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TOPICS FROM MATHEMATICS AND MECHANICS

Matematički institut SANU

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PREFACE

The aim of *Zbornik radova* is to foster further growth of pure and applied mathematics, publishing papers which contain new ideas and scopes in the mathematics. The papers have to be prepared in such a manner that they can inform readers in a favourable way, introducing them in a narrower field of mathematical theories pointing at research possibilities. It can be for the individual use or for discussions in College or University seminars.

We are open for contacts and cooperations.

Bogoljub Stanković
Editor-in-Chief

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Kosta Došen

AN INTRODUCTION TO ADJUNCTION

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Abstract. This is an introduction to the notion of adjunction from an abstract point of view. A systematic survey is made of various definitions of this notion, including two definitions not recorded in the literature. A similar survey is also made of the definitions of comonad, which also includes new material. Finally, the relationship between the notion of adjunction and the notion of comonad is explained through two adjunctions involving the category of adjunctions and the category of comonads, where the latter category is isomorphic to a full subcategory of the former. The standard presentation of this relationship, through the category of resolutions of a comonad, is a corollary of this new presentation of the matter.

Adjunction is one of the most important notions of mathematics, which category theory has taught us to recognize everywhere. To put it roughly, adjunction is half of an equivalence of categories, but taken wisely, in a "diagonal" way (cf. 2.2). This means that, though the two categories need not be equivalent—one may be *richer* than the other—something essential is not lost in passing from the richer category to the poorer one: the two categories share a common core. A formal theorem concerning the equivalence of subcategories of categories in an adjoint situation reflects this fact (see [Lambek & Scott 1986, Part 0, sections 3-4] and [Lambek 1981], where rather "obscure" antecedents are found for this important principle).

A typical adjunction is when we have, on the one hand, a category $\mathcal{A}$ whose objects are some algebras, like groups or vector spaces over a fixed field, with arrows being homomorphisms (in the case of vector spaces these are linear transformations), and on the other hand, the category $\mathcal{B}$ whose objects are sets, with functions as arrows. From $\mathcal{A}$ to $\mathcal{B}$ goes a *forgetful* functor G, which assigns to an algebra the underlying set of elements, and to a homomorphism the underlying function. This functor has a left-adjoint functor F from $\mathcal{B}$ to $\mathcal{A}$ that assigns to a set B the free algebra generated by B (with vector spaces, $F(B)$ is the vector space with basis B). In passing with G from the richer category to the poorer one, not all information about the algebras is lost: something essential is preserved. The set $G(A)$ still carries some information about the algebra A from the category $\mathcal{A}$. When we apply next the adjoint functor F to $G(A)$, the algebra $F(G(A))$ is not the same as the initial algebra A, but it is comparable to it: there is a homomorphism φ_A from $F(G(A))$ to A defined by mapping the free generators to themselves, which is a component of a natural transformation called the *counit* of the adjunction. Similarly, for a set B, the set $G(F(B))$ is comparable to B: there is a function γ_B from B to $G(F(B))$ amounting to inclusion, which is a component of a natural transformation called the *unit* of the adjunction. The categories $\mathcal{A}$ and $\mathcal{B}$ would be equivalent if $F(G(A))$ were isomorphic to A, and $G(F(B))$ were isomorphic to B. Of these isomorphisms, we have only halfs, chosen "diagonally", in opposite directions: the homomorphism φ_A from $F(G(A))$ to A and the function γ_B from B to $G(F(B))$. Moreover, arrows derived from the unit composed with arrows derived from the counit give identity arrows: $F(\gamma_B)$ composed with $\varphi_{F(B)}$

is the identity homomorphism on $F(B)$, and $\gamma_{G(A)}$ composed with $G(\varphi_A)$ is the identity function on $G(A)$.

This way, the forgetting of the forgetful functor is controlled. Some conclusions we may reach by reasoning in $\mathcal{B}$ can be transferred back to $\mathcal{A}$. However, it seems that the point of describing an adjoint situation is not so much to provide a tool for proving new theorems, but rather to illuminate, clarify and systematize already known results.

The ability to forget in a controlled manner is an important trait of rationality—perhaps the most important one. We should forget the unessential, so as not to be encumbered by it, and move more easily in our thoughts. But this forgetting should be controlled: what is essential shouldn't be forgotten. There should be a way back: conclusions reached in the simpler context, where the unessential is forgotten, should be applicable to the original, more complicated, context. Controlled forgetting, which exists in abstraction, but not only there, is certainly a major character of mathematical rationality, and an embodiment of it is found in the concept of adjunction.

We can take it as a rule of thumb that behind theorems of the "if and only if" type we should look for adjunctions. In important theorems of this type, where in passing from one side to the other there is a gain, and where, typically, one direction of the theorem is easy to prove and the other difficult, there should be an adjunction that does not amount to equivalence of categories, but obtains between a richer and a poorer category.

We should say, however, that not every adjunction not amounting to equivalence need hold between a richer and a poorer category. Two functors going from a category to this same category may be adjoints without the unit and counit giving isomorphisms. If we have a functor H from a category $\mathcal{A}$ to a category $\mathcal{B}$ that has both a left adjoint F from $\mathcal{B}$ to $\mathcal{A}$ and a right adjoint G from $\mathcal{B}$ to $\mathcal{A}$, then the composite functors FH and GH from $\mathcal{A}$ to $\mathcal{A}$ are adjoints, FH being left-adjoint and GH right-adjoint (analogously, the composite functors HF and HG from $\mathcal{B}$ to $\mathcal{B}$ are adjoints, HF being left-adjoint and HG right-adjoint). Various examples of adjunction may be found in Mac Lane's book [1971].

This introduction to the notion of adjunction will, however, not dwell much on examples. We shall rather try to decipher the abstract, logical, structure of this notion. The work will be divided into five parts. After the first part, devoted to preliminaries of category theory, we shall consider in the second part the adjunction underlying the notion of function. Then in the third part we consider definitions of the notion of adjunction. The fourth part is about the related notion of comonad (we could as well have chosen to deal with monads, also called *triples*). In the final, fifth part, we explain the relationship between the notions of adjunction and comonad.

1. Preliminaries

Before embarking upon our consideration of adjunction, we have to review first some elementary notions of category theory.

1.1. Foundations. Category theory is sometimes taken as providing mathematics with foundations alternative to set theory. This point of view often leads into discussions about the size of classes, i.e., about the distinction between sets and proper classes. Such matters are, otherwise, rather foreign to the spirit of results about categories, which are more about structure than about size. (A presumably germane point is made when, using ancient philosophical terminology, categories are said to be about form, rather than substance; cf. [Lawvere 1964].) So these discussions are usually limited to a preamble of a typical work in category theory (such is the case, too, in the most widely cited text about categories—Mac Lane's book [1971]). In general, they don't leave much trace on the mathematics in the main body of the work, except a tendency to distinguish results that hold only for *small* categories, i.e., those whose objects and arrows, not being too numerous, can be collected into sets. These distinctions often don't have much to do with the import of the results, and can be somewhat distracting.

We are here approaching categories with a logical background, but we shall neglect foundational matters. In fact, this neglect may be explained just by this background. If we were asked about foundations, we would rely on standard set-theoretical foundations, as they have become crystallized within logic. The objects of the category of sets would be for us all the sets that are the elements of the domain of a given model of first-order axiomatic set theory. Since such a domain is itself a set, there is no problem in conceiving of the category of sets as being itself small. So we restrict our attention to small categories only. Bigger categories than these maybe exist, but they shall not be our concern.

1.2. Morphisms and naturalness. The dominant opinion is that the guiding principle of category theory is to look concerning every mathematical object for structure-preserving maps. When the object has no structure, when it is simply a set, then the maps are all functions from sets to sets. When the object has structure, then it may be an algebra, in which case the maps are homomorphisms, or it may be a set with a binary relation, in which case the maps are monotonic functions. Many other sorts of structure can be envisaged.

In model theory, stress is often put on relational or functional structures with a single domain; i.e., relations are defined on a single set and functions are operations on a set. Category theory, on the other hand, is concerned much more with a plurality of domains.

Let us consider the case of relations, and let us generalize monotonicity to relations between two sets. So let A and B be sets and let $R \subseteq A \times B$. If we have another relation $R' \subseteq A' \times B'$, then a structure-preserving map from R to R' would be a pair of functions $f : A \to A'$ and $g : B \to B'$ such that for every a in A and every b in B

$$\text{if } a\,R\,b, \text{ then } f(a)\;R'\;g(b)$$

(of course, $a\,R\,b$ means $(a,b) \in R$). When $A = B$, $A' = B'$ and $f = g$, then we obtain the ordinary monotonicity condition.

The standard approach is to take a function as a special kind of relation, but we may also take the notion of function as being more primitive. Every relation $R \subseteq A \times B$ is associated to a function f from A to the power set of B such that $a\,R\,b$ iff $b \in f(a)$. To understand structure-preserving maps we shall then concentrate on the notion of function.

Let a ***function pair*** from a pair of sets (A_1, A_2) to a pair of sets (A_1', A_2') be a pair of functions (g_1, g_2) such that $g_1 : A_1 \to A_1'$ and $g_2 : A_2 \to A_2'$. A structure-preserving map from a function $f : A_1 \to A_2$ to a function $f' : A_1' \to A_2'$ is a function pair (g_1, g_2) from (A_1, A_2) to (A_1', A_2') such that for every x in A_1 and every y in A_2

$$\text{if } f(x) = y, \text{ then } f'(g_1(x)) = g_2(y).$$

This implication is equivalent to requiring that for every x in A_1

$$g_2(f(x)) = f'(g_1(x)),$$

which means that for the composite functions the following ***naturalness*** equality holds:

$$g_2 f = f' g_1.$$

We use the term *morphism* for function pairs that satisfy naturalness; so (g_1, g_2) is a morphism from f to f' iff naturalness holds. This defines *morphisms between functions*. (Note that some authors use the term "morphism" for arrows in a category.)

This terminology accords rather well with standard usage. For a binary operation $f : A \times A \to A$ and another binary operation $f' : A' \times A' \to A'$, the function pair that is an obvious candidate for a morphism from f to f' is $(g \times g, g)$ where $g : A \to A'$ and $(g \times g)(x_1, x_2)$ is defined as $(g(x_1), g(x_2))$. Such a function pair $(g \times g, g)$ is a morphism from f to f' iff g is a homomorphism in the ordinary sense.

However, we shall speak of morphisms in other situations, too, where the structure mapped is not only that of a function, but something more complicated, involving several functions, which are moreover of a special kind. Then morphisms will not be simply function pairs, but something more involved, though analogous. In particular cases, we shall introduce special names for the morphisms in question. The guiding idea will always be to impose the naturalness condition for every function involved. Since many, if not all, important structures of mathematics can be expressed in terms of functions, and often gain in clarity by being expressed so, we shall find the notion of structure-preserving map appropriate to these structures by looking for naturalness conditions.

1.3: Graphs, graph-morphisms and transformations. A graph is a function pair (S, T) from (X, X) to (Y, Y). So, S and T are both functions from X to

Y. To help imagination, we call X the set of *arrows*, Y the set of *objects*, S the *source* function and T the *target* function. With that terminology, the denomination "graph" becomes justified. (In graph theory, the corresponding notion is sometimes called "directed multigraph with loops".)

For objects of graphs we use the letters $A, B, C, \dots$, and for arrows $f, g, h, \dots$, with indices if needed. We write $f : A \to B$ to indicate that the source of the arrow f is A and its target B; we say that $A \to B$ is the *type* of f. For graphs we use the script letters $\mathcal{G}, \mathcal{H}, \dots$ A *hom-set* $\mathcal{G}(A, B)$ in a graph $\mathcal{G}$ is $\{f \mid f : A \to B \text{ is an arrow of } \mathcal{G}\}$.

An alternative way to define a graph is to identify it with a single function $\mathcal{F}$ from X to $Y \times Y$. To pass from a graph (S, T) to a graph $\mathcal{F}$, we have the definition

$$\mathcal{F}_{S,T}(f) \stackrel{\text{def}}{=} (S(f), T(f)).$$

Conversely, if we are given $\mathcal{F}$, and p^1 and p^2 are, respectively, the first and second projection function, then we define S and T by

$$S_{\mathcal{F}}(f) \stackrel{\text{def}}{=} p^1(\mathcal{F}(f)), \qquad T_{\mathcal{F}}(f) \stackrel{\text{def}}{=} p^2(\mathcal{F}(f)).$$

It is clear that if we start from a graph (S, T), define $\mathcal{F}_{S,T}$, and then define $S_{\mathcal{F}_{S,T}}$ and $T_{\mathcal{F}_{S,T}}$,we obtain that S is equal to $S_{\mathcal{F}_{S,T}}$ and T is equal to $T_{\mathcal{F}_{S,T}}$. Analogously, $\mathcal{F}_{S_{\mathcal{F}},T_{\mathcal{F}}}$ is equal to $\mathcal{F}$.

We shall say that the two notions of graph, the (S, T) notion and the $\mathcal{F}$ notion, are *equivalent*. (This we do because there is an equivalence, actually an isomorphism, between the category of (S, T) graphs and the category of $\mathcal{F}$ graphs, as we shall see in 1.5.) The equivalence of two notions does not always mean that the two notions are *coextensive*, i.e., that they cover exactly the same objects, as the notions of equilateral and equiangular triangles are coextensive. The (S, T) graphs and the $\mathcal{F}$ graphs are strictly speaking different objects, though they are in one-to-one correspondence. On the other hand, equivalence is more than just this one-to-one correspondence. The concept of equivalence of notions will be explained in detail in 1.5 (after we have introduced the notion of equivalence of categories).

A binary relation on Y may be identified with a graph $\mathcal{F}$ that is a *one-one* function. We can then forget about X, and consider just the image of $\mathcal{F}$, i.e., a subset of $Y \times Y$. If a binary relation is a *set* of ordered pairs, a graph is a family of ordered pairs indexed by the arrows, a family where the same ordered pair may occur several times with different indices. In other words, a graph is a *multiset* of ordered pairs.

If a graph is a function pair (S, T), then the appropriate notion of morphism is the following. Suppose S and T are functions from X to Y, while S' and T' are functions from X' to Y'. Then as a morphism from $\mathcal{G} = (S, T)$ to $\mathcal{H} = (S', T')$ we can take a function pair (M_X, M_Y) from (X, Y) to (X', Y') such that naturalness is satisfied, i.e.

$$M_Y(S(f)) = S'(M_X(f)), \qquad M_Y(T(f)) = T'(M_X(f)).$$

This means that arrows $f : A \to B$ of $\mathcal{G}$ are mapped to arrows $M_X(f) : M_Y(A) \to MY(B)$ of $\mathcal{H}$. As usual, we shall omit the subscripts from M_X and M_Y, referring to both by M. We shall also find it handy to omit parentheses from $M(A)$ and $M(f)$; instead we write MA and Mf.

So a *graph-morphism* M from $\mathcal{G}$ to $\mathcal{H}$ will be a pair of functions, both written M, assigning, respectively, to every object A of $\mathcal{G}$ an object MA of $\mathcal{H}$, and to every arrow $f : A \to B$ of $\mathcal{G}$ an arrow $Mf : MA \to MB$ of $\mathcal{H}$.

A graph-morphism M from $\mathcal{G}$ to $\mathcal{H}$ is *faithful* iff for every pair (A, B) of objects of $\mathcal{G}$ and for every pair $(f : A \to B,\ g : A \to B)$ of arrows of $\mathcal{G}$ if $Mf = Mg$ in $\mathcal{H}$, then $f = g$ in $\mathcal{G}$; this means that M restricted to the hom-sets $\mathcal{G}(A,B)$ and $\mathcal{H}(MA, MB)$ is one-one. A graph-morphism M from $\mathcal{G}$ to $\mathcal{H}$ is *full* iff for every pair (A, B) of objects of $\mathcal{G}$ and for every arrow $g : MA \to MB$ of $\mathcal{H}$ there is an arrow $f : A \to B$ of $\mathcal{G}$ such that $g = Mf$; this means that M restricted to the hom-sets $\mathcal{G}(A,B)$ and $\mathcal{H}(MA, MB)$ is onto. Note that if a graph-morphism is one-one on objects, then it is faithful iff it is one-one on arrows, and if it is onto on objects, then it is full iff it is onto on arrows.

A graph-morphism is an *embedding* iff it is one-one both on objects and on arrows, and it is an *isomorphism* iff it is a bijection both on objects and on arrows.

A graph $\mathcal{G}$ is a *subgraph* of a graph $\mathcal{H}$ iff there is a graph-morphism M from $\mathcal{G}$ to $\mathcal{H}$ that is the inclusion function both on objects and on arrows; M is called the *inclusion graph-morphism* from $\mathcal{G}$ to $\mathcal{H}$. This means that the objects of $\mathcal{G}$ are included among the objects of $\mathcal{H}$ and the arrows of $\mathcal{G}$ among the arrows of $\mathcal{H}$, and for every object A of $\mathcal{G}$ the object MA of $\mathcal{H}$ is A, while for every arrow f of $\mathcal{G}$ the arrow Mf of $\mathcal{H}$ is f. Moreover, since M is a graph-morphism, the arrows of $\mathcal{G}$ have in $\mathcal{H}$ the same sources and targets as in $\mathcal{G}$. The inclusion graph-morphism M is an embedding, and a fortiori it is faithful. A subgraph is *full* iff the inclusion graph-morphism is full.

The *identity graph-morphism* $I_{\mathcal{G}}$ from a graph $\mathcal{G}$ to $\mathcal{G}$ is the identity function both on objects and on arrows. If we have a graph-morphism M from a graph $\mathcal{G}$ to a graph $\mathcal{H}$ and a graph-morphism N from a graph $\mathcal{H}$ to a graph $\mathcal{J}$, then we have the *composite graph-morphism* NM from $\mathcal{G}$ to $\mathcal{J}$ obtained by composing the functions M and N, on objects and on arrows.

Let M and N be graph-morphisms from a graph $\mathcal{G}$ to a graph $\mathcal{H}$. A *transformation* from M to N is a family τ of arrows $\tau_A : MA \to NA$ of $\mathcal{H}$, indexed by the objects A of $\mathcal{G}$. More precisely, a transformation τ is a function from the set of objects of $\mathcal{G}$ to the set of arrows of $\mathcal{H}$, with values $\tau(A)$, which is written τ_A, of type $MA \to NA$. Note that a transformation need not be one-one (i.e., for different objects A and B of $\mathcal{G}$, the arrows τ_A and τ_B may be equal, provided MA is MB and NA is NB).

A slightly more general notion than transformation is obtained by assuming that M and N are only functions from the objects of $\mathcal{G}$ to the objects of $\mathcal{H}$, everything else being as for transformations. We shall have two occasions to rely on this notion of *objectual transformation* (see 3.6 and 4.5).

1.4. Deductive systems, functors, natural transformations and categories. An *identity* $\mathbf{1}$ in a graph $\mathcal{G}$ is a family of arrows $\mathbf{1}_A : A \rightarrow A$ of $\mathcal{G}$, indexed by the objects A of $\mathcal{G}$. In other words, $\mathbf{1}$ is a transformation from $I_{\mathcal{G}}$ to $I_{\mathcal{G}}$. The arrows $\mathbf{1}_A$ are called *identity arrows*.

A *composition* $\circ$ in $\mathcal{G}$ is a function that to every pair $(f : A \rightarrow B,\ g : B \rightarrow C)$ of arrows of $\mathcal{G}$ assigns an arrow $g \circ f : A \rightarrow C$ of $\mathcal{G}$.

A *deductive system* is a triple $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ where $\mathcal{D}$ is a graph, $\mathbf{1}$ is an identity in $\mathcal{D}$ and $\circ$ is a composition in $\mathcal{D}$. The identity and composition of different deductive systems will always be denoted by the same symbols $\mathbf{1}$ and $\circ$, assuming it is clear from the context to which deductive system they belong. (The term "deductive system" was introduced by Lambek because of an obvious analogy with logical consequence. This analogy, which is not superficial, is at the base of *categorial proof theory*; see [Lambek & Scott 1986] and [D. 1996, 1997].)

A *functor* F from a deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ to a deductive system $\langle \mathcal{E}, \mathbf{1}, \circ \rangle$ is a graph-morphism from $\mathcal{D}$ to $\mathcal{E}$ that satisfies

$$(fun1) \qquad F\mathbf{1}_A = \mathbf{1}_{FA},$$
$$(fun2) \qquad F(g \circ f) = Fg \circ Ff.$$

These two conditions are just naturalness conditions for morphisms of identities (where identities are understood as functions) and morphisms of compositions.

An *embedding of deductive systems* is a graph-morphism that is a functor and an embedding, and an *isomorphism of deductive systems* is a graph-morphism that is a functor and an isomorphism. A deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ is a *subsystem* of a deductive system $\langle \mathcal{E}, \mathbf{1}, \circ \rangle$ iff there is a functor from $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ to $\langle \mathcal{E}, \mathbf{1}, \circ \rangle$ that is an inclusion graph-morphism from $\mathcal{D}$ to $\mathcal{E}$. As for subgraphs in general, a subsystem is *full* iff the inclusion graph-morphism is full.

It is clear that the identity graph-morphism $I_{\mathcal{D}}$ on the graph $\mathcal{D}$ of a deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ is a functor; it is called the *identity functor*. It is also clear that the composite graph-morphism GF is a functor when F and G are functors.

Let M and N be graph-morphisms from a graph $\mathcal{G}$ to a graph $\mathcal{H}$. If $\mathcal{H}$ has a composition $\circ$, and, a fortiori, if $\mathcal{H}$ is the graph of a deductive system $\langle \mathcal{H}, \mathbf{1}, \circ \rangle$, then a transformation from M to N is *natural* iff the following equality holds for every arrow $f : A \rightarrow B$ of $\mathcal{G}$:

$$(nat) \qquad \tau_B \circ Mf = Nf \circ \tau_A.$$

If Mf, Nf, τ_A and τ_B are functions and $\circ$ is functional composition, (nat) is the naturalness condition for the morphism (τ_A, τ_B) from Mf to Nf.

A deductive system is a *category* iff the following equalities hold between its arrows:

$$(\text{cat1right}) \qquad f \circ \mathbf{1}_A = f,$$
$$(\text{cat1left}) \qquad \mathbf{1}_B \circ f = f,$$
$$(\text{cat2}) \qquad (h \circ g) \circ f = h \circ (g \circ f).$$

A *subcategory* is a subsystem of a category.

Often, we denote a deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ simply by $\mathcal{D}$, taking the identity and composition for granted, provided it is clear from the context that we have in mind a deductive system, rather than simply a graph. We do the same for categories. If, however, we need to emphasize the difference between a deductive system and its graph, we use the notation $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$.

Note that our notion of functor is slightly more general than the usual notion, which is given for categories only, whereas ours apply to arbitrary deductive systems. Note also that our notion of natural transformation is likewise more general than the usual notion, which is given for functors M and N from a category $\mathcal{G}$ to a category $\mathcal{H}$.

1.5. Equivalence of categories. If a graph is a function pair (S,T), then a possible notion of morphism between graphs is not only our notion of graph-morphism, but also a more general notion, which we shall now introduce.

Let (f,h) be a function pair from (A_1,B_1) to (A_2,B_2) and (f',h') a function pair from (A_1',B_1') to (A_2',B_2'). A morphism from (f,h) to (f',h') is then simply two function pairs, (g_1,g_2) from (A_1,A_2) to (A_1',A_2'), which is a morphism from f to f', and (k_1,k_2) from (B_1,B_2) to (B_1',B_2'), which is a morphism from h to h'. If (f,h) and (f',h') are graphs, then $A_1 = B_1 = X$, $A_2 = B_2 = Y$, $A_1' = B_1' = X'$, $A_2' = B_2' = Y'$, but we could keep the same notion of morphism. Let us call these morphisms of graphs *double morphisms*.

A graph-morphism as we have defined it in 1.3 is a double morphism where $g_1 = k_1$ and $g_2 = k_2$. With double morphisms in general we would have a function pair (M_X, M_Y) that in virtue of naturalness preserves sources, i.e., $M_Y(S(f)) = S'(M_X(f))$, and another function pair (N_X, N_Y) that in virtue of naturalness preserves targets, i.e. $N_Y(T(f)) = T'(N_X(f))$.

On the other hand, if a graph is a function $\mathcal{F}$ from X to $Y \times Y$, then a possible notion of morphism is not only our notion of graph-morphism, but also another generalization of this notion. Namely, we would have a function pair $(M_X, M_{Y\times Y})$, where M_X is, as before, a function from X to X', but $M_{Y\times Y}$ is a function from $Y \times Y$ to $Y' \times Y'$. So pairs of objects are mapped to pairs of objects. The required naturalness condition is

$$M_{Y\times Y}(\mathcal{F}(f)) = \mathcal{F}'(M_X(f)).$$

Let us call these morphisms of graphs *single morphisms*. A graph-morphism is a single morphism where $M_{Y\times Y}$ is defined as $M_Y \times M_Y$ in terms of a function M_Y from Y to Y; for $M_Y \times M_Y$ we have

$$(M_Y \times M_Y)(A,B) = (M_Y(A), M_Y(B)).$$

The notion of graph-morphism is a common denominator of double and single morphisms, which can serve for either notion of graph.

An arrow $f : A \to B$ in a category is an ***isomorphism*** iff there is an arrow $g : B \to A$, called the ***inverse*** of f, such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$. Two objects A and B are ***isomorphic*** iff there is an isomorphism f of the type $A \to B$. A natural transformation τ is a ***natural isomorphism*** iff τ_A is an isomorphism for every A.

Two categories $\mathcal{A}$ and $\mathcal{B}$ are ***equivalent*** iff there is a functor F from $\mathcal{B}$ to $\mathcal{A}$ and a functor G from $\mathcal{A}$ to $\mathcal{B}$ such that there is in $\mathcal{A}$ a natural isomorphism from FG to $I_{\mathcal{A}}$ and there is in $\mathcal{B}$ a natural isomorphism from GF to $I_{\mathcal{B}}$. An equivalence of categories where these natural isomorphisms are identities boils down to isomorphism of categories as we have defined it in the preceding section.

It is easy to show that the category of graphs in the (S,T) sense (i.e., the category whose objects are these graphs) with graph-morphisms as arrows is isomorphic to the category of $\mathcal{F}$ graphs with graph-morphisms as arrows. Hence, these categories are also equivalent. This justifies our saying that the two notions of graph are equivalent. In general, two notions are to be called ***equivalent*** iff they cover objects of two categories that are equivalent.

When two notions are equivalent, it is common to say that we have just two *formulations* of the same notion, or that the same notion is defined in alternative ways. Formulations are then called equivalent, rather than notions. We will often speak in this less formal way, too.

Consider, now, the category of (S,T) graphs with double morphisms as arrows and the category of $\mathcal{F}$ graphs with single morphisms as arrows. These two categories are not equivalent, and neither of them is equivalent to the category of (S,T) graphs with graph-morphisms as arrows, or the category of $\mathcal{F}$ graphs with graph-morphisms as arrows. So, to determine whether two notions are equivalent, it is not enough to find a bijection between the objects that fall under these notions. We also have to find the appropriate morphisms, and prove an equivalence of categories.

With the notions that will be found equivalent later in this work we will find mostly isomorphisms of categories, rather than simply equivalences. We stick, however, to the terminology of "equivalent notions", because this way of speaking is more common ("isomorphic notions" would be a neologism), and because equivalence of categories catches well the intuitive idea of equivalence of notions.

2. Functions redefined

The notion of adjunction presupposes the more elementary notion of function, whose importance and ubiquity in mathematics are, of course, not necessary to mention, let alone justify. We want to show, however, that underlying the notion of function there is an adjunction, and that this adjunction characterizes completely the notion of function. This will serve as another corroboration of the slogan that adjointness arises everywhere.

The standard definitions of the general notions of function, *onto* function and *one-one* function don't exhibit clearly the regularities and symmetries of these notions. It is not immediately clear from these definitions, without some deducing, that

(1) the property of being a function is made of two components exactly dual to the *onto* and *one-one* properties (they go in the opposite direction),
(2) the *onto* and *one-one* properties are dual to each other.

There are definitions of these notions that exhibit immediately (1) and (2), but these definitions are rarely and cryptically mentioned (the earliest reference for them I know of is [Riguet 1948, p. 127]). On their own, these definitions are quite simple. I believe that their ingredients belong to the folklore and sometimes crop up as exercises in textbooks. However, the general picture they provide seems to be missing in the standard textbook approach. Many students of mathematics probably stay pretty much in the dark about (2), and perhaps even (1); many are probably surprised when, after having known for some time about onto functions and one-one functions, they learn about (2) via the cancellation properties of epi and mono arrows in category theory.

I don't wish to suggest that these nonstandard definitions should supplant the standard ones—especially not for a first exposure to the defined notions. I suppose, however, that at some point in the study of mathematics one should get a systematic picture such as will occupy us here.

2.1. The standard definition of function. A *binary relation* is a set of ordered pairs R together with some specified domain D and codomain C such that $R \subseteq D \times C$. We speak here only about "relations", the epithet "binary" being tacitly presupposed, and, as usual, we write $x\,R\,y$ for $(x,y) \in R$.

A *function* from D to C is a relation $R \subseteq D \times C$ such that for every x in D there is *exactly one* y in C for which $x\,R\,y$. It is easy to deduce that $R \subseteq D \times C$ is a function iff

(*left-total*) for every x in D there is ***at least one*** y in C such that $x\,R\,y$,

(*right-unique*) for every x in D there is ***at most one*** y in C such that $x\,R\,y$.

A function $R \subseteq D \times C$ is *onto* iff

(*right-total*) for every y in C there is ***at least one*** x in D such that $x\,R\,y$,

and it is *one-one* iff

(*left-unique*) for every y in C there is ***at most one*** x in D such that $x\,R\,y$.

For a relation $R \subseteq D \times C$ the conjunction of (*right-total*) and (*left-unique*) is equivalent to asserting that for every y in C there is *exactly one* x in D such that $x\,R\,y$. So, after a little bit of deducing, we obtained (1): the *onto* and *one-one* properties are the two components of functionality, but going from the codomain to the domain; functionality in the direction from the domain to the codomain is made of two completely analogous, dual, components.

What is still not quite evident is (2); namely, that the *onto* and *one-one* properties are also dual to each other. That "at least one" is dual to "at most one" may be gathered from the fact we can express that a set A is a singleton by the conjunction of "for some x_1 and x_2 in A, $x_1 = x_2$" (which amounts to "there is at least one member of A") and "for every x_1 and x_2 in A, $x_1 = x_2$" (which amounts to "there is at most one member of A"). When we deal specifically with functions, the duality between the *onto* and *one-one* properties is exhibited in category theory by showing that the first property amounts to cancellability on the right in functional composition, while the second property amounts to cancellability on the left. However, as we shall see in 2.3, if we assume functionality neither for $R \subseteq D \times C$ nor for the converse set of ordered pairs, we cannot exhibit in this manner the duality between (*right-total*) and (*left-unique*), or between (*left-total*) and (*right-unique*).

2.2. The square of functions. The definitions below will enable us to see the duality mentioned at the end of the preceding section in a different, more basic, manner—without extra assumptions concerning $R \subseteq D \times C$. They will also display clearly the connection between the *onto* and *one-one* properties and functionality.

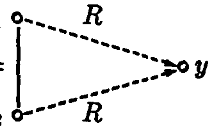
Let $R^{\leftarrow} \subseteq C \times D$ be the relation converse to $R \subseteq D \times C$, i.e., $R^{\leftarrow}$ is $\{(y,x) \mid x\,R\,y\}$, and let $R_2 \circ R_1$ be $\{(x,y) \mid \text{for some } z,\ x\,R_1\,z \text{ and } z\,R_2\,y\}$. (For the composition of R_1 with R_2 we write $R_2 \circ R_1$, rather than $R_1 \circ R_2$, so as not to deviate from standard usage when we come to functional composition. This standard usage is unfortunate—it clashes with our inclination to read other things from left to right—but it is hard to fight against. Anyway, what we have to say about functions does not depend upon reforming the notation for functional composition.) Next, for every set A, let 1_A be $\{(x,x) \mid x \in A\}$.

Then consider the following properties a relation $R \subseteq D \times C$ might have:

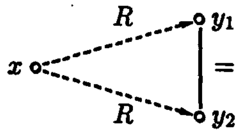
(*left-total*)	$1_D \subseteq R^{\leftarrow} \circ R$	(*right-total*)	$R \circ R^{\leftarrow} \supseteq 1_C$
(*left-unique*)	$1_D \supseteq R^{\leftarrow} \circ R$	(*right-unique*)	$R \circ R^{\leftarrow} \subseteq 1_C$

We pass from left to right in this square by replacing R by its converse $R^{\leftarrow}$ (of course, $R^{\leftarrow\leftarrow}$ is equal to R). We pass from the upper row to the lower row by replacing an inclusion by the converse inclusion.

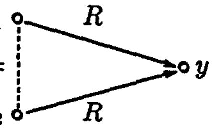
The diagrams in the figure below illustrate the four properties in the square. Solid lines are in the antecedents and dotted lines in the consequents. For example, the upper left diagram is read as follows: "If x_1 is equal to x_2, then we have arrows going from them to the right towards a point y." Since every point is equal to itself, this means that for every point in the domain we have at least one arrow going towards the codomain whose source is this point—the two dotted R arrows become one. The lower right diagram is read: "If we have arrows with the same source x, then their targets y_1 and y_2 are equal." So at most one arrow can start from a point of the domain—the two solid R arrows become one. It is easy to check with the help of these diagrams that the properties in the square are equivalent to the previously introduced properties that bear the same names.



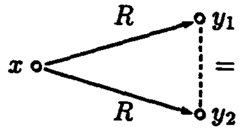
(*left-total*)



(*right-total*)



(*left-unique*)



(*right-unique*)

Functions are defined by the properties in the upper left and lower right corners of our square. With *onto* functions we cover the upper row and the lower right corner, and with *one-one* functions the lower row and the upper left corner. A relation $R \subseteq D \times C$ satisfies the properties in the upper right and lower left corner iff the converse relation $R^{\leftarrow} \subseteq C \times D$ is a function. Our square displays the duality between the *onto* and *one-one* properties, as well as the way how these properties are connected with functionality.

Each corner of the square is "one quarter" of a bijection, i.e., *one-to-one* correspondence. The notion of function involves half of these corners in a diagonal way. An explanation for this judicious choice is given in 2.4 below, when we talk of adjunction.

2.3. Cancellability of relations. Let us now consider how the properties from the square are connected with cancellation properties for relations in relational composition. A relation $R \subseteq D \times C$ may satisfy the property

(*right-cancellable*) for every S_1 and S_2, if $S_1 \circ R \subseteq S_2 \circ R$, then $S_1 \subseteq S_2$,

where $S_1 \subseteq C \times A$ and $S_2 \subseteq C \times A$ for some set A, or the property

(*left-cancellable*) for every S_1 and S_2, if $R \circ S_1 \subseteq R \circ S_2$, then $S_1 \subseteq S_2$,

where $S_1 \subseteq A \times D$ and $S_2 \subseteq A \times D$ for some set A. Note that (*right- cancellable*) and (*left-cancellable*) are equivalent, respectively, to the properties obtained by replacing $\subseteq$ in them by $=$ (to show that, we may use $(S_1 \cup S_2) \circ R = (S_1 \circ R) \cup (S_2 \circ R)$ and $R \circ (S_1 \cup S_2) = (R \circ S_1) \cup (R \circ S_2)$; with $\cup$ replaced by $\cap$ we have the inclusions from left to right of these two distributions, but the converse inclusions may fail).

Since for every relation R we have $R \subseteq R \circ R^{\leftarrow} \circ R$, it is easy to verify that (*right-cancellable*) implies (*right-total*), but for the converse implication we only have that the conjunction of (*right-unique*) and (*right-total*) implies (*right-cancellable*); neither (*right-unique*) alone nor (*right-total*) alone does so. (Let $D = \{d\}$, $C = \{c_1, c_2\}$ and $A = \{a\}$; then for $R = \{(d, c_2)\}$, $S_1 = \{(c_1, a)\}$ and $S_2 = \emptyset$, we have that (*right-unique*) holds, while neither (*right-total*) nor (*right-cancellable*) does, and for $R = \{(d,c_1),(d,c_2)\}$, $S_1 = \{(c_1,a)\}$ and $S_2 = \{(c_2,a)\}$, we have that (*right-total*) holds, while neither (*right-unique*) nor (*right-cancellable*) does hold.) We also have that the conjunction of (*right-unique*) and (*left-cancellable*) implies (*left-unique*), whereas (*left-cancellable*) alone does not (provided A is allowed to be empty). Of course, we obtain something quite analogous if in all these implications we replace everywhere "*right*" by "*left*" and "*left*" by "*right*".

So if R is a function, then (*right-cancellable*) is equivalent to (*right-total*) and (*left-cancellable*) is equivalent to (*left-unique*), but if R is not a function, these equivalences may fail.

2.4. Function and adjunction. Finally, let us try to justify the choice of properties from the square that enter into the definition of function. For $R \subseteq D \times C$ a relation, A a subset of D and B a subset of C, let $\mathbf{R}(A)$ be the set $\{y \in C \mid \text{for some } x \in A,\ x\,R\,y\}$ and $\mathbf{R}^{\leftarrow}(B)$ the set $\{x \in D \mid \text{for some } y \in B,\ x\,R\,y\}$. If $\mathcal{P}(X)$ is the power set of a set X, then for every relation $R \subseteq D \times C$, we have two functions $\mathbf{R} : \mathcal{P}(D) \to \mathcal{P}(C)$ and $\mathbf{R}^{\leftarrow} : \mathcal{P}(C) \to \mathcal{P}(D)$, monotonic with respect to $\subseteq$. We can easily verify that (*left-total*) is equivalent to

$$(\gamma) \qquad \text{for every } A \subseteq D,\ A \subseteq \mathbf{R}^{\leftarrow}(\mathbf{R}(A)),$$

while (*right-unique*) is equivalent to

$$(\varphi) \qquad \text{for every } B \subseteq C,\ \mathbf{R}(\mathbf{R}^{\leftarrow}(B)) \subseteq B.$$

On the other hand, (γ) is equivalent to the left-to-right implication and (φ) to the right-to-left implication of the equivalence

$$(*) \qquad \text{for every } A \subseteq D \text{ and every } B \subseteq C,\ \mathbf{R}(A) \subseteq B \text{ iff } A \subseteq \mathbf{R}^{\leftarrow}(B).$$

So, $\mathbf{R}$ and $\mathbf{R}^{\leftarrow}$ establish a covariant Galois connection between $\langle \mathcal{P}(D), \subseteq \rangle$ and $\langle \mathcal{P}(C), \subseteq \rangle$ iff $R \subseteq D \times C$ is a function. In more general terms, for the preorders $\langle \mathcal{P}(D), \subseteq \rangle$ and $\langle \mathcal{P}(C), \subseteq \rangle$ understood as categories (objects are subsets of D and C, and arrows exist between these objects whenever inclusion obtains), the functors $\mathbf{R}$ and $\mathbf{R}^{\leftarrow}$ together with the natural transformations induced by (γ) and (φ) make an adjunction, where $\mathbf{R}$ is left-adjoint and $\mathbf{R}^{\leftarrow}$ right-adjoint, the natural transformations of (γ) and (φ) being, respectively, the unit and counit of the adjunction. We have this adjunction if and only if $R \subseteq D \times C$ is a function. (The "if" part of this equivalence is stated in [Mac Lane 1971, p. 94].)

For every relation $R \subseteq D \times C$ we have that

$$(**) \quad \text{for every } A \subseteq D \text{ and every } B \subseteq C,\ \mathbf{R}(A) \subseteq B \text{ iff } A \subseteq D - \mathbf{R}^{\leftarrow}(C - B).$$

(I am indebted to Aleksandar Lipkovski for having drawn my attention to (**) with his note [1995], where it appears in the equivalent form

$$\text{for every } A \subseteq D \text{ and every } B \subseteq C,\ \mathbf{R}^{\leftarrow}(B) \subseteq A \text{ iff } B \subseteq C - \mathbf{R}(D - A).)$$

We also have that $R \subseteq D \times C$ is a function iff

(***) $\qquad$ for every $B \subseteq C$, $\mathbf{R}^{\leftarrow}(B) = D - \mathbf{R}^{\leftarrow}(C - B)$.

So, underlying the Galois connection of (*) there is a Galois connection of wider scope, but less pleasing. (The equivalence (**) is implicitly present in temporal logic through the connection between future necessity and past possibility. The equality (***) is also to be found in modal logic, when the functionality of the accessibility relation of Kripke models makes necessity and possibility coincide; see, for example, [Hughes & Cresswell 1996].)

The equivalence "$R \subseteq D \times C$ is a function iff (*)" may hardly serve as an alternative definition of the notion of function, since this notion is presupposed in the definitions of the mappings, or functors, $\mathbf{R}$ and $\mathbf{R}^{\leftarrow}$. However, the adjunction in this equivalence may help to explain why the notion of function, rather than some other notion (for example, the notion of *partial* function, without left totality, or the notion of *onto* function, with right totality), is so important in mathematics. Conversely, if we are already convinced of the importance of the notion of function—as we should be—our equivalence may explain why Galois correspondence and adjointness are important.

3. Definitions of adjunction

We shall now survey the standard definitions of adjunction. However, rather than simply rehash familiar matters, we present also two presumably new definitions of this notion.

One is a definition that does not economize on primitives. It takes as primitive notions the two adjoint functors, F and G, and both the natural transformations that are the counit and unit of the adjunction and the two bijections between the hom-sets $\mathcal{A}(FB, A)$ and $\mathcal{B}(B, GA)$. Usually, if the counit and unit are primitive, the bijections are defined, and vice versa. Having both kinds of notions primitive, together with the adjoint functors, enables us to formulate the specific equalities between arrows one finds in adjointness as a series of equalities defining one of these notions in terms of two remaining notions. These definitional equalities make a regular pattern, which should clarify standard definitions of adjunction.

We shall compare this uneconomical, but regular and simple, definition to standard definitions of adjunction (like those that may be found in MacLane's book [1971, IV]), and show that the notions defined are equivalent. Among the standard definitions we favour those that, like the uneconomical definition, are equationally presented. We also envisage defining adjunction in a more general kind of context—in particular, a context where F and G may fail to be functors because they don't satisfy (fun1), but only (fun2). That is, F and G are only *semifunctors* (cf. 3.4 below).

In 3.7 we consider the other nonstandard definition of adjunction. This one is, on the contrary, an economical definition, where only the functions F and G

on objects and the bijections between the hom-sets $\mathcal{A}(FB, A)$ and $\mathcal{B}(B, GA)$ are primitive. So neither of the adjoint functors F and G is taken as primitive. This economical definition simplifies one of the standard definitions.

3.1. Primitive notions in adjunction. Let $\mathcal{A}$ and $\mathcal{B}$ be two graphs. The objects of $\mathcal{A}$ will be designated by $A, A_1, A_2, \ldots$, and the arrows of $\mathcal{A}$ by $f, f_1, f_2, \ldots$, while the objects of $\mathcal{B}$ will be designated by $B, B_1, B_2, \ldots$, and the arrows of $\mathcal{B}$ by $g, g_1, g_2, \ldots$

Let F be a graph-morphism from $\mathcal{B}$ to $\mathcal{A}$ and G a graph-morphism from $\mathcal{A}$ to $\mathcal{B}$. When we need it for emphasis, we shall write F^a and G^a for the functions on arrows, and F^o and G^o for the functions on objects, of the graph-morphisms F and G. However, in most cases we will, as usual, omit these superscripts.

Let φ be a transformation from the composite graph-morphism FG to the identity graph-morphism $I_{\mathcal{A}}$ and γ a transformation from the identity graph-morphism $I_{\mathcal{B}}$ to the composite graph-morphism GF. (Remember that, as defined in 1.3, a transformation is a family of arrows like a natural transformation for which we don't assume (nat).)

Finally, for every pair of objects (A, B) (where, according to our convention, A is from $\mathcal{A}$ and B is from $\mathcal{B}$), let $\Phi_{B,A}$ be a function assigning to an arrow $g : B \to GA$ of $\mathcal{B}$ the arrow $\Phi_{B,A}g : FB \to A$ of $\mathcal{A}$, and let $\Gamma_{B,A}$ be a function assigning to an arrow $f : FB \to A$ of $\mathcal{A}$ the arrow $\Gamma_{B,A}f : B \to GA$ of $\mathcal{B}$. We denote by Φ the family of all the functions $\Phi_{B,A}$ and by Γ the family of all the functions $\Gamma_{B,A}$; we call the functions in these families the *seesaw* functions.

Consider now the following six notions we have just introduced:

the functions on arrows F^a and G^a,

the transformations φ and γ,

the families of seesaw functions Φ and Γ.

If $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are deductive systems, each of these notions can be defined in terms of other two notions from the list (with the help of the identities and compositions of $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$) by the following equalities:

for $g : B_1 \to B_2$

(F^a) $\quad Fg = \Phi_{B_1,FB_2}(\gamma_{B_2} \circ g)$,

(φ) $\quad \varphi_A = \Phi_{GA,A}G\mathbf{1}_A$,

for $g : B \to GA$

(Φ) $\quad \Phi_{B,A}g = \varphi_A \circ Fg$,

for $f : A_1 \to A_2$

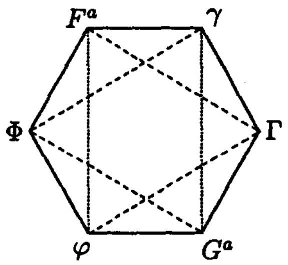
(G^a) $\quad Gf = \Gamma_{GA_1,A_2}(f \circ \varphi A_1)$,

(γ) $\quad \gamma_B = \Gamma_{B,FB}F\mathbf{1}_B$,

for $f : FB \to A$

(Γ) $\quad \Gamma_{B,A}f = Gf \circ \gamma_B$.

The definitional dependences among these notions can be read off from the following hexagonal figure.



The notion in each vertex is definable in terms of the two notions in the neighbouring vertices on the left and on the right. For example, F^a is definable in terms of γ and Φ, while Φ is definable in terms of F^a and φ, etc. On the left-hand side of the hexagon we have F and its Greek correlates, while on the right-hand side we have G with its Greek correlates. Vertices on the big, undrawn, diagonals have labels of the same type: (F^a, G^a), (Φ, Γ) and (φ, γ).

The small, dotted, diagonals are drawn to indicate possible choices of primitives, in terms of which all the six notions can be defined. In the following table we indicate with $+$ the notions taken as primitive by the choice named in the leftmost column.

	F^a	G^a	φ	γ	Γ	Φ
hexagonal	+	+	+	+	+	+
rectangular ||	+	+	+	+		
rectangular \\	+	+			+	+
rectangular //			+	+	+	+
triangular ▷	+		+		+	
triangular ◁		+		+		+

Besides these choices, there are six uneconomical pentagonal choices, with five primitives, and six more uneconomical choices with four primitives, obtained by adding a vertex to one of the triangular choices (so, altogether, we have 18 choices). What can be said about these additional uneconomical choices should be easy to infer from what is said below about the rectangular and triangular choices; so we shall not consider them separately. (In 3.7 below, we shall find one more choice, very economical, with only Φ and Γ primitive; however, this choice is based on slightly different definitional equalities.)

The hexagonal definitional pattern above becomes even more regular if we take into account the identities and compositions of the deductive systems $\mathcal{A}$ and $\mathcal{B}$. For the composition of $\mathcal{A}$, let us introduce the following notation

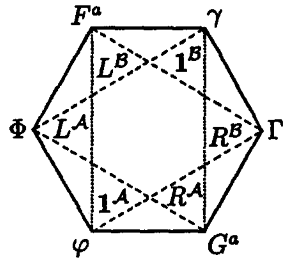
$$L^{\mathcal{A}}_{f_2}(f_1) = R^{\mathcal{A}}_{f_1}(f_2) = f_2 \circ f_1,$$

and analogously for the composition of $\mathcal{B}$. Next, let $\mathbf{1}^{\mathcal{A}}$ and $\mathbf{1}^{\mathcal{B}}$ be the identities of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Then the definitional equalities above become

$$\begin{array}{llll} (F^a) & Fg = \Phi L^B_\gamma g, & (G^a) & Gf = \Gamma R^A_\varphi f, \\ (\varphi) & \varphi = \Phi G 1^A, & (\gamma) & \gamma = \Gamma F 1^B, \\ (\Phi) & \Phi g = L^A_\varphi F g, & (\Gamma) & \Gamma f = R^B_\gamma G f, \end{array}$$

where, to make matters clearer, we have omitted parentheses and subscripts referring to objects. Our hexagonal figure with these additional notions involved in the definitions looks as follows.



3.2. Hexagonal adjunction.

The hexagonal choice of primitives of the preceding section is interesting because we can define adjunction as follows. The conditions

$\langle A, \mathbf{1}, \circ\rangle$ and $\langle B, \mathbf{1}, \circ\rangle$ are categories,
F and G are functors,
φ and γ are natural transformations,
Φ and Γ are families of seesaw functions,
$(F^a), (G^a), (\varphi), (\gamma), (\Phi)$ and (Γ) hold

are satisfied iff the functors F and G are *adjoint*, F being left adjoint and G right adjoint. The natural transformations φ and γ are respectively the *counit* and *unit* of the adjunction (often written ε and η).

In the next sections we shall verify that this notion of adjunction is indeed equivalent to the more usual ones, behind which stand more economical choices of primitives from the table above.

Note that if we replace the equalities (φ) and (γ) by the equalities

$$(\varphi^\circ) \quad f \circ \varphi_{A_1} = \Phi_{GA_1,A_2} Gf, \qquad\qquad (\gamma^\circ) \quad \gamma_{B_2} \circ g = \Gamma_{B_1,FB_2} Fg,$$

for $f : A_1 \to A_2$ and $g : B_1 \to B_2$, then the condition that the transformations φ and γ are natural becomes redundant. The equalities (φ°) and (γ°) are an immediate consequence of (Φ), (Γ) and (nat) for φ and γ. On the other hand, these two equalities yield (φ) and (γ) in the presence of (cat1left) and (cat1right). However, (φ°) and (γ°) are not exactly definitions of the transformations φ and γ, but rather definitions of composing with φ on the right and γ on the left (i.e., of $R^{\mathcal{A}}_{\varphi}$ and $L^{\mathcal{B}}_{\gamma}$).

3.3. Rectangular | | adjunction.

Suppose

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F^a and G^a satisfy (fun2),
(Φ) and (Γ) hold.

Then the equalities (F^a), (G^a), (φ) and (γ) are interderivable with the equalities

$$(\varphi\gamma F) \quad \varphi_{FB} \circ F\gamma_B = F\mathbf{1}_B, \qquad (\varphi\gamma G) \quad G\varphi_A \circ \gamma_{GA} = G\mathbf{1}_A,$$
$$(\varphi\mathbf{1}) \quad \varphi_A \circ FG\mathbf{1}_A = \varphi_A, \qquad (\gamma\mathbf{1}) \quad GF\mathbf{1}_B \circ \gamma_B = \gamma_B,$$

from which Φ and Γ are absent.

Let us first derive the latter equalities from the former. For $(\varphi\gamma F)$ we have

$$\begin{aligned} \varphi_{FB} \circ F\gamma_B &= \Phi_{B,FB}\gamma_B, \text{ by}(\Phi) \\ &= F\mathbf{1}_B, \text{ by (cat1right) and } (F^a). \end{aligned}$$

For $(\varphi\mathbf{1})$ we have

$$\begin{aligned} \varphi_A \circ FG\mathbf{1}_A &= \Phi_{GA,A}G\mathbf{1}_A, \text{ by } (\Phi) \\ &= \varphi_A, \text{ by } (\varphi). \end{aligned}$$

We proceed analogously for $(\varphi\gamma G)$ and $(\gamma\mathbf{1})$.

Conversely, we derive (F^a) as follows:

$$\begin{aligned} \Phi_{B_1,FB_2}(\gamma_{B_2} \circ g) &= (\varphi_{FB_2} \circ F\gamma_{B_2}) \circ Fg, \text{by } (\Phi), \text{(fun2) and (cat2)} \\ &= F\mathbf{1}_{B_2} \circ Fg, \text{by } (\varphi\gamma F) \\ &= Fg, \text{ by (fun2) and (cat1left).} \end{aligned}$$

For (φ) we use (Φ) and $(\varphi\mathbf{1})$, and we proceed analogously for (G^a) and (γ).

In the standard definition of adjunction with the rectangular | | choice of primitives we have that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F and G are functors,
φ and γ are natural transformations,
Φ and Γ may be defined by (Φ) and (Γ),
$(\varphi\gamma F)$ and $(\varphi\gamma G)$ hold.

In fact, instead of the equalities $(\varphi\gamma F)$ and $(\varphi\gamma G)$ we usually have the equalities obtained from them by replacing the right-hand sides with $\mathbf{1}_{FB}$ and $\mathbf{1}_{GA}$, respectively. These other equalities clearly amount to $(\varphi\gamma F)$ and $(\varphi\gamma G)$ in the presence of (fun1) for F^a and G^a.

With this standard definition of adjunction, the equalities (φ1) and (γ1) follow either from (fun1) for F^a and G^a, or from the assumption that φ and γ are natural transformations (together with (cat1right) and (cat1left)). This is enough to conclude that the notion of adjunction of the preceding section is indeed equivalent to the standard notion with the rectangular || choice of primitives.

3.4. Rectangular \\ adjunction. Suppose

$\langle \mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F^a and G^a satisfy (fun2),
(φ) and (γ) hold.

Then the equalities (F^a), (G^a), (Φ), (Γ) and (nat) for φ and γ are interderivable with the following equalities (in which, since we have (cat2), we don't write parentheses in compositions, and the subscripts of Φ and Γ are omitted so as not to encumber notation excessively; these subscripts can be recovered from the context):

$(\Phi\Gamma)$	$\Phi(Gf_3 \circ \Gamma f_2 \circ g_1) = f_3 \circ f_2 \circ Fg_1,$	$(\Gamma\Phi)$	$\Gamma(f_3 \circ \Phi g_2 \circ Fg_1) = Gf_3 \circ g_2 \circ g_1,$
$(\Phi\Phi)$	$\Phi(Gf_3 \circ g_2 \circ g_1) = f_3 \circ \Phi g_2 \circ Fg_1,$	$(\Gamma\Gamma)$	$\Gamma(f_3 \circ f_2 \circ Fg_1) = Gf_3 \circ \Gamma f_2 \circ g_1,$
(ΦF)	$\Phi g \circ F\mathbf{1}_B = \Phi g,$	(ΓG)	$G\mathbf{1}_A \circ \Gamma f = \Gamma f.$

In these equalities φ and γ don't occur.

Equalities like these were considered in [Hayashi 1985] and [Hoofman 1993], which deal with notions of adjoint semifunctors, i.e., graph-morphisms satisfying only (fun2), and not necessarily also (fun1). (Note that at the beginning of the preceding section we also didn't assume (fun1) to find equalities without Φ and Γ equivalent to (F^a), (G^a), (φ) and (γ).)

In the standard definition of adjunction with the rectangular \\ choice of primitives we have that

$\langle \mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F and G are functors,
Φ and Γ are families of seesaw functions,
φ and γ may be defined by (φ) and (γ),
the following equalities hold:

$(\Phi\Gamma')$ $\Phi\Gamma f = f,$ $(\Gamma\Phi')$ $\Gamma\Phi g = g,$
$(\Phi\Phi')$ $\Phi(g_2 \circ g_1) = \Phi g_2 \circ Fg_1,$
$(\Phi\Phi'')$ $\Phi(Gf \circ g) = f \circ \Phi g.$

The equalities $(\Phi\Phi')$ and $(\Phi\Phi'')$ can be replaced by

$(\Gamma\Gamma')$ $\Gamma(f_2 \circ f_1) = Gf_2 \circ \Gamma f_1,$
$(\Gamma\Gamma'')$ $\Gamma(f \circ Fg) = \Gamma f \circ g.$

It is easy to see that, due to the presence of (fun1) for F^a and G^a, the equalities $(\Phi\Gamma')$, $(\Gamma\Phi')$, $(\Phi\Phi')$ and $(\Phi\Phi'')$ amount to $(\Phi\Phi)$, $(\Gamma\Gamma)$, $(\Phi\Gamma)$, $(\Gamma\Phi)$, (ΦF) and (ΓG).

In this standard definition of rectangular \\ adjunction, $(\Phi\Phi')$ can be replaced by

$$\Phi g = \Phi\mathbf{1}_{GA} \circ Fg,$$

an equality that in the presence of (φ) and (fun1) for G^a amounts to (Φ). Analogously, $(\Gamma\Gamma')$ can be replaced by an equality that in the presence of (γ) and (fun1) for F^a amounts to (Γ):

$$\Gamma f = Gf \circ \Gamma\mathbf{1}_{FA}.$$

The equalities $(\Phi\Phi')$ and $(\Phi\Phi'')$ can be replaced by the implication

$$\text{if } g_2 \circ g_1 = Gf \circ g, \text{ then } \Phi g_2 \circ Fg_1 = f \circ \Phi g$$

(to show that we use (cat1right), (cat1left) and (fun1) for F^a and G^a). Analogously, $(\Gamma\Gamma')$ and $(\Gamma\Gamma'')$ can be replaced by the implication

$$\text{if } f_2 \circ f_1 = f \circ Fg, \text{ then } Gf_2 \circ \Gamma f_1 = \Gamma f \circ g.$$

With these implications, which are involved in Lawvere's definition of adjunction as an isomorphism of comma categories (see [Mac Lane 1971, p. 84, Exercise 2, and p. 53]), we abandon, however, the equational style of defining adjunction favoured here.

3.5. Rectangular // adjunction. If $\mathcal{A}$ and $\mathcal{B}$ are deductive systems that satisfy (cat1right) and (cat1left), and (F^a) and (G^a) hold, then it is clear that the equalities (φ), (γ), (Φ) and (Γ) are interderivable with the equalities

$$\varphi_A = \Phi_{GA,A}\Gamma_{GA,A}\varphi_A, \qquad \gamma_B = \Gamma_{B,FB}\Phi_{B,FB}\gamma_B,$$
$$\Phi_{B,A}g = \varphi_A \circ \Phi_{B,FGA}(\gamma_{GA} \circ g), \qquad \Gamma_{B,A}f = \Gamma_{GFB,A}(f \circ \varphi_{FB}) \circ \gamma_B,$$

from which the functions F^a and G^a are absent. (The equalities in the first line are instances of $(\Phi\Gamma')$ and $(\Gamma\Phi')$, respectively.) However, there doesn't seem to be a standard definition of adjunction with the rectangular // choice of primitives, which would be based on equalities such as these. Standard definitions take the adjoint functors F and G, or at least one of them, as primitive. In 3.7, we shall consider a definition of adjunction where neither of the functions F^a and G^a is primitive.

3.6. Triangular adjunction. Suppose

$\langle\mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle\mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F^a satisfies (fun2),
φ satisfies (nat),
(G^a), (γ) and (Φ) hold.

Then the equalities (F^a), (φ), (Γ), (fun2) for G^a and (nat) for γ are interderivable with the equalities

$$(\beta) \qquad \varphi_A \circ F\Gamma_{B,A}f = f \circ F\mathbf{1}_B,$$
$$(\Gamma\Gamma'') \qquad \Gamma_{B_1,A}(f \circ Fg) = \Gamma_{B_2,A}f \circ g,$$

from which G^a, γ and Φ are absent. Note that again we have not assumed (fun1) for F^a (nor for G^a).

The equality (β) could be replaced above by

$$\varphi_A \circ F(\Gamma_{B,A}f \circ g) = f \circ Fg,$$

while in the presence of the assumptions that $\mathcal{A}$ is a category, that F^a satisfies (fun2) and that (G^a) and (β) hold, the equality (nat) for φ is replaceable by

$$\varphi_A \circ F\mathbf{1}_{GA} = \varphi_A.$$

This last equality follows, of course, from (cat1right) and (fun1) for F^a.

In the standard definition of adjunction with the triangular choice of primitives we have that

$\langle\mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle\mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F is a functor and G^o is a function on objects,
φ is an objectual transformation,
Γ is a family of seesaw functions,
G^a, γ and Φ may be defined by (G^a), (γ) and (Φ),
the following equalities hold:

$$(\beta') \qquad \varphi_A \circ F\Gamma_{B,A}f = f,$$
$$(\eta) \qquad \Gamma_{B,A}(\varphi_A \circ Fg) = g.$$

Remember that an "objectual transformation", as specified in 1.3, is like a transformation between functions on objects, instead of graph-morphisms. We didn't assume that G^o belongs to a graph-morphism; so, to be precise, we can say only that φ is an objectual transformation from the composite function FG on the objects of $\mathcal{A}$ to the identity function on the objects of $\mathcal{A}$.

Note that in the presence of (Φ), the equalities (β') and (η) can be written as ($\Phi\Gamma'$) and ($\Gamma\Phi'$). (The names "β" and "η" come from the adjunction of cartesian closed categories, where the corresponding equalities are related to β and η conversion in the typed lambda calculus.)

It is clear that with (cat1right) and (fun1) for F^a the equality (β) amounts to (β'). On the other hand, ($\Gamma\Gamma''$), (cat1left) and

$$(\Gamma\varphi) \qquad \Gamma_{GA,A}\varphi_A = \mathbf{1}_{GA}$$

yield (η), while, conversely, from (β'), (η), (cat2) and (fun2) for F^a we obtain $(\Gamma\Gamma'')$, and from (η), (cat1right) and (fun1) for F^a we obtain $(\Gamma\varphi)$. The equality (β') implies in the presence of (G^a) that φ satisfies (nat).

The equality (η) is replaceable by the implication

$$\text{if } \varphi_A \circ Fg = f, \text{ then } g = \Gamma_{B,A} f,$$

which together with (β') is tantamount to asserting that there is a unique g such that $\varphi_A \circ Fg = f$. The definition of adjunction via a solution to a universal arrow problem is based on that (see [Mac Lane 1971, IV.1, p. 81, Theorem 2(iv)]).

Since (β') is replaceable by the converse implication, and since we have (Φ), we could assume instead of (β') and (η) the equivalence

$$g = \Gamma_{B,A} f \quad \text{iff} \quad \Phi_{B,A} g = f,$$

which is another way of assuming $(\Phi\Gamma')$ and $(\Gamma\Phi')$. However, with these implications and this equivalence we abandon the equational style of defining adjunction favoured here.

For the definition of adjunction with the triangular $\triangleleft$ choice of primitives we would have completely analogous considerations.

3.7. Seesaw adjunction. The rectangular \\ and rectangular // choices of primitives are not minimal for defining adjunction if we change slightly the defining equalities (F^a), (G^a), (φ) and (γ). The transformations φ and γ may be defined as follows in terms of Φ and Γ without F^a and G^a:

$$(\varphi') \quad \varphi_A = \Phi_{GA,A} \mathbf{1}_{GA}, \qquad (\gamma') \quad \gamma_B = \Gamma_{B,FB} \mathbf{1}_{FB},$$

which serves to transform (F^a) and (G^a) into the following definitions of F^a and G^a in terms of Φ and Γ without φ and γ:

$$(F^{a\prime}) \quad Fg = \Phi_{B_1,FB_2}(\Gamma_{B_2,FB_2} \mathbf{1}_{FB_2} \circ g),$$
$$(G^{a\prime}) \quad Gf = \Gamma_{GA_1,A_2}(f \circ \Phi_{GA_1,A_1} \mathbf{1}_{GA_1}).$$

We then have a definition of adjunction where

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F^o and G^o are functions on objects,
Φ and Γ are families of seesaw functions,
F^a, G^a, φ and γ may be defined by $(F^{a\prime})$, $(G^{a\prime})$, (φ') and (γ'),
the following equalities hold:

$$(\Phi\Gamma') \quad \Phi\Gamma f = f, \qquad (\Gamma\Phi') \quad \Gamma\Phi g = g,$$
$$(\Phi\Phi''') \quad \Phi(g_2 \circ g_1) = \Phi g_2 \circ \Phi(\Gamma \mathbf{1} \circ g_1)$$

(with the subscripts of Φ, Γ and $\mathbf{1}$ omitted).

We could replace $(\Phi\Phi''')$ by

$$(\Gamma\Gamma''') \qquad \Gamma(f_2 \circ f_1) = \Gamma(f_2 \circ \Phi\mathbf{1}) \circ \Gamma f_1.$$

To verify that this notion of adjunction is equivalent to the usual ones it suffices to show that it is equivalent to the notion with the rectangular \\ choice of primitives of 3.4. For that we have first to check that F^a and G^a defined by $(F^{a\prime})$ and $(G^{a\prime})$ satisfy (fun1) and (fun2). Next, the equalities $(\Phi\Phi''')$ and $(\Gamma\Gamma''')$ amount to the equalities $(\Phi\Phi')$ and $(\Gamma\Gamma')$ of 3.4 in the presence of $(F^{a\prime})$ and $(G^{a\prime})$, while equalities corresponding to $(\Phi\Phi'')$ and $(\Gamma\Gamma'')$ are now derivable. Here is a derivation of $(\Phi\Phi'')$:

$$\begin{aligned}\Phi(Gf \circ g) &= \Phi\Gamma(f \circ \Phi\mathbf{1}) \circ \Phi(\Gamma\mathbf{1} \circ g), \quad \text{by } (G^{a\prime}) \text{ and } (\Phi\Phi''') \\ &= f \circ \Phi g, \quad \text{by } (\Phi\Gamma'),\ (\Phi\Phi'''),\ \text{(cat1left) and (cat2)}\end{aligned}$$

(cf. [D. 1996, section 3.1]).

