. Let $P(D)$ be a differential operator with constant coefficients of the elliptic type. Denote by $\mathbf{A}^P = \{u \in \mathbf{A};\ P(D)u = 0\}$ then $0 \to \mathbf{A}^P \to \mathbf{B} \overset{P(D)}{\longrightarrow} \mathbf{B} \to 0$ is a flabby resolution of $\mathbf{A}^P$ [30].

Definition 2.2. An infinite-order differential operator

$$J(D) = \sum_{|\alpha|\geq 0} b_\alpha D^\alpha, \quad (|\alpha| = \alpha_1 + \ldots + \alpha_n),$$

with coefficients satisfying $\lim_{|\alpha|\to\infty} \sqrt[|\alpha|]{b_\alpha \alpha!} = 0$, is called a local operator with constant coefficients.

By properties of holomorphic functions the series

$$J(D)F = \sum_{|\alpha|\geq 0} b_\alpha D^\alpha F, \quad F \in \mathbf{O}(U)$$

converges locally uniformly in U. Hence a local operator is an endomorphism of the sheaf $\mathbf{O}$ and induces also an endomorphism of the sheaf $\mathbf{B}$.

Moreover, a hyperfunction f with support only at the origin is uniquely expressible as

$$f = J(D)\delta = \sum_{|\alpha|\geq 0} b_\alpha D^\alpha \delta,$$

where $J(D)$ is an appropriate local operator (see [7, p. 156]).

2.2. Hyperfunctions defined by boundary value representation

2.2.1. Definition and main properties. In the next definition of hyperfunctions we need the notion of infinitesimal wedge.

Definition 2.3. Let Ω be an open set in $\mathbf{R}^n$ and Γ an open cone in $\mathbf{R}^n$. An open set $W \subset \mathbf{C}^n$ is called an infinitesimal wedge (for short i.w.) of type $\Omega + i\Gamma 0$ if it satisfies the following conditions:

a) $W \subset \Omega + i\Gamma$;

b) For every proper subcone $\Gamma', \Gamma' \subset\subset \Gamma$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that $W \supset \Omega_\epsilon + i(\Gamma' \cap \{y; \|y\| < \delta\})$, where $\Omega_\epsilon = \{x \in \Omega; d(x, \partial\Omega) > \epsilon\}$; Ω is the edge of this i.w.

There are infinitely many infinitesimal wedges of type $\Omega + i\Gamma 0$; such an i.w. we denote by the same symbol $\Omega + i\Gamma 0$ or by $\Omega + iI$. We also express by $F \in \mathbf{O}(\Omega + i\Gamma 0)$ the fact that F is holomorphic on one of such i.w. of type $\Omega + i\Gamma 0$.

Consider $X(\Omega) = \oplus_\Gamma \mathbf{O}(\Omega + i\Gamma 0)$, where Γ ranges over all open cones V in $\mathbf{R}$. By the local Bochner theorem, if F is holomorphic on an i.w. $\Omega + iI$ of the type $\Omega + i\Gamma 0$, then it is also holomorphic on $\Omega + i\tilde{I}$, where $\tilde{I}$ is the convex hull of I. Thus we can assume, without loss of generality, that every Γ is convex.

$X(\Omega)$ is a $\mathbf{C}$-vector space with the $\mathbf{C}$-linear operation: $\lambda \bigoplus_{i=1}^n F_i + \eta \bigoplus_{j=1}^m G_j = \lambda F_1 \oplus \cdots \oplus \lambda F_n \oplus \eta G_1 \oplus \ldots \oplus \eta G_m$, where $F_i \in \mathbf{O}(\Omega + i\Gamma_i 0), i = 1, \ldots, n$, and

$G_j \in \mathbf{O}(\Omega + i\Gamma_j'0)$, $j = 1, \dots, m$. Using the notation $+$ in place of $\oplus$, consider the C-vector space $Y(\Omega)$ generated by the elements of $X(\Omega)$ of the following form: $F_1' + F_2' - F_3'$, where $F_j' \in \mathbf{O}(\Omega + i\Gamma_j'0)$, $j = 1, 2, 3$ and $\Gamma_1' \cap \Gamma_2' \supset \Gamma_3'$; $F_1'(z) + F_2'(z) = F_3'(z)$ holds on the common domain. In particular if $F \in \mathbf{O}(\Omega + i\Gamma 0)$ and $\Gamma' \subset \Gamma$, then the difference of F and its restriction on i.w. of type $\Omega + i\Gamma' 0$ also belong to $Y(\Omega)$.

Definition 2.4. The mapping

$$\Omega \to X(\Omega)/Y(\Omega), \tag{2.6}$$

where Ω is an open set in $\mathbf{R}^n$, defines a presheaf on $\mathbf{R}^n$; we denote it by $\tilde{\mathbf{B}}$. (If $\Omega' \subset \Omega$, then the restriction $r_{\Omega'\Omega} : \tilde{\mathbf{B}}(\Omega) \to \tilde{\mathbf{B}}(\Omega')$ is defined as usually via restriction of functions).

Denote by $F(x + i\Gamma 0)$ an element of the quotient space $X(\Omega)/Y(\Omega)$ determined by $F \in \mathbf{O}(\Omega + iI)$, where $\Omega + iI$ is an i.w. of the type $\Omega + i\Gamma 0$. Any element of $\tilde{\mathbf{B}}(\Omega)$ is represented by

$$f(x) = \sum_{j=1}^{m} F_j(x + i\Gamma_j 0) \tag{2.7}$$

where $\{F_j; j = 1, \dots, m\}$ is the set which gives the defining function of f.

To prove the next proposition we need the assertions of a lemma cited below. The proof of this lemma is easy and one can find it in [7, p. 332].

Lemma 2.1. *Suppose that the vectors $\eta^0, \eta^1, \dots, \eta^n$ belong to $\mathbf{R}^n$ and that the open half spaces determined by them: $E_{\eta^i} = \{y \in \mathbf{R}^n; (\eta^i, y) > 0\}$, $i = 0, 1, \dots, n$ satisfy*

$$E_{\eta^0} \cup E_{\eta^1} \cup \dots \cup E_{\eta^n} = \mathbf{R}^n \setminus \{0\}. \tag{2.8}$$

Then the following statements hold:

a) $E_{\eta^0} \cap E_{\eta^1} \cap \dots \cap E_{\eta^n} = \emptyset$

b) Any n vectors of $\eta^0, \eta^1, \dots, \eta^n$ are linearly independent. Hence the intersection of half spaces corresponding to them is a proper open convex cone.

c) Denote by $\Gamma_j = E_{\eta^0} \cap \dots \cap \hat{E}_{\eta^j} \cap .. \cap E_{\eta^n}$. Let $j, k \in \{0, 1, .., n\}$. Then $\Gamma_j + \Gamma_k = E_{\eta^0} \cap \dots \cap \widehat{E}_{\eta^j} \cap \dots \cap \widehat{E}_{\eta^k} \cap \dots \cap E_{\eta^n}$, where the notation $\widehat{}$ denotes suppression of the factor under it.

Proposition 2.5. *The presheaf $\tilde{\mathbf{B}}$ defined by (2.6) is isomorphic to the n-th derived sheaf $\mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})$ as a presheaf and hence it is actually a sheaf.*

Proof. Let $\eta^0, \eta^1, \dots, \eta^n \in \mathbf{R}^n$ be such that (2.8) holds, where $E_{\eta^i} = \{y \in \mathbf{R}^n; (\eta^i, y) > 0\}$, $i = 0, 1, \dots, n$, are the open half spaces determined by η^i. Set $U_j = (\mathbf{R}^n + iE_{\eta^j}) \cap U$, $j = 0, 1, \dots, n$, and $U_{n+1} \equiv U$. $\mathbf{U} = \{U_0, U_1, \dots, U_n, U_{n+1}\}$, $\mathbf{U}' = \{U_0, U_1, \dots, U_n\}$ give a relative Stein covering of the pair of open sets $(U, U \setminus \Omega)$, where U is a Stein open set in $\mathbf{C}^n$ such that $U \cap \mathbf{R}^n = \Omega$ and Ω is relatively closed in U. Now we can follow the idea of the proof of Corollary 2.1. Just by the same reasons as in the proof of Proposition 2.2, (2.3) holds. Thus a relative n-cochain with respect to the constructed covering is of the form

$$(2.9)\quad \sum_{j=0}^{n} \operatorname{sgn} b_{0\ldots\hat{j}\ldots n+1} F_j(z),\ \ F_j \in \mathbf{O}(U_0 \cap \ldots \cap \widehat{U}_j \cap \ldots \cap U_{n+1}),\ \ j = 0, 1, \ldots, n.$$

(The notation $\widehat{\ }$ denotes suppression of the factor under it).

By Lemma 2.1 a), $E_{\eta^0} \cap E_{\eta^1} \cap \ldots \cap E_{\eta^n} = \emptyset$. It follows that there exist no relative $(n+1)$-cochains and (2.9) is necessarily a relative cocycle.

A relative $(n-1)$-cochain is of the form

$$\sum_{j<k} \operatorname{sgn} b_{0\ldots\hat{j}\ldots\hat{k}\ldots n+1} F_{jk}(z),$$

$$F_{jk} \in \mathbf{O}(U_0 \cap \ldots \cap \widehat{U}_j \cap \ldots \cap \widehat{U}_k \cap \ldots \cap U_{n+1}),\quad j,k = 0, \ldots, n,$$

and its boundary is

$$\sum_{j=0}^{n} b_{0\ldots\hat{j}\ldots n+1}\Big(\sum_{k>j}(-1)^k F_{jk}(z) + \sum_{k<j}(-1)^{k+1} F_{kj}(z)\Big).$$

Denote by $\Gamma_j = E_{\eta^0} \cap \ldots \cap \widehat{E}_{\eta^j} \cap \ldots \cap E_{\eta^n}$. By Lemma 2.1 b) and c), Γ_j is a proper cone in $\mathbf{R}^n$ and $U_0 \cap \ldots \cap \widehat{U}_j \cap \ldots \cap U_{n+1} = (\mathbf{R}^n + i\Gamma_j) \cap U$; $U_0 \cap \ldots \cap \widehat{U}_j \cap \ldots \cap \widehat{U}_k \cap \ldots \cap U_{n+1} = (\mathbf{R}^n + i(\Gamma_j + \Gamma_k)) \cap U$.

As in Proposition 2.2 and Corollary 2.1 we conclude that

$$(2.10)\qquad \mathbf{B}(\Omega) \cong \sum_{j=0}^{n} \mathbf{O}((\mathbf{R}^n + i\Gamma_j) \cap U) \Big/ \sum_{j<k} \mathbf{O}((\mathbf{R}^n + i(\Gamma_j + \Gamma_k)) \cap U).$$

Now we can define a C-linear mapping $\mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega)$ which is consistent with restrictions so that it is a presheaf homomorphism: Suppose that the functions

$$F_j \in \mathbf{O}(U_0 \cap \ldots \cap \widehat{U}_j \cap \ldots \cap U_n),\ \ j = 0, 1, \ldots, n.$$

We associate with the element $f \in \mathbf{B}(\Omega)$, given by $(F_0, \ldots, F_n)$, the element

$$(2.11)\qquad \sum_{j=0}^{n} (-1)^j F_j(x + i\Gamma_j 0) \in \tilde{\mathbf{B}}(\Omega).$$

We have to construct the inverse correspondence to this one. Take an element $F(x + i\Gamma 0) \in \tilde{\mathbf{B}}(\Omega)$ given by $F \in \mathbf{O}(\Omega + i\Gamma 0)$. Determine $n+1$ vectors $\eta^0, \eta^1, \ldots, \eta^n \in \mathbf{R}^n$ in such a way that $E_{\eta^1} \cap \ldots \cap E_{\eta^n} \subset\subset \Gamma$ and that (2.8) holds. We also assume that the n-simplex formed by $\eta^1, \ldots, \eta^n$ is compatible with the orientation of $\mathbf{R}^n$. Choose a Stein open set $U \subset \mathbf{C}^n$, $U \cap \mathbf{R}^n = \Omega$ such that Ω is relatively closed in U and that $F(z)$ is holomorphic on the i.w. $(\Omega + i(E_{\eta^1} \cap \ldots \cap E_{\eta^n})) \cap U$. Now we can construct the relative covering $\mathbf{U} = \{U_0, \dots, U_{n+1}\}$, $\mathbf{U}' = \{U_0, \dots, U_n\}$ of the pair $(U, U \setminus \Omega)$, where $U_j = (\Omega + iE_{\eta^j}) \cap U$, $j = 0, 1, \dots, n$, and $U_{n+1} \equiv U$. With this relative covering the function F defines an element of $H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O})$ and an element of $H^n_\Omega(U, \mathbf{O}) = \mathbf{B}(\Omega)$ as in the first part of the proof. Morimoto (see [7, p. 335]) proved that this element does not depend on the choice of the vectors $\eta^0, \eta^1, \dots, \eta^n$. To the obtained element, by C-linear mapping $\mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega)$ defined in the second part of the proof, it corresponds $F(x + i\Gamma_0 0)$, where $\Gamma_0 = E_{\eta^1} \cap \dots \cap E_{\eta^n} \subset\subset \Gamma$. By the definition of the equivalence class in $X(\Omega)$, $F(x + i\Gamma_0 0) = F(x + i\Gamma 0)$. Consequently, the composition of homomorphisms just defined, $\tilde{\mathbf{B}}(\Omega) \to \mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega)$ is the identity mapping. Analogously, it can be proved that the composition $\mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega) \to \mathbf{B}(\Omega)$ is the identity mapping, as well. □

In one-dimensional case there is only two open cones with vertex at zero: $\Gamma_+ = \mathbf{R}_+$ and $\Gamma_- = \mathbf{R}_-$. If $U \subset \mathbf{C}$ is an open set such that $U \cap \mathbf{R} = \Omega$, Ω is relatively closed in U, then $U_+ = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z > 0\}$ and $U_- = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z < 0\}$ are infinitesimal wedges. Now (2.7) can be given as follows

$$f(x) = F_+(x + i\mathbf{R}_+ 0) - F_-(x + i\mathbf{R}_- 0),$$

where $F_+ \in \mathbf{O}(\Omega + i\mathbf{R}_+ 0)$ and $F_- \in \mathbf{O}(\Omega + i\mathbf{R}_- 0)$. We write for short

$$f(x) = F_+(x + i0) - F_-(x - i0). \tag{2.12}$$

(F_+, F_-) is **the pair of defining functions of f.*

Remark. After Proposition 2.5 we can identify $\mathbf{B}$ and $\tilde{\mathbf{B}}$ and we shall write only $\mathbf{B}$ for the both sheaves. The definition of hyperfunctions via $\tilde{\mathbf{B}}$ is said to be "intuitive" definition or definition by boundary value representation. The "intuitive" definition is easier to understand and to apply in solving mathematical models. But theoretically it is in some sense incomplete. First, expression (2.7) is not invariant under coordinate transformations. Secondly, it is not easy to check that a given hyperfunction is zero in a neighbourhood of a point.

The elementary operations, we gave for the elements of $\mathbf{B}(\Omega)$, can be easily transfered if these elements have the form given in (2.7). Let

$$f(x) = \sum_j F_j(x + i\Gamma_j 0) \text{ and } g(x) = \sum_k G_k(x + i\Gamma'_k 0)$$

be elements of $\mathbf{B}(\Omega)$ given in the form as in (2.7) then:

$$\lambda f(x) + \eta g(x) = \sum_j \lambda F_j(x + i\Gamma_j 0) + \sum_k \eta G_k(x + i\Gamma'_k 0), \quad \eta, \lambda, \mu \in \mathbf{C}$$

$$(\varphi f)(x) = \sum_j (\varphi F_j)(x + i\Gamma_j 0), \quad \varphi \in \mathbf{A}(\Omega)$$

$$D^\alpha_x f(x) = \sum_j (D^\alpha_z F_j)(x + i\Gamma_j 0).$$

2.2.2. Microfunctions. When we investigate solutions of mathematical models focusing our attention on points in which these solutions have their singularities, we need not the sheaf **B** but the sheaf of microfunctions $\mathcal{R}$. The theory and applications of microfunctions are significantly developed within last years (cf. [13]). For the further study of microfunctions and micro-local operators one can derive a profit from the book [10]; see also [24].

The construction of the sheaf $\mathcal{R}$ we shall give in one-dimensional case because of simplicity, but our intention is to explain the concept and the idea of microfunctions theory in many-dimensional case, too. According to this purpose we shall adapt the more general notation than are really needed in one-dimensional case.

Definition 2.5. Let $S^0 = \{\pm 1\}$ and denote a point (x, ξ) of $\mathbf{R} \times S^0$ by $(x, (\xi/i)dx\infty)$ for convenience. A hyperfunction f is said to be microanalytic at the point $(x, (1/i)dx\infty)$ if a pair of defining functions (F_+, F_-) of f can be both analytically continued to $U_+ = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z > 0\}$, where U is a suitable complex neighbourhood of x. Similarly, f is said to be microanalytic at the point $(x, -(1/i)dx\infty)$ if F_+ and F_- can be both analytically continued to $U_- = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z < 0\}$.

From the definition of the set of points, where f is microanalytic, it follows that this set is an open set in $\mathbf{R} \times S^0$.

Definition 2.6. The set of all points where the hyperfunction f is not microanalytic is called the singular spectrum of f (for short $\mathrm{SS}\, f$).

If $\pi : \mathbf{R} \times S^0 \to \mathbf{R}$ is a natural projection, then $\pi(\mathrm{SS}\, f) = \operatorname{sing\,supp} f$. The linear differential operator with real analytic coefficients does not enlarge the singular spectrum of a hyperfunction.

The first idea to investigate local properties of hyperfunctions required the construction of the quotient sheaf $\mathbf{B}/\mathbf{A}$. But for the singular spectrum of a hyperfunction, $\mathbf{B}/\mathbf{A}$ was still incomplete. So we have to introduce an other quotient space.

Definition 2.7. Let $h : \mathbf{X} \to \mathbf{Y}$ be a continuous mapping of a topological space $\mathbf{X}$ into a topological space $\mathbf{Y}$. Let U be an open set in $\mathbf{X}$ and V be any open set belonging to $\mathbf{Y}$ and containing $h(U)$. For a sheaf $\mathbf{G}$ on $\mathbf{Y}$, the correspondence $U \to \varinjlim_{V \supset h(U)} \mathbf{G}(V)$ is a presheaf on $\mathbf{X}$. Its associated sheaf is called the inverse sheaf of $\mathbf{G}$ by h and is denoted by $h^{-1}\mathbf{G}$.

In particular when f is an open function, if for every $y \in Y$ and open set $U \subset \mathbf{X}$, $U \cap f^{-1}(y)$ is connected, then $f^{-1}\mathbf{G}(U) = \mathbf{G}(f(U))$ holds.

Let us apply the construction of the inverse sheaf to the canonical projection $\pi : \mathbf{R} \times S^0 \to \mathbf{R}$. Let $\Omega_1 \times \{idx\infty\} \cup \Omega_2 \times \{-idx\infty\}$ be an open set in $\mathbf{R} \times S^0$ (Ω_1 and Ω_2 are open in $\mathbf{R}$). Then we have

$$\pi^{-1}\mathbf{B}(\Omega_1 \times \{idx\infty\} \cup \Omega_2 \times \{-idx\infty\}) = \mathbf{B}(\Omega_1) \oplus \mathbf{B}(\Omega_2)$$

Definition 2.8. We have the following two sheaves over $\mathbf{R} \times S^0$;

1. The subsheaf $\mathbf{A}^*$ of $\pi^{-1}\mathbf{B}$ defined by

$$\mathbf{A}^*(\Omega_1 \times \{idx\infty\} \sqcup \Omega_2 \times \{-idx\infty\} = \\ = \{f \in \mathbf{B}(\Omega_1);\ \mathrm{SS}\, f \cap \Omega_1 \times \{idx\infty\} = \emptyset\} \\ \oplus \{f \in \mathbf{B}(\Omega_2);\ \mathrm{SS}\, f \cap \Omega_2 \times \{-idx\infty\} = \emptyset\}.$$

2. The sheaf of microfunctions: $\mathcal{R} = \pi^{-1}\mathbf{B}/\mathbf{A}^*$.

From 2 we have the exact sequence: $0 \to \mathbf{A}^* \to \pi^{-1}\mathbf{B} \to \mathcal{R} \to 0$.

The sheaf $\mathcal{R}$ has the following main properties:

Proposition 2.6. *1. $\mathcal{R}$ is a flabby sheaf.*

2. For any open set $U \subset \mathbf{R} \times S^0$, $\mathcal{R}(U) = \pi^{-1}\mathbf{B}(U)/\mathbf{A}^*(U)$, or equivalently

$$0 \to \mathbf{A}^*(U) \to \pi^{-1}\mathbf{B}(U) \to \mathcal{R}(U) \to 0$$

is an exact sequence.

3. The linear differential operator with real analytic coefficients induces a sheaf endomorphism $\mathcal{R} \to \mathcal{R}$.

For the proof see for example [7, pp. 53–55].

Let $F \in \mathbf{O}(U)$, where $U \subset \mathbf{C}$ is a domain (open and connected set) and a neighbourhood of a point a. Define D^{-1} by

$$D^{-1}F(z) = \int_a^z F(\zeta)d\zeta \tag{2.13}$$

with an appropriate path connecting a and z. Consider the infinite series of operators

$$Q(z, D_z) = \sum_{k=1}^{\infty} b_k(z) D_z^{-k}. \tag{2.14}$$

Definition 2.9. Operator (2.14) whose coefficients satisfy the following condition

1. $b_k(z)$ are holomorphic in a complex domain $U \subset \mathbf{C}$;

2. $\limsup_{k\to\infty} \sqrt[k]{\sup_{z\in K} |b_k(z)|/k!} < \infty$

holds for every compact set $K \subset U$, is called a **pseudo-differential operator* or a *micro-differential operator of order* ≤ 0.

A pseudo-differential operator of order ≤ 0 defines a sheaf endomorphism of $\mathcal{R}$ in a special way via germs (cf. [7, p. 61]).

2.3. Fourier hyperfunctions and the Fourier transform of them

2.3.1. Mainly used approaches to Fourier hyperfunctions. 1. **Sato's definition* ([27] for the proofs see also [12]). Denote by $\mathbf{D}^n$ the compactification of

$\mathbf{R}^n$, $\mathbf{D}^n = \mathbf{R}^n \sqcup S^{n-1}_\infty$, obtained by adding points at infinity in all directions. A fundamental system of neighbourhoods of a point at infinity $(a\infty)$ is

$$U_{B,r}(a\infty) = \{x \in \mathbf{R}^n; (x/\|x\|) \in B, \|x\| \geq r\} \sqcup \{x\infty; x \in B\},$$

where B is a neighbourhood of the point a in S^{n-1}. $\tilde{\mathbf{O}}$ will be the sheaf on $\mathbf{D}^n + i\mathbf{R}^n$ defined as follows: For any open set $U \subset \mathbf{D}^n + i\mathbf{R}^n$, $\tilde{\mathbf{O}}(U)$ consists of those elements of $\mathbf{O}(U \cap \mathbf{C}^n)$ which satisfy $|F(z)| \leq C_{V,\epsilon} \exp\{\epsilon |\operatorname{Re} z|\}$ uniformly for any open set $V \subset \mathbf{C}^n, \bar{V} \subset U$ and for every $\epsilon > 0$, where $\bar{V}$ is the closure of V in $\mathbf{D}^n + i\mathbf{R}^n$. If $U \subset \mathbf{C}^n$, then $\tilde{\mathbf{O}}(U) = \mathbf{O}(U)$. Hence, $\tilde{\mathbf{O}}|_{\mathbf{C}^n} = \mathbf{O}$ It is proved that $\mathbf{D}^n \subset \mathbf{D}^n + i\mathbf{R}^n$ is purely n-codimensional relative to $\tilde{\mathbf{O}}$ ([25]). The n-th derived sheaf $\mathbf{H}^n_{\mathbf{D}^n}(\tilde{\mathbf{O}})$, denoted by $\mathbf{Q}$ and regarded as a sheaf on $\mathbf{D}^n$, is called the sheaf **of Fourier hyperfunctions (of slowly increasing hyperfunctions).* $\mathbf{Q}$ is flabby sheaf on $\mathbf{D}^n$. In particular $\mathbf{Q}|_{\mathbf{R}^n} = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O}) = \mathbf{B}$. Hence the sequence

$$\mathbf{Q}(\mathbf{R}^n) \to \mathbf{B}(\mathbf{R}^n) \to 0$$

is exact.

One of the main results on the sheaf $\mathbf{Q}$ is the following proposition.

Proposition 2.7. [7] *Let $U \subset \mathbf{D}^n + i\mathbf{R}^n$ be an open set such that $U \cap \mathbf{C}^n$ is convex and $\operatorname{Im} z$ is bounded on $\partial(U \cap \mathbf{C}^n)$. Then $H^k(U, \tilde{\mathbf{O}}) = 0$ for $k \geq 1$. Hence, in particular if we choose a convex neighbourhood I of $0 \in \mathbf{R}^n$, then $U = \mathbf{D}^n + iI$, $U_j = (\mathbf{D}^n + iI) \cap \{Imz_j \neq 0\}$, $j = 1, \dots, n$, is a relative Leray covering for the pair $(\mathbf{D}^n + iI, (\mathbf{D}^n + iI) \setminus \mathbf{D}^n)$ relative to the sheaf $\tilde{\mathbf{O}}$ and the representation*

$$\mathbf{Q}(\mathbf{D}^n) = \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n)$$

is valid.

This theorem gives a possibility of another approach to the Fourier hyperfunctions. Namely, the set of Fourier hyperfunctions can be defined as

$$\tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n).$$

2. **Zharinov's definition* [35]. Denote by $T^M = \mathbf{R}^n + iM$ and by $s_M(\xi) = \sup\{-y\xi; y \in M\}$, where $y\xi = y_1\xi_1 + \cdots + y_n\xi_n$ and $M \subset \mathbf{R}^n$. Let A and B be bounded domains in $\mathbf{R}^n$. We denote by $\Phi(A, B)$ the Banach space of holomorphic functions on T^A with the norm

$$\|\varphi\|^A_{s_B} = \sup\{\exp(s_B(\xi))|\varphi(\xi + i\eta)|;\ \xi + i\eta \in T^A\}.$$

The space $\vec{\Phi}$, defined as the inductive limit over all A and B which contain zero,

$$\vec{\Phi} = \varinjlim_{A \ni 0, B \ni 0} \Phi(A, B)$$

is a DFS space. The dual space, $\vec{\Phi}'$, is an FS space (Fréchet–Schwartz). $\vec{\Phi}'$ is isomorphic to the space of Fourier hyperfunctions. The Fourier transform of $f \in \vec{\Phi}'$ is given by $\langle \mathbf{F}f, \varphi\rangle = \langle f, \mathbf{F}\varphi\rangle$, $\varphi \in \vec{\Phi}$ and is an automorphism on $\vec{\Phi}'$.

3. *"Intuitive" definition of Fourier hyperfunctions.* The systematic exposition of this approach one can find in [7]. We shall follow it in the next part.

2.3.2. "Intuitive" definition of Fourier hyperfunctions.

Let Γ_j be an open cone in $\mathbf{R}^n$ and $\mathbf{D}^n + iI_j$ an infinitesimal wedge of type $\mathbf{D}^n + i\Gamma_j 0$. $F_j \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI_j)$ means that F_j is holomorphic on $\mathbf{R}^n + iI_j$ and for every $\epsilon > 0$, $|F_j(z)| \le C_{V,\epsilon} \exp(\epsilon|\operatorname{Re} z|)$ uniformly for any open set $V \subset \mathbf{C}^n$, $\bar{V} \subset \mathbf{D}^n + iI_j$.

Consider $X = \oplus_\Gamma \tilde{\mathrm{O}}(\mathbf{D}^n + i\Gamma 0)$ where Γ ranges over all open convex cones. X is a C-vector space. We denote by Y the C-vector space generated by the elements of X of the following form: $F_1 + F_2 - F_3$, where $F_j \in \tilde{\mathrm{O}}(\mathbf{D}^n + i\Gamma_j 0)$, $j = 1, 2, 3$, and $\Gamma_1 \cap \Gamma_2 \supset \Gamma_3$; $F_1(z) + F_2(z) = F_3(z)$ holds on the common domain.

Denote by $\tilde{\mathbf{Q}} = X/Y$. This is a C-vector space too. By $F(x + i\Gamma 0)$ we denote the element of the quotient space $\tilde{\mathbf{Q}}$ determined by $F \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI)$.

If $F_2 = 0$ and F_3 can be extended to $\mathbf{D}^n + iI_1$, then F_3 can be substituted by F_1 in $\oplus_\Gamma$.

Corollary 8.5.4 in the book of Kaneko [7] asserts that $\mathbf{Q}(\mathbf{D}^n) = \tilde{\mathbf{Q}}$. The proof is just the same as the proof for Proposition 2.5. We shall prove only that there exists a homomorphism $\mathbf{Q}(\mathbf{D}^n) \to \tilde{\mathbf{Q}}$. Notice that Proposition 2.7 asserts that

$$\mathbf{Q}(\mathbf{D}^n) = \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n).$$

Then every element of $\mathbf{Q}(\mathbf{D}^n)$ is represented by $F \in \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$ and F consists of 2^n independent holomorphic functions F_σ, $F_\sigma \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI_\sigma)$ where $\mathbf{D}^n + iI_\sigma$ is an infinitesimal wedge of the form $\mathbf{D}^n + i\Gamma_\sigma 0$, Γ_σ is the σ-th orthant in $\mathbf{R}^n$. To F we associate the following element of $\tilde{\mathbf{Q}}$:

$$\sum_\sigma \operatorname{sgn} \sigma F_\sigma(x + i\Gamma_\sigma 0).$$

Any element $G_j \in \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n)$ is holomorphic across the interface $Imz_j = 0$. The pairs given by G_j in the sum $\sum_\sigma \operatorname{sgn}\sigma F_\sigma(x + i\Gamma_\sigma 0)$ cancel each other because of the definition of Y in $\tilde{\mathbf{Q}}$. Thus the mapping $\mathbf{Q}(\mathbf{D}^n) \to \tilde{\mathbf{Q}}$ is well defined and it is C-linear. □

In $\mathbf{Q}(\mathbf{D}^n)$ is defined a topology. First, we define a family of seminorms $\|\cdot\|_{K,\epsilon}$ in $\tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \equiv E$: For every compact set $K \subset\subset I \setminus \{0\}$ and $\epsilon > 0$

$$\|F\|_{K,\epsilon} = \sup_{z \in \mathbf{R}^n + iK} |F(z)| e^{-\epsilon|\operatorname{Re} z|}, \quad F \in E.$$

The set of all such seminorms reduces essentially to a countable family and $\tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$ turns out to be a Fréchet space. It is also a Montel space. Since the space

$H \equiv \sum_{j=1}^{n} \tilde{O}((D^n + iI)\#jD^n)$ is a closed subspace of $\tilde{O}((D^n + iI)\#D^n)$, the quotient space E/H admits the structure of a Fréchet and Montel space. If π is the canonical mapping: $E \to E/H$, then π is an open mapping. A family of seminorms on $\mathbf{Q}(D^n)$ is given by

$$p_{K,\epsilon}(\widehat{F}) = \inf_{h \in H} \|F + h\|_{K,\epsilon}, \quad F \in \widehat{F} \in \mathbf{Q}(D^n).$$

Since the space $\tilde{\mathbf{Q}}$ is isomorphic to the space $\mathbf{Q}(D^n)$, this isomorphism induces a topology on $\tilde{\mathbf{Q}}$. In this way the construction of $\tilde{\mathbf{Q}}$ gives an approach to the Fourier hyperfunctions, easier then the classical one, given by Sato which uses the cohomology theory. Every element $f \in \tilde{\mathbf{Q}}$ is given by

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \tag{2.15}$$

where every $F_j(x + i\Gamma_j 0)$ denotes the element of the quotient space $\tilde{\mathbf{Q}}$ determined by $F_j \in \tilde{O}(D^n + iI_j)$, $j = 1, \dots, N$. The functions F_j, $j = 1, \dots, N$ define a function F and we write $f = [F]$.

The relation between Fourier hyperfunctions and hyperfunctions is unexpected. Namely, we have a well defined mapping $\tilde{\mathbf{Q}} \to \mathbf{B}(\mathbf{R}^n)$: given $f \in \tilde{\mathbf{Q}}$ by (2.15), it can be regarded as a hyperfunction in the form (2.3) with the same defining function. Theorem 8.4.4 in Kaneko's book [7] asserts that this is a surjective mapping.

Let φ be a real analytic function such that it can be analytically continuable to a complex neighbourhood $U \subset D^n + i\mathbf{R}^n$ of D^n and such that $\varphi(z) \in \tilde{O}(U)$. If $f \in \tilde{\mathbf{Q}}$, then the multiplication is defined by: $\varphi f = [\varphi F]$, where $f = [F]$.

2.3.3. Fourier transform of Fourier hyperfunctions.

Kaneko [7] has explained Sato's fundamental ideas concerning the Fourier transform as follows. Denote by $\mathcal{F}$ the Fourier transform. Let $f \in \tilde{\mathbf{Q}}$, where $f(x) = F_+(x + i\mathbf{R}_+0) - F_-(x + i\mathbf{R}_-0)$ then $\mathcal{F}(f) = [\phi]$, where

$$\phi_+(\zeta) = \int_{-\infty}^{0} e^{-i(x+iy_+)\zeta} F_+(x + iy_+)dx - \int_{-\infty}^{0} e^{-i(x+iy_-)\zeta} F_-(x + iy_-)dx, \ \operatorname{Im}\zeta > 0,$$

$$\phi_-(\zeta) = \int_{0}^{\infty} e^{-i(x+iy_+)\zeta} F_+(x + iy_+)dx - \int_{0}^{\infty} e^{-i(x+iy_-)\zeta} F_-(x + iy_-)dx, \ \operatorname{Im}\zeta < 0,$$

where $y_+ > 0$ and $y_- < 0$ are fixed belonging to the infinitesimal wedges $R + iR_+0$ and $R + iR_-0$, respectively.

All the integrals have a meaning because of: $-i(x + iy)(\xi + i\eta) = x\eta + \xi y - i(x\xi - \eta y)$.

To give a precise definition of the Fourier transform of elements belonging to $\tilde{\mathbf{Q}}$ we need the following proposition.

Let $F \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI)$, where $\mathbf{R}^n + iI$ is an infinitesimal wedge of type $\mathbf{R}^n + i\Gamma 0$. It is said that F **decreases exponentially* outside a closed convex proper cone Δ^0 if restricting $\operatorname{Re} z$ outside any cone containing Δ^0 as a proper subcone, then F satisfies the estimate $|F(z)| = O(\exp(-\delta|\operatorname{Re} z|))$ for a suitable $\delta > 0$ and locally uniformly for $y \in I$.

Proposition 2.8. *Suppose that for an infinitesimal wedge $\mathbf{R}^n + iI$ of type $(\mathbf{R}^n + i\Gamma 0)$ the function $F \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI)$ and decreases exponentially outside a closed convex proper cone Δ^0 (Δ^0 is the dual cone to the cone Δ). Set*

$$G(\zeta) = \int\limits_{\operatorname{Im} z = y} e^{-iz\zeta} F(z)dx$$

for any $y \in I$. Then it converges locally uniformly in ζ ranging over an infinitesimal wedge $\mathbf{R}^n - iJ$ of type $\mathbf{R}^n - i\Delta 0$ and $G \in \tilde{\mathrm{O}}(\mathbf{D}^n - iJ)$. Furthermore, $G(\zeta)$ decreases exponentially outside Γ^0. Hence $\mathcal{F}[F(x + i\Gamma 0)] = G(\xi - i\Delta 0)$, where $G \in \tilde{\mathbf{Q}}$, as well.

Proof. Let K be a fixed compact set belonging to $-\Delta$. Choose the cone Δ'_K containing Δ^0 such that $\operatorname{Re}(-iz\zeta) = x\eta + y\xi \le -c_K|x| + y\xi$ for $\eta \in K$, $x \in \Delta'_K$, where $c_K > 0$. We can now analyse the function $G(\zeta)$.

$$G(\zeta) = \int\limits_{\operatorname{Im} z = y \in I} e^{-iz\zeta} F(z)dx =$$

$$= \int\limits_{\Delta'_K} e^{-i(x+iy)\zeta} F(x + iy)dx + \int\limits_{\mathbf{R}^n \setminus \Delta'_K} e^{-(x+iy)\zeta} F(x + iy)dx.$$

The first integral converges locally uniformly in $\zeta \in \mathbf{R}^n + iK$ because $F \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI)$ and $\operatorname{Re}(-iz\zeta) \le -c_K|x| + y\xi$. For the second integral we can use that $|F(z)| = O(e^{-\delta|x|})$ locally uniformly for $y \in I$. If we suppose that $\eta \in K \cap \{|\eta| < \delta_K\}$ for a suitable chosen δ_K, then the second integral converges locally uniformly on $\mathbf{R}^n + i(K \cap \{|\eta| < \delta_K\})$. Hence, $G(\zeta)$ is a holomorphic function in ζ on an infinitesimal wedge of type $\mathbf{R}^n - i\Delta 0$. From the both integrals we can draw out the factor $e^{y\xi}$. Consequently $G \in \tilde{\mathrm{O}}(\mathbf{D}^n - iJ)$ and if ξ moves outside a cone containing Γ^0 as a proper subcone, we have $y\xi \le -\delta_y|\xi|$, $\delta_y > 0$. Thus $G(\zeta)$ decreases exponentially outside Γ^0. □

In order to define the Fourier transform of an element $f \in \tilde{\mathbf{Q}}$, $f = [F] = \sum_{m=1}^{M} F_m(x + i\Gamma_m 0)$ we shall first prove that F_m, $m = 1, \dots, M$, can be made decomposed into a finite sum of functions decreasing exponentially outside a closed convex cone. One of such decomposition can be in the following way:

Let $\sigma_k = \pm 1$, $k = 1, \dots, n$; the multi signature $\sigma = (\sigma_1, \dots, \sigma_n)$ determines the cone Γ_σ as the σ-th orthant in $\mathbf{R}^n$. Put $\chi_+(t) = e^t/(1 + e^t)$, $\chi_-(t) = 1/(1 + e^t)$ and $\chi_\sigma(z) = \chi_{\sigma_1}(z_1) \dots \chi_{\sigma_n}(z_n)$. Every $\chi_\sigma(z)$ decreases exponentially along the real axis outside any cone containing the closed σ-th orthant as a proper subcone and $\sum_\sigma \chi_\sigma(z) = 1$. These properties of χ_σ make possible the decomposition of $F_m, F_m(z) = \sum_\sigma \chi_\sigma(z)F_m(z)$, where each term $\chi(z)F_m(z)$ decreases exponentially outside the closed σ-th orthant. Consequently, the Fourier hyperfunction $f = [F]$ can be given in the form

$$f(x) = \sum_{k=1}^{N} U_k(x + i\Gamma_k 0), \tag{2.16}$$

where $U_k \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI_k)$, $\mathbf{D}^n + iI_k$ is an infinitesimal wedge of the form $\mathbf{R}^n + i\Gamma_k 0$ and $|U_k(z)| = O(\exp(-\delta|\operatorname{Re} z|))$ for a $\delta > 0$ when restricting $\operatorname{Re} z$ outside any cone containing a fixed cone Δ_k^0 but locally uniformly for $\operatorname{Im} z \in I_k$.