This economical definition of adjunction is at the opposite end of the hexagonal definition of 3.2, in which we did not economize on primitives.

To prove strictly the equivalences of various notions of adjunction considered here, we would have to introduce the appropriate morphisms between adjunctions and demonstrate equivalences of categories, which would actually be isomorphisms of categories. We shall not do that, however, since this rather straightforward matter would take too much space. We define morphisms between adjunctions in 5.1 below.

4. Definitions of comonad

We shall now survey definitions of comonad. Besides the standard definition of this notion, we shall present several alternative definitions, of equivalent notions.

The principle guiding this survey will be the adjunction between the category of our comonad and a subcategory of it, equivalent to the Kleisli category, which we will call *the delta category*. This adjunction defines the comonad, and since adjunction can be formulated in various ways, as we saw in the preceding part, we may envisage various definitions of comonad. After extracting as many interesting definitions as we could find, we compare the delta category of a comonad to its Kleisli and Eilenberg-Moore categories. These last categories play an essential role in the adjunctions involving the category of adjunctions and the category of comonads, which we shall consider in 5.3.

Of course, we could as well deal throughout with monads. Our only reason for preferring comonads is that, from a logical point of view, they seem to bear a certain primacy over monads, as the universal quantifier bears a primacy over the existential quantifier. On the other hand, from an algebraic point of view, monads bear a primacy over comonads (see [Mac Lane 1971, VI] and [Manes 1976]).

4.1. Standard definition of comonad. Suppose we are given the following:

a deductive system $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$,

a graph-morphism D from $\mathcal{A}$ to $\mathcal{A}$,

a transformation ε from D to the identity graph-morphism $I_{\mathcal{A}}$,

a transformation δ from D to the composite graph-morphism DD.

So in ε we have the arrows $\varepsilon_A : DA \to A$, and in δ the arrows $\delta_A : DA \to DDA$. Then we say that $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ is a *comonograph*. We may say that this is a comonograph *in* $\mathcal{A}$, and we use sometimes the same form of speaking with comonads, later. To simplify the notation, we don't mention the identity and composition of $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$, taking them for granted.

A *monograph* would be a comonograph with arrows reversed—sources become targets and targets sources. Note that the function D on objects in a comonograph resembles a topological interior operation, while in a monograph it would resemble a closure operation.

The appropriate morphisms between comonographs will be called *comonofunctors*. A comonofunctor from a comonograph $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ to a comonograph $\langle \mathcal{A}', D', \varepsilon', \delta' \rangle$ is a functor N from the deductive system $\mathcal{A}$ to the deductive system $\mathcal{A}'$ such that the following naturalness equalities hold:

$$ND = D'N,$$
$$N\varepsilon_A = \varepsilon'_{NA},$$
$$N\delta_A = \delta'_{NA}.$$

A *comonad* is a comonograph $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ such that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ is a category,

D is a functor,

ε and δ are natural transformations,

the following equalities hold:

$$(\varepsilon\delta) \qquad \varepsilon_{DA} \circ \delta_A = \mathbf{1}_{DA},$$
$$(\varepsilon\delta D) \qquad D\varepsilon_A \circ \delta_A = \mathbf{1}_{DA},$$
$$(\delta\delta) \qquad D\delta_A \circ \delta_A = \delta_{DA} \circ \delta_A.$$

A *monad* (also called a *triple*) is a comonad with arrows reversed.

4.2. The delta category. Let $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ be a comonad, and for an arrow $f : DA \to A'$ of $\mathcal{A}$ let the arrow $\Delta f : DA \to DA'$ be defined by

$$\Delta f \overset{\text{def}}{=} Df \circ \delta_A.$$

The operation Δ should be taken as indexed by A, and the same index is inherited by ⊚ in 4.5, but we take these indices for granted and omit them.

Then consider the subgraph $\mathcal{A}_\Delta$ of $\mathcal{A}$ whose objects are the objects of $\mathcal{A}$ of the form DA and whose arrows are the arrows of $\mathcal{A}$ of the form Δf. In $\mathcal{A}_\Delta$, there is an identity made of the arrows $\mathbf{1}_{DA}$ of $\mathcal{A}$ and the composition of $\Delta f_1 : DA_1 \to DA_2$ and $\Delta f_2 : DA_2 \to DA_3$ is defined as the arrow $\Delta f_2 \circ \Delta f_1$ of $\mathcal{A}$. To ensure that $\mathbf{1}_{DA}$ and $\Delta f_2 \circ \Delta f_1$ are indeed arrows of $\mathcal{A}_\Delta$ we check that the following equalities hold in $\mathcal{A}$:

$$(\Delta\varepsilon) \qquad \Delta\varepsilon_A = \mathbf{1}_{DA},$$
$$(\Delta\circ) \qquad \Delta(f_2 \circ \Delta f_1) = \Delta f_2 \circ \Delta f_1.$$

It is clear that $\mathcal{A}_\Delta$ is a category with this identity and this composition; namely, it is a subcategory of $\mathcal{A}$. We call $\mathcal{A}_\Delta$ the ***delta category*** of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$.

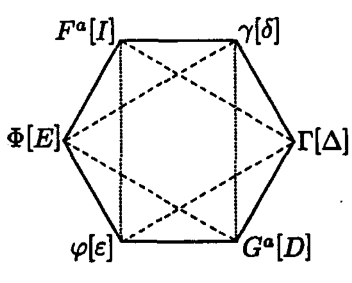
Between $\mathcal{A}$ and $\mathcal{A}_\Delta$ there is an adjunction, where the left-adjoint functor F from $\mathcal{A}_\Delta$ to $\mathcal{A}$ is inclusion I and the right-adjoint functor G from $\mathcal{A}$ to $\mathcal{A}_\Delta$ is D. To show that Df is of the form $\Delta f'$ we check that in $\mathcal{A}$ for every $f : A \to A'$ we have

$$Df = \Delta(f \circ \varepsilon_A).$$

The counit φ of this adjunction is just ε, where φ_A is ε_A, and the unit γ is δ, but with γ_{DA} being δ_A. That this adjunction obtains indeed will be shown in the next three sections.

Later, in 4.6 and 4.7, we shall compare the delta category to the Kleisli category and to the category of free coalgebras of a comonad. Before that, in the next three sections, we find the delta category handy to survey various possibilities of defining a comonad.

4.3. Primitive notions in comonad. Let us now consider how one could express the adjunction between $\mathcal{A}$ and $\mathcal{A}_\Delta$ in various ways according to the definitions of adjunction in 3. First, the primitive notions we might have to express this adjunction are displayed in square brackets in our hexagonal figure.



Besides the notions we have already encountered, we find in square brackets the seesaw functions E, corresponding to Φ, which will be defined below. The six definitional equalities of 3.1 connecting these notions would now read:

$$
\begin{array}{ll}
\text{for } \Delta f : DA' \to DA & \text{for } f : A \to A' \\
\quad (F_I^a) \quad \Delta f = E_{DA}(\delta_A \circ \Delta f), & \quad (G_D^a) \quad Df = \Delta(f \circ \varepsilon_A), \\
\quad (\varphi_\varepsilon) \quad \varepsilon_A = E_A D \mathbf{1}_A, & \quad (\gamma_\delta) \quad \delta_A = \Delta \mathbf{1}_{DA}, \\
\text{for } \Delta f : DA' \to DA & \text{for } f : DA' \to A \\
\quad (\Phi_E) \quad E_A \Delta f = \varepsilon_A \circ \Delta f, & \quad (\Gamma_\Delta) \quad \Delta f = Df \circ \delta_{A'}.
\end{array}
$$

The subscripts in Δ are unimportant now, because FA is A, but the second subscript of E, understood as Φ, matters, and this is the one we note above.

We must first settle what E stands for. The equality (Φ_E) would permit us to get rid of E in (F_I^a) and (φ_ε) if $\delta_A \circ \Delta f$ and $D\mathbf{1}_A$ were equal to arrows of the form $\Delta f'$. Now, for $D\mathbf{1}_A$ this follows immediately from (G_D^a), while for $\delta_A \circ \Delta f$ we have

$$
\begin{aligned}
\delta_A \circ \Delta f &= (\delta_A \circ Df) \circ \delta_{A'}, \quad \text{by } (\Gamma_\Delta) \text{ and (cat2)} \\
&= DDf \circ (\delta_{DA'} \circ \delta_{A'}), \quad \text{by (nat) for } \delta \text{ and (cat2)} \\
&= (DDf \circ D\delta_{A'}) \circ \delta_{A'}, \quad \text{by } (\delta\delta) \text{ and (cat2)} \\
&= \Delta\Delta f, \quad \text{by (fun2) and } (\Gamma_\Delta).
\end{aligned}
$$

So we may take that E is defined by (Φ_E).

The possible choices of primitives for our adjunction would now be the following, taking into account that F is now inclusion and doesn't figure anywhere:

hexagonal:	$\langle D, \varepsilon, \delta, E, \Delta \rangle$
rectangular | |:	$\langle D, \varepsilon, \delta \rangle$
rectangular \\\\:	$\langle D, E, \Delta \rangle$
rectangular //:	$\langle \varepsilon, \delta, E, \Delta \rangle$
triangular $\rhd$:	$\langle \varepsilon, \Delta \rangle$
triangular $\lhd$:	$\langle D, \delta, E \rangle$

The rectangular | | choice is the choice of the standard definition. The rectangular \\ choice boils down to $\langle \varepsilon, \Delta \rangle$, since ε can be defined in terms of D and E, while D can be defined in terms of ε and Δ, and E can be defined in terms of ε alone. The rectangular // choice boils down to $\langle \varepsilon, \Delta \rangle$, too, since δ can be defined in terms of Δ alone, and E can be defined in terms of ε alone. Finally, the triangular $\lhd$ choice boils down to $\langle D, \varepsilon, \delta \rangle$, since ε can be defined in terms of D and E, while E can be defined in terms of ε alone.

We should mention also the seesaw choice $\langle E, \Delta \rangle$. This boils down to $\langle \varepsilon, \Delta \rangle$, since ε_A can be defined as $E_A \mathbf{1}_{DA}$, and E is definable in terms of ε alone.

The hexagonal choice is of course full of redundances, but we shall nevertheless consider this choice in the next section. Besides that, we are left with only two interesting choices: the standard choice $\langle D, \varepsilon, \delta \rangle$ and $\langle \varepsilon, \Delta \rangle$.

4.4. Hexagonal comonads. With the hexagonal choice of primitives, we assume for a comonad $\langle \mathcal{A}, D, \varepsilon, \delta, E, \Delta \rangle$ that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ is a category,

D is a functor,

ε and δ are natural transformations,

the equalities (F_I^a), (G_D^a), (φ_ε), (γ_δ), (Φ_E) and (Γ_Δ) hold,

and, moreover, the equality $(\delta\delta)$ holds.

The equality $(\delta\delta)$ is assumed not because of the adjunction, but in order to insure that $\mathcal{A}_\Delta$ is closed under composition. It is also used in order to guarantee that E can be defined by (Φ_E) in (F_I^a), as we have shown above.

Let us show now that this hexagonal notion of comonad is equivalent to the standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion. With (Φ_E), the equality (F_I^a) reads

$$\Delta f = \varepsilon_{DA} \circ (\delta_A \circ \Delta f).$$

This equality clearly follows from $(\varepsilon\delta)$, (cat1left) and (cat2). Conversely, $(\varepsilon\delta)$ follows from this equality as follows. Since from (G_D^a) with (fun1) and (cat1left) we have $\Delta\varepsilon_A = \mathbf{1}_{DA}$ (i.e., the equality $(\Delta\varepsilon)$ mentioned above), our equality with (cat1right) will give $(\varepsilon\delta)$. Therefore, (F_I^a) amounts to $(\varepsilon\delta)$.

With (Γ_Δ), the equality (G_D^a) reads

$$Df = D(f \circ \varepsilon_A) \circ \delta_A.$$

This equality follows from $(\varepsilon\delta D)$, (fun2), (cat2) and (cat1right). Conversely, $(\varepsilon\delta D)$ immediately follows from this equality with (cat1left) and (fun1). Therefore, (G_D^a) amounts to $(\varepsilon\delta D)$. The equalities (F_I^a) and (G_D^a) are more important than the remaining four equalities (φ_ε), (γ_δ), (Φ_E) and (Γ_Δ), which boil down to definitions.

So, our hexagonal notion of comonad is equivalent to the standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion. To prove quite strictly the equivalence of these two notions, we would have to demonstrate an equivalence of categories, which would actually be an isomorphism of categories.

Note that in the hexagonal definition a comonad is defined by assuming that $\mathcal{A}$ and $\mathcal{A}_\Delta$ are categories and that the functors I and D are adjoints, I being left-adjoint and D right-adjoint. An adjunction between $\mathcal{A}$ and $\mathcal{B}$ where the left adjoint F is the inclusion functor from $\mathcal{B}$ into $\mathcal{A}$ is called a *coreflection* of $\mathcal{A}$ in its subcategory $\mathcal{B}$. So a comonad in $\mathcal{A}$ is defined by assuming that there is a coreflection of a category $\mathcal{A}$ in its subcategory $\mathcal{A}_\Delta$.

The standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion of comonad of 4.1 corresponds to the rectangular $||$ notion of adjunction of 3.3. The equality $(\varepsilon\delta)$ corresponds to $(\varphi\gamma F)$ and $(\varepsilon\delta D)$ to $(\varphi\gamma G)$, while $(\delta\delta)$ is related to (nat) for γ.

4.5. Triangular comonads. With the $\langle \varepsilon, \Delta \rangle$ choice of primitives, we can imitate the definition of triangular adjunction of 3.6 to define comonads. We define a *triangular* comonad $\langle \mathcal{A}, \varepsilon, \Delta \rangle$ by assuming that

$\mathcal{A}$ is a category,

D is a function from the objects of $\mathcal{A}$ to the objects of $\mathcal{A}$,

ε is an objectual transformation from D to the identity function on the objects of $\mathcal{A}$,

Δ is a function mapping the arrows $f : DA \to A'$ of $\mathcal{A}$ to the arrows $\Delta f : DA \to DA'$ of $\mathcal{A}$,

the following equalities hold:

$$(\varepsilon\Delta) \qquad \varepsilon_A \circ \Delta f = f, \text{ i.e., } E_A \Delta f = f,$$
$$(\Delta\circ) \qquad \Delta(f_2 \circ \Delta f_1) = \Delta f_2 \circ \Delta f_1,$$
$$(\Delta\varepsilon) \qquad \Delta\varepsilon_A = \mathbf{1}_{DA}.$$

These three equalities correspond to the equalities that were mentioned in 3.6 as a possible choice for defining triangular adjunction: $(\varepsilon\Delta)$ corresponds to (β'), while $(\Delta\circ)$ corresponds to $(\Gamma\Gamma'')$ and $(\Delta\varepsilon)$ to $(\Gamma\varphi)$. The new notion of comonad is equivalent to the standard $\langle\mathcal{A}, D, \varepsilon, \delta\rangle$ notion, via the definitions (G^{a}_{D}), (γ_δ) and (Γ_Δ). (A definition of monad analogous to this triangular notion of comonad may be found in [Manes 1976, 1.3, Exercise 12, p. 32].)

The triangular notion of comonad becomes more transparent if for $f_1 : DA_1 \to A_2$ and $f_2 : DA_2 \to A_3$ we introduce the definition given by the equality

$$(\circledast) \qquad f_2 \circledast f_1 = f_2 \circ \Delta f_1.$$

We call $\circledast$ *delta composition*. With delta composition, $(\Delta\circ)$ reads

$$(\Delta\circledast) \qquad \Delta(f_2 \circledast f_1) = \Delta f_2 \circ \Delta f_1.$$

Conversely, we may define Δ in terms of delta composition by the equality

$$(\Delta) \qquad \Delta f = \mathbf{1}_{DA} \circledast f.$$

With delta composition primitive, a comonad could be defined as being $\langle\mathcal{A}, \varepsilon, \circledast\rangle$, where $\mathcal{A}$, D and ε are as for the triangular $\langle\mathcal{A}, \varepsilon, \Delta\rangle$ notion above, $\circledast$ is a function that assigns to a pair $(f_1 : DA_1 \to A_2, f_2 : DA_2 \to A_3)$ of arrows of $\mathcal{A}$ the arrow $f_2 \circledast f_1 : DA_1 \to A_3$ of $\mathcal{A}$, and the following equalities hold:

$$(\text{cat1right}\circledast) \qquad f \circledast \varepsilon_A = f,$$
$$(\text{cat1left}\circledast) \qquad \varepsilon_A \circledast f = f,$$
$$(\text{cat2}\circledast) \qquad (f_3 \circledast f_2) \circledast f_1 = f_3 \circledast (f_2 \circledast f_1),$$
$$(\text{shift}) \qquad (f_3 \circ f_2) \circledast f_1 = f_3 \circ (f_2 \circledast f_1).$$

The first three equalities are clearly analogous to the corresponding categorial equalities, ε behaving as identity. The fourth equality can be replaced by either of the following two equalities:

$$(\text{shift1}) \qquad f_3 \circ (\mathbf{1}_{DA} \circledast f_1) = f_3 \circledast f_1,$$
$$(\text{shift}\varepsilon) \qquad (f_3 \circ \varepsilon_A) \circledast f_1 = f_3 \circledast f_1.$$

(With (shiftε), the equality (cat1left$\circledcirc$) becomes superfluous.) The $\langle \mathcal{A}, \varepsilon, \circledcirc \rangle$ notion of comonad and the triangular $\langle \mathcal{A}, \varepsilon, \Delta \rangle$ notion are equivalent, via the definitions (Δ) and ($\circledcirc$). (A definition of monad analogous to the (shiftε) variant of our $\langle \mathcal{A}, \varepsilon, \circledcirc \rangle$ notion may be found in [Manes 1976, 1.3, Definition 3.2, p. 24]; the other variants are from [D. 1996, section 4.1].)

If we don't economize on primitives, and take both Δ and delta composition as primitives, then an equivalent notion of comonad is obtained by defining it as $\langle \mathcal{A}, \varepsilon, \Delta, \circledcirc \rangle$, where $\mathcal{A}$, D, ε, Δ and $\circledcirc$ are as before and the equalities ($\varepsilon\Delta$), ($\Delta\circledcirc$) and ($\Delta\varepsilon$) hold. Now the defining equalities (Δ) and ($\circledcirc$) become derivable (this definition is in [D. 1996, section 4.1]).

Note that we are certainly not allowed to suppose that we have now exhausted all possible ways of defining comonads. But the definitions through the adjunction between $\mathcal{A}$ and $\mathcal{A}_\Delta$ are well covered, and among these definitions we find the standard definition and other definitions mentioned in the literature.

4.6. The Kleisli category. Let $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ be a comonad. Then consider the graph $\mathcal{A}_D$ whose objects are all the objects of $\mathcal{A}$, while its arrows are obtained by taking that for every object A of $\mathcal{A}$ and every arrow $f : DA \to A'$ of $\mathcal{A}$, the pair (A, f), which we abbreviate by f^A, is an arrow of $\mathcal{A}_D$ of type $A \to A'$. (Formally, we need a bijection κ that assigns to the pairs (A, f) the arrows $\kappa(A, f) : A \to A'$ of $\mathcal{A}_D$. So, $\kappa(A, f)$ may be identified with the ordered pair (A, f). We cannot identify $\kappa(A, f)$ just with f instead of (A, f), because, if D is not one-one on objects, then f could have more than one source in $\mathcal{A}_D$. Definitions of Kleisli category in the literature, including Kleisli's own definition of [1965], usually don't make this clear.)

The graph $\mathcal{A}_D$ has an identity whose arrows $\mathbf{1}_A : A \to A$ are defined as ε_A^A and composition in $\mathcal{A}_D$ is defined as follows in terms of the delta composition of $\mathcal{A}$:

$$f_2^{A_2} \circ f_1^{A_1} \overset{\text{def}}{=} (f_2 \circledcirc f_1)^{A_1}.$$

Let us call the graph $\mathcal{A}_D$ with this identity and this composition the *Kleisli deductive system* of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$. It is clear that due to (cat1right$\circledcirc$), (cat1left$\circledcirc$) and (cat2$\circledcirc$) of the preceding section, this deductive system is a category. This category is called the *Kleisli category* of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$.

A category isomorphic to $\mathcal{A}_D$ is a category $\mathcal{A}'_\Delta$ related to the delta category $\mathcal{A}_\Delta$, which is defined as follows. Its objects are again the objects of $\mathcal{A}$, while its arrows are obtained by taking that for every pair (A_1, A_2) of objects of $\mathcal{A}$ and every arrow $h : DA_1 \to DA_2$ of $\mathcal{A}$ such that

$$(\text{homo } \delta) \qquad Dh \circ \delta_{A_1} = \delta_{A_2} \circ h,$$

the triple $\langle A_1, A_2, h \rangle$, which we abbreviate by h^{A_1, A_2}, is an arrow of $\mathcal{A}'_\Delta$ of type $A_1 \to A_2$. The identity arrows $\mathbf{1}_A : A \to A$ of $\mathcal{A}'_\Delta$ are defined as $\mathbf{1}_{DA}^{A,A}$ and composition is defined by

$$h_2^{A_2, A_3} \circ h_1^{A_1, A_2} \overset{\text{def}}{=} (h_2 \circ h_1)^{A_1, A_3}.$$

The equality (homo δ), which is a kind of naturalness condition, could alternatively be written as

$$\Delta h = \Delta \mathbf{1}_{DA_2} \circ h.$$

Other conditions equivalent to (homo δ) are

$$\Delta(\varepsilon_{A_2} \circ h) = h, \text{ i.e., } \Delta E_{A_2} h = h,$$
$$\exists f(\Delta f = h).$$

The isomorphism between the categories $\mathcal{A}_D$ and $\mathcal{A}'_\Delta$ is obtained by the functor K from $\mathcal{A}_D$ to $\mathcal{A}'_\Delta$ such that $KA = A$ and for $f : DA_1 \to A_2$

$$Kf^{A_1} = (\Delta f)^{A_1,A_2}.$$

The inverse K^{-1} of K is defined by $K^{-1}A = A$ and for $h : DA_1 \to DA_2$

$$K^{-1}h^{A_1,A_2} = (\varepsilon_{A_2} \circ h)^{A_1} = (E_{A_2} h)^{A_1}.$$

If D is one-one on objects, then it is clear that the category $\mathcal{A}'_\Delta$ is isomorphic to the delta category $\mathcal{A}_\Delta$, which we have considered in 4.2. Without supposing that D is one-one on objects, we can ascertain only that $\mathcal{A}'_\Delta$ and $\mathcal{A}_\Delta$ are equivalent categories (see 1.5).

The $\langle \mathcal{A}, \varepsilon, ⊗ \rangle$ definition of comonad from the preceding section shows that we could define a comonad by assuming that its Kleisli deductive system is a category and by the the (shift) equality. This equality expresses the adjunction between $\mathcal{A}$ and $\mathcal{A}_D$, which we shall examine in 5.

4.7. The Eilenberg-Moore category. Let $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ be a comonad. Then consider the graph $\mathcal{A}^D$ whose objects are arrows $d : A \to DA$ of $\mathcal{A}$ such that

(ob1)	$\varepsilon_A \circ d = \mathbf{1}_A$,
(ob2)	$\delta_A \circ d = Dd \circ d$.

An arrow of $\mathcal{A}^D$ with source $d_1 : A_1 \to DA_1$ and target $d_2 : A_2 \to DA_2$ is made of an arrow $h : A_1 \to A_2$ of $\mathcal{A}$ such that

$$\text{(homo)} \qquad Dh \circ d_1 = d_2 \circ h.$$

To prevent the same arrow from having more then one source or more than one target, the arrow h in $\mathcal{A}^D$ should be indexed by d_1 and d_2. Formally, the arrows of $\mathcal{A}^D$ will be triples $\langle d_1, d_2, h \rangle$, but we shall take the indices d_1 and d_2 for granted and omit them (usually, they are not even mentioned).

The identity arrows of $\mathcal{A}^D$ are just $\mathbf{1}_A : A \to A$ and composition is defined as composition in $\mathcal{A}$. We can check that the equality (homo) holds when d_1 and d_2 are equal and for h we put an identity arrow; it holds also for $h_2 \circ h_1$ if it holds for h_1 and h_2. So $\mathcal{A}^D$ is a category, which is called the *Eilenberg-Moore category* of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$.

For $d : A \to DA$ and $f : A \to A'$ let

$$\Delta_d f \overset{\text{def}}{=} Df \circ d.$$

It is clear that for $f : DA \to A'$ the arrow $\Delta_{\delta_A} f$ is Δf. To define the Eilenberg-Moore category of a comonad we can assume

$$\begin{array}{ll} (\text{ob1}') & \varepsilon_A \circ \Delta_d f = f, \\ (\text{ob2}') & \delta_A \circ \Delta_d f = \Delta_d \Delta_d f, \\ (\text{homo}') & \Delta_{d_1} h = \Delta_{d_2} \mathbf{1}_{A_2} \circ h \end{array}$$

instead of (ob1), (ob2) and (homo).

The full subcategory $\mathcal{A}^D_{\text{free}}$ of $\mathcal{A}^D$ whose objects are all the arrows $\delta_A : DA \to DDA$ of $\mathcal{A}$ is called the category of *free coalgebras* of the comonad. This category is isomorphic to the delta category $\mathcal{A}_\Delta$ when there is a bijection between the objects of $\mathcal{A}$ of the form DA and the arrows δ_A of $\mathcal{A}$. This bijection exists when D is one-one on objects. When D is not such, we may still have this bijection, provided that if DA_1 is the same object as DA_2, then $\delta_{A_1} = \delta_{A_2}$ (the converse implication obtains anyway). But the bijection may also fail. (In [D. 1996, section 4.2] it is stated that it can be shown without the supposition that D is one-one on objects that $\mathcal{A}_\Delta$ and $\mathcal{A}^D_{\text{free}}$ are isomorphic. What should have been said is that this can be shown *sometimes* even without making this supposition.)

We obtain a category isomorphic to the Kleisli category $\mathcal{A}_D$ (and to $\mathcal{A}'_\Delta$) by replacing the objects δ_A of $\mathcal{A}^D_{\text{free}}$ with pairs (A, δ_A), and the arrows $h : DA_1 \to DA_2$ of $\mathcal{A}^D_{\text{free}}$ with triples $\langle A_1, A_2, h\rangle$. (In the usual presentation of Eilenberg-Moore categories, objects are said to be pairs (A, d) where A is the source of $d : A \to DA$ and d satisfies (ob1) and (ob2). These pairs are in one-to-one correspondence with the arrows d. Mentioning the source of d in the pair is not essential: it seems to be there for heuristical reasons. However, introducing A into (A, δ_A) makes a difference. Note that A is not the source DA of δ_A.)

In general, we can assert only that $\mathcal{A}^D_{\text{free}}$ is equivalent to $\mathcal{A}_\Delta$ and $\mathcal{A}_D$, without necessarily being isomorphic.

5. Adjunction between adjunctions and comonads

We shall now try to clarify the relationship between the notions of comonad and adjunction. It will appear that comonads may be understood as a special kind of adjunction, since the category of comonads (with comonofunctors as arrows) is isomorphic to a full subcategory of the category of adjunctions (with appropriate morphisms, which we shall call *junctors*, as arrows). Moreover, there are two adjunctions involving these two categories.

First, we have a functor that associates in a standard manner a comonad to an adjunction. After investigating some aspects of this functor, we show that it has a left adjoint, which associates to a comonad the adjunction with the Kleisli category, and a right adjoint, which associates to a comonad the adjunction with the Eilenberg-Moore category. At the end (5.4), we show how the usual presentation of these matters, via the category of resolutions of a comonad, where the Kleisli category is tied to the initial object and the Eilenberg-Moore category to the terminal object, is a simple corollary of our presentation.

5.1. The comonad of an adjunction. We shall first introduce the notions of *junction* and *junctor* in the rectangular || style of 3.3. A junction is a structure like an adjunction, but without the corresponding equalities between arrows. So a junction is to an adjunction what a deductive system is to a category and what a comonograph is to a comonad. A junctor is a morphism of junctions, and also a morphism of adjunctions.

Suppose we are given the following:

two deductive systems, $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$,

a graph-morphism F from $\mathcal{B}$ to $\mathcal{A}$ and a graph-morphism G from $\mathcal{A}$ to $\mathcal{B}$,

a transformation φ from FG to $I_{\mathcal{A}}$ and a transformation γ from $I_{\mathcal{B}}$ to GF.

Then $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ is a *junction*.

A *junctor* from a junction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ to a junction $\langle \mathcal{A}', \mathcal{B}', F', G', \varphi', \gamma' \rangle$ is a pair $(N_{\mathcal{A}}, N_{\mathcal{B}})$ such that $N_{\mathcal{A}}$ is a functor from the deductive system $\mathcal{A}$ to the deductive system $\mathcal{A}'$, and $N_{\mathcal{B}}$ a functor from the deductive system $\mathcal{B}$ to the deductive system $\mathcal{B}'$; moreover, the following naturalness equalities hold:

$$N_{\mathcal{A}}F = F'N_{\mathcal{B}}, \qquad N_{\mathcal{B}}G = G'N_{\mathcal{A}},$$
$$N_{\mathcal{A}}\varphi_A = \varphi'_{N_{\mathcal{A}}A}, \qquad N_{\mathcal{B}}\gamma_B = \gamma'_{N_{\mathcal{B}}B}.$$

An *adjunction* is a junction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ such that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,

F and G are functors,

φ and γ are natural transformations,

the equalities $(\varphi\gamma F)$ and $(\varphi\gamma G)$ hold (see 3.3).

To every adjunction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ we may associate the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$, where the composite functor FG is the functor D of the comonad, φ_A is ε_A and $F\gamma_{GA}$ is δ_A. (We may analogously associate to the adjunction a monad in $\mathcal{B}$.) It is routine to check that $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ is indeed a comonad. It is called *the comonad of the adjunction* $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$.

5.2. Reflections and coreflections in comonads. An adjunction between $\mathcal{A}$ and $\mathcal{B}$ where the right adjoint G is the inclusion functor from $\mathcal{A}$ into $\mathcal{B}$ is called a *reflection* of $\mathcal{B}$ in its subcategory $\mathcal{A}$. We have seen in 4.4 that a comonad in a category $\mathcal{A}$ is defined by a coreflection of $\mathcal{A}$ in its subcategory $\mathcal{A}_\Delta$, the delta category of the comonad. However, with comonads of adjunctions we may have in some interesting (and in logic rather common) cases also a reflection of a category isomorphic to $\mathcal{A}_\Delta$ in its subcategory $\mathcal{A}$. We shall now consider this matter.

Let us first prove the following proposition.

Proposition 1. *Let $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ be an adjunction where G is one-one on objects. Then the Kleisli category $\mathcal{A}_{FG}$ of the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ of the adjunction is isomorphic to the full subcategory $G(\mathcal{A})$ of $\mathcal{B}$ whose objects are all the objects of $\mathcal{B}$ of the form GA.*

Proof: First we show that for $f_1 : FGA_1 \to A_2$ and $f_2 : FGA_2 \to A_3$ in the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ we have

$$(\circledast\Phi\Gamma) \qquad f_2 \circledast f_1 = \Phi_{GA_1,A_3}(\Gamma_{GA_2,A_3} f_2 \circ \Gamma_{GA_1,A_2} f_1).$$

Indeed,

$$\begin{aligned} f_2 \circledast f_1 &= f_2 \circ (FGf_1 \circ F\gamma_{GA_1}), \text{ by definition} \\ &= f_2 \circ F\Gamma_{GA_1,A_2} f_1, \text{ by (fun2) and } (\Gamma) \text{ of 3.1,} \end{aligned}$$

and we obtain $(\circledast\Phi\Gamma)$ by applying $(\Phi\Gamma')$ and $(\Gamma\Gamma'')$ from 3.4.

We now define a functor N from $\mathcal{A}_{FG}$ to $G(\mathcal{A})$ in the following way. For every object A of $\mathcal{A}_{FG}$, which is by definition an object of $\mathcal{A}$, let NA be GA. For every arrow $f^{A_1} : A_1 \to A_2$ of $\mathcal{A}_{FG}$, for which, by definition, we have an arrow $f : FGA_1 \to A_2$ of $\mathcal{A}$, let Nf^{A_1} be $\Gamma_{GA_1,A_2} f : GA_1 \to GA_2$. To check that N is a functor we have

$$\begin{aligned} N\varphi_A^A &= \Gamma_{GA,A}\varphi_A = \mathbf{1}_{GA}, \quad \text{by } (\varphi) \text{ of 3.1, (fun1) and } (\Gamma\Phi') \text{ of 3.4,} \\ N(f_2 \circledast f_1)^{A_1} &= \Gamma_{GA_1,A_3}(f_2 \circledast f_1) \\ &= \Gamma_{GA_2,A_3} f_2 \circ \Gamma_{GA_1,A_2} f_1, \quad \text{by } (\circledast\Phi\Gamma) \text{ and } (\Phi\Gamma') \text{ of 3.4} \\ &= Nf^{A_2} \circ Nf^{A_1}. \end{aligned}$$

Relying on the fact that G is one-one on objects, we define the functor N^{-1} from $G(\mathcal{A})$ to $\mathcal{A}_{FG}$ by taking that $N^{-1}GA$ is A and that for $g : GA_1 \to GA_2$ the arrow $N^{-1}g$ is $(\Phi_{GA_1,A_2} g)^{A1}$. It remains to use the equalities $(\Phi\Gamma')$ and $(\Gamma\Phi')$ to verify that $N^{-1}Nf^{A_1} = f^{A_1}$ and $NN^{-1}g = g$.

This is an immediate corollary of Proposition 1:

Proposition 2. *Let $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ be an adjunction where G is a bijection on objects. Then the categories $\mathcal{A}_{FG}$ and $\mathcal{B}$ are isomorphic.*

We know from 4.6 that if in a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ we have that D is one-one on objects, then the Kleisli category $\mathcal{A}_D$ of the comonad is isomorphic to the subcategory $\mathcal{A}_\Delta$ of $\mathcal{A}$, the delta category of the comonad. With the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ of an adjunction, for $f : FGA_1 \to A_2$ we have

$$\Delta f = F\Gamma_{GA_1,A_2} f.$$

So $\mathcal{A}_\Delta$ will be denoted in this case by $\mathcal{A}_{F\Gamma}$. We can then state the following as a corollary of Proposition 1:

Proposition 3. *Let $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ be an adjunction where both F are G are one-one on objects. Then the categories $\mathcal{A}_{F\Gamma}$ and $G(\mathcal{A})$ are isomorphic.*

The point of this proposition is that $\mathcal{A}_{F\Gamma}$ is a subcategory of $\mathcal{A}$. So in all adjunctions $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ where F is one-one on objects and G is a bijection on objects, $\mathcal{B}$ is isomorphic to a subcategory of $\mathcal{A}$. Note that in such an adjunction $\mathcal{A}$ may actually be a subcategory of $\mathcal{B}$, so that the adjunction is a reflection of $\mathcal{B}$ in its subcategory $\mathcal{A}$. But we can assert that $\mathcal{B}$ is also isomorphic to a subcategory of $\mathcal{A}$, namely $\mathcal{A}_{F\Gamma}$, and that there is a coreflection of $\mathcal{A}$ in this subcategory.

(The situation we have just described obtains sometimes in the adjunction of *deductive completeness*, a strengthening of the deduction theorem, originally called *functional completeness* in [Lambek 1974]; see also [Lambek & Scott 1986, I.6-7] and [D. 1996]. Then $\mathcal{B}$ is the polynomial category generated by $\mathcal{A}$ and an indeterminate arrow.)

It is instructive to see that the isomorphism from $\mathcal{B}$ to $\mathcal{A}_{F\Gamma}$ above is the functor F, the left adjoint in the adjunction.

5.3. The adjunctions involving the categories of adjunctions and comonads.

Let **Adj** be the category whose objects are adjunctions, with arrows being junctors (this category should not be confused with the category bearing the same name in [Mac Lane 1971, IV.8], where arrows are adjunctions), and let **Com** be the category whose objects are comonads, with arrows being comonofunctors.

Consider now the functor **C** from **Adj** to **Com** that assigns to an adjunction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ of the adjunction, and to a junctor $(N_\mathcal{A}, N_\mathcal{B})$ the comonofunctor $N_\mathcal{A}$ (we may readily check that $N_\mathcal{A}$ is indeed a comonofunctor).

The functor **C** has a left adjoint **F** that assigns to a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the adjunction between $\mathcal{A}$ and the Kleisli category $\mathcal{A}_D$ of this comonad, namely the adjunction $\langle \mathcal{A}, \mathcal{A}_D, F_D, G_D, \varphi_D, \gamma_D \rangle$, which is defined as follows:

$$F_D A \stackrel{\text{def}}{=} DA, \qquad G_D A \stackrel{\text{def}}{=} A,$$

$$F_D f^A \stackrel{\text{def}}{=} \Delta f, \qquad G_D f \stackrel{\text{def}}{=} (f \circ \varepsilon_A)^A,$$

$$\varphi_{D_A} \stackrel{\text{def}}{=} \varepsilon_A, \qquad \gamma_{D_A} \stackrel{\text{def}}{=} (\mathbf{1}_{DA})^A.$$

If $N_\mathcal{A}$ is a comonofunctor from a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ to a comonad $\langle \mathcal{A}', D', \varepsilon', \delta' \rangle$, then $\mathbf{F}N_\mathcal{A}$ is the junctor $(N_\mathcal{A}, N_{\mathcal{A}_D})$ from the adjunction between $\mathcal{A}$ and $\mathcal{A}_D$ to the adjunction between $\mathcal{A}'$ and $\mathcal{A}'_{D'}$, where $N_{\mathcal{A}_D}$ is defined as follows:

$$N_{\mathcal{A}_D} A \stackrel{\text{def}}{=} N_\mathcal{A} A, \qquad N_{\mathcal{A}_D} f^A \stackrel{\text{def}}{=} (N_\mathcal{A} f)^{N_\mathcal{A} A}.$$

For an adjunction $J = \langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ let φ_J be the junctor $(N_{\mathcal{A}}, N_{\mathcal{B}})$ from $\mathbf{FC}J$ to J where $N_{\mathcal{A}}$ is the identity functor $I_{\mathcal{A}}$ and the functor $N_{\mathcal{B}}$ is defined by

$$N_{\mathcal{B}}A \overset{\text{def}}{=} GA, \qquad N_{\mathcal{B}}f^A \overset{\text{def}}{=} Gf \circ \gamma_{GA} = \Gamma_{GA,A'}f.$$

The arrows φ_J of $\mathbf{Adj}$ make a natural transformation φ from $\mathbf{FC}$ to $I_{\mathbf{Adj}}$. It is easy to check that for every comonad $S = \langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the comonad $\mathbf{CF}S$ is identical to S; so the identity comonofunctor $I_{\mathcal{A}}$ is an arrow from S to $\mathbf{CF}S$ in $\mathbf{Com}$. It is trivial that the arrows $I_{\mathcal{A}}$ make a natural transformation $\mathbf{I}$ from $I_{\mathbf{Com}}$ to $\mathbf{CF}$.

That $\mathbf{F}$ is left adjoint to $\mathbf{C}$ means that $\langle \mathbf{Adj}, \mathbf{Com}, \mathbf{F}, \mathbf{C}, \varphi, \mathbf{I} \rangle$ is an adjunction. In this adjunction, the unit is the identity of the category $\mathbf{Com}$. We can infer that $\mathbf{Com}$ is isomorphic by $\mathbf{F}$ to a full subcategory of $\mathbf{Adj}$ (cf. [Mac Lane 1971, IV.4, pp. 92–93]).

The functor $\mathbf{C}$ has also a right adjoint $\mathbf{G}$ that assigns to a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the adjunction between $\mathcal{A}$ and the Eilenberg-Moore category $\mathcal{A}^D$ of this comonad, namely the adjunction $\langle \mathcal{A}, \mathcal{A}^D, F^D, G^D, \varphi^D, \gamma^D \rangle$, which is defined as follows:

$$F^D d \overset{\text{def}}{=} \text{source}(d), \qquad G^D A \overset{\text{def}}{=} \delta_A,$$

$$F^D h \overset{\text{def}}{=} h, \qquad G^D f \overset{\text{def}}{=} Df,$$

$$\varphi^D_A \overset{\text{def}}{=} \varepsilon_A, \qquad \gamma^D_d \overset{\text{def}}{=} d.$$

If $N\mathcal{A}$ is a comonofunctor from a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ to a comonad $\langle \mathcal{A}', D', \varepsilon', \delta' \rangle$, then $GN_{\mathcal{A}}$ is the junctor $(N_{\mathcal{A}}, N_{\mathcal{A}_D})$ from the adjunction between $\mathcal{A}$ and $\mathcal{A}^D$ to the adjunction between $\mathcal{A}'$ and $\mathcal{A}'^{D'}$, where $N_{\mathcal{A}_D}$ is defined as follows:

$$N_{\mathcal{A}^D} d \overset{\text{def}}{=} N_{\mathcal{A}} d, \qquad N_{\mathcal{A}^D} h \overset{\text{def}}{=} N_{\mathcal{A}} h.$$

For an adjunction $J = \langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ let now γ_J be the junctor $(N_{\mathcal{A}}, N_{\mathcal{B}})$ from J to GCJ where $N_{\mathcal{A}}$ is the identity functor $I_{\mathcal{A}}$ and the functor $N_{\mathcal{B}}$ is defined by

$$N_{\mathcal{B}}B \overset{\text{def}}{=} F\gamma_B, \qquad N_{\mathcal{B}}g \overset{\text{def}}{=} Fg.$$

The arrows γ_J of $\mathbf{Adj}$ make a natural transformation γ from $I_{\mathbf{Adj}}$ to $\mathbf{GC}$. It is easy to check that for every comonad $S = \langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the comonad $\mathbf{CG}S$ is identical to S; so the identity comonofunctor $I_{\mathcal{A}}$ is an arrow from $\mathbf{CG}S$ to S in $\mathbf{Com}$. It is trivial that the arrows $I_{\mathcal{A}}$ make a natural transformation $\mathbf{I}$ from $\mathbf{CG}$ to $I_{\mathbf{Com}}$.

That $\mathbf{G}$ is right adjoint to $\mathbf{C}$ means that $\langle \mathbf{Com}, \mathbf{Adj}, \mathbf{C}, \mathbf{G}, \mathbf{I}, \gamma \rangle$ is an adjunction. In this adjunction, the counit is the identity of the category $\mathbf{Com}$. We can infer that $\mathbf{Com}$ is isomorphic by $\mathbf{G}$ to a full subcategory of $\mathbf{Adj}$ (following the terminology of [Mac Lane 1971, IV.4, pp. 92–93], the functor $\mathbf{C}$ is a left-adjoint-left-inverse of $\mathbf{G}$; the category $\mathbf{Com}$ is isomorphic to a full reflective subcategory of $\mathbf{Adj}$).

One could expect that adjunctions similar to those with $\mathbf{C}$, $\mathbf{F}$ and $\mathbf{G}$ treated in this section may be obtained by taking instead of $\mathbf{Adj}$ the category of junctions (with junctors as arrows) and instead of **Com** the category of comonographs (with comonofunctors as arrows).

5.4. The category of resolutions. Take a functor **C** from a category $\mathcal{A}$ to a category $\mathcal{B}$, and for a given object B of $\mathcal{B}$ consider the set of objects A of $\mathcal{A}$ such that $\mathbf{C}A = B$ and the set of arrows f of $\mathcal{A}$ such that $\mathbf{C}f = \mathbf{1}_B$. These two sets make the graph of a subcategory $\mathcal{A}_B$ of $\mathcal{A}$.

An object A is *initial* in a graph iff from A to every object in the graph there is exactly one arrow; A is *terminal* iff from every object to A there is exactly one arrow.

If **C** has a left adjoint **F** such that the unit of the adjunction is the identity of $\mathcal{B}$, then $\mathcal{A}_B$ has an initial object $\mathbf{F}B$, and if **C** has a right adjoint **G** such that the counit of the adjunction is the identity of $\mathcal{B}$, then $\mathcal{A}_B$ has a terminal object $\mathbf{G}B$.

To show that $\mathbf{F}B$ is initial, take an object A of $\mathcal{A}_B$; then it can be shown that $\varphi_A : \mathbf{FC}A \to A$ is the unique arrow of $\mathcal{A}_B$ from $\mathbf{F}B$ to A. For suppose there is another arrow $f : \mathbf{FC}A \to A$ in $\mathcal{A}_B$; since

$$\mathbf{C}f \circ \gamma_B = \mathbf{C}f = \mathbf{1}_B,$$

because γ_B is an identity arrow and f is in $\mathcal{A}_B$, and since

$$\mathbf{C}\varphi_A \circ \gamma_B = \mathbf{1}_B, \text{ by the equality } (\varphi\gamma G) \text{ of 3.3,}$$

we obtain

$$\varphi_{\mathbf{F}B} \circ \mathbf{F}(\mathbf{C}f \circ \gamma_B) = \varphi_{\mathbf{F}B} \circ \mathbf{F}(\mathbf{C}\varphi_A \circ \gamma_B),$$

from which with (fun2), (nat) and the equality $(\varphi\gamma F)$ of 3.3, the equality $f = \varphi_A$ follows. Analogously, in the other adjunction, the one with **G**, the arrow $\gamma_A : A \to \mathbf{GC}A$ is the unique arrow of $\mathcal{A}_B$ from A to $\mathbf{G}B$.

So by taking the functor **C** from **Adj** to **Com** and by fixing a comonad S in **Com** we obtain a subcategory $\mathbf{Adj}_S$ of **Adj**. We may call the category $\mathbf{Adj}_S$ the category of *resolutions* of S, by analogy with the terminology usual when one deals with monads instead of comonads. For a comonad $S = \langle \mathcal{A}, D, \varepsilon, \delta \rangle$, the adjunctions in $\mathbf{Adj}_S$ are all between the category $\mathcal{A}$ and a category $\mathcal{B}$, and the junctors $(N_\mathcal{A}, N_\mathcal{B})$ in $\mathbf{Adj}_S$ all have for $N_\mathcal{A}$ the identity functor on $\mathcal{A}$.

The category $\mathbf{Adj}_S$ has an initial object $\mathbf{F}S$ and a terminal object $\mathbf{G}S$, according to what we have said above. The arrow $\varphi_J : \mathbf{FC}J \to J$ is the unique arrow of $\mathbf{Adj}_S$ from $\mathbf{F}S$ to an adjunction J of $\mathbf{Adj}_S$, and $\gamma_J : J \to \mathbf{GC}J$ is the unique arrow of $\mathbf{Adj}_S$ from J to $\mathbf{G}S$. These arrows correspond to what in the case of monads is called *comparison functors*.

Suppose a functor **C** from a category $\mathcal{A}$ to a category $\mathcal{B}$ has both a left adjoint **F** and a right adjoint **G**. Then the functors **FC** and **GC** from $\mathcal{A}$ to $\mathcal{A}$ are adjoint, **FC** being left adjoint and **GC** right adjoint. (Analogously, **CF** and **CG** from $\mathcal{B}$ to $\mathcal{B}$ are adjoint, **CF** being left adjoint and **CG** right adjoint.) This is a consequence of the fact that two successive adjunctions compose to give a single adjunction (see [Mac Lane 1971, IV.8, p. 101]).

By taking that $\mathcal{A}$ is **Adj** and $\mathcal{B}$ is **Com**, we obtain that the functors **FC** and **GC** from **Adj** to **Adj** are adjoint. (The functors **CF** and **CG** are uninteresting, since they are the identity functor from **Com** to **Com**.)

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VECTORS OF THE BODY MASS MOMENTS

To the Memory of my professors

Draginja Nikolić professor of Mathematics
Prof. Dr. Ing. Math. Danilo Rašković
Academician Tatomir Anđelić

Teachers and Friends

Summary

This monograph paper introduces the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ *of the body mass inertia moment at the point N for the axis oriented by the unit vector* $\vec{n}$. *The vector is used for interpretation of the rigid body kinetic characteristics. The change of the vector of the rigid body mass inertia moment is determined in the transition from one space point to another when the axis retains its orientation which represents the Huygens-Steiner theorem translated for the defined body mass inertia moment vector. Then the change of the vector of the body mass inertia moment is defined at the given point in the case of the axis changing its orientation in the way analogous to the Cauchy equations in the Elasticity theory. Then the interpretation of the main mass inertia moments asymmetry are defined. The relation between the axis deviation load vector by the body mass inertia moment for the octahedron axis and the inertia mass asymmetry moments axis is analyzed.*

This paper defines three dynamic vectors fixed to a certain point and axis passing through the given rigid body point. These are: the vector $\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ *of the body mass at the point N for the axis oriented by the unit vector* $\vec{n}$; *the vector* $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ *of the body mass static (linear) moment at the point N for the axis oriented by the unit vector* $\vec{n}$; *and the vector* $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ *of the body mass inertia moment at the point N for the axis oriented by the unit vector* $\vec{n}$. *Also, the paper introduces the vectors:* $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ *of the material particle mass inertia moment for the pole O and the axis oriented by the unit vector* $\vec{n}$, *and* $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ *of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector* $\vec{n}$ *at the dimensional curvilinear coordinate system N.*

The rigid body kinetic parameters are interpreted by these vectors.

Future interpretation of the rigid body kinetic characteristics by means of the body mass inertia moment vector and by means of the body mass linear moment vector for the axis and the point refers to the description of the linear momentum, as well as angular momentum and kinetic energy as the functions of the body mass moment vectors and the angular velocity and the referential point velocity. The special cases of the rigid heavy body rotation are specially analyzed. The deviation part of the body mass inertia moment vector for the fixed point and for the rotation axis in view of the appearance of the dynamic pressure upon the bearings. The kinematic vector rotator is introduced as well as analyzed.

The spherical and the deviational parts of the mass inertia moment vector and of the mass inertia moment tensor are analyzed.

The conditions for dynamic balancing by means of the static mass moment vector and of the deviation load vector of the rotation axis by the rigid body mass inertia moment are shown.

The kinetic equations of a variable mass object motion rotating around a stationary axis are derived by means of the mass moment vectors for the pole and for the rotation axis: vector $\vec{\mathfrak{S}}_n^{(A)}$ *of the body mass linear moment, vector* $\vec{\mathfrak{J}}_n^{(A)}$ *of the body mass inertia moment for the pole A and for the axis oriented by the unit vector* $\vec{n}$ *and its deviational part of the vector* $\vec{\mathfrak{D}}_n^{(A)}$ *of the deviational load by the body mass inertia moment of the rotation axis through the pole A. The vectors of the reactive forces and resulting moments of the reactive forces due to the drop of the body particles are determined which are involved in the body mass change as the function of the body mass moments vector change: vector* $\vec{\mathfrak{S}}_n^{(A)}$ *of the body mass linear moment and vector* $\vec{\mathfrak{J}}_n^{(A)}$ *of the body mass inertia moment for the pole A and for the axis oriented by the unit vector* $\vec{n}$*.*

CONTENTS

CHAPTER I

I.1. Vectors of the body mass moments

I.1.1. Introduction. The idea for this monograph paper appeared during my considerations of some analogies between the models in the stress theory and the strain theory of the stressed and strained deformable bodies as they are studied or as they can be studied in Elasticity Theory (see **[15]**, **[14]**, **[28]**, **[25]**, **[34]** and **[23]**). While considering this analogy as well as the analogy between the stress tensor matrix, the relative deformation tensor-strain tensor matrix and the body mass inertia tensor matrix it occurred to me to introduce the concept of the vector $\vec{\delta}_{\vec{n}}^{(N)}$ of the total relative deformation — total strain, at the point N and for the line element drawn from that point and oriented by unit vector $\vec{n}$, as well as the concept of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N, and for the axis oriented by the unit vector $\vec{n}$ (see **[A1]**, **[A2]**, **[A6]**. For more details see **[24]**, **[30]**, **[31]**, **[A5]**, **[35]**, **[37]**, **[38]**, **[34]** and **[23]**.

In further consideration of the dynamic parameters of the rigid and deformable bodies as well as of the possibility of their interpretation by means of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N for the axis oriented by the unit vector $\vec{n}$, I came to the ideas and conclusions as well as interpretations given in my papers **[22]**, **[A2]** **[A4]**, **[24]**, **[34]** and **[23]**. The question always asked was if something like that already existed in some classic literature or not? The literature available to me which is quoted in the appendix of this paper contains no such interpretation of the rigid deformable bodies dynamic parameters by means of the mass inertia moment vector fixed to the point and to the axis.

This paper defines three dynamic vectors fixed to a certain point and axis passing through the given rigid body point. These are: the vector $\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ of the body mass at the point N for the axis oriented by the unit vector $\vec{n}$; the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ of the body mass static (linear) moment at the point N for the axis oriented by the unit vector $\vec{n}$; and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N for the axis oriented by the unit vector $\vec{n}$ (see **[A1]**, **[A2]**, **[A6]**, and **[A7]**.

The rigid body kinetic parameters are interpreted by these vectors (see **[25]**, **[26]**, **[27]** and **[41]**).

The change of the mass inertia moment vector in the transition from one rigid body point to another is determined when the axis retains its orientation which represents the modification of the Huygens-Steiner theorem expressed by means of the defined mass inertia moment vector. Then the change of the mass inertia moment vector is determined in the case of the axis changing its orientation in the way analogous to the Cauchy equations for the total stress vector in the elasticity theory. Then the interpretation of the main inertia directions are derived as well as of the main mass inertia moment asymmetry are derived. The relation between the axis deviation load vector by the material body mass inertia moment for the octahedron axis and the mass inertia moments asymmetry axis is analyzed.

Further interpretation of the kinetic parameters of the of the body by means of the body mass inertia moment vector and by means of the body mass linear (static) moment vector for the axis and the point refers to the description of the motion quantity (linear momentum) as well as motion quantity moment (angular momentum) and kinetic energy as the function of the mass moment vectors for the axis and the point and the momentary angular velocity and referential point velocity (see **[A3]**, **[32]**, **[33]**, **[36]**, **[A6]**, **[39]**, **[A7]**, **[42]**, **[43]** and **[A8]**).

I.1.2. Body mass moments vectors at point for the axis. In studying the dynamics of a rigid and solid body, geometry of mass plays an important part. In [3] and [4] there is a conclusion that it is not necessary to know all the details about the mass distribution and the masses internal structures in order to study the rigid body translatory motion under the action of the force. The properties necessary for the study of the rigid body motion as a material system are the rigid body dynamic properties. The values determining the dynamic properties are called the rigid body dynamic parameters (see [3]).

According to the given reference these parameters are taken to be: mass M of the rigid body; position vector $\vec{\rho}_C$ of the body mass center, the point C with respect to a certain point O and $\mathbf{J}^{(C)}$ the body mass inertia moment tensor matrix for the point C which is determined with six scalar dynamic parameters. In this way in the general case the dynamic rigid body characteristic ten independent scalar dynamic parameters are required. By means of these ten dynamic parameters of the rigid body the sixth order matrix of the following shape is formed:

$$\mathbf{J}^{(O)}_{\mathrm{ex}} = \begin{pmatrix} M & 0 & 0 & 0 & Mz_C & -My_C \\ 0 & M & 0 & -Mz_C & 0 & Mx_C \\ 0 & 0 & M & My_C & -Mx_C & 0 \\ 0 & -Mz_C & My_C & J_x & D_{yx} & D_{zx} \\ Mz_C & 0 & -Mx_C & D_{xy} & J_y & D_{zy} \\ -My_c & Mx_C & 0 & D_{xz} & D_{yz} & J_z \end{pmatrix} \tag{1}$$

and this matrix is given in [3] and [4] as the rigid body mass inertia matrix for the given point O and the given trihedron. This is the *matrix of the tensor expanded* in an appropriate way. The mass inertia moment matrix changes its coordinates according to the change of the reference trihedron.

In [1] the mass linear polar moment $\vec{M}^{(O)}$ of the material system or the vector static system mass moment is defined with respect to the pole O in the form:

$$\vec{M}^{(O)} = \iiint_V \vec{\rho}\, dm = \vec{\rho}_C M, \qquad dm = \sigma dV \tag{2}$$

where $\vec{\rho}$ is the vector of the rigid body points position with respect to the common pole O, V is the space region that the observed body occupies and σ is the mass density at all the body points.

There are two important properties of a certain body mass: the mass center position of a material body does not depend on the pole choice but only on the body mass distribution and the mass linear polar moment $\vec{M}^{(C)}$ with respect to the body mass center is equal to zero.

Since our aim is to consider a possibility of the interpretation of the rigid body dynamic parameters in a modified shape we are going to set, as a reference, the pole O as well as the axis oriented by the unit vector $\vec{n}$. Considering that the general case the rigid body motion can be represented by one rotation around momentary axis, that is, by the translation of the mass center velocity and the rotation around the axis through the given center we are led to the idea to define the rigid body dynamic parameters by means of the pole O as the referential point through we position an axis parallel to the momentary rotation axis (see **[41]**.

Therefore we define the following (see Fig. 1a):

1* Vector $\vec{\mathcal{M}}_{\vec{n}}^{(O)}$ of the body mass at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathcal{M}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint\limits_V \vec{n}\, dm = M\vec{n}, \quad dm = \sigma\, dV \tag{3}$$

which does not depend on the mass distribution in the body, that is, on the density. For all the space points and parallel axes it has the same values and it changes only with the axis orientation change. It is determined only with the mass quantity and the axis orientation.

2* Vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint\limits_V [\vec{n}, \vec{\rho}]\, dm, \quad dm = \sigma\, dV \tag{4}$$

where $\vec{\rho}$ is the vector of the rigid body points position of the elementary body mass dm with respect to the common pole O. For the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point O for the axis oriented by the unit vector $\vec{n}$ we can write:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{n}, \vec{\rho}_C]M = [\vec{n}, \vec{M}^{(O)}] \tag{5}$$

The illustration is given in the Figure 1a.

3* Vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of he body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form (see **[A1]**, **[A2]**, **[A6]** and **[A7]**:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint\limits_V [\vec{\rho}, [\vec{n}, \vec{\rho}]]\, dm \tag{6}$$

It can also be considered the body mass square moment vector at the point O for the axis, through the pole, oriented by the unit vector $\vec{n}$. The vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ can be decomposed into three components: the collinear with the axis $J_{\vec{n}}^{(O)}$ and the two other ones $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$ in the directions, $\vec{u}$ and $\vec{v}$, normal to the orientation axis $\vec{n}$. The collinear component represents the axial moment of the body mass inertia for the axis oriented by the unit vector $\vec{n}$ through the pole O. The other two components represent the deviational moments of the body mass for a couple of normal axes oriented by unit vectors $\vec{n}$ and $\vec{u}$, that is, $\vec{n}$ and $\vec{v}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = J_{n}^{(O)}\vec{n} + D_{nu}^{(O)}\vec{u} + D_{nv}^{(O)}\vec{v} \tag{7}$$

The definition–expression for the body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ can be obtained starting from the expression for the axial body mass inertia moment $J_{\vec{n}}^{(O)}$ for the axis oriented by unit vector $\vec{n}$ drawn through the point O and for the deviational body mass moments for the couples of the orthogonal axes oriented by unit vectors $(\vec{n},\vec{u})$ and $(\vec{n},\vec{v})$, $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$, according to **[25]**, **[38]**. By means of them we form the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{n}\iiint\limits_{V} [\vec{n},\vec{\rho}]^2\,dm + \vec{u}\iiint\limits_{V} ([\vec{n},\vec{\rho}],[\vec{u},\vec{\rho}])\,dm + \vec{v}\iiint\limits_{V} ([\vec{n},\vec{\rho}],[\vec{v},\vec{\rho}])\,dm \tag{8}$$

The rigid body axial mass inertia moment is:

$$J_{\vec{n}}^{(O)} = \iiint\limits_{V} [\vec{n},\vec{\rho}]^2\,dm \tag{8*}$$

The rigid body mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ is in the following form:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{u}\iiint\limits_{V} ([\vec{n},\vec{\rho}],[\vec{u},\vec{\rho}])\,dm + \vec{v}\iiint\limits_{V} ([\vec{n},\vec{\rho}],[\vec{v},\vec{\rho}])\,dm = \vec{T}\iiint\limits_{V} ([\vec{T},\vec{\rho}],[\vec{n},\vec{\rho}])\,dm$$

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \iiint\limits_{V} [\vec{v},[[\vec{\rho},[\vec{n},\vec{\rho}]]\vec{n}]]\,dm = [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] \tag{9}$$

By means of the previous expressions (8) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ we can write the expression identical to the expression (6) which has been set as a definition.

Figure 1a shows the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$, the rigid body mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$, the axial moment of the body mass inertia $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole O, and the other two components, $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$, the deviational moments of the body mass for a couple of normal axes oriented by unit vectors $\vec{n}$ and $\vec{u}$, that is, $\vec{n}$ and $\vec{v}$, through the pole O.

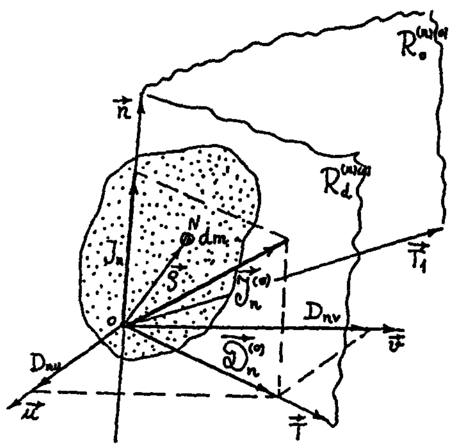

Fig. 1a

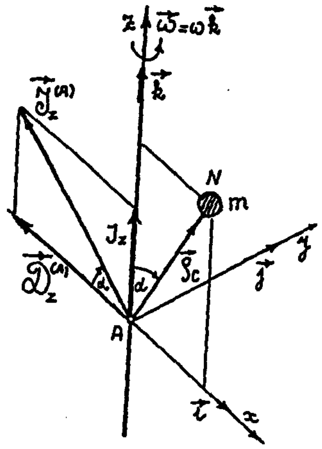

Fig. 1b

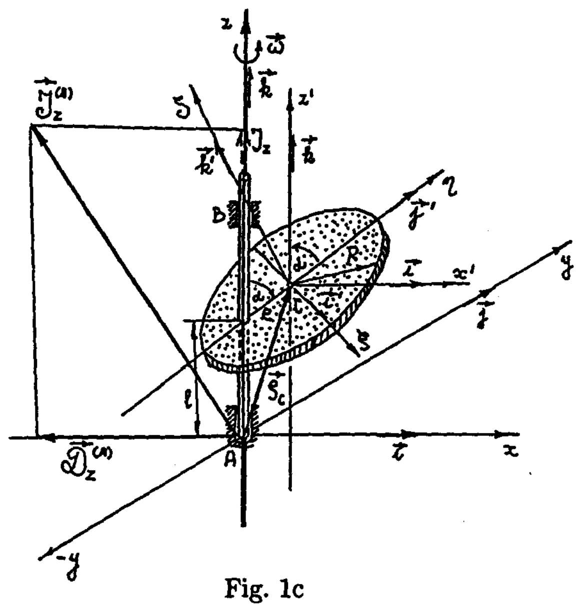

Fig. 1c

Fig. 1b shows the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the material particle mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$, the material particle mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$, the axial moment of the material particle mass inertia $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole O.

Fig. 1c shows an eccentrically skewly positioned discus respect to the axis of the shaft, as well as the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the discus mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$, the discus mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$, the axial moment of the discus mass inertia $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole O.

1.1.3. The material body mass inertia moment vectors for the two parallel axes through two referential points theorem. The Figure 2a shows the material body and two referential points - poles O and O_1 and two parallel axes through them oriented by unit vector $\vec{n}$. The same Figure also shows the denoted elementary mass dm at the point N of the rigid body and $\vec{\rho}$ and $\vec{r}$, the position vector of that point with respect to the pole O, that is, pole O_1, as well as the position vectors $\vec{\rho}_0$ of the pole O_1 with respect to pole O.

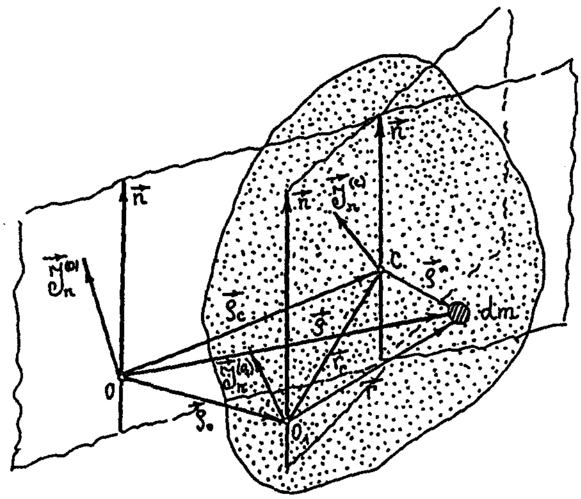

Fig. 2a

Now it is necessary to determine the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ and its relation to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O_1 for the axis oriented by the same unit vector $\vec{n}$.

This means we are interested in the change of the body mass inertia moment vector a certain axis which moves from one point to another retaining its orientation.

By using the expression (6) defining the mass inertia moment vector for a certain point and axis as well as the expression $\vec{\rho} = \vec{\rho}_O + \vec{r}$, we can write the following:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{\rho}_O + \vec{r}, [\vec{n}, \vec{\rho}_O + \vec{r}]]\, dm =$$
$$= \vec{\mathfrak{J}}_{\vec{n}}^{(O_1)} + [\vec{\rho}_O, \vec{\mathfrak{S}}_{\vec{n}}^{(O_1)}] + [\vec{M}_c^{(O_1)}, [\vec{n}, \vec{\rho}_O]] + [\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]\, M \quad (10)$$

We see that all the members in the last expression have the same structures. These structures are: $[\vec{\rho}_O, [\vec{n}, \vec{r}_C]]\, M$, $[\vec{r}_C, [\vec{n}, \vec{\rho}_O]]\, M$ and $[\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]\, M$.

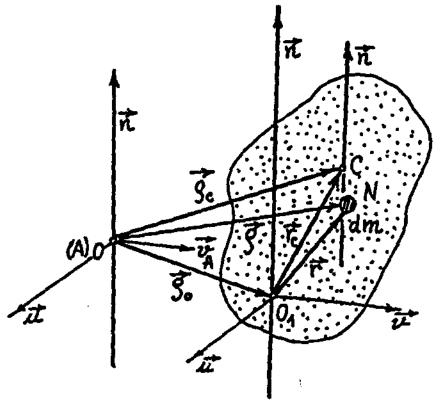

Fig. 2b

The expression (10) is the mathematical form of the theorem for the relation of the material body mass inertia moment vectors, $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ and $\vec{\mathfrak{J}}_{\vec{n}}^{(O_1)}$, for the two parallel axes through two corresponding points, pole O and pole O_1.

In the case when the pole O_1 is the center C of the body mass the vector $\vec{r}_C$ (the position vector of the masses center with respect to the pole O_1) is equal to zero, whereas the vector $\vec{\rho}_O$ turns into $\vec{\rho}_C$ so that the last expression (10) can be written in the following form (see Figure 2b):

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M \quad (11)$$

This expression (11) represents the mathematical form of the theorem of the change of the mass moment vector for the pole and the axis when the axis is translated from the pole in the mass center C to the arbitrary point, pole O.

The Huygens-Steiner theorems (see [11], [1], [3] and [4]) for the axial mass inertia moment as well as for the mass deviational moments came from this theorem (11) about the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ passing trough the mass center C and when the axis translate to the other point O.

The vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the body mass inertia moment for the body mass center C as well as for the axis oriented by unit vector $\vec{n}$ passing trough the mass center C we are going to call *the central or proper (eigen, personal) vector of the body mass inertia moment for the axis oriented by unit vector* $\vec{n}$.

The part $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} = [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M$ from the expression (11) represents the position part of the body mass inertia moment vector and we going to call it *the body mass inertia position moment vector for the point O and the axis oriented by unit vector* $\vec{n}$ *in relation to the body mass center C*. We can see that the body mass inertia moment vector for the axis trough the mass center C is the *smallest* vector since for all the other parallel axes the position part $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} = [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M$ has to be taken into consideration. This can be expressed by means of the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass linear moment for the point O and the axis oriented by unit vector $\vec{n}$ in the form $[\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}}^{(O)}]$.

The vector $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} = [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M$ is the free vector as the moment of the couple:

$$\begin{aligned}\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} &= [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M = \vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O\to C)} = [\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}}^{(O)}] = \vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(C\to O)} \\ &= [-\vec{\rho}_C, -\vec{\mathfrak{S}}_{\vec{n}}^{(O)}] = [-\vec{\rho}_C, [\vec{n}, -\vec{\rho}_C]]\, M \end{aligned} \tag{11*}$$

This vector $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)}$ can be moved from mass center C to arbitrary point O, as well as opposite from O to C, without change. This vector $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)}$ is the moment of a couple of the mass linear position moment vectors: $-\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ in the pole O and $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ in the pole mass center C.

Two vectors $-\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{n}, -\vec{\rho}_C]\, M$ and $\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{n}, \vec{\rho}_C]\, M$ having the same magnitude, parallel lines of the orientation, and opposite sense form a couple. Clearly, the sum of the moments of the two vectors about a given points, however, is not zero.

I.1.4. The change of the body mass inertia moment vector for the point and axis orientation change through the referential point. Let us now define the vectors $\vec{\mathfrak{J}}_x^{(O)}$, $\vec{\mathfrak{J}}_y^{(O)}$ and $\vec{\mathfrak{J}}_z^{(O)}$ of the body mass inertia moments at the point O and for the coordinate axes Ox, Oy and Oz. These vectors can be expressed in the form:

$$\vec{\mathfrak{J}}_x^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{i}, \vec{\rho}]]\, dm, \quad \vec{\mathfrak{J}}_y^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{j}, \vec{\rho}]]\, dm, \quad \vec{\mathfrak{J}}_z^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{k}, \vec{\rho}]]\, dm \tag{12}$$

If we denote the senses cosine of the unit vector $\vec{n}$ with $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ when the unit vector defines the orientation of the axis passing though the point O, then we can successively multiply the expressions (12) and we obtain them added:

$$\vec{\mathfrak{J}}_x^{(O)} \cos\alpha + \vec{\mathfrak{J}}_y^{(O)} \cos\beta + \vec{\mathfrak{J}}_z^{(O)} \cos\gamma = \iiint_V [\vec{\rho}, [\vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma, \vec{\rho}]]\, dm$$
$$= \iiint_V [\vec{\rho}, [\vec{n}, \vec{\rho}]]\, dm$$

From the previous expression we conclude that the body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ is equal to:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_x^{(O)} \cos\alpha + \vec{\mathfrak{J}}_y^{(O)} \cos\beta + \vec{\mathfrak{J}}_z^{(O)} \cos\gamma \tag{13}$$

The last expression is analogous to the equation for determining the total stress vector $\vec{p}_{\vec{n}}^{(O)}$ at the point O of the stressed body for the plane with normal unit vector $\vec{n}$ which is known as the Cauchy equation in the elasticity theory. There fore we are going to call it the Cauchy equation giving the relation of the body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ and the vectors $\vec{\mathfrak{J}}_x^{(O)}$, $\vec{\mathfrak{J}}_y^{(O)}$ and $\vec{\mathfrak{J}}_z^{(O)}$ of the body mass inertia moments at the point O and for the coordinate axes Ox, Oy and Oz.

I.1.5. Cauchy equations in the matrix form. Now by means of the mass inertia moment tensor matrix $\mathbf{J}^{(O)}$ the Cauchy vector equation (13) can be written in the matrix form:

$$\{\mathfrak{J}_{\vec{n}}^{(O)}\} = (\{\mathfrak{J}_x^{(O)}\}\{\mathfrak{J}_y^{(O)}\}\{\mathfrak{J}_z^{(O)}\})\{n\} = \mathbf{J}^{(O)}\{n\} \tag{14}$$

Now for the body mass axial inertia moment $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$, as well as for the body mass deviation moment $D_{\vec{n}\vec{v}}^{(O)}$ for the orthogonal axes $\vec{n}$ and $\vec{v}$ we can write the following expressions:

$$J_{\vec{n}}^{(O)} = (n)\{\mathfrak{J}_{\vec{n}}^{(O)}\} = (n)\mathbf{J}^{(O)}\{n\}, \quad D_{\vec{n}\vec{v}}^{(O)} = (v)\{\mathfrak{J}_{\vec{n}}^{(O)}\} = (v)\mathbf{J}^{(O)}\{n\} \tag{15}$$

The invariants of the body mass inertia moment state at a certain point can be determined as the first $J_1^{(O)}$, second $J_2^{(O)}$ and third $J_3^{(O)}$ scalar of the body mass inertia moment tensor matrix.

The rigid body mass inertia moment tensor matrix $\mathbf{J}^{(O)}$ for a certain pole can be separated into two matrices corresponding to the spherical $\mathbf{J}^{(O)\mathrm{sph}}$ and deviational $\mathbf{J}^{(O)\mathrm{dev}} = \mathbf{D}^{(O)\mathrm{dev}}$ part of the rigid body mass inertia moment tensor (which is analogous to the stress tensor matrix and strain (relative deformation) tensor matrix in the elasticity theory):

$$\mathbf{J}^{(O)\mathrm{sph}} = \frac{1}{3}J_1^{(O)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}J_1^{(O)} & 0 & 0 \\ 0 & \frac{1}{3}J_1^{(O)} & 0 \\ 0 & 0 & \frac{1}{3}J_1^{(O)} \end{pmatrix}$$

$$\mathbf{J}^{(O)\mathrm{dev}} = \mathbf{J}^{(O)} - \mathbf{J}^{(O)\mathrm{sph}} = (\{\vec{\mathfrak{J}}_{\vec{n}}^{(O)}\}\{\vec{\mathfrak{J}}_{\vec{u}}^{(O)}\}\{\vec{\mathfrak{J}}_{\vec{v}}^{(O)}\}) - \frac{1}{3}J_1^{(O)}I \tag{17}$$

I.1.6. Axial and deviational part of the rigid body mass inertia moment vector. The body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ can be written in the transformed form in which we separate the part $\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{aks}}$ collinear with axis oriented by unit vector $\vec{n}$ and the part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}}$ normal to the axis oriented by unit vector $\vec{n}$ as it is shown in the Figures 1a and 3.

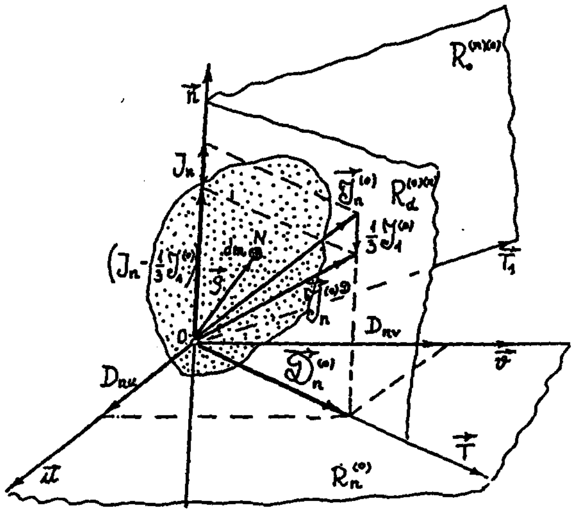

Figure 3

Now the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ can be transformed to the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{aks}} + \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{n}(\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) + [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{aks}} + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} \tag{18}$$

with components:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{aks}} = \vec{n}(\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) = \vec{n}J_{\vec{n}}^{(O)} \tag{19}$$

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] \tag{20}$$

The first part $\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{aks}}$ collinear with axis oriented by unit vector $\vec{n}$ given by formula (19) represents body mass axial inertia moment vector at the point and for the axis oriented by unit vector $\vec{n}$, and it does not depend on the pole position on the axis.

The second part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}}$ normal to the axis oriented by unit vector $\vec{n}$ given by formula (20) lies in the plane formed by the axis oriented by unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment. This plane is determined by the axis selection and by the body mass distribution with respect to the axis and the pole.

The vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ is the deviation load by the rigid body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ and it can be defined as the rigid body mass inertia moment vector component normal to the axis and in the plane which is formed by the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment. This can be seen in the Figure 1a and 3. We conclude that the vector magnitude is equal to the deviation moment of the body mass for the axis oriented by the unit vector $\vec{n}$ and the axis oriented by the unit vector $\vec{T}$ normal to the axis oriented by the unit vector $\vec{n}$, in the direction of the cutting line of the plane normal to the axis trough the pole O and of the plane formed by the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the pole and for axis oriented by the unit vector $\vec{n}$. The unit vector of this cutting line is denoted with $\vec{T}$. The unit vector normal to the unit vectors $\vec{n}$ and $\vec{T}$ is denoted with $\vec{T}_1$. We conclude that the body mass deviation moment for the axes $\vec{n}$ and $\vec{T}_1$ passing through the pole O is equal to zero. This means that for an arbitrary axis at the observed point O there can always be found at least one axis normal to it oriented by $\vec{T}_1$ for which, together with the axis oriented by the unit vector $\vec{n}$, the body mass deviation moment is equal to zero. This axis is normal to the axis oriented by the unit vector $\vec{n}$ and to the *deviation plane* formed by the unite vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$. The deviation plane we denote by R_d. Only for the mass inertia moment main axis through a retain point-pole the deviation plane is not defined nor it can be said it exists since if the axis oriented by the unit vector $\vec{n}$ through a certain point is the main axis of the body mass inertia moment then for this axis the deviation load to the axis is equal to zero. In this case the body mass inertia moment vector has only one component collinear with the axis. That is, if a certain axis through a certain point-pole is the main mass inertia moment than the vector of its deviation load by the body mass inertia moment is equal to zero.

I.1.7. Spherical and deviatorial part of the rigid body mass moment vector.

If we now follow the idea of the formation of matrices of the spherical and deviatorial part of the mass inertia moment tensor according to the analogy (see [24], [23] and [34]) with the spherical and deviatorial part of the stress tensor, that is, of the relative deformation (strain) tensor we can define two vectors (see Figure 3):

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{sph}}$ the vector spherical part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{sph}} = \frac{1}{3}J_1^{(O)}\vec{n} = \frac{1}{3}J_0^{(O)}\vec{n} \qquad (21)$$

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)D}$ the vector deviatorial part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)D} = \vec{n}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}) - \frac{1}{3} J_1^{(O)} \vec{n} + [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \vec{n}((\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}) - \frac{1}{3} J_1^{(O)} \vec{n}) + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} \quad (22)$$

Let us now consider the modification of the Huygens-Steiner theorem in its application to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ the deviation part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$, as well as the vector of the deviation load by the rigid body mass inertia moment on the axis oriented by the unit vector $\vec{n}$ in the transition from the mass center C to the pole O (see Figure 2b). We use the definition of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the mass inertia moment (18) and the formula (11) derived in the paragraph I.1.3. for the Huygens-Steiner formula modified of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$ so that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \vec{\mathfrak{D}}_{\vec{n}}^{(C)} - (\vec{n}, \vec{\rho}_C)[\vec{n}, [\vec{\rho}_C, \vec{n}]]\, M \quad (23)$$

The expression (23) represents the Huygens-Steiner Theorem modified to the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the mass inertia moment of the axis oriented by the vector $\vec{n}$ connected to the pole O. From this expression we conclude that the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the axis deviational load through an arbitrary point O oriented by the unit vector $\vec{n}$ equal to the sum of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(C)}$ of the axis deviation load through the center C of the body mass for the parallel axis and the position deviation load in the transition of the axis from the pole C-mass center to the pole – arbitrary point O determined from the expression:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(C\to O)} = [\vec{n}, [[\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\vec{n}]]\, M = -(\vec{n}, \vec{\rho}_C)[\vec{n}, [\vec{\rho}_C, \vec{n}]]\, M \quad (24)$$

If the pole O and the center C of the body mass are located on the same normal to the axis oriented by the unit vector $\vec{n}$ then the position part of the deviation load in the transition from the axis through the mass center C to the parallel axis through the pole O is equal to zero. This means that the deviation load vectors of the axis by the body mass inertia moment for the central plane points corresponding to the given axis are equal to the deviation load belonging to the central axis $\vec{\mathfrak{D}}_{\vec{n}}^{(C)}$.

I.1.8. Main mass inertia moment directions, main mass inertia moment vectors. By means of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$ we can introduce a new definition of the main mass inertia momemt axes. Through one pole O we can draw an infinite number of axes of orientations. Among them we are looking for the axis for which the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment had only one component, collinear with the axis, that is, the one for which the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load of the axis by the body mass inertia moment is equal to zero.

Using the analogy given in the papers [245] and [34] as well as the analogy with the matrix interpretation from books [14], [15], [28] and [23] as more appropriate for this case and by denoting the unit vector of the main masss inertia moment axis orientation with $\vec{n}_s$ which is in accordance with the Fig. 3a, we can write:

$$\{\mathfrak{J}_{\vec{n}_s}^{(O)}\} = \mathbf{J}^{(O)}\{n_s\} = J_s^{(O)}\{n_s\} \Rightarrow (\mathbf{J}^{(O)} - J_s^{(O)}I)\{n_s\} = \{0\} \qquad (25)$$

so that the Hamilton equation for determining the main mass inertia moments is:

$$f(J_s^{(O)}) = |\mathbf{J}^{(O)} - J_s^{(O)}I| = 0 \qquad (26)$$

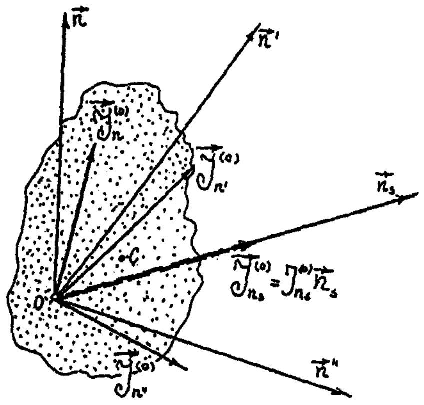

Fig. 3a

while for the senses cosines of the main mass inertia moment axes the following relations are obtained:

$$\frac{\cos\alpha_S}{\mathbf{K}_{31}^{(S)}} = \frac{\cos\beta_S}{\mathbf{K}_{32}^{(S)}} = \frac{\cos\gamma_S}{\mathbf{K}_{33}^{(S)}} = C_S, \quad \cos^2\alpha_s + \cos^2\beta_s + \cos^2\gamma_s = 1 \qquad (27)$$

where $\mathbf{K}_{3k}^{(S)}$, $k = 1,2,3$ are co-factors of the third kind elements and the corresponding matrix column, successively for the roots $J_s^{(O)}$, $s = 1,2,3$ of the Hamilton equation (26), which are the main mass inertia moments and which represent the axial mass inertia moments for the main mass inertia moments axes. There are three roots and three orthogonal main axes at every point with respect to which the rigid body mass inertia moment vectors are determined. The Hamilton equation coefficients are the first, second and third invariants of the mass inertia moment state at referent point, and they are the first, second and third scalar of the body mass inertia moment tensor matrix at referent point (see [24] or [23]).

1.1.9. Extreme values of the mass deviation moments. In [24] is given an analogy between the stress state model, the strain state model and the mass inertia moment state of the body at the observed body point. For determining the mass deviation moments extreme values we shall use this analogy which exists between the stress tensor, the strain tensor and the body mass inertia tensor, as well as between the vector $\vec{p}_{\vec{n}}^{(O)}$ of the total stress at a certain body point for the plane with the normal oriented by unit vector $\vec{n}$, the vector $\vec{\delta}_{\vec{n}}^{(O)}$ of the total strain (relative deformation) of the line element drawn from the observed point in the direction of the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the observed pole for the axis oriented by unit vector $\vec{n}$.

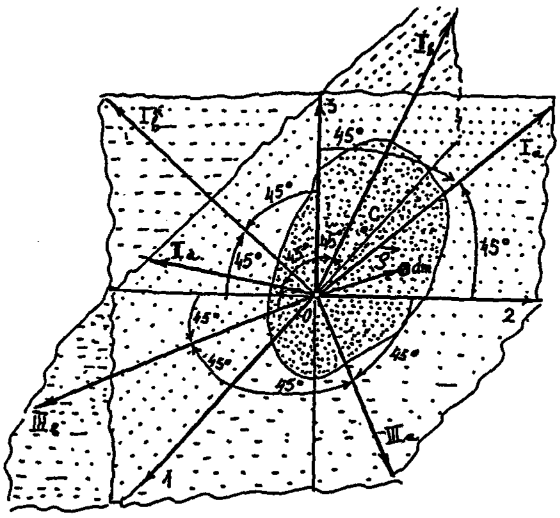

Figure 4a

On the basis of the given analogy in [24] and [23], the following conclusions are drawn, though without proofs: on the basis of the analogy between the mass deviation moments extreme values for a couple of orthogonal axes (that is, of the mass centrifugal moments) and yield stress extreme values in the orthogonal planes that pass in pair through one main stress direction and form an angle of 45° with the other two main stress direction, we conclude that the mass deviation moments extreme values appear for the axes pairs I_a and I_b, II_a and II_b, III_a and III_b that pass in pairs through the main body mass inertia moment axis trough the given point and form angles of 45° with the other two main mass inertia moment axes (see Figures 4a and 4b). For these pairs of the defined axes the mass deviation moments (the mass centrifugal moments) are equal to the semi-difference between the two main (axial) body mass inertia moment and for each axis in the corresponding pair the axial inertia moments are equal to the semi-sum of the two corresponding main moments of the body mass inertia for the given point.

The pairs of these coupled axes are the body mass inertia moments asymmetry axes since for them the mass centrifugal moments are extreme values and the axial mass inertia moments for both the axes in pair are mutually equal. The concept of "asymmetry" can be accepted since for symmetry axes the body mass centrifugal moment is equal to zero and for these axes the body mass centrifugal moment is of extreme value so that this leads to the conclusion about the asymmetry of the material body mass inertia moment properties. On the basis of the given analogy we cam write the values of the mass deviation moments and the body mass axial inertia moments of these axes (see Figures 4a and 4b):

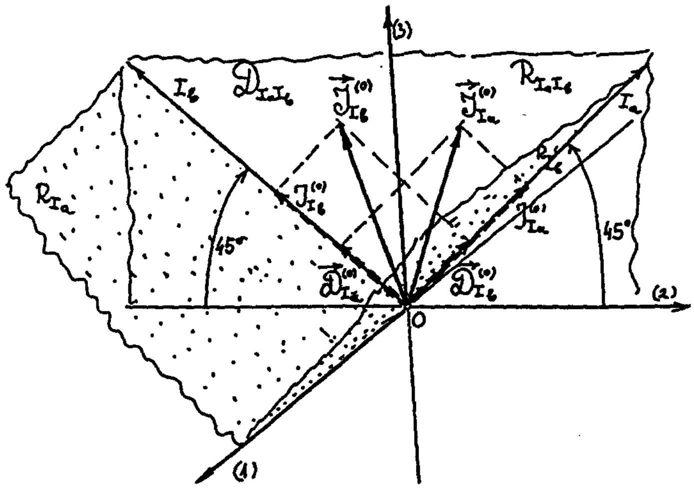

Figure 4b

$$\begin{aligned} D^{(O)}_{I_aI_b} &= \pm\frac{1}{2}(J^{(O)}_2 - J^{(O)}_3), & J^{(O)}_{I_a} &= J^{(O)}_{I_b} = \frac{1}{2}(J^{(O)}_2 + J^{(O)}_3), \\ D^{(O)}_{II_aII_b} &= \pm\frac{1}{2}(J^{(O)}_1 - J^{(O)}_3), & J^{(O)}_{II_a} &= J^{(O)}_{II_b} = \frac{1}{2}(J^{(O)}_1 + J^{(O)}_3), \\ D^{(O)}_{III_aIII_b} &= \pm\frac{1}{2}(J^{(O)}_1 - J^{(O)}_2), & J^{(O)}_{III_a} &= J^{(O)}_{III_b} = \frac{1}{2}(J^{(O)}_1 + J^{(O)}_2). \end{aligned} \tag{28}$$

In the coordinate system of the main body mass inertia directions $\vec{n}_s$, $s = 1,2,3$ the vectors $\vec{\mathfrak{J}}^{(O)}_{\vec{n}_s}$, $s = 1,2,3$ for the referential point as the pole are the body mass inertia moment vectors for the main mass inertia moment axes and we see that they have only the components collinear with the corresponding main mass inertia moment axes $\vec{\mathfrak{J}}^{(O)}_{\vec{n}_s} = J^{(O)}_s \vec{n}_s$, $s = 1,2,3$.

Let's now define the vectors $\vec{\mathfrak{J}}^{(O)}_{I_a}$, $\vec{\mathfrak{J}}^{(O)}_{II_a}$ and $\vec{\mathfrak{T}}^{(O)}_{III_a}$ of the body mass inertia moment at the observed point for the axis oriented by the unit vector $\vec{n}_{I_a}$, or $\vec{n}_{II_a}$ or $\vec{n}_{III_a}$ of the mass inertia moment asymmetry axis I_a or II_a or III_a by using the definition of this vector so that we have (see Figure 4b):

$$\vec{\mathfrak{J}}^{(O)}_{I_a} = \frac{\sqrt{2}}{2}(\vec{\mathfrak{J}}^{(O)}_{n_2} + \vec{\mathfrak{J}}^{(O)}_{n_3});$$
$$\vec{\mathfrak{J}}^{(O)}_{II_a} = \frac{\sqrt{2}}{2}(\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_3}); \qquad (29)$$
$$\vec{\mathfrak{J}}^{(O)}_{III_a} = \frac{\sqrt{2}}{2}(\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_2});$$

Let's now define the vectors $\vec{\mathfrak{J}}^{(O)}_{I_b}$, $\vec{\mathfrak{J}}^{(O)}_{II_b}$ and $\vec{\mathfrak{J}}^{(O)}_{III_b}$ of the body mass inertia moment at the observed point for the axis oriented by the unit vector $\vec{n}_{I_b}$ or $\vec{n}_{II_b}$ or $\vec{n}_{III_b}$ of the mass inertia moment asymmetry axis I_b or II_b or III_b by using the definition of this vector so that we have:

$$\vec{\mathfrak{J}}^{(O)}_{I_b} = \frac{\sqrt{2}}{2}(-\vec{\mathfrak{J}}^{(O)}_{n_2} + \vec{\mathfrak{J}}^{(O)}_{n_3});$$
$$\vec{\mathfrak{J}}^{(O)}_{II_b} = \frac{\sqrt{2}}{2}(-\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_3}); \qquad (30)$$
$$\vec{\mathfrak{J}}^{(O)}_{III_b} = \frac{\sqrt{2}}{2}(-\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_2});$$

Now we define the components of the vector $\vec{\mathfrak{J}}^{(O)}_{I_b}$. The collinear one with the body mass inertia moments symmetry axis I_a:

$$(\vec{\mathfrak{J}}^{(O)}_{I_a}, \vec{n}_{I_a}) = \frac{J^{(O)}_2 + J^{(O)}_3}{2} = J^{(O)}_{I_a} = J^{(O)}_{I_b} \qquad (31)$$

The component normal to the body mass inertia moment asymmetry axis lying in the deviation plane representing the vector $\vec{\mathfrak{D}}^{(O)}_{I_a}$ of the deviation load by the body mass inertia moment of the mass inertia moment asymmetry axis according to the previously given definition in the form:

$$\vec{\mathfrak{D}}^{(O)}_{I_a} = [\vec{n}_{I_a}, [\vec{\mathfrak{J}}^{(O)}_{I_a}, \vec{n}_{I_a}]] = \frac{J^{(O)}_2 - J^{(O)}_3}{2}\vec{n}_{I_b} = D^{(O)}_{I_a I_b}\vec{n}_{I_b} \qquad (32)$$

Analysis the expressions from (28) to (32) we conclude the following:

1* The expressions given in (28) on the analogy basis are correct;

2* Both the vectors $\vec{\mathfrak{J}}^{(O)}_{I_a}$ and $\vec{\mathfrak{J}}^{(O)}_{I_b}$ of the rigid body mass inertia moments for the pole O and the axis of the pair I of the mass inertia moment asymmetry, I_a and I_b are normal to the main mass inertia moment axis (1) and they lie in the plane $R_{I_a I_b}$ which is their mutual deviation plane. This plane is normal to the main mass inertia moment axis (1) and contains the other two main mass inertia moment directions (2) and (3);

3* The vector $\vec{\mathfrak{D}}^{(O)}_{I_a}$ of the deviation load by the body mass inertia moment of the mass inertia moment asymmetry axis oriented by unit vector $\vec{n}_{I_a}$ at given point lie in the direction of the second mass inertia moment asymmetry axis oriented by unit vector $\vec{n}_{I_b}$ of the pair I which is normal to the main mass inertia moment direction (1) and to the axis the mass inertia moment asymmetry I_a and vice versa. These two vectors, that is, $\vec{\mathfrak{D}}_{I_a}^{(O)}$ and $\vec{\mathfrak{D}}_{I_b}^{(O)}$, are the same magnitude and of the same components, of axial and deviational, and they have the same axial mass inertia moments. In a similar way the calculation can be applied to the other two pairs of the mass inertia moment asymmetry axes and the corresponding conclusions can be drawn in accordance with the expressions (28) and the previous conclusions.

I.1.10. Mass inertia moment vectors for the octahedron directions in the referential point. In analogy with defining the octahedron directions a certain point of the stressed and strained body as it is done in the elasticity or plasticity theory we shall define the octahedron directions at a certain point of the rigid body form the viewpoint of the body mass inertia moment state with respect to this pole as the direction that forms the same angles with the main inertia axes, that is, with the main inertia directions. There are eight such octahedron directions.

The vector $\vec{\mathfrak{J}}_{\text{oct}}^{(O)}$ of the mass inertia moment at the point O for the octahedron direction by using the basic definition is calculated as:

$$\vec{\mathfrak{J}}_{\text{oct}}^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{n}_{\text{oct}}, \vec{\rho}_C]]\, dm = \frac{\sqrt{3}}{3}(\vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)}) \tag{33}$$

and we can decompose it into two components.

1* The axial component in the octahedron direction in the form:

$$J_{\vec{n}_{\text{oct}}}^{(O)} = (\vec{n}_{\text{oct}}, \vec{\mathfrak{J}}_{\text{oct}}^{(O)}) = \frac{1}{3}J_1^{(O)} = \frac{2}{3}J_O^{(O)} \tag{34}$$

which represents the axial moment of the rigid body mass inertia moment for the octahedron direction axis for the given pole and it is equal to one third of the first mass inertia moment invariant or one third of the first scalar of the mass inertia polar moment for the pole O.

2* Normal component to the octahedron direction which is equal to the vector $\vec{\mathfrak{D}}_{\text{oct}}^{(O)}$ of the octahedron axis deviation load, by the body mass inertia moment and has the form:

$$\vec{\mathfrak{D}}_{\text{oct}}^{(O)} = -\frac{2\sqrt{6}}{9}(\vec{\mathfrak{D}}_{I_a}^{(O)} + \vec{\mathfrak{D}}_{II_a}^{(O)} + \vec{\mathfrak{D}}_{III_a}^{(O)}) \tag{35}$$

The vector $\vec{\mathfrak{D}}_{\text{oct}}^{(O)}$ of the deviation load by the body mass inertia moment of the octahedron axis can be expressed as the linear combination of the vectors $\vec{\mathfrak{D}}_{I_a}^{(O)}$, $\vec{\mathfrak{D}}_{II_a}^{(O)}$, $\vec{\mathfrak{D}}_{III_a}^{(O)}$ of the deviation load of the mass inertia moments asymmetry axes when it is related to one of the pair.

The intensity square of the vector $\vec{\mathfrak{D}}_{\text{oct}}^{(O)}$ of the deviation load by the body mass inertia moment of the octahedron axis can be defined by the following expression:

$$|\vec{\mathfrak{D}}_{\text{oct}}^{(O)}|^2 = \frac{4}{9}(|\vec{\mathfrak{D}}_{I_a}^{(O)}|^2 + |\vec{\mathfrak{D}}_{II_a}^{(O)}|^2 + |\vec{\mathfrak{D}}_{III_a}^{(O)}|^2) \tag{36}$$

It should be noted that there are eight axes (or four axes) at each point of the rigid body for which the mass inertia axial moments are equal to a third of the first mass inertia moment invariant and they are the octahedron directions determined with respect to the main mass inertia moment axes. The question should be asked about what sort of motion the body performs while rotating around the octahedron axis and if the conclusions can be generalized to hold for the bodies with different mass inertia moment characteristics.

If this conclusion is related to the previous section we can conclude that is: one set of eight (or four) axes for which the inertia axial moments of the body mass are mutually equal and equal to a third of the first mass inertia moment invariant: Three sets of two pairs of orthogonal axes of the inertia asymmetry for the axial inertia moments are also equal to the semi-sum of two main inertia moments each. The same stand for each body and for each pole chosen within the space or outside the space of the rigid body. Only the spherical body as the pole of all fourteen axes the axial mass inertia moment is the same and the deviation load is equal to zero.

I.2. The mass moment vectors at the dimensional coordinate system N

I.2.1. Introduction. This part introduces the vectors: $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the material particle mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, and $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$ at the dimensional curvilinear coordinate system N. The vectors can be used for the interpretation of the rigid body kinetic characteristics for the interpretation of the body dynamics at the dimensional curvilinear coordinate system N.

The change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body or particle mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, is determined in the transition from one space point to another when the axis retains its orientation which represents Huygens-Steiner theorem generalized for the defined mass inertia moment vector at the dimensional curvilinear coordinate system N.

This part gives the interpretation of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the material particles mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional curvilinear coordinate system N as well as by body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional curvilinear coordinate system N.

I.2.2. The dimensional curvilinear coordinate system N. According to the notation in the Fig. 5 the material point position vector $\vec{\rho}$, at the dimensional coordinate system n, can be written in the form:

$$\vec{\rho} = x^k \vec{g}_k \tag{37}$$

while unit vector $\vec{n}$ of the axis orientation can be written in the form:

$$\vec{n} = \lambda^k \vec{g}_k \tag{38}$$

In the previous expression $\vec{g}_k$ the basic vectors of the dimensional N of the curvilinear coordinates $\vec{g}_k = \dfrac{\partial \vec{\rho}}{\partial x^k}$ for these vectors it stands that:

$$(\vec{g}_k, \vec{g}_l) = g_{kl} \tag{39}$$

their product represents the metric tensor coordinates of the defined curvilinear coordinates system space. The position vector $\vec{\rho}$ magnitude squared is:

$$(\vec{\rho}, \vec{\rho}) = (\vec{g}_k, \vec{g}_l)\, x^l x^k = g_{kl} x^k x^l \tag{40}$$

while for the axis orientation unit vector $\vec{n}$:

$$(\vec{n}, \vec{n}) = (\vec{g}_k, \vec{g}_l)\, \lambda^l \lambda^k = g_{kl} \lambda^l \lambda^k = 1 \tag{41}$$

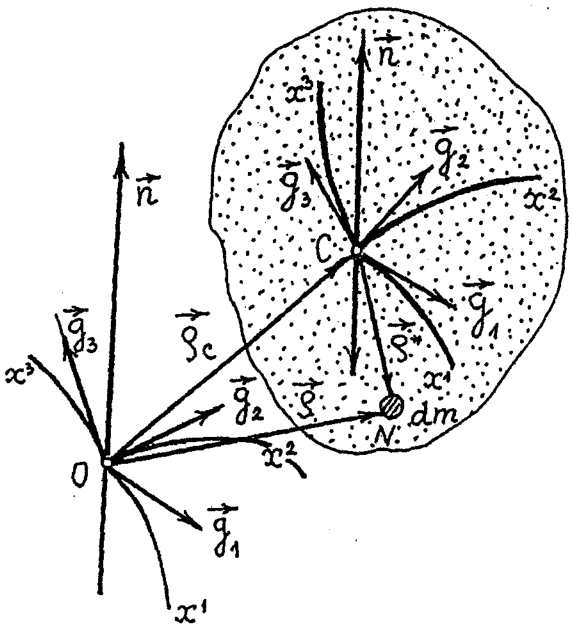

Figure 5

I.2.3. The material particle mass inertia moment vector for the pole and the axis. By introducing the expression (37) and (38) into expression (6) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ definition of the material particle mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, we obtain that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = [\vec{g}_k, [\vec{g}_l, \vec{g}_p]] x^k x^p \lambda^l m \tag{42}$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (42) can be write in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = (g_{kp} \vec{g}_l - g_{kl} \vec{g}_p) x^k x^p \lambda^l m \tag{43}$$

If we multiply scalarly the previous expression (43) with the unit vector $\vec{n}$, we obtain:

$$J_{\vec{n}}^{(O)} = (\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) = (g_{kp}g_{li} - g_{kl}g_{pi})x^k x^p \lambda^l \lambda^i m \tag{44}$$

which represent the material particle mass axial inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$. This formula is same as the formula (2.3) in [6] written by Vujičić.

If we now multiply the expression (43) twice vectorly with the unit vector $\vec{n}$, that is, according to [40], we separate the material particle mass inertia moment vector deviational part for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [[\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \{g_{kp}g_{lj}\vec{g}_i - g_{ki}g_{lj}\vec{g}_p + (g_{ki}g_{lp} - g_{kp}g_{li})\vec{g}_j\}x^k x^p \lambda^l \lambda^i \lambda^j m \tag{45}$$

The last expression represents the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the material particles mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional coordinate system N.

By introducing the expressions (37) and (38) into the expression (4) for the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ definition of the material particle mass linear moment for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain that:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{g}_i, \vec{g}_k]x^k \lambda^i m \tag{46}$$

1.2.4. The rigid body mass inertia moment vector for the pole and the axis. By introducing the expression (37) and (38) into expression (6) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ definition of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, we obtain that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint_V [\vec{g}_k, [\vec{g}_l, \vec{g}_p]]x^k x^p \lambda^l dm \tag{47}$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (47) can be written in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint_V (g_{kp}\vec{g}_l - g_{kl}\vec{g}_p)x^k x^p \lambda^l dm \tag{47*}$$

If we multiply scalarly the previous expression (48) with the unit vector $\vec{n}$, we obtain:

$$J_{\vec{n}}^{(O)} = (\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) = \iiint_V (g_{kp}g_{li} - g_{kl}g_{pi})x^k x^p \lambda^l \lambda^i dm \tag{48}$$

which represent the body mass axial inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$.

If now we multiply the expression (48) twice vectorly with the unit vector $\vec{n}$, that is, according to [40], we separate the body mass inertia moment vector deviational part for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \iiint\limits_V \{g_{kp}g_{lj}\vec{g}_i - g_{ki}g_{lj}\vec{g}_p + (g_{ki}g_{lp} - g_{kp}g_{li})\vec{g}_j\}x^k x^p \lambda^l \lambda^i \lambda^j dm \tag{49}$$

The last expression represents the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional coordinate system N.

By introducing the expressions (37) and (38) into the expression (4) for the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ definition of the body mass linear moment for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain that:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{g}_i, \vec{g}_k] x^k \lambda^i dm \tag{46*}$$

1.2.5. The Huygens-Steiner theorem. Following previous expression (11) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, the Huygens-Steiner theorem is derived which can be written in the following form for the curvilinear coordinate system (see Fig. 2a):

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]M = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{g}_k, [\vec{g}_l, \vec{g}_p]] x_C^k x_C^p \lambda^l M \tag{47*}$$

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + (g_{kp}\vec{g}_l - g_{kl}\vec{g}_p) x_C^k x_C^p \lambda^l M \tag{47**}$$

Following previous expression (23) for the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, the Huygens-Steiner theorem can be written in the following form in the curvilinear coordinate system:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{D}}_{\vec{n}}^{(C)} - g_{ij}[\vec{g}_k, [\vec{g}_l, \vec{g}_p]] x_C^j x_C^l \lambda^i \lambda^k \lambda^p M \tag{49*}$$

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{D}}_{\vec{n}}^{(C)} - g_{ij}(g_{kp}\vec{g}_l - g_{kl}\vec{g}_p) x_C^j x_C^l \lambda^i \lambda^k \lambda^p M \tag{49**}$$

which represents the expression of the Huygens-Steiner generalized to the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$.

CHAPTER II

II.1. Vector interpretations of the rigid bodies kinetic parameters

II.1.1. Rigid body kinetic energy. We shall consider the kinetic energy (see [**13**], [**7**], [**10**], and [**21**])a little with a slight modification due to the interpretation of the rigid body dynamic parameters by means of the introduced vectors $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the pole A for the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the body mass inertia moment at the pole A for the axis oriented by the unit vector $\vec{n}$. Since the velocity of each body point (see [**12**])can be defined by the two kinematic parameters of the translation velocity $\vec{v}_A$ of the referential point A and the angular velocity $\vec{\omega}$ and the unit vector $\vec{n}$ of the momentary rotation axis orientation we shall define the kinetic energy in relation to the body mass state properties with respect to the referential point translation velocity and the body mass moments state for the pole at the referential point A and for the axis oriented by the momentary rotation axis unit vector.

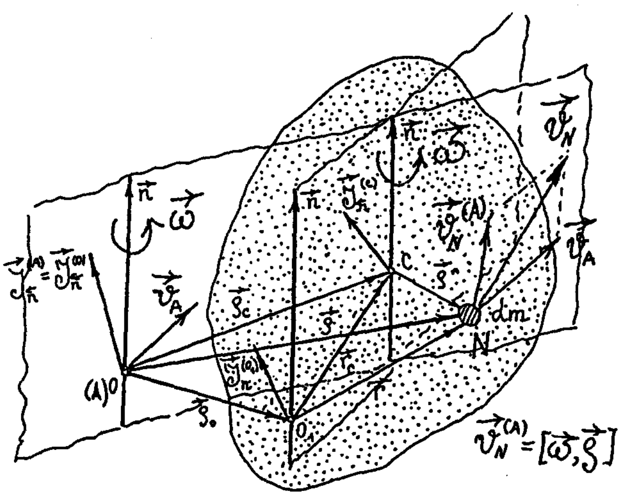

Figure 6

Using the notation in the Figures 2b and 6, for the rigid body kinetic energy we can write:

$$2E_k = \iiint\limits_V \vec{v}_N^2 dm \iiint\limits_V (\vec{v}_A + [\vec{\omega}, \vec{\rho}])^2 \, dm \tag{50}$$

that is, according to our idea the double kinetic energy is only expressed by means of the masses center velocity, body mass, momentary angular velocity and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the body mass inertia moment for the axis through the body mass center at the pole C and oriented by the momentary angular velocity unit vector $\vec{n}$:

$$2E_k = M(\vec{v}_C, \vec{v}_C) + \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) \tag{51}$$

In the case when the referential point A is not the mass center the kinetic energy can be expressed in the form:

$$2E_k = M(\vec{v}_A, \vec{v}_A) + 2\omega(\vec{v}_A, \vec{\mathfrak{S}}_{\vec{n}}^{(A)}) + \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) \tag{52}$$

which is expressed by means of the velocity $\vec{v}_A$ of the arbitrary referential point A, angular velocity $\vec{\omega}$ of the rotation around the axis through the point A and mass moment vectors as the mass inertia moment properties of the rigid body mass distribution with respect to the pole at the referential point A and for axis oriented by the momentary angular velocity $\vec{\omega}$.

We can see that the kinetic energy has a part which corresponds to the body translation of the velocity $\vec{v}_A$ of the referential point A, that is, the part corresponding to the pure rotation around the relative rotation axis that passes through the referential point A and is oriented by the vector $\vec{\omega}$ of the momentary angular velocity, as well as the mixed member which represents the coupling of the translation and rotation and can be called *"Coriolis member"* representing the double scalar products of the velocity $\vec{v}_A$ of the referential point translation and the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point A for the axis oriented by the unit vector $\vec{n}$ multiplied by the angular velocity vector magnitude. This third member represents the kinetic energy of the coupling of the translation motion of the referential pole velocity and the rotation motion around the axis through this referential point.

This *"Coriolis member"* which represents the kinetic energy of the rotatory and translatory motion coupling with respect to the referential pole is equal zero in the following cases:

1* when the translatory velocity of the referential point A is orthogonal to the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point A for the axis oriented by the unit vector $\vec{n}$, that is when the velocity $\vec{v}_A$ of the referential point A is parallel to the plane formed by the rotation axis through referential pole A and the body mass center;

2* when the referential point A is on the momentary rotation axis or at the momentary rotation pole; and

3* when the referential point A is at the body mass center C.

The expression (51) represents the modified expression of the Samuel Köning theorem for the kinetic energy, that is, the Samuel Köning theorem in new interpretation, which states that the rigid body kinetic energy is equal to the sum of the kinetic energy of its translator motion with mass center velocity and the kinetic energy of its rotation motion around the axis oriented by the momentary angular velocity through the body mass center.

If the referential point is at the momentary pole all the time, or the momentary rotation axis then the kinetic energy can be expressed as:

$$2E_k = \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(P)}) \tag{53}$$

and it has only the member corresponding to the rotation around the momentary rotation axis and is equal to the half of the product of the momentary angular velocity squared and the axial inertia moment for the momentary rotation axis as it is known.

II.1.2. Linear momentum and angular momentum of the body motion.

The classic literature (see [10], [7], [11]) gives a very well known definition of the rigid body linear momentum (motion quantity) and angular momentum (motion quantity moment). We shall consider it a little with a slight modification due to the interpretation of the rigid body dynamic parameters by means of the introduced body mass moment vectors. We are following the classic definition by using the prepositions from previous paragraph, as well as Fig. 6, so that we write for the linear momentum following expression:

$$\vec{\mathfrak{K}} = \iiint\limits_V \vec{v}_N dm = \iiint\limits_V (\vec{v}_A + [\vec{\omega}, \vec{\rho}]) dm = M\vec{v}_A + \omega \vec{\mathfrak{S}}_{\vec{n}}^{(A)} \tag{54}$$

The expression (54) of the linear momentum $\vec{\mathfrak{K}}$ of the rigid body whose points have the translation velocity $\vec{v}_A$ of the referential point A and the relative velocity $[\vec{\omega}, \vec{\rho}]$ due to the rotation around the axis oriented by the vector $\vec{\omega} = \omega\vec{n}$ through the point A has two parts: 1* the translatory one equal to the product of the referential point velocity and the body mass—the linear momentum due to the translation motion with the velocity of the referential point A; and 2* the rotatory one equal to the product of the magnitude ω of the angular velocity $\vec{\omega} = \omega\vec{n}$ and the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point A for the axis oriented by the unit vector $\vec{n}$.

If the pole A is the body mass center C then the linear momentum is equal only in the translatory part since the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment for the pole in the body mass center is equal to zero regardless of its orientation so that the linear momentum is equal to the product of this velocity $\vec{v}_C$ of the body mass center and the rigid body mass: $\vec{\mathfrak{K}} = M\vec{v}_C$. The same stands for if the pole A is not the body mass center but if the axis oriented with $\vec{\omega} = \omega\vec{n}$ trough pole A passes trough the mass center.

The second kinetic vector connected to the referential point which plays an important part (role) in the rigid body dynamics is the rigid body angular momentum (motion quantity moment) for the given pole, $\vec{\mathfrak{L}}_O$. Following the classic definition according to [1], [3] and [11] and according to the notation given in the Fig. 6 the rigid body angular momentum is calculated by means of the following expression:

$$\vec{\mathfrak{L}}_O = \iiint\limits_V [\vec{r}, \vec{v}_N] dm = \iiint\limits_V [\vec{r}_A + \vec{\rho}, \vec{v}_A + [\vec{\omega}, \vec{\rho}]] \, dm \tag{55}$$

Following the idea of this paper that at the basis of the rigid body motion interpretation there are rigid body dynamic parameters which express the mass inertia moment properties and the kinematic parameters, translation velocity $\vec{v}_A$ of the rigid body referential point and the angular velocity $\vec{\omega}$ of the relative momentary rotation around the axis oriented with $\vec{\omega}$ and through the referential point A then the angular momentum for the point A, $\vec{\mathfrak{L}}_A$ is connected not only to the pole but to the axis oriented by the momentary angular velocity vector to which we connect the vectors $\vec{\mathcal{M}}^{(A)}$ and $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the rigid body mass linear and inertia moments by connecting the body mass to the translation velocity of the referential point A. Therefore we write that it is:

$$\vec{\mathfrak{L}}_A = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \omega \vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \quad \vec{\mathcal{M}}^{(A)} = \vec{\rho}_C \mathcal{M} \tag{56}$$

that is,

$$\vec{\mathfrak{L}}_O = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \omega \vec{\mathfrak{J}}_{\vec{n}}^{(A)} + [\vec{r}_A, M\vec{v}_A + \omega \vec{\mathfrak{S}}_{\vec{n}}^{(A)}] \tag{57}$$

If the referential point A is in the body mass center than the angular momentum for the pole O is equal to:

$$\vec{\mathfrak{L}}_O = [\vec{\mathcal{M}}^{(O)}, \vec{v}_C] + \omega \vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \quad \vec{\mathcal{M}}^{(O)} = \vec{v}_C \mathcal{M} \tag{58}$$

while the angular momentum for the pole in the mass center C is:

$$\vec{\mathfrak{L}}_C = \omega \vec{\mathfrak{J}}_{\vec{n}}^{(C)} \tag{59}$$

and it is equal to the product of the magnitude of the momentary angular velocity ω and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the rigid body mass inertia moment for the central axis oriented by the vector of the momentary angular velocity $\vec{\omega}$.

The Ref. [3] has the deviation center of the body for the given direction for the material particles system and the deviation load by the linear momentum analysis. Considering that we have introduced the deviation load vector by the analysis of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the body mass inertia moment as its component normal to the axis for which it is determined we can see that the deviational part of the angular momentum vector proportional to the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviational load the body mass inertia moment of the axis around which the rigid body rotates since it is:

$$\vec{\mathfrak{L}}_A = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega[\vec{n}[\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \vec{n}]] = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega \vec{\mathfrak{D}}_{\vec{n}}^{(A)} \tag{60}$$

If the point A is the mass center then it stands for:

$$\vec{\mathfrak{L}}_C = \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) + \omega \vec{\mathfrak{D}}_{\vec{n}}^{(C)} \tag{61}$$

If the rotation axis is the main mass inertia moment axis then the angular momentum does not have any deviational part since the rotation axis is not subjected to the deviation load by the rigid body mass inertia moment and the angular momentum vector for the mass center is collinear with the rotation axis.

II.1.3. Some interpretations for the case of the rigid body rotation around the fixed axis.

Figure 7 shows the rigid body with the rotation axis around which it rotates with the angular velocity $\vec{\omega}$ which changes in time so that there appears the angular acceleration $\dot{\vec{\omega}}$ (see [A3], [32]). The kinetic energy is expressed as $2E_k = \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) == \omega^2 J_{\vec{n}}^{(A)}$. The linear momentum and angular momentum are:

$$\vec{\mathfrak{K}} = [\vec{\omega}, \vec{\rho}_C]M = \omega \vec{\mathfrak{S}}_{\vec{n}}^{(A)} \tag{62}$$

$$\vec{\mathfrak{L}}_A = \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega[\vec{n}[\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \vec{n}]] = \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega \vec{\mathfrak{D}}_{\vec{n}}^{(A)} \tag{63}$$

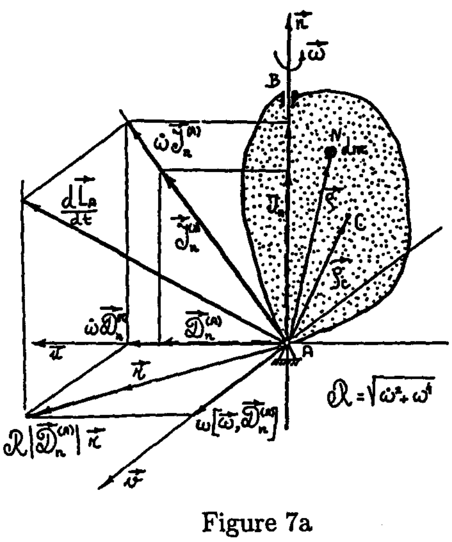

Figure 7a

Since the velocity $\vec{v}$ and the acceleration $\vec{a}$ of the body elementary mass at the point N are (see [31], [12]):

$$\vec{v} = [\vec{\omega}, \vec{\rho}], \quad \vec{a} = [\dot{\vec{\omega}}, \vec{\rho}] + [\vec{\omega}, [\vec{\omega}, \vec{\rho}]] \tag{64}$$

then for the main vector $\vec{F}_{rj}$ of the inertia force of the overall rigid body rotating around the axis with the angular velocity $\vec{\omega}$ we obtain:

$$\vec{F}_{rj} = -\iiint\limits_V \vec{a}\, dm = -\dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(A)} - \omega[\vec{\omega}, \vec{\mathfrak{S}}_{\vec{n}}^{(A)}] \tag{65}$$

For the main moment of the inertia forces of the overall rigid body rotating around the axis and for the point A we calculate the following:

$$\vec{\mathfrak{M}}_{Aj} = \iiint_V [\vec{\rho}, d\vec{F}_{rj}] = -\dot{\omega}\vec{\mathfrak{J}}_{\vec{n}}^{(A)} - \omega[\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}] \tag{66}$$

as well as:

$$\vec{\mathfrak{M}}_{Aj} = \iiint_V [\vec{\rho} d\vec{F}_{rj}] = -\frac{\dot{\omega}}{\omega}\vec{\mathfrak{L}}_A - [\vec{\omega}, \vec{\mathfrak{L}}_A] \tag{66*}$$

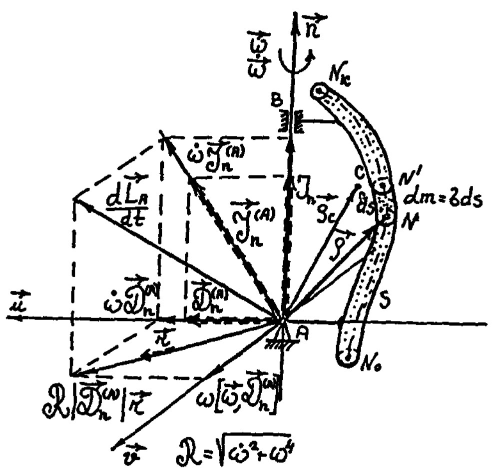

Figure 7b

The dynamic equations of the body rotation around fixed axis can be obtained by differentiating in time the expression (62) for the linear momentum and expression (54) for angular momentum on the basis of which we obtain:

$$1^* \qquad \frac{d\vec{\mathfrak{K}}}{dt} = \dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(a)} + \omega[\vec{\omega}, \vec{\mathfrak{S}}_{\vec{n}}^{(A)}] = -\vec{F}_{rj} = \vec{F}_r \tag{67}$$

$$\frac{d\vec{\mathfrak{K}}}{dt} = |\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|(\dot{\omega}\vec{u}_1 + \omega^2\vec{v}_1) = \vec{\mathfrak{R}}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| = \mathfrak{R}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|\vec{r}_1 \tag{68}$$

$$\vec{\mathfrak{R}}_1 = \mathfrak{R}\vec{r}_1, \quad \mathfrak{R} = \sqrt{\dot{\omega}^2 + \omega^4} \tag{69}$$

The rotator $\vec{\mathfrak{R}} = \mathfrak{R}\vec{r}_1$ is normal to the rotation axis and the deviation plane through the axis.

The equation (67) for the linear momentum change which is equal to the main vector (resultant) of the active and reactive forces shows that the motion linear momentum changes the vector normal to the rotation axis and has two components: one due to the angular velocity change which is normal to the rotation axis and the plane which contains the body mass center and the rotation axis, and the other component which depends on the angular velocity square which is normal to the rotation axis and lie in the plane formed by rotation axis and the rigid body mass center doing rotation.

$$2^* \qquad \frac{d\vec{\mathfrak{L}}_A}{dt} = \dot{\omega}\vec{\mathfrak{J}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}] = -\vec{\mathfrak{M}}_{Aj} = \vec{\mathfrak{M}}_A \tag{70}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \dot{\vec{\omega}}J_{\vec{n}}^{(A)} + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}] = \dot{\vec{\omega}}J_{\vec{n}}^{(A)} + |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|\vec{\mathfrak{R}} \tag{71}$$

$$\vec{\mathfrak{R}}_1 = \mathfrak{R}\vec{r}_1, \quad \mathfrak{R} = \sqrt{\dot{\omega}^2 + \omega^4} \tag{72}$$

The rotator $\vec{\mathfrak{R}} = \mathfrak{R}\vec{r}_1$ which is rotating and increasing by the angular velocity and by the angular acceleration at the same causes the inertia forces deviation moment to increase.

The equation (70) which is written on the basis of the law of the body angular momentum change which is says that the derivative in time of the body angular momentum for a certain pole in stationary bearing, equal to the moment of the active and reactive forces acting on the body for the same pole.

This form (71) immediately shows that the first component depending on the angular acceleration is collinear with the rotation axis; the second component which also depends on the angular acceleration is normal to the rotation axis and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the rigid body mass inertia moment for the pole in the fixed bearing A and for the rotation axis, that is, it is proportional to the magnitude of the angular acceleration $\dot{\vec{\omega}}$ and the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the rotation rigid body mass deviation moment of the rotation axis in the stationary bearing A and for the rotation axis; the third component is proportional to the square of the angular velocity ω^2 and to the magnitude of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the rotation rigid body mass deviation moment of the rotation axis in the stationary bearing A and for the rotation axis, whereas it is like a vector normal to the rotation axis and the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load to the rotation axis which means it is normal to the deviation plane. In the case it is the rotation with a constant angular velocity the stroke derivative components in time do not appear in the deviation plane; there is only a component normal to the deviation plane $\omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}]$.

Figure 7 shows the characteristic vectors, the rigid body mass moment vectors and the rigid body dynamics kinetic vectors in the rotation around fixed axis.

If we now return to the expressions (65) and (66) for the inertia force main vector and the inertia force main moment for the pole at the stationary bearing A we come to the following conclusion: 1* the expression (65) is equal to the one for the rigid body linear momentum derivative in time a changed sigh, while the expression (66) is equal to the angular momentum for the pole at the stationary bearing A, derivative in time, with a changed sigh so that the conclusions drawn to the expressions (67) and (53) also stand for the expression (65) and (66). These conclusions can also be defined in another way: we conclude from expression (66) that the inertia forces main moment for the rigid body rotation around the fixed axis has three components: the first one is purely rotatory around the rotation axis collinear with it if the angular acceleration is different from zero and it is proportional to the angular acceleration $\dot{\omega}$ and the body mass axial inertia moment for the rotation axis, $J_{\vec{n}}^{(A)}$; and the second deviational component is normal to the rotation axis which also depends on the angular acceleration and the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load of the rotation axis; and third component depending on the angular velocity squared of the rigid body rotation around the fixed axis and on the magnitude of the mass deviation moment vector of the rotation axis at the pole in the stationary bearing.

The derivative in time of the body angular momentum for a certain pole in stationary bearing normal to the rotation axis is:

$$\frac{d\vec{\mathfrak{L}}_A^d}{dt} = \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}] = |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|\vec{\mathfrak{R}} \tag{73}$$

By expressions (66), (68) and (73) we can write following relations:

$$\frac{|\vec{F}_{rj}|}{|\vec{\mathfrak{M}}_{Aj}|} = \frac{\left|\frac{d\vec{\mathfrak{K}}}{dt}\right|}{\left|\frac{d\vec{\mathfrak{L}}_A^d}{dt}\right|} = \frac{|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|}{|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|} = \text{constant} \tag{74}$$

II.1.4. Conditions for the dynamic balance of the rotor rotating around the fixed axis. Figure 7 shows the rotor with the main forces vector components denoted, that is, the motion linear momentum derivative in time and the inertia forces resulting moment components, that is, motion angular momentum derivative in time. In order that the effects of the dynamic balancing can appear it is necessary that bearings do not bear dynamic pressure which means that the deviational components should be equal to zero, that is, the components of the main force vector and the inertia forces resulting moment. Hence we draw the following conclusions:

1* Condition for the dynamic balancing exclusively and primarily depends on the dynamic, that is, kinetic properties of the rigid body with respect to the pole in the stationary bearing and to the rotation axis, but they do not depend on the angular velocity and the character of the acceleration;

2* Rotation axis should be the gravitational axis which is expressed by the condition that the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the rigid body mass linear moment for the rotation axis and the stationary bearing should be equal to zero;

$$|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| = 0 \tag{75}$$

3* Deviational part magnitude of the motion angular momentum derivative in time is equal to zero, that is, that the magnitude of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment for the rotation axis is equal to zero:

$$|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| = 0 \tag{76}$$

which can be reduced to the condition that the rotation axis is the main central mass inertia moment axis or that it is the symmetry axis or that it is the axis which for the point at stationary bearing represents one main direction of the rotor mass inertia moment.

II.1.5. Interpretation of the kinetic pressures on bearing by means of the mass moment vectors for the pole and the axis.

In this part the kinetic pressures of shaft bearings are expressed by means of the mass moment vectors: $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole in the stationary bearing.

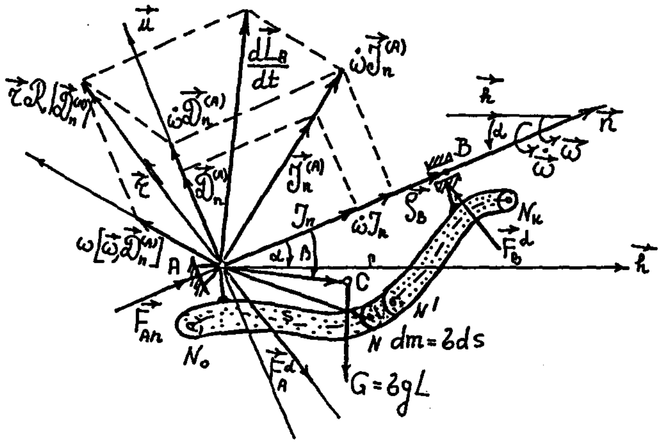

Figure 8

Figure 8 shows a rigid body that can rotate around a stationary axis is like a rigid shaft without mass supported on the stationary bearing A and on the moveable sliding one along the rotation axis. In the general case let a rigid body be subjected to a system of forces $\vec{F}_k$ whose points application N_{ko} are determined by the position vectors $\vec{\rho}_k$ with respect to the pole in the stationary bearing A.

Let's denote the rotation angle of the body around the stationary axis oriented by unit vector $\vec{n}$ with $\vec{\varphi} = \varphi\vec{n}$.

Following the expressions (67) and (70), as well as expression (68) and (71) we can write the following two vector equations:

$$\frac{d\vec{\mathfrak{K}}}{dt} = |\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|(\dot{\omega}\vec{u}_1 + \omega^2\vec{v}_1) = \vec{\mathfrak{R}}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| =$$
$$= \mathfrak{R}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|\vec{r}_1 = \sum_{k=1}^{k=N} \vec{F}_k + \vec{F}_A + \vec{F}_B \tag{77}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \dot{\vec{\omega}} J_n^{(A)} + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}] =$$
$$= \dot{\vec{\omega}} J_{\vec{n}}^{(A)} + |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|\vec{\mathfrak{R}} = \sum_{k=1}^{k=N} [\vec{\rho}_k, \vec{F}_k] + [\vec{\rho}_B, \vec{F}_B] \tag{78}$$

These two vectorial equations are kinetic equations of dynamic equilibrium in motion-rotation of the body around the stationary axis under the action of the active force system $\vec{F}_k$.

If we now multiply scalarly and vectorly these equations (77) and (78) by the unit vector $\vec{n}$ and having in mind that the $\vec{\rho}_B = \rho_B\vec{n}$, we obtain:

1* the rotation equation around the axes oriented by unit vector $\vec{n}$ in the form:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \dot{\vec{\omega}}) = \sum_{k=1}^{k=N} ([\vec{\rho}_k, \vec{F}_k], \vec{n}) \tag{79}$$

2* the equations for determining the bearings kinetic pressures, that is pressures upon the bearings, $\vec{F}_A$ and $\vec{F}_B$, that is, their components in the axis direction $\vec{n}$ and normal to the rotation axis:

$$\vec{F}_{A\vec{n}} = (\vec{F}_A, \vec{n})\vec{n} = -\vec{n} \sum_{k=1}^{k=N} (\vec{F}_k, \vec{n}) \tag{80}$$

$$\vec{F}_{AT} = -\vec{F}_B + \vec{\mathfrak{R}}_1|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| - \sum_{k=1}^{k=N} [\vec{n}, [\vec{F}_k, \vec{n}]] \tag{82}$$

$$\vec{F}_B = \frac{1}{\rho_B}\vec{\mathfrak{R}}|\vec{\mathfrak{D}}_n^{(A)}| - \frac{1}{\rho_B} \sum_{k=1}^{k=N} [\vec{n}, [[\vec{\rho}_k, \vec{F}_k], \vec{n}]] \tag{83}$$

From the expression for the bearings pressures (resistance) we select a part which is the result of the action of an external active forces and the influence of which upon the bearings resistances in possible variable in time is only due to the change of their line of application as well as the point of application with respect to the configuration of the body which is rotating such as in the case when the force of the body's own weight which retains the application line direction in relation to the rotation axis, and thus its position with respect to the body configuration, although in doing this it retains the application point constantly in the body mass center which rotates around the rotation axis together with body. The body mass center describes a circle or an arc in the plane through the mass center normal to the rotation axis.

Other part of the bearing kinetic resistance (pressures) in the body rotation around the stationary axis is the result exclusively of the kinetic-inertial body properties with respect to the rotation axis and the rotation kinematics and rigid body rotation kinematics around the stationary axis. These parts appear as parameters depending on the rotator vector $\vec{\mathfrak{R}}$ which in itself contains the angular velocity and the angular acceleration of the body rotation around the rotation axis and the rigid body mass moment properties with respect to the pole A at stationary bearing and the rotation axis expressed by the mass moment vectors: $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole A in the stationary bearing.

In order to discuss the rotor effect on the kinetic pressure upon the bearings in which the rigid body shaft axis is rotating it is necessary to know the angular acceleration $\dot{\vec{\omega}}$ and the angular velocity $\vec{\omega}$ and in order to do this it is necessary to solve the body rotation/oscillation equation around the axis (79), namely, to determine $\vec{\varphi}(t)$ and $\vec{\omega}(t)$ as well as $\omega(\varphi)$.

If the rotation axis is the central and main mass inertia moment axis and for the pole in the stationary bearing then it is a rigid body which is dynamically balanced and the member in the kinetic pressures depending on the vectors $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole A in the stationary bearing are equal to zero and are not influenced by the rotator change. Then there are only the components of the bearing resistance arising from the bearings "kvazi-static" resistances in the definite position of the active forces system and the reactive forces system during the body rotation.

If the rotation axis is the axis of the mass inertia moment asymmetry for the referential point in the stationary bearing then the kinetic pressures are the greatest both on moveable and stationary bearing. Since at each point on the rigid body there are three pairs of such mutually perpendicular axes which are in pair perpendicular to one main mass inertia moment direction and they form with the others an angle of $\frac{\pi}{4}$ each so that the mass inertia moment asymmetry axes which are perpendicular to the second main mass inertia moment direction forming angle of $\frac{\pi}{4}$ each with the first and third main mass inertia moment directions as the rotation axes will be the greatest vector of the deviation load and at the same time the greatest kinetic pressures on both the bearings.