Definition 2.10. The Fourier transform of $f = [F]$ given by (2.16) is

$$\mathcal{F}[f] = \sum_{k=1}^{N} \mathcal{F}[U_k(x + i\Gamma_k 0)].$$

By Proposition 2.8 it maps $\tilde{\mathbf{Q}}$ into $\tilde{\mathbf{Q}}$. One can prove (Lemma 8.3.3 in [7]) that $\mathcal{F}[f]$ does not depend on the decomposition of the defining function F into finite sums of hyperfunctions decreasing exponentially outside a closed convex cone.

By Proposition 2.8 it is easy to define the inverse Fourier transform $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[G](z) = \frac{1}{(2\pi)^n} \int_{\operatorname{Im}\zeta=\eta\in -J} e^{iz\zeta} G(\zeta)d\xi \equiv F(z).$$

The properties of F and G given in Proposition 2.8 make elementary the proof that $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{id}$. Hence this holds for any Fourier hyperfunction and the Fourier transform is an automorphism of $\tilde{\mathbf{Q}}$.

We saw that the mapping $\tilde{\mathbf{Q}} \to \mathbf{B}(\mathbf{R}^n)$ is surjective. In this sense every hyperfunction has the Fourier transform.

2.3.4. An other definition of the Fourier transform of Fourier hyperfunctions. First we shall define the space $\mathbf{P}_*$. Let δ be a positive constant and I an open set in $\mathbf{R}^n$ containing 0. Then $\tilde{\mathrm{O}}^{-\delta}(\mathbf{D}^n + iI)$ is defined as the set of holomorphic functions F on $\mathbf{R}^n + iI$ such that for every compact set $K \subset\subset I$ and every $\epsilon > 0$ there exists $C_{K,\epsilon} > 0$, $|F(z)| \le C_{K,\epsilon} \exp(-(\delta - \epsilon)|\operatorname{Re} z|)$ uniformly for $z \in R^n + iK$. Then

$$\mathbf{P}_* = \varinjlim_{I \ni 0} \varinjlim_{\delta \downarrow 0} \tilde{\mathrm{O}}^{-\delta}(\mathbf{D}^n + iI)$$

with the topology of inductive limit.

It is easy to prove that if $f \in \tilde{\mathrm{O}}^{-\delta}(\mathbf{D}^n + i\{|y| < \gamma\})$, the Fourier transform

$$\mathcal{F}f = \hat{f}(\zeta) = \int_{\operatorname{Im} z=y} e^{-iz\zeta} f(z)dx \in \tilde{\mathrm{O}}^{-\gamma}(\mathbf{D}^n + i\{|\eta| < \delta\}), |y| < \gamma.$$

The Fourier transform is an automorphism of $\mathbf{P}_*$. $\mathbf{P}_*$ is called the space of **rapidly decreasing real analytic functions.*

By Theorem 8.6.2 in [7], $\mathbf{P}_*$ and $\tilde{\mathbf{Q}}$ are topological dual to each other. The inner product is given by

$$\langle f,\varphi\rangle = \int\limits_{\mathbf{R}^n} f(x)\varphi(x)dx = \sum_{j=1}^{n} \int\limits_{\mathrm{Im}\, z=y^{(j)}} (\varphi F_j)(z)dx,$$

where

$$y^{(j)} \in I_j,\ \varphi \in \mathbf{P}_*,\ f = [F] = \sum_{j=1}^{N} F_j(x+i\Gamma_j 0) \in \tilde{\mathbf{Q}}.$$

The Fourier transform acts as a topological automorphism on $\tilde{\mathbf{Q}}$ and $\langle \mathcal{F}f,\varphi\rangle = \langle f,\mathcal{F}\varphi\rangle$ is valid.

Let us remark that the space $\vec{\Phi}$ in Zharinov's approach is just the space $\mathbf{P}_*$. This gives a connection between Zharinov's approach and the other two. Also the three different definitions of the Fourier transform give the same operation.

Remark. The proof that $\vec{\Phi}'$ is isomorphic to $\mathbf{Q}(\mathbf{D}^n)$ can be find in [12].

2.4. Asymptotic behaviour of Fourier hyperfunctions and its applications

Asymptotic behaviour of generalized functions has an important role in the analysis of solutions to mathematical models, to precise the asymptotics of integral transforms or to discuss some problems in the theoretical phisics.

2.4.1. Quasiasymptotics. As we cited in 2.3.1, Zharinov [35] defined the space $\vec{\Phi}'$ which is isomorphic to $\tilde{\mathbf{Q}}$ or $\mathbf{Q}(\mathbf{D}^n)$. But in the same paper he constructed the space $\vec{\Lambda}'(\mathcal{O}) \subset \vec{\Phi}'$, where $\mathcal{O}$ is a domain in $\mathbf{R}^n$. For an element of $\vec{\Lambda}'(\mathcal{O})$, he defined the quasiasymptotics.

Let Γ be a convex closed acute cone in $\mathbf{R}^n$. We denote by $\Sigma = \mathrm{int}\,\Gamma^0$, where Γ^0 is the dual cone to Γ. We will follow Zharinov's definitions and results given in [34] and [35].

Let A and B be two bounded domains in $\mathbf{R}^n$. Denote by $s_B(\xi) = \sup\{-y\xi;\ y \in B\}$ and by $\Lambda(A,B)$ the Banach space of functions holomorphic on $\mathbf{R}^n + iA$ and such that

$$\|\varphi\|^{A}_{-s_B} = \sup\{e^{-s_B(\xi)}|\varphi(\xi+i\eta)|;\ \zeta \in \mathbf{R}^n + iA\} < \infty$$

with the topology given by the norm $\|\cdot\|^{A}_{-s_B}$. It is easy to see that $\Lambda(A,B) \subset \Lambda(A',B')$, when $A' \subset A$ and $B \subset B'$. With the inclusion mapping $\rho_{AB,A'B'}$: $\Lambda(A,B) \to \Lambda(A',B')$ we can define

$$\overrightarrow{\Lambda}(\Sigma) = \varinjlim_{A\ni 0,\ B\subset\subset\Sigma} \Lambda(A,B); \quad \overleftarrow{\Lambda}(\Sigma) = \varprojlim_{B\subset\subset\Sigma,\ 0\in A} \Lambda(B,A).$$

The space $\overrightarrow{\Lambda}(\Sigma)$ is a DFS space and its dual space $\overrightarrow{\Lambda}'(\Sigma)$ is an FS space. But $\overleftarrow{\Lambda}(\Sigma)$ is an FS space. Zharinov (cf. [35]) proved that $\vec{\Phi}'_\Gamma \subset \overrightarrow{\Lambda}'(\Sigma) \subset \vec{\Phi}'$, where $\vec{\Phi}'$ is defined in 2.3.1 and $\vec{\Phi}'_\Gamma = \{g \in \vec{\Phi}';\mathrm{supp}\, g \subset \Gamma\}$.

Now we can cite the definition of the quasiasymptotics (cf. [34]).

Definition 2.11. Suppose that $g \in \vec{\Lambda}'(\Sigma)$ and that ρ is a positive and continuous function on $(0,\infty)$. If there exists

$$\lim_{t\to\infty} g(t\zeta)/\rho(t) = h(\zeta) \quad \text{in } \vec{\Lambda}'(\Sigma), \quad h \neq 0,$$

then it is said that g has the quasiasymptotics related to ρ.

Since $\vec{\Lambda}'(\Sigma)$ is an FS space, the limit in Definition 2.11 is equivalent with

$$\lim_{t\to\infty} \langle g(t\xi)/\rho(t), \varphi(\xi)\rangle = \langle h, \varphi\rangle, \; h \neq 0$$

for every $\varphi \in \vec{\Lambda}(\Sigma)$.

Similarly as for the quasiasymptotics of distributions (cf. [33]) one can prove that ρ and h in Definition 2.11 have the following properties:

1) ρ has the form $\rho(t) = t^\alpha L(t)$, $\alpha \in \mathbf{R}$ and L is Karamata's slowly varying function [9];

2) h is homogeneous of degree α.

The defined quasiasymptotic behaviour of Fourier hyperfunctions can be used to precise properties of solutions to mathematical models (partial differential equations, integral equations,...) as it is done by means of the quasiasymptotics of distributions (cf. [33]). Applications of the quasiasymptotic behaviour of Fourier hyperfunctions are not yet developed but one can expect interesting results of such investigations.

To illustrate the applications of the quasiasymptotics we cite an Abelian type theorem for the Laplace transform of Fourier hyperfunctions (cf. [34]). But first we have to define the Laplace transform of elements belonging to $\vec{\Lambda}'(\Sigma)$.

For a fixed $z \in \mathbf{R}^n + iB$, where B is a bounded subset of Σ, $e^{iz} \in \Lambda(A,B)$ for every bounded set A and $\|e^{iz}\|^{A}_{-s_B} = e^{s_A(x)}$, $z = x+iy$. Thus for every fixed $z \in \mathbf{R}^n + i\Sigma$, $e^{iz\xi} \in \vec{\Lambda}(\Sigma)$.

Definition 2.12. The Laplace transform of $g \in \vec{\Lambda}'(\Sigma)$, $\mathcal{L}g$, is defined by

$$\mathcal{L}g(z) = \langle g(\xi), e^{iz\xi}\rangle, \quad z \in \mathbf{R}^n + i\Sigma.$$

In [35] Zharinov have proved that the Laplace transform defines an isomorphism $\vec{\Lambda}'(\Sigma)$ onto $\overleftarrow{\Lambda}(\Sigma)$. With this property and the cited properties of the family of functions $\{e^{iz\xi};\ z \in \mathbf{R}^n + i\Sigma\}$ it is easy to prove the following proposition of the Abelian type.

Proposition 2.9. *Suppose that* $g, h \in \vec{\Lambda}'(\Sigma)$ *and* $\rho(t) = t^\alpha L(t)$, $\alpha \in \mathbf{R}$. *Denote by* $G = \mathcal{L}g$ *and* $H = \mathcal{L}h$, *then* $G, H \in \overleftarrow{\Lambda}(\Sigma)$. *If*

$$g(t\xi)/\rho(t) \to h(\xi), \; t \to \infty, \; \text{in } \vec{\Lambda}'(\Sigma),$$

then

$$G(z/t)/t^n\rho(t) \to H(z), \; t \to \infty, \; \text{in } \overleftarrow{\Lambda}(\Sigma).$$

In [34] one can find other properties of the quasiasymptotics of Fourier hyperfunctions.

Let us remark that Komatsu in [16] has also defined the Laplace transform of a subspace of hyperfunctions, denoted by $\mathbf{B}^{\exp}_{[a,\infty]}$, and in [17] he has related his theory with other theories of the Laplace transform of generalized functions.

2.4.2. S-asymptotics. An other asymptotic behaviour has been defined for distributions (ultradistributions) and has been applied in the quantum field theory (cf. [25], [26]). It is called the S-asymptotics. It is easy to extend it to Fourier hyperfunctions.

Definition 2.13. Suppose that c is a positive function defined on $\mathbf{R}^n$ and $f \in \mathbf{Q}(\mathbf{D}^n)$. f is said to have the S-asymptotics related to c in the cone Γ if there exists

$$\lim_{k\in\Gamma,\, \|k\|\to\infty} \frac{f(\cdot+k)}{c(k)} = h \text{ in } \mathbf{Q}(\mathbf{D}^n),\ h \neq 0. \tag{2.17}$$

Since $\mathbf{Q}(\mathbf{D}^n)$ is a Montel space, (2.17) can be given in the form:

$$\lim_{k\in\Gamma,\, \|k\|\to\infty} \langle \frac{f(x+k)}{c(k)}, \varphi(x)\rangle = \langle h, \varphi\rangle,\ \ h \neq 0, \tag{2.18}$$

for every $\varphi \in \mathbf{P}_*$.

The next examples shows that Definition 2.13 is not a trivial extension of the S-asymptotics of distributions. Let $P(D)$ be a local operator $\sum_{|\alpha|\geq 0} b^\alpha D^\alpha$, $b^\alpha \neq 0$. The Fourier hyperfunction $f = 1 + P(D)\delta$ has the S-asymptotics related to $c = 1$ in any cone Γ and with the limit $h = 1$ but f is not a distribution. For the S-asymptotics of f it is enough to prove that

$$\lim_{k\in\Gamma,\, \|k\|\to\infty} \langle P(D)\delta(x+k), \varphi(x)\rangle = 0,\ \ \varphi \in \mathbf{P}_*.$$

Since $P(D)$ maps $\mathbf{P}_*$ into $\mathbf{P}_*$,

$$\langle P(D)\delta(x+k), \varphi(x)\rangle = \langle \delta(x+k),\ P(-D)\varphi(x)\rangle = \psi(k),$$

where $\psi = P(-D)\varphi$. By the property of elements belonging to $\mathbf{P}_*$ (see 2.3.4) $\lim_{k\in\Gamma,\, \|k\|\to\infty} \psi(k) = 0$ for every cone Γ.

A hyperfunction g supported by the origin is uniquely expressible as $g = \tilde{P}(D)\delta$, where $\tilde{P}(D)$ is a local operator. In such a way, with the above, we proved that every Fourier hyperfunction with support $\{0\}$ has the limit, given in (2.17) and (2.18), equal zero.

Since $P(D)\delta = \sum_{|\alpha|\geq 0} b_\alpha D^\alpha \delta$ is a distribution if and only if $b_\alpha \neq 0$ for a finite number of α, the Fourier hyperfunction $1 + P(D)\delta$ is not a distribution, but it has the S-asymptotics related to $c = 1$.

We can also find such coefficients b_α of the local operator $P(D)$ such that $f = 1 + P(D)\delta$ is not defined by an ultradistribution belonging to the Gevrey class

$D^{(s)'}$ or $D^{\{s\}'}$, $s > 1$. Because of simplicity, we shall consider one-dimensional case. Choose $P(D)$ such that the coefficients of $P(D)$ are: $b_n = (n!)^{-(1+c_n)}$, $n \in N$, where $c_n = (10 \log n)^{-1}$. With these coefficients, $P(D)$ is a local operator. Namely,

$$\lim_{n\to\infty} \sqrt[n]{b_n n!} = \lim_{n\to\infty} (n!)^{-1/(n \log\log n)} = 0.$$

Also any ultradistribution in Gevrey class $s > 1$, supported by $\{0\}$, is of the form

$$J(D)\delta = \sum_{n=0}^{\infty} a_n D^n \delta, \;\; |a_n| \leq Ck^n/(n!)^s \tag{2.19}$$

with some constants k and C (Beurling's type) or for any $k > 0$ with a constant C (Roumieu's type). But $b_n = (n!)^{-(1+c_n)}$ does not satisfy condition for coefficients of $J(D)$ in (2.19). Namely, since $c_n \to 0$ when $n \to \infty$, for any $s > 1$, there exists n_0 such that $1 < 1 + c_n < s$, $n \geq n_0$. Thus,

$$(n!)^{-(1+c_n)} > Ck^n/(n!)^s, \quad n \geq n_0, \; k > 0.$$

Consequently, $P(D)\delta$ does not represent an ultradistribution.

However we can suppose that $P(D)\delta$ is an ultradistribution g with support $\{0\}$ in Gevrey class $s > 1$. Then we would have an ultradifferential operator $J_1(D)$ such that

$$g = J_1(D)\delta = \sum_{n=0}^{\infty} e_n D^n \delta, \; |e_n| \leq Ck^n/(n!)^s.$$

But in this case $J_1(D)$ would be a local operator, $J_1(D) \neq P(D)$. This contradicts the fact that a hyperfunction with support at $\{0\}$ is given by a unique local operator.

The defined S-asymptotics can be also used in order to precise the behaviour of solutions to mathematical models as it is done with the S-asymptotics of distributions (cf. [26]). We shall illustrate this with the problem of asymptotic behaviour of solutions to equations given by local operators.

Since a local operator maps continuously $\mathbf{Q}(\mathbf{D}^n)$ into $\mathbf{Q}(\mathbf{D}^n)$, we have:

Proposition 2.10. *Suppose that $f \in \mathbf{Q}(\mathbf{D}^n)$ and has the S-asymptotics related to c and to the cone Γ with the limit h. Then*

$$\lim_{k\in\Gamma,\, \|k\|\to\infty} \frac{P(D)f(x+k)}{c(h)} = P(D)h \;\; in \;\; \mathbf{Q}(\mathbf{D}^n).$$

Corollary. *A necessary condition that a solution of the equation $P(D)x = f$ has the S-asymptotics related to c and to the cone Γ with the limit u is that f has the limit (2.16) with $h = P(D)u$.*

If $P(D)$ fulfils some additional properties, we would have in the Corollary not only necessary, but necessary and sufficient condition. Such a case is if $P(D)y = \delta$ has a solution in $\mathbf{Q}^{-\gamma}(\mathbf{D}^n)$, $\gamma > 0$.

Let us remark that we have only first results concerning the asymptotics of Fourier hyperfunctions. Regarding the definition of the asymptotic behaviour of hyperfunction in general case we do not know that such a definition exists.

2.4.3. Asymptotics Taylor expansion. Estrada and Kanwal [2] elaborated a method of asymptotic expansions for distributions quite different in relation to the methods which can be find for distributions in [21], [26], [31] and [33]. The results of Estrada and Kanwal gave a nice confirmation that the asymptotic expansions have arisen in several fields of applications as a powerful technique. They started by considering the asymptotic Taylor expansion for distributions, its application and generalizations.

Definition 2.14. If $f \in \mathbf{D}'$, then for a fixed $\xi \in \mathbf{R}^n$ and $\epsilon \in \mathbf{R}$

$$f(x+\epsilon\xi) \sim \sum_{|k|=0}^{\infty} \frac{D^k f(x)}{k!}(\epsilon\xi)^k, \quad \text{as } \epsilon \to 0, \tag{2.20}$$

which means that for any function $w \in \mathbf{D}$ and for any $N \in \mathbf{N}$

$$\langle f(x+\epsilon\xi), w(x)\rangle = \sum_{|k|=0}^{N} \frac{\langle D^k f(x), w(x)\rangle}{k!}(\epsilon\xi)^k + O(\epsilon^{N+1}),$$

as $\epsilon \to 0$. The formal series in (2.20) is called the asymptotic Taylor expansion for f (on the straight line $\{h\xi; h \in \mathbf{R}\}$).

For any $f \in \mathbf{D}'$, (2.20) holds. Also, Definition 2.14 can be applied to any space of generalized functions defined as the dual space $\mathbf{A}'$ of a basic space $\mathbf{A}$ of smooth functions. Since the space of Fourier hyperfunctions is a space of this type, Definition 2.14 can be repeated with the space $\mathbf{Q}(\mathbf{D}^n)$ instead of $\mathbf{D}'$.

Concerning this definition a natural question arises: *What are necessary and sufficient conditions that the asymptotic Taylor expansion for a generalized function f is in the same time the Taylor series for f, convergent in the space of generalized functions.*

The answer on this question for distributions and ultradistributions one can find in [32]. For the Fourier hyperfunction we can prove

Proposition 2.11. *The asymptotic Taylor expansion* (2.20) *for* $u \in \mathbf{Q}(\mathbf{D}^n)$ *on the straight line* $\{h\xi; h \in \mathbf{R}\}$, *where* $\xi_i \neq 0$, $i = 1, \dots, n$, *is the Taylor series convergent in* $\mathbf{Q}(\mathbf{D}^n)$ *when* $\eta\xi \in B(0, \eta_0\xi)$ *for an* $\eta_0 > 0$ *if and only if there exists an* $r = (r_1, \dots, r_n)$, $r_i > 0$, $i = 1, \dots, n$, *such that u is determined by a real analytic function which can be extended as a holomorphic function on* $\{z \in \mathbf{C}^n;\ |\mathrm{Im}\, z_i| < r_i,\ i = 1, \dots, n\}$.

The proof is based on two Kaneko's results. First every Fourier hyperfunction $u \in \mathbf{Q}(\mathbf{D}^n)$ can be given in the form $u = P_1(D)f$, where $P_1(D)$ is an elliptic local operator and f is an infinitely differentiable function of infra exponential growth [7]. Second, there exist an elliptic local operator $P_2(D)$ and an infinitely differentiable function g *rapidly decreasing* ($|g(x)| \leq C\exp(-\alpha\|x\|)$, $x \in \mathbf{R}^n$ for some $\alpha > 0$) such that $\delta = P_2(D)g$ (δ is the delta distribution) [8].

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Index

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PSEUDODIFFERENTIAL OPERATORS

Contents

Part I. CLASSICAL THEORY

The aim of lecture notes is to present the basic facts of the theory of pseudodifferential operators and to give sufficiently enough motivations for further study of this very important theory. Also, in the notes authors develop the theory of pseudodifferential operators within Colombeau's new generalized functions.

Pseudodifferential operators are generalization of differential operators. They form the minimal algebra of operators in which each elliptic operator has the inverse up to a smoothing operator. Thus, the roots of the theory of pseudodifferential operators are in the theory of elliptic operators. This theory is used for microlocal analysis of equations, the hypoellipticity for example. In the second part we show this for the (hypo)elliptic pseudodifferential equations with coefficients in the space of Colombeau's generalized functions.

Part I of the notes was written when the first two authors had studied the classical theory of pseudodifferential operators, as a part of their doctoral studies, under the coordination of the third author, who prepared a seminar on that topic at the Institute of Mathematics of Novi Sad University during 1988/89 and 1990/91. The authors documented their work, writing down an extensive paper (in Serbian), proving the theorems, explaining in details various examples etc. Some parts of this unpublished material constitute these notes. The main references for Part I are monographs **[10]**, **[19]** and **[20]**.

Part II is devoted to the pseudodifferential calculus within Colombeau's space od generalized functions, $\mathcal{G}$. The idea was established by the authors during the seminar on Colombeau's theory which took place in 1989/1990. The third author made a coherent theory on pseudodifferential operators in Colombeau's sense of new generalized functions **[16]**, during his stay in Japan at the Tokyo University in the winter of 1992/1993.

It was not an easy job to present so large theory on aroumd sixty pages, the number which was predicted by the editor. Because of that our exposition is of fragmented character in some parts. We think that the reader can find in the notes enough information for further study of pseudodifferential and Fourier integral operators.

We assume that the reader is familiar with the basic notions of functional analysis, distribution theory and the theory of partial differential equations. For further study we refer to **[10]**, **[11]**, **[15]**, **[19]** **[20]**.

1. Introduction

If K is a compact subset of an open set Ω, $\Omega \subset \mathbb{R}$, and ϕ is C^∞ function, then

$$\|\phi\|_{\alpha,K} = \sup_{\substack{\|\beta\| \leq \alpha \\ x \in K}} |\partial^\beta \phi(x)|.$$

Denote by $\mathcal{D}_{\alpha,K}$ the Banach space of C^∞ functions ϕ on Ω such that $\operatorname{supp}\phi \subset K$ and $\|\phi\|_{\alpha,K} < \infty$. The projective limit of $\mathcal{D}_{\alpha,K}$, as $\|\alpha\| \to \infty$, is denoted by $\mathcal{D}_K$. The Schwartz's space of test functions $\mathcal{D}(\Omega)$ is defined as the inductive limit of spaces $\mathcal{D}_K$ as $K \subset\subset \Omega$ and the union of K's exhaust Ω. We will use the notation $C_0^\infty = \mathcal{D}(\Omega)$. (The notation $K \subset\subset \mathbb{R}$ or $K \subset\subset \Omega$ means that K is compact in $\mathbb{R}$ or $\mathbb{C}$.) The strong dual of the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ is called the Schwartz space of distributions. The space of distributions with compact supports is denoted by $\mathcal{E}(\Omega)'$. It is the strong dual of the space smooth functions on Ω with the uniform convergence of all the derivatives on compact subsets.

Schwartz's space of rapidly decreasing functions is defined by

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n),\ (\forall \alpha, \beta \in \mathbb{N}_0^n)\ (\exists c \in \mathbb{R})\ (\sup |x^\alpha (\partial^\beta u)(x)| \leq c)\}.$$

Its strong dual is the space of tempered distributions $\mathcal{S}'$.

The Fourier transformation of a function $u \in L^1$ is defined by

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx,\ \xi \in \mathbb{R}^n,$$

and the inverse transformation by

$$\mathcal{F}^{-1}(u)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} u(x)\, dx,\ \xi \in \mathbb{R}^n.$$

If u is supported by a compact set, then the Fourier–Laplace transformation is defined as above with ξ substituted by $\zeta \in \mathbb{C}^n$.

The Fourier transformation is an isomorphism of $\mathcal{S}$ (resp. $\mathcal{S}'$) onto the same space.

The Sobolev space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ consists of tempered distributions f which Fourier transform $\hat{f}$ satisfies the following condition

$$(1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^n).$$

We shall give Palley-Wiener theorem which will be used often in this work.

"Let K be a convex compact subset of $\mathbb{R}^n$ and let H be its characteristic function. If u is a distribution of order N supported by K, then for its Fourier–Laplace transformation satisfies

$$|\hat{u}(\zeta)| \leq C(1 + |\zeta|)^N e^{H(\operatorname{Im}\zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.1}$$

Every entire function on $\mathbb{C}^n$ which satisfies (1.1) is a Fourier–Laplace transformation of a distribution with the support contained in K.

If $u \in C_0^\infty(K)$, then for every $N \in \mathbb{N}$

$$|\hat{u}(\zeta)| \leq C_N(1+|\zeta|)^{-N}e^{H(\operatorname{Im}\zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.2}$$

Conversely, if (1.2) holds for an entire function and for every N, then it is a Fourier–Laplace transformation of some function $u \in C_0^\infty(K)$.

2. Elliptic operators with constant coefficients

As a motivation for the theory of pseudodifferential operators we give the construction of a parametrix for elliptic operators.

2.1. Parametrix of elliptic operator with constant coefficients. Let us consider the following equation in $\mathcal{S}'$

$$P(D)u = \sum_{|\alpha|\leq m} c_\alpha D^\alpha u = f, \tag{2.1}$$

where $f \in \mathcal{E}'$ is given $D = (D_1, D_2, \dots, D_n)$, $D_j = -\sqrt{-1}\frac{\partial}{\partial x_j}$, $c_\alpha \in \mathbb{C}$, $|\alpha| \leq m$. If a solution exists, then

$$P(\xi)\hat{u}(\xi) = \hat{f}(\xi),\ \xi \in \mathbb{R}^n,$$

and formally, $\hat{u}(\xi) = \hat{f}(\xi)/P(\xi)$. Therefore, a formal solution to problem (2.1) is given by

$$u(x) = \mathcal{F}^{-1}\Big(\frac{\hat{f}(\xi)}{P(\xi)}\Big)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\frac{\hat{f}(\xi)}{P(\xi)}\,d\xi,\ x \in \mathbb{R}^n. \tag{2.2}$$

The integral on the right-hand side in (2.2) is not defined in general because of zeros of $P(\xi)$ and the behavior of $\hat{f}(\xi)$ in infinity. There are some special cases in which a modification of (2.2) gives the solution to (2.1). We will discuss one of such cases.

Let $P(D)$ be a differential operator of order m, (i.e. the corresponding polynomial $P(\xi)$ is of order m) and let

$$P(\xi) = P_m(\xi) + Q(\xi),$$

where $P_m = \sum_{|\alpha|=m} a_\alpha D^\alpha$ and $Q(\xi)$ is polynomial of order not greater than $(m-1)$. The operator $P_m(D)$ is called the principal symbol of $P(D)$.

Note $P_m(\lambda\xi) = \lambda^m P_m(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{R}^n$, i.e. the polynomial $P_m(\xi)$ is a positive homogeneous function of order m. This implies that the set of zeros of the polynomial $P_m(\xi)$ (the variety of P_m), for $m > 0$ is a cone and it is called the characteristic cone.

Definition 2.1. A differential operator $P(D)$ of order m is elliptic if $P_m(\xi) \neq 0$, for every $\xi \in \mathbb{R}^n\backslash\{0\}$, where $P_m(D)$ is the principal symbol of the operator $P(D)$.

Example 2.1. If the dimension of the space equals one, then all the differential operators with constant coefficients are elliptic.

Example 2.2. The Laplace operator

$$\Delta = \Big(\frac{\partial}{\partial x_1}\Big)^2 + \Big(\frac{\partial}{\partial x_2}\Big)^2 + \cdots + \Big(\frac{\partial}{\partial x_n}\Big)^2$$

is elliptic. Its principal symbol is $-|\xi|^2 = -\xi_1^2 - \cdots - \xi_n^2$.

Example 2.3. For $n = 2$, the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

is elliptic, and its principal symbol is $i(\xi + i\mu)/2$.

Lemma 2.2. *Let $P(D)$ be an elliptic differential operator. Then the set of zeros of the polynomial $P(\xi)$ is compact in $\mathbb{R}^n$.*

Proof. If $P(D) = P_m(D) + Q(D)$ as above, then $P_m(\xi) \neq 0$, for $\xi \in S^{n-1}$, where S^{n-1} is the closed unit sphere in $\mathbb{R}^n$. Because of that

$$|P_m(\xi)| \geq c > 0, \quad \xi \in S^{n-1}.$$

If $0 \neq \xi \in \mathbb{R}^n$, then $\xi/|\xi| \in S^{n-1}$. This implies $|P_m(\xi/|\xi|)| \geq c$ and because of the positive homogeneity of $P_m(\xi)$ we have

$$|P_m(\xi)| \geq c|\xi|^m, \quad \xi \in \mathbb{R}^n.$$

The order of polynomial $Q(\xi)$ is not greater than $m - 1$, and therefore,

$$|Q(\xi)| \leq c_1|\xi|^{m-1}, \quad \xi \in \mathbb{R}^n, \quad |\xi| > 1.$$

Let $\xi \in \mathbb{R}^n$ satisfy $P(\xi) = 0$ and $|\xi| > 1$. Then we have

$$c|\xi|^m \leq |P_m(\xi)| = |Q(\xi)| \leq c_1|\xi|^{m-1}.$$

This implies $|\xi| \leq c_1/c$. Thus the set of zeros of $P(\xi)$ is bounded. □

Let $P(D)$ be an elliptic operator such that its variety is contained in the ball $L(0, \rho)$, with the center at zero and radius ρ and let $\kappa(\xi) \in C^\infty(\mathbb{R}^n)$ be such that $\kappa(\xi) = 0$ for $|\xi| < \rho$ and $\kappa(\xi) = 1$ for $|\xi| > \rho' > \rho$. Denote

$$v(x) = \mathcal{F}^{-1}(\hat{f}(\xi)\kappa(\xi)/P(\xi))(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\hat{f}(\xi)\kappa(\xi)/P(\xi)\,d\xi, \ x \in \mathbb{R}^n.$$

This formal integral makes sense within the space of tempered distributions. It is the Fourier transformation of a tempered distribution.

In the sequel we will use the notation which have to be understood in the distributional sense.

It will be shown that $v(x)$ is not the solution of equation (2.1), but it differs from it only by a smooth function.

Formally (in fact in the sense of the tempered distributions)

$$\begin{aligned}P(D)v(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}\hat{f}(\xi)\kappa(\xi)\,d\xi = \mathcal{F}^{-1}(\hat{f}(\xi)\kappa(\xi))(x)\\ &= (\mathcal{F}^{-1}(\hat{f}(\xi)) - \mathcal{F}^{-1}(\hat{f}(\xi)(1-\kappa(\xi)))(x) = f(x) - Rf(x),\end{aligned}$$

where $Rf = \mathcal{F}^{-1}(\hat{f}(\xi)(1-\kappa(\xi)))$.

Note that $\kappa(\xi)/P(\xi)$ is a tempered distribution on $\mathbb{R}^n$ since it is a bounded smooth function. Since

$$|P(\xi)| \geq |P_m(\xi)| - |Q(\xi)| \geq (c|\xi| - c_1)|\xi|^{m-1} > 1,$$

for large enough $|\xi|$, it follows that $\mathcal{K} = \mathcal{F}^{-1}(\kappa(\xi)/P(\xi))$ is a tempered distribution, and

$$\begin{aligned}v(x) &= \mathcal{F}^{-1}(\hat{f}(\xi)\kappa(\xi)/P(\xi))(x)\\ &= (\mathcal{F}^{-1}(\kappa(\xi)/P(\xi)) * \mathcal{F}^{-1}(\hat{f}(\xi)))(x) = (\mathcal{K} * f)(x).\end{aligned}$$

Since the function $(1-\kappa) \in C_0^\infty$, the Palley-Wiener theorem implies that its Fourier transform $h = \mathcal{F}^{-1}(1-\kappa)$ can be extended on $\mathbb{C}^n$ as an analytic function of exponential type, such that its restriction on $\mathbb{R}^n$ belongs to $\mathcal{S}$. Then $Rf = h * f$ which implies

$$P(D)(\mathcal{K} * f)(x) = f(x) - h * f(x). \tag{2.3}$$

Let us define operators $\mathbf{R}$ and $\mathbf{K}$ by

$$\begin{aligned}&\mathbf{R} : \mathcal{E}' \to C^\infty, \quad \mathbf{R} : f \to Rf,\\ &\mathbf{K} : \mathcal{E}' \to \mathcal{S}', \quad \mathbf{K} : f \to Kf := \mathcal{K} * f.\end{aligned}$$

Then, $\mathbf{R}$ is a smoothing operator i.e. a linear and continuous mapping from $\mathcal{E}'$ to C^∞.

Using this notation we write (2.3) as $P(D) = \mathbf{K} = I - \mathbf{R}$. The operator $\mathbf{K}$ is called the parametrix of the differential operator $P(D)$. If it is known, then the solution of equation

$$P(D)E = \delta \tag{2.4}$$

(the fundamental solution for $P(D)$) exists, and $u = E * f$ is the solution to problem (2.1). By the classical theory, equation $P(D)w = h$ has a solution which is an analytic function on $\mathbb{C}^n$. Solution to equation (2.4) is $E = \mathcal{K} + w$ (because $P(D)\mathcal{K} = \delta - h$ and $P(D)w = h$).

3. Integral operators

3.1. Kernel theorem. Schwartz's kernel theorem is the basis one for the theory of integral operators is based on it.

Definition 3.1. Let X_i be open subsets of $\mathbb{R}^{n_i}$, and let $u_i \in C(X_i)$, $i \in \{1,2\}$. Then the continuous function $u_1 \otimes u_2$ on $X_1 \times X_2$ defined by

$$(u_1 \otimes u_2)(x_1, x_2) = u_1(x_1)u_2(x_2), \quad x_i \in X_i,$$

is called the tensor product u_1 and u_2.

Proposition 3.2. *Let $u_i \in \mathcal{D}'(X_i)$, $i = 1,2$. Then there exists a distribution $u \in \mathcal{D}'(X_1 \times X_2)$ such that*

$$u(\phi_1 \otimes \phi_2) = u_1(\phi_1)u_2(\phi_2), \ \phi_i \in C_0^\infty(X_i), \quad i = 1,2.$$

Proof. Let us define

$$u(\phi) = u_1(u_2(\phi(x_1, x_2))), \ \phi \in C_0^\infty(X_1 \times X_2),$$

(where u_i depends only on x_i). It is clear that the assertion of the proposition holds for u and $u(\phi) = u_2(u_1(\phi))$. □

Note, if $u_i \in \mathcal{E}'$, $i = 1,2$, then $u(\phi) = u_2(u_1(\phi))$, $\phi \in C^\infty(X_1 \times X_2)$.

The distribution u is called the tensor product of u_1 and u_2 and it is denoted by $u = u_1 \otimes u_2$.

Definition 3.3. A linear and continuous operator $A : \mathcal{D}(X_2) \to \mathcal{D}'(X_1)$ is called integral operator.

Theorem 3.4. *Let $K \in \mathcal{D}'(X_1 \times X_2)$. By*

$$\langle A\phi, \psi \rangle = K(\psi \otimes \phi), \ \psi \in C_0^\infty(X_1), \ \phi \in C_0^\infty(X_2) \tag{3.1}$$

is determinated a linear operator $A : \mathcal{D}(X_2) \to \mathcal{D}'(X_1)$. It is continuous, in the sense that $A\phi_j \to 0$ in the space $\mathcal{D}'(X_1)$, when $\phi_j \to 0$ in $C_0^\infty(X_2)$, i.e. it determines an integral operator.