The kinetic pressure on the stationary bearing depends on the body mass center position with respect to the rotation axis and this can be adjusted by the choice of the inertia asymmetry axes in pair as well as by the choice of the moveable bearing position with respect to the stationary one on the definite axis of mass inertia moment asymmetry. The body mass inertia moment asymmetry axes should be avoided as the rotation axis in order to reduce the dynamic pressures upon the bearings.

For a pair of the mass inertia moment asymmetry axes as the rotation axes the axial mass inertia moment of the rotatory body is identical so that depending on the body mass center position with respect to one axis or another and on the choice of the moveable bearing an increase, that is, decrease of the kinetic pressure at a given constant value of the initial energy communicated to the rotating body.

There are four (that is, eight) axes through each point of the body which we have chosen as a stationary bearing for which the axial mass inertia moments are the same value and the vectors $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole A in the stationary bearing are proportional to the sum of the three mass deviation load vectors by the body mass inertia moment of the mass inertia moment asymmetry axes. For these octahedral axes the dynamic pressures on both the stationary and moveable bearings are the same while the pressures on the stationary bearing are different and by choosing one of the octahedral axes minimization of maximization of their value can be performed. By displacing the moveable bearing from one to another octahedral axis through the stationary bearing the kinetic pressure on the stationary bearing can be adjusted while retaining the share in the pressure on both the bearing of the part that corresponds to the deviation load vector although the rotator is going to change as well (but this can also be adjusted). The smallest pressures would appear an octahedral axis is chosen so that the body mass center is closest to the rotation axis, that is, the most favorable of all the octahedral axes for the rotation axes is the one which body mass center is closest to.

A general conclusion would be that if we cannot select in the design way the rotation axis as the rigid body main central mass inertia moment axis when the system is dynamically balanced and analysis of the mass inertia moment state should be performed at each of the possible points of the stationary bearing positioning and according to the design requirements the selection should be done of both the stationary bearing and of the rotation axis according to the analysis.

These conclusions are very important if the designer cannot change the stationary bearing but if we can change the moveable one and chose it freely in the rigid body then his choose is important since the dynamic pressures should be as small as possible (see **[33]**, **[32]**).

II.2. Interpretation of the motion equations of a variable mass object rotating around a stationary axis by means of the mass moment vector for the pole and the axis.

In this part the kinetic equations of a variable mass object motion rotating around a stationary axis are derived by means of the mass moment vectors for the pole and for the rotation axis: vector $\vec{\mathfrak{S}}_{n}^{(A)}$ of the body mass linear moment, vector $\vec{\mathfrak{J}}_{n}^{(A)}$ of the body mass inertia moment for the pole A and for the axis oriented by the unit vector $\vec{n}$ and its deviational part of the vector $\vec{\mathfrak{D}}_{n}^{(A)}$ of the deviational load by the body mass inertia moment of the rotation axis through the pole A. The vectors of the reactive forces and resulting moments of the reactive forces due to the drop of the body particles are determined which are involved in the body mass change as the function of the body mass moments vector change: vector $\vec{\mathfrak{S}}_n^{(A)}$ of the body mass linear moment and vector $\vec{\mathfrak{J}}_n^{(A)}$ of the body mass inertia moment for the pole A and for the axis oriented by the unit vector $\vec{n}$ (see [49], [45], [27]).

The bearings resistances of the shaft on which an object of the variable mass is rotation and the analysis of the kinetic pressures is performed.

II.2.1. Introduction. In the last fifty years the equations of Meschersky given in his M.Sc. theses in 1897 [9] have obtained a wide theoretical consideration and the practical application in scientific centers of many countries. Meschersky has introduced the notion of the reactive force whereas Newton has defined the dynamic object properties by means of the kinetic properties of the matter quantity as the inertia measure. In the context of the Meschersky theory [9] the object are discussed as the dynamic variable objects. If the object is subjected to the dynamic change (see [17], [18]) (change of its own mass inertia moments) then it is the dynamic variable object whose rotation around the stationary axis is discussed in this paper.

The reactive force acts on a body in motion whose mass changes in time (due to the mechanical wasting – rejection or adhesion) in the sense of action and reaction. This motion is described by the Meschersky equation and gives an expression for the reactive force while the Ciolkovsky formula (see [5], [2], [11]) determines the motion velocity due to such a force and the dependence of the mass separation velocity in the case of the mass rejection.

Beside the papers quoted above relating to the mechanics of the variable mass body and rocket-dynamics which began to develop between the two World Wars there are other publications of a famous Italian scientist Tullio Levy-Civita (1873–1941) (see [8]) who discovered these laws 31 years after Meschersky and independently of him.

In engineering practice, especially in Mechanical Engineering, an important role is played by the rotor of the variable mass so that it is of greatest to consider the dynamic equations of the motion of the variable mass rotor as well as the dynamic resistances of the shaft bearings which these rotors are rotating upon.

II.2.2. Main vector of the reactive forces and the reactive forces resulting moment. By means of the previous introduced mass moment vectors here we are going to the interpret the kinetic equations of the variable mass body rotation.

Figure 9 shows a rigid body of a variable mass rotating around the axis oriented by the unit vector $\vec{n}$ by the angular velocity $\vec{\omega}$.

We introduce the hypothesis about the knowledge of the of the law on the mass separation from the body as the absolute velocity $\vec{w}_N$ of the particles falling off which create the reactive force. Let's assume that the absolute velocity $\vec{w}_N$ of the body particles falling of is equal to the velocity of the body point which rotates around the axis by the angular velocity $\vec{\omega}$, that is, that it is: $\vec{w}_N = \lambda \vec{v}_N = \lambda[\vec{\omega}, \vec{\rho}]$, where λ is a scalar, the proportionality coefficient (see [**19**], [**27**]). The reactive force $d\vec{\mathfrak{F}}_r$, due to the elementary particle falling off is: $d\vec{\mathfrak{F}}_r = \vec{w}_N d\dot{m} = \lambda[\vec{\omega}, \vec{\rho}]d\dot{m}$.

Due to the falling off of all the particles which are involved in the body mass change the main vector of the reactive forces is:

$$\vec{\mathfrak{F}}_r = \iiint\limits_V \vec{w}_N d\dot{m} = \lambda \iiint\limits_V [\vec{\omega}, \vec{\rho}]\, d\dot{m} = \lambda\omega \frac{\overset{*}{d}}{dt} \iiint\limits_V [\vec{n}, \vec{\rho}]\, dm = \lambda\omega \frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} \tag{84}$$

we see that it is proportional to the body rotation angular velocity and to the derivative in time of the body mass static moment vector in the case when the body changes its mass in rotation. In the formula (84) the differential operator $\frac{\overset{*}{d}}{dt}$ is a derivative in the time of the body mass linear moment vector for the body mass change:

$$\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} = \frac{\overset{*}{d}}{dt} \iiint\limits_V [\vec{n}, \vec{\rho}]\, dm = \iiint\limits_V [\vec{\omega}, \vec{\rho}] d(\frac{dm}{dt}) \tag{84*}$$

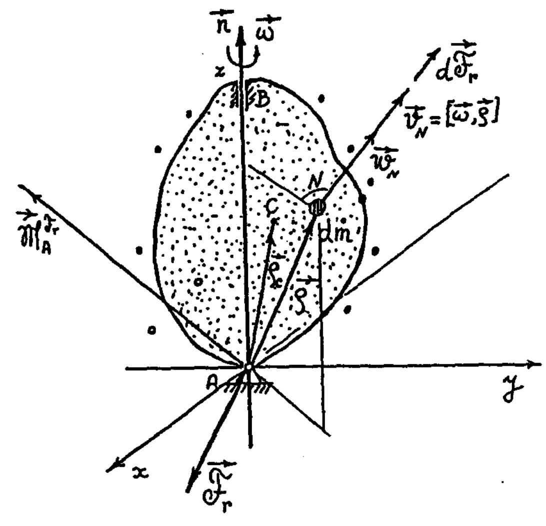

Figure 9

Due to the falling off of all the body particles which are involved in the body mass change the resulting moment of the reactive forces:

$$\vec{\mathfrak{M}}_r^{\vec{\mathfrak{F}}} = \iiint\limits_V [\vec{\rho}, d\vec{\mathfrak{F}}_r] = \lambda \iiint\limits_V [\vec{\rho}, [\vec{\omega}, \vec{\rho}]]\, d\dot{m} = \lambda\omega \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} \tag{85}$$

We see that the resulting moment of the reactive forces due to the body particles falling off, that is, of the particles involved in the body mass change for the case of the rotation is proportional to the body rotation angular velocity and to the derivative in the time of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the vector of the body mass inertia moment for the pole at A and for the rotation axis.

In (85) the differential operator $\frac{\overset{*}{d}}{dt}$ is a derivative in the time of the body mass inertia moment vector for the body mass change:

$$\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} = \frac{\overset{*}{d}}{dt}\iiint\limits_{V}[\vec{\rho},[\vec{\omega},\vec{\rho}]]\,dm = \iiint\limits_{V}[\vec{\rho},[\vec{\omega},\vec{\rho}]]\,d\dot{m} \tag{85*}$$

II.2.3. Linear momentum and angular momentum of the body rotation around the stationary axis. Following the idea of this part the linear momentum $\vec{\mathfrak{K}}$ and the angular momentum $\vec{\mathfrak{L}}_A$ for the pole in the stationary A bearing for the case of the body rotation around the stationary axis can be written by means of the previously defined vectors of the body mass moments by the expressions (4) and (6), as well as by the expressions (62) and (63), in the following form: $\vec{\mathfrak{K}} = \omega\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$, $\vec{\mathfrak{L}}_A = \omega\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$. Since for the formation of the dynamic equations is necessary to determine the derivatives in the time of the linear momentum and of the angular momentum of the body rotation, we write that it is:

$$\frac{d\vec{\mathfrak{K}}}{dt} = \dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{S}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} \tag{86}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \vec{\omega}J_{\vec{n}}^{(A)} + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{D}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} \tag{87}$$

II.2.4. Kinetic equations of a variable mass body rotation around a stationary axis. By using the basic laws of the dynamics that the linear momentum derivative in time is equal to the sum of all the active and reactive forces and that the angular momentum derivative in time for the pole in the stationary bearing is equal to the sum of all the active and reactive moments for the same pole, we can write the following two vector equations by means of the expressions (86) and (87) as well as of the expressions (84) and (85):

$$\frac{d\vec{\mathfrak{K}}}{dt} = \dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{S}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} = -\vec{F}_{rj} = \vec{F}_r + \lambda\omega\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} \tag{88}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \vec{\omega}J_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{D}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} = -\vec{\mathfrak{M}}_{Aj} = \vec{\mathfrak{M}}_{Ar} + \lambda\omega\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} \tag{89}$$

These two vector-equations are the motion kinetic ones-of the rotation of a variable mass body around the stationary axis. In these equations $\vec{F}_r$ and $\vec{\mathfrak{M}}_{Ar}$ are the main vector of the active and passive forces acting on the body as well as these forces' resulting moments for the pole at A.

If we multiply these equations (88) and (89) first scalarly and then from different sides by the vector $\vec{n}$ of the rotation having in view that we obtain:

a) Rotation equation around the axis oriented by the unit vector $\vec{n}$:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \dot{\vec{\omega}}) = ([\vec{\rho}_C, \vec{G}], \vec{n}) + \sum_{k=1}^{k=N} ([\vec{\rho}_k, \vec{F}_k], \vec{n}) + (\lambda - 1)\left(\vec{\omega}, \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt}\right) \tag{90}$$

b) Equations for the bearings kinetic resistances:

$$(\vec{F}_A, \vec{n}) + (\vec{G}, \vec{n}) + (\lambda - 1)\left(\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt}, \vec{n}\right) + \sum_{k=1}^{k=N} (\vec{F}_k, \vec{n}) = 0 \tag{91}$$

$$\vec{\mathfrak{R}}_1 |\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| = [\vec{n}, [\vec{F}_A, \vec{n}]] + [\vec{n}, [\vec{F}_B, \vec{n}]] + \sum_{k=1}^{k=N} [\vec{n}, [\vec{F}_k, \vec{n}]] + $$
$$+ [\vec{n}, [\vec{G}, \vec{n}]] + (\lambda - 1)\left[\vec{n}, \left[\frac{d\overset{*}{\vec{\mathfrak{S}}}_{\vec{n}}^{(A)}}{dt}, \vec{\omega}\right]\right] \tag{92}$$

$$\vec{\mathfrak{R}} |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| = [\vec{n}, [[\vec{\rho}_B, \vec{F}_B], \vec{n}]] + [\vec{n}, [[\vec{\rho}_B, \vec{G}] \vec{n}]] + $$
$$+ (\lambda - 1)[\vec{n}, [\frac{\overset{*}{\vec{\mathfrak{J}}}_{\vec{n}}^{(A)}}{dt}, \vec{\omega}]] + \sum_{k=1}^{k=N} [\vec{n}, [[\vec{\rho}_k, \vec{F}_k], \vec{n}]] \tag{93}$$

II.2.5. Shaft bearings resistances carried by the variable mass body. From the equations (91), (92) and (93) we determine the bearings resistances components in the form:

The stationary bearing resistance components A are:

1* The axial components in the rotation axis direction is:

$$\vec{F}_{An} = \vec{n}\left\{(\vec{G}, \vec{n}) - (\lambda - 1)[\vec{n}, \left(\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt}, \vec{n}\right) \sum_{k=1}^{k=N} (\vec{F}_k, \vec{n})\right\} \tag{94}$$

2* The deviational components perpendicular to the rotation axis are:

2.1* The component coming from the body mass center eccentricity is:

$$\vec{F}_{AN}^{*} = \vec{F}_{A\bullet}^{(\text{dev})} = \vec{\mathfrak{R}}_1 |\vec{\mathfrak{S}}_{n}^{(A)}| - \sum_{k=1}^{k=N} [\vec{n}, [\vec{F}_k, \vec{n}]] - [\vec{n}, [\vec{G}, \vec{n}]] - (\lambda - 1)\left[\vec{n}, \left[\frac{d\overset{*}{\vec{\mathfrak{S}}}_{\vec{n}}^{(A)}}{dt}, \vec{\omega}\right]\right] \tag{95}$$

2.2* The component coming from the deviational couple:

$$\vec{F}_{AN}^{**} = \vec{F}_{A**}^{(\mathrm{dev})} = -\vec{F}_B = -\frac{1}{\rho_B}\vec{\mathfrak{R}}\,|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| + \frac{1}{\rho_B}[\vec{n},[[\vec{\rho}_C,\vec{G}],\vec{n}]] + \\ + \frac{(\lambda-1)}{\rho_B}\left[\vec{n},\left[\frac{d\overset{*}{\vec{\mathfrak{J}}}{}_{\vec{n}}^{(A)}}{dt},\vec{\omega}\right]\right] + \frac{1}{\rho_B}\sum_{k=1}^{k=N}[\vec{n},[[\vec{\rho}_k,\vec{F}_k],\vec{n}]] \quad (96)$$

3* The moveable bearing resistance - sliding in the axis direction is of the deviational character:

$$\vec{F}_B = \frac{1}{\rho_B}\vec{\mathfrak{R}}\,|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| - \frac{1}{\rho_B}[\vec{n},[[\vec{\rho}_C,\vec{G}],\vec{n}]] \\ - \frac{(\lambda-1)}{\rho_B}\left[\vec{n},\left[\frac{d\overset{*}{\vec{\mathfrak{J}}}{}_{\vec{n}}^{(A)}}{dt},\vec{\omega}\right]\right] - \frac{1}{\rho_B}\sum_{k=1}^{k=N}[\vec{n},[[\vec{\rho}_k,\vec{F}_k],\vec{n}]] \quad (97)$$

in which the rotator $\vec{\mathfrak{R}}$ is determined by the formula:

$$\vec{\mathfrak{R}} = \mathfrak{R}\vec{n} = \dot{\omega}\vec{u} + \omega^2\vec{v}\,(=)\,\frac{\vec{a}}{r}; \quad \vec{u}^2 = \vec{v}^2 = 1; \quad \vec{u}\perp\vec{v}\perp\vec{n}; \quad \mathfrak{R} = \sqrt{\dot{\omega}^2+\omega^4} \quad (98)$$

From the expressions for the bearings resistances we select the part which is the result of the direct "static-dynamic" action of the active forces and a part which is the result of the rotating variable mass body kinetic properties.

We see that as the result of the rotor kinetic properties the deviational couple appears which is equal to the product of the rotator vector $\vec{\mathfrak{R}}$ and of the vector intensity $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis and it directly depends on the axis selection in the variable mass rotating body. This deviational couple causes a part of the kinetic pressures of the same intensity and perpendicular to the rotation axis in both the bearings, the stationary and the moveable one.

In the case that the rotation axis is always the main inertia axis for the pole in the stationary axis this deviational couple is equal to zero and it does not cause any pressure upon the bearings.

An additional pressure only upon the stationary bearing is formed when the masses center is outside the rotation axis and this part is proportional to the rotator vector $\vec{\mathfrak{R}}$ and to the vector intensity $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the mass linear moment for the pole in the stationary bearing and for the rotation axis $\vec{n}$.

Due to the mass changeability the kinetic pressures are formed in both the stationary and moveable bearings and they depend on the character of the body mass inertia vector change for the pole at A and for the rotation axis and they also make another deviational couple.

An additional pressures on the stationary bearing is formed due to the change of the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the mass linear moment and the angular velocity. A part of the kinetic pressures due to the reactive effect of the mass falling off from the rotator in the fact depends on the falling-off masses kinetic properties.

II.2.6. Special case of self-rotation. To illustrate let's observe a special case when there are no active forces but the rotor is only under the action of the reactive forces due to the masses separation (for instances, rotor with nozzles through which the particles are falling). Then the self- rotation equation is:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \dot{\vec{\omega}}) = (\lambda - 1)\left(\vec{\omega}, \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt}\right) \tag{99}$$

whose one first integral is: $(\vec{\mathfrak{J}}_{n}^{(A)}, \vec{\omega}) = \text{const}$.

If the rotation axis is the central rotation axis and the main inertia axis for the pole in the stationary bearing then the dynamic pressures do not effect the bearings. Then we can conclude that due to the reactive forces the body rotates around a free axis which retains its orientation. This would be a case of the body self-rotation around the central axis. In **[20]** the motion integral of the form is given which according to the Savić-Kašanin theory **[16]** represents the motion integral, that is, the self-rotation equations of celestial bodies (of the Earth, of the Sun).

II.3. Vectorial equations for the self induced rotations

Starting from the idea of Savić and Kašanin **[16]** and from idea of Vujičić **[18]**, as well as from an analogy with paper of Vujičić **[20]** and idea of **[23]**, a new form of the vectorial equation for the self-induced rotations of a rigid body is derived. That equation is:

$$\dot{\omega}\vec{\mathfrak{J}}_{n}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{n}^{(C)}] + \omega(1-\lambda)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{n}^{(C)}}{dt} = 0 \tag{100}$$

where $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ is the vector of body mass inertia moment at the point C center of mass, for the instantaneous rotation axis oriented by the unit vector $\vec{n}$ and $\vec{\omega}$ is the instantaneous angular velocity vector of the self-induced rotation, where $\omega = |\vec{\omega}|$.

II.3.1. Introduction. In the monograph **[16]** it is supposed that the rotation of a celestial body result from the expulsion of electrons from atoms: *"The expulsion of electrons from an atom has as its consequence the rotation of a celestial body, this rotation occuring at the instant in which the magnetic moment occurs-both phenomena occur concurrently with one another; both of them are the consequences of the expulsion of electrons from atoms, without which there would be neither a magnetic moment nor a rotation"*. The authors of this theory in their monograph, starting from the relation of the rotation of the plane rigid body derive a formula for calculating the angular velocity of a celestial body (see **[16**, p. 75**]**).

In [20] a new form of the tensorial equations for the self-rotation of a celestial body is derived by Vujičić. In [20] author have the following view: *"The classical mechanics have never succeeded neither could to explain the origin of rotation of celestial bodies by material point model and rigid body. The sum of interior force moments have disappeared during any analysis, and therefore the dynamics have to account the exterior forces as the cause of rotation. However the celestial mechanics have not accounted for electromagnetic forces although they are predominant in comparison with gravitational in microstructure. The gravitational forces became predominant within the large mass bodies. But the evolutional processes are much more complex the later mechanical model. So far the scientific opinion as that the formulation of stars-starts with gravitational condensation of low density hydrogen".*

II.3.2. Vectorial equations for the self-induced-rotations of bodies. According to [25] we shall introduce the notation of the mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ for the pole in the mass center C and for the axis oriented by the unit vector $\vec{n}$, defined by:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(C)} = \sum_{\nu=1}^{\nu=N} [m_\nu[\vec{r}_\nu, [\vec{n}, \vec{r}_\nu]] \tag{101}$$

where $\vec{r}_\nu$ is a position vector of mass particle m_ν, $v = 1, 2, \dots N$, relative to a fixed pole (in the mass center C. This vector is connected for the pole in the mass enter and for the self-rotation axis.

The vector $\vec{\mathfrak{S}}_{\vec{n}}^{(C)}$ for the pole in the mass center C and for the axis of the self-induced rotation, oriented by the unit vector $\vec{n}$, defined by:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(C)} = \sum_{\nu=1}^{\nu=N} m_\nu[\vec{n}, \vec{r}_\nu] = 0 \tag{102}$$

is equal to zero. In [20] author wrote: *"If we have in mind very complicated structure of celestial bodies, these results, as the one concerning the magnetic moment (see [16]), very sufficient stimulus for further work on this theory. For the purpose of mathematical generalization, it is always possible to consider any part of the body as the material points with the mass m_i, $i = 1, 2, \dots N$, if its own rotation is considered. If we separate any part of the body, even one single electron, from the original body, the mass of the body m_i changes for the mass Δm_i of the separated particles. If the mass Δm_i is separated, with the velocity $\vec{u}_i$, from the body with mass m_i, there appears a reactive impulse:*

$$\Delta m_i \vec{u}_i = \frac{\Delta m_i}{\Delta t} \vec{u}_i \Delta t \tag{103}$$

and it provokes the change of impulse $m_i \vec{v}_i$ in the original with mass m_i. Naturally if the separated particle, for example an electron, takes with itself an electrical charge it induced also the electromagnetic field, and the occurrence of a magnetic moment".

In [20] it was assumed that the observed object was composed of the set of N parts of masses m_ν, $v = 1, 2, \ldots N$. Starting from the theory of separation under the pressure, we can accept the assumption that the mass m_ν of the body changes for a differentially small amount of mass Δm_ν. At the moment of expulsion of masses dm_ν, $v = 1, 2, \ldots M$, with the corresponding absolute velocities $\vec{u}_\nu$, there appear the reactive forces $\vec{u}_\nu \frac{dm_\nu}{dt}$ which perform the work:

$$\delta A_\nu = \frac{dm_\nu}{dt}(\vec{u}_\nu, \delta\vec{r}_\nu) \tag{104}$$

on virtual displacements $\delta\vec{r}_\nu$.

The perturbation in the state of the j-th particle provokes (causes) a change in the impulse of the motion of all other particles. For such a dynamical system, the general classical principle of mechanics should be valid, and according to it, we can write:

$$\delta \int_0^1 \frac{1}{2} \sum_{\nu=1}^{\nu=N} m^v(\vec{v}_\nu, d\vec{r}_\nu) = -\int_0^1 \sum_{\nu=1}^{\nu=M} \dot{m}^v(\vec{u}_\nu, \delta\vec{r}_\nu) \tag{105}$$

where $M < N$ and $\vec{r}_\nu$ are the radius vectors of observed material points with the assumption that the eventual displacements $\delta\vec{r}_{v_0}$ and $\delta\vec{r}_{v_1}$ are equal to zero, and with the validity of the relations $\delta d\vec{r}_\nu = d\delta\vec{r}_\nu$. Now, for left-hand side of (105) we can write:

$$\frac{1}{2}\sum_{\nu=1}^{\nu=N} \int_0^1 \{(\delta(m_\nu, \vec{v}_\nu), d\vec{r}_\nu) + m_\nu(\vec{v}_\nu, \delta d\vec{r}_\nu)\} = \sum_{\nu=1}^{\nu=N} \int_0^1 m_\nu(\vec{v}_\nu, d\delta\vec{r}_\nu) =$$

$$= -\sum_{\nu=1}^{\nu=N} \int_0^1 (d(m_\nu\vec{v}_\nu), \delta\vec{r}_\nu) \tag{106}$$

because

$$\sum_{\nu=1}^{\nu=N} m_\nu(\vec{v}_\nu, \delta\vec{r}_\nu)|_0^1 = 0 \tag{107}$$

Introducing the time t, the last integral (106) transforms in the form:

$$-\int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=N} \Big(\frac{d(m_\nu\vec{v}_\nu)}{dt}, \delta\vec{r}_\nu\Big)\, dt \tag{108}$$

If we introduced the time also in the right-hand side of the relation (105), by means of $dm_\nu = \dot{m}_\nu\, dt$, where obviously $\dot{m}_\nu = \frac{dm_\nu}{dt}$ is the mass velocity (secondary change of mass) if will be

$$\int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=N} \Big(\frac{d(m_\nu\vec{v}_\nu)}{dt}, \delta\vec{r}_\nu\Big)\, dt = \int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=M} \dot{m}_\nu(\vec{u}_\nu, \delta\vec{r}_\nu)\, dt \tag{109}$$

Due to the arbitrariness in the choice of the pole of the position vector $\vec{r}_\nu$, the mass center C can be taken as the pole.

Setting the origin of an inertial reference system at the center of mass, which is always possible due to the arbitrariness of the choice of the reference point, the velocity of v-th point can be determined approximately by the relation $\vec{v}_\nu = [\vec{\omega}, \vec{r}_\nu]$. In the model of the body, the angular velocities $\vec{\omega}_\nu$, are equal to the instantaneous angular velocity vectors of a body-fixed, non inertial reference system with vector base $\vec{e}_\nu$, that is $\vec{\omega}_\nu \approx \vec{\omega}$. For the particles of a fluid medium, its velocity can be considered as an average angular velocity, for which $\vec{\omega}_\nu = \vec{\omega}$ so that the angular displacement $\delta\vec{\varphi} \approx \vec{\omega}\, dt$, within the limits of such an approximations, can connect the velocity $\vec{v}_\nu$ of the point of mass m_ν with the velocity $\vec{u}_\nu$ of an expulsive particle of mass dm_ν, that is $\vec{v} = \lambda\vec{u}$, where λ is an unknown scalar multiplier. Consequently, from the equation (105) [**20**] we can write:

$$\int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=N} \frac{d}{dt}(m_\nu[\vec{\omega}, \vec{r}_\nu], [\delta\vec{\varphi}, \vec{r}_\nu])\, dt = \int_{t_0}^{t_1} \lambda \sum_{\nu=1}^{\nu=N} \frac{dm_\nu}{dt}([\vec{\omega}, \vec{r}_\nu], [\delta\vec{\varphi}, \vec{r}_\nu])\, dt \tag{110}$$

Integration of the left-hand side of relation (110) can be transformed to (see [**20**]):

$$\frac{d}{dt} \sum_{\nu=1}^{\nu=N} m_\nu([\vec{r}_\nu, [\vec{\omega}, \vec{r}_\nu]], \delta\vec{\varphi}) = \frac{d}{dt}(\omega \vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \delta\vec{\varphi}) =$$

$$= \dot{\omega}(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \delta\vec{\varphi}) + \omega([\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}], \delta\vec{\varphi}) + \omega\left(\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi}\right) \tag{111}$$

where

$$\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} = \overset{*}{\vec{\mathfrak{J}}}{}_{\vec{n}}^{(C)} = \sum_{\nu=1}^{\nu=N} \dot{m}_\nu[\vec{r}_\nu, [\vec{n}, \vec{r}_\nu]] \tag{112}$$

The left-hand side of relation (110) can be transformed into the following form:

$$\int_{t_0}^{t_1} \left\{ \dot{\omega}(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \delta\vec{\varphi}) + \omega([\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}], \delta\vec{\varphi}) + \omega\left(\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi}\right) \right\} dt \tag{113}$$

Similarly, the right-hand side of the relation (110) can be transformed and it will have following form:

$$\int_{t_0}^{t_1} \lambda \sum_{\nu=1}^{\nu=N} \dot{m}_\nu)[\vec{r}_\nu, [\vec{\omega}, \vec{r}_\nu]], \delta\vec{\varphi})\, dt = \int_{t_0}^{t_1} \lambda\omega \left(\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi}\right) dt \tag{114}$$

Due to the transformed expressions (113) and (114), the relation (110) can be written in the form:

$$\int_{t_0}^{t_1} \left\{ \left(\dot{\omega}\vec{\mathfrak{J}}_n^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_n^{(C)}] + \omega(1-\lambda)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi} \right) \right\} dt = 0 \tag{115}$$

Hence, from here, the vectorial equation of the self-induced rotation of the body has the following form:

$$\dot{\omega}\vec{\mathfrak{J}}_{\vec{n}}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}] + \omega(1-\lambda)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} = 0 \tag{116}$$

i.e.,

$$\dot{\omega}(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \vec{n}) + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(C)}] = \omega(\lambda-1)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} \tag{117}$$

On the right-hand side of this vectorial equation there is the vector $\vec{\mathfrak{M}}_{C(R)}$ of the reactive moment:

$$\vec{\mathfrak{M}}_{C(R)} = \omega(\lambda-1)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} \tag{118}$$

on which the change in the body motions-rotation begins. In the case of appearing of a total (complete) central symmetry of expulsion of parts of mass, the sum of all components of moments of all reactive forces is equal to zero, because the moment vectors (torques) in pairs probably act in opposite directions.

From the vectorial equation (117) it follows that:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \dot{\vec{\omega}}) = (\lambda-1)\left(\vec{\omega}, \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}\right) \tag{119}$$

$$\dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(C)}] = (\lambda-1)\left[\vec{n}, \left[\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \vec{\omega}\right)\right]\right] \tag{120}$$

Now, Equation (119) can be written in the following form:

$$\frac{d\omega}{\omega} = (\lambda-1)\frac{(\vec{n}, \overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)})}{(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})} \tag{121}$$

This last equation is equivalent to the relation (2.1), which appears in [**16**, p. 75], or to the relation (2.14) which appears in [**20**, p. 99]. Thus, with the integration we will have:

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})^{\lambda-2} = \text{const} \tag{122}$$

i.e.

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) = \text{const} \tag{123}$$

where the constant of integration is to be determined from chosen initial conditions. This formula (123) is analogous with corresponding result of Savić-Kašanin from [**16**].

According to the theory applied here, at the initial time t_0, the vector of the body mass inertia moment, for the pole C and for the axis oriented by the unit vector $\vec{n}$, is $\vec{\mathfrak{J}}_{\vec{n}_0}^{(A)}$ and the instantaneous angular velocity of particles is $\vec{\omega}_0$, so we can write:

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})^{\lambda-2} = (\vec{\omega}_0, \vec{\mathfrak{J}}_{\vec{n}_0}^{(C)})(\vec{n}_0, \vec{\mathfrak{J}}_{\vec{n}_0}^{(C)})^{\lambda-2} \tag{124}$$

Therefore

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) = (\vec{\omega}_0, \vec{\mathfrak{J}}_{\vec{n}_0}^{(C)}) \tag{124*}$$

For the classical case when the mass of the body is constant, the right-hand side of the equation (121) is equal to zero, so that the vectorial equation is reduced to the equation of the rotation of a body by inertia $\vec{\omega} = \vec{\omega}_0 = \vec{c}\text{onst}$.

II.3.3. Concluding remarks. The exposed analysis of the bodies self-rotation does not aim to explain finally and describe fully the appearance of its induced self-rotation. This is only a contribution to the attempt for the mathematical vectorial descriptions of the law of motion - self-rotation by a new form of the vectorial differential equation, which are typical to the motion of rotor under the action of the reactive forces due to the masses separation (for instances, rotor with nozzles through which the particles are falling). If the rotation axis is the central rotation axis and the main inertia axis for the pole in the stationary bearing then the dynamic pressures do not effect the bearings. Then we can conclude that due to the reactive forces the body rotates around a free axis which retains its orientation. This would be a case of self-rotation of a body around the central axis.

8 * *

Main results of this monograph paper were presented on various seminars, congresses and other scientific meetings, as follows:

A1. Hedrih (Stevanović), K., *On Some interpretations of the rigid bodies kinetic parameters*, XVIII ICTAM Haifa, 1992, Abstracts.

A2. Hedrih (Stevanović), K., *New interpretation of the rigid bodies kinetic parameters*, Abstracts of 2-nd International Symposium of Ukrainian Mechanical Engineers in Lviv, State University "Lvivska Politechnika", Ukainian engineerr's Society in Lviv and Ukainian engineer's Society of America, 1995, p. 51

A3. Hedrih (Stevanović), K., *On rotation of a heavy body around a stationary axis in the fields with turbulent damping and dynamic pressures on bearings*, Abstract of lectures YUCNP Niš, 1991, pp. 38–39.

A4. Hedrih (Stevanović), K., *Vektori vezani za polove i pravu i pojmovi vektora sektorskih momenata masa i površina za pol i osu i sektorski pol*, Apstrakt, Sažeci, PRIM 94, Primenjena analiza, IX Seminar primenjene matematike, Budva, 30. maj – 1. jun 1994, Novi Sad, p. 44.

A5. Hedrih (Stevanović), K., *Energijska analiza kinetike konstrukcija za različite modele materijala*, Kratki prikazi radova naučnog skupa "Mehanika, materijali i konstrukcije", Srpska Akademija Nauka i Umetnosti, Odelenje tehničkih nauka, 1995, 106–107.

A6. Hedrih (Stevanović), K., *On New Interpretations of the rigid Bodies Kinetic Parameters and on Rotation of a Heavy body around a Stationary Axis in the Foeld* *with Turbulent Damping and Dynamic Pressures on Bearings*, Conference on Differential Geometry and Applications, Masaryk University, Brno 1995, Zbornik Apstrakata.

A7. Hedrih (Stevanović), K., *Interpretation of the Rigid Bodies Kinetics by Vectors of the Bodies Mass Moments*, Invited Lecture, Book of Abstracts, The International Conference: Stability, Control and Rigid Bodies Dynamics - ICSCD 96, Institute of Applied Mathematics and Mechanics of NAS of Ukraine, Donetsk - Mariupol, 2-6 September, 1996, pp. 35–36.

A8. Hedrih (Stevanović), K., *Nonlinear Dynamics of Rotor with a vibratin axis and sensitive dependence on initial conditions of forced vibration/rotation/stochasticlike-chaoticlike motion of a hevy rotor*, Third Bogoliubov Readings: Asymptotic and QUalitative Methods of Nonlinear Mechanics, ASYM 97, Tezi dopovidey, Institut Math. NANU, Kiev, 1997, pp. 73–74.

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Index of symbols

x^i curvilinear coordinate 68, 69, 70, 71

$\vec{\omega}$ angular velocity of the rotation around the axis 72, 73, 74, 75, 76, 77, 78, 79, 81, 82, 84, 85, 86, 87, 88, 89, 92, 93, 94

$\vec{\mathfrak{K}}$ vector of the linear momentum of the rigid body dynamic 74, 76, 77

$\vec{\mathcal{L}}_A$ vector of the angular momentum for the point A 75, 76, 78

$\vec{F}_{rj}$ the main vector of the inertia force of the rigid body rotating around the axis with the angular velocity $\vec{\omega}$ 76, 77, 79

$\vec{\mathfrak{M}}_{Aj}$ the main moment of the inertia forces of the rigid body rotating around the axis and for the point A 76, 77, 79

$\vec{\mathfrak{R}} = \mathfrak{R}\vec{r}_1$ rotator is normal to the rotation axis 77, 78, 79, 81, 87, 88

$\vec{w}_N$ absolute velocity of the body particles falling of 85

λ a scalar, the proportionality coefficient 85, 88

$d\vec{\mathfrak{F}}_r$ reactive force due to the elementary particle falling off 85

$\vec{\mathfrak{F}}_r$ main vector of the reactive forces 84, 85, 86

$\vec{\mathfrak{M}}_r^{\vec{\mathfrak{F}}}$ resulting moment of the reactive forces due the body particles falling off 85, 86

$\vec{r}_v$ a position vector of mass particle $m_v, v - 1, 2, \dots N$ 90, 91, 92

$m_v, v = 1, 2, \dots N$ mass particles 90, 91, 92

$\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ body mass deviation moment vector at the point O for the axis oriented by the unit vector $\vec{n}$ 48, 54, 56, 69, 71, 75, 76, 78, 79, 80, 81, 82, 83, 86, 87, 88

$\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ vector of the body mass linear moment 47, 48, 51, 53, 58, 69, 71, 73, 74, 76, 77, 81, 84, 85, 86, 87, 90

$\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ vector of the body mass at the point N for the axis oriented by the unit vector $\vec{n}$ 47, 51, 53

$\vec{\delta}_{\vec{n}}^{(N)}$ vector of the total relative deformation - total relative strain, at the point N and for the line element drawn from point N and oriented by unit vecto $\vec{n}$ 51

$\vec{p}_{\vec{n}}^{(O)}$ vector of the total stress at a certain body point for the plane with the normal oriented by unit vector $\vec{n}$ 59

$\mathbf{J}^{(O)}$ body mass inertia moment matrix 52 59 69

$\vec{M}^{(O)}$ the mass linear polar moment of the material system 52

$J_{\vec{n}}^{(O)}$ axial mass inertia moment 54, 59

$D_{nu}^{(O)}, D_{nv}^{(O)}$ the deviational moments of the body mass for a coulpe of normal axes orianted by unit vectors $\vec{n}$ and $\vec{u}$, thet is, $\vec{n}$ and $\vec{v}$ 54, 55

$J_1^{(O)}, J_1^{(O)}, J_1^{(O)}$ first, second and third scalar of the body mass inertia moment tensor matrix 59

$\mathbf{J}^{(O)\mathrm{sph}}, \mathbf{J}^{(O)\mathrm{dev}} = \mathbf{D}^{(O)\mathrm{dev}}$ two matrices corresponding to the spherical and deviational part of the rigid body mass inertia moment tensor 59, 71

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{aks}}$ body mass axial inertia moment vector at the point on for the axis orianted by unit vector $\vec{n}$ 60

$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{dev}}$ body mass deviation moment vector at the point and for the axis oriented by unit vector $\vec{n}$; vector of the axis deviation load 60, 61, 62, 71

$\vec{n}_S$ unit vector of the main mass inertia moment axis orientation 62, 63

$\mathrm{K}_{3k}^{(S)}, k = 1, 2, 3$ are co-factors of the third kind elements and the corresponding matrix column, successively for the roots $J_s^{(O)}, s = 1, 2, 3$; 63

I_a and I_b, II_a and II_b, III_a and III_b the axes pairs of the mass deviation moments extreme values 63, 64, 65, 66

$\vec{\mathfrak{J}}_{\vec{N}_s}^{(O)}, s = 1, 2, 3$ the body mass inertia moment vectors for the main mass inertia moment axis for the referential point 65, 66

ВЕКТОРИ МОМЕНАТА МАСА ТЕЛА

Овај монографски чланак уводи вектор $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ момента инерције масе тела за тачку N и осу орјентисану јединичним вектором $\vec{n}$. Вектор момента инерције масе крутог тела је коришћен за анализу стања момената инерције масе тела за одређену конфигурацију масе тела, као и за интерпретацију кинетичких параметера материјалног система у кретању. Промена вектора момента инерције масе тела при промени пола када оса задржава своју орјентацију је одређена и представља уопштење Huygens-Steiner-ове теореме на уведене векторе момената инерције масе тела. Изведен је израз за одређивање промене вектора момената инерције масе тела када оса мења орјентацију, што је једначина аналогна Cauchy-јевим једначинама из теорије еластичности. Показано је како се помоћу вектора момената маса одређују главни правци момената инерције маса као и правци инерционе асиметрије. Одређени су вектори момената инерције маса за октаедарске правце. Указано је на аналогије модела стања момената инерције маса тела, стања напона и стања деформације помоћу вектора везаних за тачку и осу, односно раван. Анализирани су сферна и девијациона својства вектора момената маса.

Овим чланком су уведени следећи вектори везани за тачку и осу: вектор $\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ масе тела у тачки N за осу орјентисану јединичним вектором $\vec{n}$; вектор $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ линеарног (статичког) момента масе тела у тачки N за осу орјентисану јединичним вектором $\vec{n}$; и вектор $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ момента инерције масе тела у тачки N за осу орјентисану јединичним вектором $\vec{n}$. Изведени су изрази за векторе момената маса у n-димензионалном криволинијском систему координата.

Затим су помоћу уведених вектора момената маса изражени кинетички параметри кретања крутог тела. Даље интерпретације су одредиле изразе за кинетичку енергију, количину кретања и момент количине кретања крутог тела помоћу уведених вектора момената маса тела. Специјално, за случај обртања тела око непомичне осе, одређени су изводи количине кретања и момента количине кретања у функцији тих вектора момената маса и написане кинетичке једначине ротације у векторском облику. Одређени су изрази за кинетичке притиске и уведен кинематички вектор ротатор. Показује се да коришћење вектора момената маса и вектора ротатора даје сасвим једноставне изразе за кинетичке притиске који зависе од девијационих делова вектора момената маса у односу на осу ротације и од кинематичког вектора ротатора. Услови динамичког ба-

лансирања се такође једноставно изражавају у услову да су девијациони делови вектора момената маса једнаки нули.

У чланку су изведени изрази за промене вектора момената маса при ротацији тела и за случај крутог тела променљиве масе. Изведена је векторска једначина самоcapотације крутог тела променљиве масе.

Овај монографски чланак представља преглед научних резултата које је аутор публиковано у научним часописима и/или саопштио на научним когресима и конференцијама међународног или националног значаја што се види из списка литературе која садржи ауторових 30 библиографских јединица.

Овај монографско прегледни чланак представља целину по векторској методи коју је аутор засновао на векторима везаним за пол и осу увођењем вектора момената маса тела за пол и осу којима се изражавају геометријско конфигурациона својства маса тела и кинематичких вектора ротатора који су везани за пол и осу и ротирају око ње одговарајућом угаоном брзином и убрзањем. Такође, чланак представља целину и по садржајима: комплетном интерпретацијом анализе стања момената маса тела у односу на пол уведеним векторима момената маса и комплетном интерпретацијом кинетичких параметара кретања ротора.

САДРЖАЈ

ГЛАВА I

ГЛАВА II

Svetlana Janković

INTRODUCTION TO THE THEORY OF THE ITÔ-TYPE STOCHASTIC INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS

CONTENTS

Preface

The theory of stochastic differential equations, as a part of the general theory of stochastic processes, began to develop in the fifties in the discussions of I.I. Gikhman, and independently of him, K. Itô. The accepted terminology, however, derived from Itô. In his papers [**15**], [**16**], [**17**], for special classes of stochastic processes he introduced the notion of the stochastic integral and of the stochastic differential equation with respect to a Wiener process. Following the classical theory of ordinary differential equations, he proved the fundamental theorem of the existence and uniqueness of solutions and also the Markov property of solutions. From then on this theory has developed in several aspects, mostly induced by mathematical abstractions or by many applications in technical practice, having in mind that a Gaussian white noise could be mathematically interpreted by a Wiener process.

One of the most important moments in the development of the theory of stochastic integrals and stochastic differential equations was the introduction of the notion of a martingale by Doob [**7**] and the subsequent establishment of the notion by Meyer [**34**], [**35**], [**36**]. In this way the fundamental supermartingale-decomposition theorem of Doob–Meyer [**36**] and the basic inequalities for martingales were established.

It is necessary to emphasize the notion of a stochastic integral with respect to a second-order martingale, introduced and studied by Kunita and Watanabe [**27**], which generalizes the stochastic Itô's integral. In fact, many properties of the Itô's integrals remain valid for this class of stochastic processes.

Later on, the theory of stochastic integrals and stochastic differential equations relative to other types of martingales and stochastic measures was developed ([**6**], [**29**]). Concurrently with it, the appropriate theory for a larger class of stochastic processes-semimartingales was introduced by Doléan-Dade and Meyer [**6**], and later essentially studied by Jacod [**22**] and Gikhman and Skorokhod [**11**].

The theory of stochastic differential equations had a permanent development with a large number of innovations, including some nonstandard constructions of stochastic integrals [**12**]. However, the Itô-calculus remains essential because several phenomena in technical, biological and social sciences can be modeled and described by stochastic differential equations of the Itô type. In fact, this theory is now applied in many diverse fields, which proves the flexibility of its application.

This paper represents an introduction to the study of the Itô-type calculus, as the initial information about the general theory of stochastic differential equations. Section 1 contains the basic theory of stochastic Itô's integrals, stating some important properties of stochastic indefinite integrals, introducing the stochastic differential and giving a differential formula known as the Itô's formula. In Section 2 the basic theory of Itô-type stochastic differential equations is established. The basic existence and uniqueness theorem, the Markov property and the continuous dependence on parameters of solutions are considered. Some simple examples are given to illustrate the preceding theoretical considerations.

There is a number of papers about stochastic differential equations. In the References some monographs and historically important papers are also given.

We shall restrict ourselves to the one-dimensional case for notational simplicity. The extension to the multidimensional case is not difficult in itself and it can be treated analogously.

1. Itô-type stochastic integrals

1.1. Definition of the Itô-type stochastic integral. Throughout the paper we suppose that all random variables and processes considered here are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $w = (w_t, t \in R)$ be an one-dimensional standard Wiener process, adapted to the increasing family of sub-σ-algebras $(\mathcal{F}_t, t \in R)$, i.e., for all $s \leq t$ random variables w_s are $\mathcal{F}_t$-measurable, $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ for $t_1 < t_2$ and $w_t - w_s$ is independent on $\mathcal{F}_s$ for all $t \geq s$. Further on, from this fact w is usually marked with $w = ((w_t, \mathcal{F}_t), t \in R)$.

Having in mind the definition of w, let us recall some of its more important properties: $w(0) = 0$ a.s.; $w_t - w_s : \mathcal{N}(0, |t-s|)$; it has independent increments; sample functions are continuous, but nowhere differentiable and they are of unbounded variation in every finite interval; it is a Markov process; $w_t - w_s = w_{t-s}$; $w = ((w_t, \mathcal{F}_t), t \in R)$ is a martingale, i.e., $E\{w_t | \mathcal{F}_s\} = w_s$ a.s. for $t \geq s$.

Moreover, because $((w_t^2 - t, \mathcal{F}_t), t \in R)$ is a martingale, w is a second-order martingale with the quadratic variation t. Indeed, for all $t \geq s$,

$$\begin{aligned} E\{w_t^2 - t | \mathcal{F}_s\} &= E\{w_s^2 + (w_t - w_s)^2 + 2w_s(w_t - w_s) | \mathcal{F}_s\} - t \\ &= w_s^2 + E(w_t - w_s)^2 - t = w_s^2 - s. \end{aligned}$$

Exactly, the martingale characteristics of the Wiener process play an important role in the construction of the Itô-type stochastic integrals.

In this section we shall define the Itô-type stochastic integral

$$I(\varphi) = \int_a^b \varphi(t)\, dw(t) \tag{1}$$

where $\varphi = (\varphi(t), t \in R)$ is a stochastic process, and we study its basic properties.

Since w is neither differentiable nor of the bounded variation, it is impossible to define (1) as an integral in the ordinary sense, i.e., as a Riemann–Stieltjes or a Lebesgue–Stieltjes integral. Recall that if φ is a nonrandom function, then (1) can be treated as a second-order stochastic integral (see, for example, [**27**], [**28**], [**30**], [**31**], [**39**], [**45**]). In this case only the fact that the Wiener process has orthogonal increments is used. The problem arises if φ is a random function, i.e., a stochastic process. Then the construction of the integral (1) depends on martingale properties of the Wiener process.

Furthermore, we shall suppose that the stochastic processes φ and w are independent.

Denote by $\mathcal{M}_2$ the class of stochastic processes with following properties: if $\varphi \in \mathcal{M}_2$, then

(i) φ is a measurable process, i.e., the function $(\omega, t) \to \varphi(\omega, t)$ is measurable with respect to $\mathcal{F}$ in ω and Lebesgue measurable in t;

(ii) φ is adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in R)$, i.e., for each t, $\varphi(\omega, t)$ is measurable with respect to $\mathcal{F}_t$;

(iii) $\int_a^b E|\varphi(t)|^2\, dt < \infty$.

When (i) and (ii) hold, we say that φ is *nonanticipating* with respect to $(\mathcal{F}_t, t \in R)$.

Note that every deterministic function φ is a nonanticipating function. Also, if φ is a nonanticipating function, every product-measurable function $g(t, \varphi)$, defined on $(R \times C)$ into C, is nonanticipating.

We are now in a position to define the stochastic integral of a process $\varphi \in \mathcal{M}_2$ relative to w, following the ideas of Itô [**15**]. We shall do this gradually, in two phases. In the first phase we define the stochastic integral for step functions in $\mathcal{M}_2$; in the second phase we extend this definition to the entire set $\mathcal{M}_2$, approximating an arbitrary process from $\mathcal{M}_2$ with the sequence of step functions (see [**1**], [**8**], [**9**], [**10**], [**28**], [**30**], [**45**]).

Definition 1. A stochastic process $\varphi \in \mathcal{M}_2$ is called a step function if there exists a decomposition $a = t_0 < t_1 < t_2 \cdots < t_k = b$, independent of ω, such that

$$\varphi(\omega, t) = \varphi(\omega, t_\nu) \text{ a.s.}, \qquad t_\nu \le t < t_{\nu+1}, \quad \nu = 0, 1, \dots, k-1.$$

Definition 2. The stochastic integral of the step function $\varphi \in \mathcal{M}_2$ with respect to w is the random variable

$$\int_a^b \varphi(\omega, t)\, dw(\omega, t) := \sum_{\nu=0}^{k-1} \varphi(\omega, t_\nu)\, [w(\omega, t_{\nu+1}) - w(\omega, t_\nu)].$$

The following theorem makes it possible to define the stochastic integral for every $\varphi \in \mathcal{M}_2$.

Theorem 1. *Let $w = ((w_t, \mathcal{F}_t), t \in R)$, be a standard Wiener process and $\varphi \in \mathcal{M}_2$. Then:*

(a) *There exists a sequence of step functions* $\{\varphi_n, n \in N\}$ *such that*

$$\|\varphi - \varphi_n\|^2 = \int_a^b E|\varphi(t) - \varphi_n(t)|^2\,dt \to 0 \quad \text{as} \quad n \to \infty;$$

(b) *If a sequence of step functions* $\{\varphi_n, n \in N\}$ *approximates* φ *in the sense* $\|\varphi - \varphi_n \to 0$ *as* $n \to 0$ *and if* $I(\varphi_n)$ *is defined as in Definition 1, then the sequence of random variables* $\{I(\varphi_n), n \in N\}$ *converges in q.m. as* $n \to \infty$;

(c) *If* $\{\varphi_n, n \in N\}$ *and* $\{\varphi'_n, n \in N\}$ *are two sequences of step functions such that* $\|\varphi - \varphi_n\| \to 0$, $\|\varphi - \varphi'_n\| \to 0$ *as* $n \to \infty$, *then*

$$\text{q.m.}\lim_{n\to\infty} I(\varphi_n) = \text{q.m.}\lim_{n\to\infty} I(\varphi'_n).$$

Proof. (a) If $\varphi \in \mathcal{M}_2$, but not obviously bounded a.s., we form the sequence of stochastic processes $\{f_n, n \in N\}$ such that

$$f_n(t) = \begin{cases} \varphi(t), & |\operatorname{Re}\varphi| \le n,\ |\operatorname{Im}\varphi| \le n, \\ n, & \text{otherwise} \end{cases}$$

By the dominated convergence theorem it follows that $\int_a^b E|\varphi(t) - f_n(t)|^2\,dt \to 0$ as $n \to \infty$. So, further on we can always assume φ to be bounded a.s..

Suppose that φ is q.m. continuous. Then an approximating sequence of step functions $\{\varphi_n, n \in N\}$ can be constructed by an arbitrary decomposition of the segment $[a, b]$: $a = t_0^{(n)} < t_1^{(1)} < \cdots < t_k^{(n)} = b$, such that for $t_\nu^{(n)} \le t < t_{\nu+1}^{(n)}$ we have $\varphi_n(t) = \varphi(t_\nu^{(n)})$ a.s. and $\max_\nu [t_{\nu+1}^{(n)} - t_\nu^{(n)}] \to 0$ as $n \to \infty$. Since φ is q.m. continuous, then $E|\varphi(t) - \varphi_n(t)|^2 \to 0$ as $n \to \infty$ for every $t \in [a, b]$. By the dominated convergence theorem it follows

$$\int_a^b E|\varphi(t) - \varphi_n(t)|^2\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

If φ is bounded a.s., but not obviously q.m. continuous, we shall define the sequence of stochastic processes $\{g_n, n \in N\}$, where

$$g_n(t) = \int_0^\infty e^{-\tau}\varphi\Big(t - \frac{\tau}{n}\Big)\,d\tau.$$

It is easy to conclude that $g_n \in \mathcal{M}_2$, $n \in N$ and q.m. continuous on $[a, b]$. Since

$$\begin{aligned} \int_a^b E|\varphi(t) - g_n(t)|^2\,dt &= \int_a^b E\Big|\int_0^\infty e^{-\tau}\Big[\varphi(t) - \varphi\Big(t - \frac{\tau}{n}\Big)\Big]\,d\tau\Big|^2\,dt \\ &\le \int_a^b \int_0^\infty e^{-\tau}\,d\tau \int_0^\infty e^{-\tau} E\Big|\varphi(t) - \varphi\Big(t - \frac{\tau}{n}\Big)\Big|^2\,d\tau\,dt \end{aligned}$$

and since $\int_a^b |\varphi(t) - \varphi\left(t - \frac{\tau}{n}\right)|^2\,dt \to 0$ as $n \to \infty$ whenever φ is a.s. bounded and Lebesgue measurable, then

$$\int_a^b E|\varphi(t) - g_n(t)|^2\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

Now it is clear that we can construct indirectly a sequence of step functions approximating φ by sampling q.m. continuous stochastic processes $g_n, n \in N$ at the partition points of the segment $[a, b]$, such that the partitions go to zero as $n \to \infty$.

(b) Suppose that $\{\varphi_n, n \in N\}$ is a sequence of step functions such that $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$. Using Definition 2, let us define $I(\varphi_n)$ by

$$I(\varphi_n) = \int_a^b \varphi_n(t)\, dw(t) := \sum_\nu \varphi(t_\nu^{(n)})[\, w(t_{\nu+1}^{(n)}) - w(t_\nu^{(n)})\,].$$

Denote by $\Delta_\nu^{(n)} w = w(t_{\nu+1}^{(n)}) - w(t_\nu^{(n)})$. Then

$$E|I(\varphi_n)|^2 = \sum_\nu \sum_\mu E[\,\varphi(t_\nu^{(n)})\overline{\varphi(t_\mu^{(n)})}\, \Delta_\nu^{(n)} w \Delta_\mu^{(n)} w\,].$$

Since $E[\,\Delta_\nu^{(n)} w\, \Delta_\mu^{(n)} w\,] = E\Delta_\nu^{(n)} w\, E\Delta_\mu^{(n)} w = 0$ if $\nu \neq \mu$, we get

$$\begin{aligned} E|I(\varphi_n)|^2 &= \sum_\nu E|\varphi(t_\nu^{(n)})|^2\, E|\Delta_\nu^{(n)} w|^2 \\ &= \sum_\nu E|\varphi(t_\nu^{(n)})|^2 (\, t_{\nu+1}^{(n)} - t_\nu^{(n)}\,) = \int_a^b E|\varphi_n(t)|^2\, dt. \end{aligned}$$

Also, $E|I(\varphi_n)|^2 < \infty$. Since $I(\varphi_{n+m}) - I(\varphi_n) = I(\varphi_{n+M} - \varphi_n)$ and $\varphi_{n+m} - \varphi_n$ is again a step function, it follows

$$\begin{aligned} E|I(\varphi_{n+m}) - I(\varphi_n)|^2 &= E|I(\varphi_{n+m} - \varphi_n)|^2 = \int_a^b E|\varphi_{n+m}(t) - \varphi_n(t)|^2\, dt \\ &\leq 2\int_a^b E|\varphi_{n+m}(t) - \varphi(t)|^2\, dt + 2\int_a^b E|\varphi(t) - \varphi_n(t)|^2\, dt \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Hence, $\{I(\varphi_n), n \in N\}$ converges in q.m. because every Cauchy sequence of random variables is also q.m. convergent. It means that there exists a random variable $I(\varphi)$ such that $E|I(\varphi)|^2 < \infty$ and

$$E|I(\varphi) - I(\varphi_n)|^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

(c) Let $\{\varphi_n, n \in N\}$ i $\{\varphi_n', n \in N\}$ be two sequences of step functions approximating φ, i.e., $\|\varphi - \varphi_n\| \to 0$, $\|\varphi - \varphi_n'\| \to 0$ as $n \to \infty$. Because

$$\|\varphi_n - \varphi_n'\| \leq \sqrt{2}\,(\,\|\varphi_n - \varphi\|^2 + \|\varphi - \varphi_n'\|^2\,)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty,$$

then

$$E|I(\varphi_n) - I(\varphi_n')|^2 = \int_a^b E|\varphi_n(t) - \varphi_n'(t)|^2\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, q.m. $\lim_{n\to\infty} I(\varphi_n)$ = q.m. $\lim_{n\to\infty} I(\varphi_n')$. Thus the theorem is completely proved. □

Summarizing the results of the preceding theorem, we conclude that the stochastic integral $I(\varphi)$ can be defined as q.m. limit of the sequence of random variables $\{I(\varphi_n), n \in N\}$, i.e.,

$$I(\varphi) = \int_a^b \varphi(t)\,dw(t) := \text{q.m.} \lim_{n\to\infty} \int_a^b \varphi_n(t)\,dw(t).$$

This limit is in q.m. sense uniquely determined and independent of the choice of the sequence of step functions $\{\varphi_n, n \in N\}$ for which (2) holds.

Note that if a and b are not finite, the stochastic integral is defined as q.m. limit as $a \to \infty$ or $b \to \infty$.

The next theorem summarizes some of the more important properties of the stochastic integral.

Theorem 2. *Let $\varphi, \psi \in \mathcal{M}_2$ and α, β be arbitrary numbers. Then:*

(a) $I(\alpha\varphi + \beta\psi) = \alpha I(\varphi) + \beta I(\psi)$;

(b) $EI(\varphi) = 0$;

(c) $EI(\varphi)\overline{I(\psi)} = \int_a^b E\varphi(t)\overline{\psi(t)}\,dt$.

Proof. (a) This part follows immediately from the construction of the stochastic integral of step functions.

(b) The proof is obvious if $\varphi \in \mathcal{M}_2$ is a step function. If not, let $\{\varphi_n\}$ be a sequence of step functions approximating φ in q.m., i.e., $E|\varphi(t) - \varphi_n(t)|^2 \to 0$ as $n \to \infty$ on $[a, b]$. Since by Theorem 1b

$$0 \le (EI(\varphi) - EI(\varphi_n))^2 \le E|I(\varphi) - I(\varphi_n)|^2 \to 0 \quad \text{as} \quad n \to \infty,$$

then $EI(\varphi) = 0$.

(c) It is enough to prove that $E|I(\varphi)|^2 = \int_a^b E|\varphi(t)|^2\,dt$ because

$$\begin{aligned} EI(\varphi)\overline{I(\psi)} &= \frac{1}{4}[\,E|I(\varphi+\psi)|^2 - E|I(\varphi-\psi)|^2\,] \\ &\quad + \frac{i}{4}[\,E|I(-i\varphi+\psi)|^2 - E|I(-i\varphi-\psi)|^2\,]. \end{aligned}$$

If φ is a step function, the proof directly follows from the proof of Theorem 1b. If not, let $\{\varphi_n, n \in N\}$ be a sequence of step functions approximating φ in q.m. Then

$$\begin{aligned} E|I(\varphi)|^2 &= E|I(\varphi - \varphi_n) + I(\varphi_n)|^2 \\ &= E|I(\varphi - \varphi_n)|^2 + 2\mathrm{Re}\, EI(\varphi - \varphi_n)\overline{I(\varphi_n)} + E|I(\varphi_n)|^2, \end{aligned}$$

and therefore

$$\begin{aligned} E|I(\varphi)|^2 &= \lim_{n\to\infty} E|I(\varphi_n)|^2 = \lim_{n\to\infty} \int_a^b E|\varphi_n(t)|^2\,dt \\ &= \int_a^b \lim_{n\to\infty} E|\varphi_n(t)|^2\,dt = \int_a^b E|\varphi(t)|^2\,dt. \quad \square \end{aligned}$$

The notion of the stochastic integral of the Itô type can be introduced under some weaker conditions (see, for example, [1], [8], [10], [28], [30], [39], [45]). Thus, denote by $\mathcal{P}$ a class of stochastic processes, measurable and adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in R)$, satisfying the condition

$$P\left\{ \int_a^b |\varphi(t)|^2 dt < \infty \right\} = 1.$$

Clearly, $\mathcal{M}_2 \subset \mathcal{P}$.

Theorem 3. *Let $((w_t, \mathcal{F}_t), t \in R)$ be a standard Wiener process and let $\varphi \in \mathcal{P}$. Let also φ_n be defined by*

$$\varphi_n(t) = \begin{cases} \varphi(t), & \int_a^b |\varphi(t)|^2\, dt \le n, \\ 0, & \text{otherwise} \end{cases}$$

and let $I(\varphi_n)$ denote the stochastic integral $I(\varphi_n) = \int_a^b \varphi_n(t)\, dw(t)$. Then $\{I(\varphi_n), n \in N\}$ converges in probability as $n \to \infty$.

Proof. Let φ_n be defined as the above. Then $\varphi_n \in \mathcal{M}_2$ and $I(\varphi_n)$ is well defined. Now for arbitrary $m, n \in N$ and for any $\omega \in \Omega$, such that

$$\int_a^b |\varphi(t)|^2\, dt \le \min\{m, n\},$$

we obtain $\sup_{t\in[a,b]} |\varphi_n(t) - \varphi_m(t)| = 0$. So, $\int_a^b \varphi_n(t)\, dt = \int_a^b \varphi_m(t)\, dt$ a.s. For every $\epsilon > 0$ it follows that

$$P\{\, |I(\varphi_n) - I(\varphi_m)| \ge \epsilon\} \le P\left\{ \int_a^b |\varphi(t)|^2 dt > \min\{m, n\} \right\} \to 0 \text{ as } m, n \to \infty,$$

which implies in turn that $\{I(\varphi_n), n \in N\}$ converges in probability since every Cauchy sequence of random variables also converges in probability. □

Therefore, under the conditions of the preceding theorem there exists a random variable $I(\varphi)$ such that $I(\varphi_n) \to I(\varphi)$ in probability as $n \to \infty$. In other words, we can define the stochastic integral

$$I(\varphi) := \text{p.} \lim_{n\to\infty} I(\varphi_n).$$

The notion of the Itô-type stochastic integral can be analogously generalized to the $(n \times m)$-matrix valued stochastic process $\varphi = [\varphi_{ij}]_{n\times m}$, where $\varphi_{ij} \in \mathcal{M}_2$ or $\varphi_{ij} \in \mathcal{P}$, with respect to the m-dimensional standard Wiener process $w = ((w_t, \mathcal{F}_t), t \in R)$, $w_t - w_s : \mathcal{N}(0, |t - s| I)$. The matrix φ has the norm

$$|\varphi| = \Big(\sum_{i=1}^{n} \sum_{j=1}^{m} |\varphi_{ij}|^2 \Big)^{1/2} = (\operatorname{tr} \varphi \overline{\varphi'})^{1/2}.$$

Clearly, in this case $I(\varphi)$ is the n-dimensional random variable.

1.2. The stochastic indefinite integral. Denote by $I_{\{s<t\}}$, $a \le s < t < b$, an indicator of the set $[a,t]$ which is obviously $\mathcal{F}_t$-measurable. This fact gives a possibility to introduce a notion of a stochastic indefinite integral.

Definition 3. The stochastic indefinite integral of the process $\varphi \in \mathcal{M}_2$ is the stochastic process $x = (x(t), t \in [a,b])$, defined by

$$x(t) := \int_a^b I_{\{s<t\}} \varphi(s)\, dw(s) = \int_a^t \varphi(s)\, dw(s), \quad t \in [a,b].$$

Having in mind the construction of the Itô-type stochastic integral $I(\varphi)$, the indefinite stochastic integral possesses the following important properties:

(i) x is defined uniquely up to the stochastic equivalence with its separable and measurable modification (Doob's theorem - see [7], [45]);

(ii) $x(t)$ is $\mathcal{F}_t$-measurable for every $t \in [a,b]$;

(iii) $x(a) = 0$ a.s.;

(iv) $x(t) - x(s) = \int_s^t \varphi(u)\, dw(u)$, $t, s \in [a,b]$.

Using the results of Theorem 2, for every $t \in [a,b]$ it follows:

(v) $Ex(t) = 0$;

(vi) $E|x(t)|^2 = \int_a^t E|\varphi(s)|^2\, ds$.

Theorem 4. *If $\varphi \in \mathcal{M}_2$, then $((x_t, \mathcal{F}_t), t \in [a,b])$, is a martingale.*

Proof. Let φ be a step function and $s < t_1 < t_2 < \cdots < t_n < t$. Then

$$\begin{aligned} x(t) - x(s) &= \int_s^t \varphi(u)\, dw(u) \\ &= \varphi(s)[w(t_1) - w(s)] + \varphi(t_1)[w(t_2) - w(t_1)] + \cdots + \varphi(t_n)[w(t) - w(t_n)]. \end{aligned}$$

Therefore, by successively taking conditional expectations, we obtain

$$\begin{aligned} E\{x(t) - x(s)|\mathcal{F}_s\} &= E\{E\{\cdots E\{x(t) - x(s)|\mathcal{F}_{t_n}\}|\mathcal{F}_{t_{n-1}}\}|\cdots \mathcal{F}_{t_1}\}|\mathcal{F}_s\} \\ &= \cdots = E\{x(t_1) - x(s)|\mathcal{F}_s\} = E\varphi(s)\, E(w(s) - w(s)) = 0. \end{aligned}$$

In the following part of the proof we use the well-known convergence property of conditional expectation (see [45]): for $\nu \ge 1$ if the sequence of stochastic variables $X_n \overset{\nu.m.}{\to} X$ as $n \to \infty$, then $E(X_n|\mathcal{F}) \overset{\nu.m.}{\to} E(X|\mathcal{F})$ as $n \to \infty$.

If φ is not a step function, let $\{\varphi_n, n \in N\}$ be a sequence of step functions approximating φ. Denote by $x_n(t) = \int_a^t \varphi_n(s)\, dw(s)$. Then for every $t \in [a,b]$ we have $E|x(t) - x_n(t)|^2 \to 0$ as $n \to \infty$, and therefore $E\{x(t) - x_n(t)\,|\mathcal{F}_s\} \to 0$ as $n \to \infty$. Now for all $t > s$

$$\begin{aligned} &E\{x(t) - x(s)|\mathcal{F}_s\} \\ &\quad = E\{x(t) - x_n(t)|\mathcal{F}_s\} + E\{x_n(t) - x(s)|\mathcal{F}_s\} \to 0 \quad \text{as} \quad n \to \infty. \quad \Box \end{aligned}$$

Moreover, one can be show that $((x_t, \mathcal{F}_t), t \in [a,b])$, is *a second-order martingale* with the quadratic variation

$$u(t) = \int_a^t |\varphi(u)|^2 \, du.$$

In order to do this, recall an important property of second order martingales.

Let $((z_t, \mathcal{F}_t), t \in [a,b])$ be a sample continuous second-order martingale. Then by the supermartingale-decomposition theorem of Doob and Meyer (see **[36]**, and also **[10]**, **[11]**, **[30]**, **[31]**, **[34]**, **[35]**), there exist both a sample continuous martingale $((m_t, \mathcal{F}_t), t \in [a,b])$ and a sample continuous increasing process $((u_t, \mathcal{F}_t, t \in [a,b])$ — called *the quadratic variation*, with $u(a) = 0$ a.s. and $Eu(b) < \infty$, such that

$$z^2(t) - u(t) = m(t) \quad \text{a.s.}, \quad t \in [a,b].$$

Also, the following inequality, defined first by Doob [7], and in different variations by Meyer **[36]** and others, holds: for $1 < \alpha < \infty$,

$$E\{\sup_{t\in[a,b]} |z(t)|^{\alpha}\} \le \Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha} E|z(b)|^{\alpha}.$$

For $\alpha = 2$ and $\varphi \in \mathcal{M}_2$ we get

$$\sup_{t\in[a,b]} E|x(t)|^2 \le E\{\sup_{t\in[a,b]} |x(t)|^2\} \le 4\int_a^b E|\varphi(t)|^2 \, dt < \infty.$$

Next, $u(t)$ is $\mathcal{F}_t$-measurable for every $t \in [a,b]$, non-negative and increasing a.s., $u(a) = 0$ a.s. and $Eu(t) \le Eu(b) < \infty$. For all $t \ge s$ we obtain

$$E\{x^2(t) - u(t)|\mathcal{F}_s\} = E\Big\{\Big(\int_a^t \varphi(u)\,dw(u)\Big)^2 \Big|\mathcal{F}_s\Big\} - E\Big\{\int_a^t \varphi^2(u)\,du \Big|\mathcal{F}_s\Big\}$$
$$= E\Big\{\Big(\int_a^s \varphi(u)\,dw(u)\Big)^2 \Big|\mathcal{F}_s\Big\} - E\Big\{\int_a^s \varphi^2(u)\,du \Big|\mathcal{F}_s\Big\}$$
$$+E\Big\{\Big(\int_s^t \varphi(u)\,dw(u)\Big)^2 \Big|\mathcal{F}_s\Big\} - E\Big\{\int_s^t \varphi^2(u)\,du \Big|\mathcal{F}_s\Big\} = x^2(s) - u(s).$$

So, $u(t)$ is the quadratic variation of the martingale $((x_t, \mathcal{F}_t), t \in [a,b])$.

Recall that for $\varphi \in \mathcal{M}_2$ the inequality

$$E\Big\{\sup_{t\in[a,b]} \Big|\int_a^t \varphi(s)ds\Big|\Big\} \le 4\int_a^b E|\varphi(t)|^2 \, dt \tag{3}$$

is also known as Doob's inequality for Itô-type integrals.

Example: The formal application of the classical rules for the integration by parts yields

$$\frac{1}{2}\int_0^t w(s)\,dw(s) = w^2(t).$$

Clearly, it is not correct because for $t > s$

$$E\{w^2(t)\,|\mathcal{F}_s\} = w^2(s) - t + s \neq w^2(s),$$

and therefore $w^2(t)$ is not a martingale.

Also, it can be proved (see, for example, [8], [11], [25]) that if $\varphi \in \mathcal{M}_2$ and τ is a stopping time with respect to $(\mathcal{F}_t, t \in [a,b])$, i.e., $a \leq \tau \leq b$ a.s. and $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [a,b]$, then the process

$$\int_a^{\tau\wedge t} \varphi(s)\,dw(s), \quad a \leq t \leq b,$$

is a martingale and $E\int_a^{\tau\wedge t} \varphi(s)\,dw(s) = 0$.

Theorem 5. *If $\varphi \in \mathcal{M}_2$, then $x = (x(t), t \in [a,b])$ is a continuous process.*

Proof. Let φ be a step function with a decomposition $a < t_1 < t_2 \cdots < t_n < t$. Then

$$x(t) = \varphi(a)[w(t_1) - w(a)] + \cdots + \varphi(t_n)[w(t) - w(t_n)].$$

Obviously, a.s. continuity of x follows from a.s. continuity of the Wiener process.

If φ is not a step function, let $\{\varphi_n, n \in N\}$ be a sequence of step functions approximating φ, i.e., $\int_a^b E|\varphi(t) - \varphi_n(t)|^2 dt \to 0$ as $n \to \infty$. By Chebyshev's inequality and Doob's inequality (3), it follows that

$$P\Big\{\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\,dw(s) - \int_a^t \varphi_n(s)\,dw(s)\Big| > \epsilon\Big\} \leq \frac{4}{\epsilon^2}\int_a^b E|\varphi(s) - \varphi_n(s)|^2\,ds.$$

Next, we can choose $\epsilon_k > 0$ such that $\epsilon_k \to 0$ as $n \to \infty$, and $\{n_k, k \in N\}$ in such a way that $n_k \nearrow$ if $k \to \infty$, (for example, $\epsilon_k = 2^{-k}$, $n_k = k^{-2}$), for which

$$\sum_{k=1}^{\infty} \frac{1}{\epsilon_k^2}\int_a^b E|\varphi(s) - \varphi_{n_k}(s)|^2\,ds < \infty.$$

Since

$$\sum_{k=1}^{\infty} P\Big\{\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\,dw(s) - \int_a^t \varphi_{n_k}(s)\,dw(s)\Big| > \epsilon_k\Big\} \leq \infty,$$

the Borel–Cantelli's lemma implies that

$$\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\,dw(s) - \int_a^t \varphi_{n_k}(s)\,dw(s)\Big| \leq \epsilon_k \quad \text{a.s.}$$

for all $t \in [a,b]$ if $k \geq k_0(\omega)$, i.e.,

$$\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\,dw(s) - \int_a^t \varphi_n(s)\,dw(s)\Big| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, the integral $\int_a^t \varphi(s)\,dw(s)$ is a.s. uniform limit on $[a,b]$ of the sequence of a.s. continuous stochastic processes $\left\{\int_a^t \varphi_n(s)\,dw(s), t \in [a,b], n \in M\right\}$ and, because of that, it is itself a.s. continuous.

Moreover, x has a.s. continuous sample functions (see [1], [8], [10]). □

Analogously to the stochastic indefinite integral for a process $\varphi \in \mathcal{M}_2$, it is possible to define the Itô-type indefinite integral for $\varphi \in \mathcal{P}$ with

$$x(t) := \int_a^b \varphi(s) I_{\{s<t\}}\,dw(s) = \int_a^t \varphi(s)\,dw(s).$$

In this case the process $x = (x(t), t \in [a,b])$ is measurable, adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in [a,b])$, a.s. continuous, but in general *it is not a martingale*. It can be shown that it is *a local martingale* (see, for example, [25], [30], [31], first of all [27]). Remember that if we denote by τ_n the stopping time

$$\tau_n = \min_t \left\{ \int_a^t |\varphi(s)|^2 ds \geq n \right\},$$

then since $\tau_n \nearrow b$ as $n \to \infty$, it can be proved that $((x_n(t \wedge \tau_n), \mathcal{F}_t), t \in [a,b])$ is a martingale for every $n \in N$. By definition $((x_t, \mathcal{F}_t), t \in [a,b])$ is said to be a local martingale.

1.3. The Itô's formula. In order to determine effectively some classes of stochastic indefinite integrals and to obtain explicit solutions of some types of stochastic differential equations, it is necessary to use *the Itô's formula*, so called *the Itô's differential rule.*

Let $(a(t), t \in [a,b])$ and $(b(t), t \in [a,b])$ be measurable processes adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in [a,b])$, such that

$$\int_a^b |a(t)|\,dt < \infty \quad \text{a.s.}, \qquad \int_a^b |b(t)|^2\,dt \leq \infty \quad \text{a.s.}.$$

Then the stochastic process

$$x(t) = x(a) + \int_a^t a(u)\,du + \int_a^u b(u)\,dw(u)$$

is called *the Itô's process*. It is measurable, adapted to $(\mathcal{F}_t, t \in [a,b])$ and a.s. continuous. Here $x(a)$ is a random variable, $\mathcal{F}_a$-measurable and independent of $w(t) - w(a)$ for all $t \geq a$.

Definition 4. If for every s, t such that $a \leq s < t \leq b$,

$$x(t) - x(s) = \int_s^t a(u)\,du + \int_s^t b(u)\,dw(u) \quad \text{a.s.},$$

then the stochastic process x has *the stochastic differential $dx(t)$* on $[a,b]$, given by

$$dx(t) = a(t)\,dt + b(t)\,dw(t).$$

One can easily conclude that $x(t)$ is measurable, adapted to $(\mathcal{F}_t, t \in [a,b])$ and a.s. continuous.

Theorem 6. (The Itô's formula) *Let $dx(t) = a(t)\,dt + b(t)\,dw(t)$ and let $f(t,x)$ be a nonrandom function defined on $[a,b] \times R$, continuous together with its derivatives f'_t, f'_x, f''_{xx}. Then the process $f(t,x(t))$ has the stochastic differential, given by*

$$df(t,x(t)) = f'_t(t,x(t))\,dt + f'_x(t,x(t))\,dx(t) + \frac{1}{2} f''_{xx}(t,x(t))\,b^2(t)\,dt.$$

For the proof see [1], [8], [9], [11], [28], [30], for example, and first of all [17].

In this formula the surprise is the last term because by the standard calculus formula for total derivatives the term $\frac{1}{2} f''_{xx}(t,x(t))\,b^2(t)\,dt$ would not appear. This correction term arises from the nondifferentiability of the Wiener process. Since

$$\begin{aligned} df(t,x) &\approx f(t+dt, x+dx) - f(t,x) \\ &\approx f'_t(t,x)\,dt + f'_x(t,x)\,dx + \frac{1}{2} f''_{xx}(t,x)\,(dx)^2, \end{aligned}$$

and $Ew^2(t) = t$, we obtain $(dw(t))^2 \approx dt$. So,

$$(dx(t))^2 = [\,a(t)\,dt + b(t)\,dw(t)\,]^2 \approx b^2(t)\,dt.$$

Note that the Itô's formula asserts the two processes: $f(t,x(t)) - f(a,x(a))$ and

$$\int_a^t [\,f'_s(s,x(s)) + f'_x(s,x(s))\,a(s) + \frac{1}{2} f''_{xx}(s,x(s))\,b^2(s)\,]\,ds + \int_a^t f'_x(s,x(s))\,b(s)\,dw(s),$$

which are stochastically equivalent.

Now we are in a position to find the integral $\int_0^t w(s)\,dw(s)$. Since $w(t)$ has the stochastic differential for $a \equiv 0$, $b \equiv 1$, applying the Itô's formula to the function $f(x) = x^2$, we have $dw^2(t) = dt + 2w(t)\,dw(t)$. Thus we obtain

$$\int_0^t w(t)\,dw(t) = \frac{1}{2} w^2(t) - \frac{1}{2} t,$$

which is a martingale.

The Itô's formula can be used to estimate some types of stochastic integrals. Thus, for a process $((\varphi_t, \mathcal{F}_t), t \in [0,T])$, such that $|\varphi(t)| \le K$ a.s. for all $t \in [a,b]$, by applying the Itô's formula to the function $f(x) = x^{2m}$, $m \in N$, we obtain (see [8], [30])

$$E\Big(\int_0^t \varphi(s)\,dw(s)\Big)^{2m} \le K^{2m}(2m-1)!!\,t^m.$$

If φ is unbounded a.s., but $\int_0^T E\varphi^{2m}(t)\,dt < \infty$, then (see [8], [28], [30])

$$E\Big(\int_0^t \varphi(s)\,dw(s)\Big)^{2m} \le [m(2m-1)]^m t^{m-1} \int_0^t E\varphi^{2m}(s)\,ds.$$

The Itô's formula can be easily generalized to a function $f(t, x_1, x_2, \ldots, x_n)$, defined on $[a,b] \times R^n$, continuous together with its derivatives $f'_t, f'_{x_k}, f''_{x_k x_j}$, $1 \leq k, j \leq n$. If $dx_k(t) = a_k(t)\,dt + b_k(t)\,dw(t)$, $1 \leq k \leq n$, then the process $f(t, x_1(t), \ldots, x_n(t))$ has the stochastic differential

$$df(t,x(t)) = f'_t(t,x(t))\,dt + \sum_{k=1}^{n} f'_{x_k}(t,x(t))\,dx_k(t) + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} f''_{x_k x_j}(t,x(t))\,b_k(t)\,b_j(t)\,dt,$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ (see earlier cited references).

Thus, if the stochastic processes $x_1(t)$ and $x_2(t)$ have the stochastic differentials $dx_i(t) = a_i(t)\,dt + b_i(t)\,dw(t)$, $i = 1,2$, then the product $x_1(t)x_2(t)$ has the stochastic differential

$$d(x_1(t)x_2(t)) = x_1(t)\,dx_2(t) + x_2(t)\,dx_1(t) + b_1(t)\,b_2(t)\,dt \tag{4}$$
$$= [\,x_1(t)\,a_2(t) + x_2(t)\,a_1(t) + b_1(t)\,b_2(t)\,]\,dt + [\,x_1(t)\,b_2(t) + x_2(t)\,b_1(t)\,]\,dw(t).$$

The most important role of the Itô's calculus is that it can be generalized to a stochastic integral, replacing the Wiener process by a more general one. For example, let $((z_t, \mathcal{F}_t), t \in [a,b])$ be a sample-continuous second-order martingale. Then by the supermartingale-decomposition theorem of Meyer (see [**36**]) there exists a sample-continuous a.s. increasing process $((u_t, \mathcal{F}_t), t \in [a,b])$ with $u(a) = 0$ a.s., such that for a stochastic process $(\varphi(t), t \in [a,b])$, measurable, adapted to $(\mathcal{F}_t, t \in [a,b])$ and

$$\int_a^b \varphi^2(t)\,du(t) < \infty \quad \text{a.s.},$$

analogously to the procedure in Theorem 1, the Itô's integral (see [**27**])

$$I(\varphi) = \int_a^b \varphi(t)\,dz(t)$$

can be defined with the help of step functions φ_n, as

$$I(\varphi_n) := \sum_{\nu} \varphi(t_\nu^{(n)})[\,z(t_{\nu+1}^{(n)}) - z(t_\nu^{(n)})\,],$$

where $I(\varphi_n) \xrightarrow{P} I(\varphi)$ as $n \to \infty$.

The stochastic indefinite integral $x(t) = \int_a^t \varphi(s)\,dz(s)$ can be defined adequately.

If the process $x(t)$ has the stochastic differential $dx(t) = a(t)\,dt + b(t)\,dz(t)$, then the analogue Itô's formula, first proved in [**27**], has the form

$$df(t,x(t)) = f'_t(t,x(t))\,dt + f'_x(t,x(t))\,dx(t) + \frac{1}{2}\,f''_{xx}(t,x(t))\,b^2(t)\,du(t).$$

2. Stochastic differential equations

2.1. Definition of the Itô-type stochastic differential equation. The stochastic differential equation (shorter SDE) of an unknown n-dimensional process $x = (x(t), t \in [t_0, T])$ with the initial value η is given by

$$dx(t) = a(t, x(t))\, dt + b(t, x(t))\, dw(t), \quad x(t_0) = \eta \text{ a.s.}, \quad t \in [t_0, T], \tag{5}$$

where $w = (w_t, t \in R)$ is an m-dimensional Wiener process, η is an n-dimensional random variable, stochastically independent of w in the sense that random variables w_t and η are stochastically independent for all t, and $a : [t_0, T] \times R^n \to R^n$, $b : [t_0, T] \times R^n \to R^n \times R^m$ are non-random functions, Borel-measurable on their domains.

Because of simplicity, we shall confine ourselves to the one-dimensional case. So, x, w and η are one-dimensional, and $a : [t_0, T] \times R \to R$, $b : [t_0, T] \times R \to R$.

Denote by $\mathcal{F}_t$ the σ-algebra generated by η and w_t, i.e. the smallest σ-algebra with respect to which η and the random variables $w_s, s \le t$, are measurable, such that $w_t - w_s$ is independent on $\mathcal{F}_s$ for all $t \ge s$. Thus the Wiener process w is adapted with respect to the increasing family of sub-σ-algebras $(\mathcal{F}_t, t \in [t_0, T])$, and η is $\mathcal{F}_{t_0}$-measurable.

Denote by $\mathcal{P}$ the space of stochastic processes $\varphi = (\varphi(t), t \in [t_0, T])$, measurable and adapted to $(\mathcal{F}_t,\ t \in [t_0, T])$, such that

$$P\Big\{ \int_{t_0}^{T} |\varphi(t)|^2 dt < \infty \Big\} = 1.$$

Definition 5. The measurable stochastic process $x = (x(t), t \in [t_0, T])$ is a strong solution of the SDE (5) if:

(i) $x(t)$ is $\mathcal{F}_t$-measurable for each $t \in [t_0, T]$;

(ii) $\bar{a}(t) = a(t, x(t))$, $\bar{b}(t) = b(t, x(t))$, such that

$$\int_{t_0}^{T} |\bar{a}(t)|\, dt < \infty, \qquad \int_{t_0}^{T} |\bar{b}(t)|^2\, dt < \infty \quad \text{a.s.};$$

(iii) $x(t_0) = \eta$ a.s.;

(iv) the equation (5) holds a.s. for each $t \in [t_0, T]$.

Since $dx(t) = \bar{a}(t)\, dt + \bar{b}(t)\, dW(t)$ a.s. for all $t \in [t_0, T]$, this is, therefore, the stochastic differential of the process x.

The SDE (5) has the equivalent integral form

$$x(t) = \eta + \int_{t_0}^{t} a(s, x(s))\, ds + \int_{t_0}^{t} b(s, x(s))\, dw(s), \quad t \in [t_0, T]. \tag{6}$$

Because of (i) and (ii) from Definition 5, the integrals on the right-hand side of (6) are well defined: since $\bar{b} \in \mathcal{P}$, then $\int_{t_0}^{t} \bar{b}(s)\, dW(s)$ is the Itô-type stochastic integral; since $\overline{a}$ is measurable and absolutely integrable random function adapted to $(\mathcal{F}_t, t \in [t_0, T])$, $\int_{t_0}^t \overline{a}(s)\, d(s)$ exists as the Lebesgue integral with the parameter ω. Both integrals are defined uniquely up to the stochastic equivalence and therefore the solution x is also determined up to the stochastic equivalence.

Moreover, since both integrals in (6) are a.s. continuous, then x is a.s. continuous. For this, by Doob's theorem [7] we shall always assume that we have chosen a measurable, separable and a.s. continuous version of the strong solution.

Definition 6. The SDE (6) has a unique strong solution if for any two strong solutions x_1 and x_2,

$$P\{\,\omega : x_1(t) = x_2(t),\ t \in [t_0, T]\,\} = 1.$$

This is equivalent to $P\{\sup_{t\in[t_0,T]} |x_1(t) - x_2(t)| > 0\} = 0$.

Example. Solving formally the SDE

$$dx(t) = x(t)\, dw(t), \quad x(0) = \eta \text{ a.s.}, \quad t \geq 0,$$

as an ordinary differential equation, we obtain $x(t) = \eta e^{w(t)}$. By applying the Itô's formula, we get

$$dx(t) = \eta e^{w(t)}\, dw(t) + \frac{1}{2}\,\eta e^{w(t)}\, dt \neq x(t)\, dw(t).$$

Therefore, the solution must have some other form. We shall express as $x(t) = \eta e^{w(t)+\varphi(t)}$, where φ is an unknown function. Using again the Itô's formula, we obtain

$$\begin{aligned} dx(t) &= \eta e^{w(t)+\varphi(t)}\, \varphi'(t)\, dt + \eta e^{w(t)+\varphi(t)}\, dw(t) + \frac{1}{2}\,\eta e^{w(t)+\varphi(t)}\, dt \\ &= x(t)[\,\varphi'(t) + 1/2\,]dt + x(t)\, dw(t). \end{aligned}$$

So, $\varphi'(t) + 1/2 = 0$, i.e., $\varphi(t) = -1/2 + c$, $c =$ const. The initial condition easily gives $c = 0$. Thus, $x(t) = \eta e^{w(t)-t/2}$, $t \geq 0$.

2.2. Existence and uniqueness of a solution. Following the ideas of Itô [16] we give the basic existence and uniqueness theorem of a solution of the SDE (6).

Theorem 7. *Let $w = (w_t, t \in R)$ be a standard Wiener process and η be a random variable, independent of w. Let also $a : [t_0, T] \times R \to R$ and $b : [t_0, T] \times R \to R$ be Borel-measurable functions, satisfying the Lipschitz condition and the condition on the restriction on growth on the last argument respectively, i.e. for all $(t, x), (t, y) \in [t_0, T] \times R$ there exists a constant $L > 0$ such that*

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq L|x - y|, \tag{7}$$

$$|a(t,x)|^2 + |b(t,x)|^2 \leq L^2(1 + x^2). \tag{8}$$

Then there exists a unique a.s. continuous strong solution of the SDE (6).

Proof. The theorem can be proved by Picard-Lindelöf method of iterations, modeled after the corresponding proof for ordinary differential equations (see, for example, [1], [8], [9], [14], [16], [30], [45]). For the proof here we shall apply the Banach fixed point theorem (see [10]).

First, let us suppose that $E|\eta|^2 < \infty$. Denote by $\mathcal{B}$ a space of measurable processes x, defined on $[t_0, T]$, adapted to the nondecreasing family of sub-σ-algebras $(\mathcal{F}_t, t \in [t_0, T])$, satisfying the condition $\sup_{t_0 \le t \le T} E|x(t)|^2 < \infty$. Then $\mathcal{B}$ is the Banach space with the norm

$$\|x\| = \Big(\sup_{t_0 \le t \le T} E|x(t)|^2 \Big)^{1/2}.$$

Let us define an operator S such that for $x \in S$,

$$Sx(t) = \eta + \int_{t_0}^{t} a(s, x(s))\, ds + \int_{t_0}^{t} b(s, x(s))\, dw(s), \quad t \in [t_0, T]. \tag{9}$$

Since a and b are Borel-measurable functions and x is a measurable process, adapted to $(\mathcal{F}_t, t \in [t_0, T])$, it follows that the processes $\bar{a}(t) = a(t, x(t))$ and $\bar{b}(t) = b(t, x(t))$ also have these properties. Moreover, Schwarz inequality and (8) imply

$$\begin{aligned} E\Big| \int_{t_0}^{T} a(s, x(s))\, ds \Big|^2 & \\ &\le (T - t_0) \int_{t_0}^{T} E|a(s, x(s))|^2\, ds \le \alpha + \beta \sup_{t_0 \le t \le T} E|x(t)|^2 < \infty; \end{aligned}$$

$$\begin{aligned} \sup_{t_0 \le t \le T} E\Big| \int_{t_0}^{t} b(s, x(s))\, dw(s) \Big|^2 & \\ &= \sup_{t_0 \le t \le T} \int_{t_0}^{t} E|b(s, x(s))|^2\, ds \le \gamma + \delta \sup_{t_0 \le t \le T} E|x(t)|^2 < \infty, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are some constants depending on L, t_0 and T. Accordingly, since $a^{\frac{1}{2}}, b \in \mathcal{P}$, the integrals in (9) are well defined.

Let us prove that $S : \mathcal{B} \to \mathcal{B}$. If $x \in \mathcal{B}$, then $Sx(t)$ is a measurable process, $\mathcal{F}_t$-measurable for every $t \in [t_0, T]$ and a.s. continuous. Also,

$$\begin{aligned} E|Sx(t)|^2 &\le 3E|\eta|^2 + 3(T - t_0) \int_{t_0}^{t} E|a(s, x(s))|^2\, ds + 3 \int_{t_0}^{t} |b(s, x(s))|^2\, ds \\ &\le 3E|\eta|^2 + 3(T - t_0 + 1)L^2 \int_{t_0}^{t} (1 + E|x(s)|^2)\, ds \\ &\le 3E|\eta|^2 + 3(T - t_0 + 1)L^2 (T - t_0)(1 + \|x\|^2) = M. \end{aligned}$$

Thus,

$$\|Sx\| = \Big(\sup_{t_0 \le t \le T} E|Sx(t)|^2 \Big)^{1/2} < \infty,$$

and therefore $S : \mathcal{B} \to \mathcal{B}$.

In the next step of the proof we shall show that there exists a unique fixed point of the operator S. Indeed, for every $x_1, x_2 \in \mathcal{B}$ we have

$$\begin{aligned}
&E|Sx_1(t) - Sx_2(t)|^2 \\
&\leq 2E\Big|\int_{t_0}^{t} [a(s,x_1(s)) - a(s,x_2(s))]\,ds\Big|^2 + 2E\Big|\int_{t_0}^{t} [b(s,x_1(s)) - b(s,x_2(s))]\,dws\Big|^2 \\
&\leq 2(T-t_0)L^2 \int_{t_0}^{t} E|x_1(s) - x_2(s)|^2\,ds + 2L^2 \int_{t_0}^{t} E|x_1(s) - x_2(s)|^2\,ds \\
&\leq K\|x_1 - x_2\|^2\,(t - t_0),
\end{aligned}$$

where $K = 2(T - t_0 + 1)L^2$. Now it is easy to prove by induction that

$$\begin{aligned}
E|S^n x_1(t) - S^n x_2(t)|^2 &\leq K \int_{t_0}^{t} E|S^{n-1}x_1(s) - S^{n-1}x_2(s)|^2 ds \\
&\leq \cdots \leq \frac{K^n(t-t_0)^n}{n!}\|x_1 - x_2\|^2, \quad t \in [t_0, T] \quad n \in N,
\end{aligned}$$

such that

$$\|S^n x_1 - S^n x_2\|^2 \leq \frac{K^n(T-t_0)^n}{n!}\|x_1 - x_2\|^2, \quad n \in N.$$

Since $K^n(T-t_0)^n/n! \to 0$ as $n \to \infty$, then there exists $n_0 \in N$ such that $K^{n_0}(T-t_0)^{n_0}/n_0! = q < 1$. Thus S^{n_0} is a contraction. By one version of the Banach fixed point theorem it follows that the operator S has a unique fixed point $x \in \mathcal{B}$, i.e., $x = Sx$. On the other hand,

$$x(t) = \eta + \int_{t_0}^{t} a(s,x(s))\,ds + \int_{t_0}^{t} b(s,x(s))\,dw(s) \text{ a.s.}, \quad t \in [t_0, T].$$

Since $x(t_0) = \eta$ a.s., from Definition 5 holds that x is a unique strong solution of the SDE (6), moreover satisfying $\sup_{t_0 \leq t \leq T} E|x(t)|^2 < \infty$. Also, it is easy to show that

$$\sup_{t_0 \leq t \leq T} E|x(t)|^2 \leq 3E|\eta|^2 e^{3K(T-t_0)}.$$

Let us prove now the existence of a solution of the SDE (6) without the assumption $E|\eta|^2 < \infty$. Denote by $I_\eta^N = I_{\{|\eta| \leq N\}}$ and $\eta^N = \eta I_\eta^N$. Obviously, η^N is a random variable, independent with respect to w and $\mathcal{F}_{t_0}$-measurable. Since $E|\eta^N|^2 \leq N^2 < \infty$, the SDE

$$x^N(t) = \eta^N + \int_{t_0}^{t} a(s,x^N(s))\,ds + \int_{t_0}^{t} b(s,x^N(s))\,dw(s), \quad t \in [t_0, T] \tag{10}$$

has a unique solution. For $N' > N$ it follows that

$$\begin{aligned}
x^{N'}(t) - x^N(t) = \eta^{N'} - \eta^N &+ \int_{t_0}^{t} [a(s,x^{N'}(s)) - a(s,x^N(s))]\,ds \\
&+ \int_{t_0}^{t} [b(s,x^{N'}(s)) - b(s,x^N(s))]\,dW(s).
\end{aligned}$$

Since $(\eta^{N'}-\eta^N)I_\eta^N=\eta^{N'}I_\eta^N-\eta^N I_\eta^N=0$, we obtain

$$\begin{aligned}\sup_{t_0\le t\le u}&\Big(x^{N'}(t)-x^N(t)\Big)^2 I_\eta^N\\ &\le 2\sup_{t_0\le t\le u}\Big(I_\eta^N\int_{t_0}^t[a(s,x^{N'}(s))-a(s,x^N(s))]\,ds\Big)^2\\ &\quad+2\sup_{t_0\le t\le u}\Big(I_\eta^N\int_{t_0}^t[b(s,x^{N'}(s))-b(s,x^N(s))]\,dw(s)\Big)^2\\ &\le 2(T-t_0)\int_{t_0}^u I_\eta^N|a(s,x^{N'}(s))-a(s,x^N(s))|^2\,ds\\ &\quad+2\sup_{t_0\le t\le u}\Big|\int_{t_0}^t I_\eta^N[b(s,x^{N'}(s))-b(s,x^N(s))]\,dw(s)\Big|^2.\end{aligned}$$

By applying the Lipschitz condition (7) and Doob's inequality (3), we finally get

$$E\sup_{t_0\le t\le u}|x^{N'}(t)-x^N(t)|^2I_\eta^N\le 2(T-t_0+4)L^2\int_{t_0}^u E\sup_{t_0\le v\le s}|x^{N'}(v)-x^N(v)|^2ds.$$

Now we need the well-known Gronwall's lemma: if $u:[a,b]\to R$ and $v:[a,b]\to R$ are non-negative integrable functions and $L=\text{const}>0$, then

$$u(t)\le v(t)+L\int_a^t u(s)ds\implies u(t)\le v(t)+L\int_a^t e^{L(t-s)}v(s)ds,\quad t\in[a,b].$$

Especially, if $v(t)\equiv\text{const}=u(a)$, then

$$u(t)\le u(a)+L\int_a^t u(s)ds\implies u(t)\le u(a)e^{L(t-a)},\quad t\in[a,b].$$

If $u(a)=0$, then $u(t)=0$ for all $t\in[a,b]$.

By applying the preceding lemma, it follows that

$$E\sup_{t_0\le t\le T}|x^{N'}(t)-x^N(t)|^2I_\eta^N=0,$$

which implies $P\{\sup_{t_0\le T\le t}|x^{N'}(t)-x^N(t)|^2=0\}=0$. Now,

$$P\{\sup_{t_0\le t\le T}|x^{N'}(t)-x^N(t)|^2>0\}\le P\{|\eta|>N\}\to 0\quad\text{as}\quad N',N\to\infty.$$

Therefore, $\{x^N(t)\}$ is a Cauchy sequence converging in probability for all $t\in[t_0,T]$. So, there exists $\mathcal{F}_t$-measurable process $(x(t),t\in[t_0,T])$, such that $\sup_{t_0\le t\le T}|x^N(t)-x(t)|\overset{\text{rm p.}}{\to}0$ as $N\to\infty$. Since

$$\begin{aligned}\int_{t_0}^T&|a(s,x(s))-a(s,x^N(s))|^2\,ds+\int_{t_0}^T|b(s,x(s))-b(s,x^N(s))|^2\,ds\\ &\le 2L^2\int_{t_0}^T\sup_{t_0\le u\le s}|x(u)-x^N(u)|^2\,ds\\ &\le 2L^2(T-t_0)\sup_{t_0\le t\le T}|x(t)-x^N(t)|^2\overset{\text{p.}}{\to}0\quad\text{as}\quad N\to\infty,\end{aligned}$$

then for every fixed $t \in [t_0, T]$,

$$\int_{t_0}^{t} a(s, x^N(s))\, ds \xrightarrow{P} \int_{t_0}^{t} a(s, x(s))\, ds,$$

$$\int_{t_0}^{t} b(s, x^N(s))\, dw(s) \xrightarrow{P} \int_{t_0}^{t} b(s, x(s))\, dw(s).$$

holds. The limits in probability on both sides of the equation (10) show that $x(t)$ satisfies the SDE (6) a.s. and, therefore, it is its strong solution.

It remains to prove a uniqueness of a solution of the SDE (6) if $E|\eta|^2 < \infty$ does not hold.

Let x_1 and x_2 be two solutions of this equation. Then for every $t \in [t_0, T]$,

$$x_1(t) - x_2(t) = \int_{t_0}^{t} [a(s, x_1(s)) - a(s, x_2(s))]\, ds + \int_{t_0}^{t} [b(s, x_1(s)) - b(s, x_2(s))]\, dw(s)$$

holds a.s. Denote

$$I_N(t) = \begin{cases} 1, & |x_1(s)| \le N, |x_2(s)| \le N, \quad s \in [t_0, t], \\ 0, & \text{otherwise} \end{cases}$$

Since $I_N(t) I_N(s) = I_N(t)$ for all $s \le t$, then

$$I_N(t)[x_1(t) - x_2(t)] = I_N(t) \int_{t_0}^{t} I_N(s)[a(s, x_1(s)) - a(s, x_2(s))]\, ds + I_N(t) \int_{t_0}^{t} I_N(s)[b(s, x_1(s)) - b(s, x_2(s))]\, dw(s).$$

Thus

$$I_N(s)|a(s, x_1(s)) - a(s, x_2(s))| \le I_N(s) L |x_1(s) - x_2(s)| \le 2LN, \quad \text{a.s.},$$

and analogously for b. If we apply the dominated convergence theorem, we obtain

$$\begin{aligned} E I_N(t)|x_1(t) - x_2(t)|^2 &\le 2(t - t_0) \int_{t_0}^{t} E\{I_N(s)|a(s, x_1(s)) - a(s, x_2(s))|^2\}\, ds \\ &\quad + 2 \int_{t_0}^{t} E\{I_N(s)|b(s, x_1(s)) - b(s, x_2(s))|^2\}\, ds \\ &\le 2(T - t_0 + 1) L^2 \int_{t_0}^{t} E\{I_N(s)|x_1(s) - x_2(s)|^2\}\, ds. \end{aligned}$$

Applying now the Gronwall's lemma we get $E\{I_N(t)|x_1(t) - x_2(t)|^2\} = 0$ for all $t \in [t_0, T]$, which implies $P\{ I_N(t) x_1(t) = I_N(t) x_2(t) \} = 1$. From there we easily conclude

$$P\{x_1(t) \ne x_2(t)\} \le P\{\sup_{t_0 \le s \le t} |x_1(s)| > N\} + P\{\sup_{t_0 \le s \le t} |x_2(s)| > N\}.$$

Since x_1 i x_2 are a.s. continuous on $[t_0,T]$, they are a.s. bounded. It means that the right-hand side of the last inequality goes to zero by taking $N \to \infty$ and, therefore $P\{x_1(t) \neq x_2(t)\} = 0$ for all $t \in [t_0,T]$, i.e.

$$P\{\sup_{t_0 \le t \le T} |x_1(t) - x_2(t)| > 0\} = 0.$$

Thus the proof is complete. □

Clearly, Theorem 7 gives only sufficient conditions for the existence and uniqueness of a solution of the SDE (6). Note that if the functions a and b are defined on $[t_0,\infty) \times R$ and if the assumptions of Theorem 7 hold on every finite subinterval $[t_0,T] \subset [t_0,\infty)$, then the SDE (6) has a unique solution, defined on the entire half-line $[t_0,\infty)$, called *a global solution*. Naturally, in some cases the SDE (6) could have *a local solution*, particularly if the assumptions of Theorem 7 do not hold, as in the following example.

Indeed, the coefficients od the SDE

$$dx(t) = -\frac{1}{2}e^{-2x(t)}dt + e^{-x(t)}dw(t), \quad x(t_0) = \eta \text{ a.s}, \quad t \ge t_0,$$

do not satisfy any Lipschitz condition or any growth condition for $x < 0$. However, there exists a unique local solution $x(t) = \ln[w(t) - w(t_0) + e^{\eta}]$, defined on the random interval $[t_0,\tau)$, where the random variable τ is determined with $\tau = \inf\{t : w_t - w_{t_0} + e^{\eta} < 0\}$ (see [1], [32]). Naturally, we use the Itô's formula to prove that $x(t)$ is the solution of this equation.

The next theorem, known as *the local uniqueness theorem*, plays a very important role in the study of stochastic differential equations (see, for example, [1], [8], [9]).

Theorem 8. *Let the functions a_i and b_i, $i = 1,2$, satisfy the assumptions of Theorem 7 and let there exist $N > 0$ such that $a_1(t,x) = a_2(t,x)$, $b_1(t,x) = b_2(t,x)$ for all $(t,x) \in [-N,N]$. Let $x_i(t)$, $i = 1,2$, be a solution of the SDE*

$$dx_i(t) = a_i(t,x_i(t))\,dt + b_i(t,x_i(t))\,dw(t), \quad x_i(t_0) = \eta \text{ a.s.}, \quad t \in [t_0,T].$$

Denote by τ_i the first time, after t_0, such that $x_i(t)$ intersects $R \setminus [-N,N]$ if such time $t \in [t_0,T]$ exists, and $\tau_i = T$ otherwise. Then

$$P\{\tau_1 = \tau_2\} = 1 \quad \text{and} \quad P\{\sup_{t_0 \le t \le \tau_1} |x_1(t) - x_2(t)| = 0\} = 1.$$

Proof: Denote by

$$\psi_1(t) = \begin{cases} 1, & \sup_{t_0 \le t \le t} |x_1(t)| \le N, \\ 0, & \text{otherwise,} \end{cases}$$

Let $\psi_1(t) = 1$. Then $\psi_1(s) = 1$ for all $t_0 \le s \le t \le \tau_1$ and here $a_1(s,x_1(s)) = a_2(s,x_1(s))$ a.s., $b_1(s,x_1(s)) = b_2(s,x_1(s))$ a.s.. From integral form of the SDE-s it is easy to obtain

$$\psi_1(t)[x_1(t) - x_2(t)]^2 \le 2\Big\{\int_{t_0}^{t} \psi_1(s)[a_2(s,x_1(s)) - a_2(s,x_2(s))]\,ds\Big\}^2$$
$$+ 2\Big\{\int_{t_0}^{t} \psi_1(s)[b_2(s,x_1(s)) - b_2(s,x_2(s))]\,dw(s)\Big\}^2.$$

By applying the Lipschitz condition (7), it follows that

$$E\psi_1(t)\,[\,x_1(t)-x_2(t)\,]^2 \le c\int_{t_0}^{t} E\psi_1(s)[\,x_1(s)-x_2(s)\,]^2\,ds,$$

where c is a constant depending on L, T and t_0. Then from Gronwall's lemma

$$E\psi_1(t)\,[\,x_1(t)-x_2(t)\,]^2 = 0, \quad t\in[t_0,\tau_1],$$

holds. From that,

$$P\{\sup_{t_0\le t\le\tau_1}|\,x_1(t)-x_2(t)\,|=0\}=1,$$

and therefore $x_1(t) = x_2(t)$ a.s. for $t \in [t_0, \tau_1]$. Consequently, $P\{\tau_2 \ge \tau_1\} = 1$. Analogously we get $P\{\tau_1 \le \tau_2\} = 1$, which completes the proof. □

Theorem 8 makes it possible to express the next stronger existence and uniqueness theorem.

Theorem 9. *Let $a : [t_0,T]\times R \to R$, $b : [t_0,T]\times R \to R$ be measurable functions satisfying the assumptions:*

(i) there exists a constant $K > 0$ such that for all $(t,x) \in [t_0,T]\times R$,

$$|a(t,x)|^2 + |b(t,x)|^2 \le L^2(1+|x|^2);$$

(ii) for any $N > 0$ there exists a constant $L_N > 0$ such that for all $(t,x),(t,y) \in [t_0,T]\times[-N,N]$,

$$|a(t,x)-a(t,y)| + |b(t,x)-b(t,y)| \le L_N|x-y|.$$

If a standard Wiener process w and a random variable η are independent and $E|\eta|^2 < \infty$, there exists a unique solution $(x(t), t\in[t_0,T])$ of the SDE (6), satisfying the initial value $x(t_0) = \eta$ a.s.

The proof can be found in [9].

Let us give some important notions. Remark that Theorem 7 can be extended to the SDE, similar to the SDE (6), in which the coefficients $a : \Omega\times[t_0,T]\times R \to R$ and $b : \Omega\times[t_0,T]\times R \to R$ are random functions, Borel measurable on their domains, adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t\in[t_0,T])$ generated by w, such that the stochastic integrals in this SDE exist in the sense of Definition 5-(ii).

Theorem 10. *Let $(\eta(t), t\in[t_0,T])$ be a stochastic process, independent of w, adapted to $(\mathcal{F}_t, t\in[t_0,T])$, such that $\sup_{t\in[t_0,T]} E|\eta(t)|^2 < \infty$. Let also the random functions a and b satisfy a.s. the Lipschitz condition (7) and the condition of the restriction on growth (8). Then there exists a unique solution $(x(t), t\in[t_0,T])$ of the SDE*

$$x(t) = \eta(t) + \int_{t_0}^{t} a(\omega,s,x(s))\,ds + \int_{t_0}^{t} b(\omega,s,x(s))\,dw(s), \quad t\in[t_0,T],$$

with $\sup_{t\in[t_0,T]} E|x(t)|^2 < \infty$. *Moreover, if the process* $\eta(t)$ *is a.s. continuous, then the solution* $x(t)$ *is a.s. continuous.*

A theorem analogous to Theorem 9 can also be proved.

Note that the approach given by Theorems 7, 8, 9 and 10 is appropriatelly extended to analyze the existence and uniqueness problem for special classes of stochastic differential equations, stochastic functional differential equations, stochastic integral and integrodifferential equations containing the Itô's integrals (see [3], [4], [5], [9], [11], [25], [26], [30], [37], for example, and many others).

Remember again that Theorem 7 gives only sufficient conditions for the existence and uniqueness of the solution of the SDE (6). In fact, there is a number of papers in which various sufficient conditions, essentially other than the conditions (7) and (8), are considered. Note that many new theorems present a direct extension of the corresponding deterministic results (see, for example, [3], [4], [5], [9], [18], [22], [28], [45], [46]). In many papers different kinds of contractions are used instead of the Lipschitz condition, for example in [24], [38].

Naturally, the permanently current problem is the relationship between ordinary and stochastic differential equations, especially for applications to stochastic control problems and to stochastic filtering problems (see [30], [42], [43], [44], for example).

An important fact is that the problem of the existence and uniqueness of solutions of the Itô-type stochastic differential equations can be extended to stochastic differential equations with respect to martingales and stochastic measures (see, for example, [6], [10], [11], [14], [25], [27], [29], [31], [34], [41], [47]), and also to stochastic differential equations with semimartingales (see [22], [33], [47]).

One of the most important problems in qualitative analysis of solutions for different classes of stochastic differential equations is the stability problem, including the asymptotic behavior of solutions when $t \to \infty$ and the existence of singular solutions (see [1], [2], [3], [4], [5], [13], [14], [37], [46], for example). By using the concept of Lyapunov function and the theory of stochastic and deterministic inequalities, several comparison theorems are developed in many papers and books (see, for example, [9], [13], [14], [28], [46]).

2.3. Stochastic differential equations depending on parameters. Now we give the basic theorem which describes the stochastic differential equation of the Itô type depending on a parameter $\alpha \in A$, where A is a parameter set. This theorem shows that the change in the solution can be made arbitrarily small by making the change in the parameter sufficiently small.

Theorem 11. *Let the random functions* η_α, a_α, b_α *satisfy the assumptions of Theorem 10 for any parameter* $\alpha \in A$, *with the same constant* L *in (7) and (8). Let also the process* $(\eta_\alpha(t), t \in [t_0, T])$ *be a.s. continuous and* $\sup_{t\in[t_0,T]} E|\eta_\alpha(t)|^2 < c$ *for all* $\alpha \in A$, c *=const.. Suppose that for any* $N > 0$, $\alpha_0 \in A$, $\epsilon > 0$ *and* $t \in [t_0, T]$,

$$\lim_{\alpha\to\alpha_0} P\{ \sup_{|x|\le N} [\,|a_\alpha(\omega,t,x) - a_{\alpha_0}(\omega,t,x)| + |b_\alpha(\omega,t,x) - b_{\alpha_0}(\omega,t,x)|\,] > \epsilon\} = 0$$

and

$$\lim_{\alpha\to\alpha_0} \sup_{t\in[t_0,T]} E|\eta_\alpha(t)-\eta_{\alpha_0}(t)|^2 = 0.$$

If $(x_\alpha(t), t\in[t_0,T])$ *is a solution of the SDE*

$$x_\alpha(t) = \eta_\alpha(t) + \int_{t_0}^t a_\alpha(\omega,s,x_\alpha(s))\,ds + \int_{t_0}^t b_\alpha(\omega,s,x_\alpha(s))\,dw(s),\ t\in[t_0,T],\quad ()$$

then $\lim \alpha\to\alpha_0 \sup_{t\in[t_0,T]} E|x_\alpha(t)-x_{\alpha_0}(t)|^2 = 0.$

Proof. Denote

$$x_\alpha(t) - x_{\alpha_0}(t) = \xi_\alpha(t) + \int_{t_0}^t [\,a_\alpha(\omega,s,x_\alpha(s)) - a_\alpha(\omega,s,x_{\alpha_0}(s))\,]\,ds$$
$$+ \int_{t_0}^t b_\alpha(\omega,s,x_\alpha(s)) - b_\alpha(\omega,s,x_{\alpha_0}(s))\,]\,dw(s),$$

where

$$\xi_\alpha(t) = \eta_\alpha(t) - \eta_{\alpha_0}(t) + \int_{t_0}^t [\,a_\alpha(\omega,s,x_{\alpha_0}(s)) - a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,]\,ds$$
$$+ \int_{t_0}^t [\,b_\alpha(\omega,s,x_{\alpha_0}(s)) - b_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,]\,dw(s).$$

Using the Lipschitz condition (7) on the first identity and applying the usual stochastic isometry, we easily obtain

$$E|x_\alpha(t)-x_{\alpha_0}(t)|^2 \le 3|\xi_\alpha(t)|^2 + K\int_{t_0}^t E|x_\alpha(s)-x_{\alpha_0}(s)|^2\,ds,$$

where $K = 3(T-t_0+1)L^2$. By Gronwall's lemma it follows that

$$E|x_\alpha(t)-x_{\alpha_0}(t)|^2 \le 3E|\xi_\alpha(t)|^2 + K\int_{t_0}^t e^{K(t-s)}E|\xi_\alpha(s)|^2\,ds.$$

Therefore, it follows from the last inequality that the theorem will be proved if we show that $\sup_{t\in[t_0,T]} E|\xi_\alpha(t)|^2 \to 0$ as $\alpha\to\alpha_0$.

Since

$$E\Big|\int_{t_0}^t [\,a_\alpha(\omega,s,x_{\alpha_0}(s)) - a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,]\,ds\Big|^2$$
$$\le (t-t_0)\int_{t_0}^t E|\,a_\alpha(\omega,s,x_{\alpha_0}(s)) - a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,|^2\,ds,$$

by applying the condition (8) we obtain that the last integrand is bounded by $2L^2(1+|x_{\alpha_0}(t)|^2)$. Since $E\int_{t_0}^T(1+|x_{\alpha_0}(t)|^2\,dt<\infty$, it follows from the conditions of the theorem that this integrand also converges to zero in probability, as $\alpha \to \alpha_0$. So, by the Lebesgue bounded convergence theorem we conclude

$$\sup_{t\in[t_0,T]} E\left|\int_{t_0}^{t}[a_\alpha(\omega,s,x_{\alpha_0}(s)) - a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))]ds\right|^2$$
$$\leq (T-t_0)\int_{t_0}^{T} E|\,a_\alpha(\omega,s,x_{\alpha_0}(s)) - a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,|^2 ds \to 0 \quad \text{as} \quad \alpha \to \alpha_0.$$

Similarly, using Doob's inequality (3) and the previous arguments, we have

$$E\sup_{t\in[t_0,T]}\left\{\int_{t_0}^{t}[b_\alpha(\omega,s,x_{\alpha_0}(s)) - b_{\alpha_0}(\omega,s,x_{\alpha_0}(s))]dw(s)\right\}^2$$
$$\leq 4\int_{t_0}^{T} E|\,b_\alpha(\omega,s,x_{\alpha_0}(s)) - b_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,|^2 ds \to 0 \quad \text{as} \quad \alpha \to \alpha_0.$$

This completes the proof, because $\sup_{t\in[t_0,T]} E|\eta_\alpha(t) - \eta_{\alpha_0}(t)|^2 \to 0$ as $\alpha \to \alpha_0$. □

Note that there are suitable versions of the preceding theorem for different classes of stochastic differential equations. So, for the SDE (6) one can state a theorem which ensures the continuous dependence of the solution on the initial value (t_0,η) (see [9], [24]).

The more important application of Theorem 11 is for a discrete parameter set, i.e., if $A = \{\alpha_n, n = 0,1,\dots\}$ and $\alpha_n \to \alpha_0$ as $n \to \infty$. Then the following theorem holds:

Theorem 12. *Let the random functions $\eta_n(t)$, $a_n(\omega,t,x)$, $b_n(\omega,t,x)$, $n = 0,1,2,\dots$, satisfy all conditions of Theorem 11 for n and 0 instead of α and α_0 respectively. If $(x_n(t), t\in[t_0,T])$ is the solution of the SDE*

$$x_n(t) = \eta_n(t) + \int_{t_0}^{t} a_n(\omega,s,x_n(s))\,ds + \int_{t_0}^{t} b_n(\omega,s,x_n(s))\,dw(s),\ t\in[t_0,T], \quad ()$$

then

$$\lim_{n\to\infty}\sup_{t\in[t_0,T]} E|x_n(t) - x_0(t)|^2 = 0.$$

From purely theoretical point of view, and much more from the point of view of various applications, this theorem gives a possibility to study the solution $x_0(t)$ of the SDE (12) for $n = 0$ by finding at least an approximate solution $x_{n_0}(t)$ of the SDE (12) for $n = n_0$.

This theorem enables the construction of some iterative methods for solving the SDE (6), or the SDE (12) for $n = 0$, and to estimate an error of the n-th approximation of the solution of the original equation. There is a number of papers in which various sufficient conditions of closeness of the random or non-random functions η_0, a_0, b_0 with the functions η_n, a_n, b_n respectively, are given, such that $x_n(t) \to x(t)$ as $n \to \infty$ in probability or in p-th mean sense or with probability one (see, for example, [3], [9], [11], [23], [26], [45]).

2.4. The Markov property. Now we describe in short one of the most important properties of the solutions of the SDE (6), known as *the Markov property.*

Having in mind that a solution $x(t)$ of the SDE (6) must be $\mathcal{F}_t$-measurable, it can be interpreted as a stochastic process determined by non-random functions a and b and by random elements η and $w_s, s \leq t$. So, $x(t)$ depends on η and $w_s, s \leq t$. Moreover, the construction of $x(t)$, especially the construction of a solution by Picard–Lindelöf method of iterations, shows that it depends only on $w_s - w_{t_0}$ for $t_0 \leq s \leq t$ (see [1], [8]). Thus, $x(t)$ can be expressed as a functional

$$x(t) = f(\eta; w_s - w_{t_0}, t_0 \leq s \leq t).$$

This fact makes possible a description of the Markov property of the solution of the SDE (6).

Definition 7. The stochastic process $(x(t), t \in [t_0, T])$ is said to be a Markov process with respect to $(\mathcal{F}_t, t \in [t_0, T])$ if for all $t_0 \leq s \leq t \leq T$ and for any set $A \in \mathcal{B}$

$$P\{\, x(t) \in A | \mathcal{F}_s\} = P\{\, x(t) \in A | x(s)\} \quad \text{a.s.}$$

holds.

Theorem 13. *Let the conditions of Theorem 7 hold with* $E|\eta|^2 < \infty$ *and let* $(\mathcal{F}_t, t \in [t_0, T])$ *be the increasing family of the sub-σ-algebras generated by* η *and* w. *Then the unique solution* $(\, x(t), t \in [t_0, T])$ *of the SDE (6) is a Markov process with respect to* $(\mathcal{F}_t, t \in [t_0, T])$.

For a detailed proof see [8], for example. We give only a short survey of the proof.

Together with the SDE (6) we consider the same equation, now on an interval $[s, T] \subset [t_0, T]$, i.e., for $t \in [s, T]$ we have

$$x(t) = x + \int_s^t a(u, x(u))\, du + \int_0^t b(u, x(u))\, dw(u), \quad x(s) = x \ \text{ a.s.} \qquad ()$$

For the given initial value $x(s) = x$ a.s., let $(\, x_{s,x}(t), t \in [s, T]\,)$ be a solution of the SDE (13). From the fact that the SDE (6) has a unique solution $(x(t), t \in [t_0, T])$, it follows that $x(t) = x_{s,x}(t)$ a.s. for all $t \in [s, T]$. Also, for $t \in [s, T]$, $x_{s,x}(t)$ is completely determined as a functional $x_{s,x}(t) = f(x; w_u - w_s, u \in [s, T])$. Moreover, since $x(s)$ is $\mathcal{F}_s$-measurable and increments $w_u - w_s, u \in [s, t]$, are independent on $\mathcal{F}_s$, for any set $A \in \mathcal{B}$ it follows that $P\{x(t) \in A | \mathcal{F}_s\} = P\{x(t) \in A | x(s)\}$ a.s.. Therefore, the solution of the SDE (6) is a Markov process.

For $t_0 \leq s \leq t \leq T$ and for any set $A \in \mathcal{B}$, the function

$$p(s, x; t, A) = P\{x(t) \in A | x(s) = x\}$$

is called *the transition probability function.* Clearly, considering s and x fixed, $p(s, x; t, A)$ is precisely the distribution of the solution $x_{s,x}(t)$ of the equation (13). Also, $p(s, x; t, A)$ has the following properties: it is Borel measurable in x for fixed

s, t, A; it is a probability measure in A for fixed s, x, t; the function p satisfies the Chapman–Kolmogorov equation: for all $x \in R$ and $s < u < t$,

$$p(s, x; t, A) = \int_{-\infty}^{\infty} p(s, x; u, dy)\, p(u, y; t, A)$$

holds.

Recall that a Markov process is said to be *homogeneous* if the transition probability functions are stationary, i.e., $p(s, x; t, A) = \phi(t - s, x, A)$.

It is easy to see that if the SDE (6) is *autonomous*, that is $a(t, x) = a(x)$, $b(t, x) = b(x)$, then its solution will be a homogeneous Markov process.

Moreover, in addition to the conditions of Theorem 7, if the functions $a(t, x)$ and $b(t, x)$ are supposed to be continuous, then a solution of the SDE (6) is *a diffusion process*, i.e., a stochastic process with continuous sample functions whose transition probability functions $p(s, x; t, A)$ have certain infinitesimal properties as $t \to s$ (see, for example, [1], [8], [9], [10], [45]).

The density function of the transition probability function is called *the transition density function.* Under some very strict conditions of differentiability of the functions a and b, beginning from the Chapman–Kolmogorov equations one comes to the well-known *backward and forward parabolic equations*, alternatively called *diffusion equations*, whose solutions are transition density functions. Note that the forward equation is also known as *the Fokker–Planck equation.* Naturally, the solution of the SDE (6) is completely described if the transition probability functions, i.e., the transition density functions, are known.

Emphasize an important fact that the theory of diffusion processes is applied to study several phenomena in physics, astronomy, biology, etc. The modern theory of the Markov processes, primarily a semigroup theory, is engaged in the studies of the solutions of diverse classes of stochastic differential equations, which are diffusion processes.

2.5. Solvable stochastic differential equations. We say that the SDE (6) is *explicitly solvable* if its solution can be represented by quadratures, i.e., in terms of ordinary (Lebesgue) and Itô's stochastic integrals.

I. Just as with ordinary differential equations, a lot of theory is developed to describe solutions of linear Itô-type stochastic differential equations, first of all analytic properties of the solutions, including the overall behavior of sample functions on the interval $[t_0, \infty)$. Now we give the procedure to obtain explicit solutions of homogeneous and non-homogeneous linear stochastic differential equations.

Let $a : [t_0, \infty) \to R$ and $b : [t_0, \infty) \to R$ be Borel-measurable bounded functions. Then the equation

$$dx(t) = a(t)x(t)\, dt + b(t)x(t)\, dw(t), \quad x(t_0) = \eta = \text{const. a.s.}, \quad t \geq t_0.$$

is said to be the homogeneous linear SDE. If $\eta = 0$ a.s., this equation has a trivial solution $x(t) = 0$ a.s. Since the conditions of Theorem 7 hold, then there exists a unique solution such that $x(t) > 0$ a.s. for $\eta > 0$ a.s.; $x(t) < 0$ a.s. for $\eta < 0$ a.s.

If we put $y(t) = \ln x(t)$ for $\eta > 0$ a.s., or $y(t) = \ln(-x(t))$ for $\eta < 0$ a.s., by Itô's formula we have

$$dy(t) = \frac{1}{x(t)}\,dx(t) + \frac{1}{2}\Big(-\frac{1}{x(t)}\Big)b^2(t)x(t)\,dt$$

i.e.,

$$dy(t) = \Big[a(t) - \frac{1}{2}b^2(t)\Big]\,dt + b(t)\,dw(t), \quad y(t_0) = \ln\eta \text{ a.s.}$$

Thus we obtain the stochastic differential of the process $y(t)$ and, therefore

$$y(t) = \ln\eta + \int_{t_0}^{t}\Big[a(s) - \frac{1}{2}b^2(s)\Big]\,ds + \int_{t_0}^{t} b(s)\,dw(s), \quad t \ge t_0.$$

From that the homogeneous linear SDE has the solution

$$x(t) = \eta\exp\Big\{\int_{t_0}^{t}\Big[a(s) - \frac{1}{2}b^2(s)\Big]\,ds + \int_{t_0}^{t} b(s)\,dw(s)\Big\}, \quad t \ge t_0.$$

Especially, the Langevin SDE

$$dx(t) = -\alpha x(t)\,dt + \beta\,dw(t), \quad x(0) = \eta \text{ a.s.}, \quad t \ge 0,$$

where $\alpha > 0$ and β are constants, has the solution

$$x(t) = e^{-\alpha t}\Big[\eta + \int_0^t e^{\alpha s}\beta\,dw(s)\Big], \quad t \ge 0.$$

For normally distributed or constant η, the solution $x(t)$ is a Gaussian process, the so-called Ornstein–Uhlenbech velocity process (see [1], [8]).

The non-homogeneous linear SDE

$$dx(t) = [\,\alpha(t) + a(t)x(t)\,]\,dt + [\,\beta(t) + b(t)x(t)\,]\,dw(t), \qquad (14)$$
$$x(t_0) = \eta \text{ a.s.}, \quad t \ge t_0,$$

can be solved analogously, putting $y(t) = \Phi^{-1}(t)\,x(t)$, where $\Phi^{-1}(t)$ is a particular solution of the corresponding homogeneous linear SDE with the initial value $\Phi(t_0) = 1$. So,

$$\Phi^{-1}(t) = \exp\Big\{-\int_{t_0}^{t}[\,a(s) - \frac{1}{2}b^2(s)\,]\,ds - \int_{t_0}^{t} b(s)\,dw(s)\Big\}.$$

Applying the Itô's formula we have

$$d\Phi^{-1}(t) = \Phi^{-1}(t)\Big\{[\,a(s) - \frac{1}{2}b^2(s)\,]\,ds - b(s)\,dw(s)\Big\}.$$

Applying again the Itô's formula on the product $\Phi^{-1}(t)x(t)$, from (4) we obtain

$$\begin{aligned} dy(t) &= d\,(\,\Phi^{-1}(t)\,x(t)\,) \\ &= \Phi^{-1}(t)\,dx(t) + x(t)d\Phi^{-1}(t) - [\,\beta(t) + b(t)x(t)\,]\,\Phi^{-1}(t)b(t)\,dt. \end{aligned}$$

By replacing $dx(t)$ and $d\Phi^{-1}(t)$ with the corresponding differentials, we obtain finally

$$dy(t) = \Phi^{-1}(t)\{[\alpha(t) - \beta(t)b(t)] + \beta(t)\,dw(t)\}$$

and, therefore

$$y(t) = \eta + \int_{t_0}^{t} \Phi^{-1}(s)\,[\alpha(s) - \beta(s)b(s)\,]\,ds + \int_{t_0}^{t} \Phi^{-1}(s)\beta(s)\,dw(s).$$

Thus the explicit solution of the non-homogeneous linear SDE (14) is given as

$$x(t) = \Phi(t)\left[\eta + \int_{t_0}^{t} \Phi^{-1}(s)\,[\alpha(s) - \beta(s)b(s)\,]\,ds + \int_{t_0}^{t} \Phi^{-1}(s)\beta(s)\,dw(s)\right].$$

II. In general, in order to transform the SDE (6) on a solvable form, we introduce a change of variables $y = h(t,x)$, where a smooth function $h(t,x)$ has an inverse $k(t,y)$, such that $h(t,k(t,y)) \equiv y$, $k(t,h(t,x)) \equiv x$.

According to the Itô's formula, the process $y(t) = h(t,x(t))$ satisfies the SDE

$$dy(t) = f(t,y(t))\,dt + g(t,y(t))\,dw(t), \quad y(t_0) = h(t_0,\eta) \text{ a.s.},$$

where

$$f(t,y) = \left[h'_t + a\,h'_x + \frac{1}{2}\,b^2\,h''_{xx}\right](t,k(t,y)), \tag{15}$$

$$g(t,y) = [\,b\,h'_x\,](t,k(t,y)). \tag{16}$$

The SDE (6) is said to be *reducible* if such a function h can be found so that the functions f and g, given by (15) and (16) respectively, are independent of y. Thus, the change of variables $y = h(t,x)$ permits the explicit representation of the solution $x(t)$ of the SDE (6) as

$$x(t) = k(t,y(t)),$$

where

$$y(t) = h(t_0,\eta) + \int_{t_0}^{t} f(s)\,ds + \int_{t_0}^{t} g(s)\,dw(s).$$

In other words, the SDE (6) is reducible if a sufficiently smooth invertible function $h(t,x)$ and functions $f(t)$ and $g(t)$, exist, such that

$$\left[\frac{\partial h}{\partial t} + a\,\frac{\partial h}{\partial x} + \frac{1}{2}\,b^2\,\frac{\partial^2 h}{\partial x^2}\right](t,x) \equiv f(t), \tag{17}$$

$$\left[b\,\frac{\partial h}{\partial x}\right](t,x) \equiv g(t). \tag{18}$$

Under the assumptions that $b \neq 0$ and a and b possess the indicated derivatives, one can obtain the necessary and sufficient conditions so that the SDE (6) be reducible. Indeed, differentiating (17) with respect to x gives

$$\frac{\partial^2 h}{\partial x \partial t} + \frac{\partial}{\partial x}\left\{a\,\frac{\partial h}{\partial x} + \frac{1}{2}\,b^2\,\frac{\partial^2 h}{\partial x^2}\right\} \equiv 0. \tag{19}$$

Since from (18) we get

$$\frac{\partial h(t,x)}{\partial x} \equiv \frac{g(t)}{b(t,x)}, \tag{20}$$

then the following derivatives hold

$$\frac{\partial^2 h}{\partial t \partial x} \equiv \frac{b(t,x)g'(t) - g(t)\,\partial b(t,x)/\partial t}{b^2(t,x)}, \quad \frac{\partial^2 h}{\partial x^2} \equiv -\frac{g(t)\,\partial b(t,x)/\partial x}{b^2(t,x)}.$$

By substituting the appropriate derivatives into (19), we obtain finally

$$g' = g\,b\left[\frac{1}{b^2}\frac{\partial b}{\partial t} - \frac{\partial}{\partial x}\left(\frac{a}{b}\right) + \frac{1}{2}\frac{\partial^2 b}{\partial x^2}\right] \equiv 0. \tag{21}$$

Since the left side of this identity is independent of x, then

$$\frac{\partial}{\partial x}\left\{b\left[\frac{1}{b^2}\frac{\partial b}{\partial t} - \frac{\partial}{\partial x}\left(\frac{a}{b}\right) + \frac{1}{2}\frac{\partial^2 b}{\partial x^2}\right]\right\} \equiv 0. \tag{22}$$

If (22) holds, the function g, $g \not\equiv 0$, can be found as a solution to the ordinary differential equation (21). The function h, which is at least locally invertible since $\partial h/\partial x \not\equiv 0$, can be determined from (20), and the function f from (17). Then (21) is equivalent with (19) and thus the functions f and g are independent of x. Therefore, *the SDE* (6) *is reducible if and only if f and g satisfy* (22).