Conversely, for every integral operator A there exists one and only one distribution K such that (3.1) holds. It is called the kernel of the operator A.

We refer to **[10]** for the proof.

Example 3.1. The kernel of the identity operator $\mathcal{D}(X) \to \mathcal{D}'(X)$, $A\varphi = \varphi$, where X is an open set in $\mathbb{R}^n$, is given by

$$\langle K, \phi \rangle = \int_X \phi(x,x)\,dx, \quad \phi \in C_0^\infty(X \times X),$$

i.e. $K(x,y) = \delta(x-y)$. It has the support on the diagonal.

We will use the following notation. If $A \subset X$ and $B \subset X \times Y$ then

$$A \circ B := \{y \in Y, (\exists x \in A)((x,y) \in B)\}.$$

If $A \subset Y$ and $B \subset X \times Y$, then

$$B \circ A := \{x \in X, (\exists y \in A)((x,y) \in B)\}. \tag{3.2}$$

Note that if A is a compact set and B is closed, then $B \circ A$ is a closed set.

In the following proposition we assume that $\operatorname{supp} K = B \subset X_1 \times X_2$, $A = \operatorname{supp} u \subset X_2$.

Proposition 3.5. *If $K \in \mathcal{D}'(X_1 \times X_2)$ is the kernel of the integral operator $A : \mathcal{D}(X_1) \to \mathcal{D}'(X_2)$, then $\operatorname{supp} Au \subset \operatorname{supp} K \circ \operatorname{supp} u$, $u \in C_0^\infty(X_2)$.*

Proof. Let us suppose that $x_1 \notin (\operatorname{supp} K \circ \operatorname{supp} u)$. Then there exists a neighborhood V of x_1 such that $V \cap (\operatorname{supp} K \circ \operatorname{supp} u) = 0$ because the set $\operatorname{supp} K \circ \operatorname{supp} u$ is closed. If $v \in C_0^\infty(V)$, then

$$(\operatorname{supp}(v \otimes u)) \cap \operatorname{supp} K = 0,$$

and therefore $\langle Au, v\rangle = 0$, i.e. $Au = 0$ on V, and $x_1 \notin \operatorname{supp} Au$. □

3.2. Proper integral operators. Let E and F be topological spaces and f be a continuous mapping of E into F. The mapping f is proper if for every compact set $K \subset F$ the set $f^{-1}(K)$ is compact in E.

Definition 3.6. Let X and Y be open sets in $\mathbb{R}^n$. An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is proper if the mappings $\pi_1 : \operatorname{supp} K_A(x, y) \to X$ and $\pi_2 : \operatorname{supp} K_A(x, y) \to Y$ are proper, where $K_A(x, y)$ is the kernel of A and π_1 and π_2 are the first and the second projection, respectively.

Proposition 3.7. *An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is proper if and only if distributions $K_A(x, y)\varphi(y)$ and $K_A(x, y)\phi(x)$ have compact supports in $X \times Y$ for arbitrary functions $\phi \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$.*

Proof. Let A be a proper integral operator, $\phi \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$. Since

$$\operatorname{supp} K_A(x, y)\varphi(y) \subset \operatorname{supp} K_A(x, y) \cap \pi_2^{-1}(\operatorname{supp} \varphi(y)),$$

it follows that $\operatorname{supp} K_A(x, y)\varphi(y)$ is a compact set. Analogously $K_A(x, y)\phi(y) \in \mathcal{E}'(X \times Y)$.

Assume that for every $\phi \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$ the distributions $K_A(x, y)\varphi(y)$ and $K_A(x, y)\phi(y)$ belong to $\mathcal{E}'(X \times Y)$. We will show that for arbitrary compact sets K_1 and K_2 of X and Y, respectively, the sets

$$\operatorname{supp} K_A \cap \pi_2^{-1}(K_2) \text{ and } \operatorname{supp} K_A \cap \pi_1^{-1}(K_1)$$

are compact in $X \times Y$. Let $\phi \in C_0^\infty(Y)$ and $\phi(y) = 1$ in some neighborhood of the set K_2. It follows

$$\operatorname{supp} K_A \cap \pi_2^{-1}(K_2) \subset \operatorname{supp} K_A(x, y)\phi(y),$$

which implies the compactness of the set $\operatorname{supp} K_A \cap \pi_2^{-1}(K_2)$. Analogously one can prove the compactness of the set $\operatorname{supp} K_A \cap \pi_1^{-1}(K_1)$. □

Proposition 3.8. *If an integral operator A is proper, then its transpose operator ${}^tA$ is proper, as well.*

Proof. Theorem 3.4 implies that there exists $K_A(x,y) \in \mathcal{D}'(X \times Y)$ and $K_{{}^tA}(y,x) \in \mathcal{D}'(Y \times X)$, such that

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle$$
$$\langle {}^tAv, u\rangle = \langle K_{{}^tA}(y,x), v(x)u(y)\rangle,$$

for every $u \in C_0^\infty(Y)$ and $v \in C_0^\infty(X)$

Since $\langle Au, v\rangle = \langle u, {}^tAv\rangle$, it follows

$$\langle K_A(x,y), u(y)v(x)\rangle = \langle K_{{}^tA}(y,x), v(x)u(y)\rangle,$$

i.e. $K_A(x,y) = K_{{}^tA}(y,x)$ in $\mathcal{D}(X,Y)$. Thus it follows that ${}^tA$ is a proper operator if A is a proper operator. □

Example 3.2. Let $P : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a continuous linear operator. Let $(\phi_j)_{j\in J}$, and $(\varphi_i)_{i\in I}$ be sequences in $C_0^\infty(X)$ and $C_0^\infty(Y)$ respectively. Let the families of sets $(\operatorname{supp}\phi_j)_{j\in J}$ and $(\operatorname{supp}\varphi_i)_{i\in I}$ be locally finite. (A family $(A_\alpha)_{\alpha\in\Lambda}$ of subsets of $\mathbb{R}^n$ is locally finite if for every $x \in \mathbb{R}^n$ and a bounded neighbourhood B of X, $B \cap A_\alpha \neq \emptyset$ only for finitely many $\alpha \in \Lambda$.) The mapping $u \mapsto Qu$, where

$$(Qu)(x) = \sum_{j\in J} \phi_j(x) P(\varphi_j(y)u(y))(x),\ u \in C_0^\infty(Y),\ x \in X \tag{3.3}$$

is a proper integral operator.

Because of the local finiteness of the family $(\phi_j)_{j\in J}$ the above sum is finite for every fixed x. One can simply check that $Q : C_0^\infty(Y) \to \mathcal{D}'(X)$ is an integral operator. Let us show that it is proper. Let $\psi \in C_0^\infty(X)$. Since P is an integral operator, Theorem 3.4 implies that there exists a kernel $K_P(x,y) \in \mathcal{D}'(X \times Y)$, such that

$$\begin{aligned}\langle (Qu)(x), \psi(x)\rangle &= \Big\langle \sum_{j\in J} \phi_j(x)\langle K_P(x,y), \varphi_j(y)u(y)\rangle, \psi(x)\Big\rangle \\ &= \sum_{j\in J} \langle\langle K_P(x,y), \varphi_j(y)u(y)\rangle, \phi_j(x)\psi(x)\rangle \\ &= \sum_{j\in J} \langle K_P(x,y), \varphi_j(y)u(y)\phi_j(x)\psi(x)\rangle \\ &= \Big\langle \sum_{j\in J} K_P(x,y)\varphi_j(y)\phi_j(x), u(y)\psi(x)\Big\rangle.\end{aligned}$$

Here we have used the fact that the sums are finite. The kernel of the integral operator Q equals

$$\sum_{j\in J} K_P(x,y)\varphi_j(y)\phi_j(x).$$

As $\varphi \in C_0^\infty(Y)$ (analogously $\phi \in C_0^\infty(X)$) the set $\operatorname{supp}\sum_{j\in J} K_P(x,y)\varphi_j(y)\phi_j(x)\varphi(y)$ ($\operatorname{supp}\sum_{j\in J} K_P(x,y)\varphi_j(y)\phi_j(x)\phi(x)$) is compact, since the sum is finite. From Theorem 3.7 it follows that Q is a proper integral operator.

Note that (3.3) is well defined for $u \in C^\infty(Y)$.

Proposition 3.9. *If* $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ *is a proper integral operator, with the kernel* K_A *and if* $u \in C_0^\infty(Y)$, *then*

$$\operatorname{supp}(Au) \subset (\operatorname{supp} K_A) \circ (\operatorname{supp} u) \tag{3.4}$$

and $(\operatorname{supp} K_A) \circ (\operatorname{supp} u)$ *is compact.*

Proof. By Proposition 3.5, $\operatorname{supp} Au \subset \operatorname{supp} K_A \circ \operatorname{supp} u$. We have

$$\langle Au, \psi\rangle = \langle K_A(x,y), u(y)\psi(x)\rangle = \langle K_A(x,y)u(y), \psi(x)\rangle.$$

Let us denote $T = \operatorname{supp} K_A$, $R = \operatorname{supp} u$. Since R is a compact set, it follows that $T \circ R$ is a closed set. Let $W = Y \setminus (T \circ R)$ and assume $\psi \in C_0^\infty(W)$. This means that $T \cap (\operatorname{supp}\psi \times R) = \emptyset$. The kernel theorem and the fact $\langle Au, \psi\rangle = 0$ imply that (3.4). Let us prove that $T \circ R$ is a compact set. From (3.2) it follows

$$(\operatorname{supp} K_A) \circ (\operatorname{supp} u) = \pi_1(\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} u)).$$

The set $\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} u)$ is compact, since $\operatorname{supp} u$ is a compact set and $\pi_2 : \operatorname{supp} K_A \to Y$ is a proper mapping. Therefore $\pi_1(\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} u))$ is a compact set as a continuous image of a compact set. □

Theorem 3.10. *If* $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ *is a proper integral operator, then it can be continuously and linearly extended to an operator* $A : C^\infty(Y) \to \mathcal{D}'(X)$.

Proof. Let $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a proper integral operator, $u \in C_0^\infty(Y)$, $v \in C_0^\infty(X)$, by Theorem 3.4, there exists $K_A(x,y) \in \mathcal{D}'(X \times Y)$ such that

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle.$$

Let $\{\varphi_j\}_{j\in J\subset\mathbf{N}}$ be a partition of unity with the properties

(1) $\varphi_j \subset C_0^\infty(X \times Y)$, $j \in J$, and the collection of supports $\{\operatorname{supp}\varphi_j\}_{j\in J}$ is locally finite,

(2) $\sum_{j\in J} \varphi_j(x,y) = 1$ for every$(x,y) \in X \times Y$,

(3) $\varphi_j(x,y) \geq 0$ for every$(x,y) \in X \times Y$ and $j \in J$.

Let

$$\kappa(x,y) = \sum_{j\,:\,\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq \emptyset} \varphi_j(x,y)$$

Clearly, $\kappa(x,y) \in C^\infty(X \times Y)$. Define the operator $\tilde{A} : C^\infty(Y) \to \mathcal{D}'(X)$ by

$$\langle \tilde{A}u(x), v(x)\rangle = \langle K_A(x,y), \kappa(x,y)u(y)v(x)\rangle, \; u \in C^\infty(Y), v \in C^\infty(X).$$

The set $\operatorname{supp} \kappa(x,y)u(y)v(x)$ is compact. Namely $\operatorname{supp} K_A(x,y)v(x)$ is compact and it implies that a family of functions φ_j such that $\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq \emptyset$, is finite. Therefore $\tilde{A}$ is well defined. From the definition it follows that $\tilde{A}$ is a continuous linear operator. Also, if $u \in C_0^\infty(Y)$, then

$$\langle \tilde{A}u(x), v(x)\rangle = \langle K_A(x,y)\kappa(x,y), u(y)v(x)\rangle = \langle K_A(x,y), u(y)v(x)\rangle,$$

since $\kappa = 1$ on $\operatorname{supp} K_A$. We conclude that $\tilde{A}$ is a linear continuous extension of the operator A. □

Theorem 3.11. *An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is proper if and only if:*

(1) *For every compact subset M of Y there exists a compact subset M_1 of X such that if $\operatorname{supp} u \subset M$ then $\operatorname{supp} Au \subset M_1$, where $u \in C_0^\infty(Y)$.*
(2) *For every compact subset L of X there exists a compact subset S of Y such that if $\operatorname{supp} v \subset L$, then $\operatorname{supp} {}^tAu \subset S$, where $v \in C_0^\infty(X)$.*

Proof. Let us prove that condition (2) is equivalent with the following one

(2*) For every compact subset L of X there exists a compact subset S of Y such that if $u = 0$ on S, then $Au = 0$ on L for $u \in C_0^\infty(Y)$.

Assume that (2*) does not hold, i.e. there exists a compact set L_0 such that for every compact set $\tilde{S}$ there exits $u \in C_0^\infty(Y)$ such that $\operatorname{supp} u \subset Y \setminus \tilde{S}$, $\langle Au, \tilde{v}\rangle \neq 0$ for some $\tilde{v}$ with $\operatorname{supp} \tilde{v} \subset L_0$. Let (2) holds and let S_1 be related to the set L_0 by condition (2). For every $v \in C_0^\infty(X)$, with $\operatorname{supp} v \subset L_0$, it follows that support of ${}^tAv$ is in S_1. Let $u \in C_0^\infty(Y)$ and let the support of u be in the complement of S_1. We should have that $\langle \tilde{v}, Au\rangle \neq 0$ for some $\tilde{v} \in C_0^\infty$ with support in L_0, but it is not true, since $\langle \tilde{v}, Au\rangle = \langle {}^tA\tilde{v}, u\rangle$ and $\langle {}^tA\tilde{v}, u\rangle = 0$, for every $\tilde{v}$ with $\operatorname{supp} \tilde{v} \subset L_0$.

Analogously one can prove that (2*) implies (2).

Let us suppose (1) and (2*). We will show that the mapping $\pi_2 : \operatorname{supp} K_A(x, y) \to Y$ is proper. Suppose that M is an arbitrary compact subset of Y and N is a compact subset of X, which is related to the first one by (1). Then we will prove

$$\pi_2^{-1}(M) \cap \operatorname{supp} K_A \subset N \times M. \tag{3.5}$$

Let $(x_0, y_0) \in (X \setminus N) \times M$, and let a function $\omega(x, y) = v(x)u(y)$ be such that $\operatorname{supp} v \subset X \setminus N$, $\operatorname{supp} u \subset M$, $\omega \neq 0$ in some neighborhood of the point (x_0, y_0) and $\omega \in C_0^\infty(X \times Y)$. We have $\langle K_A, \omega\rangle = \langle Au(x), v(x)\rangle = 0$ which implies that $(x_0, y_0) \notin \pi_2^{-1}(M) \cap (\operatorname{supp} K_A)$. This implies (3.5). The proof that the mapping $\pi_1 : \operatorname{supp} K_A(x, y) \to X$ is proper is similar

Let A be a proper integral operator. Condition (1) follows immediately from the properties of a proper integral operator and condition (2) follows from the fact that ${}^tA$ is a proper integral operator. □

3.3 Smoothing operators.

Definition 3.12. A continuous linear operator $A : \mathcal{E}'(X_2) \to C^\infty(X_1)$, X_1 and X_2 are open in $\mathbb{R}^n$, is called a smoothing operator.

If a distribution $K(x_1, x_2)$ belongs to the space $C^\infty(X_1 \times X_2)$, then the operator A defined on $\mathcal{E}'(X_2)$ by

$$(A(u(x_2)))(x_1) = \langle K(x_1, x_2), u(x_2)\rangle, \quad x_1 \in X_1,\ u \in \mathcal{E}'(X_2)$$

is a smoothing operator. To prove it we need the following lemma.

Lemma 3.13. *The lineal L of the set of translations of δ-distribution ($L = \{\sum_{i=1}^{p} a_i\delta(x - x_i),\ a_i \in \mathbb{C},\ x_i \in X\}$) is dense in the space $\mathcal{E}'(X)$.*

Proof. We will show the assertion for $n = 1$ and $X = \mathbb{R}$. For $n > 1$ and $X \subset \mathbb{R}^n$ the proof is analogous. Let $\phi \in C_0^\infty(\mathbb{R})$. We have

$$\langle \phi, \psi \rangle = \int_\alpha^\beta \phi(x)\psi(x)\,dx,$$

for every $\psi \in C^\infty(\mathbb{R})$. The integral on the right-hand side is equal to the limit value of Riemann's sum i.e.

$$\langle \phi, \psi \rangle = \lim_{n\to\infty} \sum_{i=1}^{n} \phi(x_i)\psi(x_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \langle \delta(x - x_i), \psi(x) \rangle,$$

where $a_i = \phi(x_i)\Delta x_i$. This implies that $\sum_{i=1}^{n} a_i\delta(x - x_i)$ converges to $\phi \in C_0^\infty(\mathbb{R})$ in $\mathcal{E}'(\mathbb{R})$, i.e. that the set of finite linear combinations of delta distributions is dense in $C_0^\infty(\mathbb{R})$. Since $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{E}'(\mathbb{R})$, it follows that this set is dense in $\mathcal{E}'(\mathbb{R})$. □

Theorem 3.14. *An operator $A : \mathcal{E}'(X_2) \to C^\infty(X_1)$ is a smoothing operator if and only if there exists a distribution $K(x_1, x_2) \in C^\infty(X_1 \times X_2)$ such that*

$$(A(u(x_2)))(\cdot) = \langle K(\cdot, x_2), u(x_2) \rangle,\ u \in \mathcal{E}'(X_2).$$

Proof. Let $A : \mathcal{E}'(X_2) \to C^\infty(X_1)$ be a smoothing operator. Denote

$$K(x_1, a) = A(\delta(\cdot - a))(x_1), a \in X_2,\ x_1 \in X_1.$$

Let a be fixed and $K(x_1, a)$ be a function of x_1. It is an element of $C^\infty(X_1)$. We will show that for every fixed $x_1 \in X_1$, $K(x_1, \cdot)$ is a function in $C^\infty(X_2)$. This will imply $K(x_1, x_2) \in C^\infty(X_1 \times X_2)$. Thus, let x_1 be fixed, $\{a_n\}_{n\in\mathbb{N}} \subset X_2$ and $\lim_{n\to\infty} a_n = a \in X_2$. Then

$$\lim_{n\to\infty} A(\delta(x_2 - a_n))(x_1) = A(\delta(x_2 - a))(x_1)$$

(which is equivalent to $\lim_{n\to\infty} K(x_1, a_n) = K(x_1, a)$), because of the continuity of A and the fact that $\delta(x_2 - a_n) \to \delta(x_2 - a)$ in $\mathcal{E}'(X_2)$ as $n \to \infty$. Therefore, $K(x_1, a)$ is continuous with respect to the variable a. We have

$$\begin{aligned}\frac{K(x_1, a + h) - K(x_1, a)}{h} &= \frac{A(\delta(x_2 - a - h))(x_1) - A(\delta(x_2 - a))(x_1)}{h} \\ &= A\Big(\frac{\delta(x_2 - a - h) - \delta(x_2 - a)}{h}\Big)(x_1).\end{aligned}$$

Since

$$\frac{\delta(x_2 - a - h) - \delta(x_2 - a)}{h} \to \delta'(x_2 - a) \text{ in } \mathcal{E}'(X_2), \quad h \to 0,$$

the continuity of A implies

$$\lim_{h\to 0} \frac{K(x_1, a + h) - K(x_1, a)}{h} = A(\delta'(x_2 - a)).$$

Analogously one can continue the proof for all the derivatives. This means that the mapping

$$(x_1, x_2) \mapsto K(x_1, x_2) = A(\delta(t - x_2))(x_1)$$

is in $C^\infty(X_1 \times X_2)$

It remains to prove that $K_A = K$, where K_A is the kernel of A. Since $\langle K_A(x_1, x_2), u(x_2)\rangle \in C^\infty(X_1)$, it is enough to prove

$$\langle K_A(x_1, x_2), u(x_2)\rangle = \int_{X_2} K(x_1, x_2)u(x_2)\, dx_2, \quad u \in C_0^\infty(X_2).$$

As we have shown in Lemma 3.13, L is dense in $C_0^\infty(X_2)$. Thus, there exists $\sum_{i=1}^{p_r} a_i^r \delta(x_2 - x_{2i}^n)$ in L which converges to u in $\mathcal{E}'(X_2)$ as $r \to \infty$. From above it follows $(x_1 \in X_1)$

$$\begin{aligned}
\langle K_A(x_1, x_2), u(x_2)\rangle &= \lim_{n\to\infty} A\Big(\sum_{i=1}^{p_n} a_i^n \delta(x_2 - x_{2i}^n)\Big)(x_1) \\
&= \lim_{n\to\infty} \sum_{i=1}^{p_n} a_i^n K(x_1, x_{2i}^n) = \lim_{n\to\infty} \Big\langle K(x_1, x_2), \sum_{i=1}^{p_n} a_i^n \delta(x_2 - x_{2i}^n)\Big\rangle \\
&= \Big\langle K(x_1, x_2), \lim_{n\to\infty} \sum_{i=1}^{p_n} a_i^n \delta(x_2 - x_{2i}^n)\Big\rangle = \langle K(x_1, x_2), u(x_2)\rangle \\
&= \int_{X_2} K(x_1, x_2)u(x_2)\, dx_2 \qquad \Box
\end{aligned}$$

4. Oscillatory integrals

The notion of oscillatory integral is the crucial one for the theory of pseudo-differential and Fourier integral operators.

In oder to explain the oscillatory integrals we will consider the definition of generalized Fourier transformation of continuous functions $u(x)$ for which there exists positive real number c and $m \in \mathbb{N}$ such that

$$(4.1) \qquad |u(x)| \leq c(1 + |x|)^m, \ x \in \mathbb{R}^n.$$

In other words we will give the meaning to the right-hand side of equality

$$(4.2) \qquad \langle \hat{u}, \phi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\xi}u(x)\phi(\xi)\, dx\, d\xi, \ \phi \in \mathcal{S}(\mathbb{R}^n),$$

when a continuous function u satisfies (4.1). Later on we shall give a method which will be applied in the general case.

Let $k \in \mathbb{N}$ and $\psi \in \mathcal{S}$ If $u \in \mathcal{S}$, then the integral (4.2) makes sense, since

$$e^{-ix\xi} = (1 + |x|^2)^{-k}(1 - D_{\xi_1}^2 - \cdots - D_{\xi_n}^2)^k e^{-ix\xi}.$$

Then we have

$$\langle \hat{u}, \psi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)\psi(\xi)(1+|x|^2)^{-k}(1 - D_{\xi_1}^2 - \cdots - D_{\xi_n}^2)^k e^{-ix\xi}\, dx\, d\xi.$$

The integration by parts implies

$$\langle \hat{u}, \psi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi}(1+|x|^2)^{-k}u(x)(1 - D_{\xi_1}^2 - \cdots - D_{\xi_n}^2)^k \psi(\xi)\, dx\, d\xi. \tag{4.3}$$

The right-hand side of (4.3) is defined not only when $u \in \mathcal{S}(\mathbb{R}^n)$ but as well as when u satisfies (4.1) and $k > m + n$.

Let us suppose (4.1). Since $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is the isomorphism, it follows $\hat{u}(\xi) \in \mathcal{S}'(\mathbb{R}^n)$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(0) = 1$ and

$$I_{\phi,\varepsilon}(\psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi}\phi(\varepsilon x)u(x)\psi(\xi)\, dx\, d\xi,\ \psi \in \mathcal{S}(\mathbb{R}^n),\ \varepsilon > 0.$$

where the integral on the right-hand side converges because of (4.1). Analogously as above, for $k \in \mathbb{N}_0^n$, we obtain

$$I_{\phi,\varepsilon} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi}\phi(\varepsilon x)(1+|x|^2)^{-k}u(x)(1 - D_{\xi_1}^2 - \cdots - D_{\xi_n}^2)^k \psi(\xi)\, dx\, d\xi.$$

Let $k > m + n$. By the Lebesgue theorem, it follows that there exists $I \in \mathbb{R}$ such that $\lim_{\varepsilon \to 0} I_{\phi,\varepsilon} = I$. Note that the integral in (4.3) does not depend on k for which $k > m + n$. We define the mapping $\mathcal{S}(\mathbb{R}^n) \ni \psi \mapsto \langle \hat{u}, \psi \rangle = I(\psi)$ which gives the definition of $\hat{u}$ as an element of $\mathcal{S}'(\mathbb{R})$.

4.1. Space of symbols $S^m_{\rho,\delta}(X, \mathbb{R}^N)$. Let X be an open set in $\mathbb{R}^n$ and let (formally)

$$I_\phi(au) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\, dx d\xi,\ u \in C_0^\infty(X), \tag{4.4}$$

where functions ϕ and a are the phase function and the symbol defined as follows.

Definition 4.1. A real valued function ϕ which is of the class $C^\infty(X \times (\mathbb{R}^N \backslash \{0\}))$ positively homogeneous of order 1 with respect to the variable ξ (i.e. $\phi(x, t\xi) = t\phi(x,\xi)$ for every $x \in \mathbb{R}^n, \xi \in \mathbb{R}^N, t \in \mathbb{R},\ t > 0$) and which does not have characteristic points on $X \times (\mathbb{R}^N \backslash \{0\})$ (i.e. $0 \neq d\phi(x,\xi) = (\phi_{x_1}, \cdots, \phi_{x_n}, \phi_{\xi_1}, \cdots, \phi_{\xi_N})$ for $\xi \neq 0$), is called a phase function.

Definition 4.2. Let $m,\ \rho,\ \delta \in \mathbb{R}$, $0 < \rho \leq 1$, $0 \leq \delta < 1$.

Elements of the space $S^m_{\rho,\delta}(X, \mathbb{R}^N)$, which are called symbols, are functions $a(x,\xi) \in C^\infty(X \times \mathbb{R}^N)$ such that for arbitrary multi-indices α and β and arbitrary compact set $K \subset X$ there exists a constant $c_{\alpha,\beta,K} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq c_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},\ x \in K,\ \xi \in \mathbb{R}^N$$

Example 4.1. $(1+|\xi|)^m \in S^m_{1,0}$.

We will use the following notations

$$S^m(X,\mathbb{R}^N) = S^m_{1,0}(X,\mathbb{R}^N), \quad S^m_{\rho,\delta} = S^m_{\rho,\delta}(X,\mathbb{R}^N),$$

$$S^{\infty}_{\rho,\delta}(X,\mathbb{R}^N) = \bigcup_m S^m_{\rho,\delta}(X,\mathbb{R}^N), \quad S^{-\infty}_{\rho,\delta}(X,\mathbb{R}^N) = \bigcap_m S^m_{\rho,\delta}(X,\mathbb{R}^N).$$

The space S^m is called the space of standard symbols.

Let us introduce the topology in the space $S^m_{\rho,\delta}$. Suppose that $(K_\nu)_{\nu\in\mathbb{N}}$ is a sequence of compact sets such that

$$K_1 \subset K_2 \cdots \subset K_\nu \subset \cdots \subset X, \quad \bigcup_{\nu=1}^{\infty} K_\nu = X.$$

For $a(x,\xi) \in S^m_{\rho,\delta}$ define

$$\|a(x,\xi)\|_\nu = \sup_{x\in K_\nu, \xi\in\mathbb{R}^N, |\alpha|<\nu, |\beta|<\nu} |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)|(1+|\xi|)^{-m+\rho\alpha-\delta\beta}.$$

It is clear that $\|\cdot\|_\nu$, $\nu \in \mathbb{N}$ is a growing sequence of seminorms; it defines the topology on $S^m_{\rho,\delta}$ such that $S^m_{\rho,\delta}$ is Freshet's space.

One can simply prove:

Proposition 4.3. *If $a \in S^m_{\rho,\delta}(X,\mathbb{R}^n)$, then $\partial^\alpha_\xi \partial^\beta_x a \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(X,\mathbb{R}^N)$. If $a \in S^m_{\rho,\delta}(X,\mathbb{R}^N)$ and $b \in S^{m'}_{\rho,\delta}(X,\mathbb{R}^N)$ then $a\cdot b \in S^{m+m'}_{\rho,\delta}(X,\mathbb{R}^N)$.*

The right-hand side in (4.4), where $a(x,\xi) \in S^m_{\rho,\delta}(X,\mathbb{R}^N)$ and $\phi(x,\xi)$, is a phase function, is called an oscillatory integral. Our aim will be to give the meaning to the integral, which in the general case does not converge absolutely.

Theorem 4.4. *Let $\phi(x,\xi)$, $(x,\xi) \in X\times\mathbb{R}^N$, be a phase function. There exists an operator*

$$L = \sum_{j=1}^{N} a_j(x,\xi)\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n} b_k(x,\xi)\frac{\partial}{\partial x_k} + c(x,\xi) \tag{4.5}$$

such that $a_j(x,\xi) \in S^0(X,\mathbb{R}^N)$, $b_k(x,\xi), c(x,\xi) \in S^{-1}(X,\mathbb{R}^N)$ and that for its transpose operator (determined by $\int(L\varphi)\psi = \int \varphi({}^tL\psi)$, $\varphi,\psi \in C_0^\infty$)

$${}^tLu(x,\xi) = -\sum_{j=1}^{N}\frac{\partial}{\partial\xi_j}(a_j u) - \sum_{k=1}^{n}\frac{\partial}{\partial x_k}(b_k u) + c(x,\xi)$$

there holds ${}^tLe^{i\phi} = e^{i\phi}$.

Note that the operator L is not uniquely determined.

Proof. Since $\frac{\partial}{\partial\xi_j}e^{i\phi} = i\frac{\partial\phi}{\partial\xi_j}e^{i\phi}$, $\frac{\partial}{\partial x_k}e^{i\phi} = i\frac{\partial}{\partial x_k}e^{i\phi}$, we have

$$\Big(\sum_{j=1}^{N} -i|\xi|^2\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n} -i\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\Big)e^{i\phi}$$

$$=\Big(\sum_{j=1}^{N}|\xi|^2\left|\frac{\partial\phi}{\partial\xi_j}\right|^2 + \sum_{k=1}^{n}\left|\frac{\partial\phi}{\partial x_k}\right|^2\Big)e^{i\phi} = \frac{e^{i\phi}}{\psi},$$

where $\psi(x,\xi) \in C^\infty(X \times (\mathbb{R}^n\backslash\{0\}))$ is a positively homogeneous of order -2 as a function of variable ξ. It follows

$$-i\psi\Big(\sum_{j=1}^{N}|\xi|^2\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n}\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\Big)e^{i\phi} = e^{i\phi},$$

and it remains only to take care of the singularity in $\xi = 0$.

Let $\kappa(\xi) \in C_0^\infty(\mathbb{R}^n)$ be such that $\kappa(\xi) = 1$ for $|\xi| < 1/4$ and $\kappa(\xi) = 0$ for $|\xi| > 1/2$. Let us define

$$M = -i(1-\kappa)\psi\Big[\sum_{j=1}^{N}|\xi|^2\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n}\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\Big] + \kappa.$$

Note $Me^{i\phi} = e^{i\phi}$. By using Proposition 4.3 one can prove that the coefficients of ${}^tM = L$ satisfy the asserted conditions. Since ${}^tM = L$, it follows ${}^tL = M$. □

For $m' > m$ we have $S^m_{\rho,\delta} \subset S^{m'}_{\rho,\delta}$ and the identity mapping $I : S^m_{\rho,\delta} \to S^{m'}_{\rho,\delta}$ is continuous.

Theorem 4.5. *Let $m' > m$ and let B be a bounded subset in $S^m_{\rho,\delta}$. The topologies in B induced by*

(a) topology of pointwise convergence on $S^{m'}_{\rho,\delta}$,

(b) the topology of the uniform convergence on compact sets (topology from $\mathcal{E}(X,\mathbb{R}^n)$) on $S^{m'}_{\rho,\delta}$ and

(c) the topology of the space $S^{m'}_{\rho,\delta}$ are the same.

Proof. We will give the proof of this assertion from [**15**]. Let us recall that a convergence satisfies the Urysohn condition if the following holds:

A sequence is convergent if and only if its every subsequence has a convergent subsequence.

It is obvious that all of the mentioned topologies are Hausdorff, that they fulfill the Urysohn axiom (because they are topological convergencies) and that the first two are weaker that the third one on B.

We will show that the set B is relatively compact in $S^{m'}_{\rho,\delta}$ (every sequence in B has a convergent subsequence in the sense of the convergence in $S^{m'}_{\rho,\delta}$). Since B is a bounded subset of $S^m_{\rho,\delta}$, a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset B$ is bounded in the sense of convergence in $\mathcal{E}(X \times \mathbb{R}^N)$. Therefore it has a convergent subsequence ϕ_{k_n} which converges to $\phi \in C^\infty(X \times \mathbb{R}^N)$.

Note that for every compact set K and $\alpha, \beta \in \mathbb{N}_0^n$

$$|\partial_\xi^\alpha \partial_x^\beta \phi_{k_n}(x,\xi)| \leq c_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},\ x \in K,\ \xi \in \mathbb{R}^N,$$

where $c_{K,\alpha,\beta}$ does not depend on the subsequence. It implies

$$|\partial_\xi^\alpha \partial_x^\beta \phi(x,\xi)| \leq c_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},\ x \in K,\ \xi \in \mathbb{R}^N.$$

Therefore $\phi \in S^m_{\rho,\delta}$. We have

$$(1+|\xi|)^{-m'+\rho|\alpha|-\delta|\beta|}|\partial_\xi^\alpha \partial_x^\beta (\phi_{k_n}(x,\xi) - \phi(x,\xi))| \leq 2c_{K,\alpha,\beta}(1+|\xi|)^{m-m'},$$
$$x \in K,\ \xi \in \mathbb{R}^N,$$

for fixed compact set $K \subset X$, $\alpha \in \mathbb{N}_0^N$, $\beta \in \mathbb{N}_0^n$. Therefore, there exists $a > 0$ such that for $|\xi| > a$ the left-hand side of the inequality is less than $\varepsilon > 0$ independently of k_n.

For $|\xi| \leq a$ the set $K \times \{\xi,\ |\xi| \leq a\}$ is compact. Since the sequence ϕ_{k_n} converges to ϕ in the sense of convergence in $\mathcal{E}$, it follows

$$(1+|\xi|)^{-m'+\rho|\alpha|-\delta|\beta|}|\partial_\xi^\alpha \partial_x^\beta (\phi_{k_n}(x,\xi) - \phi(x,\xi))| < \varepsilon,$$

for some $n_0 \in \mathbb{N}, k_n > n_0, (x,\xi) \in K \times \{\xi,\ |\xi| \leq a\}$. Thus, every sequence in B has a convergent subsequence in $S^{m'}_{\rho,\delta}$.

Now we will prove that (a) implies (c). Let a sequence in B be pointwisely convergent. We have proved that every subsequence of it has a convergent subsequence in $S^{m'}_{\rho,\delta}$. From Urysohn's condition follows the assertion. □

4.2. The oscillatory integral and its properties. Let $u \in C_0^\infty(X), a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$, X is open in $\mathbb{R}^n$ and $m < -N$. Note, if $a \in S^m_{\rho,\delta}$ and $s = \min(\rho, 1-\delta)$, then the properties of L (cf. (4.5)) and a imply that there exists $C > 0$ such that

$$|L^k(a(x,\xi)u(x))| \leq C(1+|\xi|)^{m-ks}, \quad x \in X, \xi \in \mathbb{R}^N.$$

With the above assumptions the integral on the right-hand side of (4.4) makes sense. Moreover

$$I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} L^k(a(x,\xi)u(x))\,dx\,d\xi \tag{4.6}$$

and

$$|I_\phi(ua)| \leq c\sup\{(1+|\xi|)^{-m}|a(x,\xi)|,\ x \in \operatorname{supp} u,\ \xi \in \mathbb{R}^N\},$$

where

$$c = \int_X |u(x)|\,dx \int_{\mathbb{R}^N}(1+|\xi|)^m\,d\xi.$$

This implies that $a \mapsto I_\phi(au)$ is a continuous mapping $S^m_{\rho,\delta} \to \mathbb{C}$. In the following theorem we shall show that this mapping has a continuous extension on $S^\infty_{\rho,\delta} = \bigcup_{m>0} S^m_{\rho,\delta}$. This extension is called the oscillation integral and it is denoted by

$$I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\,dxd\xi \text{ [osc]}. \tag{4.7}$$

Theorem 4.6. *Let $\rho \in (0,1]$, $\delta \in [0,1)$ and ϕ be a phase function. For a fixed $u \in C_0^\infty(X)$ define $I_\phi(\cdot u)$ by*

$$a \mapsto I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\,dxd\xi, \quad a \in S^\infty_{\rho,\delta} = \bigcup_{m,\rho,\delta} S^m_{\rho,\delta}(X,\mathbb{R}^N)$$

when that integral is absolutely convergent. Then $I_\phi(\cdot u)$ can be extended uniquely on the whole $S^\infty_{\rho,\delta}$ such that the mapping $u \mapsto I_\phi(au)$, $a \in S^m_{\rho,\delta}(X,\mathbb{R}^N)$, is continuous and linear (i.e. it is a distribution).

Proof. Let $\kappa(\xi) \in C_0^\infty(\mathbb{R}^N)$, $\kappa(\xi) = 1$ in a neighborhood of zero and $\kappa_\nu(\xi) = \kappa(\xi/\nu)$, $\nu \in \mathbb{N}$. The set $\{\kappa_\nu(\xi)a(x,\xi),\ \nu \in \mathbb{N}\}$ is bounded in $S^m_{\rho,\delta}(X,\mathbb{R}^N)$, therefore $\kappa_\nu(\xi)a(x,\xi)$ converges to $a(x,\xi)$ in $S^{m'}_{\rho,\delta}(X,\mathbb{R}^N)$, as $\nu \to \infty$ for $m' > m$. Also it converges pointwise. This follows from Theorem 4.5. The integral is absolutely convergent because κ_ν and u are compactly supported and therefore

$$\begin{aligned} I_\phi(a(x,\xi)\kappa_\nu(\xi)u(x)) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)\kappa_\nu(\xi)u(x)\,dx\,d\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(a(x,\xi)\kappa_\nu(\xi)u(x))\,dxd\xi, \quad u \in C_0^\infty(X) \end{aligned} \tag{4.8}$$

(cf. (4.6)). It is clear that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(a(x,\xi)\kappa_\nu(\xi)u(x))\,dxd\xi$$

converges to

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(a(x,\xi)u(x))\,dxd\xi, \tag{4.9}$$

as $\nu \to \infty$, since $a(x,\xi)\kappa_\nu(\xi)$ converges to $a(x,\xi)$ in $S^{m'}_{\rho,\delta}(X,\mathbb{R}^N)$ and L^k maps $S^{m'}_{\rho,\delta}(X,\mathbb{R}^n)$ continuously in $S^{m'-ks}_{\rho,\delta}$, for $s = \min(\rho, 1-\delta)$. This implies the convergence of the integral in (4.8). Let us denote this limit by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\,dx\,d\xi \text{ [osc]}. \tag{4.10}$$

Since for fixed ν, $\mathcal{D} \ni u \mapsto I_\phi(a\kappa_\nu u)$ defines a distribution and $I_\phi(a\kappa_\nu u)$ converges to $I_\phi(au)$ for every $u \in \mathcal{D}$. By the sequential completeness of $\mathcal{D}'$, it follows that $u \mapsto I_\phi(au)$ is a distribution.

Therefore (4.10) is defined by (4.9). Clearly, in (4.8) the operator L can be substituted by any other one which has the properties as in Theorem 4.4 and we can take any k such that $m - ks < -N$. This implies that (4.10) does not depend on L and k, i.e.

$$\mathcal{D}(X) \ni u \mapsto I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} a(x,\xi) u(x)\, dx d\xi \text{ [osc.]},$$

is an element of the space $\mathcal{D}'(X)$.

The same proof show that (4.8) does not depend on the choice of $\kappa_\nu(\xi)$ with the prescribed properties. □

Example 4.2. Let us show that

$$\delta(x) = (2\pi)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\cdot 1\, d\xi \text{ [osc.]}.$$

Note, $1 \in S^0_{\rho,\delta}$. Let $\kappa \in C_0^\infty(X)$, $\kappa(\xi) = 1$ in a neighborhood of zero and $u \in C_0^\infty$. Then $\kappa(\xi/t) \to 1$ in $S^m_{1,0}$ as $t \to \infty$ and

$$\begin{aligned}\Big\langle (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\, d\xi, u(x)\Big\rangle &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\xi} u(x)\, dx\, d\xi \text{ [osc.]}\\ = \lim_{t\to\infty}(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\xi}\kappa(\xi/t)u(x)\, dx\, d\xi &= \lim_{t\to\infty}\int_{\mathbb{R}^n}\kappa(\xi/t)\mathcal{F}^{-1}(u)(\xi)\, d\xi\\ &= \kappa(0)\int_{\mathbb{R}^n}\mathcal{F}^{-1}(u)(\xi)\, d\xi = u(0).\end{aligned}$$

We have used $\mathcal{F}(\mathcal{F}^{-1}(u(\xi))(x) = u(x)$, which implies $\mathcal{F}(\mathcal{F}^{-1}(u(\xi))(0) = u(0)$.

4.3. Singularities of an oscillatory integral. Let X be open in $\mathbb{R}^n$ and

$$C_\phi = \{(x,\xi),\ x \in X, \xi \in \mathbb{R}^N\backslash\{0\}, \phi_\xi(x,\xi) = 0\},\ S_\phi = \pi_1 C_\phi,\ R_\phi = X\backslash S_\phi,$$

where $\pi_1 : (X \times \mathbb{R}^N\backslash\{0\}) \to X$ being the first projection of the set $(X \times \mathbb{R}^N\backslash\{0\})$. Since S_ϕ is closed, R_ϕ is open.

The set C_ϕ is a cone with respect to ξ, because $\phi(x,\xi)$ is homogeneous function of ξ of order 1 and $\partial\phi/\partial\xi$ is homogeneous of order 0.

Theorem 4.7. *Denote by A the distribution defined by $\langle A, u\rangle = I_\phi(au)$, $u \in C_0^\infty(X)$. Then* $\operatorname{Sing\,supp} A \subset S_\phi$.

Recall, $\operatorname{Sing\,supp} A$ is the complement of the maximal open set where A is smooth.

Proof. We will show that there exists $A \in C^\infty(R_\phi)$ such that

$$I_\phi(au) = \int_X A(x)u(x)\, dx, \quad u \in C_0^\infty(R_\phi).$$

We shall show that there exists $L = \sum a_j(x,\xi)\frac{\partial}{\partial \xi_j} + c(x,\xi)$, where $a_j \in S^0(R_\phi, \mathbb{R}^N)$, $c \in S^{-1}(R_\phi, \mathbb{R}^N)$ such that $Le^{i\phi} = e^{i\phi}$. Put

$$M = -i(1-\kappa)\psi\Big[\sum_{j=1}^{N} |\xi|^2 \frac{\partial \phi}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\Big] + \kappa.$$

where ψ satisfies $-i\psi\sum_{j=1}^{N}|\xi|^2|\frac{\partial\phi}{\partial\xi_j}|^2 e^{i\phi} = e^{i\phi}$, $\xi \neq 0$, $\kappa \in C_0^\infty(\mathbb{R}^N)$ and $\kappa(\xi) = 1$ for $|\xi| < 1$. Then $Me^{i\phi} = e^{i\phi}$ and put $Le^{i\phi} = {}^tMe^{i\phi}$. Thus

$$Lu = \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j}\Big(i(1-\kappa)\psi|\xi|^2\frac{\partial\phi}{\partial\xi}u\Big) + \kappa u.$$

Let $\kappa \in C_0^\infty(\mathbb{R}^n)$, $\kappa(0) = 1$ and $\kappa_\nu(\xi) = \kappa(\xi/\nu)$, $\nu \in \mathbb{N}$. Note, for every $K \subset\subset X$

$$|M^k a(x,\xi)| \le C(1+|\xi|)^{m-k},\ \xi \in \mathbb{R}^n,\ x \in K.$$

$$\begin{aligned}\langle A, u\rangle &= \lim_{\nu\to\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(\kappa_\nu(\xi)a(x,\xi)u(x))\,d\xi\,dx\\ &= \lim_{\nu\to\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}(\kappa_\nu(\xi)a(x,\xi)u(x))\,d\xi\,dx\\ &= \int_{\mathbb{R}^n}(\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k a(x,\xi)\,d\xi)u(x)\,dx.\end{aligned}$$

Therefore

$$A(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k a(x,\xi)\,d\xi\ [\text{osc}].$$

(It does not depend on k.) For large enough k the integral exists in ordinary sense and the function A is continuous. Moreover, we can differentiate $A(x)$ by differentiating the function under the integral sign. This is the consequence of the fact that $\phi(x,\xi)$ is a homogeneous function of ξ of order 1 as well as all its derivatives with respect to x. Note, if a function $r(\xi)$ is homogeneous of order 1, then

$$|r(\xi)| < \text{const}\cdot(1+|\xi|),\ \xi \in \mathbb{R}^N.$$

This implies that by taking large enough k differentiation under the integral is legitimate. Thus for any $p \in \mathbb{N}_0$ we have $A(x) \in C^p(R_\phi)$. □

Analogously one can prove:

Proposition 4.8. *If* $a \in S^m_{\rho,\delta}(X, \mathbb{R}^N)$ *and* $a = 0$ *in some conic neighborhood of the set* C_ϕ, *then* $A \in C^\infty(X)$, *where* A *is defined by* $\langle A, u\rangle = I_\phi(au)$.

5. Fourier integral operators

We shall give some introductory facts which are useful for the theory of pseudodifferential operators.

5.1. Definition and the basic properties. Let X and Y be open sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, $\rho > 0$, $\delta < 1$. Let

$$Au(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y)\, dy\, d\xi,\ u \in C_0^\infty(Y),\ x \in X,\ \text{[osc.]}$$

where $\phi(x,y,\xi)$ is a phase function on $(X \times Y) \times \mathbb{R}^N$ and $a(x,y,\xi) \in S^m_{\rho,\delta}(X \times Y, \mathbb{R}^N)$. Under these conditions the integral

$$(5.1) \qquad \langle Au, v\rangle = \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy\, d\xi, \quad v \in C_0^\infty(X)$$

is defined as an oscillatory integral. For fixed u the right-hand side in (5.1) defines a distribution $Au \in \mathcal{D}'(X)$ (see Theorem 4.6).

Remark 5.1. In the sequel we will not write explicitly [osc.] for integrals which are defined as oscillatory integrals. It will clear from the context.

Definition 5.1. An operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ defined by (5.1) is called a Fourier integral operator with a phase function $\phi(x,y,\xi)$ and an amplitude $a(x,y,\xi)$.

Every smoothing integral operator can be written in the form of a Fourier integral operator:

Theorem 5.2. *An integral operator* $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ *is a smoothing operator if and only if there exists a phase function* $\phi(x,y,\xi)$ *and amplitude* $\tilde{a}(x,y,\xi) \in S^{-\infty}_{1,0}$ *such that*

$$(5.2) \qquad Au(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} \tilde{a}(x,y,\xi) u(y)\, dy\, d\xi.$$

Proof. Let A be of the form (5.2). If $\tilde{a}(x,y,\xi) \in S^{-\infty}_{1,0}$ it is clear that the kernel of the operator

$$\int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} \tilde{a}(x,y,\xi)\, d\xi$$

is of the class $C^\infty(X \times Y)$.

Conversely, by Theorem 3.14 there exists $K(x,y) \in C^\infty(X \times Y)$ such that

$$\begin{aligned} Au(x) &= \langle K(x,y), u(y)\rangle = \int_Y K(x,y) u(y)\, dy \\ &= \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} (K(x,y) e^{-i\phi(x,y,\xi)} \kappa(\xi)) u(y)\, dy\, d\xi,\ u \in C_0^\infty,\ x \in X, \end{aligned}$$

where ϕ is an arbitrary phase function, $\kappa \in C_0^\infty(\mathbb{R}^N)$, $\int \kappa(\xi)\, d\xi = 1$ and $\kappa(\xi) = 0$ in some neighbourhood of zero.

Since $\tilde{a}(x,y,\xi) = K(x,y) e^{-i\phi(x,y,\xi)} \kappa(\xi) \in S^{-\infty}_{1,0}$, the assertion follows. □

A distribution $K_A \in \mathcal{D}'(X \times Y)$ defined as the oscillatory integral

$$\langle K_A, w\rangle = \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) w(x,y)\, dx\, dy\, d\xi \,[\text{osc.}],$$

$w \in C_0^\infty(X \times Y)$, is the kernel of the operator A. It is the kernel of the operator A since

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle, u \in C_0^\infty(Y), \quad v \in C_0^\infty(X).$$

Proposition 5.3. *Let A be a Fourier integral operator given by (5.1), and let K_A be its kernel. Then $K_A \in C^\infty(R_\phi)$, where*

$$R_\phi = \{(x,y),\ \forall \xi \in \mathbb{R}^N \backslash \{0\}, \phi'_\xi(x,y,\xi) \neq 0\}.$$

If $a(x,y,\xi) = 0$ in a conic neighbourhood of the set

$$C_\phi = \{(x,y,\xi),\ \phi'_\xi(x,y,\xi) = 0\},$$

then $K_A \in C^\infty(X \times Y)$.

Proof. It follows immediately from Theorem 4.7 and Proposition 4.8. □

Remark 5.2. Different pairs ϕ_1, a_1 and ϕ_2, a_2 may define the same operator A of the form (5.1). Moreover, a function $a(x,y,\xi)$ is not completely determined by the operator A, even when the phase function ϕ is fixed.

Let $A : C_0^\infty(X) \to \mathcal{D}'(X)$ be a Fourier integral operator given by (5.1). We shall evaluate the form of ${}^tA$ and A^*. Recall, ${}^tA : C_0^\infty(X) \to \mathcal{D}'(X)$ such that

$$\langle Au, v\rangle = \langle u, {}^tAv\rangle, \quad u \in C_0^\infty(X),\ v \in C_0^\infty(X)$$

i.e.

$$\langle Au, v\rangle = \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy d\xi = \langle u, {}^tAv\rangle.$$

We have

$$({}^tAv(x))(y) = \int_X \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) v(x)\, dx\, d\xi,$$

for $y \in Y = X$. By the change of the variables $x \mapsto y$ and $y \mapsto x$, we obtain

$$({}^tAv(y))(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(y,x,\xi)} a(y,x,\xi) v(y)\, dy\, d\xi. \tag{5.3}$$

Therefore, for $x \in X$

$$({}^tAv(y))(x) = \int_X \int_{\mathbf{R}^N} e^{i\tilde{\phi}(x,y,\xi)} \tilde{a}(x,y,\xi) v(y)\, dy\, d\xi. \tag{5.4}$$

(The above integrals are oscillatory integrals.) This proves

Proposition 5.4. *The phase function and the amplitude of ${}^tA$ are defined by $\tilde{\phi}(x,y,\xi) = \phi(y,x,\xi)$ and $\tilde{a}(x,y,\xi) = a(y,x,\xi)$.*

The operator A^* is determined by $\langle Au, v\rangle = (u, A^*v)$, $A^* : C_0^\infty(X) \to \mathcal{D}'(X)$. Therefore

$$\langle u, {}^tAv\rangle = \langle Au, v\rangle = (Au, \overline{v}) = (u, A^*\overline{v}) = \langle u, \overline{A^*\overline{v}}\rangle,$$

for $u \in C_0^\infty(X)$ and $v \in C_0^\infty(X)$ i.e.

$$ {}^tAv(x) = \overline{A^*\overline{v}}(x) = \int_Y \int_{\mathbf{R}^N} e^{-i\phi(y,x,\xi)} \overline{a(y,x,\xi)\overline{v(y)}}\, dy\, d\xi, \tag{5.5}$$

and for $y \in Y = X$

$$ (A^*v(x))(y) = \int_X \int_{\mathbf{R}^N} e^{i\varphi(x,y,\xi)} b(x,y,\xi) v(x)\, dx\, d\xi. \tag{5.6}$$

Proposition 5.5. *The phase function and the amplitude of A^* are given by $\varphi(x,y,\xi) = \phi(y,x,\xi)$ and $b(x,y,\xi) = \overline{a(y,x,\xi)}$.*

5.2. Fourier integral operator with operator phase function.

Definition 5.6. Phase function $\phi(x,y,\xi)$, $x \in X$, $y \in Y$, X, Y are open in $\mathbb{R}^n$, is an operator phase function if the following holds

$$ \phi'_{y,\xi}(x,y,\xi) = (\phi_{y_1}, \dots, \phi_{y_n}, \phi_{\xi_1}, \dots, \phi_{\xi_n}) \neq 0 \text{ for } \xi \neq 0, x \in X, y \in Y, \tag{5.7}$$

$$ \phi'_{x,\xi}(x,y,\xi) \neq 0 \text{ for } \xi \neq 0, \quad x \in X, y \in Y. \tag{5.8}$$

Proposition 5.7. *If (5.7) holds then the operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, determined by (5.1), continuously map $C_0^\infty(Y)$ into $C^\infty(X)$.*

Proof. From (5.7) it follows that $\phi(x,y,\xi)$, considered as function of (y,ξ), is a phase function (x is a parameter). By Theorem 4.7 there exists an operator L (which does not contain $\partial/\partial x$) such that ${}^tLe^{i\phi} = e^{i\phi}$. Analogously as in the proof of Theorem 4.7 (with operator L instead of M) we obtain

$$\begin{aligned}\langle Au, v\rangle &= \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy\, d\xi \\ &= \int_X \Big(\int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} L^k(a(x,y,\xi)u(y))\, dy\, d\xi \Big) v(x)\, dx,\end{aligned}$$

for $u \in C_0^\infty(Y)$ and $v \in C_0^\infty(X)$. Therefore, as in Theorem 4.7

$$(Au(y))(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} L^k_{y,\xi}(a(x,y,\xi)u(y))\, dy\, d\xi, \ x \in X,$$

we can prove that Au is a smooth function. □

Proposition 5.8. *If (5.8) holds, then the operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, given by (5.2), can be linearly and continuously extended to $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$, where the topologies in $\mathcal{E}'(Y)$ and $\mathcal{D}'(X)$ are weak topologies.*

Proof. The transpose operator ${}^t\!A : C_0^\infty(X) \to \mathcal{D}'(Y)$ of the operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is given by

$$({}^t\!Av(x))(y) = \int_X \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) v(x)\, dx\, d\xi, v \in C_0^\infty(X).$$

From (5.8) by the previous theorem it follows ${}^t\!A : C_0^\infty(X) \to C^\infty(Y)$. Therefore ${}^t({}^t\!A) : \mathcal{E}'(Y) \to \mathcal{D}'(X)$. Since ${}^t({}^t\!A)|_{C_0^\infty(Y)} = A$ and ${}^t({}^t\!A) : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ is a linear and continuous mapping, the assertion of the proposition follows. □

From the previous two proposition it follows

Theorem 5.9. *Let $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a Fourier integral operator with an operator phase function ϕ. Then*

a) $A : C_0^\infty(Y) \to C^\infty(X)$,

b) A can be linearly and continuously extended to $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$,

c) ${}^t\!A : C_0^\infty(X) \to C^\infty(Y)$,

d) ${}^t\!A$ can be linearly and continuously extended to ${}^t\!A : \mathcal{E}'(X) \to \mathcal{D}'(Y)$.

For the singular support the following estimation holds.

Theorem 5.10. *Let $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ be a Fourier integral operator with an operator phase function ϕ. Then*

$$\operatorname{Sing\,supp} Au \subset S_\phi \circ \operatorname{Sing\,supp} u, u \in \mathcal{E}'(Y),$$

where $R_\phi = \{(x,y),\ \phi_\xi(x,y) \neq 0$ for every $\xi \in \mathbb{R}^N \setminus \{0\}\}$ and $S_\phi = (X \times Y)\setminus R_\phi$.

Proof. Let $u_1 \in \mathcal{E}'(U)$, where U is fixed neighbourhood of $K = \operatorname{Sing\,supp} u$ such that $u = u_1$ on some neighbourhood of $K \subset U$. Then for $u_2 = u_1 - u$ we have $\operatorname{supp} u_2 \subset Y \setminus K$. Since $u_2 \in C_0^\infty(Y)$ and $A : C_0^\infty(Y) \to C^\infty(X)$, it follows $Au_2 \in C^\infty(X)$. If we show that

$$\operatorname{Sing\,supp} Au_1 \subset M = S_\phi \circ \operatorname{supp} u_1, \tag{5.9}$$

it will means that $\operatorname{Sing\,supp} Au \subset \operatorname{Sing\,supp} Au_1 \subset M \subset S_\phi \circ U$. By letting $U \to K$, we will have

$$\operatorname{Sing\,supp} Au \subset S_\phi \circ K = S_\phi \circ \operatorname{Sing\,supp} u.$$

Let us prove (5.9). Let $K_0 = \operatorname{supp} u_1$, $K' \subset X$ such that $K' \times K_0 \subset R_\phi$ ($K' \subset X \setminus M$)) and let $X' \times X \subset R_\phi$ be a neighbourhood of $K' \times K_0$. We have

$$\langle Ah, k\rangle = I_\phi(ahk),$$

for $h \in C_0^\infty(X)$ and $k \in C_0^\infty(X')$. By Theorem 4.7

$$\operatorname{Sing\,supp} A \subset S_\phi.$$

It follows $A \in C^\infty(X' \times X)$. Therefore $\operatorname{Sing\,supp} Au_1 \subset X \setminus K'$, which implies the theorem. □

Example 5.1. Let

$$A = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha,$$

where $a_\alpha(x) \in C_0^\infty(X)$, $X \subset \mathbb{R}^n$. Using the Fourier transform we obtain

$$D^\alpha u(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \xi^\alpha e^{i(x-y)\xi} u(y)\, dy\, d\xi.$$

It implies,

$$Au(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \sigma_A(x,\xi) u(y)\, dy\, d\xi,$$

where $\sigma_A = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ and is called the symbol of the operator A. Since $\sigma_A(x,\xi) \in S^m(X \times \mathbb{R}^n)$, A is a Fourier integral operator.

Example A solution to the Cauchy problem

$$c^{-2}\frac{\partial^2 E}{\partial t^2} - \Delta E = 0,\ E(0,x) = 0,\ \frac{\partial}{\partial t}E(0,x) = \delta(x), \tag{5.10}$$

$E = E(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, is given by

$$(2\pi)^{-1}E(t,x) = \int \frac{e^{i(ct|\xi|+x\xi)} - e^{-i(ct|\xi|+x\xi)}}{2i|\xi|c}\, d\xi \text{ [osc]}. \tag{5.11}$$

Let us prove it. Applying the Fourier transformation on equation (5.10) we obtain

$$c^{-2}\frac{\partial^2 \tilde{E}}{\partial t^2} + |\xi|^2 \tilde{E}(t,\xi) = 0,$$

where $\tilde{E}(t,\xi) = \mathcal{F}(E(t,x))(\xi)$. Let us fix ξ. We obtain an ordinary differential equation (with respect to the variable t) which solution is $\tilde{E}(t,\xi) = c_1 e^{-itc|\xi|} + c_2 e^{itc|\xi|}$. It follows

$$\begin{aligned} E(0,x) = 0 &\Rightarrow \tilde{E}(0,\xi) = 0 \Rightarrow c_1 + c_2 = 0 \\ &\Rightarrow \frac{\partial}{\partial t}\tilde{E}(0,\xi) = 1 = \mathcal{F}(\delta(x)) \Rightarrow -c_1 + c_2 = 1/ic|\xi|. \end{aligned}$$

Therefore (5.11) holds.

Example 5.3. Pseudodifferential operators.

If $n_1 = n_2 = N = n$ and $X = Y$, then a Fourier integral operator with a phase function $\phi(x,y,\xi) = (x-y)\xi$ is called a pseudodifferential operator (ΨDO).

6. Pseudodifferential operators

Pseudodifferential operators generalizes differential and singular integral operators. In this section we shall analyze the basic properties of pseudodifferential operators.

6.1. Definition and the properties. Let X be an open set in $\mathbb{R}^n$; then a Fourier integral operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ given by

$$Au(x) = \int_X \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x, y, \xi) u(y)\, dy\, d\xi \tag{6.1}$$

is called a pseudodifferential operator, for short ΨDO.

Example 6.1. An example of a pseudodifferential operator which is not a differential operator is a singular operator in $\mathbb{R}^n$ given by

$$\begin{aligned} A &= a(x)u(x) + \text{v.p.} \int \frac{L(x, (x-y)/|x-y|)}{|x-y|^n} u(y)\, dy \\ &= a(x)u(x) + \lim_{\varepsilon \to 0} \int_{|y-x| \ge \varepsilon} \frac{L(x, (x-y)/|x-y|)}{|x-y|^n} u(y)\, dy, \end{aligned}$$

where $a \in C^\infty(\mathbb{R}^n)$, $L = L(x, \omega) \in C^\infty(\mathbb{R}^n \times S^{n-1})$ (S^{n-1} is a unit sphere in $\mathbb{R}^n$) such that

$$\int_{S^{n-1}} L(x, \omega) d\omega = 0, \; x \in \mathbb{R}^n.$$

With accuracy up to the operator with a smooth kernel, the operator A has an amplitude $a(x, \xi) = a(x) + \chi(\xi) g(x, y)$, where $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$, for $|\xi| \ge 1$, $\chi(\xi) = 0$, for $|\xi| \le 1/2$ and $g = \frac{1}{|x-y|^n} L\left(x, \frac{x-y}{|x-y|}\right)$.

Theorem 6.1. *Let $A : C_0^\infty(X) \to \mathcal{D}'(X)$ be a ΨDO, K_A be the kernel of the operator A and let Δ be the diagonal in $X \times X$. Then*

a) $K_A \in C^\infty((X \times X) \setminus \Delta)$.

b) *Operator A defines linear and continuous mappings $A : C_0^\infty(X) \to C^\infty(X)$, $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$. If $u \in \mathcal{E}'(X)$, then* $\operatorname{Sing\,supp} Au \subset \operatorname{Sing\,supp} u$. *(This property is called the pseudolocality of the operator A.)*

c) *The operators ${}^tA$ and A^* define linear and continuous mappings*

$$\begin{aligned} &{}^tA : C_0^\infty(X) \to C^\infty(X), \quad {}^tA : \mathcal{E}'(X) \to \mathcal{D}'(X); \\ &A^* : C_0^\infty(X) \to C^\infty(X), \quad A^* : \mathcal{E}'(X) \to \mathcal{D}'(X). \end{aligned}$$

Proof. a) The phase function for a ΨDO A is $\phi(x, y, \xi) = (x - y)\xi$. Therefore $R_\phi = X \times X \setminus \Delta$, since $\phi_\xi = (x - y)$. By putting $X = Y$, Proposition 5.3 immediately implies $K_A \in C^\infty((X \times X) \setminus \Delta)$.

b) The following conditions are fulfilled for phase function of the operator A

$$\begin{aligned} \phi'_{y,\xi}(x, y, \xi) &= (-\xi_1, \dots, -\xi_n, x_1 - y_1, \dots, x_n - y_n) \ne 0, \\ \phi'_{x,\xi}(x, y, \xi) &= (\xi_1, \dots, \xi_n, x_1 - y_1, \dots, x_n - y_n) \ne 0, \end{aligned}$$

for $\xi \ne 0$, $x, y \in X$. Therefore $A : C_0^\infty(X) \to \mathcal{D}'(X)$ is a Fourier integral operator with the operator phase function. The assertions a) and c) follow from Theorem

5.9. Since $S_\phi = \Delta$, where S_ϕ is the set attached to the operator A determined in Theorem 5.10. From this theorem it follows that $\operatorname{Sing\,supp} Au \subset \Delta \circ \operatorname{Sing\,supp} u = \operatorname{Sing\,supp} u$.

c) Let $A : C_0^\infty(X) \to \mathcal{D}'(X)$ be a ΨDO given by (6.1). We shall evaluate the forms of ${}^tA : C_0^\infty(X) \to \mathcal{D}'(X)$ and $A^* : C_0^\infty(X) \to \mathcal{D}'(X)$. Note that these operators are again pseudodifferential operators and the assertion *c)* follows from *b)*

From (5.3) it follows

$$ {}^tAv(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)(-\xi)} a(y,x,\xi)v(y)\,dy\,d\xi, $$

for $v \in C_0^\infty(X)$. By changing of variables $-\xi \to \xi$, we obtain

$$ {}^tAv(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} a(y,x,-\xi)v(y)\,dy\,d\xi, $$

for $v \in C_0^\infty(X)$, i.e.

$$ \text{(6.2)} \qquad {}^tAv(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} \tilde{a}(x,y,\xi)v(y)\,dy\,d\xi, \quad v \in C_0^\infty(X). $$

where $\tilde{a}(x,y,\xi) = a(y,x,-\xi)$. From (5.5) it follows

$$ A^*v(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} \overline{a(y,x,\xi)} v(y)\,dy\,d\xi, \quad v \in C_0^\infty(X), $$

i.e.

$$ \text{(6.3)} \qquad A^*v(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} b(x,y,\xi)v(y)\,dy\,d\xi, \quad v \in C_0^\infty(X), $$

where $b(x,y,\xi) = \overline{a(y,x,\xi)}$. □

Remark 6.1. Linear differential operators fulfills the condition of locality ($\operatorname{supp} Au \subset \operatorname{supp} u$, $u \in C_0^\infty(X)$), which for ΨDO's in general case do not hold.

6.2. Algebra of pseudodifferential operators and its symbols.

6.2.1. Proper pseudodifferential operators.

Definition 6.2. Pseudodifferential operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$, X is open in $\mathbb{R}^n$, is proper if it is proper as an integral operator.

For example, linear differential operators (5.10) are proper ΨDO.

Theorem 6.3. *Let A be a proper ΨDO. Then, A defines linear and continuous mapping $A : C_0^\infty(X) \to C_0^\infty(X)$ which can be linearly and continually continued to mappings*

$$ A : \mathcal{E}'(X) \to \mathcal{E}'(X), \quad A : C^\infty(X) \to C^\infty(X), \quad A : \mathcal{D}'(X) \to \mathcal{D}'(X). $$

Proof. By Theorem 6.1 $A : C_0^\infty(X) \to C^\infty(X)$ and by Proposition 3.9

$$\text{supp}(Au) \subset (\text{supp}\, K_A) \circ (\text{supp}\, u), \quad u \in C_0^\infty, \tag{6.4}$$

where K_A is the kernel of A. The set on the right-hand side of (6.4) is compact. This immediately implies $A : C_0^\infty(X) \to C_0^\infty(X)$. Continuity of the operator $A : C_0^\infty(X) \to C_0^\infty(X)$ easily follows. Since (6.2) holds and ${}^tA$ defines a proper ΨDO, it follows ${}^tA : C_0^\infty(X) \to C_0^\infty(X)$, and

$${}^t({}^tA) : \mathcal{D}'(X) \to \mathcal{D}'(X).$$

Since ${}^t({}^tA)|_{C_0^\infty} = A$, we have that $A : C_0^\infty(X) \to C_0^\infty(X)$ can be linearly and continuously extended to a mapping $A : \mathcal{D}'(X) \to \mathcal{D}'(X)$.

By Theorem 6.1, the operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ can be linearly and continuously continued to mapping $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$. Then (6.4) holds for $u \in \mathcal{E}'(X)$, as well. The proof follows from the fact that $C_0^\infty(X)$ is dense in $\mathcal{E}'(X)$. This means that the continuation (6.2) maps $\mathcal{E}'(X)$ in $\mathcal{E}'(X)$. □

Proposition 6.4. *Let A be a proper ΨDO. Then ${}^tA : C_0^\infty(X) \to C_0^\infty(X)$ can be linearly and continuously extended to the mappings*

$${}^tA : \mathcal{E}'(X) \to \mathcal{E}'(X), \;\; {}^tA : C^\infty(X) \to C^\infty(X), \;\; {}^tA : \mathcal{D}'(X) \to \mathcal{D}'(X).$$

Proof. The proof is analogous to the proof of the previous theorem because of the duality of operators A and ${}^tA$. □

We will prove that the space of pseudodifferential operators is an algebra with respect to operation of composition.

From Theorem 6.3 it follows that the composition of two proper ΨDO defines a linear and continuous operator on every one of the spaces $C_0^\infty(X)$, $\mathcal{E}'(X)$, $C^\infty(X)$ or $\mathcal{D}'(X)$.

Definition 6.5. It is said that $a(x,y,\xi) \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ is an amplitude with a proper support if the projections

$$\pi_1 : \text{supp}_{x,y}\, a(x,y,\xi) \to X, \quad \pi_2 : \text{supp}_{x,y}\, a(x,y,\xi) \to X$$

are proper for every $\xi \in \mathbb{R}^n$.

Theorem 6.6. *Let*

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,y,\xi) u(y)\, dy\, d\xi \text{ [osc]}, \quad u \in C_0^\infty(X)$$

be a proper pseudodifferential operator, where $a(x,y,\xi) \in S^m_{\rho,\delta}$. Then A can be defined by the formula

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} b(x,y,\xi) u(y)\, dy\, d\xi \text{ [osc]}, u \in C_0^\infty,$$

where $b(x,y,\xi) \in S^m_{\rho,\delta}$ is an amplitude with a proper support.

Proof. Let the functions $\kappa(x,y)$ and $\varphi_j(x,y)$ be the ones constructed in the proof of Theorem 3.10, K_A be a kernel of the operator A. It can be easily seen that

$$\kappa(x,y) = \sum_{\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq \emptyset} \varphi_j(x,y) \in C^\infty(X \times X)$$

and that $\operatorname{supp}\kappa(x,y)$ is contained in some neighbourhood of $\operatorname{supp} K_A$. Let us show that $\pi_1 : \operatorname{supp}\kappa(x,y) \to X$ is a proper mapping, i.e. that for every compact subset K in X the set $\operatorname{supp}\kappa(x,y) \cap \pi_1^{-1}(K)$ is compact. We have

$$\begin{aligned}\operatorname{supp}\kappa(x,y) \cap \pi_1^{-1}(K) &\subset \operatorname{supp}\Big(\sum_{\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq 0} \varphi_j(x,y)\Big) \cap \pi_1^{-1}(K)\\ &\subset \bigcup_{\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq 0} (\operatorname{supp}\varphi_j(x,y) \cap \pi_1^{-1}(K)).\end{aligned}$$

This union is finite, because A is a proper ΨDO, $\pi_1^{-1}(K) \cap \operatorname{supp} K_A$ is compact and a family $\operatorname{supp}\varphi_j$ is locally finite. Since $\operatorname{supp}\kappa(x,y) \cap \pi_1^{-1}(K)$ is a closed subset of a finite union of compact sets, it is compact, too. In the same way, it can be shown that the mapping $\pi_2 : \operatorname{supp}\kappa(x,y) \to X$ is proper.

We will show that the amplitude $b(x,y,\xi) = \kappa(x,y)a(x,y,\xi)$ belongs to the space of symbols $S^m_{\rho,\delta}$ and that it has a proper support. It has a proper support, because $\operatorname{supp}_{x,y} b(x,y,\xi) \subset \operatorname{supp}\kappa(x,y)$ and the first and second projections of $\operatorname{supp}\kappa$ are proper mappings. From $a(x,y,\xi) \in S^m_{\rho,\delta}$ it follows $b(x,y,\xi) \in S^m_{\rho,\delta}$. For every $u \in C_0^\infty(X)$ and $v \in C_0^\infty(X)$

$$\begin{aligned}\langle Au(x), v(x)\rangle &= \langle K_A(x,y), u(y)v(x)\rangle = \langle K_A(x,y), \kappa(x,y)u(y)v(x)\rangle\\ &= \iiint e^{i(y-x)\xi} a(x,y,\xi)\kappa(x,y)v(x)u(y)\, dx\, dy\, d\xi \text{ [osc.]}.\end{aligned}$$

This proves the last part of the assertion. □

Let us note that if $a(x,y,\xi)$ has a proper support, then the integral (6.1) is defined for every $u \in C^\infty(X)$. More precisely, we have

Theorem 6.7. *A proper pseudodifferential operator continuously and linearly maps $C^\infty(X)$ into $C^\infty(X)$.*

Theorem 6.8. *Every pseudodifferential operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ is of the form $A = A_1 + A_2$, where A_1 is a proper operator and A_2 is a smoothing one.*

Proof. Let A be an arbitrary pseudodifferential operator and let for $u \in C_0^\infty(X)$

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,y,\xi)u(y)\, dy\, d\xi.$$

Then

$$\begin{aligned}Au(x) &= (2\pi)^{-n} \iint e^{i(x-y)\xi} \kappa(x,y)a(x,y,\xi)u(y)\, dy\, d\xi\\ &\quad + (2\pi)^{-n} \iint e^{i(x-y)\xi}(1 - \kappa(x,y))a(x,y,\xi)u(y)\, dy\, d\xi,\end{aligned}$$

where $\kappa(x,y)$ is smooth and $\kappa(x,y) = 1$ in some neighbourhood of the diagonal Δ such that the both projection $\pi_1 : \operatorname{supp}\kappa(x,y) \to X$ and $\pi_2 : \operatorname{supp}\kappa(x,y) \to X$ are proper mapping. (The construction of a function κ is given in the proof of Theorem 3.10.)

The operator A_1, defined by

$$(A_1 u)(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} \kappa(x,y) a(x,y,\xi) u(y)\, dy\, d\xi.$$

is proper. The proof is analogous to a part of the proof of Theorem 6.6. The function $e^{i(x-y)\xi}(1 - \kappa(x,y))\; a(x,y,\xi)$ equals zero in some neighbourhood of the diagonal and out of the diagonal it is C^∞. So the operator A_2 defined by

$$(A_2 u)(x) = \iint e^{i(x-y)\xi}(1 - \kappa(x,y)) a(x,y,\xi) u(y)\, dy\, d\xi$$

for $u \in C_0^\infty(X)$ is a smoothing operator by Theorem 6.1. □

6.2.2. The symbol of a proper pseudodifferential operator.

Definition 6.9 Let A be a proper ΨDO. The function $\sigma_A(x,\xi)$ defined on $X \times \mathbb{R}^n$, X is open in $\mathbb{R}^n$, by

$$\sigma_A(x,\xi) = e^{ix\xi}(Ae^{iy\xi})(x), \tag{6.5}$$

where $e_\xi(x) = e^{ix\xi}$, is called a symbol of the pseudodifferential operator A.

If $\sigma_A(x,\xi)$ is a symbol of a proper ΨDO, then $\sigma_A(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$, because A is a linear and continuous mapping $C^\infty(X) \to C^\infty(X)$ and $\xi \mapsto e^{i\xi x}$ is C^∞-function with respect to ξ with values in $C^\infty(X)$. Let us write $u \in C_0^\infty(X)$ as

$$u(y) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{iy\xi} \hat{u}(\xi)\, d\xi.$$

The continuity of A and the fact that

$$\sum_\nu e^{i\xi_\nu y} \hat{u}(\xi_\nu) \Delta\xi_\nu \to \int_{\mathbf{R}^n} e^{iy\xi} \hat{u}(\xi)\, d\xi, \quad \nu \to \infty,$$

in $\mathcal{E}(\mathbb{R}^n)$ (where on the left-hand side we have a sequence of integral sums) imply

$$(Au(y))(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} (Ae^{iy\xi})(x) \hat{u}(\xi)\, d\xi = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\xi} \sigma_A(x,\xi) \hat{u}(\xi)\, d\xi,$$

$u \in C_0^\infty(X)$, i.e.

$$(Au(y))(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(x-y)\xi} \sigma_A(x,\xi) u(y)\, dy\, d\xi,\ x \in X. \tag{6.6}$$

From (6.5) and (6.6) it follows that the symbol $\sigma_A(x,\xi)$ determines the operator A.