Let us suppose that (22) holds. Then:

(i) If $g \equiv 1$, then $h(t,x) = \int_{x_0}^{x} \frac{du}{b(t,u)}$, x_0 =const.;

(ii) If $f \equiv 0$, then h must be a solution of the partial differential equation $h'_t + a h'_x + \frac{1}{2} b^2 h''_{xx} = 0$;

(iii) If the SDE (6) is autonomous, i.e., $a(t,x) = a(x)$, $b(t,x) = b(x)$, then it is reducible if and only if

$$b\left[\frac{1}{2}b'' - \left(\frac{a}{b}\right)'\right] = c, \quad c = \text{const.}$$

From (21) and (18) we obtain $g(t) = e^{ct}$, $h(t,x) = e^{ct}\int_{x_0}^{x} \frac{du}{b(u)}$ respectively.

Note that, in general, linear SDE-s are not reducible. For the SDE (14) the reducibility condition becomes

$$\beta(t)b'(t) - [\,\alpha(t)b(t) - a(t)\beta(t) + \beta'(t)\,]\,b(t) \equiv 0,$$

until the homogeneous linear SDE is always reducible.

III. Let us present now a very strict type of reducibility, illustrated by the autonomous SDE. The fact that the linear SDE (14) is solvable motivates us to find an invertible transformation $y = h(x)$, such that the transformed equation be linear with constant coefficients. In other words, we require the existence of the constants $\alpha, \beta, \gamma, \delta$, $\delta \neq 0$, such that the conditions (15) and (16) become

$$a(x)h'(x) + \frac{1}{2}b^2(x)h''(x) \equiv \alpha + \beta h(x), \qquad b(x)h'(x) \equiv \gamma + \delta h(x). \tag{23}$$

If we assume $b \neq 0$, then $h(x)$ is a solution of the linear ordinary differential equation $b(x)h' - \delta h = \gamma$. Thus,

$$h(x) = c\, e^{\delta B(x)} - \gamma/\delta,$$

where $B(x) = \int_{x_0}^{x} \frac{du}{b(u)}$ and x_0 and c are some constants. The substitution of $h(x)$ into (23) gives finally

$$\left\{ \left[\frac{a(x)}{b(x)} - \frac{1}{2}\, b'(x) \right] \delta + \frac{1}{2}\,\delta^2 - \beta \right\} e^{\delta B(x)} \equiv \frac{\alpha\gamma - \beta\delta}{c\gamma}.$$

Replacing $A(x) = \frac{a(x)}{b(x)} - \frac{1}{2}\, b'(x)$ in the last identity and differentiating results, we have

$$\left\{ \left[A(x)\delta + \frac{1}{2}\,\delta^2 - \beta \right] \frac{1}{b(x)} + A'(x) \right\} \delta\, e^{\delta B(x)} \equiv 0.$$

Differentiating again we finally obtain

$$\delta\, A'(x) + (\, b(x)\, A'(x)\,)' \equiv 0.$$

From that

$$A'(x) \equiv 0 \quad \text{or} \quad \left(\frac{(\, b(x)\, A'(x)\,)'}{A'(x)} \right)' \equiv 0 \tag{24}$$

follows. Conversely, if the last condition in (24) is satisfied, then the transformation

$$h(x) = c e^{\delta B(x)}, \quad \text{where} \quad \delta = -\frac{(\, b(x)\, A'(x)\,)'}{A'(x)},$$

reduces the autonomous SDE to the linear form. Also, for $\delta = 0$ the simple choice $h(x) = \gamma B(x) + c$ leads to the reducibility condition

$$(\, b(x)\, A'(x)\,)' \equiv 0.$$

At the end, let us indicate briefly how to apply the foregoing results to find the explicit solution of the autonomous nonlinear SDE

$$dx(t) = \lambda x(t) \Big(1 - \frac{x(t)}{k} \Big)\, dt + \mu x(t)\, dw(t), \quad x(0) = \eta \text{ a.s.}, \quad t \geq 0,$$

where $\lambda,\ k, \mu$ are constants. This equation is reducible in the previous sense, because the condition (24) is valid. It is easy to conclude that $\delta = -\mu,\ h(x) = 1/x$, and from (23) that $\alpha = \lambda/k,\ \beta = -\lambda + \mu^2,\ \gamma = 0$. So, the original SDE is transformed to the linear form

$$dy(t) = \Big[\frac{\lambda}{k} + (-\lambda + \mu^2) y(t) \Big] dt - \mu y(t)\, dw(t), \quad y(0) = \eta^{-1} \text{ a.s.}, \quad t \geq 0.$$

Now it is easy to obtain the explicit solution of the original equation,

$$x(t) = \frac{1}{y(t)} = \frac{\exp\{(\lambda - \mu^2/2)t + \mu w(t)\}}{\eta^{-1} + \frac{\lambda}{k} \int_0^t \exp\{(\lambda - \mu^2/2)s + \mu w(s)\} ds}, \quad t \geq 0.$$

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SUR LES SYSTÈMES EN INVOLUTION DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU PREMIER ORDRE ET D'ORDRE SUPÉRIEUR. L'APPLICATION DES SYSTÈMES DE CHARPIT

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I Chapitre

SUR LA THÉORIE NOUVELLE DES CARACTÉRISTIQUES DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU PREMIER ORDRE

Il s'agit dans ce chapitre de considérer la théorie nouvelle de *R. Courant* des caractéristiques des équations aux dérivées partielles du premier ordre, [1] et de faire la généralisation correspondante sur les systèmes en involution des équations aux dérivées partielles du premier ordre [2]. Nous allons nous servir du système correspondant de *Charpit*, établi dans les recherches de *N. Saltykov*, [3], pour faire la généralisation mentionnée.

1. Sur la théorie des caractéristiques

Considérons l'équation aux dérivées partielles du premier ordre dépendante explicitement de la fonction inconnue z des variables indépendantes $x_1, \ldots, x_n$

(1) $$F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0.$$

Il est bien connue le théorème d'existence et d'unicité de la solution de *Cauchy*, [4]:

Théorème. *Soit $f(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ de la classe C^2 sur un certain ouvert E_{2n+1} de l'espace à $2n+1$ dimensions, dont les coordonnées du point sont les variables: $x(x_1, \ldots, x_n)$, z, $p(p_1, \ldots, p_n)$. Soit $(x^0, z^0, p^0) \in E_{2n+1}$. Soit*

(2) $$x = a(t), \quad t = (t_1, \ldots, t_{n-1})$$

une pièce de la hypersurface de la classe C^2 définie pour t voisin de

$$t^0\left(t_1^0, \ldots, t_{n-1}^0\right) \text{ et } a(t^0) = x^0.$$

Soit $b(t)$ une fonction de la variable t de la classe C^2 voisine de t^0 et $b(t^0) = z^0$. Aussi, les conditions suivantes

(3) $$F(x^0, z^0, p^0) = 0,$$

(4) $$p^0 \cdot \frac{\partial a(t^0)}{\partial t_i} = \frac{\partial b(t^0)}{\partial t_i}, \quad (i = 1, \ldots, n-1),$$

(5) $$\det\left[\frac{\partial a(t^0)}{\partial t_i},\ F_p(x^0, z^0, p^0)\right]\neq 0$$

soient satisfaites. Cela étant, dans le voisinage E_n de $x=x^0$ il existe une solution unique: $z=z(x)$ de la classe C^2 du problème de Cauchy (1—2) *et* $z[a(t)]=b(t)$.

C'est la méthode de *Cauchy* des caractéristiques que réduit ce problème à la théorie des équations différentielles ordinaires, [1], [4].

R. Courant, [1], a utilisé la théorie des systèmes de *Charpit* pour établir une méthode nouvelle des caractéristiques pour l'équation (1).

Pour cela il a associé à l'équation (1) le système suivant de *Charpit*

(6) $$\begin{gathered}\sum_{s=1}^{n}\frac{\partial F}{\partial p_s}\frac{\partial p_i}{\partial x_s}+\frac{\partial F}{\partial z}p_i+\frac{\partial F}{\partial x_i}=0, \quad (i=1, \ldots, n)\\ \sum_{s=1}^{n}\frac{\partial F}{\partial p_s}\frac{\partial z}{\partial x_s}-\sum_{s=1}^{n}\frac{\partial F}{\partial p_s}p_s=0\end{gathered}$$

et de même aussi, grâce au système (6), le système équivalent des caractéristiques

(7) $$\frac{dx_i}{d\tau}=\frac{\partial F}{\partial p_i},\quad \frac{dp_i}{d\tau}=-\frac{\partial F}{\partial x_i}-p_i\frac{\partial F}{\partial z},\quad \frac{dz}{d\tau}=\sum_{s=1}^{n}p_s\frac{\partial F}{\partial p_s} \qquad (i=1, \ldots, n)$$

et il a fait la conclusion suivante:

Un problème initial convenablement choisi pour le système de *Charpit* (6) est équivalent au problème correspondant de *Cauchy* de l'équation (1). Cette conclusion offre une base nouvelle pour résoudre le problème de *Cauchy* relatif à l'équation (1) à l'aide des équations des caractéristiques (7).

Posons alors pour le système (6) le problème suivant:

Déterminer les fonctions z, p_i pour que les conditions suivantes

(8) $$F=0, \quad dz=\sum_{s=1}^{n}p_s\,dx_s$$

soient toujours satisfaites sur une multiplicité non caractéristique M_{n-1} donnée, ou:

Déterminer la surface

(9) $$z=z(x)$$

que contient la multiplicité M_{n-1} donnée et vérifié sur M_{n-1} les conditions (8).

Notre but est de démontrer que la surface (9) est aussi de même une intégrale correspondante de *Cauchy* de l'équation (1).

Il ne s'agit donc à présent que de vérifier sur la surface (9) les identités suivantes

(10) $$F(x_1, \ldots, x_n, z, p_1, \ldots, p_n)\equiv f(x_1, \ldots, x_n)\equiv 0$$

(11) $$p(x_1, \ldots, x_n)-\frac{\partial z(x_1, \ldots, x_n)}{\partial x_i}\equiv P_i(x_1, \ldots, x_n)\equiv 0, \quad (i=1, \ldots, n).$$

On a immédiatement, grâce à la dernière des équations (6) les relations suivantes

$$\sum_{s=1}^{n} \frac{\partial F}{\partial p_s} P_s = 0. \tag{6'}$$

et

$$\sum_{s=1}^{n} \frac{\partial F}{\partial p_s} \frac{\partial P_s}{\partial x_i} + \sum_{s=1}^{n} A_{si} P_s = 0, \tag{6''}$$

où nous avons désigné par A_{si} les dérivées suivantes: $A_{si} \equiv \partial^2 F / \partial p_s \partial x_i$.

Il est aisé de mettre la dérivée de la fonction f

$$\frac{\partial f}{\partial x_i} = \sum_{s=1}^{n} \frac{\partial F}{\partial p_s} \frac{\partial p_s}{\partial x_i} + \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x_i},$$

en vertu des équations (6) et des relations suivantes

$$\frac{\partial p_i}{\partial x_s} - \frac{\partial p_s}{\partial x_i} = \frac{\partial P_i}{\partial x_s} - \frac{\partial P_s}{\partial x_i},$$

sous la forme suivante

$$\frac{\partial f}{\partial x_i} = \sum_{s=1}^{n} \frac{\partial F}{\partial p_s} \left(\frac{\partial P_s}{\partial x_i} - \frac{\partial P_i}{\partial x_s} \right) - \frac{\partial F}{\partial z} P_i. \tag{12}$$

Multipliant les deux membres des égalités écrites respectivement par les dérivées $\partial F / \partial p_i$, sommons les résultats obtenus; il en résulte

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial F}{\partial p_i} = 0 \tag{13}$$

et en y substituant les valeurs des dérivées $\partial F / \partial p_i$ tirées des équations des caractéristiques (7), on a $df/d\tau = 0$. Grâce à la proposition que la première condition de (8) est satisfaite sur la multiplicité donnée M_{n-1}, on peut conclure qu'il y a lieu la condition (10), c'est-à-dire $f \equiv 0$.

Quant à la vérification des conditions (11), on peut d'abord partir des relations (12) écrites sous la forme nouvelles

$$\sum_{s=1}^{n} \frac{\partial F}{\partial p_s} \left(\frac{\partial P_i}{\partial x_s} - \frac{\partial P_s}{\partial x_i} \right) + \frac{\partial F}{\partial z} P_i = 0, \tag{12'}$$

ou, en vertu des équations (6″) et (7), sous la forme évidente

$$\frac{dP_i}{d\tau} + \sum_{s=1}^{n} A_{si}^{*} P_s = 0, \qquad (i = 1, 2, \ldots, n), \tag{14}$$

A_{si}^{*} désignant les coefficients connus. Grâce au système linéaire et homogène (14) et à la deuxième des conditions (8) sur M_{n-1} on a définitivement $P_i \equiv 0$.

2. Sur une variante dans la théorie des caractéristiques du système des équations aux dérivées partielles du premier ordre en involution

Considérons un système des m équations aux dérivées partielles du premier ordre d'une fonction z inconnue, contenant explicitement cette dernière:

$$F_k(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0, \quad (k = 1, 2, \ldots, m < n) \tag{15}$$

en involution

$$[F_k, F_j] \equiv \sum_i \left(\frac{\partial F_k}{\partial p_i} \frac{dF_j}{dx_i} - \frac{\partial F_j}{\partial p_i} \frac{dF_k}{dx_i} \right) = 0, \quad (j, k = 1, \ldots, m) \tag{16}$$

et le déterminant fonctionnel

$$\Delta \equiv \mathcal{D}\left(\frac{F_1, \ldots, F_m}{p_1, \ldots, p_m} \right) \neq 0 \tag{17}$$

ne s'annulant pas.

Il y a plusieurs des méthodes différentes de démontrer l'existence et l'unicité de la solution de *Cauchy* du système (15).

E. Goursat, [5], avait donné sa démonstration en utilisant les transformations connues de *Mayer*. Sur la méthode employée, dans un cas spécial, était fait un exemple contraire par *L. Bieberbach*, [6]. Une autre démonstration de *Goursat*, [5], est fondée à la conclusion de $(m-1)$ à m équations. Suivant une opinion de *E. Kamke*, [7], la démonstration dernière est plus compliquée et aussi n'est pas convenable à estimer le domain d'existence de la solution.

C. Carathéodory, [8], avait donné la démonstration de son théorème en employant pour le système normal en involution

$$p_k = f_k(x_1, \ldots, x_n, z, p_{m+1}, \ldots, p_n), \quad (k = 1, \ldots, m) \tag{15'}$$

les idées et les définitions antérieures de la théorie des caractéristiques de *Cauchy* et le système correspondant d'équations aux différentielles totales des caractéristiques.

Dans l'article mentionné *E. Kamke*: 1) avait donné — dans une forme explicite — le domain d'existence de la solution, 2) avait proposé que les fonctions données f_k possédaient les dérivées des ordres supérieurs, 3) avait établi la dépendance de la solution considérée aux paramètres donnés, et 4) il avait donné sa démonstration en utilisant la méthode de la réduction à une (single) équation correspondante.

P. Hartman, [4], en exposant la méthode de *Cauchy* des caractéristiques (dans le cas $m = 1$ d'après la méthode citée le problème de *Cauchy* pour l'équation (1) est réduit à la théorie des équations différentielles ordinaires) avait noté qu'il n'y a analogie aucune de cette méthode pour le cas du système (15).

N. Saltykow avait étudié plus antérieurement la théorie des caractéristiques en utilisant les systèmes correspondants de *Charpit*, [3], [9]. Il avait démontré que par le système linéaire

$$[F_k, f] = 0, \quad (k = 1, 2, \ldots, m)$$

étaient définies les équations aux dérivées partielles des caractéristiques sous la forme d'un système suivant de *Charpit*

(CH)
$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial x_{m+j}}{\partial x_r} = \frac{\partial F_k}{\partial p_{m+j}}, \qquad (j=1, \ldots, n-m)$$
$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial p_s}{\partial x_r} = -\frac{dF_k}{dx_s}, \qquad (s=1, \ldots, n)$$
$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial z}{\partial x_r} = \sum_{i=1}^{n} p_i \frac{\partial F_k}{\partial p_i}, \qquad (k=1, \ldots, m).$$

Notre but dans ce paragraphe est d'utiliser une méthode que réduit le problème de *Cauchy* du système (15) au problème initial correspondant du système de *Charpit* (CH), [2].

Considérons le système (15) en involution des équations aux dérivées partielles du premier ordre d'une fonction réelle $z=z(x)$ des n variables réelles indépendantes $x=(x_1, \ldots, x_n)$, $F_k(x, z, p)$, $p=(p_1 \equiv \partial z/\partial x_1, \ldots, p_n \equiv \partial z/\partial x_n)$ désignant m fonctions réelles des $n+1+n$ variables de la classe C^2, $F_k \in C^2(E_{2n+1})$ sur un ensemble ouvert E_{2n+1}. Soit $(x^0, z^0, p^0) \in E_{2n+1}$ et $x^0=(x_1^0, \ldots, x_n^0)$, $p^0=(p_1^0, \ldots, p_n^0)$. Une solution de (15) est une fonction $z=z(x)$ de la classe C^1 sur un x-ensemble E_n tel que $[x, z(x), z_x(x)] \in E_{2n+1}$ pour $x \in E_n$ et (15) devient les identités par rapport à x, c'est-à-dire: $F_k[x, z(x), z_x(x)] \equiv 0$, $x \in E_n$.

Une solution de *Cauchy* est une fonction $z(x)$ que devient une fonction donnée

$$z=\zeta(x_{m+1}, \ldots, x_n) \text{ pour } x_i=x_i^0, \qquad (i=1, \ldots, m)$$

ζ étant de la classe C^2 sur un ensemble ouvert E_{n-m} dans un voisinage des $x_{m+1}^0, \ldots, x_n^0$, ou dans la forme paramétrique correspondante

(18)
$$\begin{cases} x_i=x_i^0, \quad (i=1, \ldots, m), \; x_{m+j}=t_j, \; (j=1, \ldots, n-m), \\ z=\zeta(t), \quad t=(t_1, \ldots, t_{n-m}) \end{cases}$$

où $\zeta(t)$ est de la classe C^2 dans un voisinage de $t=t^0$, $t^0=(t_1^0, \ldots, t_{n-m}^0)$ et $\zeta(t^0)=z^0$.

La solution du problème (15)—(18) ne peut exister qu'il y a une fonction $p(t)$ dans un voisinage de $t=t^0$ et $p^0=p(t^0)$.

Notons que la relation suivante

$$z(\xi^0, t)=\zeta(t),$$

où l'on a posé $\xi=(x_1, \ldots, x_m)$, $\xi^0=(x_1^0, \ldots, x_m^0)$, définit les composantes suivantes

$$p_{m+s}(t)=\frac{\partial \zeta(t)}{\partial t_s}, \qquad (s=1, \ldots, n-m)$$

que sont de la classe C^1 dans un voisinage de t^0.

Si la condition (17) est remplie dans un voisinage de $(x^0, z^0, p^0) \in E_{2n+1}$, les relations

$$F_k\left(\xi^0, t, \zeta(t), p_1, \ldots, p_m, \frac{\partial \zeta}{\partial t_1}, \ldots, \frac{\partial \zeta}{\partial t_{n-m}}\right) = 0 \qquad (k = 1, \ldots, m)$$

déterminent les autres composantes $p_i(t)$, $(i = 1, \ldots, m)$ comme les fonctions uniformes dans un voisinage de $t = t^0$.

De cette manière nous avons les fonctions initiales

$$\xi = \xi^0, \; x_{m+j} = t_j, \; (j = 1, \ldots, n-m), \; z = \zeta(t), \; p = p(t) \tag{19}$$

où $\zeta(t)$ est de la classe C^2 et $p(t)$ est de la classe C^1 dans un voisinage de $t = t^0$.

Supposons que l'intégrale de *Cauchy* du système de *Charpit* (CH)

$$\begin{aligned} x_{m+j} &= \varphi_j(\xi, t), \qquad (j = 1, \ldots, n-m) \\ z &= \psi(\xi, t) \\ p_i &= \pi_i(\xi, t), \qquad (i = 1, \ldots, n) \end{aligned} \tag{20}$$

vérifie les conditions initiales (19), c'est-à-dire

$$\begin{aligned} x_{m+j} &= \varphi_j(\xi^0, t) \equiv t_j, \qquad (j = 1, \ldots, n-m) \\ z &= \psi(\xi^0, t) \equiv \zeta(t) \\ p_i &= \pi_i(\xi^0, t) \equiv p_i(t), \qquad (i = 1, \ldots, n). \end{aligned} \tag{21}$$

Grâce à la proposition faite que F_k soient de la classe C^2 dans un voisinage de (x^0, z^0, p^0), l'intégrale de *Cauchy* (20) est unique et les fonctions φ_j, ψ, π_i sont de la classe C^1 et les dérivées du seconde ordre existent avec les propriétés

$$\frac{\partial}{\partial x_k}\left(\frac{\partial \varphi_j}{\partial t_i}\right) = \frac{\partial}{\partial t_i}\left(\frac{\partial \varphi_j}{\partial x_k}\right), \quad \frac{\partial}{\partial x_k}\left(\frac{\partial \psi}{\partial t_i}\right) = \frac{\partial}{\partial t_i}\left(\frac{\partial \psi}{\partial x_k}\right)$$

dans un voisinage de (ξ^0, t^0).

Grâce aux propriétés d'intégrale générale du système (CH), [2], on a les relations suivantes

$$F_k[\xi, \varphi(\xi, t), \psi(\xi, t), \pi(\xi, t)] = 0 \tag{1'}$$

remplies pour (ξ, t) voisin à (ξ^0, t^0), avec $\varphi = (\varphi_1, \ldots, \varphi_{n-m})$, $\pi = (\pi_1, \ldots, \pi_n)$.

Les équations différentielles des caractéristiques (CH) sont identiquement satisfaites par les valeurs (20), et les identités provenant des équations de deux premières lignes (20) nous donnent les identités suivantes

$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r}\left(\frac{\partial \psi}{\partial x_r} - \pi_r - \sum_{j=1}^{n-m} \pi_{m+j} \frac{\partial \varphi_j}{\partial x_r}\right) = 0. \tag{22}$$

Le déterminant (17) étant distinct de zéro, les égalités (22) nous donnent

$$(23)\qquad \frac{\partial \psi(\xi, t)}{\partial x_r} \equiv \pi_r(\xi, t) + \sum_{j=1}^{n-m} \pi_{m+j}(\xi, t) \frac{\partial \varphi_j(\xi, t)}{\partial x_r}, \qquad (r = 1, \ldots, m).$$

En vertu des équations de la première ligne de (21) on a

$$\left. \frac{\partial(\varphi_1, \ldots, \varphi_{n-m})}{\partial(t_1, \ldots, t_{n-m})} \right|_{\xi=\xi^0} \neq 0$$

et grâce à la continuité il s'ensuit

$$(24)\qquad \frac{\partial(\varphi_1, \ldots, \varphi_{n-m})}{\partial(t_1, \ldots, t_{n-m})} \neq 0$$

pour (x, t) voisine à (x^0, t^0). De cette manière on a une transformation unique

$$t_s = \tau_s(x_1, \ldots, x_n), \quad (s = 1, \ldots, n-m)$$

ou

$$(25)\qquad t = \tau(x), \quad \tau = (\tau_1, \ldots, \tau_{n-m}).$$

$\tau(x)$ étant de la classe C^1 dans un voisinage de $x^0 \in E_n$. Donc, dans ce voisinage on a aussi

$$(25_1)\qquad x_{m+j} \equiv \varphi_j(\xi, \tau), \quad (j = 1, \ldots, n-m)$$

et

$$(26)\qquad t_s \equiv \tau_s(\xi^0, t), \quad (s = 1, \ldots, n-m).$$

Supposons que l'élimination des variables t_s de (20) donne les relations nouvelles

$$(27)\qquad z = a(x)$$

$$(28)\qquad p = b(x), \quad (b_1, \ldots, b_n) = b$$

pour x voisine à x^0, où l'on a posé

$$(29)\qquad a(x) = \psi[\xi, \tau(x)],$$

$$(30)\qquad b(x) = \pi[\xi, \tau(x)].$$

Les fonctions $a(x)$ et $b(x)$ sont de la classe C^1 dans un voisinage de x^0.

Il est facile de vérifier

$$(29_1)\qquad A \equiv a(\xi, \varphi) = \psi(\xi, t),$$

$$(30_1)\qquad B_i \equiv b_i(\xi, \varphi) \equiv \pi_i(\xi, t).$$

Il faut maintenant démontrer dans des voisinages de x^0 et t^0 l'existence des relations suivantes

$$(31)\qquad \text{a)}\quad F_k[x, a(x), b(x)] = 0,$$

$$(32)\qquad \text{b)}\quad a(\xi^0, t) = \zeta(t),$$

$$(33)\qquad \text{c)}\quad \partial a/\partial x_i = b_i(x), \quad (i = 1, \ldots, n).$$

Démonstration. ad a) Les formules (1′) ont lieu pour chaque (ξ, t) voisine à (ξ^0, t^0), on a donc pour $t=\tau(x)$, c'est-à-dire

$$F_k[\xi, \varphi(\xi, \tau), \psi(\xi, \tau), \pi(\xi, \tau)]=0$$

et les dernières relations, grâce aux (25_1), (29) et (30), nous donnent (31) dans un voisinage de x^0.

ad b) Considérons (29) pour $\xi=\xi^0$ et $x_{m+j}=t_j$, les relations (29), (26), (21) nous donnent les relations requises (32)

$$a(\xi^0, t)=\psi[\xi^0, \tau(\xi^0, t)]=\psi(\xi^0, t)=\zeta(t)$$

qui sont satisfaites dans un voisinage de t^0.

ad c) Il est facile de voire que l'existence des relations (33) peut être réduite à l'existence des relations suivantes

$$U_s(\xi, t)\equiv\frac{\partial\psi}{\partial t_s}-\sum_{j=1}^{n-m}\pi_{m+j}\frac{\partial\varphi_j}{\partial t_s}=0, \quad (s=1, \dots, n-m) \tag{34}$$

dans un voisinage de (ξ^0, t^0).

c_1) Montrons que si l'on a

$$b_{m+j}=\frac{\partial a}{\partial x_{m+j}} \quad (j=1, \dots, n-m) \tag{33_1}$$

pour x voisin à x^0, alors

$$b_r=\frac{\partial a}{\partial x_r}, \quad (r, \dots, m). \tag{33_2}$$

Calculons, en effet, les dérivées partielles des relations (29_1) par rapport à x_r

$$\frac{\partial\psi}{\partial x_r}=\frac{\partial A}{\partial x_r}+\sum_{j=1}^{n-m}\frac{\partial A}{\partial x_{m+j}}\frac{\partial\varphi_j}{\partial x_r}.$$

et grâce aux (23) on a

$$\pi_r-\frac{\partial A}{\partial x_r}+\sum_{j=1}^{n-m}\left(\pi_{m+j}-\frac{\partial A}{\partial x_{m+r}}\right)\frac{\partial\varphi_j}{\partial x_r}=0$$

ou, pour $t=\tau(x)$

$$b_r-\frac{\partial a}{\partial x_r}+\sum_{j=1}^{n-m}\left[\frac{\partial\varphi_j}{\partial x_r}\right]\left(b_{m+j}-\frac{\partial a}{\partial x_{m+j}}\right)=0, \quad (r=1, \dots, m). \tag{35}$$

Donc, si l'on a (33_1), les dernières identités nous donnent (33_2). Il ne s'agit donc que d'être satisfaites les relations (33_1).

c_2) Montrons à présent si les relations suivantes

$$\frac{\partial\psi}{\partial t_s}-\sum_{j=1}^{n-m}\pi_{m+j}\frac{\partial\varphi_j}{\partial t_s}=0 \tag{36}$$

sont satisfaites, alors (33_1) ont lieu.

Grâce aux dérivées partielles des relations (29_1) par rapport à t_s

$$\frac{\partial \psi}{\partial t_s} - \sum_{j=1}^{n-m} \frac{\partial A}{\partial x_{m+j}} \frac{\partial \varphi_j}{\partial t_s} = 0$$

et aux (36), on a

$$\sum_{j=1}^{n-m} \frac{\partial \varphi_j}{\partial t_s} \left(\pi_{m+j} - \frac{\partial A}{\partial x_{m+j}} \right) = 0, \quad (s = 1, \ldots, n-m)$$

ou, pour $t = \tau(x)$

$$\sum_{j=1}^{n-m} \left[\frac{\partial \varphi_j}{\partial t_s} \right] \left(b_{m+j} - \frac{\partial a}{\partial x_{m+j}} \right) = 0.$$

Donc, en utilisant (24) on a (33_1). Il suffit, donc, à présent de satisfaire: $U_s(\xi, t) = 0$ pour (ξ, t) dans un voisinage de (ξ^0, t^0).

c_3) Il nous reste de démontrer à présent qu'il est toujours possible de satisfaire aux conditions (36).

Calculons les dérivées partielles des fonctions U_s par rapport à x_r

$$\frac{\partial U_s}{\partial x_r} = \frac{\partial^2 \psi}{\partial t_s \partial x_r} - \sum_{j=1}^{n-m} \left(\frac{\partial \pi_{m+j}}{\partial x_r} \frac{\partial \varphi_j}{\partial t_s} + \pi_{m+j} \frac{\partial^2 \varphi_j}{\partial t_s \partial x_r} \right)$$

et les dérivées des relations (23) par rapport à t_s

$$\frac{\partial^2 \psi}{\partial x_r \partial t_s} = \frac{\partial \pi_r}{\partial t_s} + \sum_{j=1}^{n-m} \left(\frac{\partial \pi_{m+j}}{\partial t_s} \frac{\partial \varphi_j}{\partial x_r} + \pi_{m+j} \frac{\partial^2 \varphi_j}{\partial x_r \partial t_s} \right),$$

on a

$$\frac{\partial U_s}{\partial x_r} = \frac{\partial \pi_r}{\partial t_s} + \sum_{j=1}^{n-m} \left(\frac{\partial \pi_{m+j}}{\partial t_s} \frac{\partial \varphi_j}{\partial x_r} - \frac{\partial \pi_{m+j}}{\partial x_r} \frac{\partial \varphi_j}{\partial t_s} \right).$$

Grâce au système (CH) on a aussi les relations évidentes

$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial \varphi_j}{\partial x_r} = \frac{\partial F_k}{\partial p_{m+j}}, \qquad (j = 1, \ldots, n-m)$$

$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial \pi_i}{\partial x_r} = -\frac{dF_k}{dx_i}, \qquad (i = 1, \ldots, n)$$

et on peut former la somme suivante

$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial U_s}{\partial x_r} = \sum_{i=1}^{n} \frac{\partial F_k}{\partial p_i} \frac{\partial \pi_i}{\partial t_s} + \sum_{j=1}^{n-m} \frac{dF_k}{dx_{m+j}} \frac{\partial \varphi_j}{\partial t_s}.$$

En utilisant la somme dernière et aussi les dérivées partielles par rapport aux t_s des relations (1') on obtient

$$\sum_{r=1}^{m} \frac{\partial F_k}{\partial p_r} \frac{\partial U_s}{\partial x_r} + U_s \frac{\partial F_k}{\partial z} = 0 \qquad \begin{matrix} (k = 1, \ldots, m) \\ (s = 1, \ldots, n-m). \end{matrix} \tag{37}$$

C'est un système linéaire et homogène des fonctions U_s. Les valeures initiales des ces dernières fonctions sont

$$U_s(\xi^0, t) \equiv \frac{\partial \zeta}{\partial t_s} - \sum_{j=1}^{n-m} p_{m+j}(t) \frac{\partial t_j}{\partial t_s} = \frac{\partial \zeta}{\partial t_s} - p_{m+s}(t) \equiv 0$$

pour t voisin à t^0. Donc, les relations (36) ont lieu dans un voisinage de (ξ^0, t^0).

Par conséquent, l'idée de la démonstration, en utilisant le système (CH), était

$$U_s(\xi^0, t) = 0 \Rightarrow U_s(\xi, t) = 0 \Rightarrow b_{m+j} = \frac{\partial a}{\partial x_{m+j}} \Rightarrow b_i = \frac{\partial a}{\partial x_i} \quad \begin{matrix} (j = 1, \dots, n-m) \\ i = 1, \dots, n). \end{matrix}$$

Théorème d'existence et d'unicité. Soient $F_k(x, z, p)$ de la classe C^2 sur un domaine ouvert E_{2n+1}. Soit $(x^0, z^0, p^0) \in E_{2n+1}$ et $F_k(x^0, z^0, p^0) = 0$. Soit $\zeta(t)$, $t = (t_1 = x_{m+1}, \dots, t_{n-m} = x_n)$ de la classe C^2 pour t voisin à t^0 et $z^0 = \zeta(t^0)$. Enfin, si les conditions (16) et (17) sont vérifiées, il existe dans un voisinage E_n de x^0 une solution unique $a(x)$ de la classe C^2 et vérifiant les conditions initiales

$$x_i = x_i^0, \quad (i = 1, \dots, m), \qquad z = \zeta(x_{m+1}, \dots, x_n),$$

et le système (15).

II Chapitre

SUR LE SYSTÈME EN INVOLUTION DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU SECOND ORDRE

Les systèmes des équations aux dérivées partielles du second ordre *en involution de Darboux-Lie ou en involution de l'intégrabitité complète* admettent d'établir plusieurs propriétés qui sont analogues aux propriétés de la théorie des équations aux dérivées partielles du premier ordre.

1. Sur l'involution de Darboux-Lie

Considérons l'équation aux dérivées partielles du second ordre

(1) $$A(x, y, z, p, q, r, s, t)=0$$

d'une fonction réelle z des deux variables indépendantes en utilisant les désignations habituelles

$$p=\frac{\partial z}{\partial x}, \quad q=\frac{\partial z}{\partial y}, \quad r=\frac{\partial^2 z}{\partial x^2}, \quad s=\frac{\partial^2 z}{\partial x \partial y}, \quad t=\frac{\partial^2 z}{\partial y^2}.$$

Introduisons une équation auxiliaire en l'écrivant sous la forme

(2) $$B(x, y, z, p, q, r, s, t)=0$$

de sorte que l'on ait

(3) $$\mathcal{D}\left(\frac{A, B}{r, t}\right)\neq 0.$$

Considérons les variables

(4) $$z, p, q, r, s, t$$

comme les fonctions des deux variables indépendantes x et y, liées par les relations

(4') $$\begin{aligned} dz &= p\,dx + q\,dy, \\ dp &= r\,dx + s\,dy, \\ dq &= s\,dx + t\,dy. \end{aligned}$$

Formons les équations dérivées du premier ordre du système (1), (2) respectivement par rapport aux variables indépendantes x et y

$$
\begin{aligned}
&A_r\,\alpha + A_s\,\beta + A_t\,\gamma + D_x A = 0,\\
(5)\qquad &A_r\,\beta + A_s\,\gamma + A_t\,\delta + D_y A = 0,\\
&B_r\,\alpha + B_s\,\beta + B_t\,\gamma + D_x B = 0,\\
&B_t\,\beta + B_s\,\gamma + B_t\,\delta + D_y B = 0,
\end{aligned}
$$

α, β, γ, δ désignant les dérivées partielles du troisième ordre de la fonction z

$$\alpha = \frac{\partial^3 z}{\partial x^3}, \quad \beta = \frac{\partial^3 z}{\partial x^2 \partial y}, \quad \gamma = \frac{\partial^3 z}{\partial x\, \partial y^2}, \quad \delta = \frac{\partial^3 z}{\partial y^3}$$

et en posant

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q},$$

$$D_y = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q}.$$

Éliminant de la première et de la troisième équation (5) respectivement d'abord γ et ensuite α, on obtient les deux équations suivantes

$$(6)\qquad J_1\,\alpha + J_2\,\beta + J_4 = 0, \qquad J_3\,\beta - J_1\,\gamma + J_5 = 0.$$

On obtient les autres deux équations analogues, en éliminant de la seconde et de la quatrième équation (5) d'abord δ et ensuit β

$$(7)\qquad J_1\,\beta + J_2\,\gamma + J_6 = 0 \qquad J_3\,\gamma - J_1\,\delta + J_7 = 0,$$

$J_1, \ldots, J_7$ désignant les déterminants fonctionels suivants

$$J_1 \equiv \mathcal{D}\left(\frac{A, B}{r, t}\right), \quad J_2 \equiv \mathcal{D}\left(\frac{A, B}{s, t}\right), \quad J_3 \equiv \mathcal{D}\left(\frac{A, B}{s, r}\right), \quad J_4 \equiv \mathcal{D}\left(\frac{A, B}{x, t}\right),$$

$$J_5 \equiv \mathcal{D}\left(\frac{A, B}{x, r}\right), \quad J_6 \equiv \mathcal{D}\left(\frac{A, B}{y, t}\right), \quad J_7 \equiv \mathcal{D}\left(\frac{A, B}{y, r}\right),$$

où les dérivées par rapport aux variables x et y sont prises totalement.

En vertu de la définition bien connue de l'involution de *Darboux-Lie*, les équations (6) et (7) ne doivent point être résolubles par rapport aux dérivées du troisième ordre. Il s'ensuit que la seconde équation de (6) et la première équation de (7) se confondent.

On en tire les conditions d'involution de *Darboux-Lie* sous la forme suivante

$$(8)\qquad \frac{J_3}{J_1} = \frac{-J_1}{J_2} = \frac{J_5}{J_6}.$$

Les systèmes des quatre équations (6) et (7), par conséquent revient au système des trois équations suivantes

$$J_1 \frac{\partial r}{\partial x} + J_2 \frac{\partial r}{\partial y} + J_4 = 0,$$

(9)
$$J_1 \frac{\partial s}{\partial x} + J_2 \frac{\partial s}{\partial y} + J_6 = 0,$$

$$J_1 \frac{\partial t}{\partial x} + J_2 \frac{\partial t}{\partial y} + J_6 J_7 : J_5 = 0.$$

Complétons ces dernières équations par les égalités suivantes

$$J_1 \frac{\partial z}{\partial x} + J_2 \frac{\partial z}{\partial y} - J_1 p - J_2 q = 0,$$

(10)
$$J_1 \frac{\partial p}{\partial x} + J_2 \frac{\partial p}{\partial y} - J_1 r - J_2 s = 0,$$

$$J_1 \frac{\partial q}{\partial x} + J_2 \frac{\partial q}{\partial y} - J_1 s - J_2 t = 0.$$

L'ensemble obtenu d'équations (9) et (10) représente un système du type de *Charpit*.

A ce système de *Charpit* on peut associer le système suivant d'équations différentielles ordinaires des caracté istiques

$$(CH) \qquad \frac{dx}{J_1} = \frac{dy}{J_2} = \frac{dz}{J_1 p + J_2 q} = \frac{dp}{J_1 r + J_2 s} = \frac{dq}{J_1 s + J t_2} = \frac{dr}{-J_4} = \frac{ds}{-J_6} = \frac{dt}{-J_6 J_7 : J_5}.$$

2. L'application des conditions d'involution de Darboux-Lie

On peut appliquer les conditions (8) d'involution de *Darboux-Lie* pour chercher une fonction $B(x, y, z, p, r, s, t)$ que soit en involution de *Darboux-Lie* avec la fonction donnée $A(x, y, z, p, q, r, s, t)$, [10].

Les conditions (8) peuvent être écrites de la manière suivante

(11) $$J_1^2 + J_2 J_3 = 0,$$

(12) $$J_1 J_6 + J_2 J_5 = 0.$$

Grâce aux valeurs citées antéreurement pour J_i on peut la condition (11) mettre sous la forme

(11') $$(A_r B_t - A_t B_r)^2 + (A_s B_t - A_s B_s)(A_s B_r - A_r B_s) = 0.$$

En vertu de l'hypothèse $A_r \neq 0$ et des notations suivantes

$$A_s : A_r = m, \qquad A_t : A_r = n$$

on obtient

$$(B_t - B_r n)^2 + (m B_t - n B_s)(m B_r - B_s) = 0$$

ou

$$2 B_t = m B_s + (2n - m^2) B_r + R$$

où l'on a posé

$$R = \pm (m^2 - 4n)^{1/2} (B_s - m B_r).$$

En introduisant la désignation k_{12} pour la racine de l'équation algébrique du second ordre

$$A_r k^2 - A_s k + A_t = 0$$

on peut la condition (11) étudiée mettre sous la forme

(11″) $$B_t - k_{12} B_s + k_{12}^2 B_r = 0.$$

Quant à la condition (12), en nous servant des désignations pour les valeurs m et n et posant

$$\frac{D_x A}{A_r} \equiv \mu, \quad \frac{D_y A}{A_r} \equiv \nu$$

on mettra la condition mentionnée sous la forme cherchée (en vertu de B_s, tirée de (11″))

$$2 \nu B_t = N + n \nu B_r \pm (N - n \nu B_r), \quad N \equiv k_{12} D_x B + n D_y B - \mu k_{12} B_r.$$

On prendre de deux formules celle qui correspond au signe supérieur. Il en résulte la seconde condition cherchée

(12′) $$k_{12} D_x B + n D_y B - \mu k_{12} B_r - \nu B_t = 0.$$

De cette manière on vient d'obtenir deux équations (11″) (12′) qui sont linéaires par rapport aux dérivées partielles du premier ordre de la fonction B. Les formules obtenues, [10], seront utiles pour chercher une fonction B qui soit en involution de *Darboux-Lie* avec la fonction donnée: A.

3. Sur l'involution de l'intégrabilité complète et la méthode de N. Saltykow

Considérons à présent les équations aux dérivées partielles du second ordre que l'on écrira sous la forme

(13) $$r + a(x, y, z, p, q, s, t) = 0$$

Introduisant une équation auxiliaire en l'écrivant sous la forme

(14) $$t + b(x, y, z, p, q, s) = 0.$$

Différentiant les équations (13) et (14) respectivement par rapport aux variables x et y on obtient les équations suivantes

$$(15)\qquad \begin{cases} \dfrac{\partial r}{\partial x}+a_s\dfrac{\partial s}{\partial x}+a_t\dfrac{\partial t}{\partial x}+D_x a=0,\\ \dfrac{\partial r}{\partial y}+a_s\dfrac{\partial s}{\partial y}+a_t\dfrac{\partial t}{\partial y}+D_y a=0,\\ \dfrac{\partial t}{\partial x}+b_s\dfrac{\partial s}{\partial x}+D_x b=0,\\ \dfrac{\partial t}{\partial y}+b_s\dfrac{\partial s}{\partial y}+D_y b=0. \end{cases}$$

En utilisant que les valeurs des quantités

$$(16)\qquad z(x,y),\quad p(x,y),\quad q(x,y),\quad r(x,y),\quad s(x,y),\quad t(x,y)$$

vérifient les conditions

$$(17)\qquad \frac{\partial r}{\partial y}=\frac{\partial s}{\partial x},\qquad \frac{\partial s}{\partial y}=\frac{\partial t}{\partial x}$$

on peut les relations (15) transformer et mettre sous la forme nouvelle

$$(18)\qquad \begin{cases} \dfrac{\partial r}{\partial x}+(a_s-a_t b_s)\dfrac{\partial r}{\partial y}-a_t D_x b+D_x a=0,\\ \dfrac{\partial s}{\partial x}+(a_s-a_t b_s)\dfrac{\partial s}{\partial y}-a_t D_y b+D_y a=0,\\ \dfrac{\partial s}{\partial y}+b_s\dfrac{\partial s}{\partial x}+D_x b=0,\\ \dfrac{\partial t}{\partial y}+b_s\dfrac{\partial t}{\partial x}+D_y b=0. \end{cases}$$

La seconde et la troisième équation (18) donnent les dérivées de la fonction s

$$(19)\qquad \frac{\partial s}{\partial x}=-\frac{\Delta_1}{\Delta},\qquad \frac{\partial s}{\partial y}=-\frac{\Delta_2}{\Delta}$$

où l'on vient de poser

$$(20)\qquad \Delta\equiv\begin{vmatrix} b_s & 1\\ 1 & a_s-a_t b_s \end{vmatrix}.$$

Δ_1 et Δ_2 représentent ce qui devient le déterminant Δ en y remplaçant respectivement les élements des colonnes par les termes des equations considérées indépendantes des dérivées de la fonction s.

La condition d'involution d'intégrabitité complète du système (13) et (14) est exprimée par la condition nécessaire de la compatibilité des équations (19) sous la forme suivante

$$\frac{d}{dy}\left(\frac{\Delta_1}{\Delta}\right)=\frac{d}{dx}\left(\frac{\Delta_2}{\Delta}\right). \tag{21}$$

La condition (21) est, par rapport à la fonction inconnue b, linéaire aux dérivées partielles du second ordre par rapport aux variables x, y, z, p, q, s.

Dans le cas du système des équations (13) et (14) les conditions de l'involution de *Darboux-Lie* sont exprimées par les relations suivantes (voir les conditions générales (8)):

$$\begin{aligned} &a_t b_s^2 - a_s b_s + 1 = 0, \quad \text{ou} \quad \Delta = 0, \\ &D_x b + b_s (a_t D_y b - D_y a) = 0. \end{aligned} \tag{22}$$

Pour le système de deux équations de la forme suivante

$$\begin{aligned} &r + H(x, y, z, p, q, s) = 0, \\ &t + \Phi(x, y, z, p, q, s) = 0 \end{aligned} \tag{23}$$

les conditions d'involution de *Darboux-Lie* sont de la forme

$$\begin{aligned} &H_s \Phi_s = 1, \\ &D_x \Phi = \Phi_s D_y H \end{aligned} \tag{24}$$

et la condition de *l'intégrabilité complète* est de la forme

$$\frac{d}{dy}\left(\frac{D_y H - H_s D_x \Phi}{H_s \Phi_s - 1}\right) = \frac{d}{dx}\left(\frac{D_x \Phi - \Phi_s D_y H}{H_s \Phi_s - 1}\right). \tag{25}$$

Au cas du système (23) les relations (18) ont les formes suivantes

$$\left\{\begin{aligned} &\frac{\partial r}{\partial x} + H_s \frac{\partial r}{\partial y} + D_x H = 0, \\ &\frac{\partial s}{\partial x} + H_s \frac{\partial s}{\partial y} + D_y H = 0, \\ &\frac{\partial s}{\partial y} + \Phi_s \frac{\partial s}{\partial x} + D_x \Phi = 0, \\ &\frac{\partial t}{\partial y} + \Phi_s \frac{\partial t}{\partial x} + D_y \Phi = 0. \end{aligned}\right. \tag{26}$$

En utilisant les raisonnements les plus élémentaires, analogues à la théorie de *Lagrange-Charpit* pour les équations partielles du première ordre, *N. Saltykow*, [11], démontrait les deux théorèmes suivants.

T h é o r è m e 1. Si les valeurs (16) de r, s, t vérifient les conditions (17), alors les fonctions (16) satisfont aux équations (26).

T h é o r è m e 2. Si les fonctions (16) vérifient les équations (23), (26), les conditions (17) en découlent comme conséquence immédiate.

Cela étant, l'intégration du système (23) dépend de l'existence de la solution du système des équations (23) et (26).

En effet, si la solution des équations (23) et (26) existe, elle donne, en vérifiant les conditions (17), le système des trois équations aux différentielles totales

$$dz = p\,dx + q\,dy, \quad dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy \tag{27}$$

dont l'intégration définit la fonction cherchée.

Donc, la résolution du problème posé dépend de l'intégrabilité des équations (26).

Les valeurs de r et t sont déterminées par le système donné (23). Il ne reste qu'à trouver la fonction s qui doivent être compatible avec les deux fonctions connues r et t. Pour déterminer la fonction s on a les deux équations

$$\begin{aligned} &\frac{\partial s}{\partial x} + H_s \frac{\partial s}{\partial y} + D_y H = 0, \\ &\frac{\partial s}{\partial y} + \Phi_s \frac{\partial s}{\partial x} + D_x \Phi = 0. \end{aligned} \tag{26'}$$

Il est nécessaire de distinguer trois cas suivants, [11]:

1) Il existe la condition (25) de l'intégrabilité complète du système (23).

L'intégration du système (23) s'achève en intégrant le système des quatre équations aux différentielles totales, formé par (27) et l'équation équivalent au système jacobien (26') (en supposant que les valeurs de r et t soient données par (23)).

2) Ils existent les conditions (24) d'involution de *Darboux-Lie* et les équations (26') se confondent.

3) La condition (25) n'est pas vérifiée identiquement en vertu des relations (23) et (26'). Il en résulte une nouvelle relation entre les variables (16), ainsi que x et y. On a dans ce cas dernier de s'assurer si cette nouvelle relation est compatible ou non avec le système donné (23).

Citons quelques exemples des systèmes intégrables par la méthode *N. Saltykow*:

$$\left.\begin{aligned} r + xs &= 0, \\ t - a &= 0, \end{aligned}\right\} \quad z = 1/2\,ay^2 + C_1 x\ (y - x^6/6) + C_2 x + C_3 y + C_4;$$

$$\left.\begin{aligned} r + xs &= 0, \\ t + ys &= 0, \end{aligned}\right\} \quad z = C_1\,(xy - x^3/6 - y^3/6) + C_2 x + C_3 y + C_4,$$

où les C désignent dans les intégrales complètes correspondantes quatre constantes arbitraires.

4. Sur la notion et les propriétés de l'intégrale complète

Il s'agit à présent de poser le problème suivant: établir les conditions qui doivent être satisfaites par l'intégrale complète du système (23) aux cas de l'involution de *Darboux-Lie* et de l'involution de l'intégrabilité complète.

E. Goursat, [12], en partant de l'équation

$$r+2\,sm+t\,m^2+2\,\psi\ (x,y,z,p,q,m)=0$$

m étant un paramètre, et l'étudiant d'un point géométrique, établit les conditions dites dans un cas spécial, [13].

Le résultat récemment acquis sur le problème posé, [13], s'obtient d'une méthode purement analytique dans le cas général du système (23).

Considérons le système donné suivant

$$\begin{aligned} r+H\,(x,y,z,p,q,s)&=0,\\ t+\Phi\,(x,y,z,p,q,s)&=0, \end{aligned} \tag{23}$$

qui est en involution de *Darboux-Lie*

$$H_s\,\Phi_s=1, \tag{24$_1$}$$

$$D_y\,H=H_s\,D_x\,\Phi. \tag{24$_2$}$$

Nous partons de l'équation

$$z=V\,(x,y,C_1,C_2,C_3,C_4) \tag{28}$$

où C_1 sont les constantes arbitraires distinctes et indépendantes des variables x et y. Supposons que $V\in C^3(D)$, D désignant un domaine de x,y,C_1,C_2,C_3,C_4. Formons les équations dérivées

$$\begin{aligned} p&=V_x\ (x,y,C_1,C_2,C_3,C_4),\\ q&=V_y\ (x,y,C_1,C_2,C_3,C_4),\\ s&=V_{xy}\,(x,y,C_1,C_2,C_3,C_4). \end{aligned} \tag{29}$$

$$\begin{aligned} r&=V_{xx}\,(x,y,C_1,C_2,C_3,C_4),\\ t&=V_{yy}\,(x,y,C_1,C_2,C_3,C_4) \end{aligned} \tag{30}$$

sous la condition suivante dans le domaine D

$$\Delta_{xy}\equiv\mathcal{D}\left(\frac{V,V_x,V_y,V_{xy}}{C_1,C_2,C_3,C_4}\right)\neq 0. \tag{31}$$

Si le résultat de l'élimination des paramètres C_i parmi les équations (28), (29) et (30) ne donne que les équations (23), nous dirons dans ce cas que *l'intégrale complète* du système (23) est définit par l'équation considérée (28).

Dans le cas quand le système (23) est en involution de *Darboux-Lie*, son intégrale complète (28) doit satisfaire aux conditions complémentaires.

En effet, les équations (28) et (29), grâce à (31), sont équivalentes dans D aux équations

$$F_k(x, y, z, p, q, s) = C_k, \quad (k = 1, 2, 3, 4). \tag{32}$$

En vertu des équations précédentes et (23) on a les égalités suivantes:

$$-H(x, y, z, p, q, s) = V_{xx}(x, y, F_1, F_2, F_3, F_4) \equiv \overline{V}_{xx}, \tag{33}$$

$$-\Phi(x, y, z, p, q, s) = V_{yy}(x, y, F_1, F_2, F_3, F_4) \equiv \overline{V}_{yy}. \tag{34}$$

En différentiant les relations évidentes

$$\overline{F}_k \equiv F_k(x, y, V, V_x, V_y, V_{xy}) = C_k, \quad (k = 1, 2, 3, 4) \tag{35}$$

par rapport aux C_i on obtient

$$\frac{\partial \overline{F}_k}{\partial z}\frac{\partial V}{\partial C_i} + \frac{\partial \overline{F}_k}{\partial p}\frac{\partial^2 V}{\partial x \partial C_i} + \frac{\partial \overline{F}_k}{\partial q}\frac{\partial^2 V}{\partial y \partial C_i} + \frac{\partial \overline{F}_k}{\partial s}\frac{\partial^3 V}{\partial x \partial y \partial C_i} \equiv \frac{\partial C_k}{\partial C_i},$$

$$(i, k = 1, 2, 3, 4)$$

Grâce à (31), il en résulte

$$(36)\quad \begin{cases} \dfrac{\partial \overline{F}_k}{\partial z} = \dfrac{1}{\Delta_{xy}} \mathcal{D}\left(\dfrac{C_k, V_x, V_y, V_{xy}}{C_1, C_2, C_3, C_4}\right), & \dfrac{\partial \overline{F}_k}{\partial p} = \dfrac{1}{\Delta_{xy}} \mathcal{D}\left(\dfrac{V, C_k, V_y, V_{xy}}{C_1, C_2, C_3, C_4}\right), \\[2ex] \dfrac{\partial \overline{F}_k}{\partial q} = \dfrac{1}{\Delta_{xy}} \mathcal{D}\left(\dfrac{V, V_x, C_k, V_{xy}}{C_1, C_2, C_3, C_4}\right), & \dfrac{\partial \overline{F}_k}{\partial s} = \dfrac{1}{\Delta_{xy}} \mathcal{D}\left(\dfrac{V, V_x, V_y, C_k}{C_1, C_2, C_3, C_4}\right) \end{cases}$$

$$(k = 1, 2, 3, 4).$$

En différentiant l'égalité (33) par rapport à s on obtient

$$-H_s = \sum_{k=1}^{4} \overline{V}_{xx}\, C_k \frac{\partial F_k}{\partial s},$$

ou

$$-(H_s) = \sum_{k=1}^{4} V_{xx}\, C_k \frac{\partial \overline{F}_k}{\partial s},$$

où la parenthèse signifit le résultat de la substitution de z, p, q, s respectivement par leurs valeurs V, V_x, V_y, V_{xy}. Grâce à l'égalité dernière de (36) on a

$$-(H_s) = \frac{\Delta_{xx}}{\Delta_{xy}}. \tag{37}$$

désignant par Δ_{xx} le déterminant fonctionnel

(31′) $$\Delta_{xx} \equiv \mathcal{D}\left(\frac{V, V_x, V_y, V_{xx}}{C_1, C_2, C_3, C_4}\right).$$

D'une manière analogue en utilisant les relations (34), (35) et (36) on peut aussi obtenir

$$-(\Phi_s) \equiv \frac{\Delta_{yy}}{\Delta_{xy}},$$

(38) $$(D_y H) \equiv -V_{xxy} + \frac{\Delta_{xx}}{\Delta_{xy}} V_{xyy}$$

$$(D_x \Phi) \equiv -V_{yyx} + \frac{\Delta_{yy}}{\Delta_{xy}} V_{xxy},$$

où l'on a posé

(31″) $$\Delta_{yy} \equiv \mathcal{D}\left(\frac{V, V_x, V_y, V_{yy}}{C_1, C_2, C_3, C_4}\right)$$

et $V_{xxy} = \partial^3/\partial x\, \partial xy$, etc.

La condition (24_1) nous donne

$(24'_1)$ $$(H_s) \cdot (\Phi_s) = 1,$$

où les parenthèses ont les significations antérieurement établies. Grâce aux égalités (37), (38), $(24_1')$ on a

$$(H_s) \cdot (\Phi_s) \equiv (\Delta_{xx}/\Delta_{xy})\ (\Delta_{yy}/\Delta_{xy}) = 1$$

ou

(39) $$\Delta_{xx}\ \Delta_{yy} - \Delta^2_{xy} = 0.$$

La relation (24_2) ou la relation suivante

$$(D_y H) = (H_s) \cdot (D_x \Phi)$$

est vérifiée identiquement en vertu de la condition (39), à savoir

$$(D_y H) - (H_s)\ (D_x \Phi) \equiv -V_{xxy} + \frac{\Delta_{xx}}{\Delta_{xy}} V_{xyy} + \frac{\Delta_{xx}}{\Delta_{xy}}\left(-V_{yyx} + \frac{\Delta_{yy}}{\Delta_{xy}} V_{xxy}\right) = 0.$$

Donc, la relation (24_2) n'impose pas des conditions nouvelles à la fonction V.

On peut démontrer que les conditions (31) et (39) sont aussi suffisantes.

Donc, l'équation (28) définit une intégrale complète du système (23) en involution de *Darboux-Lie* si la fonction V admet les conditions nécessaires et suffisantes

$$\Delta_{xy} \neq 0, \quad \Delta_{xx}\, \Delta_{yy} - \Delta^2_{xy} = 0$$

Δ_{xx}, Δ_{yy}, Δ_{xy} désignant les déterminants fonctionnels (31), (31′), (31″).

Étudions à présent le cas du système (23) en involution de l'intégrabilité complète:

(25′) $$\frac{d}{dy}\left[\frac{D_1}{D}\right]=\frac{d}{dx}\left[\frac{D_2}{D}\right],$$

où l'on a les désignations abrégées suivantes

(41) $$D\equiv H_s\Phi_s-1,\quad D_1\equiv H_s D_x\Phi-D_y H,\quad D_2\equiv\Phi_s D_y H-D_x\Phi.$$

Grâce aux relations (38) et suivantes

$$(D)\equiv\Delta_{xx}\,\Delta_{yy}/\Delta^2_{xy}-1,\quad (D_1)/(D)=-V_{xxy},\quad (D_2)/(D)=-V_{xy}.$$

les parenthèses désignant le résultat de la substitution de z, p, q, s respectivement par les fonctions V, V_x, V_y, V_{xy}, la condition

(25″) $$\frac{d}{dy}\left[\frac{(D_1)}{(D)}\right]=\frac{d}{dx}\left[\frac{(D_2)}{(D)}\right],\quad \Delta_{xx}\,\Delta_{yy}/\Delta^2_{xy}-1\neq 0$$

devient

$$\frac{\partial}{\partial y}(V_{xxy})=\frac{\partial}{\partial x}(V_{xyy}).$$

En partant de la condition

$$\Delta_{xx}\,\Delta_{yy}/\Delta^2_{xy}-1\neq 0$$

et en substituant les C_i dans (25″) par les fonctions (32) on a (25′).

Donc, en vertu des considérations précédentes on peut distinguer les intégrales complètes des systèmes d'équations en involution de *Darboux-Lie* et aussi, d'autre part, d'équations qui se trouvent en involution d'intégrabilité complète:

La formule $z=V(x, y, C_1, C_2, C_3, C_4)$ sous l'hypothèse $\Delta_{xy}\neq 0$ est une intégrale complète du système (23) en involution de *Darboux-Lie* si l'expression

$$\Delta_{xx}\,\Delta_{yy}-\Delta^2_{xy}$$

est identiquement nulle, ou l'on a introduit les notations suivantes

$$\Delta_{xx}\equiv\mathfrak{D}\left(\frac{V, V_x, V_y, V_{xx}}{C_1, C_2, C_3, C_4}\right),\quad \Delta_{yy}\equiv\mathfrak{D}\left(\frac{V, V_x, V_y, V_{yy}}{C_1, C_2, C_3, C_4}\right),\quad \Delta_{xy}\equiv\mathfrak{D}\left(\frac{V, V_x, V_y, V_{xy}}{C_1, C_2, C_3, C_4}\right).$$

Si par la formule $z=V$ est défini l'intégrale complète du système en involution d'intégrabilité complète, l'expression

$$\Delta_{xx}\,\Delta_{yy}-\Delta^2_{xy}$$

est distincte du zéro.

Considérons le système suivant

(42) $$\begin{cases} s=F(x, y, z, p, q, r),\\ t=\Phi(x, y, z, p, q, r),\end{cases}$$

et les expressions suivantes

$$\delta \equiv \Phi_r - F_r^2,$$
$$\delta_1 \equiv D_y F + F_r D_x F - D_x \Phi,$$
$$\delta_2 \equiv F_r D_y F + \Phi_r D_x^{\cdot} F - F_r D_x^{\cdot} \Phi.$$

Le système (42) est en involution de *Darboux-Lie* si on a lieu identiquement les conditions suivantes

$$\delta = 0, \quad \delta_1 = 0$$

et le système (42) est en involution d'intégrabilité complète au cas

$$\frac{d}{dy}\left(\frac{\delta_1}{\delta}\right) = \frac{d}{dx}\left(\frac{\delta_2}{\delta}\right).$$

D'une manière analogue comme au cas du système (23) on peut établir: Par la formule

$$z = V(x, y, C_1, C_2, C_3, C_4)$$

sous l'hypothèse $\Delta_{xx} \neq 0$ est défini une intégrale complète du système (42) en involution de *Darboux-Lie* ou en involution d'intégrabilité complète selon que l'expression

$$\Delta_{xx} \Delta_{yy} - \Delta^2_{xy}$$

est égale identiquement ou distincte du zéro.

Ces propriétés caractéristiques de l'intégrale complète du système (23) ou (42) sont importantes pour l'étude des systèmes considérés que l'on étudiera en suite.

5. Le problème de Cauchy — au moyen de la méthode de la variation des constantes

La formation de l'intégrale de *Cauchy* des systèmes en involution de *Darboux-Lie* à l'aide de l'intégrale complète va traiter dans ce paragraphe.

Il est bien connu qu'on peut pour le système en involution de *Darboux-Lie*

$$\text{(43)} \qquad \begin{cases} s = F(x, y, z, p, q, r), \\ t = \Phi(x, y, z, p, q, r) \end{cases}$$

établir le théorème suivant:

Soient $x_0, y_0, z_0, p_0, q_0, r_0$ *un système des valeurs des variables* x, y, z, p, q, r *dans les voisinages desquelles les fonctions* F *et* Φ *sont holomorphes; soit, de plus,* $\Pi(x)$ *une fonction holomorphe dans le voisinage du point* x_0 *telle que l'on ait* $\Pi(x_0) = z_0$, $\Pi'(x_0) = p_0$, $\Pi''(x_0) = r_0$. *Il existe une intégrale des équations* (43), *régulière dans le voisinage des valeurs* x_0, y_0, *se réduisant à* $\Pi(x)$ *pour* $y = y_0$, *et pour laquelle la dérivée* $\partial z/\partial y$ *prend la valeur* q_0, *pour* $x = x_0$, $y = y_0$.

Ce théorème se démontre d'une manière comme dans le cas général de *Cauchy*, [14].

D'autre part, on peut déterminer l'intégrale de *Cauchy* à l'aide des multiplicités caractéristiques et la représenter comme une surface passant par une courbe arbitraire dont le plan tangent est donné arbitrairement en un point de la courbe mentionnée, [14]. Cette méthode des caractéristiques était complétée par la théorie des fonctions caractéristiques, [10].

Notre but est de former l'intégrale de *Cauchy* par la méthode de la variation des constantes dans l'intégrale complète. Pour résoudre ce problème nous allons utiliser les propriétés connues de l'intégrale complète, [13], [15].

Considérons, pour fixer les idées, le système en involution de *Darboux-Lie* (43), vérifiant les conditions d'existence de l'intégrale complète, [14], admettant l'intégrale dite sous la forme suivante

$$z = V(x, y, C_1, C_2. C_3, C_4). \tag{44}$$

Supposons que $V \in C^4(D'), D'$ désignant un domain de x, y, C_1, C_2, C_3, C_4, et que soit

$$\Delta_{xx} \equiv \mathcal{D}\left(\frac{V, V_x, V_y, V_{xx}}{C_1, C_2, C_3, C_4}\right) \neq 0 \tag{45}$$

dans le domain D'.