We shall show in Theorem 7.1 that if A has an amplitude in $S^m_{\rho,\delta}$ and $\delta < \rho$, then $\sigma_A(x,\xi) \in S^m_{\rho,\delta}$, so the integral on the left-hand side in (6.6) can be considered as the oscillating one.

If A is an arbitrary ΨDO on X, then the function $\sigma_A(x,\xi)$, which is a symbol of a proper ΨDO A_1 on X such that $A - A_1$ is smoothing, is called the symbol of A. In this case a symbol is not uniquely determinated and two symbols differs by a function $r(x,\xi) \in S^{-\infty}$.

6.2.3. Asymptotic decomposition in $S^m_{\rho,\delta}$

Definition 6.10. Let $a_j(x,\xi) \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^n)$, $j = 1, 2, \dots$, $\lim_{j\to\infty} m_j = -\infty$, $a(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$. Then a is an asymptotic sum of a_k,

$$a(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi),$$

if for every integer $r \geq 2$ there holds

$$a(x,\xi) - \sum_{j=1}^{r-1} a_j(x,\xi) \in S^{\overline{m}_r}_{\rho,\delta}(X, \mathbb{R}^N),$$

where $\overline{m}_r = \max_{j \geq r} m_j$. Note $a \in S^{\overline{m}_r}_{\rho,\delta}(X, \mathbb{R}^N)$.

Theorem 6.11. *Let $a_j \in S^{m_j}_{\rho,\delta}(X, \mathbb{R}^N)$, $j \in \mathbb{N}$, $\lim_{j\to\infty} m_j = -\infty$. Then there exists a function $a(x,\xi)$ such that*

$$a(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi).$$

If there exists another function a' with the same property

$$a'(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi),$$

then $a - a' \in S^{-\infty}(X \times \mathbb{R}^n)$.

The proof will be given in the case $\rho = 1$, $\delta = 0$. We follow the proof given in [11]. First, we shall prove the following two lemmas.

Lemma 6.12. *Let $\kappa \in C_0^\infty(\mathbb{R}^n)$, $\kappa(\xi) = 1$ in some neighbourhood of $\xi = 0$ and $\kappa_\lambda(\xi) = \kappa(\lambda\xi)$. Then the set $\{\lambda^{-k}(1-\kappa_\lambda)\}_{0<\lambda\leq 1}$ is bounded in $S^k_{10}(X, \mathbb{R}^n)$ for every $k \geq 0$.*

Proof. Let us prove that the functions

$$\mathbb{R}^n \ni \xi \mapsto (1+|\xi|)^{|\beta|-k} \lambda^{-k} \left(\frac{\partial}{\partial \xi}\right)^{\beta} (1 - \kappa_\lambda(\xi)),\ \beta \in \mathbb{N}_0^n,$$

are bounded independently on $\lambda \in (0,1]$. Since for κ there holds

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}(1-\kappa)\Big| \le c_\beta,$$

we have

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}(1-\kappa_\lambda)\Big| \le \Big|\lambda^{|\beta|}\Big(\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}(1-\kappa)\Big)_\lambda\Big| \le \lambda^{|\beta|}c_\beta,\ \lambda \in (0,1].$$

First, we shall prove it for $\beta = 0$. Let $R > 1$ be large enough such that $\kappa(\xi) = 0$ for $|\xi| > R$ and $\kappa(\xi) = 1$ for $|\xi| < 1/R$. If $0 < \lambda \le 1$ and $(1-\kappa_\lambda) \ne 0$, then $|\xi| \ge 1/\lambda R$,

$$((1+|\xi|)\lambda)^{-k} \le ((1/R\lambda)\lambda)^{-k} \le R^k$$

and

$$|\lambda^{-k}(1-\kappa_\lambda)| \le c_0 R^k(1+|\xi|)^k.$$

If $(\frac{\partial}{\partial\xi})^{\beta}(1-\kappa_\lambda) \ne 0$ for $\beta \ne (0,0,\dots 0)$, then $|\xi| \le R/\lambda$. This implies

$$((1+|\xi|)\lambda)^{|\beta|} \le ((1+R/\lambda)\lambda)^{|\beta|} \le (R+1)^{|\beta|}$$

and

$$\Big|\lambda^{-k}(\frac{\partial}{\partial\xi})^{\beta}(1-\kappa_\lambda)\Big| \le c_\beta R^k(R+1)^{|\beta|}(1+|\xi|)^{k-|\beta|}. \qquad \square$$

Lemma 6.13. *Let $\{F_k\}$ be a sequence of Frechét spaces such that $F_{k+1} \subset F_k$ and the topology in F_{k+1} is stronger than the topology induced by F_k. For every k, let (a_k^m) be a sequence of elements in F_k which converges to 0 as $m \to \infty$. Then there exists a sequence m_k such that for every N the series $\sum_{k\ge N} a_k^{m_k}$ converges in F_N.*

Proof. Let $p_k^l (l \in \mathbb{N})$ be a fundamental sequence of seminorms in F_k, $k \in \mathbb{N}$, such that $p_k^l \le p_k^{l+1}$, $l \in \mathbb{N}$. By a simple procedure one can substitute a sequence with equivalent one such that there holds $p_k^l \le p_{k+1}^l$, $k, l \in \mathbb{N}$. For example, p_k^l can be substituted by

$$\sup_{k'\le k} p_{k'}^l|_{F_k}.$$

Since $\lim_{m\to\infty} a_k^m = 0$, let us chose m_k (increase as k increases) such that $p_k^k(a_k^{m_k}) \le 2^{-k}$. Then for $l \le k$ there holds

$$p_k^l(a_k^{m_k}) \le p_k^k(a_k^{m_k}) \le 2^{-k},$$

so, for every $l \ge 0$ the series $\sum_{k=N}^{\infty} p_k^l(a_k^{m_k})$ converges. Since F_N is Frechét space it follows that $\sum_{k=N}^{\infty} a_k^{m_k}$ converges in F_N. $\square$

Proof of Theorem 6.11. One can suppose that $a_k \in S_{10}^{-k}(X \times \mathbb{R}^n)$ when $k \ge 1$. This can be achieved by summing elements in the sequences if it is necessary. Let $a_k^m = (1-\kappa_{1/m})a_k$, where $\kappa_{1/m}$ is defined in the proof of Theorem 4.7. The sequence $(1-\kappa_{1/m})$ converges to zero in S^1 and a_k^m converges to zero in S^{-k+1} as $m \to \infty$.

Lemma 6.13 implies that one can chose a sequence m_k such that for every $N \geq 1$ the series $\sum_{k=N}^{\infty} a_k^{m_k}$ converges in S^{-N+1}. Let $a = \sum_{k=0}^{\infty} a_k^{m_k}$. Then a is a symbol and

$$a - \sum_{k<N} a_k = \sum_{k<N} (a_k^{m_k} - a_k) + \sum_{k=N}^{\infty} a_k^{m_k} \in S^{-N+1},$$

because $(a_k^{m_k} - a_k) = -\kappa_{1/m} a_k \in S^{-\infty}$ for every k. So $a \approx \sum a_k$. Second part of the assertion is obvious.

Theorem 6.14. *Let* $a_j \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^n), \lim_{j\to\infty} m_j = -\infty, a \in C^\infty(X \times \mathbb{R}^n)$. *Assume:*

1) For every compact set $K \subset X$ and for all multi-indices α, β there exist constants $\mu = \mu(\alpha, \beta, K)$ and $C = C(\alpha, \beta, K)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C(1 + |\xi|)^\mu, x \in K. \tag{6.7}$$

2) If for every compact set $K \subset X$ there exists a sequence of real numbers $\mu_l = \mu_l(K), l \in \mathbb{N}$, and a sequence of constants $C_l = C_l(K)$ such that $\mu_l \to -\infty$ for $l \to \infty$ and

$$\Big|a(x,\xi) - \sum_{j=1}^{l-1} a_j(x,\xi)\Big| \leq C_l(1 + |\xi|)^{\mu_l}, \quad x \in K. \tag{6.8}$$

Then

$$a(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi).$$

Proof. First we will prove the following assertion. Let the function $f(t)$ has continuous derivatives $f'(t)$ and $f''(t)$ in $[-1,1]$. Let us denote $A_j = \sup_{-1\leq t\leq 1} |f^{(j)}(t)|, j = 0, 2$. Then

$$|f'(0)|^2 \leq 4A_0(A_0 + A_2). \tag{6.9}$$

By Lagrange's theorem,

$$|f'(t) - f'(0)| \leq A_2|t|, \quad t \in [-1,1].$$

Because of that,

$$|f'(t)| \geq \frac{1}{2}|f'(0)|, \quad \text{if } A_2|t| \leq \frac{1}{2}|f'(0)|, \ |t| \leq 1.$$

Let us denote $\Delta = \min\{\frac{1}{2A_2}|f'(0)|, 1\}$. There holds

$$|f'(t)| \geq \frac{1}{2}|f'(0)|, \quad t \in [-\Delta, \Delta]$$

and

$$2A_0 \geq |f(\Delta) - f(-\Delta)| \geq 2\Delta\frac{1}{2}|f'(0)|.$$

It follows

$$|f'(0)| \le 2A_0/\Delta = 2A_0 \max\{2A_2/|f'(0)|, 1\},$$

which implies (6.9). Now by (6.9) we have the estimate needed for the proofs of theorem.

Also, we need the following estimate. Let K_1 and K_2 be compact sets in $\mathbb{R}^n$ such that $K_1 \subset \operatorname{int} K_2$. Then there exists a constant $C > 0$ such that for every C^∞-function f in a neighbourhood of K_2

$$(6.10) \quad \Big(\sup_{x\in K_1} \sum_{|\alpha|=1} |D^\alpha f(x)|\Big)^2 \le c \sup_{x\in K_2} |f(x)| + \Big(\sup_{x\in K_2} |f(x)| + \sup_{x\in K_2} \sum_{|\alpha|=2} |D^\alpha f(x)|\Big).$$

Now we give the proof of the assertion in the theorem. Let $b \approx \sum_{j=1}^\infty a_j(x,\xi)$ (such b exists by Theorem 6.11) and let $d(x,\xi) = a(x,\xi) - b(x,\xi)$. By the assumptions, for arbitrary compact set $K \subset X$ there holds

$$|\partial_\xi^\alpha \partial_x^\beta d(x,\xi)| \le C(1+|\xi|)^\mu, x \in K.$$

where C and μ depend on α, β, K and

$$(6.11) \qquad |d(x,\xi)| \le C_r(1+|\xi|)^{-r}, x \in K, r > 0,$$

where $C_r = C_r(K)$. Let us denote $d_\xi(x,\vartheta) = d(x,\xi+\vartheta)$. Then

$$\partial_\vartheta^\alpha \partial_x^\beta d_\xi(x,\vartheta)|_{\vartheta=0} = \partial_\xi^\alpha \partial_x^\beta d(x,\xi).$$

By (6.10), for $K_1 = K \times \{0\}, K_2 = \hat{K} \times \{|\xi| \le 1\}$, where $\hat{K}$ is a compact set in X such that $K \subset \operatorname{int} \hat{K}$ and from (6.11) it follows that for $\vartheta = 0$ there holds

$$\sup_{x\in K} \sum_{|\alpha|+|\beta|\le 1} |\partial_\xi^\alpha \partial_x^\beta d(x,\xi)|)^2 \le C(1+|\xi|)^{-r}[(1+|\xi|)^{-r} + (1+|\xi|)^\mu],$$

where r is arbitrary, $\mu = \mu(\alpha,\beta,K)$ and $C = C(\alpha,\beta,K,r)$. Moreover, for $x \in K$ and $|\alpha| + |\beta| \le 1$ the function $\partial_\xi^\alpha \partial_x^\beta d(x,\xi)$ decreases faster than each power of $|\xi|$ as $|\xi| \to \infty$. By induction, it follows that $d \in S^{-\infty}(X, \mathbb{R}^n)$. □

7. Calculus with symbols

The simplicity of the calculus with symbols is the central point of the theory of ΨDO. The main ideas of their calculus are given in Theorems 7.1 and (7.6) below.

7.1. Symbol of a proper ΨDO.

Let $\delta < \rho$. This will be a permanent assumption in the rest of the notes.

Theorem 7.1. *Let A be a proper ΨDO given by (6.1) and $\sigma_A(x,\xi)$ be its symbol. Then*

$$(7.1) \qquad \sigma_A(x,\xi) \approx \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x},$$

where the asymptotic sum is taken over all the multi-indices.

Remark that $\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x} \in S^{m-(\rho-\delta)|\alpha|}$.

Proof. We will apply Theorem 6.14. We can assume that the amplitude $a(x,y,\xi)$ is properly supported. Then by (6.5)

(7.2)

$$\sigma_A(x,\xi) = e^{-i\xi x} A(e^{iy\xi})(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} a(x,y,\xi) e^{i(x-y)\vartheta} e^{i(y-x)\xi}\, dy d\vartheta \text{ [osc]},$$

(for fixed x the integration by y is made over a compact set). If K is a compact subset of X, then for $x \in K$ (7.2) determines the oscillating integral depending on the parameter x. Let us change the variables by $z = y - x, \eta = \vartheta - \xi$. Then

$$(2\pi)^n \sigma_A(x,\xi) = \iint_{\mathbb{R}^{2n}} a(x, x+z, \xi+\eta) e^{-iz\eta}\, dz d\eta. \tag{7.3}$$

Expand $a(x, x+z, \xi+\eta)$ into the Taylor series at $\eta_0 = 0$ with the powers of η. Then,

$$a(x, x+z, \xi+\eta) = \sum_{|\alpha| \le N-1} \partial_\xi^\alpha a(x, x+z, \xi) \eta^\alpha / \alpha! + r_N(x, x+z, \xi, \eta), \tag{7.4}$$

where

$$r_N(x, x+z, \xi, \eta) = \sum_{|\alpha|=N} \frac{N\eta^\alpha}{\alpha!} \int_0^1 (1-t)^{N-1} \partial_\xi^\alpha a(x, x+z, \xi+t\eta)\, dt. \tag{7.5}$$

Let us note that for every $\xi \in \mathbb{R}^n$ and $x \in K$, $a(x, x+z, \xi)$ is compactly supported with respect to variable z. By the Fourier transform

$$\begin{aligned}(2\pi)^{-n} \iint_{\mathbb{R}^{2n}} & \partial_\xi^\alpha a(x, x+z, \xi) \eta^\alpha e^{-iz\eta}\, dz\, d\eta \\ &= \mathcal{F}^{-1}(\mathcal{F}(i^{-|\alpha|} \partial_\xi^\alpha \partial_z^\alpha a(x, x+z, \xi))(\eta))(z)|_{z=0} \\ &= \partial_\xi^\alpha D_z^\alpha a(x, x+z, \xi)|_{z=0}.\end{aligned} \tag{7.6}$$

This gives

$$\sigma_A(x,\xi) = \sum_{|\alpha| \le N} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x} + (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{-iz\eta} r_N(x, x+z, \xi, \eta)\, dz\, d\eta.$$

Integration by parts gives, from (7.3),

$$\sigma_A(x,\xi) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{-iz\eta} (1 - D_{z_1}^2 - \cdots - D_{z_n}^2)^{\nu/2} a(x, x+z, \xi+\eta)(1+|\eta|)^{-\nu/2}\, dz d\eta,$$

where ν is a even and nonnegative number. By using

$$(1 + |\xi+\eta|^2)^{1/2} \le (1 + |\xi|^2)^{1/2} (1 + |\eta|^2)^{1/2}$$

the above equality implies

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \le C(1+|\xi|^2)^{(p+\delta\nu)/2} \int_{\mathbf{R}^n} (1+|\eta|^2)^{(p-(1-\delta)\nu)/2}\, d\eta,$$

where $p = \max(m - \rho|\alpha| + \delta|\beta|, 0)$, $x \in K$ and ν is large enough. Thus, we obtain estimates of the form (6.7).

Let us estimate the rest of the series. Substitute a and r_N (defined by (7.5) in (7.3). After the change of the order of integration (first by t, then by z and η), let us note that we have to estimate the integral

$$R_{\alpha,t}(x,\xi) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{-iz\eta} \eta^\alpha \partial_\xi^\alpha a(x, x+z, \xi + t\eta)\, dz\, d\eta,$$

where $|\alpha| = N$, uniformly over $t \in (0,1]$ and $x \in K$. Integration by parts gives

$$R_{\alpha,t}(x,\xi) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{-iz\eta} \partial_\xi^\alpha D_z^\alpha a(x, x+z, \xi + t\eta)\, dz d\eta.$$

Let

$$R_{\alpha,t}(x,\xi) = R'_{\alpha,t}(x,\xi) + R''_{\alpha,t}(x,\xi),$$

where $R'_{\alpha,t}(x,\xi)$ is the integral over the set $\{(z,\eta),\ |\eta| \le |\xi|/2\}$ and $R''_{\alpha,t}(x,\xi)$ over its complement. (Recall, z belongs to a compact set.) If $|\eta| \le |\xi|/2$, then $|\xi|/2 \le |\xi + t\eta| \le 3|\xi|/2$. Since the measure of the domain of the integration of $R'_{\alpha,t}(x,\xi)$ with respect to η variable is less or equal to $C|\xi|^n$, then

$$|R'_{\alpha,t}(x,\xi)| \le C(1+|\xi|^2)^{(m-(\rho-\delta)N+n)/2},$$

where C does not depend on ξ and t. Let us estimate $R''_{\alpha,t}(x,\xi)$. By using

$$(1+|\eta|^2)^{-\nu/2}(1 - D_{z_1}^2 - \cdots - D_{z_n}^2)^{\nu/2} e^{-iz\eta} = e^{-iz\eta},$$

where ν is even positive integer, let us integrate by parts. Then $R''_{\alpha,t}(x,\xi)$ is a finite sum of terms of the form

$$R_{\alpha,\beta,t}(x,\xi) = (2\pi)^{-n} \iint_{|\eta|>|\xi|/2} e^{-iz\eta}(1+\eta^2)^{-\nu/2} \partial_\xi^\alpha D_z^{\alpha+\beta} a(x, x+z, \xi+t\eta)\, dz\, d\eta,$$

where $|\beta| \le \nu$. Since x and z belong to a compact set for $|\eta| > |\xi|/2$ there holds

$$|\partial_\xi^\alpha D_z^{\alpha+\beta} a(x, x+z, \xi+t\eta)| \le C(1+|\eta|^2)^{(m-(\rho-\delta)N+\delta\nu)/2},$$

for $m-(\rho-\delta)N+\delta\nu \ge 0$, i.e. $|\partial_\xi^\alpha D_z^{\alpha+\beta} a(x, x+z, \xi+t\eta)| \le C$ for $m-(\rho-\delta)N+\delta\nu < 0$. In both cases C does not depend on ξ, η and t. For large enough ν there holds

$$|R_{\alpha,\beta,t}(x,\xi)| \le C \int_{|\eta|>|\xi|/2} (1+|\eta|^2)^{(p-(1-\delta)\nu)/2}\, d\eta,$$

where $p = \max(m - (\rho-\delta)N, 0)$. If $p - (1-\delta)\nu + n + 1 < 0$, then

$$\begin{aligned} |R_{\alpha,\beta,t}(x,\xi)| &\le C(1+|\xi|^2)^{(p-(1-\delta)\nu+n+1)/2} \int_{\mathbf{R}^n} (1+|\eta|^2)^{(n+1)/2}\, d\eta \\ &\le C(1+|\xi|^2)^{(p-(1-\delta)\nu+n+1)/2}, \end{aligned}$$

where C does not depend on x, ξ, t if $x \in K$ and $t \in (0,1]$. For ν large enough we have

$$|R_{\alpha,t}(x,\xi)| \leq C(1+|\xi|^2)^{(m-(\rho-\delta)N+n)/2}, \quad x \in K, \ t \in (0,1].$$

By Theorem 6.14 the proof follows. Note that the assumption $\rho > \delta$ is crucial for the proof. □

Proposition 7.2 *Let A be a proper ΨDO, $\sigma_A(x,\xi)$ its symbol and $\sigma'_A(x,\xi)$ a symbol of ${}^tA$. Then,*

$$\sigma'_A(x,\xi) \approx \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \sigma_A(x,-\xi).$$

Proof. By (6.6)

$${}^tAv(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} \sigma_A(y,-\xi) v(y)\, dy\, d\xi.$$

The assertion follows from (7.1). □

Analogously, one can prove the following assertion.

Proposition 7.3. *Let A be a proper ΨDO with a symbol $\sigma_A(x,\xi)$ and A^* its adjoint operator. If $\sigma_A^*(x,\xi)$ is a symbol of adjoint operator, then*

$$\sigma_A^*(x,\xi) \approx \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{\sigma_A}.$$

Definition 7.4 A dual symbol $\tilde{\sigma}_A(x,\xi)$ for A is given by

$$\tilde{\sigma}_A(x,\xi) = \sigma'_A(x,-\xi).$$

By using ${}^t({}^tA) = A$ we obtain

$$Au(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \tilde{\sigma}_A(y,\xi) u(y)\, dy\, d\xi. \tag{7.7}$$

The following proposition follows immediately.

Proposition 7.5. $\tilde{\sigma}_A(x,\xi) \approx \sum (-\partial_\xi)^\alpha D_x^\alpha \sigma_A(x,\xi)/\alpha!$.

7.2. Composition of proper ΨDO's.

Theorem 7.6. *Let A and B be proper ΨDO's in $X \subset \mathbb{R}^n$, $\sigma_A(x,\xi)$, $\sigma_B(x,\xi)$ their symbols and $C = BA$. Then C is a proper ΨDO, with the symbol $\sigma_{BA}(x,\xi)$ which is given by*

$$\sigma_{BA}(x,\xi) \approx \sum \partial_\xi^\alpha \sigma_B(x,\xi) D_x^\alpha \sigma_A(x,\xi)/\alpha!.$$

Recall $D_x^j = \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}\right)^j$.

Proof. Note, the dual symbol is used for representation (7.7):

$$\widehat{(Au)}(\xi) = \int_{\mathbf{R}^n} e^{-iy\xi}\tilde{\sigma}_A(y,\xi)u(y)\,dy.$$

By (6.6)

$$Bu(x) = \int_{\mathbf{R}^n} e^{ix\xi}\sigma_B(x,\xi)\hat{u}(\xi)\,d\xi,$$

which implies

$$Cu(x) = (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} e^{i(x-y)\xi}\sigma_B(x,\xi)\tilde{\sigma}_A(y,\xi)u(y)\,dy\,d\xi.$$

Clearly, if $\sigma_A \in S^{m_1}_{\rho,\delta}$ and $\sigma_B \in S^{m_2}_{\rho,\delta}$, then $\sigma_B\tilde{\sigma}_A \in S^{m_1+m_2}_{\rho}$ and thus the symbol of C is in $S^{m_1+m_2}_{\rho,\delta}$. Analogously, we have that the symbol of ${}^tC = {}^tA\,{}^tB$ is in $S^{m_1+m_2}_{\rho,\delta}$. By Theorem 3.11 it follows that C is proper.

Let us find the symbol for C. By using Theorem 7.1 and Proposition 7.5 we have

$$\begin{aligned}
\sigma_{BA}(x,\xi) &\approx \sum_{\alpha}\partial_\xi^\alpha D_y^\alpha\sigma_B(x,\xi)\tilde{\sigma}_A(y,\xi)/\alpha!|_{y=x}\\
&= \sum_{\alpha}\partial_\xi^\alpha[\sigma_B(x,\xi)D_x^\alpha\tilde{\sigma}_A(x,\xi)]/\alpha!\\
&\approx \sum_{\alpha,\beta}\partial_\xi^\alpha[\sigma_B(x,\xi)(-\partial_\xi)^\beta D_x^{\alpha+\beta}\sigma_A(x,\xi)]/\alpha!\beta!.
\end{aligned}\tag{7.8}$$

Leibnitz formula implies

$$\begin{aligned}
\sigma_{BA}(x,\xi) &\approx \sum_{\alpha,\beta}\sum_{\gamma,\delta,\gamma+\delta=\alpha}(-1)^{|\beta|}[\partial_\xi^\gamma\sigma_B(x,\xi)][\partial_\xi^{\beta+\delta}D_x^{\alpha+\beta}\sigma_A(x,\xi)]/\delta!\beta!\gamma!\\
&= \sum_{\beta,\gamma,\delta}(-1)^{|\beta|}[\partial_\xi^\gamma\sigma_B(x,\xi)][\partial_\xi^{\beta+\delta}D_x^{\beta+\gamma+\delta}\sigma_A(x,\xi))]/\delta!\beta!\gamma!\\
&= \sum_{\gamma}\sum_{\kappa}\Big(\sum_{\beta+\delta=\kappa}(-1)^{|\beta|}\frac{1}{\beta!\delta!}\Big)[\partial_\xi^\gamma\sigma_B(x,\xi)][\partial_\xi^\kappa D_x^{\kappa+\gamma}\sigma_A(x,\xi))]/\gamma!.
\end{aligned}\tag{7.9.}$$

We shall use the following identity

$$(x-y)^\alpha = (x_1-y_1)^{\alpha_1}\dots(x_n-y_n)^{\alpha_n} = \sum_{\beta+\delta=\alpha}\frac{\alpha!}{\beta!\delta!}(-1)^{|\delta|}x^\beta y^\delta$$

for $x = (1,\dots,1)$, $y = (1,\dots,1)$. This gives

$$\sum_{\beta+\delta=\alpha}(-1)^{|\beta|}\frac{1}{\beta!\delta!} = 1, \qquad \sum_{\beta+\delta=\alpha}(-1)^{|\beta|}\frac{1}{\beta!\delta!} = \begin{cases}0, & \alpha\neq 0\\ 1, & \alpha = 0\end{cases}$$

and by substituting this in (7.8), the assertion of theorem follows. □

Proposition 7.7. *Let $\sigma_A \in S^{m_1}_{\rho,\delta}(X,\mathbb{R}^n), \sigma_B \in S^{m_2}_{\rho,\delta}(X,\mathbb{R}^n), 0 \leq \delta < \rho \leq 1$ and let B be a proper operator. Then the operators AB, BA are determined by the symbols in $S^{m_1+m_2}_{\rho,\sigma}(X,\mathbb{R}^n)$.*

Proof. Let $A = A_1 + R$, where A is a proper ΨDO and R has a kernel $K_R(x,\xi) \in C^\infty(X \times X)$. Then, BR and RB have smooth kernels. Let us prove this for BR. Let $\varphi \in C_0^\infty$. We have

$$\begin{aligned} B(R\varphi)(x) &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi}(R\varphi)(y) b(x,y,\xi)\, dy\, d\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \left(\int_{\mathbb{R}^n} K_R(y,t)\varphi(t)\, dt\right) b(x,y,\xi)\, dy\, d\xi \\ &= \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} K_R(y,t) b(x,y,\xi)\varphi(t)\, dy\, d\xi\right) \varphi(t)\, dt. \end{aligned}$$

Thus the kernel of BR equals

$$\iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} K_R(y,t) b(x,y,\xi)\, dy\, d\xi = \iint_{\mathbb{R}^{2n}} \frac{(-1)^r}{|x-y|^{2r}} \Delta^r e^{i(x-y)\xi} K_R(y,t) b(x,y,\xi)\, dy\, d\xi.$$

Since $|x-y| > d > 0$ by taking enough large r, we obtain that the kernel of BR is smooth with respect to x and t. The same holds for RB. □

7.3. Classical symbols and pseudodifferential operators.

Definition 7.8. A classical symbol is a function $a(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$, X is open in $\mathbb{R}^n$ which has an asymptotic expansion

$$a(x,\xi) \approx \sum_{j=0}^{\infty} a_{m-j}(x,\xi), \tag{7.10}$$

for some complex m, where $a_{m-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n \setminus \{0\}))$ are positively homogeneous with respect to ξ of order $m-j$, $j = 0,1,\ldots$. The set of such symbols is denoted by $CS^m(X \times \mathbb{R}^n)$ and the corresponding pseudodifferential operators are called classical pseudodifferential operators. a_m is called the main symbol.

Note a_{m-j} is not smooth for $\xi = 0$ and should be cuted off in an appropriate way.

If $a_k(x,\xi)$ is positive homogeneous with respect to ξ of order k, then $\partial^\alpha_\xi \partial^\beta_x a_k(x,\xi)$ is positive homogeneous with respect to ξ of order $k - |\alpha|$. Because of that,

$$CS^m(X \times \mathbb{R}^n) \subset S^{\mathrm{Re}(m)}(X \times \mathbb{R}^n).$$

The following proposition can be easily proved

Proposition 7.9. a) *If A and B are proper classical pseudodifferential operators determinated by the symbols in CS^{m_1} and CS^{m_2}, then BA is a classical pseudodifferential operator with the symbol in $CS^{m_1+m_2}(X)$.*

b) *If A is a classical operator then ${}^tA$ and A^* are also classical with the symbols in the same class.*

7.4. Hypoellipticity and ellipticity. Parametrix.

As we already said, ΨDO are founded in the development for the theory of elliptic and hypoelliptic operators. The construction of a parametrix for a given hypoelliptic operator which is to follow is the most important application of the pseudodifferential calculus. Note that in the first section we gave the motivation of the whole theory by considering elliptic operators.

Definition 7.10. A function $\sigma(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$, where X is open in $\mathbb{R}^n$, is hypoelliptic symbol if the following holds.

a) There exist reals m and m_0 such that for every compact set $K \subset X$ there exist positive constants R, C_1, C_2 such that

$$C_1|\xi|^{m_0} \le |\sigma(x,\xi)| \le C_2|\xi|^m, |\xi| \ge R, x \in K. \tag{7.11}$$

b) There exist $\rho, \delta, 0 \le \delta < \rho \le 1$ such that for every compact set $K \subset X$ there exists a constant R such that for every pair of multi-indices α, β there exists a constant $C_{\alpha,\beta,K}$ such that

$$|(\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi))\sigma^{-1}(x,\xi)| \le C_{\alpha,\beta,K}|\xi|^{-\rho|\alpha|+\delta|\beta|}, |\xi| \ge R, x \in K. \tag{7.12}$$

The class of hypoelliptic symbols is denoted by $HS_{\rho,\delta}^{m,m_0}(X \times \mathbb{R}^n)$. From (7.11) and (7.12) it follows $HS_{\rho,\delta}^{m,m_0}(X \times \mathbb{R}^n) \subset S_{\rho,\delta}^m(X \times \mathbb{R}^n)$.

Definition 7.11. ΨDO A is called hypoelliptic if there exists a proper ΨDO A_1 with the symbol $HS_{\rho,\delta}^{m,m_0}(X \times \mathbb{R}^n)$, such that $A = A_1 + R_1$, where R_1 is smoothing.

If $m = m_0$ then σ is called elliptic, i. e. A is called elliptic ΨDO.

Let us note that in the decomposition of a hypoelliptic operator $A = A_1 + R_1$, where R_1 is smoothing and A_1 is a proper ΨDO, it follows that its symbol belongs to $HS_{\rho,\delta}^{m,m_0}(X \times \mathbb{R}^n)$.

Recall, $A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ is called elliptic, if its principal symbol satisfies

$$a_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ne 0, (x,\xi) \in X \times (\mathbb{R}^n \setminus \{0\}). \tag{7.13}$$

Example 7.1. Examples of hypoelliptic operators.

The Heat operator $\partial_t - \sum_{i=1}^n \partial_{x_i}^2$ is an example of a hypoelliptic and not elliptic operator.

(2) Differential operator $D_y^2 + y^2 D_x^2 + \lambda D_x$, $\mathrm{Re}\,\lambda = 0$ is hypoelliptic if and only if $\lambda \ne 2k+1$, $k \in \mathbb{Z}$, while

$$D_y + iay^r D_x, \ \mathrm{Re}\, a \ne 0,$$

is hypoelliptic if and only if $r = 2k$, $k \in \mathbb{N}$.

(3) Pseudodifferential operator $D_y + iay^r\sqrt{D_x^2 + D_y^2}$, $\operatorname{Re} a \neq 0$, is hypoelliptic if r is even or r is odd and $\operatorname{Re} a > 0$, and the pseudodifferential operator given by the symbol $P(x,\xi) = 1 + |x|^{2\nu}|\xi|^{2\mu}$ is hypoelliptic for $\mu/\nu < 1$.

Remark 7.1. The change of the variables does not preserve the hypoellipticity. For example, the change of variables

$$y_i = x_i,\ i = 1,\dots,n,\ \tau = t + x_1^2/2$$

in the heat operator, gives a non-hypoelliptic operator.

Proposition 7.12. *For a differential operator A the following two conditions are equivalent*

a) *A is elliptic.* b) *The symbol of A is in $HS_{1,0}^{m;m}(X \times \mathbb{R}^n)$.*

Proof. The implication *b)* $\Rightarrow$ *a)* is obvious. For the another part of the proof we note that the symbol of A is

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha. \tag{7.14}$$

If a) holds, then

$$a(x,\xi)/a_m(x,\xi) = 1 + b_{-1}(x,\xi) + \dots + b_{-m}(x,\xi),$$

where $b_{-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n\backslash\{0\}))$ are homogeneous in respect to ξ of order $-j$. This implies (7.11), while (7.12) follows in the same manner. □

Definition 7.13. A classical operator A is called elliptic if its main symbol $a_m(x,\xi) \in CS^m(X \times \mathbb{R}^n)$ satisfies (7.13).

Proposition 7.12 holds for a classical ΨDO. More precisely, if a symbol of A satisfies (7.11) for $m = m_0$ then it satisfies (7.12), too. This means that in the case of the symbols of elliptic operators we can omit the condition (7.12) for them. This follows from the following proposition.

Proposition 7.14. *Let $\sigma(x,\xi) \in HS_{\rho,\delta}^{m,m_0}(X \times \mathbb{R}^n)$. Then*

$$\sigma^{-1}(x,\xi) \in HS_{\rho,\delta}^{-m,-m_0}(X \times \mathbb{R}^n)$$

for ξ large enough, $|\xi| > \xi_0 > 0$. Further on, for any pair of multi-indices $\alpha,\beta \in \mathbb{N}_0^n$,

$$\partial_\xi^\alpha\partial_x^\beta\sigma(x,\xi)/\sigma(x,\xi) \in S_{\rho,\delta}^{-\rho|\alpha|+\delta|\beta|}, \tag{7.15}$$

for ξ large enough.

Proof. One can simply prove (7.15) for $|\alpha| = |\beta| = 1$. Let $p \in \mathbb{N}_0^{2n}$. By induction with respect to $|p|$, it can be shown that

$$\partial^p\frac{\partial^{(\alpha,\beta)}\sigma(x,\xi)}{\sigma(x,\xi)} = \sum_{k=0}^{|p|}\sum_{p_0+\dots+p_k=p}\frac{\partial^{p_0+(\beta,\alpha)}\sigma(x,\xi)}{\sigma(x,\xi)}\prod_{l=1}^{k}\frac{\partial^{p_l}\sigma(x,\xi)}{\sigma(x,\xi)}. \tag{7.16}$$

Now (7.15) follows from (7.16) by induction. □

Theorem 7.15. *Let A be a proper pseudodifferential operator with a symbol in $HS^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$, $\delta < \rho$. Then there exists a proper pseudodifferential operator B with a symbol in $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ such that*

$$BA = I + R_1, \quad AB = I + R_2, \tag{7.17}$$

where R_1, R_2 are smoothing operators and I is the unity operator.

If B' is an operator with the same property, then $B - B'$ is a smoothing operator.

Proof. Let σ_A be the symbol of the operator A. Chose $b_0(x,\xi) \in HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ such that $b_0(x,\xi) = \sigma_A^{-1}(x,\xi)$ for large enough ξ and a proper pseudodifferential operator B_0 with a symbol in $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ such that $\sigma_{B_0} - b_0 \in S^{-\infty}(X \times \mathbb{R}^n)$. Let us show that

$$B_0 A = I - R_0,$$

where the symbol of R_0 is in $S^{-(\rho-\delta)}_{\rho,\delta}(X \times \mathbb{R}^n)$. By Theorem 7.6 it follows that

$$\sigma_{B_0 A}(x,\xi) \approx 1 + \sum_{|\alpha|\geq 1} \partial_\xi^\alpha \sigma_A^{-1} D_x^\alpha \sigma_A/\alpha! = 1 + \sum_{|\alpha|\geq 1} \partial_\xi^\alpha \sigma_A^{-1} D_x^\alpha \sigma_A/(\alpha! \sigma_A^{-1} \sigma_A)$$

for large enough ξ. Proposition 7.14 implies that R_0 has the symbol in $S^{-(\rho-\delta)}_{\rho,\delta}$. Let C_0 be a proper ΨDO which satisfies

$$C_0 \approx \sum_{j=0}^{\infty} (-1)^j R_0^j, \quad \text{i.e.} \tag{7.18}$$

$$\sigma_{C_0} \approx \sum_{j=0}^{\infty} (-1)^j \sigma_R^j. \tag{7.19}$$

From (7.18) immediately follows that the operator $C_0(I + R_0) - I$ is smoothing, so, if we put $B_1 = C_0 B_0$ we obtain

$$B_1 A = I + R_1, \tag{7.20}$$

where R_1 is smoothing. It is clear from the construction that the symbol of B_1 belongs to $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$. Analogously, we obtain that the symbol of the operator B_2 is in $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ for which

$$AB_2 = I + R_2, \tag{7.21}$$

where R_2 is smoothing.

Let B_1 and B_2 be a pair of ΨDO's for which (7.20) and (7.21) hold. We can suppose that they are proper operators. By multiplying the right-hand side of

(7.20) with B_2 (in fact by applying B_2) and by using (7.21) we obtain $B_1 - B_2 = R_1B_2 - B_1R_2$ and $R_1B_2 - B_1R_2$ is smoothing. □

Definition 7.16. The operator B satisfying (7.17) is called a parametrix of the operator A.

Note that an elliptic operator A with a symbol belonging to $S^m_{\rho,\sigma}(X \times \mathbb{R}^n)$ has a parametrix B with a symbol in $HS^{-m,-m}_{\rho,\delta}(X \times \mathbb{R}^n)$.

Proposition 7.17. *Let A be a hypoelliptic ΨDO. Then*

$$\text{Sing supp}\, Au = \text{Sing supp}\, u, \quad u \in \mathcal{E}'(X). \tag{7.22}$$

If A is also a proper operator, then (7.22) holds for every $u \in \mathcal{D}'(X)$.

Proof. The relation $\text{Sing supp}\, Au \subset \text{Sing supp}\, u$ follows from the pseudolocality of the operator A. Let B be a proper ΨDO which is a parametrix of the operator A. Then from the equation $u = B(Au) - R_1u$ and pseudolocality of the operator B it follows that

$$\text{Sing supp}\, u \subset \text{Sing supp}\, Au \cup \text{Sing supp}\, R_1u.$$

Since $R_1u \in C^\infty(X)$ (and $\text{Sing supp}\, R_1u = \emptyset$) the assertion follows. □

This was a global aspect of hypoellipticity of ΨDO's. Now, we shall give few assertions about a local hypoellipticity.

Definition 7.18. A class of symbols in $HS^{m;m_0}_{\rho,\sigma}(x_0,\xi_0)$ consists of symbols in $S^m_{\rho,\delta}$, which are hypoelliptic at (x_0,ξ_0), i.e. which satisfies the conditions of Definition 7.10 in the set of the form $U \times \Gamma_{R,\eta}$ where U is a neighbourhood of the point x_0 and $\Gamma_{R,\eta} = \left\{\xi,\ \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \eta,\ |\xi| > R\right\}$.

A ΨDO A is called hypoelliptic at x_0 (locally hypoelliptic at x_0) if there exists a proper ΨDO A_1 with a symbol in $HS^{m;m_0}_{\rho,\delta}(x_0,\xi)$ for every $\xi \in \mathbb{R}^n$ such that $A = A_1 + R_1$, where R_1 is smoothing in a neighbourhood of x_0. Locally elliptic ΨDO are analogously defined.

The following assertion can be proved in the same way as in Theorem 7.15.

Proposition 7.19. *Let an operator A be hypoelliptic at x_0 (and proper). Then there exists an operator B, hypoelliptic at x_0 (and proper) such that*

$$BA = I + R_1, \quad AB = I + R_2, \tag{7.23}$$

where R_1, R_2 are smoothing operators in a neighbourhood of x_0, and I denotes identity operator. If B' is an operator with the same property as B, then $B - B'$ is smoothing in a neighbourhood of x_0.

Let A be a classical elliptic ΨDO with a symbol $a(x,\xi)$ such that

$$a(x,\xi) \approx \sum_{j=0}^{\infty} a_{m-j}(x,\xi),$$

where $a_{m-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n \backslash \{0\}))$, $a_{m-j}(x,\xi)$ is positively homogeneous with respect to ξ of order $m-j$, $j \in \mathbb{N}$, and $a_m(x,\xi) \neq 0$, $x \in X$, $\xi \neq 0$.

Let B be a parametrix of A given by the symbol $b(x,\xi)$. We shall prove that $b(x,\xi)$ has an asymptotic expansion

$$b(x,\xi) \approx \sum_{j=0}^{\infty} b_{-m-j}(x,\xi),$$

where $b_{-m-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n \backslash \{0\}))$, $b_{-m-j}(x,\xi)$ is positively homogeneous with respect to ξ of order $-m-j$, $j \in \mathbb{N}$. The formula for composition implies

$$\sum_{\alpha} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi)/\alpha! \approx 1 \quad \text{or}$$

$$\sum_{\alpha,k,j} \partial_\xi^\alpha a_{m-k}(x,\xi) D_x^\alpha b_{-m-j}(x,\xi)/\alpha! \approx 1. \tag{7.24}$$

By factoring the expression with respect to the degree of homogeneity we obtain the following system of equations

$$a_m b_{-m} = 1, a_m b_{-m-j} + \sum_{\substack{k+l+|\alpha|=j \\ l<j}} (\partial_\xi^\alpha a_{m-k})(D_x^\alpha b_{-m-l})/\alpha! = 0, \tag{7.25}$$

$$j = 1,2,\ldots.$$

The functions $b_{-m-j}(x,\xi)$ in (7.25) are uniquely determinated and we have to find a proper ΨDO B such that $\sigma_B(x,\xi) - b(x,\xi) \in S^{-\infty}(X \times \mathbb{R}^n)$. Such B is the solution to the system.

8. Wave front sets and ΨDO

The notion of the wave front set was introduced by Hörmander [**10**] and, independently, by Sato (he called it singular spectrum). It is a basic notion of microlocal analysis.

Pseudodifferential operators do not increase the wave front set and this is one of the most important property of this class of operators. For example, if we apply the method of parametrix on elliptic operators, then the set of microlocal singularities will not be changed.

8.1. Sobolev spaces and the wave front set. First we recall some properties of Sobolev spaces.

A distribution f belongs to $H^s(\mathbb{R}^n)$ if and only if $(1-\Delta)^{s/2} f \in L^2(\mathbb{R}^n)$.

Note that $(1-\Delta)^{s/2}$ is an elliptic ΨDO of order s. (Note in this section we deal with operators with symbols in $S^s = S^s_{1,0}$, $s \in \mathbb{R}$.)

Let X be an open set in $\mathbb{R}^n$. Then $H^s_{loc}(X)$ is the space of distributions $f \in \mathcal{D}(X)$ such that $Af \in L^2_{loc}(X)$ where A is proper elliptic pseudodifferential operator of order s. Note, $f \in L^2_{loc}(X)$ if and only if for every $\varphi \in C_0^\infty(X)$, $f\varphi \in L^2(X)$.

Proposition 8.1. (1) *Let $f \in \mathcal{D}'(X)$. Then $f \in H^s_{loc}(X)$ if and only if $Af \in L^2_{loc}(X)$ for every proper ΨDO A of order s.*

(2) *$A(H^s_{loc}(X)) \subset H^{s-m}_{loc}(X)$ for every ΨDO A of order m.*

Proof. (1) Let A be a proper ΨDO. Then $Sf \in L^2_{loc}$ since $Af = AB^{-1}Bf$, where B is a proper elliptic operator of order s. Thus the assertion follows.

(2) Since the composition of two proper operators of orders m_1 and m_2 is a proper one of order $m_1 + m_2$, Part 1 implies that $A(H^s_{loc}) \subset H^{s-m}_{loc}$, where A is a proper pseudodifferential operator of order m. □

Note that $U \to H^s_{loc}(U)$ is a sheaf with respect to the restrictions. (For the definition of a sheaf we refer to next section)

Definition 8.2 Let K be a compact subset of X. Define $H^s_K = H^s_{loc}(X) \cap \mathcal{D}'_K$ (where $\mathcal{D}'_K$ denotes the space of distributions with supports in K).

With the appropriate scalar product, $H^s_K(X)$ is a Hilbert space ($H^s_{loc}(X)$ is a Frechét space).

The following assertion is important for the microlocal analysis of distributions.

Theorem 8.3. *Let A be a proper elliptic pseudodifferential operator of order m on X and $f \in \mathcal{D}'(X)$. If $Af|_{X'} \in H^s_{loc}(X')$, then $f|_{X'} \in H^{s+m}_{loc}(X')$, where $X' \subset X$, X is an open set.*

Proof. Let B be a proper operator in $HS^{-m_1,-m}$ which is a parametrix for A ($BA = I + R$, where R is a smoothing operator). We have shown in Proposition 7.9 that for every $f \in \mathcal{D}'(X)$, $BAf - f \in C^\infty(X)$. Let $x \in X$ and $g = \phi Af$, where $\phi \in C_0^\infty(X)$, and $\phi = 1$ in a compact neighbourhood of x. Then $g \in H^s_{loc}(X)$ and $g - Af|_V = 0$, where $V = \operatorname{int} K$. Moreover, $(Bg - BAf)|_V, (Bg - f)|_V \in C^\infty(V)$ and since B is of the order $-m$, by Proposition 8.1, (2) it follows that $Bg \in H^{s+m}_{loc}(X)$. So $f|_V \in H^{s+m}(V)$. This holds for every $x \in X'$ and this implies $f|_{X'} \in H^{s+m}_{loc}(X')$. □

Definition 8.4. Let X be open in $\mathbb{R}^n$, $(x_0, \xi_0) \in X \times (\mathbb{R}^n \backslash \{0\})$ and $u \in \mathcal{D}'(X)$. Then (x_0, ξ_0) is not in $\mathrm{WF}(u)$ if there exists $v \in \mathcal{E}'(X)$ such that $u = v$ in a neighbourhood of x_0 and there exists $\varepsilon > 0$ such that for every $N > 0$ there exists $C_N > 0$ such that

$$|\hat{v}(\xi)| \le C_N (1 + \xi^2)^{-N/2} \text{ for} \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon, \tag{8.1}$$

that is, $\hat{v}(\xi)$ rapidly decreases in a conic neighbourhood of ξ_0. In this case it is said that u is microlocally regular in (x_0, ξ_0).

The closed conic set $\mathrm{WF}(u) \subset X \times (\mathbb{R}^n \backslash \{0\})$ (closure in $X \times \mathbb{R}^n \setminus \{0\}$ of the complement of the set of all microlocally regular points) is called the wave front set of the distribution u.

Theorem 8.5. *If (x_0, ξ_0) is not in* WF(u), *then* (x_0, ξ_0) *is not in* WF(φu) *for every* $\varphi \in C_0^\infty$.

Proof. Let Γ_ε be an open cone of the form $\Gamma_\varepsilon = \{\eta \mid \frac{\eta}{|\eta|} - \frac{\xi_0}{|\xi_0|} < \varepsilon\}$ and $v \in \mathcal{E}'$, $u = v$ in a neighbourhood of x_0. Then

$$\widehat{(\varphi v)}(\xi) = \int_{|\eta| \leq R} \hat{v}(\xi - \eta)\hat{\varphi}(\eta)\, d\eta + \int_{|\eta| > R} \hat{v}(\xi - \eta)\hat{\varphi}(\eta)\, d\eta,$$

for $\xi \in \Gamma_\varepsilon$, where R will be determined later. We have

$$|\widehat{(\varphi v)}(\xi)| \leq \bar{C} \sup_{|\eta| \leq R} |\hat{v}(\xi - \eta)| + CC_L \int_{|\eta| > R} (1 + |\xi - \eta|)^p (1 + |\eta|)^{-L}\, d\eta$$

where we have used the Paley-Wiener theorem for $v \in \mathcal{E}'$ and $\varphi \in C_0^\infty$ $\left(|\hat{v}(\xi)| \leq C(1 + |\xi|)^p,\ |\hat{\varphi}(\xi)| \leq \frac{C_L}{(1+|\xi|)^L}\right)$. This implies

$$\begin{aligned} |\hat{\varphi v}(\xi)| &\leq \bar{C} \sup_{|\eta| \leq R} |\hat{v}(\xi - \eta)| + CC_L(1 + |\xi|)^p \int_{|\eta| > R} (1 + |\eta|)^{p-L}\, d\eta \\ &\leq \bar{C}C \sup_{|\eta| \leq R} |\hat{v}(\xi - \eta)| + CC_L(1 + |\xi|)^p R^{n+p-L}. \end{aligned}$$

Put $R = |\xi|^{1/2}$. If ξ belongs to a cone $\Gamma_{\varepsilon'}$, $\varepsilon' < \varepsilon$, then $\xi - \eta \in \Gamma_\varepsilon$ for large enough ξ and $|\eta| < R$. Beside that, $|\xi - \eta| \approx |\xi|$ and $R^{n+p-L} \approx |\xi|^{(n+p-L)/2}$. For large enough L we obtain that $\widehat{(\varphi v)}(\xi)$ rapidly decreases when $|\xi| \to \infty, \xi \in \Gamma_{\varepsilon'}$. □

By this theorem it follows that in Definition 7.10 we can take $v = \varphi u, \varphi \in C_0^\infty(X), \varphi = 1$ in a neighbourhood of x_0.

Example 8.1. 1. $\mathrm{WF}(\delta(x)) = \{(0, \xi),\ \xi \in \mathbb{R}^n \setminus \{0\}\}$. 2. Since $\delta(x_1) = \delta(x_1) \otimes \kappa_{\mathbb{R}^{n-1}}$, where $\mathbb{R}^{n-1} = \{x' = (x_2, \ldots, x_n)\}$ and $\kappa_{\mathbb{R}^{n-1}} = 1$ for $x' \in \mathbb{R}^{n-1}$, it follows that $\mathrm{WF}(\delta(x_1)) = \{((0, x'), (\xi_1, 0)),\ x' \in \mathbb{R}^{n-1},\ \xi_1 \in \mathbb{R} \setminus \{0\}\}$.

Proposition 8.6. *Let $\pi : X \times (\mathbb{R}^n \backslash \{0\}) \to X$ be the natural projection and let $u \in \mathcal{D}'(X)$. Then $\pi\, \mathrm{WF}(u) = \mathrm{Sing\,supp}\, u$.*

Proof. If x_0 is not element of $\mathrm{Sing\,supp}\, u$, then by taking $\varphi \in C_0^\infty(X), \varphi(x) = 1$ in a neighbourhood of x_0, and $\varphi(x) = 0$ in a neighbourhood of $\mathrm{Sing\,supp}\, u$, we obtain that $\varphi u \in C_0^\infty(X)$. This implies $\widehat{(\varphi v)} \in \mathcal{S}(\mathbb{R}^n)$ and thus x_0 is not in $\pi\, \mathrm{WF}(u)$.

Let $x_0 \notin \pi\, \mathrm{WF}(u)$. For every $\xi_0 \in S^{n-1}$ there exist $\varphi_{\xi_0} \in C_0^\infty(X)$ and a conic neighbourhood Γ_{ξ_0} of ξ_0 such that $\varphi_{\xi_0}(x) = 1$ in a neighbourhood of x_0 and $\widehat{(\varphi_{\xi_0} u)}(\xi)$ rapidly decreases in Γ_{ξ_0}. Since S^{n-1} is compact there exist finitely many points $\xi_1, \ldots, \xi_N$ such that S^{n-1} is covered by $\Gamma_{\xi_1} \cap S^{n-1}, \ldots, \Gamma_{\xi_N} \cap S^{n-1}$. Thus, $\Gamma_{\xi_1}, \ldots, \Gamma_{\xi_N}$ cover $\mathbb{R}^n \backslash \{0\}$. Then, by putting $\varphi = \prod_{j=1}^N \varphi_{\xi_j}$, we obtain that $\widehat{(\varphi u)}$ rapidly decreases, and this means $\varphi u \in C_0^\infty(X)$, i.e. $u \in C^\infty$ in a neighbourhood of x_0. So, x_0 is not in $\mathrm{Sing\,supp}\, u$. □

Proposition 8.7. *Let $u \in \mathcal{D}'(X)$ and $(x_0, \xi_0) \notin \mathrm{WF}(u)$. Then there exists a classical ΨDO A of order 0 such that $\sigma_A = 1 (\mathrm{mod}\, S^{-\infty})$ in a conic neighbourhood of (x_0, ξ_0) and $Au \in C_0^\infty(X)$.*

(Recall that a conic neighbourhood of (x_0, ξ_0) is of the form $U \times \Gamma_{R,\xi_0}$, where U is a neighbourhood of x_0 and Γ_{R,ξ_0} a cone around ξ_0 (cf. Definition 7.18)

Proof. Let $\varphi \in C_0^\infty(X), \varphi = 1$ around x_0. Then $\widehat{(\varphi u)}(\xi)$ rapidly decreases in a conic neighbourhood of ξ_0. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n), \chi(t\xi) = \chi(\xi)$ for $t \geq 1$, $|\xi| \geq 1$ (χ is homogeneous of order 0 for $|\xi| \geq 1$.), $\chi(\xi) = 1$ in some small enough neighbourhood of ξ_0. This means that $\chi(\xi)\widehat{(\varphi u)}(\xi)$ rapidly decreases, so $\chi(D)(\varphi(x)u(x)) \in C^\infty(X)$. But then $\psi(x)\chi(D)(\varphi(x)u(x)) \in C_0^\infty(X)$ if $\psi(x) \in C_0^\infty(X)$. We can take ψ such that $\psi(x) = 1$ in a neighbourhood of x_0. Then $A = \psi(x)\chi(D)\varphi(x)$ satisfies all assertions of the proposition. □

Note that the operator $A = \psi(x)\chi(D)\varphi(x)$ from the previous proposition is locally elliptic (see Definition 7.18).

Theorem 8.8. *Let $u \in \mathcal{D}'(X), (x_0, \xi_0) \in X \times (\mathbb{R}^n \backslash \{0\})$ be given as well as the classical operator A defined by the principal symbol $a_m(x, \xi) \in CS^m(X \times \mathbb{R}^n)$. Let either $u \in \mathcal{E}'(X)$ or A be proper. Suppose that $a_m(x_0, \xi_0) \neq 0$ and $Au \in C^\infty(X)$. Then $(x_0, \xi_0) \notin \mathrm{WF}(u)$.*

Proof. By Proposition 7.19 and Section 7.4, we can make the parametrix for a classical elliptic operators. So there exists a classical pseudodifferential operator B with the symbol in $CS^{-m}(X \times \mathbb{R}^n)$, such that $\sigma_{BA} = 1(\mathrm{mod}\, S^{-\infty})$. Since $BAu \in C^\infty(X)$ we can assume that $\sigma_A = 1(\mathrm{mod}\, S^{-\infty})$ in a conic neighbourhood of (x_0, ξ_0).

Let $\chi(\xi) = 1$ in a neighbourhood of ξ_0, $\chi(\xi) \in C^\infty(\mathbb{R}^n), \chi(\xi)$ is homogeneous of zero order with respect to ξ for $|\xi| \geq 1$ and let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ in a neighbourhood of x_0. Let the supports of φ, χ be chosen such that

$$\chi(\xi)\varphi(x)\sigma_A(x, \xi) = \chi(\xi)\varphi(x) \quad (\mathrm{mod}\ S^{-\infty}).$$

Then $\chi(D)\varphi(x)A - \chi(D)\varphi(x)$ is smoothing operator, and since $\chi(D)\varphi(x)Au \in C^\infty(X)$, it follows

$$\chi(D)\varphi(x)u \in C^\infty(\mathbb{R}^n). \tag{8.2}$$

If we prove that

$$\chi(D)\varphi(x)u \in \mathcal{S}(\mathbb{R}^n), \tag{8.3}$$

then it would follow that $\chi(\xi)\widehat{(\varphi u)}(\xi) \in \mathcal{S}(\mathbb{R}^n)$, and specially, $\widehat{(\varphi u)}(\xi)$ would rapidly decrease in a conic neighbourhood of ξ_0, what we are aimed to prove.

The implication (8.2) $\Leftarrow$ (8.3) follows from the following lemma, which is formulated separately because it has a more general meaning. □

Lemma 8.9. *Let $v \in \mathcal{E}'(\mathbb{R}^n), \chi(\xi) \in S^m_{\rho,0}$, $\rho > 0$. Then for every $N > 0$ and $\alpha \in \mathbb{N}_0^n$ there exists $C_{\alpha,N}$ such that*

$$|D^\alpha \chi(D)v(x)| \leq c_{\alpha,N}|x|^{-N}, \; x \in \mathbb{R}^n, \; d(x, \operatorname{supp} v) \geq 1. \tag{8.4}$$

Proof. We can consider only the case $\alpha = 0$, because $\xi^\alpha \chi(\xi) \in S^{m+|\alpha|}_{\rho,0}$. Also, we may assume that v is continuous because every element $v \in \mathcal{E}'(\mathbb{R}^n)$ is of the form $v = \sum_{|\gamma|\leq p} D^\gamma v_\gamma, v_\gamma \in C(\mathbb{R}^n)$. We have

$$\chi(D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}\chi(\xi)v(y)\,dyd\xi. \tag{8.5}$$

Integration by parts gives

$$|x-y|^{-2N}(-\Delta_\xi)^N e^{i(x-y)\xi} = e^{i(x-y)\xi}.$$

From (8.5), with $d(x, \operatorname{supp} v) \geq 1$, we have

$$\chi(D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}|x-y|^{-2N}((-\Delta_\xi)^N\chi(\xi))v(y)\,dyd\xi. \tag{8.6}$$

By choosing large enough N, such that $(-\Delta_\xi)^N\chi(\xi) \in S^{-n-1}_{\rho,0}$, one can see that the integral in (8.6) converges absolutely and satisfies $C(1+x^2)^{-N}$. □

Definition 8.10. Let A be a classical pseudodifferential operator with the symbol in $CS^m(X \times \mathbb{R}^n)$. Then

$$\operatorname{char}(A) = \{(x,\xi) \in X \times (\mathbb{R}^n \backslash \{0\}),\; a_m(x,\xi) = 0\}.$$

Theorem 8.8 directly implies the following important (and practical) characterization of the wave front set.

Theorem 8.11. (1) *Let $u \in \mathcal{E}'(X)$ and A be a classical ΨDO with a symbol in $CS^m(X \times \mathbb{R}^n)$. If $Au \in C^\infty(X)$, then* $\mathrm{WF}(u) \subset \operatorname{char}(A)$.

(2) *Let $u \in \mathcal{E}'(X)$. Then $\mathrm{WF}(u) = \bigcap \operatorname{char}(A)$, where the intersection is taken over all classical operators of the order zero (with the symbols in $CS^0(X \times \mathbb{R}^n)$) for which $Au \in C^\infty(X)$.*

(3) *Let $u \in \mathcal{D}'(X)$. Then $\mathrm{WF}(u) = \bigcap \operatorname{char}(A)$, where the intersection is taken over all proper classical operators of the order zero for which $Au \in C^\infty(X)$.*

(4) *Let A be a proper ΨDO with the symbol in $CS^m(X \times \mathbb{R}^n)$, $u \in \mathcal{D}'$, or $u \in \mathcal{E}'(X)$. If $a_m(x_0,\xi_0) \neq 0$ and $(x_0,\xi_0) \notin \mathrm{WF}(Au)$, then $(x_0,\xi_0) \notin \mathrm{WF}(u)$. This means*

$$\mathrm{WF}(u) \subset \operatorname{char}(A) \cup \mathrm{WF}(Au). \tag{8.7}$$

The importance of the second assertion is that the definition of $\mathrm{WF}(u)$ makes sense if X is a manifold (see Section 9.1). This theorem gives us the estimate of the propagation of singularities of a pseudodifferential equation.

Theorem 8.12. (Microlocallity of ΨDO's) *Let $u \in \mathcal{D}'$, A be a ΨDO with symbol in $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$, $0 \leq \delta < \rho \leq 1$ and let A be proper or $u \in \mathcal{E}'(X)$. If $(x_0, \xi_0) \notin \mathrm{WF}(Au)$. In other words*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{8.8}$$

Proof. The condition $(x_0, \xi_0) \notin \mathrm{WF}(u)$ is equivalent to the existence of a proper classical ΨDO of order 0 such that $Pu \in C^\infty(X)$ and $\sigma_P = 1 (\mathrm{mod}\, S^{-\infty})$ in a conic neighbourhood of (x_0, ξ_0). Let Q be a proper classical ΨDO of order zero such that $q_0(x_0, \xi_0) \neq 0$ (q_0 is the main symbol of Q) and $\sigma_Q \in S^{-\infty}$ outside some small conic neighbourhood of (x_0, ξ_0) and

$$PQ = Q \text{ and } QP = Q \quad \text{(mod smoothing operators)}.$$

We shall show that $QAu \in C^\infty(X)$ because $\sigma_P = 1$ (mod $S^{-\infty}$) in a conic neighbourhood of (x_0, ξ_0) and $\sigma_{QA} - \sigma_{QAP} \in S^{-\infty}$ in this neighbourhood, and $\sigma_Q \in S^{-\infty}$ out of it. We have that $QA - QAP$ is smoothing. So, it is enough to verify that $QAPu \in C^\infty(X)$. But this follows immediately, because $Pu \in C^\infty(X)$. The fact that $(x_0, \xi_0) \notin \mathrm{WF}(Au)$ follows from the previous theorem. □

From the two previous theorems we have the following theorem.

Theorem 8.13. *If $u \in \mathcal{E}'(X)$ and A is a classical ΨDO with a symbol in $CS^m(X \times \mathbb{R}^n)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \mathrm{char}(A).$$

With the assumption that A is proper, the assertion holds for $u \in \mathcal{D}'(X)$. Specially, if the operator A is elliptic, then $\mathrm{WF}(Au) = \mathrm{WF}(u)$.

8.2. Microfunctions. In this section we shall present the notion of a microfunction by following [**11**]. Microfunctions are the equivalence classes in the space of distributions whose representatives are determined only with their singularities.

First, we shall present some of the basic facts of sheaf theory.

Let X be a topological space, U be an open set in X. Let $\{\mathcal{F}(U)\}_{U \text{open set in} X}$ be a family of vector spaces. For U such that $V \subset U$ there exists a linear mapping $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\mathcal{F}(U)$ is a vector space of the functions on U and

$$\rho_{UU} = \mathrm{id} \text{ and } \rho_{WV} \circ \rho_{VU} = \rho_{WU},$$

for $W \subset V \subset U$.

The family $\{\mathcal{F}(U), \rho_{U,V}, U, V \subset X\}$ is a presheaf. $\mathcal{F}(U)$ is called the set of sections. In the sequel we shall consider the case when $\mathcal{F}(U)$ is a subspace of $\mathcal{F}(V)$ and if $\rho_{U,V}$ is a restriction of $f \in \mathcal{F}(U)$, then $\rho_{U,V} f = f_{|V}$ is a restriction of f to V for $V \subset U$.

Presheaf is a sheaf if the following two conditions are satisfied.

(i) Let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ (all sets are open) and $f \in \mathcal{F}(U)$. If for every $\lambda \in \Lambda$ $f|_{U_\lambda} = 0$, then $f|_U = 0$.

(ii) Let $f_\lambda \in \mathcal{F}(U_\lambda)$ and let for every $\lambda, \mu \in \Lambda$, $f_\lambda = f_\mu$ in $U_\lambda \cap U_\mu$. Then there exists $f \in \mathcal{F}(\bigcup_{\lambda\in\Lambda} U_\lambda)$ such that $f|_{U_\lambda} = f_\lambda$.

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves or sheaves on a topological space X. The family $h = \{h_U\}$ of linear mappings $\mathcal{F}(U) \to \mathcal{G}(U)$ is a (pre)sheaf homomorphism if the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{h_U} & \mathcal{G}(U) \\ \rho^{\mathcal{F}}_{VU}\Big\uparrow & & \Big\uparrow\rho^{\mathcal{G}}_{VU} \\ \mathcal{F}(V) & \xrightarrow[h_V]{} & \mathcal{G}(V) \end{array}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space X. Then $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$ if for every open set U there exists associated inclusion $i_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the family $i = \{i_U\}$ is a presheaf homomorphism. In the same way we define a subsheaf.

Let $\mathcal{F}$ be a (pre)sheaf on X and $x \in X$. Then $\mathcal{F}_x = \operatorname{limind}_{x\in U} \mathcal{F}(U)$ is called a stalk in x. An element in $\mathcal{F}_x$ is called a section germ or a germ of $\mathcal{F}$ in x.

For a presheaf $\mathcal{F}$ one can construct a sheaf $\overline{\mathcal{F}}$ with the same stalks as in $\mathcal{F}$. This sheaf is called the associated sheaf for presheaf $\mathcal{F}$. If a presheaf $\mathcal{F}$ satisfies condition (i) for sheaves, then its associated sheaf is simply defined:

$$\overline{\mathcal{F}}(U) = \operatorname{limind}_{\{U_\lambda\}} \{(s_\lambda) | s_\lambda \in \mathcal{F}(U_\lambda), s_\lambda|_{U_\lambda\cap U_\mu} = s_\mu|_{U_\lambda \cap U_\mu}\},$$

where U_λ are open subsets of U.

Now we shall present the definition of a microfunction.

Let X be an open set in $\mathbb{R}^n$ and $SX = X \times S^{n-1}$. Let U be an open set in SX and CU be a cone generated by U in $X \times \mathbb{R}^n$:

$$CU = \{(x, \lambda\xi) | (x, \xi) \in U, \lambda > 0\}.$$

Let us define

$$O^m(U) = S^m(CU)/S^{-\infty}(CU) \text{ and } O(U) = \bigcup_{m\in\mathbb{N}} O^m(U) = S^\infty(CU)/S^{-\infty}(CU).$$

The elements of these sets are called classes of pseudodifferential operators (of order m) on U. If there are no misunderstandings, we shall omit the word "class".

Let us define

$$\operatorname{Sing}(X) = \mathcal{D}'(X)/C^\infty(X).$$

This is a space of singularities on X. The family $\operatorname{Sing}(X)$, $X \subset \mathbb{R}$, is a sheaf. For $f \in \mathcal{D}'$, the support of f in $\operatorname{Sing}(X)$ is $\operatorname{Sing\,supp} f$ in $\mathcal{D}'$. ΨDO acts as a local operator on the space of singularities, which means that it does nod increase the singular support of the distribution (pseudolocality).

Definition 8.14. Let $f \in \mathcal{D}'(X)$ and $(x, \xi) \in X \times \mathbb{R}^n \setminus \{0\}$. It is said that f is a C^∞-function in (x, ξ) if there exists a proper ΨDO A, elliptic in (x, ξ), such that $Af \in C^\infty(X)$. Singular spectrum of f, $\operatorname{Sing Sp} f$, is the closure of the set of all points (x,ξ) in $X \times \mathbb{R}^n$ in which f is not C^∞.

Definition 8.15. Let $f \in \mathcal{D}'(X)$ and U be an open set in SX. We say that $f \in C^\infty(U)$ if $\operatorname{Sing Sp} f \cap U = \emptyset$. The microfunction defined by f in U is a class of f modulo the space $C^\infty(U)$.

By Section 8.2, one can see that the notions of WF and Sing Sp are equivalent, so in the sequel we shall use only the notion of the wave front set instead of singular spectrum.

We shall define the sheaf of the microfunctions in one dimension.

Recall, $\mathcal{D}'(X), X \subset \mathbb{R}$ is a sheaf of the distributions and $S^0 = \{-1,1\}$ is the unit circle in $\mathbb{R}$.

We say that $u \in \mathcal{D}'(X)$ is microanalytical in $(x,1)$ (resp. $(x,-1)$), $x \in X$ if there exists a neighbourhood U of x and $v \in \mathcal{E}'$, $u = v$ on U such that for every $N \in \mathrm{N}$ there exists a constant C_N such that

$$|\mathcal{F}(v)(\xi)| < C_N(1+\xi^2)^{-N/2},\ \xi > 0 \text{ (resp. } \xi < 0).$$

The point (x,ξ_0) (where $\xi_0 = 1$ or -1) is in $\operatorname{WF} u$ if and only if it is not microanalytical in (x,ξ_0).

Let us define a subsheaf $C^{\infty *}$ of the sheaves $\mathcal{D}'(X) \times \{-1\} \oplus \mathcal{D}'(X) \times \{1\}$ in the following way. *Definition* 8.16. Let

$$C^{\infty *} = \{f \in \mathcal{D}'(X);\ \operatorname{WF}(u) \cap X \times \{-1\} = \emptyset\}$$
$$\oplus \{f \in \mathcal{D}'(X);\ \operatorname{WF}(u) \cap X \times \{1\} = \emptyset\}.$$

The associated sheaf for a presheaf $\mathcal{D}'(X) \times \{-1\} \oplus \mathcal{D}'(X) \times \{1\}/C^{\infty *}$ is denoted by $\mathcal{C}$ and it is called the sheaf of microfunctions.

Intuitively, $f \in \mathcal{D}'(X)$ defines a germ in (x,ξ_0) ($\xi_0 = \pm 1$) modulo germs of any $C^\infty_{(x,\xi_0)}$-function which are microlocal in (x,ξ_0).

The support of a microfunction is a wave front set of a distribution which defines it.

9. Change of variables

Let $(y,\eta) \to (x,\xi)$, $(y,\eta) \in V$, $(x,\xi) \in U$, be a diffeomorphism where U and V are conic regions in $\mathbb{R}^n \times \mathbb{R}^N$ and $\mathbb{R}^{n_1} \times \mathbb{R}^N$, respectively, $x = x(y,\eta)$, $\xi = \xi(y,\eta)$, where $x(y,\eta)$ is positively homogeneous of order 0 and $\xi(y,\eta)$ positively homogeneous of order 1 with respect to η. Let $b(y,\eta) = a(x(y,\eta),\xi(y,\eta))$.

Theorem 9.1. *Let $a(x,\xi) \in S^m_{\rho,\delta}(U)$. Assume that one of conditions*
a) $\rho+\delta = 1$; b) $\rho+\delta \geq 1$ and $x = x(y)$; c) $x = x(y)$, $\xi = \xi(\eta)$;
holds. Then $b \in S^m_{\rho,\delta}(V)$.

Let us consider the oscillating integral

$$I_\phi(au) = \iint e^{i\phi(x,\xi)} a(x,\xi)u(x)\,dx\,d\xi\ [\mathrm{osc}] = \langle A(x), u(x)\rangle,$$

where

$$A(x) = \int e^{i\phi(x,\xi)} a(x,\xi)\, d\xi \text{ [osc]}$$

and where we use the same notions as in the Section 5.

A phase function $\phi(x,\xi)$ is called regular if $d(\partial\phi/\partial\xi_j)$, $j = 1,\dots,N$, is a linearly independent set in C_ϕ, i. e. if the range of the matrix $(\phi_{\xi\xi}\phi_{\xi x})_{N\times(N+n)}$ equals to N.

(We shall use the notation

$$d(\partial\phi/\partial\xi_j) = \sum_{k=1}^{N} \frac{\partial\phi}{\partial\xi_k\partial\xi_j}\, d\xi_k + \sum_{k=1}^{n} \frac{\partial\phi}{\partial x_k\partial\xi_j}\, dx_k,$$

and let us remind that $C_\phi = \{(x,\xi),\ \phi_\xi(x,\xi) = 0\}$ and $R_\phi = X \setminus \mathrm{C}$.)

Let $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and $a = 0$ in a conic neighbourhood of C_ϕ. Then $A \in C^\infty(\mathbb{R}_\phi)$ and one can simply prove that $A \in C^\infty(X)$.

The following lemma is interesting in its own. It is called Hadamard's lemma.

Lemma 9.2. *Let $\phi_1(x,\xi),\dots,\phi_k(x,\xi)$ be in $C^\infty(U)$ and let them be positively homogeneous of order 0 with respect to ξ. Let $d\phi_1,\dots,d\phi_k$ be linearly independent on the set $C = \{(x,\xi) \in U | \phi_j(x,\xi) = 0,\ j = 1,\dots,k\}$ and $a \in S^m_{\rho,\delta}(U)$, $a|_C = 0$ and $\rho + \delta = 1$. Then there exists $a_j(x,\xi) \in S^{m+\delta}_{\rho,\delta}(U)$, $j = 1,\dots,k$ such that*

$$a = \sum_{j=1}^{k} a_j\phi_j. \tag{9.1}$$

If $a(x,\xi)$ has a zero of infinite order on C, then the same holds for all $a_j(x,\xi)$ on C as well.

Theorem 9.3. *Let ϕ be a regular phase function, $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and let one of the following conditions hold:*

1) $\rho > \delta$ and $\rho + \delta = 1$, 2) $\rho > \delta$ and ϕ is linear with respect to ξ.

Then: a) If a has a zero of infinite order in C_ϕ, then $A(x) \in C^\infty(X)$.

b) If $a = 0$ in C_ϕ, then there exists $b \in S^{m-(\rho-\delta)}_{\rho,\delta}(X \times \mathbb{R}^N)$ such that

$$I_\phi(au) = I_\phi(bu) \text{ for every } u \in C_0^\infty(X).$$

Proof. Suppose that 1) holds. If $a|_C = 0$, by previous lemma, we can write

$$a = \sum_{j=1}^{N} a_j\phi_j,\ a_j \in S^{m+\delta}_{\rho,\delta}(U), \tag{9.2}$$

where $\phi_j = \partial\phi/\partial\xi_j$. By using the fact that $\phi_j e^{i\phi} = -i\frac{\partial}{\partial\xi_j}e^{i\phi}$ and integrating by parts, we obtain

$$I_\phi(au) = \sum_{j=1}^{N} I_\phi\Big(i\frac{\partial a_j}{\partial\xi_j}u\Big).$$

Since $\frac{\partial a_j}{\partial \xi_j} \in S^{m+\delta-\rho}_{\rho,\delta}(U)$, we have proved the assertion a). This implies that if a has an infinite order zero in C_ϕ, then b can be chosen such that it has the same property. Thus, we can transfer the assertion a) to the case $a(x,\xi) \in S^{-M}_{\rho,\delta}(X \times \mathbb{R}^N)$, where M is arbitrary large. But, then the integral

$$A(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\xi)} a(x,\xi)\, d\xi$$

absolutely uniformly converges with respect to x, as well as all the integrals which can be obtained from it by differentiating the integrand with respect to x of order up to $l(M)$, where $l(M) \to \infty$ as $M \to \infty$, which implies the smoothness of $A(x)$. □

Let $\kappa : X \to X_1$ (X and X_1 are open), $x = \kappa(t), x \in X_1 \subset \mathbb{R}^n, t \in X \subset \mathbb{R}^n$ be a diffeomorphism. Then the induced mapping, the pull back, $\kappa^* : C^\infty(X_1) \to C^\infty(X)$ is defined by $(\kappa^*\psi)(t) = (\psi \circ \kappa)(t) = \psi(\kappa(t))$.