Rappelons d'abord le résultat connu, [15]:

Il existe la relation

$$\mathcal{D}\left(\frac{\alpha, \beta}{x, y}\right) \equiv V_i(\Delta_{xx} \Delta_{yy} - \Delta^2_{xy})/(i, j, k)^3 = 0 \tag{46}$$

où l'on a

$$V_i \equiv \frac{\partial V}{\partial C_i}, \quad (i, j, k) \equiv \mathcal{D}\left(\frac{V, V_x, V_y}{C_i, C_j, C_k}\right), \quad \alpha \equiv \left(\frac{(i, k, l)}{(i, j, k)}\right), \quad \beta \equiv \left(\frac{(i, j, l)}{(i, j, k)}\right). \tag{47}$$

En variant les constantes C_i, dans la formule (44), posons

$$\sum_{i=1}^{4} \frac{\partial V}{\partial C_i} \frac{\partial C_i}{\partial \xi_j} = 0, \quad \sum_{i=1}^{4} \frac{\partial^2 V}{\partial x \partial C_i} \frac{\partial C_i}{\partial \xi_j} = 0, \quad \sum_{i=1}^{4} \frac{\partial^2 V}{\partial y \partial C_i} \frac{\partial C_i}{\partial \xi_j} = 0 \tag{48}$$

avec $\xi_1 \equiv x$, $\xi_2 \equiv y$. L'intégrale générale, à une fonction arbitraire, du système (43) est défini par la formule (44) et les suivantes

$$C_{i+1} = \varphi_i(C_1), \quad (i = 1, 2, 3) \tag{49}$$

$$\begin{gathered} V_1 + \sum_{i=1}^{3} V_{i+1} \varphi_i' = 0, \qquad V_{x1} + \sum_{i=1}^{3} V_{x, i+1} \varphi_i' = 0, \\ V_{y1} + \sum_{i=1}^{3} V_{y, i+1} \varphi_i' = 0, \end{gathered} \tag{50}$$

avec $V_{xi} \equiv \partial^2 V / \partial x \, \partial C_i$, $V_{yi} \equiv \partial^2 V / \partial y \, \partial C_i$.

Sous l'hypothèse $(2, 3, 4) \neq 0$ écrivons les équations (50) à la forme suivante

$$\varphi'_1 = -\frac{(1, 3, 4)}{(2, 3, 4)}, \quad \varphi'_2 = -\frac{(2, 1, 4)}{(2, 3, 4)}, \quad \varphi'_3 = -\frac{(1, 2, 3)}{(2, 3, 4)}.$$

Grâce à la condition (46), on en tire

$$\varphi'_i = F_i(C_1, \varphi_1, \varphi_2, \varphi'_3, \varphi_3), \quad (i=1, 2). \tag{51}$$

Donc, il n'y a, entre les fonctions φ_i, qu'une fonction arbitraire.

On va chercher la solution particulière $z = \psi(x, y)$ du système (43) parmi les solutions que l'on obtient de l'intégrale complète par la variation des constantes (les fonctions $C_i(x, y)$ vérifient les conditions (48)).

Donc, on va appliquer les résultats précédents pour établir l'intégrale du système (43) satisfaisant aux conditions

$$z = \Pi(x) \text{ pour } y = y_0; \quad \frac{\partial z}{\partial y} = a \text{ pour } x = x_0, \; y = y_0; \tag{44'}$$

x_0, y_0, a étant les valeurs constantes.

Posons

$$V(x, y_0, C_1, C_2, C_3, C_4) = \Pi(x) \tag{52}$$

et cherchons les fonctions C_i qui doivent satisfaire aux conditions (48). On a d'abord

$$\sum_{i=1}^{4} V_i(x, y_0, C_1, C_2, C_3, C_4)\frac{\partial C_i}{\partial x} = 0, \quad \sum_{i=1}^{4} V_{xi}(\cdot\,\cdot)\frac{\partial C_i}{\partial x} = 0, \quad \sum_{i=1}^{4} V_{yi}(\cdot\,\cdot)\frac{\partial C_i}{\partial x} = 0. \tag{53}$$

En différentiant la relation (52) par rapport à x, on obtient

$$V_x(\cdot\,\cdot) + \sum_{i=1}^{4} V_i(\cdot\,\cdot)\frac{\partial C_i}{\partial x} = \Pi'(x)$$

ou, en vertu de la première condition (53), aussi

$$V_x(x, y_0, C_1, C_2, C_3, C_4) = \Pi'(x). \tag{54}$$

Appliquant le même procédé à la relation (54) et en utilisant la seconde condition (53), on a

$$V_{xx}(x, y_0, C_1, C_2, C_3, C_4) = \Pi''(x). \tag{55}$$

Supposant que les équations (52), (54), (55) soient résolubles, grâce à la condition (45), par rapport à trois des constantes C_i, on a

$$\mathcal{D}\left(\frac{V, V_x, V_{xx}}{C_1, C_3, C_4}\right) \neq 0, \quad C_{i+1} = a_i\,(x, y_0, C_1). \qquad (i=1, 2, 3) \tag{56}$$

Pour déterminer C_1 on a, en vertu de la dernière condition (53) (et de la première de (56)), l'équation différentielle ordinaire

$$(57) \qquad V_{y1}[x, y_0, C_1, a_1, a_2, a_3]\frac{dC_1}{dx}+\sum_{i=1}^{3} V_{y,i+1}[..]\frac{\partial a_i}{\partial x}=0,$$

parce que, grâce aux conditions (45) et (56), le coefficient de dC_1/dx est distincte du zéro. Soit l'inégrale générale de l'équation (57)

$$(58) \qquad C_1 = a_4(x, y_0, b)$$

b étant une constante arbitraire. Enfin, pour déterminer b on a la condition

$$V_y\{x_0, y_0, a_4(x_0, y_0, b), a_1[x_0, y_0, a_4(x_0, y_0, b)], a_2[..], a_3[..]\} = a.$$

Dans le cas général on peut mettre les équations (56) et (58) sous les formes suivantes

$$(59) \qquad C_{i+1} = \psi_i(C_1), \quad (i=1, 2, 3).$$

Pour déterminer les fonctions $C_i(x, y)$, on a, d'après (50), les conditions

$$V_1(x, y, C_1, \psi_1, \psi_2, \psi_3)+\sum_{i=1}^{3} V_{i+1}(..)\psi_i'(C_1)=0,$$

$$V_{x1}(x, y, C_1, \psi_1, \psi_2, \psi_3)+\sum_{i=1}^{3} V_{x,i+1}(..)\psi_i'(C_1)=0.$$

$$V_{y1}(x, y, C_1, \psi_1, \psi_2, \psi_3)+\sum_{i=1}^{3} V_{y,i+1}(..)\psi_i'(C_1)=0.$$

De ces trois équations il ne reste qu'une équation pour définir C_1 en fonction de x et y

$$(60) \qquad C_1 = C_1(x, y),$$

parce que les deux autres, d'après (51), sont équvalentes aux relations

$$\psi_i' = \overline{F}_i(C_1, \psi_1, \psi_2, \psi_3'), \quad (i=1, 2)$$

qui, étant toujours identiquement vérifiées — grâce à la méthode même de la formation des fonctions ψ_i — ne produisent pas des conditions nouvelles pour C_1.

De cette manière les formules (44), (59) et (60) nous donnent l'intégrale de *Cauchy* vérifiant les conditions (44'). Cette intégrale est contenue dans l'intégrale générale du système (43).

Dans le cas où les équations (56) et (58) nous ne donnent que les valeurs constantes pour les quantités C_i, on a avec la formule (44) une intégrale de *Cauchy* vérifiant les conditions (44') et qui est contenue dans l'intégrale complète du système (43).

Exemples

1) Déterminer l'intégrale de *Cauchy* pour le système de *Goursat*

(61) $$r + s^3/3 = 0, \quad t - 1/s = 0$$

avec les conditions:

a) $z = 4/3\ (y-1)^{3/2}$ pour $x=1$, et $z_x = 1$ pour $x = y = 1$;

b) $z = 3/2\, y^{3/2}$ pour $x=1$, et $z_x = 49/48$ pour $x=1$, $y = 1/16$.

Une intégrale complète de ce système est

(61′) $$z = 4/3\,(y-C_1)^{3/2}\ x^{1/2} + C_2 x + C_3 (y - C_1) + C_4 .$$

ad a) Les équations (52), (54), (55) (en y changeant x par y) sont

$$4/3\,(y-C_1)^{3/2} + C_2 + C_3 (y-C_1) + C_4 = 4/3\,(y-1)^{3/2},$$

$$2\,(y-C_1)^{1/2} + C_3 = 2\,(y-1)^{1/2},$$

$$(y-C_1)^{-1/2} = (y-1)^{-1/2}.$$

On en tire $C_1 = 1$, $C_2 + C_4 = 0$, $C_3 = 0$. L'intégrale correspondante (58) est $C_2 = b$ et en vertu de la condition $z_x(1, 1) = 1$, on a $b = 1$. Donc, $C_2 = 1$ et $C_4 = -1$. L'intégrale cherchée de *Cauchy* devient

(62) $$z = 4/3\, x^{1/2}\,(y-1)^{3/2} + x - 1.$$

ad b) Dans ce cas les équations (52), (54) et (55) nous donnent: $C_1 = -3\,y$, $C_3 = -3\,y^{1/2}$, $C_2 + C_4 = 2\,y^{3/2}$. L'équation différentielle ordinaire pour déterminer C_2 est

$$6\,y^{1/2} + C'_{2,}(y) = 0,$$

et grâce à la condition $z_x(1, 1/16) = 49/48$ on a $C_2 = -4\,y^{3/2} + 1$. Les équations (59) et (60) deviennent

$$C_2 = -4\,(-C_1/3)^{3/2} + 1,\ \ C_3 = -3\,(-C_1/3)^{1/2},$$

$$C_4 = 6\,(-C_1/3)^{3/2} - 1,\ \ C_1 = 3\,y : (3 - 4\,x).$$

L'intégrale cherchée de *Cauchy* devient

(63) $$z = 2/3\,(4\,x - 3)^{1/2}\ y^{3/2} + x - 1$$

et elle est contenue dans l'intégrale générale.

6. Le problème de Cauchy — au moyen du système correspondant de Charpit

Dans le *Chapitre I* nous avons considéré la théorie nouvelle des caractéristiques des équations aux dérivées partielles du premier ordre. Dans ce paragraphe nous allons traiter, [34], un problème analogue concernant le système des équations aux dérivées partielles du second ordre en involution de *Darboux-Lie*:

$$(23)\qquad \begin{cases} r+H(x, y, z, p, q, s)=0, \\ t+\Phi(x, y, z, p, q. s)=0 \end{cases}$$

sous les conditions

$$(24)\qquad H_s\Phi_s=1, \quad D_yH=H_sD_x\Phi.$$

On peut associer au système (23) un système de *Charpit* de la forme suivante

$$(64)\qquad \begin{cases} \dfrac{\partial z}{\partial x}+H_s\dfrac{\partial z}{\partial y}=p+H_sq, \\ \dfrac{\partial p}{\partial x}+H_s\dfrac{\partial p}{\partial y}=r+H_ss \\ \dfrac{\partial q}{\partial x}+H_s\dfrac{\partial q}{\partial y}=s+H_st \\ \dfrac{\partial r}{\partial x}+H_s\dfrac{\partial r}{\partial y}=-D_xH, \\ \dfrac{\partial s}{\partial x}+H_s\dfrac{\partial s}{\partial y}=-D_yH, \\ \dfrac{\partial t}{\partial x}+H_s\dfrac{\partial t}{\partial y}=-H_sD_y\Phi \end{cases}$$

des fonctions inconnues z, p, q, r, s, t de deux variables indépendantes x, y ou un système équivalent des caractéristiques

$$(65)\qquad dx=\frac{dy}{H_s}=\frac{dz}{p+H_sq}=\frac{dp}{r+H_ss}=\frac{dq}{s+H_st}=-\frac{dr}{D_xH}=\frac{ds}{-D_yH}=\frac{dt}{-H_sD_y\Phi}$$

en posant

$$D_x=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q}, \qquad H_s=\frac{\partial H}{\partial s},$$

$$D_y=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}, \qquad \Phi_s=\frac{\partial \Phi}{\partial s}.$$

Les valeurs initiales des variables z, p, q, r, s, t sont définies le long de la courbe donnée non caractéristique

$$(C)\qquad x=x_0, \quad y=\tau, \quad z=z(\tau),$$

de telle manière qu'on a le long de la courbe (C) les conditions suivantes

(66) $$\begin{cases} r+H=0, \ t+\Phi=0 \\ dz=p\,dx+q\,dy, \ dp=r\,dx+s\,dy, \ dq=s\,dx+t\,dy, \end{cases}$$

(67) $$p=a, \ x=x_0, \ y=y_0.$$

Pour le système (64) nous posons:

Problème I. Déterminer telles intégrales des caractéristiques (65) *contenant la courbe (C), avec les valeurs correspondantes initiales pour les variables z, p, q, r, s, t. Ces intégrales forment une surface*

(68) $$z=z(x, y),$$

contenant la courbe (C) et définissent les fonctions: z (x, y), p (x, y), q (x, y) r (x, y), s (x, y), t (x, y) qui représentent la solution du problème initial du système de Charpit (64).

Nous n'insistons pas ici sur la formulation des propriétés précis des fonctions H, Φ, $z(\tau)$ admettant le procédé suivant employé pour résoudre le problème posé.

Nous supposons d'abord que les fonctions H, Φ soient telles qu'il existe les sept intégrales premières distinctes du système des caractéristiques (65)

$$f_i(x, y, z, p, q, s), \quad (i=1, \ldots, 5),$$

$$f_6 \equiv r+H(x, y, z, p, q, s),$$

$$f_7 \equiv t+\Phi(x, y, z, p, q, s)$$

et grâce à ces intégrales on peut écrire l'intégrale générale du système de *Charpit* (64) sous la forme

(69) $$f_{i+1}=\Pi_i(f_1), \ (i=1, \ldots, 6)$$

Π_i étant des fonctions arbitraires.

Ensuite, nous supposons encore que les fonctions: $p(\tau)$, $q(\tau)$, $r(\tau)$, $s(\tau)$, $t(\tau)$ soient bien déterminées, grâce aux conditions (66) et (67), par les équations suivantes

$$r(\tau)+H[x_0, \tau, z(\tau), p(\tau), q(\tau), s(\tau)]=0, \ t(\tau)+\Phi[\ldots]=0,$$

$$z'(\tau)=q(\tau), p'(\tau)=s(\tau), q'(\tau)=t(\tau)$$

et par la condition $p=a$ pour $x=x_0$, $y=y_0=\tau$.

Introduisions, à présent, les notations suivantes

$$\lambda_i(\tau) \equiv f_i[x_0, \tau, z(\tau), p(\tau), q(\tau), r(\tau), s(\tau), t(\tau)]$$

et les paramètres auxiliaires u_i par les relations

(70) $$\lambda_i(\tau)=u_i \qquad (i=1, \ldots, 7)$$

En éliminant le paramètre τ entre des équations (70) on obtient les six relations entre des paramètres u_i

$$u_{i+1} = p_i(u_1), \qquad (i = 1, 2, \ldots, 6) \tag{71}$$

Donc, les fonctions arbitraires Π_i doivent avoir les formes p_i dans le cas des propositions du problème I, et la solution cherchée du problème I, c'est-à-dire les fonctions

$$z(x, y), \quad p(x, y), \quad q(x, y), \quad r(x, y), \quad s(x, y) \quad t(x, y) \tag{72}$$

sont définies par les formules

$$f_{i+1} = p_i(f_1), \qquad (i = 1, \ldots, 6). \tag{72'}$$

Posons, maintenant, pour le système (23):

Problème II. La solution (72) *du système de Charpit* (64) *représente l'intégrale de Cauchy du système* (23) *sous les conditions:* (C) *et* (67).

Il suffit de démontrer que les conditions suivantes

$$\begin{cases} r(x, y) + H[x, y, z(x, y), p(x, y), q(x, y), s(x, y)] \equiv v(x, y) \equiv 0, \\ t(x, y) + \Phi[x, y, z(x, y), p(x, y), q(x, y), s(x, y)] \equiv w(x, y) \equiv 0, \end{cases} \tag{73}$$

$$\begin{cases} p(x, y) - \dfrac{\partial z(x, y)}{\partial x} \equiv P(x, y) \equiv 0, \\ q(x, y) - \dfrac{\partial z(x, y)}{\partial y} \equiv Q(x, y) \equiv 0, \end{cases} \tag{74}$$

$$\begin{cases} r(x, y) - \dfrac{\partial p(x, y)}{\partial x} \equiv R(x, y) \equiv 0, \\ s(x, y) - \dfrac{\partial p(x, y)}{\partial y} \equiv s(x, y) - \dfrac{\partial q(x, y)}{\partial x} \equiv S(x, y) \equiv 0, \\ t(x, y) - \dfrac{\partial q(x, y)}{\partial y} \equiv T(x, y) \equiv 0 \end{cases} \tag{75}$$

sont remplies sur la surface (68).

Démontrons, par exemple, qu'il existe (73) et (75).

Grâce à l'introduction des dernières fonctions P, Q, R, S, T les trois premières équations (64) vont s'exprimer de la manière suivante:

$$\begin{cases} R + H_s Q = 0, \\ R + H_s S = 0, \\ S + H_s T = 0, \end{cases} \tag{76}$$

parce que les fonctions (72) sont les intégrales du système (64).

Formons les équations dérivées respectivement par rapport à x et y des équations (76):

$$(77)\quad \begin{cases} \dfrac{\partial P}{\partial x}+H_s\dfrac{\partial Q}{\partial x}+H_{s_1}\ Q=0, \\ \dfrac{\partial R}{\partial x}+H_s\dfrac{\partial S}{\partial x}+H_{s_1}\ S=0, \\ \dfrac{\partial S}{\partial x}+H_s\dfrac{\partial T}{\partial x}+H_{s_1}\ T=0, \end{cases}$$

et

$$(78)\quad \begin{cases} \dfrac{\partial P}{\partial y}+H_s\dfrac{\partial Q}{\partial y}+H_{s_2}\ Q=0, \\ \dfrac{\partial R}{\partial y}+H_s\dfrac{\partial S}{\partial y}+H_{s_2}\ S=0, \\ \dfrac{\partial S}{\partial y}+H_s\dfrac{\partial T}{\partial y}+H_{s_2}\ T=0, \end{cases}$$

en introduisant les désignations

$$H_{s_1}\equiv\frac{\partial H_s}{\partial x}, \qquad H_{s_2}\equiv\frac{\partial H_s}{\partial y},$$

on obtient, en vertu des relations définissant les variables v et w, les relations évidentes

$$(79)\quad \begin{cases} \dfrac{\partial v}{\partial x}=\dfrac{\partial r(x,y)}{\partial x}+\dfrac{\partial H}{\partial x}+\dfrac{\partial H}{\partial z}\dfrac{\partial z(x,y)}{\partial x}+\dfrac{\partial H}{\partial p}\dfrac{\partial p(x,y)}{\partial x}+ \\ \qquad +\dfrac{\partial H}{\partial q}\dfrac{\partial q(x,y)}{\partial x}+\dfrac{\partial H}{\partial s}\dfrac{\partial s(x,y)}{\partial x}, \\ \dfrac{\partial v}{\partial y}=\dfrac{\partial r(x,y)}{\partial y}+\dfrac{\partial H}{\partial y}+\dfrac{\partial H}{\partial z}\dfrac{\partial z(x,y)}{\partial y}+\dfrac{\partial H}{\partial p}\dfrac{\partial p(x,y)}{\partial y}+ \\ \qquad +\dfrac{\partial H}{\partial q}\dfrac{\partial q(x,y)}{\partial y}+\dfrac{\partial H}{\partial s}\dfrac{\partial s(x,y)}{\partial y}, \\ \dfrac{\partial w}{\partial x}=\dfrac{\partial t(x,y)}{\partial x}+\dfrac{\partial \Phi}{\partial x}+\dfrac{\partial \Phi}{\partial z}\dfrac{\partial z(x,y)}{\partial x}+\dfrac{\partial \Phi}{\partial p}\dfrac{\partial p(x,y)}{\partial x}+ \\ \qquad +\dfrac{\partial \Phi}{\partial q}\dfrac{\partial q(x,y)}{\partial x}+\dfrac{\partial \Phi}{\partial s}\dfrac{\partial s(x,y)}{\partial x}, \\ \dfrac{\partial w}{\partial y}=\dfrac{\partial t(x,y)}{\partial y}+\dfrac{\partial \Phi}{\partial y}+\dfrac{\partial \Phi}{\partial z}\dfrac{\partial z(x,y)}{\partial y}+\dfrac{\partial \Phi}{\partial p}\dfrac{\partial p(x,y)}{\partial y}+ \\ \qquad +\dfrac{\partial \Phi}{\partial q}\dfrac{\partial q(x,y)}{\partial y}+\dfrac{\partial \Phi}{\partial s}\dfrac{\partial s(x,y)}{\partial y} \end{cases}$$

et écrivons, en vertu des équations (64), les identités suivantes

$$\frac{\partial r(x,y)}{\partial x}+H_s\frac{\partial r(x,y)}{\partial y}+\frac{\partial H}{\partial x}+\frac{\partial H}{\partial z}p(x,y)+\frac{\partial H}{\partial p}r(x,y)+\frac{\partial H}{\partial q}s(x,y)=0,$$

$$\frac{\partial s(x,y)}{\partial x}+H_s\frac{\partial s(x,y)}{\partial y}+H_s\left[\frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial z}p(x,y)+\frac{\partial \Phi}{\partial p}r(x,y)+\frac{\partial \Phi}{\partial q}s(x,y)\right]=0,$$

(80) $$\frac{\partial t(x,y)}{\partial x}+H_s\frac{\partial t(x,y)}{\partial y}+H_s\left[\frac{\partial \Phi}{\partial y}+\frac{\partial \Phi}{\partial z}q(x,y)+\frac{\partial \Phi}{\partial p}s(x,y)+\frac{\partial \Phi}{\partial q}t(x,y)\right]=0,$$

$$\frac{\partial s(x,y)}{\partial x}+H_s\frac{\partial s}{\partial y}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}q(x,y)+\frac{\partial H}{\partial p}s(x,y)+\frac{\partial H}{\partial q}t(x, y=0$$

Éliminant les dérivées $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial y}$, $\frac{\partial \Phi}{\partial x}$, $\frac{\partial \Phi}{\partial y}$ respectivement entre les équations (79) et (80) et utilisant les relations évidentes

$$\frac{\partial s(x,y)}{\partial x}-\frac{\partial r(x,y)}{\partial y}=\frac{\partial S}{\partial x}-\frac{\partial R}{\partial y},\quad \frac{\partial t(x,y)}{\partial x}-\frac{\partial s(x,y)}{\partial y}=\frac{\partial T}{\partial x}-\frac{\partial S}{\partial y}$$

on obtient

(81) $$\begin{cases} \dfrac{\partial v}{\partial x}=H_s\left(\dfrac{\partial S}{\partial x}-\dfrac{\partial R}{\partial y}\right)-H_zP-H_pR-H_qS, \\ \dfrac{\partial v}{\partial y}=\dfrac{\partial R}{\partial y}-\dfrac{\partial S}{\partial x}-H_zQ-H_pS-H_qT; \end{cases}$$

(82) $$\begin{cases} \dfrac{\partial w}{\partial x}=\dfrac{\partial T}{\partial x}-\dfrac{\partial S}{\partial y}-\Phi_zP-\Phi_pR-\Phi_qS, \\ H_s\dfrac{\partial w}{\partial y}=\dfrac{\partial S}{\partial y}-\dfrac{\partial T}{\partial x}-H_s(\Phi_zQ+\Phi_pS+\Phi_qT). \end{cases}$$

Or, en vertu de l'équation $H_s=dy/dx$, (65), les équations (81) et (82) nous donnent

(81′) $$\frac{dv}{dx}=-H_z(P+H_sQ)-H_p(R+H_sS)-H_q(S+H_sT),$$

(82′) $$\frac{dw}{dx}=-\Phi_z(P+H_sQ)-\Phi_p(R+H_sS)-\Phi_q(S+H_sT),$$

ou, grâce aux relations (76), on aura

$$\frac{dv}{dx}=0,\quad \frac{dw}{dy}=0.$$

Dans le point initial de la courbe (C) on a

$$v=0, \quad w=0,$$

donc

$$v(x, y)\equiv 0, \quad w(x, y)\equiv 0$$

et les conditions (73) sont remplies.

Quant aux conditions (75), nous supposons que les conditions (73) et (74) (nous n'insistons pas ici sur la démonstration de l'existence des conditions (74)) soient remplies. Dans ce cas les relations (81) et (82) nous donnent

$$H_s\left(\frac{\partial S}{\partial x}-\frac{\partial R}{\partial y}\right)-H_p R-H_q S=0, \tag{83}$$

$$\frac{\partial R}{\partial y}-\frac{\partial S}{\partial x}-H_p S-H_q T=0, \tag{84}$$

$$\left\{\begin{aligned} &\frac{\partial T}{\partial x}-\frac{\partial S}{\partial y}-\Phi_p R-\Phi_q S=0,\\ &\frac{\partial S}{\partial y}-\frac{\partial T}{\partial x}=H_s(\Phi_p T+\Phi_q S). \end{aligned}\right. \tag{85}$$

En utilisant les équations dérivées (77) et (78), les équations (83), (84) et (85) nous donnent

$$\left\{\begin{aligned} &\frac{dR}{dx}+H_p R+(H_q+H_{s1})S=0,\\ &\frac{dS}{dx}+(H_p+H_{s2})\,S+H_q T=0,\\ &\frac{dT}{dx}-\Phi_p' R-\Phi_q S+H_{s2} T=0. \end{aligned}\right. \tag{86}$$

Par conséquent, les fonctions R, S, T sont définies par un système linéaire et homogène des équations différentielles ordinaires. Le long de la courbe (C) on a

$$R=0, \quad S=0, \quad T=0$$

donc

$$R(x, y)\equiv 0, \quad S(x, y)\equiv 0, \quad T(x, y)\equiv 0$$

et les conditions (75) sont remplies.

Exemple. Résoudre le problème de *Cauchy* pour le système de *Goursat*:

$$r+s^3/3=0, \quad t-1/s=0 \tag{61}$$

avec les conditions

(C) $$x=1,\ y=\tau,\ z=4(\tau-1)^{3/2}:3$$

et

(67) $$p=1 \quad \text{pour} \quad x=1,\ y=1.$$

Dans ce cas les valeurs initiales des variables p, q, r, s, t sont

$$p(\tau)\equiv 2/3\,(\tau-1)^{3/2}+1,\quad q(\tau)\equiv 2\,(\tau-1)^{1/2},$$

$$r(\tau)\equiv -1/3\,(\tau-1)^{3/2},\quad s(\tau)\equiv(\tau-1)^{1/2},\quad t(\tau)\equiv(\tau-1)^{-1/2}$$

et les intégrales des caractéristiques s'obtiennent sous la forme suivante

$$f_1\equiv s,\ f_2\equiv y-s^2x,\ f_3\equiv p-2/3\,s^3x,\ f_4\equiv q-2\,sx,$$

$$f_5\equiv z-4/3\,s^3x^2-[s^2(q-2\,sx)+p-2/3\,s^3x]\,x,$$

$$f_6\equiv r+s^3/3,\ f_7\equiv t-1/s.$$

L'intégrale de *Cauchy* pour le système de *Charpit* (64) est définie d'après (72'), par les formules

$$y-s^2x=1,\ z-4/3\,s^3x^2-x=-1,\ p=2/3\,s^3x+1,$$

$$q=2\,sx,\ r=-1/3\,s^3,\ t=1/s$$

et l'on en tire les fonctions (72):

$$z(x,y)\equiv 4/3\,[x\,(x-y)^3]^{1/2}+x-1,$$

$$p(x,y)\equiv 2/3\left[\frac{(y-1)^3}{x}\right]^{1/2}+1,\quad q(x,y)\equiv 2\,[x\,(y-1)]^{1/2}$$

$$r(x,y)\equiv -1/3\left(\frac{y-1}{x}\right)^{3/2},\ s(x,y)\equiv\left(\frac{y-1}{x}\right)^{1/2},\ t(x,y)\equiv\left(\frac{x}{y-1}\right)^{1/2}.$$

Donc, la surface intégrale cherchée (68) est la suivante

$$z=4/3\,[x\,(y-1)^3]^{1/2}+x-1.$$

7. Théorème de Jacobi pour le système d'équations en involution de Darboux-Lie

Dans ce paragraphe nous allons traiter le problème suivant: *former l'intégrale générale des caractéristiques à l'aide de l'intégrale complète pour un système d'équations aux dérivées partielles du second ordre en involution de Darboux-Lie.*

Jacobi, [16], a étudié ce problème concernant une équation aux dérivées partielles du premier ordre. Les résultats de ces recherches sont généralisés par *N. Saltykow*, [17], [18], *Th. De-Donder*, etc.

Pour le cas d'un système d'équations aux dérivées partielles du second ordre en involution de *Darboux-Lie* ce problème avait été traité par *E. Goursat*, [14], Cependant son procédé peut être simplifié et mis sous la forme analogue à celle de la théorie des équations aux dérivées partielles du premier ordre. On y réussit, [15], grâce à la condition qui doit satisfaire l'intégrale complète du système considéré que nous venons d'exposer dans le paragraphe 4 de ce chapitre.

Considérons le système en involution de *Darboux-Lie*

$$\begin{aligned} r+H(x, y, z, p, q, s)&=0, \\ t+\Phi(x, y, z, p, q, s)&=0 \end{aligned} \tag{23}$$

et le système correspondant des caractéristiques

$$dx=\frac{dy}{H_s}=\frac{dz}{p+H_s q}=\frac{dp}{-H+sH_s}=\frac{dq}{s-\Phi H_s}=\frac{ds}{-D_y H}. \tag{65}$$

E. Goursat, [14], partant de l'intégrale complète

$$F(x, y, z, C_1, C_2, C_3, C_4)=0 \tag{87}$$

en tire, grâce aux considérations géométriques, une courbe caractéristique moyenant l'équation (87) et l'équation suivante

$$\frac{\partial F}{\partial C_1}+\frac{\partial F}{\partial C_2}\frac{dC_2}{dC_1}+\frac{\partial F}{\partial C_3}\frac{dC_3}{dC_1}+\frac{\partial F}{\partial C_4}\frac{dC_4}{dC_1}=0. \tag{88}$$

L'équation (88) définit la fonction y de la variable x. Parce que les multiplicités correspondantes des caractéristiques des éléments du second ordre satisfont aux équations différentielles des caractéristiques, *E. Goursat* obtient deux relations

$$\Phi_i\left(C_1, C_2, C_3, C_4, \frac{dC_2}{dC_1}, \frac{dC_3}{dC_1}, \frac{dC_4}{dC_1}\right)=0, \quad (i=1, 2) \tag{89}$$

entre le sept paramètres $C_1, C_2, C_3, C_4, dC_2/dC_1, dC_3/dC_1, dC_4/dC_1$ figurant dans l'équations (89). Éliminant, grâce aux relations (89), deux paramètres, la courbe caractéristique ne dépend que de cinq paramètres. En pratique la détermination des relations (89) et les éliminations correspondantes présentent souvent de très grandes difficultés.

Mettons l'intégrale complète du système (23) sous la forme

$$z=V(x, y, C_1, C_2, C_3, C_4) \tag{28}$$

en supposant

$$\Delta_{xy}\equiv\mathfrak{D}\left(\frac{V, V_x, V_y, V_{xy}}{C_1, C_2, C_3, C_4}\right)\neq 0. \tag{31}$$

Il est aisé de démontrer que les formules

$$z=V, \ p=V_x, \ q=V_y, \ s=V_{xy} \tag{29}$$

définissent les quatre premières intégrales du système (65). En effet, on a

$$\frac{dz}{dx}=V_x+V_y\frac{dy}{dx},\quad \frac{dp}{dx}=V_{xx}+V_{xy}\frac{dy}{dx},$$

$$\frac{dq}{dx}=V_{xy}+V_{yy}\frac{dy}{dx},\quad \frac{ds}{dx}=V_{xxy}+V_{xyy}\frac{dy}{dx}$$

et en vertu des formules connues

(37)
$$(H_s)=-\frac{\Delta_{xx}}{\Delta_{xy}},$$

(38)
$$(D_y H)=-V_{xxy}+\frac{\Delta_{xx}}{\Delta_{xy}}V_{xyy}$$

et aussi

$$\frac{dy}{dx}=(H_s)$$

le symbole ( ) désignant le résultat de la substitution de z, p, q, s respectivement par V, V_x, V_y, V_{xy} et Δ_{xx}, Δ_{yy} les déterminants fonctionels (31′) et (31″), on obtient, en éliminant les constantes C_i définies par les formules (29), les équations suivantes

$$\frac{dz}{dx}=p+qH_s,\quad \frac{dp}{dx}=-H+sH_s$$

$$\frac{dq}{dx}=s-\Phi H_s,\quad \frac{ds}{dx}=-D_y H.$$

En différentiant les identités évidentes

$$\frac{\partial^2 V}{\partial x^2}+H\left(x, y, V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x\partial y}\right)=0,$$

$$\frac{\partial^2 V}{\partial y^2}+\Phi\left(x, y, V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x\partial y}\right)=0,$$

par rapport à C_k on obtient

(90)
$$\frac{\partial^3 V}{\partial x^2\partial C_k}+\left(\frac{\partial H}{\partial z}\right)\frac{\partial V}{\partial C_k}+\left(\frac{\partial H}{\partial p}\right)\frac{\partial^2 V}{\partial x\partial C_k}+\left(\frac{\partial H}{\partial q}\right)\frac{\partial^2 V}{\partial y\partial C_k}+\left(\frac{\partial H}{\partial s}\right)\frac{\partial^3 V}{\partial x\partial y\partial C_k}=0,$$

(91)
$$\frac{\partial^3 V}{\partial y^2\partial C_k}+\left(\frac{\partial \Phi}{\partial z}\right)\frac{\partial V}{\partial C_k}+\left(\frac{\partial \Phi}{\partial p}\right)\frac{\partial^2 V}{\partial x\partial C_k}+\left(\frac{\partial \Phi}{\partial q}\right)\frac{\partial^2 V}{\partial y\partial C_k}+$$

$$+\left(\frac{\partial \Phi}{\partial s}\right)\frac{\partial^3 V}{\partial x\partial y\partial C_k}=0,\quad (k=1, 2, 3, 4).$$

Grâce à l'involution du système (23) et aux équations des caractéristiques (5) on a les relations

$$(92)\qquad \frac{dy}{dx}=(H_s)=\frac{1}{(\Phi_s)}.$$

Il est connu que l'intégrale (28) vérifie identiquement la condition suivante

$$(39)\qquad \Delta_{xx}\,\Delta_{yy}-\Delta_{xy}^2=0.$$

En vertu de (31), on conclut de (39)

$$(93)\qquad \Delta_{xx}\neq 0,\quad \Delta_{yy}\neq 0.$$

En éliminant $(\partial H/\partial z)$, $(\partial H/\partial p)$, $(\partial H/\partial q)$, $(\partial \Phi/\partial z)$, $(\partial \Phi/\partial p)$, $(\partial \Phi/\partial q)$ des équations (90) et (91) on obtient, en vertu des (31), (93), (92) deux relations pour définir la fonction y

$$(94)\qquad \Delta_{xx}\,dx+\Delta_{xy}\,dy=0;$$

$$(95)\qquad \Delta_{xy}\,dx+\Delta_{yy}\,dy=0.$$

Or, elle se confondent, grâce à la condition (39).

Introduisons les symboles suivants

$$(i,j,k)=\mathcal{D}\left(\frac{V,\,V_x,\,V_y}{C_i,\,C_j,\,C_k}\right),\quad (i,j)=\mathcal{D}\left(\frac{V_x,\,V_y}{C_i,\,C_j}\right),\quad \underline{(i,j)}=\mathcal{D}\left(\frac{V,\,V_x}{C_i,\,C_j}\right),$$

$$\overline{(i,j)}=\mathcal{D}\left(\frac{V,\,V_y}{C_i,\,C_j}\right)$$

et considérons les trois hypothèses suivantes:

a) Supposons, d'abord, que le système (23) n'admet pas d'intégrales intérmédiaires avec une constante arbitraire, c'est-à-dire que $(i,j,k)\neq 0$, $(i,j,k=$ $=1,2,3,4)$. Utilisons les identités évidentes

$$\underline{(i,l)}\,(i,j,k)_x-\underline{(i,k)}\,(i,j,l)_x+\underline{(i,j)}\,(i,k,l)_x=V_i\,\Delta_{xx}$$

$$\underline{(i,l)}\,(i,j,k)_y-\underline{(i,k)}\,(i,j,l)_y+\underline{(i,j)}\,(i,k,l)_y=V_i\,\Delta_{xy}$$

$$\overline{(i,l)}\,(i,j,k)_x-\overline{(i,k)}\,(i,j,l)_x+\overline{(i,j)}\,(i,k,l)_x=V_i\,\Delta_{xy}$$

$$\overline{(i,l)}\,(i,j,k)_y-\overline{(i,k)}\,(i,k,l)_y+\overline{(i,j)}\,(i,k,l)_y=V_i\,\Delta_{yy}$$

où $(\ )_x$ et $(\ )_y$ désignant les dérivées partielles correspondantes par rapport aux x et y. On a de plus identités

$$\underline{(i,l)}\,\overline{(i,k)}-\underline{(i,k)}\,\overline{(i,l)}=-V_i\,(i,k,l),$$

$$\underline{(i,j)}\,\overline{(i,k)}-\underline{(i,k)}\,\overline{(i,j)}=-V_i\,(i,j,k).$$

Formons, ensuite, les identités suivantes

$$\underline{(i,k)}\ \Delta_{xy}-\overline{(i,k)}\ \Delta_{xx}=(i,k,l)\ (i,j,k)_x-(i,j,k)\ (i,k,l)_x,$$

$$\underline{(i,k)}\ \Delta_{yy}-\overline{(i,k)}\ \Delta_{xy}=(i,k,l)\ (i,j,k)_y-(i,j,k)\ (i,k,l)_y$$

qui donnent, grâce à la condition (39)

$$(39') \qquad \frac{\dfrac{\partial}{\partial x}\left[\dfrac{(i,k,l)}{(i,j,k)}\right]}{\Delta_{xx}}=\frac{\dfrac{\partial}{\partial y}\left[\dfrac{(i,k,l)}{(i,j,k)}\right]}{\Delta_{xy}}.$$

L'équation (94), introduisant la désignation

$$\alpha\equiv\frac{(i,k,l)}{(i,j,k)}$$

admet l'intégrale sous la forme suivante

$$\alpha=\text{const},\ \alpha_x\neq0,\quad \alpha_y\neq0.$$

En effet, grâce aux conditions (31), (39), (93), (39') on peut toujours choisir les index i, j, k, l de telle manière que la fonction α soit dépendue des variables x et y.

On a de même, d'une manière analogue, les intégrales de l'équation mentionnée (94):

$$\beta\equiv\frac{(i,j,l)}{(i,j,k)}=\text{const.},\qquad \gamma\equiv\frac{(j,k,l)}{(i,j,k)}=\text{const.}$$

Or, α, β, γ ne sont pas distinctes par rapport aux variables x et y. Démontrons, par exemple, cela pour les fonctions α, β. Considérons le déterminant

$$A\equiv\mathcal{D}\left(\frac{\alpha,\beta}{x,y}\right).$$

Substituant y les expressions pour α et β, on obtient

$$(97) \qquad A\equiv\frac{1}{(i,j,k)^3}\begin{vmatrix}(i,j,k) & (i,j,k)_x & (i,j,k)_y\\ (i,j,l) & (i,j,l)_x & (i,j,l)_y\\ (i,k,l) & (i,k,l)_x & (i,k,l)_y\end{vmatrix}.$$

D'autre part, grâce aux identités (96), on obtient la relation suivante

$$V_i\left(\Delta_{xx}\Delta_{yy}-\Delta_{xy}^2\right)=\begin{vmatrix}(i,j,k) & (i,j,k)_x & (i,j,k)_y\\ (i,j,l) & (i,j,l)_x & (i,j,l)_y\\ (i,k,l) & (i,k,l)_x & (i,k,l)_y\end{vmatrix}$$

et l'on a en vertu de (97) et (32)

$$A \equiv \frac{V_i(\Delta_{\lambda\lambda}\Delta_{yy}-\Delta_{xy}^2)}{(i, k, l)^3}=0.$$

De cette manière on peut prendre

$$\alpha = \text{const.}, \quad \alpha_x \alpha_y \neq 0 \tag{98}$$

pour une intégrale cherchée des caractéristiques.

b) Passons, à présent, à la seconde hypothèse. Si la fonction inconnue ne figure pas dans le système (23) et C_k représente la constante arbitraire additive dans l'intégrale complète, alors, en vertu de (98), l'intégrale des caractéristiques admet la forme suivante

$$\frac{(i, l)}{(i, j)}=\text{const.}$$

c) Supposons, enfin, conservant l'hypothèse sous b), que le système (23) ait une intégrale intermédiaire ayant la constante C_l, c'est-à-dire que l'on ait $(j, k, l)=0$. Dans ce cas l'intégrale cherchée des caractéristiques, admet la forme

$$\frac{V_{xl}}{V_{xj}}=\text{const.} \quad \frac{V_{yl}}{V_{yj}}=\text{const.}$$

C o n c l u s i o n. L'intégrale générale des caractéristiques est définie, en vertu de l'intégrale complète, par les formules suivantes

(99)

$$z=V, \ p=V_x, \ q=V_y, \ s=V_{xy}, \ \alpha \equiv \mathcal{D}\left(\frac{V, V_x, V_y}{C_i, C_j, C_k}\right) : \mathcal{D}\left(\frac{V, V_x, V_y}{C_i, C_l, C_k}\right)=C_5 \ \alpha_x \alpha_y \neq 0.$$

8. L'intégrale générale mixte

Dans ses Leçons sur la théorie des équations aux dérivées partielles du second ordre, [19], était introduite par *N. Saltykow* la notion de l'intégrale générale mixte pour une équation ou pour un système d'équations aux dérivées partielles du second ordre.

On démontre dans ce paragraphe que les systèmes

$$\begin{aligned} & r+H x, y, z, p, q, s)=0, \\ & t+\Phi(x, y, z, p, q, s)=0 \end{aligned} \tag{23}$$

admettant une intégrale générale mixte sont:

a) en involution de *Darboux-Lie*,

b) linéaires par rapport aux r, s, t,

c) possédent une intégrale intermédiaire.

Considérons une équation de la forme

$$z=V[x, y, C_1, C_2, f(\omega)] \tag{100}$$

où f étant une fonction arbitraire de $\omega(x, y, C_1, C_2)$, et ω une fonction donnée des variables indépendantes x et y et des constantes arbitraires C_1, C_2.

On dit que par l'équation (100) est définie une intégrale générale mixte de l'équation donnée aux variables du second ordre

$$F(x, y, z, p, q, r, s, t)=0 \tag{101}$$

si le résultat de l'élimination des quantités C_1, C_2, f, f', f'' parmi l'équation (100) et les équations correspondantes dérivées par rapport aux x et y — du premier et second ordre — ne donne que l'équation (101).

D'une manière analogue on peut définir l'intégrale générale mixte pour le système (23).

Pour que l'équation

$$z=V[x, y, C, f(\omega)] \tag{102}$$

à une fonction f arbitraire de la fonction donnés $\omega(x, y, C)$ et une constante C arbitraire, soit l'intégrale générale mixte du système (23), ce dernier doit s'obtenir par l'éliminations des quantités arbitraires: C, f, f', f'' entre l'équation (102) et les équations dérivées suivantes

$$\begin{aligned} &p=V_x+V_f\omega_x f'\equiv d_x V, \quad q=V_y+V_f\omega_y f'\equiv d_y V, \\ &r=V_{xx}+2V_{xf}\omega_x f'+V_{ff}\omega_x^2 f'^2+V_f\omega_x^2 f''+V_f\omega_{xx}f', \\ &s=V_{xy}+(V_{xf}\omega_y+V_{fy}\omega_x)f'+V_{ff}\omega_x\omega_y f'^2+V_f\omega_x\omega_y f''+V_f\omega_{xy}f', \\ &t=V_{yy}+2V_{yf}\omega_y f'+V_{ff}\omega_y^2 f'^2+V_y\omega_y^2 f''+V_f\omega_{yy}f'. \end{aligned} \tag{103}$$

Par l'élimination menntionée on obtient évidemment un système (23) qui est linéaire par rapport aux r, s, t.

Grâce à l'élimination de f' entre les équations $p=d_x V$ et $q=d_y V$ on a

$$\frac{p-V_x}{\omega_x}=\frac{q-V_y}{\omega_y}. \tag{104}$$

Ensuite, par l'élimination de f entre (104) et (102) on obtiendra une relation linéaire par rapport aux p et q:

$$A(x, y, z, p, q, C)\equiv a(x, y, z, C)p+b(x, y, z, C)q+c(x, y, z, C)=0. \tag{105}$$

La relation obtenue (105) est une intégrale intermédiaire première à une constante arbitraire C du système (23).

Supposons maintenant que l'on ait dans un domain $D(x, y, z, C)$ bien déterminé la condition suivante: $\partial A/\partial C\neq 0$ et l'équation équivalente à (105)

$$C=m(x, y, z, p, q).$$

En vertu de l'identité évidente $A(x, y, z, p, q, m) \equiv 0$ on peut obtenir les relations suivantes

$$\begin{aligned}(106)\qquad & A_{xm} + A_{mm} m_x = 0, \quad A_{ym} + A_{mm} m_y = 0, \\ & A_{zn} + A_{mm} m_z = 0, \quad A_{pm} + A_{mm} m_p = 0, \\ & A_{qm} + A_{mm} m_q = 0.\end{aligned}$$

On peut obtenir le système (23) aussi et à l'aide de l'intégrale intermédiaire (105):

$$\Phi \equiv (A_x) + p(A_z) + r(A_p) + s(A_q) = 0,$$
$$W \equiv (A_y) + q(A_z) + s(A_p) + t(A_q) = 0,$$

les parenthèses désignant le résultat de la substitution de la valeur C avec la fonction $m(x, y, z, p, q)$.

Grâce aux identités (106) il est aisé de voir que les équations précédentes vérifient les conditions bien connues de l'involution de *Darboux-Lie* (8)

$$\frac{X}{Y} = -\frac{Z}{X} = -\frac{R}{T}$$

en utilisant les désignations suivantes

$$X \equiv \mathcal{D}\left(\frac{\Phi, W}{r, t}\right), \qquad Y \equiv \mathcal{D}\left(\frac{\Phi, W}{s, t}\right), \qquad Z \equiv \mathcal{D}\left(\frac{\Phi, W}{s, r}\right),$$

$$R \equiv \begin{vmatrix} D_x \Phi & D_x W \\ \Phi_r & W_r \end{vmatrix}, \qquad T \equiv \begin{vmatrix} D_y \Phi & D_y W \\ \Phi_t & W_t \end{vmatrix}.$$

Donc, le système admettant l'intégrale générale mixte (102) est linéaire par rapport aux r, s, t et en involution de *Darboux-Lie.*

Exemples.

a) Le système

$$st + x(rt - s^2)^2 = 0, \qquad 2x(rt - s^2) = pt - qs$$

admettant l'intégrale générale mixte

$$z = 4/3\ Cx^{3/2} + f(y - C^2 x)$$

est équivalente au système linéaire suivant en involution de *Darboux-Lie*

$$r = \frac{xp^2 s - xp \pm R}{x(pq - 2x \pm 2R)}, \qquad t = \frac{pq - 2x \pm 2R}{p^2} s$$

$$R \equiv x^2 - xpq$$

avec l'intégrale intermédiaire

$$p^2 + C^2 q = 2Cx^{1/2}.$$

b) Le système linéaire en involution de *Darboux-Lie*

$$s=\frac{p-xr}{y}, \qquad [t=\left(\frac{x}{y}\right)^2 r-\frac{2}{y^2}(z-yq)$$

a l'intégrale mixte

$$z=Cy-x^2 f(y/x)$$

et l'intégrale première intermédiaire

$$xp+yq=2z-Cy.$$

On peut former l'intégrale générale mixte et aussi par la méthode de la variations des constantes dans l'intégrale complète

$$z=V(x, y, C_1, C_2, C_3, C_4) \tag{44}$$

du système (23) en involution de *Darboux-Lie*, [20].

En variant les contantes dans la foimule (44) on peut obtenir les conditions suivantes

$$\begin{aligned}&\sum_{i=1}^{4}\frac{\partial V}{\partial C_i}\frac{\partial C_i}{\partial x}=0, \quad \sum_{i=1}^{4}\frac{\partial^2 V}{\partial x\partial C_i}\frac{\partial C_i}{\partial x}=0, \quad \sum_{i=1}^{4}\frac{\partial^2 V}{\partial y\partial C_i}\frac{\partial C_i}{\partial x}=0,\\ &\sum_{i=1}^{4}\frac{\partial V}{\partial C_i}\frac{\partial C_i}{\partial y}=0, \quad \sum_{i=1}^{4}\frac{\partial^2 V}{\partial x\partial C_i}\frac{\partial C_i}{\partial y}=0, \quad \sum_{i=1}^{4}\frac{\partial^2 V}{\partial y\partial C_i}\frac{\partial C_i}{\partial y}=0.\end{aligned} \tag{48}$$

pour les fonctions inconnues $C_i(x, y)$.

Utilisons les désignations

$$(i, j, k)\equiv\mathcal{D}\left(\frac{V, V_x, V_y}{C_i, C_j, C_k}\right)$$

pour les déterminants fonctionels correspondants.

En supposant qu'on ait: $(2, 3, 4)=0$, et que les autres déterminants de la forme (i, j, k) soient distincts du zéro, on peut les conditions (48) mettre sous une de la forme suivante

$$\frac{\partial C_1}{\partial\xi_j}=0, \quad \frac{\partial C_2}{\partial\xi_j}=-\frac{(1, 4, 3)}{(1, 2, 3)}\frac{\partial C_4}{\partial\xi_j}, \quad \frac{\partial C_3}{\partial\xi_j}=-\frac{(1, 2, 4)}{(1, 2, 3)}\frac{\partial C_4}{\partial\xi_j}, \tag{48$_1$}$$

$$\frac{\partial C_i}{\partial\xi_j}=0, \quad \frac{\partial C_2}{\partial\xi_j}=-\frac{(1, 3, 4)}{(1, 2, 4)}\frac{\partial C_3}{\partial\xi_j}, \quad \frac{\partial C_4}{\partial\xi_j}=-\frac{(1, 2, 3)}{(1, 2, 4)}\frac{\partial C_3}{\partial\xi_j}, \tag{48$_2$}$$

$$\frac{\partial C_1}{\partial\xi_j}=0, \quad \frac{\partial C_3}{\partial\xi_j}=-\frac{(1, 2, 4)}{(1, 3, 4)}\frac{\partial C_2}{\partial\xi_j}, \quad \frac{\partial C_4}{\partial\xi_j}=-\frac{(1, 3, 2)}{(1, 3, 4)}\frac{\partial C_2}{\partial\xi_j}, \tag{48$_3$}$$

avec $\xi_1\equiv x$, $\xi_2\equiv y$.

En vertu des équations citées et de la condition (46) on peut obtenir le système correspondant de *Charpit*

$$(107) \qquad \frac{\partial}{\partial y}\left[\frac{(1, 2, 4)}{(1, 2, 3)}\right]\frac{\partial C_{i+1}}{\partial x}=\frac{\partial}{\partial x}\left[\frac{(1, 2, 4)}{(1, 2, 3)}\right]\frac{\partial C_{i+1}}{\partial y} \qquad (i=1, 2, 3)$$

pour déterminer les fonctions $C_{i+1}(x, y)$ inconnues. L'intégrale générale de ce système est

$$C_{i+1}=f_i\left[\frac{(1, 2, 4)}{(1, 2, 3)}\right], \qquad (i=1, 2, 3),$$

f_i étant trois fonctions arbitraires. Grâce aux système (48), par exemple au système (48_1) on a

$$(108) \qquad f_1'(u)=a[C_1, u, f_i]f_3', \; f_2'(u)=-uf_3', \; u\equiv(1, 2, 4)/(1, 2, 3)$$

et une des fonctions f_i reste arbitraire.

Donc, par les équations

$$z=V[x, y, C_1, f_1(u), f_2(u), f_3(u)]$$

et (108) est défini l'intégrale générale mixte avec une constante et une fonction arbitraire.

a) Reprenons l'équation *d'Ampère*

$$st+x(rt-s^2)=0$$

que avec l'équation

$$2x(rt-s^2)=pt-qs$$

représente un système en involution de *Darboux-Lie*. On voit immédiatement que ce système admet l'intégrale complète

$$z=4/3\, C_1 x^{3/2}+C_2(y-C_1{}^2x)^2+C_3(y-C_1{}^2x)+C_4$$

vérifiant la condition $(2, 3, 4)=0$. Le système correspondant de *Charpit*

$$\frac{\partial C_{i+1}}{\partial x}+C_1{}^2\frac{\partial C_{i+1}}{\partial y}=0, \quad (i=1, 2, 3), \quad C_1=\text{const.}$$

a l'intégrale générale

$$C_{i+1}=f_i(u), \quad (i=1, 2, 3), \quad u\equiv y-C_1^2x.$$

Les fonctions f_i, dans ce cas, vérifient les conditions suivantes

$$f_1'=-\frac{1}{2u}f_2', \qquad f_3'=-\frac{1}{2}uf_2'.$$

En prennant $f_2' = u\alpha'''$ (α désignant une fonction arbitraire de u) et en substituant les constantes C_{i+1} dans l'intégrale complète par les fonctions correspondantes f_i, on obtient l'intégrale générale mixte cherchée dans la forme

$$z = 4/3\, C_1 x^{3/2} - \alpha(y - C_1^2 x)$$

antérieurement citée.

b) Le second système suivant en involution de *Darboux-Lie*

$$s = \frac{p - xr}{y}, \qquad t = \left(\frac{x}{y}\right)^2 r - \frac{2}{y^2}(z - yq)$$

a l'intégrale complète

$$z = C_1(C_2 + C_3)\, y + C_4^2 xy + C_3 x^2 + C_2 C_4 y^2$$

admettant les conditions $(i, j, k) \neq 0$. En suivant un procédé de *C. Orloff*, [21] on définit les constantes nouvelles a_i par les relations suivantes

$$a_1 = C_1(C_2 + C_3), \quad C_2 C_4 = a_2, \quad C_3 = a_3, \quad C_4^2 = a_4$$

et on peut mettre l'intégrale complète considérée sous la forme

$$z = a_1 y + a_4 xy + a_3 x^2 + a_2 y^2$$

vérifiant les conditions suivantes

$$(1, 2, 3) = -2xy^2, \quad (1, 2, 4) = -y^3, \quad (1, 3, 4) = x^2 y, \quad (2, 3, 4) = 0.$$

Le système correspondant de *Charpit*

$$\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\frac{\partial a_{i+1}}{\partial x} - \frac{\partial}{\partial x}\left(\frac{y}{x}\right)\frac{\partial a_{i+1}}{\partial y} = 0,$$

admet l'intégrale générale

$$a_{i+i} = f_i(u), \quad u = y/x, \quad (i = 1, 2, 3, 4).$$

Les conditions suivantes

$$f_1' = -f_3'/2u, \quad f_2' = -uf_3'/2$$

donnent

$$f_1 = -\alpha''/2, \quad f_2 = -u^2\alpha''/2 + u\alpha' - \alpha, \quad f_3 = u\alpha'' - \alpha';$$

$\alpha(u)$ étant une fonction arbitraire, et l'intégrale générale mixte devient

$$z = a_1 y - x^2 \alpha(y/x)$$

avec la constante arbitraire: a_1 et la fonction arbitraire: α.

9. L'intégrale générale de Lagrange

Considérons le système en involution de *Darboux-Lie*

$$(23) \qquad \begin{cases} r + H(x, y, z, p, q, s) = 0, \\ t + \Phi(x, y, z, p, q, s) = 0 \end{cases}$$

et ajoutons au (23) le système correspondant de *Charpit*

$$(64)\qquad \begin{cases} \dfrac{\partial z}{\partial x}+H_s\dfrac{\partial z}{\partial y}=p+H_s q, \\ \dfrac{\partial p}{\partial x}+H_s\dfrac{\partial p}{\partial y}=r+H_s s, \\ \dfrac{\partial q}{\partial x}+H_s\dfrac{\partial q}{\partial y}=s+H_s t, \\ \dfrac{\partial r}{\partial x}+H_s\dfrac{\partial r}{\partial y}+D_x H=0, \\ \dfrac{\partial s}{\partial x}+H_s\dfrac{\partial s}{\partial y}+D_y H=0, \\ \dfrac{\partial t}{\partial x}+H_s\dfrac{\partial t}{\partial y}+H_s D_y \Phi=0 \end{cases}$$

que joue le rôle du système des caractéristiques.

Posons à présent le problème de la formation de l'intégrale générale, contenant des fonctions arbitraires, du système donné (23) à l'aide de l'intégrale générale du système de *Charpit*. Ce problème peut être résolu de la manière analogue à celle de *Lagrange* concernant les équations aux dérivées partielles du premier ordre, [22], [23].

Le système de *Charpit* (64), outre les premiers membres des équations (23), admet encore cinq intégrales distinctes que l'on va désigner: f, f_1, f_2, f_3, f_4 et que l'on va supposer indépendantes des variables r et t, en vertu des équations (23).

Par conséquant, l'intégrale générale du système de *Charpit*, vérifiant de plus les conditions (23), sera représentée par l'ensemble de ces dernières deux équations et des quatre équations suivantes

$$(108)\qquad f_i=\prod\nolimits_i(f), \quad (i=1,\,2,\,3,\,4).$$

On va chercher les solutions du système (23) vérifiant les conditions

$$(109)\qquad dz=p\,dx+q\,dy, \quad dp=r\,dx+s\,dy, \quad dq=s\,dx+t\,dy$$

sous l'hypothèse que les valeurs de r et de t sont déterminées par le système (23).

Comme les fonctions $\prod_i$ sont arbitraires, il est impossible de résoudre les équations (108) par rapport aux variables qui y figurent. Pour éviter les difficultés mentionnées nous posons

$$f=u, \quad f_i=u_i, \quad (i=1,\,2,\,3,\,4).$$

Grâce aux propriétés du système (64) les équations précédentes sont résoluble par rapport aux variables y, z, p, q de sort que l'on obtient

$$\begin{aligned} y&=K_1(x, u, u_1, u_2, u_3, u_4),\\ z&=K_2(x, u, u_1, u_2, u_3, u_4),\\ p&=K_3(x, u, u_1, u_2, u_3, u_4),\\ q&=K_4(x, u, u_1, u_2, u_3, u_4),\\ s&=K_5(x, u, u_1, u_2, u_3, u_4), \end{aligned}$$

Les variables r et t s'expriment de même comme fonctions de x, u, u_1, u_2, u_3, u_4 grâce aux équations (23).

Cela posé, les équations (109) deviennent en nouvelles variables

$$(109') \qquad \sum_{i=0}^{4} A_{ik}\, du_i=0, \quad (k=1, 2, 3)$$

en posant $u_0=u$, le coefficient de dx s'annule identiquement grâce aux équations (64).

Or, les formules (108) imposent les relations nouvelles

$$u_i=\prod_i(u), \quad (i=1, 2, 3, 4).$$

Par conséquent, les équations (109') devient

$$\left(A_{0k}+\sum_{i=1}^{4} A_{ik}\prod_i'\right) du=0 \quad (k=1, 2, 3).$$

Comme la différentielle du ne peut pas s'annuler, on en tire trois relations liant les fonctions arbitraires $\prod_i'$, à savoir

$$(110) \qquad A_{ok}+\sum_{i=1}^{4} A_{ik}\prod_i'=0, \quad (k=1, 2, 3).$$

Donc, la solution cherchée du système (23) est définie par l'ensemble des équations (108) contenant quatre fonctions arbitraires qui sont liées par trois relations (110) de sorte qu'il ne reste qu'une seule fonction arbitraire. Par conséquent le problème cité ci-dessus est résolu d'une manière analogue comme le problème posé par *Lagrange*, [22], dans la théorie des équations aux dérivées partielles du premier ordre.

10. Sur les systèmes d'équations aux dérivés partielles du second ordre à trois variables indépendantes réductibles à ceux de Charpit

On étudie un tel système d'équations aux dérivées partielles du second ordre à une fonction inconnue de trois variables indépendantes qui soit réductible à un système de *Charpit* dont l'intégration se ramène à celle d'un système des équations différentielles ordinaires. La détermination des conditions lesquelles doivent vérifier le système en question. Les conditions nécessaires et suffisantes pour l'intégrale complète. La généralisation de la notion des fonctions des caractéristiques de *N. Saltykow*, [18] pour le système considéré et leur application à la formation de l'intégrale complète. L'intégrale générale du système correspondant de *Charpit* et la formation des solutions du système étudié en généralisant la méthode de *Lagrange*, [24].

A. Considérons le système de trois équations aux dérivées partielles du second ordre à trois variables indépendantes x_i

$$p_{ii}+f_i(x_1, x_2, x_3, z, p_1, p_2, p_3, p_{12}, p_{13}, p_{23})=0, \quad (i=1, 2, 3) \tag{111}$$

avec les notations usuelles

$$p_i=\frac{\partial z}{\partial x_i}, \quad p_{ij}=\frac{\partial^2 z}{\partial x_i \partial x_j}$$

et avec les conditions d'indépendance de l'ordre de la différentiation de la fonction z par rapport aux x_i.

Formons les équations dérivées

$$\frac{\partial p_{ii}}{\partial x_k}+\frac{\partial f_i}{\partial p_{12}}\frac{\partial p_{12}}{\partial x_k}+\frac{\partial f_i}{\partial p_{13}}\frac{\partial p_{13}}{\partial x_k}+\frac{\partial f_i}{\partial p_{23}}\frac{\partial p_{23}}{\partial x_k}+D_k f_i=0 \quad (i, k=1, 2, 3) \tag{112}$$

le symbol D_k désignant l'opérateur suivant

$$D_k \equiv \frac{\partial}{\partial x_k}+p_k\frac{\partial}{\partial z}+\sum_s p_{sk}\frac{\partial}{\partial p_s}.$$

Si l'on introduit les conditions suivantes:

$$\frac{\partial f_1}{\partial p_{23}}=\frac{\partial f_2}{\partial p_{13}}=\frac{\partial f_3}{\partial f_{12}}=0 \tag{113}$$

$$\frac{1}{\dfrac{\partial f_2}{\partial p_{12}}}=\frac{\dfrac{\partial f_1}{\partial p_{12}}}{1}=\frac{\dfrac{\partial f_1}{\partial p_{13}}}{\dfrac{\partial f_2}{\partial p_{23}}}=\frac{D_2 f_1}{D_1 f_2}, \tag{114}$$

$$\frac{1}{\dfrac{\partial f_3}{\partial p_{13}}}=\frac{\dfrac{\partial f_1}{\partial p_{13}}}{1}=\frac{\dfrac{\partial f_1}{\partial p_{12}}}{\dfrac{\partial f_3}{\partial p_{23}}}=\frac{D_3 f_1}{D_1 f_3}, \tag{115}$$

$$\frac{1}{\dfrac{\partial f_3}{\partial p_{23}}}=\frac{\dfrac{\partial f_2}{\partial p_{23}}}{1}=\frac{\dfrac{\partial f_2}{\partial p_{12}}}{\dfrac{\partial f_3}{\partial p_{13}}}=\frac{D_3 f_2}{D_2 f_3} \tag{116}$$

des neuf équations ili ne reste que six équations distinctes suivantes

$$
(117)\quad
\begin{gathered}
\frac{\partial p_{11}}{\partial x_1}+\frac{\partial f_1}{\partial p_{12}}\frac{\partial p_{11}}{\partial x_2}+\frac{\partial f_1}{\partial p_{13}}\frac{\partial p_{11}}{\partial x_3}+D_1 f_1=0,\\
\frac{\partial p_{22}}{\partial x_1}+\frac{\partial f_1}{\partial p_{12}}\frac{\partial p_{22}}{\partial x_2}+\frac{\partial f_1}{\partial p_{13}}\frac{\partial p_{22}}{\partial x_3}+\frac{1}{\dfrac{\partial f_2}{\partial p_{12}}}D_2 f_2=0,\\
\frac{\partial p_{33}}{\partial x_1}+\frac{\partial f_1}{\partial p_{12}}\frac{\partial p_{33}}{\partial x_2}+\frac{\partial f_1}{\partial p_{13}}\frac{\partial p_{33}}{\partial x_3}+\frac{1}{\dfrac{\partial f_3}{\partial p_{13}}}D_3 f_3=0,\\
\frac{\partial p_{12}}{\partial x_1}+\frac{\partial f_1}{\partial p_{12}}\frac{\partial p_{12}}{\partial x_2}+\frac{\partial f_1}{\partial p_{13}}\frac{\partial p_{12}}{\partial x_3}+D_2 f_1=0,\\
\frac{\partial p_{13}}{\partial x_1}+\frac{\partial f_1}{\partial p_{12}}\frac{\partial p_{13}}{\partial x_2}+\frac{\partial f_1}{\partial p_{13}}\frac{\partial p_{13}}{\partial x_3}+D_3 f_1=0,\\
\frac{\partial p_{23}}{\partial x_1}+\frac{\partial f_1}{\partial p_{12}}\frac{\partial p_{23}}{\partial x_2}+\frac{\partial f_1}{\partial p_{13}}\frac{\partial p_{23}}{\partial x_3}+\frac{1}{\dfrac{\partial f_2}{\partial p_{12}}}D_3 f_2=0.
\end{gathered}
$$

Si l'on ajoute aux équations (117) les équations suivantes

$$
(118)\quad
\begin{cases}
\dfrac{\partial z}{\partial x_1}+\dfrac{\partial f_1}{\partial p_{12}}\dfrac{\partial z}{\partial x_2}+\dfrac{\partial f_1}{\partial p_{13}}\dfrac{\partial z}{\partial x_3}-p_1-\dfrac{\partial f_1}{\partial p_{12}}p_2-\dfrac{\partial f_1}{\partial p_{13}}p_3=0\\
\dfrac{\partial p_i}{\partial x_1}+\dfrac{\partial f_1}{\partial p_{12}}\dfrac{\partial p_i}{\partial x_2}+\dfrac{\partial f_1}{\partial p_{13}}\dfrac{\partial p_i}{\partial x_3}-p_{i1}-\dfrac{\partial f_1}{\partial p_{12}}p_{i2}-\dfrac{\partial f_1}{\partial p_{13}}p_{i3}=0,\\
\qquad (i=1,2,3)
\end{cases}
$$

le système (117) et (118) est alors un système de *Charpit* des fonctions z p_i, p_{ii}, p_{ij}. Le système correspondant aux équations différentielles ordinaires devient:

$$
(119)\quad
\begin{cases}
dx_1=\dfrac{dx_2}{\dfrac{\partial f_1}{\partial p_{12}}}=\dfrac{dx_3}{\dfrac{\partial f_1}{\partial p_{13}}}=\dfrac{dz}{p_1+\dfrac{\partial f_1}{\partial p_{12}}p_2+\dfrac{\partial f_1}{\partial p_{13}}p_3}=\dfrac{dp_i}{p_{i1}+\dfrac{\partial f_1}{\partial p_{12}}p_{i2}+\dfrac{\partial f_1}{\partial p_{13}}p_{i3}}=\\
=\dfrac{dp_{11}}{-D_1 f_1}=\dfrac{dp_{22}}{-D_2 f_2\Big/\dfrac{\partial f_2}{dp_{12}}}=\dfrac{dp_{33}}{-D_3 f_3\Big/\dfrac{\partial f_3}{\partial p_{13}}}=\dfrac{dp_{12}}{-D_2 f_1}=\\
=\dfrac{dp_{33}}{-D_3 f_2\Big/\dfrac{\partial f_2}{\partial p_{12}}}=\dfrac{dp_{13}}{-D_3 f_1}.
\end{cases}
$$

Ces dernières équations jouent, par rapport au système (111) avec les conditions (113)—(116), le même rôle que les équations des caractéristiques dans la théorie des équations aux dérivées partielles du premier ordre.