Let A be a ΨDO on X. We define $A_1 : C_0^\infty(X_1) \to C^\infty(X_1)$ by the diagram

$$\begin{array}{ccc} C_0^\infty(X) & \xrightarrow{\ A\ } & C^\infty(X) \\ {\scriptstyle \kappa^*}\uparrow & & {\scriptstyle \kappa^*}\uparrow\downarrow{\scriptstyle \kappa_1^*} \\ C_0^\infty(X_1) & \xrightarrow[A_1]{} & C^\infty(X_1) \end{array}$$

where $\kappa_1 = \kappa^{-1}$. Then

$$A_1 u = (A(u \circ \kappa)(x)) \circ \kappa_1, \text{ i.e.}$$

$$A_1 u(x) = (2\pi) \iint_{\mathbb{R}^{2n}} e^{i(\kappa_1(x)-p)\xi} a(\kappa_1(x), p, \xi) u(\kappa(p))\, dp\, d\xi.$$

If we change the variables by $p = \kappa_1(y)$, then

$$A_1 u(x) = (2\pi) \iint_{\mathbb{R}^{2n}} e^{i(\kappa_1(x)-\kappa_1(y))\xi} a(\kappa_1(x), \kappa_1(y), \xi) u(y) \left|\frac{\partial \kappa_1}{\partial y}\right| dy\, d\xi, \tag{9.3}$$

where $\partial p/\partial y = \partial \kappa_1/\partial y$ and $|\partial \kappa_1/\partial y|$ is Jacobian.

This means that A_1 is a Fourier integral operator with the phase function $\phi(x,y,\xi) = (\kappa_1(x) - \kappa_1(y))\xi$.

Theorem 9.4. *With the above notation, A_1 is a pseudodifferential operator for $1 - \rho \le \delta < \rho$.*

This will be a special case of the following theorem.

Theorem 9.5. *Let ϕ be a phase function on $X \times X \times \mathbb{R}^n$ such that*
1) $\phi(x,y,\xi)$ is linear with respect to ξ.
2) $\phi'_\xi(x,y,\xi) = 0$ if and only if $x = y$.

Let A be a Fourier integral operator

$$Au(x) = \iint_{\mathbb{R}^{2n}} e^{i\phi(x,y,\theta)} a(x,y,\theta) u(y)\, dy d\theta, \tag{9.4}$$

where $a \in S^m_{\rho,\delta}$ and $1-\rho \le \delta < \rho$. Then A is a pseudodifferential operator with an amplitude in $S^m_{\rho,\delta}$.

We need the following lemma for the proof.

Lemma 9.6. *Let assumptions 1) and 2) of Theorem 9.5 hold. Then there exists a neighbourhood X of the diagonal Δ and a C^∞ mapping $\psi : X \to \mathrm{Gl}\,(n,\mathbb{R})$ (regular matrices of order n on $\mathbb{R}$) such that:*

a) $\phi(x,y,\psi(x,y)\xi) = \langle (x-y), \xi\rangle, (x,y) \in X^2$.

b) $\det \psi(x,x) \cdot \det \phi''_{x\xi}(x,y,\xi)_{|y=x} = 1$.

Proof. By 1),

$$\phi(x,y,\theta) = \sum_{j=1}^{n} \phi_j(x,y)\theta_j.$$

Now, by 2) we have $\phi_j(x,x) = 0$ and if $\phi_j(x,y) = 0$ for $j = 1,\dots,n$, then $x = y$. Note,

$$(\phi'_x, \phi'_y) = \phi'_{x,\theta} = \Big(\sum_{j=1}^{n} \frac{\partial\phi}{\partial x_1}\theta_j, \dots, \sum_{j=1}^{n} \frac{\partial\phi}{\partial x_n}\theta_j, \phi_1, \dots, \phi_n\Big).$$

By differentiating the expression $\phi(x,x,\theta) = 0$ with respect to x, it follows

$$\phi'_x(x,y,\theta)_{|x=y} + \phi'_y(x,y,\theta)_{|x=y} = 0, \quad \text{i.e.}$$

$$\phi'_x(x,x,\theta) = -\phi'_y(x,x,\theta). \tag{9.5}$$

From $\phi'_\theta(x,x,\theta) = 0$ and $\phi'_{x,y,\theta}(x,y,\theta)_{|x=y} \neq 0$ it follows $\phi'_x(x,y,\theta)_{|x=y} \neq 0$. If this is not true, then (9.5) implies $\phi'_y(x,y,\theta)_{|x=y} = 0$, i. e. $\phi'_{x,y,\theta}(x,y,\theta)_{|x=y} = 0$, which gives a contradiction. This means that there exists $k \in \{1,\dots,n\}$ such that $\sum_{j=1}^{n} \frac{\partial \phi_j}{\partial x_k}\theta_{j|x=y} \neq 0$, so

$$\det\Big(\frac{\partial\phi_j}{\partial x_k}(x,y)\Big)\Big|_{x=y} \neq 0. \tag{9.6}$$

By Hadamard's lemma (Lemma 9.2), for close enough x and y we have

$$\phi_j(x,y) = \sum_{k=1}^{n} \phi_{kj}(x,y)(x_k - y_k),$$

where $\phi_{kj} \in C^\infty(X'), X'$ is some neighbourhood of the diagonal in $X \times X$. We also have

$$\phi_{kj}(x,x) = \frac{\partial\phi_j(x,y)}{\partial x_k}\Big|_{x=y}. \tag{9.7}$$

Denote by $\phi(x,y)$ the matrix $(\phi_{kj}(x,y))$. From (9.6) and (9.7) it follows that there exists a neighbourhood Ω of the diagonal in $X \times X$ such that $\det\phi(x,y) \neq 0$ for $(x,y) \in \Omega$. Let

$$\psi(x,y) = \phi(x,y)^{-1} \quad \text{(the inverse of } \phi) \tag{9.8}$$

Since

$$\phi(x,y,\theta) = \sum_{j,k=1}^{n} \phi_{kj}(x,y)\theta_j(x_k - y_k) = \langle (x-y), \phi(x,y)\theta\rangle,$$

by putting $\phi(x,y)\theta = \xi$ we obtain a), while b) follows from (9.7) and (9.8). □

Proof of Theorem 9.5. We assume that $a(x,y,\theta)$ equals 0 for $(x,y) \in X \times X \setminus \Omega'$, where $\Omega' \subset \Omega$ and Ω' is a neighbourhood of Δ. By putting $\theta = \phi(x,y)^{-1}\xi$ in (9.4) we obtain

$$Au(x) = \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\psi(x,y)\xi)|\det\psi(x,y)|u(y)\,dy\,d\xi.$$

From Theorem 9.1 it follows that $a_1(x,y,\xi) = a(x,y,\psi(x,y)\xi)$ is in $S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$.

9.1. Pseudodifferential operators on C^∞-manifolds. We will give the definition of pseudodifferential operators on a manifold, but before that we shall recall the definitions of the theory of generalized functions on a manifold. Let us remind that Hausdorff topological space M is locally Euclidean of dimension n if every point in M has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

If φ is a homomorphism of an open set $U \subset M$ on an open subset of $\mathbb{R}^n$, φ is called the coordinate mapping and (U,φ) is called the coordinate system or coordinate section. Recall, a differentiable structure $\mathcal{F}$ of the class C^k, $k \in [1,\infty]$, on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha,\ \varphi_\alpha), \alpha \in A\}$ which satisfies:

(i)] $\bigcup_{\alpha\in A} U_\alpha = M$.

(ii) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is of the class C^k in $\varphi_\beta(U_\alpha \cap U_\beta)$ for every $\alpha,\beta \in A$.

(iii) The collection $\mathcal{F}$ is maximal with respect to (ii) which means that if (U,φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are of the class C^k for every $\alpha \in A$, then $(U,\varphi) \in \mathcal{F}$.

If $\mathcal{F}_0 = \{(U_\alpha,\ \varphi_\alpha), \alpha \in A\}$ is an arbitrary collection of coordinate systems satisfying (i) and (ii), then there exists a unique differentiable structure $\mathcal{F}$ containing $\mathcal{F}_0$. $\mathcal{F}_0$ is called the atlas of a manifold M.

In the sequel we shall consider only C^∞-manifolds. Let M and N be C^∞-manifolds.

Let $O \subset M$ be open. Then $F : O \to \mathbb{R}$ is a C^∞-function on O, $(f \in C^\infty(O))$ if $f \circ \varphi^{-1}|_{\varphi(U\cap O)}$ is a C^∞-function for every coordinate section (U,φ).

A mapping $\psi : M \to N$ is of the class C^∞ if for every two coordinate sections (U,φ) on M and (U_1,φ_1) on N, $\varphi_1 \circ \psi \circ \varphi^{-1}|_{\varphi(U)}$ is a C^∞-function.

The important construction in the analysis on manifolds is the partition of unity. Let M be a manifold and $\mathcal{U} = \{U_\alpha,\ \alpha \in A\}$ be a cover of M. Then there exists a C^∞ partition of unity $\{\varphi_i,\ i \in \mathbb{N}\}$ corresponding to the cover $\mathcal{U}$ such that $\operatorname{supp}\varphi_i$ is compact for every $i \in \mathbb{N}$ and $\operatorname{supp}\varphi_i \subset U_\alpha$ for some $\alpha \in$A.

If $v \in C_0^\infty(\tilde{U})$ (where $\tilde{U} = \varphi(U)$), then we define

$$u = \begin{cases} v \circ \varphi, & \text{in } U, \\ 0, & \text{otherwise.} \end{cases}$$

The definition is the same if $v \in C^k(\tilde{U})$ or $v \in L^p(\tilde{U})$. We shall use the notation $u = v \circ \varphi$.

Let $u \in C^k(M)$ and $u_k = u \circ \varphi_k^{-1}$, where (U_k, φ_k) is an arbitrary coordinate section. There holds:

(a) $u = u_k \circ \varphi_k = u_{k'} \circ \varphi_{k'}$ on $U_k \cap U_{k'}$.

(b) $u_k = u_{k'} \circ (\varphi_{k'} \circ \varphi_k^{-1})$, which is denoted by $(\varphi_{k'} \circ \varphi_k^{-1})^* u_{k'} = u_k$.

Conversely, if (b) holds for arbitrary sections (U_k, φ_k) and $(U_{k'}, \varphi_{k'})$, then there exists a unique function $u \in C^k(M)$ satisfying (a).

Definition 9.7. Let $\mathcal{F} = \{(U_k,\ \varphi_k), k \in A\}$ be a differentiable structure of a manifold M. If there exists a distribution u_k in $\mathcal{D}'(\varphi_k(U_k))$ for every coordinate section (U_k, φ_k) and if

(c) $u_k = u_{k'} \circ (\varphi_{k'} \circ \varphi_k^{-1})$ on $\varphi_k(U_k \cap U_{k'})$,

then $\{u_k,\ k \in A\}$ is a distribution in M. We shall denote it by $u \in \mathcal{D}'(M)$, and that is in fact the notation for the family $\{u_k,\ k \in A\}$. We shall use the notation $u_k = u \circ \varphi_k^{-1}$.

This definition generalizes the definition of a function in $C^k(M)$. The proof of the next theorem is omitted.

Theorem 9.8. *Let $\mathcal{F} = \{(U_k,\ \varphi_k),\ k \in A_0\}$ be an atlas for M. If $\{u_k,\ k \in A_0\}$ is a family of distributions in $\mathcal{D}'(\varphi_k(U_k))$ satisfying (c) for every k, k' in A_0, Then there exist one and only one distribution $u \in \mathcal{D}'(M)$ such that*

$$u \circ \varphi_k^{-1} = u_k \text{ for every } k \in A_0.$$

There appears a natural question: Why one can not define the distribution on a manifold M as a continuous linear function on $C_0^\infty(M)$? The reason is that there does not exist an invariant procedure for the definition of the integral $\int f\phi, f \in C(M), \phi \in C_0^\infty(M)$ such that f can be identified with a continuous linear functional.

Let u be a continuous linear functional on $C_0^\infty(M)$. For every (U_k, φ_k), by

$$u_k(\phi) = u(\phi \circ \varphi_k), \phi \in C_0^\infty(\tilde{U}_k)$$

is defined an element in $\mathcal{D}'(\varphi_k(U_k))$. But $\{u_k,\ k \in A\}$ does not satisfy condition (c).

Let $\phi \in C_0^\infty(\varphi_k(U_k \cap U_{k'}))$. Then

$$\langle u_k, \phi \rangle = \langle u, \phi \circ \varphi_k \rangle = \langle u, \phi \circ \varphi_k \circ \varphi_{k'}^{-1} \circ \varphi_{k'} \rangle = \langle u_{k'}, \phi \circ \varphi_k \circ \varphi_{k'}^{-1} \rangle.$$

By the change of variables: $t = \varphi_k \circ \varphi_{k'}^{-1}(x)$ we obtain

$$\langle u_k(t), \phi(t)\rangle = \langle u_{k'}(x), \phi\circ\varphi_k\circ\varphi_{k'}^{-1}(x)\rangle = \Big\langle u_{k'}(\varphi_{k'}\circ\varphi_k^{-1}(t)), \phi(t)\Big|\frac{\partial\varphi_{k'}\circ\varphi_k^{-1}(x)}{\partial t}\Big|\Big\rangle,$$

i.e.

$$u_k = \Big|\frac{\partial\varphi_{k'}\circ\varphi_k^{-1}(x)}{\partial t}\Big| u_{k'}\circ\varphi_{k'}\circ\varphi_k^{-1}. \tag{9.9}$$

This is similar to condition (c), but now we have an additional multiplication by Jacoby's determinant which equals $|\partial\varphi_k\circ\varphi_{k'}^{-1}(x)/\partial t|$.

A family $\{u_k,\ k\in A\}$ of elements in $\mathcal{D}'(\varphi_k(U_k))$ satisfying (9.9) is called a distributional density.

In the same way we define a C^k-density by (9.9).

If a is a strictly positive C^∞-density on M, and $u\in\mathcal{D}'(M)$, then au is the distributional density, and the mapping $u\to au$ is a bijection of the space of the distributions to the space of distributional densities.

Let u be a distributional density and $x=\varphi(y)$. There holds

$$\begin{aligned}\langle\varphi_* u(x), \psi(x)\rangle &= \langle u(y), \varphi^*\psi(y)\rangle = \langle u(y), \psi(\varphi(y))\rangle \\ &= \langle u(\varphi^{-1}(x)), |J|\psi(x)\rangle,\end{aligned} \tag{9.10}$$

where $|J|$ is a Jacoby's determinant. This formula will be useful for the definition of a pseudodifferential operator on a manifold which acts on distributions with compact supports.

Let A be a linear operator, $A: C_0^\infty(M)\to C^\infty(M)$, where M is an n-dimensional C^∞-manifold. Let (U,φ) be a coordinate section of the manifolds. Then the commutative diagram

$$\begin{array}{ccc} C_0^\infty(U) & \xrightarrow{A} & C^\infty(U) \\ {\scriptstyle\varphi^*}\uparrow & & \uparrow{\scriptstyle\varphi^*} \\ C_0^\infty(\tilde U) & \xrightarrow[A_1]{} & C^\infty(\tilde U) \end{array}$$

uniquely defines the operator A_1.

Definition 9.9. $A: C_0^\infty(M)\to C^\infty(M)$ is a ΨDO on M if for every coordinate section the operator A_1 defined above, is a ΨDO on U_1.

By using (9.10) and the analogous procedure as in the case of ordinary ΨDO's and like in the previous definition we have that A is a ΨDO on a manifold if A_1 is a ΨDO on U, where A_1 is defined by the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}'(U) & \xrightarrow{A} & \mathcal{D}'(U) \\ {\scriptstyle\varphi_*^{-1}}\uparrow & & \downarrow{\scriptstyle\varphi_*} \\ \mathcal{E}'(\tilde U) & \xrightarrow[A_1]{} & \mathcal{D}'(\tilde U) \end{array}$$

Theorem 9.5 ensures that a ΨDO on an open set $X \subset \mathbb{R}^n$ can be considered as ΨDO on manifold X. Theorem 9.1 shows that $S^m_{\rho,\delta}(T^*M)$ is well defined for $1-\rho \le \delta < \rho$.

Part II. COLOMBEAU GENERALIZED FUNCTIONS AND ΨDO

In this part we present the basic concept of the pseudodifferential calculus in the frame of Colombeau's generalized functions. It is developed in **[16]**, **[17]**, **[18]**, **[12]** as well as by Oberguggenberger **[14]**.

10. Basic notions

We recall in this section the notation and notions in Colombeau's theory.

$\mathcal{A}_0(\mathbb{R}^n)$ denotes the set of the functions ϕ in $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(t)\,dt = 1$, $\mathcal{A}_q(\mathbb{R}^n) = \{\phi \in \mathcal{A}_0,\ \int_{\mathbb{R}^n} t^i\phi(t)\,dt = 0,\ 0 < |i| \le q\}$, $q \in \mathbb{N}$, where $t^i = t_1^{i_1} \cdots t_n^{i_n}$.

Obviously, if $\phi \in \mathcal{A}_q$, $q \in \mathbb{N}_0$, then for every $\varepsilon > 0$, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, $x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$.

If $\phi \in \mathcal{A}_0$, then it's support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x|,\ \phi(x) \neq 0\}.$$

In the sequel we assume that ϕ in $\mathcal{A}_0$ has the support number equals one, $d(\phi) = 1$, i.e. $\operatorname{supp}\phi \subset B(0,1)$.

Denote by $\mathcal{E}[\Omega]$ the set of the functions $R : \mathcal{A}_0 \times \Omega \to \mathbb{C}$, $(\phi, x) \mapsto R(\phi, x)$, which are in $C^\infty(\Omega)$ for every fixed ϕ. Note that for any $f \in C^\infty(\Omega)$, the mapping $(\phi, x) \mapsto f(x)$, $(\phi, x) \in \mathcal{A}_0 \times \Omega$, defines an element in $\mathcal{E}[\Omega]$ which does not depend on ϕ.

The space of functions $R : \mathcal{A}_0 \to \mathbb{C}$ (resp.$\mathbb{R}$) is denoted by $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$). It is an algebra and it is subalgebra of $\mathcal{E}[\Omega]$ in the sense of natural identification of $R \in \mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$), $R : (\phi, x) \mapsto C(\phi) \in \mathbb{C}$ (resp. $\mathbb{R}$).

A function $R \in \mathcal{E}[\Omega]$ is called moderate if for every $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}_0^n$ there exists $N \in \mathbb{N}_0$ such that, for every $\phi \in \mathcal{A}_N$, there exist $\eta > 0$ and $C > 0$ such that

$$|\partial^\alpha R(\phi_\varepsilon, x)| \le C\varepsilon^{-N},\ x \in K,\ 0 < \varepsilon < \eta.$$

The set of all moderate elements is denoted by $\mathcal{E}_M[\Omega]$.

The set of all moderate elements in $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$), denoted by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$), consists of elements $R \in \mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$) which satisfy: There exists $N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $\eta > 0$ and $C > 0$ such that

$$|R(\phi_\varepsilon)| < C\varepsilon^{-N},\ \ 0 < \varepsilon < \eta.$$

Clearly $\mathcal{E}_M[\Omega]$ and $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$) are associative subalgebras of $\mathcal{E}[\Omega]$ and $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$).

Denote by Γ the set of sequences $\{a_q\}$ of positive numbers which strictly increase to infinity.

An element $R \in \mathcal{E}_M[\Omega]$ is called a null element if for every $K \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$ there exists $\eta > 0$ and $C > 0$ such that

$$|\partial^\alpha R(\phi_\varepsilon, x)| \leq C\varepsilon^{a_q - N}, \ x \in K, \ 0 < \varepsilon < \eta.$$

The space of null elements is denoted by $\mathcal{N}[\Omega]$.

The space of null elements of $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$) denoted by $\mathcal{N}_0(\mathbb{C})$ (resp. $\mathcal{N}_0(\mathbb{R})$) consists of all the elements $R \in \mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$) with the following property: There exist $N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$ there exists $\eta > 0$ and $C > 0$ such that

$$|R(\phi_\varepsilon)| \leq C\varepsilon^{a_q - N}, \ \ 0 < \varepsilon < \eta.$$

Clearly, $\mathcal{N}[\Omega]$ and $\mathcal{N}_0(\mathbb{C})$ (resp. $\mathcal{N}_0(\mathbb{R})$) are ideals of $\mathcal{E}_M[\Omega]$ and $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

The spaces of generalized functions on Ω, $\mathcal{G}(\Omega)$, generalized complex numbers $\overline{\mathbb{C}}$ and generalized real numbers $\overline{\mathbb{R}}$ are defined by

$$\mathcal{G}(\Omega) = \mathcal{E}_M[\Omega]/\mathcal{N}[\Omega], \ \ \overline{\mathbb{C}} = \mathcal{E}_{0M}(\mathbb{C})/\mathcal{N}_0(\mathbb{C}), \ \ \overline{\mathbb{R}} = \mathcal{E}_{0M}(\mathbb{R})/\mathcal{N}_0(\mathbb{R}).$$

$\Omega \mapsto \mathcal{G}(\Omega)$ is a sheaf. This implies the natural definition of the support, $\text{supp}_g\, \mathcal{G}$.

Note that $\overline{\mathbb{C}}$ and $\overline{\mathbb{R}}$ are not fields and $\overline{\mathbb{C}} = \overline{\mathbb{R}} + i\overline{\mathbb{R}}$. Because of that, from now on, we shall use the symbols $\mathcal{E}_{0M} = \mathcal{E}_{0M}(\mathbb{C})$ and $\mathcal{N}_0 = \mathcal{N}_0(\mathbb{C})$.

The classical complex numbers are embedded in $\overline{\mathbb{C}}$ by

$$\mathbb{C} \ni z \mapsto R(\phi) = z, \ \phi \in \mathcal{A}_0.$$

Let $g \in \mathcal{D}'$. Then $\text{Cd}(g) \in \mathcal{G}$ is given by the representative $g * \check{\phi}_\varepsilon$, where $\check{\phi}(y) = \phi(-y)$.

$\mathcal{E}_t$ is the set of all elements $G \in \mathcal{E}$ with the following property: For every $\beta \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}$, $a \in \mathbb{R}$ and $\gamma > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G(\phi_\varepsilon, x)| \leq C(1 + |x|)^\gamma \varepsilon^a, \text{ for } \varepsilon < \eta, \ x \in \mathbb{R}^n.$$

$\mathcal{N}_t$ is the set of elements $G \in \mathcal{E}_t$ with the property: For every $\beta \in \mathbb{N}_0^n$ there exist $\gamma > 0$, $N \in \mathbb{N}$ and $g \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \geq N$, there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G(\phi_\varepsilon, x)| \leq C(1 + |x|)^\gamma \varepsilon^{g(q) - N}, \text{for} \varepsilon < \eta, \ x \in \mathbb{R}^n.$$

It is an ideal of $\mathcal{E}_t$.

Colombeau's space of tempered generalized functions is defined by $\mathcal{G}_t = \mathcal{E}_t/\mathcal{N}_t$.

It is said that $G \in \mathcal{G}$ ($G \in \mathcal{G}_t$) is equal to $H \in \mathcal{G}$ ($H \in \mathcal{G}_t$) in generalized distribution sense, $G = H(g.d.)$ (in generalized tempered distribution sense, $G = H(g.t.d.)$), if for every $\psi \in \mathcal{D}$ ($\psi \in \mathcal{S}$)

$$\langle G - H, \psi \rangle = 0 \text{ in } \overline{\mathbb{C}}.$$

$A \in \overline{\mathbb{C}}$ is associated to $c \in \mathbb{C}$, $A \approx c$, if there exists $N \in \mathbb{N}$ such that $\lim_{\varepsilon \to 0} A_{\phi,\varepsilon} = c$ for every $\phi \in \mathcal{A}_N$.

$G \in \mathcal{G}$ is associated to $H \in \mathcal{G}$, $G \approx H$, if for every $\psi \in \mathcal{D}$ there exists $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$

$$\lim_{\varepsilon \to 0} \langle G(\phi_\varepsilon, \cdot) - H(\phi_\varepsilon, \cdot), \psi \rangle = 0.$$

The definition of t-association is obtained if one takes $\psi \in \mathcal{S}$ instead of $\psi \in \mathcal{D}$ above.

For the microlocal analysis of Colombeau's generalized functions we shall define a subalgebra $\mathcal{G}^\infty(\Omega)$ by following Oberguggenberger [**13**].

$\mathcal{G}^\infty(\Omega)$ is the set of all $G \in \mathcal{G}(\Omega)$ which have representatives $G(\phi, x) \in \mathcal{E}_M[\Omega]$ with the property: For every $K \subset\subset \Omega$ there is $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0$, there is $M \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there are $C > 0$ and $\eta > 0$ such that

$$\sup_{x \in K} |G^{(\alpha)}(\phi_\varepsilon, x)| \le C\varepsilon^{-N}, \ 0 < \varepsilon < \eta.$$

One can prove that $\mathcal{G}^\infty(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$.

Proposition 10.1. *1.* $\mathcal{G}^\infty(\Omega) \bigcap \mathcal{D}'(\Omega) = C^\infty(\Omega)$ *([**13**]).*

2. $\mathcal{G}^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ *is a sheaf.*

3. $G \in \mathcal{G}(\Omega)$ *is* $\mathcal{G}^\infty$ *in* $\Omega_1 \subset \Omega$ *if it is* $\mathcal{G}^\infty$ *at every point of* Ω_1.

The last assertion means: For every $x \in \Omega_1$ *there are open sets* U *and* V *such that*

$$x \in U, \ \bar{U} \subset\subset V, \ V \subset\subset \Omega_1$$

and a function $\psi \in C_0^\infty(V)$, $\psi \equiv 1$ *on* $\bar{U}$, *such that* $\psi G \in \mathcal{G}^\infty(\Omega_1)$.

Definition 10.2. Let $G \in \mathcal{G}(\Omega)$. The complement of the largest open set of Ω in which G is $\mathcal{G}^\infty$ is called the singular support of G, $\operatorname{Sing\,supp}_g G$.

Recall, it is said that G is $\mathcal{G}^\infty$ in $\Omega_1 \subset \Omega$ if $G|_{\Omega_1} \in \mathcal{G}^\infty(\Omega_1)$.

The set $\operatorname{Sing\,supp}_g G$, $G \in \mathcal{G}(\Omega)$ is defined to be the complement of the largest open set $\Omega' \subset \Omega$ such that $G|_{\Omega'} = 0$.

From Proposition 10.1 we have that for distributions

$$\operatorname{Sing\,supp} f = \operatorname{Sing\,supp}_g \operatorname{Cd} f, \ f \in \mathcal{D}'(\Omega).$$

Denote by $\mathcal{G}_c(\Omega)$ the set of all elements in $\mathcal{G}(\Omega)$ which have compact supports.

If $G \in \mathcal{G}_c(\Omega)$, then it belongs to $\mathcal{G}_c(\mathbb{R}^n)$ by defining

$$G(\phi, x) = 0,\ \phi \in \mathcal{A}_0,\ x \in \mathbb{R}^n \setminus \operatorname{supp}_g G$$

(and by using the sheaf property of $\mathcal{G}(\mathbb{R}^n)$).

For every ψ_1 and ψ_2 in $C_0^\infty(\mathbb{R}^n)$, where ψ_1 and ψ_2 equals one on corresponding neighborhoods of $\operatorname{supp}_g G$,

$$\psi_1(\cdot)(G(\phi_\varepsilon, \cdot) + \mathcal{N}[\mathbb{R}^n]) \text{ and } \psi_2(\cdot)(G(\phi_\varepsilon, \cdot) + \mathcal{N}[\mathbb{R}^n])$$

determine the same element in $\mathcal{G}_t(\mathbb{R}^n)$. They are equal in $\mathcal{G}(\mathbb{R}^n)$.

Thus the mapping $\mathcal{M} : \mathcal{G}_c(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)$, symbolically written by

$$G(\phi_\varepsilon, \cdot) + \mathcal{N}(\mathbb{R}^n) \mapsto \psi(\cdot)(G(\phi_\varepsilon, \cdot) + \mathcal{N}[\mathbb{R}^n]) + \mathcal{N}_t[\mathbb{R}^n],$$

is linear, multiplicative and injective, which will enable us to consider $\mathcal{G}_c(\mathbb{R}^n)$ as a subspace of $\mathcal{G}_t(\mathbb{R}^n)$.

If $G \in \mathcal{G}_t(\mathbb{R}^n)$, then $G|_\omega \in \mathcal{G}(\omega)$ is defined by a representative $G(\phi_\varepsilon, \cdot)|_\omega$, where $G(\phi_\varepsilon, \cdot)$ is a representative of G.

If $f \in S'(\mathbb{R}^n)$ then $\mathrm{Cd}_t\, f$ denotes the corresponding element in $\mathcal{G}_t(\mathbb{R}^n)$ defined by

$$(f * \check{\phi})(x) + G(\phi, x), \text{ where } G(\phi, x) \in \mathcal{N}_t[\mathbb{R}^n].$$

Let $G \in \mathcal{G}_t(\mathbb{R}^n)$ and ω be an open set. If $G|_\omega$ determines an element in $\mathcal{G}^\infty(\omega)$, then we say that it is $\mathcal{G}_t^\infty$ in ω, where we use this notation to emphasize that the generalized function in consideration is from $\mathcal{G}_t(\mathbb{R}^n)$.

Let $G \in \mathcal{G}(\Omega)$ and if $G|_\omega \in \mathcal{G}^\infty(\omega)$ where ω is a bounded open set in Ω. Then $\mathcal{M}(\kappa G)$, where $\kappa \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $\overline{\omega}$, is $\mathcal{G}_t^\infty$ in ω.

(Recall $\mathcal{M}(\kappa G) = \kappa_1(x)(\kappa(x)G(\phi, x) + \mathcal{N}[\Omega]) + \mathcal{N}_t[\mathbb{R}^n]$.)

Thus the singular support of $G \in \mathcal{G}_t(\mathbb{R}^n)$ is the singular support of G considered as an element of $\mathcal{G}(\mathbb{R}^n)$. We define the subalgebra $\mathcal{G}_t^\infty(\mathbb{R}^n)$ as follows.

$\mathcal{G}_t^\infty(\mathbb{R}^n)$ is the set of all $U \in \mathcal{G}_t(\mathbb{R}^n)$ which have a representative $G(\phi, x) \in \mathcal{E}_t[\mathbb{R}^n]$ with the property: There is $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^n$ there is $M \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_M$ there are $C > 0$ and $\eta > 0$ such that

$$|G^{(\alpha)}(\phi, x)| \leq C(1 + |x|)^M \varepsilon^{-N},\ 0 < \varepsilon < \eta.$$

Note, if $G \in \mathcal{G}_t^\infty(\mathbb{R}^n)$ then $G \in \mathcal{G}^\infty(\mathbb{R}^n)$.

Let $\mu \in C_0^\infty(\mathbb{R}^n)$ such that $\mu = 1$ in some neighborhood of zero. Then $\mu_\varepsilon(x) = \mu(x\varepsilon)$, $\varepsilon \in (0, 1)$, is called a unit net.

Let μ_ε be a unit net, B a measurable subset of $\mathbb{R}^n$ and $G \in \mathcal{G}_t$. Then we define

$$\int_B^{t,\mu} G(x)\, dx \in \overline{\mathbb{C}} \text{ by its representative } \int_B G(\phi_\varepsilon, x)\mu_\varepsilon(x)\, dx \in \mathcal{E}_{0,M}.$$

If $B = \mathbb{R}^n$ then the symbol $\int^{t,\mu}$ is used. One can easily prove that $G(\phi_\varepsilon, \cdot) \in \mathcal{N}_t$ implies $\int_B G(\phi_\varepsilon, x)\mu_\varepsilon(x)\,dx \in \mathbb{C}_0$. Thus the definition of the integral in $\mathcal{G}_t$ makes sense.

11. Pseudodifferential operators

We will give the simplest definitions of an amplitude of type $\rho = 1$, $\sigma = 0$.

Definition 11.1. The set of amplitudes $S_g^m = S_g^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,1])$, $m \in \mathbb{R}$, is the set of functions $a(x,y,\xi,\varepsilon)$, smooth in $(x,y,\xi) \in (\mathbb{R}^n)^3$ for every $\varepsilon \in (0,1]$, continuous in $\varepsilon \in (0,1]$ for every $(x,y,\xi) \in (\mathbb{R}^n)^3$, such that there exists $N \in \mathbb{N}_0$ such that for every $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exists $C = C(\alpha,\beta,\gamma) > 0$ such that

$$(11.1) \qquad |\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi,\varepsilon)| \le \frac{C}{\varepsilon^N}(1+|\xi|)^{m-|\alpha|},\ (x,y,\xi) \in (\mathbb{R}^n)^3, \varepsilon \in (0,1].$$

If there exists $N \in \mathbb{N}_0$ such that for every $m \in \mathbb{R}$ and every $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exists $C = C(\alpha,\beta,\gamma,m) > 0$ such that (11.1) holds, then $a(x,y,\xi,\varepsilon) \in S_g^{-\infty}$.

The following set of amplitudes is suitable for the calculus in the frame of Colombeau's generalized functions.

Definition 11.2. The set of amplitudes $S_{gt}^m = S_{gt}^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,1])$, $m \in \mathbb{R}$, is the set of functions a with the same regularity properties as in Definition 11.1 but which satisfies the following:

There exists $N \in \mathbb{N}_0$ such that for every $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exist $C = C(\alpha,\beta,\gamma)$ and $k = k(\alpha,\beta,\gamma)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi,\varepsilon)| \le \frac{C}{\varepsilon^N}(1+|x|)^k(1+|\xi|)^{m-|\alpha|},\ (x,y,\xi) \in (\mathbb{R}^n)^3, \varepsilon \in (0,1].$$

Elements of $S_{gt}^{-\infty}$ are appropriately defined. In this case constants C and k depend also on m.

We will use Definition 11.1 in Section 11 and later in order to avoid a lot of technical difficulties which may appear.

Definition 11.3. Let $a \in S_{gt}^m$, $r \in \mathbb{N}_0$ and $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$ be unit nets from $C_0^\infty(\mathbb{R}_\xi^n)$ and $C_0^\infty(\mathbb{R}_y^n)$, respectively. Let $G \in \mathcal{G}_t(\mathbb{R}^n)$.

We define $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ on $\mathcal{G}_t(\mathbb{R}^n)$ by

$$(11.2) \quad A_{\mu_2 r}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} \frac{e^{i\langle x-y,\xi\rangle}}{(1+|\xi|^2)^{[(|m|+n)/2]+r}}(1-\Delta_y)^{[(|m|+n)/2]+r} \times (a(x,y,\xi,\varepsilon)\mu_{2\varepsilon}(y)G(\phi_\varepsilon,y)\,dy\,d\xi,\ (\phi,x) \in \mathcal{A}_0 \times \mathbb{R}$$

where $[(|m|+n)/2]$ is the integer part of $(|m|+n)/2$, and by

$$(11.3) \quad A_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y)G(\phi_\varepsilon,y)\,dy\,d\xi.$$

Note, if $m < -n$ then we take $r = 0$ in (11.2) and

$$A_{\mu_2}G(\phi_\varepsilon, x) = A_{\mu_2}G(\phi_\varepsilon, x)$$
$$= \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{2\varepsilon}(y)G(\phi_\varepsilon, y)\,dy\,d\xi.$$

Theorem 11.4. *1. $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ are linear mappings from $\mathcal{G}_t(\mathbb{R}^n)$ to $\mathcal{G}_t(\mathbb{R}^n)$.*

2. For every $\mu_{1\varepsilon}(\xi), \mu_{2\varepsilon}(y), r$ and $G \in \mathcal{G}_t(\mathbb{R}^n)$, $A_{\mu_2 r}G$ and $A_{\mu_1\mu_2}G$ are equal in (g.t.d.) sense.

3. For every $\mu_{1\varepsilon}(\xi), \tilde{\mu}_{1\varepsilon}(\xi), \mu_{2\varepsilon}(y), \tilde{\mu}_{2\varepsilon}(y)$ and $G \in \mathcal{G}_t(\mathbb{R}^n)$, $A_{\mu_1\mu_2}G$ and $A_{\tilde{\mu}_1\tilde{\mu}_2}G$ are equal in (g.t.d.) sense.

Proof. The proof of 1 is obvious. Note that (11.3) is equal to

$$A_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} \frac{e^{i\langle x-y,\xi\rangle}}{(1+|\xi|^2)^{[(|m|+n)/2]+r}}(1-\Delta_y)^{[(|m|+n)/2]+r}$$
$$\times (a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y)G(\phi_\varepsilon, y))\,dy\,d\xi.$$

Since the proof of 3 is typical for the calculus we will collect here the equalities and the estimations which will be used in the sequel.

There holds

$$(1-\Delta_x)^s e^{i\langle x-y,\xi\rangle} = (1+|\xi^2|)^s e^{i\langle x-y,\xi\rangle},$$
$$\frac{(1-\Delta_\xi)^s(1-\Delta_x)^p e^{i\langle x-y,\xi\rangle}}{(1+|y^2|)^s(1+|\xi^2|)^p} = e^{i\langle x-y,\xi\rangle}, \tag{11.5}$$
$$(1-\Delta_x)^s e^{i\langle x-y,\xi\rangle} = (1+\Delta_y)^s e^{i\langle x-y,\xi\rangle}.$$

A unit net $\mu_\varepsilon(\xi)$, $\varepsilon \in (0,1]$, where $\mu(\xi) = 1$, $|\xi| \leq A$, $\mu(\xi) = 0$, $|\xi| \geq B > A$, satisfies the following estimation. Let $\alpha \in \mathbb{N}_0^n$. Since

$$|\partial^\alpha \mu_\varepsilon(\xi)| = |\varepsilon^{|\alpha|}\partial^\alpha \mu(\varepsilon\xi)|, \quad A \leq |\varepsilon\xi| \leq B$$

it follows

$$|\partial^\alpha \mu_\varepsilon(\xi)| \leq C_\alpha \varepsilon^{|\alpha|} \leq \frac{\beta^{|\alpha|}C_\alpha}{|\xi|^{|\alpha|}}, \; |\xi| > A/\varepsilon, \; |\partial^\alpha \mu_{\varepsilon(\xi)}| = 0, \; |\xi| \leq A/\varepsilon. \tag{11.6}$$

If μ_ε and $\tilde{\mu}_\varepsilon$ are unit nets determined by different functions μ_1 and μ_2 then, by the above notation,

$$|\mu_\varepsilon(\xi) - \tilde{\mu}_\varepsilon(\xi)| = 0, \;\text{ for } |\xi| \leq \frac{\min\{A,\tilde{A}\}}{\varepsilon} \text{ and } |\xi| \geq \frac{\max\{B,\tilde{B}\}}{\varepsilon}. \tag{11.7}$$

Now, we will give the proof of 3. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $I = \int_{\mathbb{R}}(A_{\mu_1\mu_2}G(\phi_\varepsilon, x) - A_{\tilde{\mu}_1\tilde{\mu}_2}G(\phi_\varepsilon, x))\psi(x)\,dx$. By (11.5), for enough large s and p, we have

$$I = \iiint_{\mathbb{R}^{3n}} e^{i\langle x-y,\xi\rangle}\frac{1}{(1+|y|^2)^s}(1-\Delta_\xi)^s(1-\Delta_x)^p$$
$$\times \left(\frac{a(x,y,\xi,\varepsilon)}{(1+|\xi|^2)} + G(\phi_\varepsilon, y)\left(\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y) - \tilde{\mu}_{1\varepsilon}(\xi)\tilde{\mu}_{2\varepsilon}(y)\right)\psi(x)\right)\,dx\,dy\,dz.$$

Note that the differentiation with respect to y is changed by differentiation with respect to x. By using the identity

$$\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y) - \tilde{\mu}_{1\varepsilon}(\xi)\tilde{\mu}_{2\varepsilon}(y) = (\mu_{1\varepsilon}(\xi) - \tilde{\mu}_{1\varepsilon}(\xi))\mu_{2\varepsilon}(y) + \tilde{\mu}_{1\varepsilon}(\xi)(\mu_{2\varepsilon}(y) - \tilde{\mu}_{2\varepsilon}(y))$$

we have that I is smaller than the linear combination of factors of the form

$$\iiint_{\mathbf{R}^{3n}} \frac{1}{(1+|y|^2)^s}\left|\partial_x^q\partial_\xi^r \frac{a(x,y,\xi)}{(1+|\xi|^2)^p}\right| \cdot |\partial_\xi^t(\mu_{1\varepsilon}(\xi) - \tilde{\mu}_{1\varepsilon}(\xi))| \\ \times |\mu_{2\varepsilon}(y)| \cdot |G(\phi_\varepsilon, y)| \cdot |\partial^h\psi(x)|\, dx\, dy\, dz$$

$$\iiint_{\mathbf{R}^{3}} \frac{1}{(1+|y|^2)^s}\left|\partial_x^q\partial_\xi^r \frac{a(x,y,\xi)}{(1+|\xi|^2)^p}\right| \cdot |\partial_\xi^t\tilde{\mu}_{1\varepsilon}(\xi)| \\ \times |\mu_{2\varepsilon}(y) - \tilde{\mu}_{2\varepsilon}(y)| \cdot |G(\phi_\varepsilon, y)| \cdot |\partial^h\psi(x)|\, dx\, dy\, dz,$$

where $|q|, |h| \leq 2p$, $|r|, |t| \leq 2s$. The properties of $a(x,y,\xi,\varepsilon)$ imply that for suitable constants

$$\left|\partial_x^q\partial_\xi^r \frac{a(x,y,\xi)}{(1+|\xi|^2)^p}\right| \cdot |\partial_\xi^t(\mu_{1\varepsilon}(\xi) - \tilde{\mu}_{1\varepsilon}(\xi))| \leq \\ C(1+|x|)^k(1+|\xi|^2)^{-p+m-s} \leq C_1\varepsilon^{p+s-m}(1+|x|)^k.$$

since the left side is equal to 0 for $|\xi| <$ const$/s$. Note that

$$\int_{\mathbf{R}^n} |G(\phi_\varepsilon, y)\mu_{2\varepsilon}(y)|\, dy \leq \frac{C}{\varepsilon^{N_G}} \int_{\mathbf{R}^n} (1+|y|)^{p_G}\mu_{2\varepsilon}(y)\, dy \leq C\varepsilon^{-N_G-p_G-n}.$$

By choosing enough large p and s, this implies that for every $d > 0$ the members of the form (11.8) are $o(\varepsilon^d)$, $\varepsilon \to 0$.

To prove that the members of the form (11.8) are $o(\varepsilon^d)$, $\varepsilon \to 0$, for every $d > 0$, we have to estimate the factor

$$\frac{1}{(1+|y|^2)^s|\mu_{2\varepsilon}(y) - \tilde{\mu}_{2\varepsilon}(y)|},$$

which is different from zero if $|y| > \frac{\text{const}}{\varepsilon}$, and to take sufficiently large s.

This proves 3. The proof of 2 is almost the same. □

The relation $\stackrel{\text{g.t.d.}}{=}$ is the relation of equivalence in $\mathcal{G}_t(\mathbb{R}^n)$. So, the mappings $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ are equal if they are considered as the mappings from $\mathcal{G}_t(\mathbb{R}^n)$ into $\mathcal{G}_t(\mathbb{R}^n)/\stackrel{\text{g.t.d.}}{=}$.

Definition 11.5. The mappings $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ are the representatives of the mapping

$$A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/\stackrel{\text{g.t.d.}}{=}$$

which is called the pseudodifferential operator which corresponds to $a \in S^m_{gt}$.

Proposition 11.6. *If $a \in S_{gt}^{-\infty}$, then for every $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$ and $r \in \mathbb{N}_0$ the operators $A_{\mu_2 r}G(\phi_\varepsilon, x)$ and $A_{\mu_1\mu_2}G(\phi_\varepsilon, x)$ are in $\mathcal{G}_t^\infty(\mathbb{R}^n)$.*

Proof. Let $r = 0$. We will prove that

$$A_{\mu_2}G(\phi_\varepsilon, x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon) G(\phi_\varepsilon, y)\mu_{2\varepsilon}(y)\, dy\, d\xi$$

is in $\mathcal{G}_t^\infty(\mathbb{R}^n)$. Other parts of the proposition may be proved in a similar way by using (11.2).

Let $\alpha \in \mathbb{N}_0^n$. Then

$$|\partial^\alpha(A_{\mu_2}G(\phi_\varepsilon, x))| =$$
$$\left|\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}\sum_{j\le\alpha}\binom{\alpha}{j} i^{|\alpha-j|}\xi^{\alpha-j}\partial_x^\alpha a(x,y,\xi,\varepsilon)G(\phi_\varepsilon,y)\mu_{2\varepsilon}(y)\, dy\, d\xi\right|.$$

By using

$$|\partial_x^\alpha a(x,y,\xi,\varepsilon)| \le C_{m,\alpha}\varepsilon^{-N}(1+|x|)^k(1+|\xi|)^m, \quad x,y,\xi \in \mathbb{R}^n,\ \varepsilon \in (0,1],$$

which holds for enough large $-m$ (where N does not depend on m and α),

$$|G(\phi_\varepsilon, y))| \le \tilde{C}\varepsilon^{-N_1}(1+|y|)^{N_1}, \quad y \in \mathbb{R}^n,\ \varepsilon \in (0,\eta),\ \phi \in \mathcal{A}_N$$

and

$$\left|\int_{\mathbb{R}^n}(1+|y|)^{N_1}\mu_{2\varepsilon}(y)\, dy\right| \le \tilde{\tilde{C}}\varepsilon^{-N_1-n}, \quad \varepsilon \in (0,1],$$

we obtain:

If $N_2 = N + N_1 + n$, then for every $\alpha \in \mathbb{N}_0^n$ there is $M \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_M$ there are $C > 0$ and $\eta > 0$ such that

$$|\partial^\alpha(A_{\mu_2}G(\phi_\varepsilon, x))| \le C\varepsilon^{-N_2}(1+|x|))^M, \quad 0 < \varepsilon < \eta. \qquad \square$$

If an amplitude $a \in S_{gt}^m$ does not depend on ε, i.e. $a = a(x,y,\xi)$, then it determines a convenient pseudodifferential operator which will be denoted by $\mathbb{A}$:

$$\mathbb{A}\varphi(x) = (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}a(x,y,\xi)\varphi(y)\, dy\, d\xi,\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

It can be extended on $\mathcal{S}'(\mathbb{R}^n)$ to be linear and continuous mapping from $\mathcal{S}'(\mathbb{R}^n)$ into itself.

In fact,

$$({}^t\mathbb{A}\varphi)(x) = (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}\tilde{a}(x,y,\xi)\varphi(y)\, dy\, d\xi$$

where $\tilde{a}(x,y,\xi) = a(y,x,\xi)$ is continuous and linear: $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathbb{A} = {}^t({}^t\mathbb{A})$ is continuous and linear: $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

We will compare A and A but before that we need the following definition and proposition.

Definition 11.7. If $a \in S^m_{gt}$ and $G \in \mathcal{G}_c(\mathbb{R}^n)$, then $AG = A(\mathcal{M}G)$. Let $G(\phi_\varepsilon, x), (\phi_\varepsilon, x) \in \mathcal{A}_0 \times \mathbb{R}^n$, be a representative of G and $\kappa \in C_0^\infty(\mathbb{R}^n), \kappa \equiv 1$ on $\text{supp}_g G$. Then,

$$\kappa(x)G(\phi_\varepsilon, x), x \in \mathbb{R}^n, \phi \in \mathcal{A}_0,$$

is a representative of $\mathcal{M}G \in \mathcal{G}_t(\mathbb{R}^n)$. $AG = A(\mathcal{M}G)$ is defined by 2 in Theorem 11.4

$$A_{\mu_1\mu_2}(\mathcal{M}G)(\phi_\varepsilon, x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\kappa(y)G(\phi_\varepsilon, y)\, dy\, d\xi.$$

From the next proposition it follows that this definition does not depend on κ.

Proposition 11.8. *If $\kappa_1, \kappa_2 \in C_0^\infty$ are equal to 1 on $\text{supp}_g G$ then*

$$(2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)(\kappa_1(y) - \kappa_2(y))G(\phi_\varepsilon, y)\, dy\, d\xi = 0$$

in $\mathcal{G}_t(\mathbb{R}^n)$.

The following proposition also can be proved.

Proposition 11.9. *Let $a \in S^m_{gt}$ be independent on ε and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $A(\text{Cd}_t f) = \text{Cd}_t(\text{A}f)$ in $\mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$.*

12. Pseudolocal property and the microlocalization

Denote $\mathcal{G}_c^\infty(\Omega) = \mathcal{G}^\infty(\Omega) \cap \mathcal{G}_c(\Omega)$. Clearly, if $G \in \mathcal{G}_c^\infty(\Omega)$ then $\mathcal{M}G \in \mathcal{G}_t^\infty(\mathbb{R}^n)$. In the sequel we will consider $\mathcal{G}_c(\Omega)$ and $\mathcal{G}_c^\infty(\Omega)$ as subspaces of $\mathcal{G}_t(\mathbb{R}^n)$.

Without the proof we give the following theorem.

Theorem 12.1. *Let $a \in S^m_{gt}$ and $G \in \mathcal{G}_c^\infty(\mathbb{R}^n)$. Then, $AG \in \mathcal{G}_t^\infty(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$. More precisely, for every $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$ and $r \in \mathbb{N}_0$, $A_{\mu_2 r}G(\phi_\varepsilon, x)$ and $A_{\mu_1\mu_2}G(\phi_\varepsilon, x)$ are in $\mathcal{G}_t^\infty(\mathbb{R}^n)$ and they are equal in (g.t.d.) sense.*

Definition 12.2. Let $G \in \mathcal{G}_t(\mathbb{R}^n)$ and A be a pseudodifferential operator. It is said that AG is regular at $x \in \mathbb{R}^n$ if there exists an open set $\omega \ni x$ such that for every unit nets $\mu_{1\varepsilon}$, $\mu_{2\varepsilon}$ and $r \in \mathbb{N}_0$,

$$A_{\mu_1\mu_2}(G)|_\omega \text{ and } A_{\mu_2 r}(G)|_\omega \text{ belong to } \mathcal{G}^\infty(\omega).$$

The singular support of AG, Sing $\text{supp}_g AG$, is the complement of a set of points in which AG is regular. If x (resp. any point of ω) does not belong to $\text{Sing}\,\text{supp}_g\, AG$, then it is said that AG is $\mathcal{G}_t^\infty/ \overset{\text{g.t.d.}}{=}$ in x (resp. in ω).

Proposition 12.3. *Let $G \in \mathcal{G}_c(\mathbb{R}^n)$, $a \in S^m_{gt}$. Then,*

$$\text{Sing}\,\text{supp}_g\, AG \subset \text{Sing}\,\text{supp}_g\, G.$$

More precisely, for every $\mu_{1\varepsilon}(\xi), \mu_{2\varepsilon}(y)$ *and* $r \in \mathbb{N}_0$,

$$\operatorname{Sing\,supp}_g A_{\mu_{2\varepsilon} r} G \subset \operatorname{Sing\,supp}_g G,$$
$$\operatorname{Sing\,supp}_g A_{\mu_{1\varepsilon}\mu_{2\varepsilon}} G \subset \operatorname{Sing\,supp}_g G.$$

Proof. Let G be $\mathcal{G}^\infty$ in a neighborhood ω of x_0. We shall show that $AG = A(\mathcal{M}G)$ is in $\mathcal{G}_t^\infty / \overset{\text{g.t.d.}}{=}$ in some open set $\omega_1 \ni x_0$, such that $\overline{\omega}_1 \subset\subset \omega$.

Let $\kappa_1 \in C_0^\infty(\omega)$ such that $\kappa_1 \equiv 1$ on $\overline{\omega}_1$ and let $\kappa_2 \in C_0^\infty(\omega)$ such that $\kappa_2 \equiv 1$ on $K_1 = \operatorname{supp} \kappa_1$.

For small enough ε, we have

$$\begin{aligned}
&\kappa_1(x) A_{\mu_1\mu_2} G(\phi_\varepsilon, x) = \\
&\quad \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\kappa_1(x)\kappa(y)\kappa_2(y) G(\phi_\varepsilon, y)\, dy\, d\xi \\
&\quad + \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\kappa_1(x)(1-\kappa_2(y))\kappa(y) G(\phi_\varepsilon, y)\, dy\, d\xi \\
&\quad = I_1 + I_2.
\end{aligned}$$

As earlier we have that I_1 is $\mathcal{G}_t^\infty$ in $\mathbb{R}^n$. So we have to prove the same for I_2. Let $k \in \mathbb{N}$. Then

$$\begin{aligned}
&I_2 = \\
&\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int \frac{e^{i\langle x-y,\xi\rangle}}{|x-y|^{2k}} (-\Delta_\xi)^k (a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi))\kappa_1(x)(1-\kappa_2(y))\kappa(y) G(\phi_\varepsilon, y)\, dy\, d\xi.
\end{aligned}$$

By using (11.3) and Leibniz's rule one can prove that

$$\begin{aligned}
|\Delta_\xi^k (a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi))| &\le \frac{C_k}{\varepsilon^N}(1+|x|)^{r_k}(1+|\xi|)^{m-2nk} \\
&\le \frac{\tilde{C}_k}{\varepsilon^N}(1+|\xi|)^{m-2nk}, \quad y,\xi \in \mathbb{R}^n, \ x \in \operatorname{supp} \kappa_1,
\end{aligned}$$

where C_k and $\tilde{C}_k$ are suitable constants. By taking large enough k we can apply the same procedure as in the proof of Proposition 11.9. This implies that $I_2 \in \mathcal{G}_t^\infty(\mathbb{R}^n)$. □

The notion of the wave front for Colombeau's generalized functions has been introduced by Scarpalezos [18] as a natural generalization of the wave front for distributions.

Definition 12.4. A tempered generalized function G is called $\mathcal{G}^\infty$-rapidly decreasing if it has a representative $G(\phi_\varepsilon, x)$ with the following property. There exists $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^n$ and $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_{n_0}$ there are $C > 0$ and $\delta > 0$ such that

$$|D^\alpha G(\phi_\varepsilon, x)| \le C\varepsilon^{-N}(1+|x|^2)^{-p/2}, \ x \in \mathbb{R}^n.$$

Clearly, if $G \in \mathcal{G}^\infty(\mathbb{R}^n) \cap \mathcal{G}_c(\mathbb{R}^n)$, then $\mathcal{M}G$ is $\mathcal{G}^\infty$-rapidly decreasing. If $G \in \mathcal{G}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, then we will denote $\mathcal{M}(\varphi G)$ simply by φG.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $G(\phi_\varepsilon, \cdot)$ be a representative of G. We define $\mathcal{F}(\varphi G) \in \mathcal{G}_t(\mathbb{R}^n)$ by a representative

$$\mathcal{F}_t(\varphi G)(\phi_\varepsilon, \xi) = \mathcal{F}(\varphi(x)G(\phi_\varepsilon, x))(\xi), \quad \xi \in \mathbb{R}^n, \tag{12.1}$$

where $\mathcal{F}$ denotes the Fourier transformation in $L^1(\mathbb{R}^n)$. One can prove easily that this definition makes sense. Also, the following proposition is simple.

Proposition 12.5. *If representative (12.1) has the properties given in Definition 12.4, then $\varphi G \in \mathcal{G}_c^\infty$.*

We denote by Γ a convex open cone in $\mathbb{R}^n$ which does not contain a straight line.

Let $(x_0, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. The following functions will be used.

$$\begin{cases} \text{(a) } \varphi \in C_0^\infty(\Omega),\ \varphi = 1 \text{ in a neighbourhood of } x_0; \\ \operatorname{supp} \psi \subset \Gamma,\ \psi \text{ is positive-homogeneous} \\ \text{of degree zero in } \Gamma \text{ and } \psi = 1 \text{ in a neighbourhood of} \xi_0. \end{cases} \tag{12.2}$$

Definition 12.6. It is said that $G \in \mathcal{G}(\mathbb{R}^n)$ is $\mathcal{G}^\infty$-rapidly decreasing in a cone Γ if for every $\xi_0 \in \Gamma$ there is ψ with the properties in (12.2)(b) such that ψG is $\mathcal{G}^\infty$-rapidly decreasing.

The cone $\sum_g(G)$ is the set of all $\eta \in \mathbb{R}^n \setminus \{0\}$ for which does not exist ψ with the properties in (12.2)(b) such that ψG is $\mathcal{G}^\infty$-rapidly decreasing.

Definition 12.7. It is said that $G \in \mathcal{G}(\Omega)$ is microlocally regular in an open conic set $\gamma \subset \Omega \times \mathbb{R}^n$ (conic in the second variable) if for every $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ there exist an open neighborhood Ω_0 of x_0, a conic neighborhood Γ_0 of ξ_0, and functions φ and ψ with the properties in (12.2) (with Ω_0 and Γ_0 instead Ω and Γ) such that $\psi(\xi)\mathcal{F}_t(\varphi G)(\xi)$ is $\mathcal{G}^\infty$-rapidly decreasing. The wave front of $G \in \mathcal{G}$ denoted by $\mathrm{WF}_g\, G$, is the complement of the union of all conic open sets γ where G is microlocally regular.

By using functions φ and ψ satisfying (12.2) and a unit net μ_ε we define operator $\psi(D)_\mu \varphi$ on $\mathcal{G}_t(\mathbb{R}^n)$ by $G \to \psi(D)_\mu(\varphi G)$, where

$$\psi(D)_\mu(\varphi G)(\phi_\varepsilon, x) = (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} \psi(\xi)\varphi(y)G(\phi_\varepsilon, y)\mu_\varepsilon(\xi)d\mu\, d\xi.$$

Clearly $\psi(D)_\mu\varphi(\cdot)$ maps $\mathcal{G}_t(\mathbb{R}^n)$ into itself and it defines a pseudodifferential operator. Because of (11.2), (11.3), (11.4) and the estimate

$$|\partial^\alpha \psi(\xi)| \le C_\alpha |\xi|^{-\alpha}, |\xi| > R,$$

one can prove that $\psi(D)_{\mu_1}(\varphi G)$ and $\psi(D)_{\mu_2}(\varphi G)$ are equal in (g.t.d.) sense for every unit nets $\mu_{1\varepsilon}$ and $\mu_{2\varepsilon}$. The amplitude of $\psi(D)\varphi$ is $a(x, y, \xi, \varepsilon) = \psi(\xi)\varphi(y)$.

Proposition 12.8. *A point* $(x_0, \xi_0) \notin \mathrm{WF}_g G$, $G \in \mathcal{G}(\Omega)$, *if and only if there exist smooth functions* φ, ψ *with the properties in (12.2) and a unit net* μ_ε *such that* $\psi(D)_\mu(\varphi G) \in \mathcal{G}_t^\infty$.

The proofs of the following propositions are similar to the classical one in distribution theory and because of that they are omitted.

Proposition 12.9 *If* $h \in C_0^\infty(\mathbb{R}^n)$ *and* $G \in \mathcal{G}(\mathbb{R}^n)$, *then* $\mathrm{WF}_g(hG) \subset \mathrm{WF}_g(G)$.

This proposition implies

Corollary 12.10. $\mathrm{WF}_g G = \{(x,\xi),\ \xi \in \Sigma_{gx}(G) = \bigcap_h \Sigma_g(hG)\}$, *where the intersection is taken over all* $h \in C_0^\infty$.

Denote $T^*\Omega = \Omega \times \mathbb{R}^n$ and $\pi : T^*\Omega \to \Omega$ the first projection.

Proposition 12.11. $\pi\,\mathrm{WF}_g G = \mathrm{Sing\,supp}_g\, G$

Proposition 12.12. *Let* $f \in \mathcal{D}'(\Omega)$. *Then* $\mathrm{WF}\, f = \mathrm{WF}_g\, \mathrm{Cd}\, f$.

For the propagation of singularities of a pseudodifferential operator we need the following definition.

Definition 12.13. $\mathrm{WF}_g AG$, $G \in \mathcal{G}_t$, is the complement of the set of points $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ such that for every unit nets $\mu_{1\varepsilon}$, $\mu_{2\varepsilon}$ and $r \in \mathbb{N}_0$,

$$A_{\mu_1\mu_2}(G)|_\omega \text{ and } A_{\mu_2 r}(G)|_\omega$$

are microlocally regular at (x_0, ξ_0).

Proposition 12.14. *Let* $G \in \mathcal{G}_c(\Omega)$ *and* A *be a pseudodifferential operator. Then*

$$\mathrm{WF}_g AG \subset \mathrm{WF}_g G.$$

13. Composition of pseudodifferential operators

The results of the sections which are to follow are proved in **[12]**. We shell present only the definitions and assertions without proofs.

First, we define properly supported pseudodifferential operators.

Let $a \in S_g^m$ and $h \in C_0^\infty(\mathbb{R})$ such that $h(t) = 1$, $|t| \le t_0$, $h(t) = 0$, $|t| > t_1 > t_0$. We decompose a representative $A_{\mu_1\mu_2}$ of A as follows:

$$A_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \dot{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x) + \tilde{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x),\ G \in \mathcal{G}_t(\mathbb{R}^n),$$

where

$$\dot{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} h(|x-y|) a(x,y,\xi,\varepsilon) \eta_{1\varepsilon}(\xi)\eta_{2\varepsilon}(y) G(\phi_\varepsilon, y)\, dy\, d\xi$$

and $\tilde{A}_{\mu_1\mu_2}$ has $(1-h(|x-y|))$ instead of $h(|x-y|)$ in the double integral.

Let (ξ,ε) be arbitrary, but fixed. Then the function

$$(x,y)\mapsto h(|x-y|)a(x,y,\xi,\varepsilon)$$

is properly supported which means that the inverses for the first and second projection of a compact set in $\mathbb{R}^n$ intersect the support of this function over the compact sets.

One can easily prove that $h(|x-y|)a(x,y,\xi,\varepsilon)\in S_g^m$.

Definition 13.1. Pseudodifferential operator corresponding to $a\in S_g^m$ satisfying the property that for every $(\xi,\varepsilon)\in\mathbb{R}^n\times(0,1]$,

$$(\mathbb{R}^n)^2\ni(x,y)\mapsto a(x,y,\xi,\varepsilon),$$

is properly supported, is called a properly supported pseudodifferential operator.

Pseudodifferential operator which maps $\mathcal{G}_c(\mathbb{R}^n)$ into $\mathcal{G}_t^\infty(\mathbb{R}^n)/\overset{\text{g.t.d}}{=}$ is called the smoothing pseudodifferential operator.

As in Proposition 11.6 one can prove

Proposition 13.2. $\tilde{A}_{\mu_1\mu_2}:\mathcal{G}_c(\mathbb{R}^n)\to\mathcal{G}_t^\infty(\mathbb{R}^n)$.

So, for every pseudodifferential operator

$$A:\mathcal{G}_t(\mathbb{R}^n)\to\mathcal{G}_t(\mathbb{R}^n)/\overset{\text{g.t.d}}{=},$$

there exists a properly supported pseudodifferential operator

$$\dot{A}:\mathcal{G}_t(\mathbb{R}^n)\to\mathcal{G}_t(\mathbb{R}^n)/\overset{\text{g.t.d}}{=}$$

such that $A-\dot{A}$ is a smoothing pseudodifferential operator.

Remark The extension of a properly supported pseudodifferential operator on $\mathcal{G}(\mathbb{R}^n)$ may be done as follows. Let $\dot{A}$ be properly supported with the properly supported amplitude $a\in S_g^m$ and let $\{\kappa_i,\ i\in\mathbb{N}\}$ be a partition of unity with elements in $C_0^\infty(\mathbb{R}^n)$.

Let $G\in\mathcal{G}(\mathbb{R}^n)$. Put

$$\dot{A}G(\phi_\varepsilon,x)=\sum_{i\in\mathbb{N}}\dot{A}(\kappa_iG)(\phi_\varepsilon,x).$$

Since $\kappa_iG\in\mathcal{G}_c(\mathbb{R}^n)$, any member in the sum is well defined.

One can prove easily that

$$\dot{A}_{\mu_1\mu_2}G(\phi_\varepsilon,x)\in\mathcal{E}_M[\mathbb{R}^n]$$

for every unit nets $\mu_{1\varepsilon}$ and $\mu_{2\varepsilon}$, and that for different unit nets the corresponding elements are equal in (g.t.d.) sense.

Let $a \in S_g^m$, $b \in S_g^{m'}$ determine operators A and B by representatives $A_{\mu_1\mu_2}$ and $B_{\tilde{\mu}_1\tilde{\mu}_2}$, where $\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tilde{\mu}_{1\varepsilon}$ and $\tilde{\mu}_{2\varepsilon}$ are unit nets. Put

$$(A_{\mu_1\mu_2} \circ B_{\tilde{\mu}_1\tilde{\mu}_2})G = A_{\mu_1\mu_2}((B_{\tilde{\mu}_1\tilde{\mu}_2})G), \; G \in \mathcal{G}_t(\mathbb{R}^n).$$

The following proposition shows that the composition of properly supported pseudodifferential operators AB defined by a representative given above is well defined and it can be proved by a direct calculation.

Proposition 13.3. *Let $a \in S_g^m$. For every eight unit nets $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$, $\tilde{\mu}_{1\varepsilon}(\xi)$, $\tilde{\mu}_{2\varepsilon}(y)$, $\mu_{3\varepsilon}(\xi)$, $\mu_{4\varepsilon}(y)$, $\tilde{\mu}_{3\varepsilon}(\xi)$, $\tilde{\mu}_{4\varepsilon}(y)$ and $G \in \mathcal{G}_t(\mathbb{R}^n)$*

$$A_{\mu_1\mu_2}(B_{\tilde{\mu}_1\tilde{\mu}_2}G) \;\text{ and } A_{\mu_3\mu_4}(B_{\tilde{\mu}_3\tilde{\mu}_4}G)$$

are equal in (g.t.d.) sense in $\mathcal{G}_t(\mathbb{R}^n)$.

From now on we shall assume that amplitudes are defined by Definition 11.1.

Properly supported amplitudes will be indicated by $\dot{a}$. By $\dot{A}$ is denoted the corresponding pseudodifferential operator.

Theorem 13.4 *Let $\dot{a} \in S_g^m$, $\dot{b} \in S_g^{m'}$. Then the composition of $\dot{A}$ and $\dot{B}$ is represented by*

$$(13.1) \qquad A_{\mu_1\mu_2}B_{\tilde{\mu}_1\tilde{\mu}_2}G(\phi_\varepsilon, x) = \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}\dot{k}(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)G(\phi_\varepsilon, y)\,dy\,d\xi,$$

where

$$(13.2) \quad \dot{k}(x,y,\xi,\varepsilon) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle y-z,\xi-\eta\rangle}a(x,z,\xi,\varepsilon)b(z,y,\eta,\varepsilon)\tilde{\mu}_{2\varepsilon}(\eta)\,dz\,d\eta,$$
$$x,y,z,\xi,\eta \in \mathbb{R}^n, \; \varepsilon \in (0,\eta_0), \; (\eta_0 = \eta_0(\phi)).$$

Moreover, $\dot{k}(x,y,\xi,\varepsilon) \in S_g^{m+m'}$ and it is properly supported.

14. Calculus with symbols. Hypoelliptic operator

Definition 14.1. By $S_{sg}^m = S^m(\mathbb{R}^n \times \mathbb{R}^n \times [0,1))$ is denoted the subspace of $S_g^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1))$ consisting of amplitudes $a(x,\xi,\varepsilon)$ independent of y for which (11.1) holds. By $S_{sg}^{-\infty}$ is denoted the set of elements from $S_g^{-\infty}$ which do not depend on y. Elements of S_{sg}^m are called the symbols of degree m.

As before, it can be proved that every $a \in S_{sg}^m$ defines a pseudodifferential operator $A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$.

Definition 14.2 A formal symbol is a sequence of symbols $a_j \in S_{sg}^{m_j}$, $j \in \mathbb{N}_0$, such that $m_j \to -\infty$ strictly, and $N_j \le N < \infty$ (N_j are exponents of ε for a_j). It is denoted by

$$\sum_{j=0}^{\infty} a_j(x,\xi,\varepsilon).$$

As in standard theory one can make the construction of the true symbol:

Proposition 14.3. *There exists $a \in S_{sg}^{m_0}$ such that for every $j_0 \in \mathbb{N}_0$,*

$$a - \sum_{j<j_0} a_j \in S_{sg}^{m_{j_0}}.$$

It is determinated uniquely modulo $S_{sg}^{-\infty}$.

Theorem 14.4. *For every amplitude $\tilde{a}(x, y, \xi, \varepsilon) \in S_g^m$ there exists the symbol $a(x, \xi, \varepsilon) \in S_{sg}^m$ which determinates the same pseudodifferential operator $A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$ modulo the smoothing pseudodifferential operator $\tilde{A}$.*

Thus,

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_y^\alpha \tilde{a}(x, y, \xi, \varepsilon)_{|y=x}$$

determines $a \in S_{sg}^m$.

For example,

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \psi(\xi) \partial_x^\alpha \varphi(x)$$

is the symbol for $\psi(D)\varphi$.

Theorem 14.5. *Let A and B be pseudodifferential operators with the symbols $a \in S_{sg}^m$ and $b \in S_{sg}^{m'}$ and let $\dot{A}$ and $\dot{B}$ be the corresponding properly supported operators. The symbol of the properly supported operator $\dot{A}\dot{B}$ is given by*

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, \xi, \varepsilon) \partial_x^\alpha b(x, \xi, \varepsilon).$$

We are going to give the microlocal analysis of solutions of a pseudodifferential equation. For this we need the next definition.

Definition 14.6. A pseudodifferential operator A in Ω is smoothing in $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ if there exist $\varphi \in C_0^\infty(\Omega)$, $\varphi = 1$ in a neighborhood of x_0, and a convex open cone Γ, a neighborhood of ξ_0, such that the symbol $a(x, \xi, \varepsilon)$ of A has the following property:

There is $N > 0$ such that for every $\alpha, \beta \in \mathbb{N}_0^n$ and $M \in \mathbb{N}_0$ there is $C_{\alpha,\beta,M} \geq 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \varphi(x) a(x, \xi, \varepsilon)| \leq C_{\alpha,\beta,M} \varepsilon^{-N} (1 + |\xi|)^{-M}, \; x \in \Omega, \; \xi \in \Gamma_R, \; |\xi| > R.$$

A pseudodifferential operator A in Ω is said to be smoothing in a conic open subset γ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$ if it is smoothing in every point of γ.

The complement in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ of the union of all conic open sets in which A is regularizing is called the microsupport of A and it is denoted by $\mu \operatorname{supp}_g A$.

Proposition 14.7. *Let $G \in \mathcal{G}_c(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. Then $(x_0, \xi_0) \notin \mathrm{WF}_g\, G$ if and only if there is a conic open neighborhood γ of (x_0, ξ_0) in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ such that $B_{\mu_1\mu_2} G \in \mathcal{G}_t^\infty(\Omega)$ $(B_{\mu_1 r} G \in \mathcal{G}_t^\infty(\Omega))$ for any pseudodifferential operator B in Ω whose microsupport is contained in γ.*

Theorem 14.8. *Let A be a pseudodifferential operator which is smoothing in a conic open set γ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$. If the wave front of $G \in \mathcal{G}_c(\Omega)$ is contained in $\gamma' \subset \gamma$, then $\mathrm{Sing\,supp}_g\, A(G)$ is empty.*

The previous two propositions simply imply the following important assertion.

Proposition 14.9. *Let A be a properly supported pseudodifferential operator in Ω and $G \in \mathcal{G}(\Omega)$. Then*

$$\mathrm{WF}_g(AG) \subset (\mathrm{WF}_g\, G) \cap \mu\,\mathrm{supp}_g\, A.$$

Definition 14.10. A proper pseudodifferential operator P with a symbol $[p(x, \xi, \varepsilon)]$ is called hypoelliptic if the following holds:

(1) There exists $N \in \mathbb{N}$ such that for every compact set $K \subset \mathbb{R}^n$ there exist $\xi_0 > 0$ and $M > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$C^{-1}(1 + |\xi|)^{-M}\varepsilon^N \leq |p(x, \xi, \varepsilon)| \leq C(1 + |\xi|)^M \varepsilon^{-N}, \tag{14.1}$$

for $x \in K$, $|\xi| \geq \xi_0$, $\varepsilon < \eta$.

(2) There exists $N \in \mathbb{N}$ such that for every compact set $K \subset \mathbb{R}^n$ there exists $\xi_0 > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C_{\alpha,\beta} > 0$ and $\eta > 0$ such that

$$\left|\frac{D_\xi^\alpha D_x^\beta p(x, \xi, \varepsilon)}{p(x, \xi, \varepsilon)}\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|},\ x \in K,\ |\xi| \geq \xi_0,\ \varepsilon < \eta. \tag{14.2}$$

Without a proof we give

Theorem 14.11. *(i) Let P be a proper pseudodifferential operator with symbol $p(x, \xi, \varepsilon)$ which satisfies Definition 14.6. Then the following holds: There exists $N \in \mathbb{N}$ such that for every compact set $K \subset \mathbb{R}^n$ there exists $\xi_0 > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C'_{\alpha,\beta} > 0$ and $\eta > 0$ such that*

$$\left|\frac{D_\xi^\alpha D_x^\beta p(x, \xi, \varepsilon)^{-1}}{p(x, \xi, \varepsilon)^{-1}}\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|},\ x \in K,\ |\xi| \geq \xi_0,\ \varepsilon < \eta. \tag{14.3}$$

(ii) For every hypoelliptic pseudodifferential operator P there exists a proper pseudodifferential operator Q such that $PQ - I \in S^{-\infty}$, and $QP - I \in S^{-\infty}$.

Proposition 14.12. *Let P be a hypoelliptic pseudodifferential operator. Then*

$$\mathrm{WF}_g(PG) = \mathrm{WF}_g\, G$$

for every $G \in \mathcal{G}$.

Pseudodifferential operator P is called elliptic with a classical amplitude if its symbol $p(x, \xi, \varepsilon)$ satisfies the following inequality

$$C^{-1}(1 + |\xi|)^{-M} \le |p(x, \xi, \varepsilon)| \le C(1 + |\xi|)^M \tag{14.4}$$

instead of (14.1). One can prove that (14.4) implies (14.2) and that means that there exist a parametrix for such pseudodifferential operators, too.

Pseudodifferential operator is called elliptic if

$$C^{-1}(1 + |\xi|)^{-M}\varepsilon^{-N} \le |p(x, \xi, \varepsilon)| \le C(1 + |\xi|)^M\varepsilon^{-N} \tag{14.5}$$

holds instead of (14.1). As in the previous case, one can prove that then (14.5) implies (14.2), and this implies the existence of the parametrix for A.

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