B. Dans la théorie du système (111) nous utiliserons la notion de l'intégrale complète:

(120) $$z = V(x_1, x_2, x_3, C_1, \dots, C_7)$$

où l'on a la condition

(121) $$\Delta \equiv D\left(\frac{V, V_1, V_2, V_3, V_{12}, V_{13}, V_{23}}{C_1, C_2, \dots\dots\dots\dots, C_7}\right) \neq 0$$

introduisant les notations suivantes

$$V_i = \frac{\partial V}{\partial x_i}, \qquad V_{ij} \equiv \frac{\partial^2 V}{\partial x_i \partial x_j},$$

C_i désignant les constantes arbitraires distinctes; de plus le résultat de l'élimination des équations: $z = V$, $p = V_i$, $p_{ii} = V$, $p_{ij} = V_{ij}$ de toutes les constantes arbitraires C_i ne donne que les équations (111).

Dans le cas où le système (111) est réductible à un système de *Charpit* (117)—(118), son intégrale complète (120) doit satisfaire à des conditions complémentaires qu'on va établir. En effet, les équations suivantes

$$z = V(x_1, x_2, x_3, C_1, \dots, C_7)$$

$$p_i = V_i\,(\dots\dots\dots\dots\dots\dots), \quad (i = 1, 2, 3)$$

$$p_{ij} = V_{ij}\,(\dots\dots\dots\dots\dots\dots), \quad (j = 1, 2, 3, i \neq j)$$

sont équivalentes, grâce à (121), aux équations

(122) $$F_k(x_1, x_2, x_3, z, p_1, p_2, p_3, p_{12}, p_{13}, p_{23}) = C_k, \quad (k = 1, 2, 3, \dots, 7).$$

On a de plus les relations $p_{ii} = V_{ii}$, $(i = 1, 2, 3)$ que l'on peut remplacer aisément, en vertu des équations précédentes, par les égalités suivantes

(123) $$-f_i(x_1, x_2, x_3, z, p_1, p_2, p_3, p_{12}, p_{13}, p_{23}) = V_{ii}(x_1, x_2, x_3, F_1, \dots, F_7) \equiv \overline{V}_{ii}, \quad (i = 1, 2, 3).$$

Introduisons, enfin, les désignations

(122') $$\overline{F}_k = F_k(x_1, x_2, x_3, V, V_1, V_2, V_3, V_{12}, V_{13}, V_{23}) = C_k, \quad (k = 1, 2, \dots, 7).$$

En différentiant ces dernières relations par rapport aux C_i on obtient

(124) $$\frac{\partial \overline{F}_k}{\partial z}\frac{\partial V}{\partial C_i} + \sum_{s=1}^{3} \frac{\partial \overline{F}_k}{\partial p_s}\frac{\partial^2 V}{\partial x \partial C_i} + \frac{\partial \overline{F}_k}{\partial p_{12}}\frac{\partial^3 V}{\partial x_1 \partial x_2 \partial C_i} + \frac{\partial \overline{F}_k}{\partial p_{13}}\frac{\partial^3 V}{\partial x_1 \partial x_3 \partial C_i} + \frac{\partial \overline{F}_k}{\partial p_{23}}\frac{\partial^3 V}{\partial x_2 \partial x_3 \partial C_i} = \frac{\partial C_k}{\partial C_i},$$

$$(i, k = 1, 2, \dots, 7).$$

Il en résulte, grâce à (121):

$$
(125)\quad \begin{cases}
\dfrac{\partial \overline{F}_k}{\partial z}=\dfrac{1}{\Delta}D\left(\dfrac{C_k, V_1, V_2, V_3, V_{12}, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right), & \dfrac{\partial \overline{F}_k}{\partial p_{12}}=\dfrac{1}{\Delta}D\left(\dfrac{V, V_1, V_2, V_3, C_k, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right) \\
\dfrac{\partial \overline{F}_k}{\partial p_1}=\dfrac{1}{\Delta}D\left(\dfrac{V, C_k, V_2, V_3, V_{12}, V_{13}, V_{23}}{C_1, C_2 \ldots\ldots\ldots\ldots, C_7}\right), & \dfrac{\partial \overline{F}_k}{\partial p_{13}}+\dfrac{1}{\Delta}D\left(\dfrac{V, V_1, V_2, V_3, V_{12}, C_k, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right) \\
\dfrac{\partial \overline{F}_k}{\partial p_2}=\dfrac{1}{\Delta}D\left(\dfrac{V, V_1, C_k, V_3, V_{12}, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right), & \dfrac{\partial \overline{F}_k}{\partial p_{13}}=\dfrac{1}{\Delta}D\left(\dfrac{V, V_1, V_2, V_3, V_{12}, V_{13}, C_k}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right) \\
\dfrac{\partial \overline{F}_k}{\partial p_3}=\dfrac{1}{\Delta}D\left(\dfrac{V, V_1, V_2, C_k, V_{12}, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right). &
\end{cases}
$$

D'autre part, différentiant les identités (123), on trouve:

$$
(126)\qquad -\frac{\partial f_i}{\partial \mu}=\sum_{k=1}^{5}\overline{V}_{iiC_k}\frac{\partial F_k}{\partial \mu}, \quad (i=1, 2, 3)
$$

$$
(127)\qquad -\frac{\partial f_i}{\partial x_s}=\frac{\partial \overline{V}_{ii}}{\partial x_s}+\sum_{k=1}^{5}\overline{V}_{iiC_k}\frac{\partial F_k}{\partial x_s}, \quad (i, s=1, 2, 3),
$$

où l'on remplace μ respectivement par p_{ij}, p_s, z. En désignant, par les paranthèses ( ), le résultat de la substitution z, p_i, p_{ij} respectivement par leur valeurs V, V_i, V_{ij} et par $\Delta_{12}^{(i)}$, les valeurs des déterminants fonctionnels

$$
\Delta_{12}^{(i)}\equiv D\left(\frac{V, V_1, V_2, V_3, V_{ii}, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots C_7}\right), \qquad (i=1, 2, 3)
$$

on met les équations (126) sous la forme suivante, par exemple, pour $\mu=p_{12}$

$$
-\frac{\partial f_i}{\partial p_{12}}=\sum_{k=1}^{5}V_{iiC_k}\frac{\partial \overline{F}_k}{\partial p_{12}}=\frac{1}{\Delta}\sum_{k=1}^{5}V_{iiC_k}D\left(\frac{V, V_1, V_2, V_3, C_k, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right),
$$

ou bien

$$
(128)\qquad -\left(\frac{\partial f_i}{\partial p_{12}}\right)=\frac{\Delta_{12}^{(i)}}{\Delta}, \quad (i=1, 2, 3)
$$

et de la même manière aussi

$$
-\left(\frac{\partial f_i}{\partial p_{13}}\right)=\frac{\Delta_{13}^{(i)}}{\Delta}, \qquad -\left(\frac{\partial f_i}{\partial p_{23}}\right)=\frac{\Delta_{23}^{(i)}}{\Delta}, \quad (i=1, 2, 3)
$$

$$
-\left(\frac{\partial f_i}{\partial z}\right)=\frac{\Delta_z^{(i)}}{\Delta}, \qquad -\left(\frac{\partial f_i}{\partial p_s}\right)=\frac{\Delta_s^{(i)}}{\Delta}, \quad (i=1, 2, 3)
$$

où l'on vient de poser

$$\Delta_{13}^{(i)} \equiv D\left(\frac{V, V_1, V_2, V_3, V_{12}, V_{ii}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right), \quad \Delta_{23}^{(i)} \equiv D\left(\frac{V, V_1, V_2, V_3, V_{12}, V_{13}, V_{ii}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right),$$

$$\Delta_z^{(i)} \equiv D\left(\frac{V_{ii}, V_1, V_2, V_3, V_{12}, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right), \quad \Delta_1^{(i)} \equiv D\left(\frac{V, V_i, V_2, V_3, V_{12}, V_{13}, V_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots, C_7}\right), \ldots$$

D'autre part en différentiant les identités (122') par rapport aux x_s on obtient

$$-\frac{\partial \overline{F}_k}{\partial x_s} = \frac{\partial \overline{F}_k}{\partial z} V_s + \sum_{i=1}^{3} \frac{\partial \overline{F}_k}{\partial p_i} V_{is} + \frac{\partial \overline{F}_k}{\partial p_{12}} V_{12s} + \frac{\partial \overline{F}_k}{\partial p_{13}} V_{13s} + \frac{\partial \overline{F}_k}{\partial p_{23}} V_{23s} = 0$$

et grâce aux formules (125) et (127) on a

$$-\left(\frac{\partial f_i}{\partial x_s}\right) = V_{is} - \frac{1}{\Delta}\left(V_s \Delta_z^{(i)} + \sum_{j=1}^{3} V_{js} \Delta_j^{(i)} + V_{12s} \Delta_{12}^{(i)} + V_{13s} \Delta_{13}^{(i)} + V_{23s} \Delta_{23}^{(i)}\right).$$

En vertu des formules établies on aura

(130)
$$D_s f_i = -V_{iis} + V_{12s} \frac{\Delta_{12}^{(i)}}{\Delta} + V_{13s} \frac{\Delta_{13}^{(i)}}{\Delta} + V_{23s} \frac{\Delta_{23}^{(i)}}{\Delta}.$$

$(i, s = 1, 2, 3)$.

Écrivons les conditions (114)—(116) sous la forme suivante

(114')
$$\frac{\partial f_1}{\partial p_{12}} \frac{\partial f_2}{\partial p_{12}} = 1, \quad \frac{\partial f_1}{\partial p_{12}} \frac{\partial f_2}{\partial p_{23}} = \frac{\partial f_1}{\partial p_{13}}, \quad D_2 f_1 = \frac{\partial f_1}{\partial p_{12}} D_1 f_2,$$

(115')
$$\frac{\partial f_1}{\partial p_{13}} \frac{\partial f_3}{\partial p_{13}} = 1, \quad \frac{\partial f_1}{\partial p_{13}} \frac{\partial f_3}{\partial p_{23}} = \frac{\partial f_1}{\partial p_{12}}, \quad D_3 f_1 = \frac{\partial f_1}{\partial p_{13}} D_1 f_3,$$

(116')
$$\frac{\partial f_2}{\partial p_{23}} \frac{\partial f_3}{\partial p_{23}} = 1, \quad \frac{\partial f_2}{\partial p_{23}} \frac{\partial f_3}{\partial p_{13}} = \frac{\partial f_2}{\partial p_{12}}, \quad D_3 f_2 = \frac{\partial f_2}{\partial p_{23}} D_2 f_3,$$

où les deux dernières de la deuxième colonne sont les conséquences des précédentes. Les conditions (113) et (114')—(116') sont satisfaites identiquement aussi dans le cas de la substitution de z, p_i, p_{ik} respectivement par V, V_i, V_{ik} de sorte que l'on obtient les conditions cherchées:

(113_1) $\Delta_{23}^{(1)} = \Delta_{13}^{(2)} = \Delta_{12}^{(3)} = 0$

(114_1) $\Delta_{12}^{(1)} \Delta_{12}^{(2)} - \Delta^2 = 0, \quad \Delta_{12}^{(1)} \Delta_{23}^{(2)} + \Delta \Delta_{13}^{(1)} = 0,$

(115_1) $\Delta_{13}^{(1)} \Delta_{13}^{(3)} - \Delta^2 = 0, \quad \Delta_{13}^{(1)} \Delta_{23}^{(2)} + \Delta \Delta_{12}^{(1)} = 0,$

(116_1) $\Delta_{23}^{(2)} \Delta_{23}^{(3)} - \Delta^2 = 0, \quad \Delta_{23}^{(2)} \Delta_{13}^{(1)} + \Delta \Delta_{12}^{(2)} = 0.$

Les deux dernières conditions de la seconde colonne sont les conséquences de toutes les précédentes. Les relations de la troisième colonne (114')—(116') n'imposent pas des conditions nouvelles à la fonction V. En effet, en prennant, par exemple, la relation

$$(D_2 f_1) = \left(\frac{\partial f_1}{\partial p_{12}}\right) D_1 f_2$$

il est aisé de démontrer que cette relation est vérifiée identiquement, grâce aux conditions précédentes, à savoir

$$-V_{112}\left(1-\frac{\Delta_{12}^{(1)}\Delta_{12}^{(2)}}{\Delta_2}\right)+V_{123}\left(\frac{\Delta_{13}^{(1)}}{\Delta}+\frac{\Delta_{12}^{(1)}\Delta_{23}^{(2)}}{\Delta^2}\right)+V_{223}\frac{\Delta_{23}^{(1)}}{\Delta}+V_{113}\frac{\Delta_{13}^{(2)}}{\Delta}\equiv 0.$$

On trouve de même que les conditions (113_1)—(116_1), sauf celles qui viennent d'être mentionnées, sont non seulement nécessaires mais aussi suffisantes pour que $z=V$ soit une intégrale du système (111) réductible à celui de *Charpit* (117)—(118).

Il est intéressant de poser la question: est ce qu'il est possible d'étendre sur le système (119) le théorème de Jacobi concernant les équations aux dérivées partielles du premier ordre et généralisé sur le système d'équations aux dérivées du second ordre, en involution de *Darboux-Lie*, [13], [15], [20]?

C. Posons maintenant le problème de la formation de l'intégrale complète du système (111) sous les conditions (113)—(116), à l'aide de l'intégrale générale du système des équations différentielles ordinaires (119). L'intégrale générale de ce système de 12 équations est composée de trois équations données (111) et de neuf nouvelles intégrales distinctes. La variable x_1 étant principale, cherchons l'intégrale générale sous la forme suivante:

$$(131)\qquad \begin{cases} x_2 = m(x_1, C_1, \dots, C_9) \\ x_3 = n(x_1, C_1, \dots, C_9) \\ z = Z(x_1, C_1, \dots, C_9) \end{cases} \qquad \begin{aligned} &p_i = P_i(x_1, C_1, \dots, C_9) \\ &p_{ii} = P_{ii}(x_1, C_1, \dots, C_9) \\ &p_{ik} = P_{ik}(x_1, C_1, \dots, C_9),\ P_{ik} = P_{ki} \end{aligned}$$

$$(i, k = 1, 2, 3)$$

Supposons que l'on ait

$$(132)\qquad D\left(\frac{m, n}{C_8, C_9}\right) \neq 0,$$

alors les deux premières équations (131) se mettent sous la forme résoluble par rapport à C_8 et C_9:

$$(133)\qquad \begin{aligned} C_8 &= a(x_1, x_2, x_3, C_1, \dots, C_7) \\ C_9 &= b(x_1, x_2, x_3, C_1, \dots, C_7). \end{aligned}$$

En éliminant C_8 et C_9 des autres équations (131), à l'aide de (133), on obtient

$$(134)\quad \begin{cases} z=\overline{Z}(x_1, x_2, x_3, C_1, \ldots, C_7) \\ p_i=\overline{P}_i(x_1, x_2, x_3, C_1, \ldots, C_7) \\ p_{ii}=\overline{P}_{ii}(x_1, x_2, x_3, C_1, \ldots, C_7), \ (i=1, 2, 3) \\ p_{ik}=\overline{P}_{ik}(x_1, x_2, x_3, C_1, \ldots, C_7), \ (i=1, 2, 3\text{: } i\neq k). \end{cases}$$

Puisque les équations (131) sont les intégrales du système (119), les identités suivantes ont lieu

$$(135)\quad \begin{cases} \dfrac{\partial m}{\partial x_1}=\dfrac{\partial f_1}{\partial p_{12}}, \quad \dfrac{\partial n}{\partial x_1}=\dfrac{\partial f_1}{\partial p_{13}}, \\ \dfrac{\partial Z}{\partial x_1}=P_1+\dfrac{\partial m}{\partial x_1}P_2+\dfrac{\partial n}{\partial x_1}P_3, \\ \dfrac{\partial P_i}{\partial x_1}=P_{i1}+\dfrac{\partial m}{\partial x_1}P_{i2}+\dfrac{\partial n}{\partial x_1}P_{i3}, \ (i=1, 2, 3). \end{cases}$$

Considérons les identités évidentes résultant des formules (131) et (134),

$$(136)\quad \begin{cases} Z(x_1, C_1, \ldots, C_9)=\overline{Z}(x, m, n, C_1, \ldots, C_7), \\ P_1(x_1, C_1, \ldots, C_9)=\overline{P}_i(x_1, m, n, C_1, \ldots, C_7), \quad (i=1, 2, 3) \end{cases}$$

et leurs formules dérivées

$$(137)\quad \begin{cases} \dfrac{\partial Z}{\partial x_1}=\left(\dfrac{\partial \overline{Z}}{\partial x_1}\right)+\left(\dfrac{\partial \overline{Z}}{\partial x_2}\right)\dfrac{\partial m}{\partial x_1}+\left(\dfrac{\partial \overline{Z}}{\partial x_3}\right)\dfrac{\partial n}{\partial x_1}, \\ \dfrac{\partial P_i}{\partial x_1}=\left(\dfrac{\partial \overline{P}_i}{\partial x_1}\right)+\left(\dfrac{\partial \overline{P}_i}{\partial x_2}\right)\dfrac{\partial m}{\partial x_1}+\left(\dfrac{\partial \overline{P}_i}{\partial x_3}\right)\dfrac{\partial n}{\partial x_1}, \quad (i=1, 2, 3) \end{cases}$$

les parenthèses désignant le résultat de la substitution de x_2 et x_3 respectivement par les fonctions m et n. En soustrayant (137) de (135) on a

$$\left(\frac{\partial \overline{Z}}{\partial x_1}\right)-P_1+\left[\left(\frac{\partial \overline{Z}}{\partial x_2}\right)-P_2\right]\frac{\partial m}{\partial x_1}+\left[\left(\frac{\partial \overline{Z}}{\partial x_3}\right)-P_3\right]\frac{\partial n}{\partial x_1}=0.$$

$$\left(\frac{\partial \overline{P}_i}{\partial x_1}\right)-P_{i1}+\left[\left(\frac{\partial \overline{P}_i}{\partial x_2}\right)-P_{i2}\right]\frac{\partial m}{\partial x_1}+\left[\left(\frac{\partial \overline{P}_i}{\partial x_3}\right)-P_{i3}\right]\frac{\partial n}{\partial x_1}=0, \qquad (i=1, 2, 3).$$

et l'on a tiré les relations nouvelles

$$(138)\quad \begin{cases} \dfrac{\partial \overline{Z}}{\partial x_1}-\overline{P}_1+\left(\dfrac{\partial \overline{Z}}{\partial x_2}-\overline{P}_2\right)\left(\dfrac{\partial m}{\partial x_1}\right)+\left(\dfrac{\partial \overline{Z}}{\partial x_3}-\overline{P}_3\right)\left(\dfrac{\partial n}{\partial x_1}\right)=0, \\ \dfrac{\partial \overline{P}_i}{\partial x_1}-\overline{P}_{i1}+\left(\dfrac{\partial \overline{P}_i}{\partial x_2}-\overline{P}_{i2}\right)\left(\dfrac{\partial m}{\partial x_1}\right)+\left(\dfrac{\partial \overline{P}_i}{\partial x_3}-\overline{P}_{i3}\right)\left(\dfrac{\partial n}{\partial x_1}\right)=0. \quad (i=1, 2, 3) \end{cases}$$

En vertu de la première des équations (138) et des relations

(139) $$\bar{P}_2=\frac{\partial \bar{Z}}{\partial x_2}\qquad \bar{P}_3=\frac{\partial \bar{Z}}{\partial x_3}$$

on conclut

(140) $$\bar{P}_1=\frac{\partial \bar{Z}}{\partial x_1}.$$

Supposons que l'on tire de (139) et (140) les relations

$$\frac{\partial \bar{P}_1}{\partial x_2}=\frac{\partial \bar{P}_2}{\partial x_1},\quad \frac{\partial \bar{P}_1}{\partial x_3}=\frac{\partial \bar{P}_3}{\partial x_1},\quad \frac{\partial \bar{P}_2}{\partial x_3}=\frac{\partial \bar{P}_3}{\partial x_2}.$$

Grâce aux conditions $\frac{\partial f_1}{\partial p_{12}}\neq 0$ et $\frac{\partial f_1}{\partial p_{13}}\neq 0$, aux équations (138) et les conditions

(141) $$\frac{\partial \bar{P}_1}{\partial x_2}=\bar{P}_{12},\quad \frac{\partial \bar{P}_1}{\partial x_3}=\bar{P}_{13},\quad \frac{\partial \bar{P}_3}{\partial x_2}=\bar{P}_{23}$$

on obtient les conditions

(142) $$\frac{\partial \bar{P}_1}{\partial x_1}=\bar{P}_{11},\quad \frac{\partial \bar{P}_2}{\partial x_2}=\bar{P}_{22},\quad \frac{\partial \bar{P}_3}{\partial x_3}=\bar{P}_{33}.$$

Donc, les conditions (139) et (141) produisent les conditions (140) et (142). Les dérivées des identités (136) donnent les nouvelles identités

(143) $$\frac{\partial Z}{\partial C_\mu}=\left(\frac{\partial \bar{Z}}{\partial x_2}\right)\frac{\partial m}{\partial C_\mu}+\left(\frac{\partial \bar{Z}}{\partial x_3}\right)\frac{\partial n}{\partial C_\mu}+\left(\frac{\partial \bar{Z}}{\partial C_\mu}\right),\qquad (\mu=1,2,\dots,7),$$

(144) $$\frac{\partial Z}{\partial C_\nu}=\left(\frac{\partial \bar{Z}}{\partial x_2}\right)\frac{\partial m}{\partial C_\nu}+\left(\frac{\partial \bar{Z}}{\partial x_3}\right)\frac{\partial n}{\partial C_\nu},\qquad (\nu=8,9),$$

(145) $$\frac{\partial P_i}{\partial C_\mu}=\left(\frac{\partial \bar{P}_i}{\partial x_2}\right)\frac{\partial m}{\partial C_\mu}+\left(\frac{\partial \bar{P}_i}{\partial x_3}\right)\frac{\partial n}{\partial C_\mu}+\left(\frac{\partial \bar{P}_i}{\partial C_\mu}\right),\qquad (\mu=1,2,\dots,7),$$

(146) $$\frac{\partial P_i}{\partial C_\nu}=\left(\frac{\partial \bar{P}_i}{\partial x_2}\right)\frac{\partial m}{\partial C_\nu}+\left(\frac{\partial \bar{P}_i}{\partial x_3}\right)\frac{\partial n}{\partial C_\nu},\qquad (\nu=8,9)$$

les parenthèses ayant la désignation antérieurement établie. En admettant que l'on ait

$$\frac{\partial \bar{Z}}{\partial C_\mu}\neq 0,\qquad (\mu=1,2,\dots,7)$$

on obtient, grâce aux (139), (143)—(144) les conditions suivantes

(147) $$U_{C_\mu}\neq 0,\quad (\mu=1,2,\dots,7),\quad U_{C_\nu}=0,\quad (\nu=8,9)$$

où nous introduisons des nouvelles notions U_C des fonctions que nous dirons „caractéristiques“ du système (111)

$$U_C \equiv \frac{\partial Z}{\partial C} - P_2 \frac{\partial m}{\partial C} - P_3 \frac{\partial n}{\partial C}.$$

D'autre part, sous l'hypothèse

$$\frac{\partial P_i}{\partial C_\mu} \neq 0 \quad (i=1, 2, 3;\ \mu=1, 2, \ldots, 7)$$

et en vertu de (141) et (145)—(146), on conclut

$$(148) \qquad W^i_{C_\mu} \neq 0, \quad (\mu=1, 2, \ldots, 7;\ i=1, 2, 3), \quad W^i_{C_\mu}=0, \quad (\mu=8, 9)$$

en appelant W^i_C les autres nouvelles fonctions caractéristiques qui signifient

$$W^i_C \equiv \frac{\partial P_i}{\partial C} - P_{i2} \frac{\partial m}{\partial C} - P_{i3} \frac{\partial n}{\partial C}.$$

Si les formules (134) définissent l'intégrale complète du système (111) sous l'hypothèse (113)—(116), alors les conditions (147) et (148) ont lieu. Donc les conditions citées sont nécessaires. Démontrons que ces conditions sont aussi suffisantes pour la formation de l'intégrale complète. Donc, en admettant que les conditions (147) et (148) existent, il est aisé de former les différences suivantes des formules (147) et ((143)—(144)

$$(149) \qquad \left[\left(\frac{\partial \overline{Z}}{\partial x_2}\right) - P_2\right] \frac{\partial m}{\partial C_\nu} + \left[\left(\frac{\partial \overline{Z}}{\partial x_3}\right) - P_3\right] \frac{\partial n}{\partial C_\nu} = 0, \quad (\nu=8, 9)$$

$$(150) \qquad \left[\left(\frac{\partial \overline{Z}}{\partial x_2}\right) - P_2\right] \frac{\partial m}{\partial C_\mu} + \left[\left(\frac{\partial \overline{Z}}{\partial x_3}\right) - P_3\right] \frac{\partial n}{\partial C_\mu} + \left(\frac{\partial \overline{Z}}{\partial C_\mu}\right) \neq 0, \quad (\mu=1, ., 7)$$

et celles des formules (148) et (145)—(146)

$$(151) \qquad \left[\left(\frac{\partial \overline{P}_i}{\partial x_2}\right) - P_{i2}\right] \frac{\partial m}{\partial C_\nu} + \left[\left(\frac{\partial \overline{P}_i}{\partial x_3}\right) - P_{i3}\right] \frac{\partial n}{\partial C_\nu} = 0, \quad (\nu=8, 9;\ i=1, 2, 3)$$

$$(152) \qquad \left[\left(\frac{\partial \overline{P}_i}{\partial x_2}\right) - P_{i2}\right] \frac{\partial m}{\partial C_\mu} + \left[\left(\frac{\partial \overline{P}_i}{\partial x_3}\right) - P_{i3}\right] \frac{\partial n}{\partial C_\mu} + \left(\frac{\partial \overline{P}_i}{\partial C_\mu}\right) \neq 0,$$

$$(i=1, 2, 3;\ \mu=1, ., 7)$$

D'après (132) les équations (149) donnent les conditions (139) et par conséquent la condition (140). En vertu de l'équation (150) on obtient

$$\frac{\partial \overline{Z}}{\partial C_\mu} \neq 0, \quad (\mu=1, 2, ., 7).$$

Il suffit de prendre les deux premières équations (151). Ces équations, d'après (132), donnent les conditions (141), et alors les conditions (142) ont lieu. Les équations (152) ne donnent que les relations

$$\frac{\partial \overline{P}_i}{\partial C_\mu} \neq 0, \quad i=1, 2, 3; \ \mu=1, \ldots 7)$$

qui ne sont cependant nécessaires.

Il résulte des considérations exposées que pour former l'intégrale complète du système (111) à l'aide de l'intégrale générale (131) du système (119) les conditions nécessaires et suffisantes, exprimées par les fonctions caractéristiques suivantes, doivent avoir lieu

$$(147') \qquad \begin{cases} U_{C_\mu} \neq 0, \ (\mu=1, \ldots, 7) \ U_{C_\nu}=0, \ (\nu=8, 9), \\ W^i_{C_\nu}=0, \ (\nu=8, 9: \ i=1, 2). \end{cases}$$

De cette manière il est démontré le rôle important que jouent les fonctions caractéristiques introduites dans la théorie du système étudié (111), sous l'hypothèse (113)—(116)

D. Citons quelques propriétés des fonctions caractéristiques. Les formules (131) s'obtient de neuf intégrales distinctes du système (119), c'est-à-dire ces équations sont résolubles par rapport aux neuf constantes arbitraires $C_1, C_2, \ldots, C_9$ de sorte que l'on a la condition suivante.

$$D\left(\frac{Z, P_1, P_2, P_3, P_{12}, P_{13}, P_{23}}{C_1, C_2, \ldots\ldots\ldots\ldots\ldots\ldots, C_7}\right) \neq 0.$$

On peut en former les déterminants dans lesquels les colonnes 3ᵉ, 4ᵉ, 5ᵉ, 6ᵉ sont remplacées respectivement par les U_C, W^i_C $(i=1, 2, 3)$. Ces déterminants sont distincts du zéro. Il en résulte qu'au moins l'une des fonctions caractéristiques dans chaque groupe: U_{C_i}, $W^1_{C_i}$, $W^2_{C_i}$, $W^3_{C_i}$, est distincte du zéro. Démontrons que les fonctions caractéristiques sont homogènes et linéaires de leurs valeurs initiales: U^0_C, $(W^i_C)^0$. En partant des identités (135) et des expressions qui définissent des fonctions caractéristiques on obtient l'équation

$$\frac{\partial U_C}{\partial x_1} = W^1_C + \frac{\partial m}{\partial x_1} W^2_C + \frac{\partial n}{\partial x_1} W^3_C.$$

Pour obtenir les expressions des dérivées par rapport à x_1 des autres fonctions caractéristiques nous procédérons de la manière suivante. Substituons dans les formules

$$\frac{\partial W^1_C}{\partial x_1} = \frac{\partial}{\partial C}\left(\frac{\partial P_1}{\partial x_1} - P_{12}\frac{\partial m}{\partial x_1} - P_{13}\frac{\partial n}{\partial x_1}\right) + \frac{\partial P_{12}}{\partial C}\frac{\partial m}{\partial x} +$$
$$+ \frac{\partial P_{13}}{\partial C}\frac{\partial n}{\partial x} - \frac{\partial P_{12}}{\partial x_1}\frac{\partial m}{\partial C} - \frac{\partial P_{13}}{\partial x_1}\frac{\partial n}{\partial C}$$

les formules

$$\frac{\partial P_{12}}{\partial x_1} = -(D_2 f_1), \qquad \frac{\partial P_{13}}{\partial x_1} = -(D_3 f_1)$$

$$\frac{\partial P_{11}}{\partial C} + (D_2 f_1)\frac{\partial m}{\partial C} + (D_3 f_1)\frac{\partial n}{\partial C} + \frac{\partial f_1}{\partial z} U_c + \frac{\partial f_1}{\partial p_1} V_C^1 + \frac{\partial f_1}{\partial p_2} V_C^2 +$$

$$+ \frac{\partial f_1}{\partial p_3} V_C^3 + \frac{\partial f_1}{\partial p_{12}} \frac{\partial P_{12}}{\partial C} + \frac{\partial f_1}{\partial p_{13}} \frac{\partial P_{13}}{\partial C} + \frac{\partial f_1}{\partial p_{23}} \frac{\partial P_{23}}{\partial C} = 0$$

ainsi que les relations (113), (116) et (135). On obtient de cette manière le résultat

$$(154) \qquad \frac{\partial W_C^1}{\partial x} = -\left[\left(\frac{\partial f_1}{\partial z}\right) U_C + \left(\frac{\partial f_1}{\partial p_1}\right) V_C^1 + \left(\frac{\partial f_1}{\partial p_2}\right) V_C^2 + \left(\frac{\partial f_1}{\partial p_3}\right) V_C^3\right].$$

D'une manière analogue on obtient encore les deux équations nouvelles

$$(155) \qquad \frac{\partial W_C^2}{\partial x_1} = -\frac{\partial m}{\partial x_1}\left[\left(\frac{\partial f_2}{\partial z}\right) U_C + \left(\frac{\partial f_2}{\partial p_1}\right) V_C^1 + \left(\frac{\partial f_2}{\partial p_2}\right) V_C^2 + \left(\frac{\partial f_2}{\partial p_3}\right) V_C^3\right],$$

$$(156) \qquad \frac{\partial W_C^3}{\partial x_1} = -\frac{\partial n}{\partial x_1}\left[\left(\frac{\partial f_3}{\partial z}\right) U_C + \left(\frac{\partial f_3}{\partial p_1}\right) V_C^1 + \left(\frac{\partial f_3}{\partial p_2}\right) V_C^2 + \left(\frac{\partial f_3}{\partial p_3}\right) V_C^3\right].$$

Les équations (153)—(156) représentent les équations différentielles linéaires et homogènes des fonctions U_C et W_C^i. Les solutions de telles équations s'expriment linéairement et homogènement par les valeurs initiales des fonctions considérées.

On peut utiliser les propriétés démontrées des fonctions caractéristiques pour vérifier les conditions (147′), pour former l'intégrale complète du système (111), sous l'hypothèse (113)—(116) à l'aide de l'intégrale générale du système (119).

E. Posons le problème de la formation de intégrale contenant des fonctions arbitraires du système donné (111) à l'aide de l'intégrale générale du système de *Charpit*. Outre de trois intégrales particulières définies par les équations données (111), le système (119) a encore neuf intégrales ψ_1, ψ_2, $F_1, \ldots, F_7$. L'intégrale générale du système de *Charpit* (117)—(118) qui vérifie les équations (111), sous l'hypothèse (113)—(116), est définie par les équations citées et les équations suivantes

$$(157) \qquad F_j = \varphi_j(\psi_1, \psi_2), \quad (j = 1, \ldots, 7)$$

φ_j désignant les fonctions arbitraires. On pose pour déterminer les valeurs de variables paramétriques x_{i+1}, p_i, p_{ik}

$$(158) \qquad f_i \equiv \alpha_i, \quad (i = 1, 2); \qquad F_j \equiv \beta_j, \quad (j = 1, 2, \ldots, 7).$$

Alors les variables mentionnées sont définies par les équations

$$x_{i+1}=\Phi_i(x_1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_7), \qquad (i=1, 2)$$
$$Z=\Phi_3(x_1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_7),$$
$$p_j=\Phi_{j+3}(x_1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_7), \qquad (j=1, 2, 3)$$
$$p_{12}=\Phi_7(x_1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_7),$$
$$p_{13}=\Phi_8(x_1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_7),$$
$$p_{23}=\Phi_9(x_1, \alpha_1, \alpha_2, \beta_1, \dots, \beta_7),$$

x_1 désignant la variable indépendante principale. On va chercher les solutions du système (111) en vérifiant de plus les conditions:

$$dz=\sum_{k=1}^{3} p_k\, dx_k, \quad dp_s=\sum_{k=1}^{3} p_{sk}\, dx_k, \quad (s=1, 2, 3)$$

p_{ii} étant déterminées par les équations (111). Grâce aux conditions introduites on a les conditions nouvelles

$$\sum_{i=1}^{2} A_{ki} d\alpha_i+\sum_{j=1}^{7} B_{kj}\, d\beta_j+K_k\, dx_1=0, \quad (k=1, \ldots, 4)$$

avec

$$A_{1i}\equiv\frac{\partial \Phi_3}{\partial \alpha_i}-\Phi_5\frac{\partial \Phi_1}{\partial \alpha_i}-\Phi_6\frac{\partial \Phi_2}{\partial \alpha_i}, \qquad B_{1j}\equiv\frac{\partial \Phi_3}{\partial \beta_j}-\Phi_5\frac{\partial \Phi_1}{\partial \beta_j}-\Phi_6\frac{\partial \Phi_2}{\partial \beta_j},$$
$$A_{2i}\equiv\frac{\partial \Phi_4}{\partial \alpha_i}-\Phi_7\frac{\partial \Phi_1}{\partial \alpha_i}-\Phi_8\frac{\partial \Phi_2}{\partial \alpha_i}, \qquad B_{2j}\equiv\frac{\partial \Phi_4}{\partial \beta_j}-\Phi_7\frac{\partial \Phi_1}{\partial \beta_j}-\Phi_8\frac{\partial \Phi_2}{\partial \beta_j},$$
$$A_{3i}\equiv\frac{\partial \Phi_5}{\partial \alpha_i}-\bar{p}_{22}\frac{\partial \Phi_1}{\partial \alpha_i}-\Phi_9\frac{\partial \Phi_2}{\partial \alpha_i}, \qquad B_{3j}\equiv\frac{\partial \Phi_5}{\partial \beta_j}-\bar{p}_{22}\frac{\partial \Phi_1}{\partial \beta_j}-\Phi_9\frac{\partial \Phi_2}{\partial \beta_j},$$
$$A_{4i}\equiv\frac{\partial \Phi_6}{\partial \alpha_i}-\Phi_9\frac{\partial \Phi_1}{\partial \alpha_i}-\bar{p}_{33}\frac{\partial \Phi_2}{\partial \alpha_i}, \qquad B_{4j}\equiv\frac{\partial \Phi_6}{\partial \beta_j}-\Phi_9\frac{\partial \Phi_1}{\partial \beta_j}-\bar{p}_{33}\frac{\partial \Phi_2}{\partial \beta_j},$$
$$(i=1, 2) \qquad\qquad (j=1, 2, \dots, 7)$$

$$K_1\equiv\frac{\partial \Phi_3}{\partial x_1}-\Phi_5\frac{\partial \Phi_1}{\partial x_1}-\Phi_6\frac{\partial \Phi_2}{\partial x_1}-\Phi_4,$$
$$K_2\equiv\frac{\partial \Phi}{\partial x_1}-\Phi_1\frac{\partial \Phi_1}{\partial x_1}-\Phi_8\frac{\partial \Phi_2}{\partial x_1}-\bar{p}_{11},$$
$$K_3\equiv\frac{\partial \Phi_5}{\partial x_1}-\bar{p}_{22}\frac{\partial \Phi_2}{\partial x_1}-\Phi_9\frac{\partial \Phi_2}{\partial x_1}-\Phi_7$$
$$K_4\equiv\frac{\partial \Phi_6}{\partial x_1}-\Phi_9\frac{\partial \Phi_1}{\partial x_1}-\bar{p}_{33}\frac{\partial \Phi_2}{\partial x_1}-\Phi_8$$
$$p_{ii}\equiv-f_i(x_1, \Phi_1, \ldots, \Phi_7), \quad (i=1, 2, 3).$$

Grâce aux équations (119) ils existent les identités suivantes

$$K_k\equiv 0 \quad (k=1, 2, 3, 4).$$

En vertu de (157) et (158) on a

$$\beta_j = \varphi_j(\alpha_1, \alpha_2), \qquad (j=1, \ldots, 7)$$

et les conditions (159) deviennent

$$\left(\sum_{i=1}^{2} A_{ki} + \sum_{i=1}^{7} B_{kj} \frac{\partial \Phi_j}{\partial \alpha_i}\right) d\alpha_i = 0 \quad (k=1, 2, 3, 4).$$

Les différentiels $d\alpha_1$ étant distincts du zéro, on établit les relations

$$A_{ki} + \sum_{i=1}^{7} B_{kj} \frac{\partial \varphi_j}{\partial \alpha_i} = 0, \quad (k=1, \ldots, 4; \ (i=1, 2)$$

lesquelles doivent satisfaire les fonctions φ_j pour que (157) détermine la solution du système (111). Par conséquent le problème cité ci-dessus est résolu d'une manière analogue comme le problème posé par Lagrange [22] dans la théorie des équations aux derivées partielles du premier ordre.

Les résultats obtenus dans cet article se généralisent aisément sur les systèmes des $n\,(n>3)$ équations aux dérivées partielles du second ordre à n variables indépendantes.

11. Sur une classe des équations aux dérivées partielles du second ordre d'une fonction inconnue avec trois variables indépendantes

En suivant les idées de *N. Saltykow*, [11], [24—27] on donne dans ce paragraphe une classe des équations aux dérivées partielles du second ordre d'une fonction inconnue avec trois variables indépendantes pour laquelle on peut faire la correspondance avec un système de *Charpit*, [28].

Considérons l'équation

(160) $$f(x_1, x_2, x_3, z, p_1, p_2, p_3, p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33}) = 0$$

avec les notations usuelles

$$p_i = \frac{\partial z}{\partial x_i}, \qquad p_{ij} = \frac{\partial z}{\partial x_i \partial x_j}$$

et avec les conditions d'indépendance de l'ordre de la différentiation de la fonction z par rapport aux x_i.

On peut les équations dérivées

$$\frac{\partial f}{\partial p_{11}} \frac{\partial p_{11}}{\partial x_i} + \frac{\partial f}{\partial p_{12}} \frac{\partial p_{12}}{\partial x_i} + \frac{\partial f}{\partial p_{13}} \frac{\partial p_{13}}{\partial x_i} + \frac{\partial f}{\partial p_{22}} \frac{\partial p_{22}}{\partial x_i} + \frac{\partial f}{\partial p_{23}} \frac{\partial p_{23}}{\partial x_i} + $$
$$+ \frac{\partial f}{\partial p_{33}} \frac{\partial p_{33}}{\partial x_i} + D_i f = 0, \quad (i=1, 2, 3),$$

les symboles D_i désignant l'opérateur suivant

(161) $$D_i \equiv \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{s=1}^{3} p_{si} \frac{\partial}{\partial p_s},$$

mettre sous la forme suivante

$$(162)\quad \begin{cases} \dfrac{\partial f}{\partial p_{11}}\dfrac{\partial p_{11}}{\partial x_1}+\dfrac{\partial f}{\partial p_{12}}\dfrac{\partial p_{11}}{\partial x_2}+\dfrac{\partial f}{\partial p_{13}}\dfrac{\partial p_{11}}{\partial x_3}+D_1 f+ \\ \qquad +\left(\dfrac{\partial f}{\partial p_{22}}\dfrac{\partial p_{22}}{\partial x_1}+\dfrac{\partial f}{\partial p_{23}}\dfrac{\partial p_{23}}{\partial x_1}+\dfrac{\partial f}{\partial p_{33}}\dfrac{\partial p_{33}}{\partial x_1}\right)=0, \\ \dfrac{\partial f}{\partial p_{12}}\dfrac{\partial p_{22}}{\partial x_1}+\dfrac{\partial f}{\partial p_{22}}\dfrac{\partial p_{22}}{\partial x_2}+\dfrac{\partial f}{\partial p_{23}}\dfrac{\partial p_{22}}{\partial x_3}+D_2 f+ \\ \qquad +\left(\dfrac{\partial f}{\partial p_{11}}\dfrac{\partial p_{11}}{\partial x_2}+\dfrac{\partial f}{\partial p_{13}}\dfrac{\partial p_{13}}{\partial x_2}+\dfrac{\partial f}{\partial p_{33}}\dfrac{\partial p_{33}}{\partial x_2}\right)=0, \\ \dfrac{\partial f}{\partial p_{13}}\dfrac{\partial p_{33}}{\partial x_1}+\dfrac{\partial f}{\partial p_{23}}\dfrac{\partial p_{33}}{\partial x_2}+\dfrac{\partial f}{\partial p_{33}}\dfrac{\partial p_{33}}{\partial x_3}+D_3 f+ \\ \qquad +\left(\dfrac{\partial f}{\partial p_{11}}\dfrac{\partial p_{11}}{\partial x_3}+\dfrac{\partial f}{\partial p_{12}}\dfrac{\partial p_{12}}{\partial x_3}+\dfrac{\partial f}{\partial p_{22}}\dfrac{\partial p_{22}}{\partial x_3}\right)=0. \end{cases}$$

Ajoutons aux équations (162) les trois nouvelles

$$(163)\quad \begin{cases} \dfrac{\partial p_{12}}{\partial x_1}-m\dfrac{\partial p_{12}}{\partial x_2}-n\dfrac{\partial p_{12}}{\partial x_3}=0, \\ \dfrac{\partial p_{13}}{\partial x_1}-m\dfrac{\partial p_{13}}{\partial x_2}-n\dfrac{\partial p_{13}}{\partial x_3}=0, \\ \dfrac{\partial p_{23}}{\partial x_1}-m\dfrac{\partial p_{23}}{\partial x_2}-n\dfrac{\partial p_{23}}{\partial x_3}=0. \end{cases}$$

Les fonctions $m(x_1, x_2, x_3, z, p_1, p_2, p_3, p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33})$, $n(\ldots)$, seront déterminées d'une manière convenable.

Faisons les éliminations suivantes:

a) Éliminons de la première équation (162) les dérivées $\dfrac{\partial p_{22}}{\partial x_1}$, $\dfrac{\partial p_{23}}{\partial x_1}$, $\dfrac{\partial p_{33}}{\partial x_1}$ en utilisant les relations évidentes

$$(163_1)\quad \begin{cases} \dfrac{\partial p_{22}}{\partial x_1}=\dfrac{1}{m}\dfrac{\partial p_{12}}{\partial x_1}-n\dfrac{\partial p_{23}}{\partial x_2}-\dfrac{n^2}{m}\dfrac{\partial p_{23}}{\partial x_3}, \\ \dfrac{\partial p_{23}}{\partial x_1}=m\dfrac{\partial p_{23}}{\partial x_2}+n\dfrac{\partial p_{23}}{\partial x_3}, \\ \dfrac{\partial p_{33}}{\partial x_1}=\dfrac{1}{n}\dfrac{\partial p_{13}}{\partial x_1}-\dfrac{m^2}{n}\dfrac{\partial p_{23}}{\partial x_2}-m\dfrac{\partial p_{23}}{\partial x_3}, \end{cases}$$

b) Éliminons de la seconde équation (163) les dérivées $\frac{\partial p_{11}}{\partial x_2}, \frac{\partial p_{13}}{\partial x_2}, \frac{\partial p_{33}}{\partial x_2}$ en utilisant les relations évidentes

$$(163_2)\qquad \left\{\begin{aligned} &\frac{\partial p_{11}}{\partial x_2}=m\frac{\partial p_{12}}{\partial x_2}-\frac{n^2}{m}\frac{\partial p_{13}}{\partial x_3}+\frac{n}{m}\frac{\partial p_{13}}{\partial x_1},\\ &\frac{\partial p_{13}}{\partial x_2}=\frac{1}{m}\frac{\partial p_{13}}{\partial x_1}-\frac{n}{m}\frac{\partial p_{13}}{\partial x_3},\\ &\frac{\partial p_{33}}{\partial x_2}=\frac{1}{mn}\frac{\partial p_{13}}{\partial x_1}-\frac{1}{m}\frac{\partial p_{13}}{\partial x_3}-\frac{m}{n}\frac{\partial p_{23}}{\partial x_2}.\end{aligned}\right.$$

c) Éliminons de la troisième équation (163) les dérivées suivantes: $\frac{\partial p_{11}}{\partial x_3}$, $\frac{\partial p_{12}}{\partial x_3}, \frac{\partial p_{22}}{\partial x_3}$ en utilisant les relations suivantes

$$(163_3)\qquad \left\{\begin{aligned} &\frac{\partial p_{11}}{\partial x_3}=\frac{m}{n}\frac{\partial p_{12}}{\partial x_1}-\frac{m^2}{n}\frac{\partial p_{12}}{\partial x_2}+n\frac{\partial p_{13}}{\partial x_3},\\ &\frac{\partial p_{12}}{\partial x_3}=\frac{1}{n}\frac{\partial p_{12}}{\partial x_1}-\frac{m}{n}\frac{\partial p_{12}}{\partial x_2},\\ &\frac{\partial p_{22}}{\partial x_3}=\frac{1}{mn}\frac{\partial p_{12}}{\partial x_1}-\frac{n}{m}\frac{\partial p_{23}}{\partial x_3}-\frac{1}{n}\frac{\partial p_{12}}{\partial x_2}.\end{aligned}\right.$$

Après les éliminations indiquées le système (162) obtient la forme nouvelle

$$(164)\qquad \left\{\begin{aligned} &\frac{\partial f}{\partial p_{11}}\frac{\partial p_{11}}{\partial x_1}+\left(\frac{\partial f}{\partial p_{12}}+\frac{1}{m}\frac{\partial f}{\partial p_{22}}\right)\frac{\partial p_{11}}{\partial x_2}+\left(\frac{\partial f}{\partial p_{13}}+\frac{1}{n}\frac{\partial f}{\partial p_{33}}\right)\frac{\partial p_{11}}{\partial x_3}+\\ &\qquad +D_1f+K_1\left(m\frac{\partial p_{23}}{\partial x_2}+n\frac{\partial p_{23}}{\partial x_3}\right)=0,\\ &\left(\frac{\partial f}{\partial p_{12}}+m\frac{\partial f}{\partial p_{11}}\right)\frac{\partial p_{22}}{\partial x_1}+\frac{\partial f}{\partial p_{22}}\frac{\partial p_{22}}{\partial x_2}+\left(\frac{\partial f}{\partial p_{23}}-\frac{m}{n}\frac{\partial f}{\partial p_{33}}\right)\frac{\partial p_{22}}{\partial x_3}+\\ &\qquad +D_2f+\frac{1}{m}K_2\left(\frac{1}{n}\frac{\partial p_{13}}{\partial x_1}-\frac{\partial p_{23}}{\partial x_3}\right)=0,\\ &\left(\frac{\partial f}{\partial p_{13}}+n\frac{\partial f}{\partial p_{11}}\right)\frac{\partial p_{33}}{\partial x_1}+\left(\frac{\partial f}{\partial p_{23}}-\frac{n}{m}\frac{\partial f}{\partial p_{22}}\right)\frac{\partial p_{33}}{\partial x_2}+\frac{\partial f}{\partial p_{33}}\frac{\partial p_{33}}{\partial x_3}+\\ &\qquad +D_3f+\frac{1}{n}K_3\left(\frac{1}{m}\frac{\partial p_{22}}{\partial x_1}-\frac{\partial p_{12}}{\partial x_2}\right)=0\end{aligned}\right.$$

en utilisant les désignations suivantes

$$(165)\quad \begin{cases} K_1 \equiv \dfrac{\partial f}{\partial p_{23}} - \dfrac{n}{m}\dfrac{\partial f}{\partial p_{22}} - \dfrac{m}{n}\dfrac{\partial f}{\partial p_{33}}, \\ K_2 \equiv n^2\dfrac{\partial f}{\partial p_{11}} + n\dfrac{\partial f}{\partial p_{13}} + \dfrac{\partial f}{\partial p_{33}}, \\ K_3 \equiv m^2\dfrac{\partial f}{\partial p_{11}} + m\dfrac{\partial f}{\partial p_{12}} + \dfrac{\partial f}{\partial p_{22}}. \end{cases}$$

Il est aisé de s'en persuader que les équations (164) forment un système de *Charpit* si les conditions suivantes sont remplies

$$(166)\quad \begin{cases} \left(\dfrac{\partial f}{\partial p_{12}} + \dfrac{1}{m}\dfrac{\partial f}{\partial p_{22}}\right) : \dfrac{\partial f}{\partial p_{12}} = \dfrac{\partial f}{\partial p_{22}} : \left(\dfrac{\partial f}{\partial p_{12}} + m\dfrac{\partial f}{\partial p_{11}}\right) = \\ \qquad = \left(\dfrac{\partial f}{\partial p_{23}} - \dfrac{n}{m}\dfrac{\partial f}{\partial p_{22}}\right) : \left(\dfrac{\partial f}{\partial p_{13}} + n\dfrac{\partial f}{\partial p_{11}}\right) = -m, \\ \left(\dfrac{\partial f}{\partial p_{13}} + \dfrac{1}{n}\dfrac{\partial f}{\partial p_{33}}\right) : \dfrac{\partial f}{\partial p_{11}} = \left(\dfrac{\partial f}{\partial p_{23}} - \dfrac{m}{n}\dfrac{\partial f}{\partial p_{33}}\right) : \left(\dfrac{\partial f}{\partial p_{12}} + m\dfrac{\partial f}{\partial p_{11}}\right) = \\ \qquad = \dfrac{\partial f}{\partial p_{33}} : \left(\dfrac{\partial f}{\partial p_{13}} + n\dfrac{\partial f}{\partial p_{11}}\right) = -n, \end{cases}$$

ou

$$(166_1)\quad m^2\frac{\partial f}{\partial p_{11}} + m\frac{\partial f}{\partial p_{12}} + \frac{\partial f}{\partial p_{22}} = 0, \quad n^2\frac{\partial f}{\partial p_{11}} + n\frac{\partial f}{\partial p_{13}} + \frac{\partial f}{\partial p_{33}} = 0,$$

$$(162_2)\quad \begin{aligned} &\frac{\partial f}{\partial p_{23}} - \frac{m}{n}\frac{\partial f}{\partial p_{33}} + n\frac{\partial f}{\partial p_{12}} + mn\frac{\partial f}{\partial p_{11}} = 0, \\ &\frac{\partial f}{\partial p_{23}} - \frac{n}{m}\frac{\partial f}{\partial p_{22}} + m\frac{\partial f}{\partial p_{13}} + mn\frac{\partial f}{\partial p_{11}} = 0. \end{aligned}$$

Grâce aux équations (166_1) les équations (166_2) ne donnent qu'une condition qu'on peut mettre sous la forme suivante

$$\frac{\partial f}{\partial p_{23}} - \frac{n}{m}\frac{\partial f}{\partial p_{22}} - \frac{m}{n}\frac{\partial f}{\partial p_{33}} = 0$$

C'est ainsi que l'on parvient à mettre les conditions (166) sous la forme plus symétrique

$$(167)\quad K_i = 0, \quad (i = 1, 2, 3).$$

Donc, en vertu des conditions (167) le système (164) devient

$$(168)\qquad \begin{aligned} &\frac{\partial p_{11}}{\partial x_1}-m\frac{\partial p_{11}}{\partial x_2}-n\frac{\partial p_{11}}{\partial x_3}+D_1 f\Big/\frac{\partial f}{\partial p_{11}}=0,\\ &\frac{\partial p_{22}}{\partial x_1}-m\frac{\partial p_{22}}{\partial x_2}-n\frac{\partial p_{22}}{\partial x_3}+D_2 f\Big/\left(\frac{\partial f}{\partial p_{12}}+\frac{\partial f}{\partial p_{11}}m\right)=0,\\ &\frac{\partial p_{33}}{\partial x_1}-m\frac{\partial p_{33}}{\partial x_2}-n\frac{\partial p_{33}}{\partial x_3}+D_3 f\Big/\left(\frac{\partial f}{\partial p_{13}}+\frac{\partial f}{\partial p_{11}}n\right)=0.\end{aligned}$$

Ajoutons aux équations (163) et (168) les identités évidentes

$$(169)\qquad \begin{cases}\dfrac{\partial p_i}{\partial x_1}-m\dfrac{\partial p_i}{\partial x_2}-n\dfrac{\partial p_i}{\partial x_3}=p_{i1}-mp_{i2}-np_{i3},\\[2mm] \dfrac{\partial z}{\partial x_1}-m\dfrac{\partial z}{\partial x_2}-n\dfrac{\partial z}{\partial x_3}=p_1-mp_2-np_3.\end{cases}$$

Les équations (163), (168) et (169) forment un système du type de *Charpit* On peut associer au système dit un système équivalent des caractéristiques

$$(170)\qquad \begin{cases} dx_1=\dfrac{dx_2}{-m}=\dfrac{dx_3}{-n}=\dfrac{dz}{p_1-mp_2-np_3}=\dfrac{dp_i}{p_{i1}-mp_{i2}-np_{i3}}=\\[2mm] =\dfrac{dp_{11}}{-D_1 f\Big/\dfrac{\partial f}{\partial p_{11}}}=\dfrac{dp_{22}}{-D_2 f\Big/\left(\dfrac{\partial f}{\partial p_{12}}+m\dfrac{\partial f}{\partial p_{11}}\right)}=\\[2mm] =\dfrac{dp_{33}}{-D_3 f\Big/\left(\dfrac{\partial f}{\partial p_{13}}+n\dfrac{\partial f}{\partial p_{11}}\right)}=\dfrac{dp_{ik}}{0}.\end{cases}$$

Grâce aux conditions: $K_2=0$, $K_3=0$ on peut déterminer les fonctions inconnues: m et n et alors en vertu de la condition $K_1=0$ on obtient la condition cherchée pour la fonction f

$$(171)\qquad \begin{aligned}&\left\{\frac{\partial f}{\partial p_{23}}\pm\left[\left(\frac{\partial f}{\partial p_{23}}\right)^2-4\frac{\partial f}{\partial p_{22}}\frac{\partial f}{\partial p_{33}}\right]^{1/2}\right\}:2\frac{\partial f}{\partial p_{33}}=\\ &=\left\{\frac{\partial f}{\partial p_{12}}\mp\left[\left(\frac{\partial f}{\partial p_{12}}\right)^2-4\frac{\partial f}{\partial p_{11}}\frac{\partial f}{\partial p_{22}}\right]^{1/2}\right\}\cdot\left\{\frac{\partial f}{\partial p_{13}}\mp\left[\left(\frac{\partial f}{\partial p_{13}}\right)^2-4\frac{\partial f}{\partial p_{11}}\frac{\partial f}{\partial p_{33}}\right]^{1/2}\right\}\end{aligned}$$

Autrement dit si la fonction f admet la condition (171) alors on peut à l'équation donnée (160) associer le système de *Charpit* (163), (168), (169) ou le système des équations différentielles ordinaires des caractéristiques (170).

Pour former la solution correspondante de l'équation (160) (admettant la condition (171)) au moyen de l'intégrale générale du système de *Charpit* (163), (168), (169) on peut construire la théorie que serait analogue à la théorie concernant le système (23) en involution de *Darboux-Lie*.

Considérons une équation aux dérivées partielles du second ordre d'une fonction z inconnue:

(172) $$u = F(x_1, x_2, x_3, z, v)$$

F étant une fonction arbitraire et les quantités u et v sont définies par les relations données

(173) $$u = \varphi\left(x_1, x_2, x_3, z, v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3}\right), \quad v = \psi\left(x_1, x_2, x_3, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}\right).$$

Grâce aux (173) on peut mettre l'équation (172) sous la forme suivante

$$f \equiv \varphi\left[x_1, x_2, x_3, z, \psi(x_1, x_2, x_3, z, p_1, p_2; p_3), \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_3}\right] - $$
$$- F[x_1', x_2, x_3, z, \psi(x_1, x_2, x_3, z, p_1, p_2, p_3)] = 0.$$

La fonction considérée f admet la condition (171) et on peut appliquer la théorie exposée à l'équation considérée: $f=0$.

12. Sur les résultats de D. H. Parsons

Soit l'équation

(174) $$f(x_1, \ldots, x_n, z, p_{11}, \ldots, p_{nn}) = 0$$

dans laquelle z est la fonction des n variables indépendantes $x_1, \ldots, x_n$

$$p_i = \frac{\partial z}{\partial x_i}, \quad p_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j}$$

et telle que f soit analytique pour toutes variables.

D. H. Parsons, [29], [30], [31], définissait le rang de l'équation donnée (174) comme le rang de la matrice suivante

$$\begin{bmatrix} \partial f/\partial p_{11} & 1/2\ \partial f/\partial p_{12} & \ldots & 1/2\ \partial f/\partial p_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ 1/2\ \partial f/p_{n1} & \ldots & \ldots & \partial f/\partial p_{nn} \end{bmatrix}.$$

Il démontrait aussi les propositions suivantes:

Si l'équation (174) est de rang 1, elle a une famille des caractéristiques du premier ordre;

Si l'équation (174) est du rang 2, elle a deux familles diverses et si l'équation donnée est du rang 3, elle n'a pas des caractéristiques de cette espèce;

Le rang de l'équation donnée est invariant sous la transformation de contact.

Dans son oeuvre, [29], *D. H. Parsons* appliquait la théorie classique des caractéristiques, la théorie des équations en involution et la mèthode de *Darboux*, [32], [14], à l'équation (174), $n=3$, quand elle est du rang 2 ou 1.

On ne peut pas généraliser la méthode de *Darboux* sans la condition suffisante et nécessaire: l'équation (174) est du rang 1 ou 2.

D. H. Parsons illustrait sa théorie avec l'exemple suivant

$$p_{11}+\frac{p_{12}\,p_{13}}{1-p_{23}}+\left(\frac{p_{12}}{1-p_{23}}\right)^2=0$$

ajoutant les conditions

$$\text{pour } x_1=0\colon\ z=1/2\,x_3^2,\ \ p_1=x_2^2+x_2x_3;\ \text{ ou}$$

$$\text{pour } x_1=0\colon\ z=0,\ \ p_1=x_2x_3.$$

La méthode exposés de *Parsons* peut être généralisé au cas $n\geq 3$ et aussi aux équations correspondantes aux dérivées partielles des ordres supérieurs.

III Chapitre

SUR LE SYSTÈME DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES EN INVOLUTION DE DARBOUX DU TROISIÈME ORDRE

Les équations aux dérivées partielles d'une fonction à deux variables indépendantes en involulion de *Darboux* du troisième ordre admettent d'établir plusieurs propriétés qui sont analogues aux propriétés de la théorie des équations aux dérivées partielles du premier ordre et des systèmes des équations aux dérivées partielles du second ordre en involution de *Darboux-Lie.*

On peut établir pour les équations en involution de *Darboux* du troisième ordre, par exemple, les propriétés suivantes:

1) on peut former le système des équations différentielles jouant le rôle d'un système des caractéristiques,

2) on peut étendre la notion de l'intégrale complète et préciser les conditions nécessaires et suffisantes concernant cette intégrale,

3) on peut résoudre deux problèmes de *Jacobi* et d'une manière bien déterminée faire la liaison intime entre l'intégrale générale du système des caractéristiques et l'intégrale complète du système des équations aux dérivées partielles,

4) on peut traiter la théorie de *Lagrange* des intégrales des systèmes considérés, et

5) on peut aussi poser le problème da la formation d'une intégrale de *Cauchy* à l'aide d'une intégrale complète donnée. ([10], [13], [15], [33]).

1. L'involution de Darboux du troisième ordre

Considérons une équation aux dérivées partielles du second ordre, utilisant les désignations habituelles, sous la forme suivante

$$r+f(x, y, z, p, q, s, t)=0. \tag{1}$$

On supposera pour la fonction f que $f\in C^2(G)$, où G est un domaine déterminé des variables x, y, z, p, q, s, t. A l'équation (1) faisons correspondre une autre équation aux dérivées partielles du troisième ordre suivant une loi qui sera déterminée plus tard, et écrivons

$$z_{xyy}+\Phi(x, y, z, p, q, s, t, z_{yyy})=0. \tag{2}$$

On supposera que $\Phi \in C^1(G_1)$, où G_1 est un domaine des variables $x, y, z, p, q, s, t, z_{yyy}$ et $G \subset G_1$.

Formons les équations dérivées du second ordre de l'équation (1) et du premier ordre de l'équation (2) respectivement par rapport aux variables indépendantes x et y:

$$
\begin{aligned}
& z_{xxxx} + f_s z_{xxxy} + f_t z_{xxyy} + D_{xx} f = 0, \\
& z_{xxxy} + f_s z_{xxyy} + f_t z_{xyyy} + D_{yx} f = 0, \\
(3) \qquad & z_{xxyy} + f_s z_{xyyy} + f_t z_{yyyy} + D_{yy} f = 0, \\
& z_{xxyy} + \Phi_\delta z_{xyyy} + D_x \Phi = 0, \\
& z_{xyyy} + \Phi_\delta z_{yyyy} + D_y \Phi = 0,
\end{aligned}
$$

δ, Φ_δ, f_s, f_t, z_{xxxx}, ... désignant respectivement les dérivées partielles

$\partial^3 z/\partial y^3$, $\partial\Phi/\partial\delta$, $\partial f/\partial s$, $\partial f/\partial t$, $\partial^4 z/\partial x^4$, ...; quant à D_x, D_y, D_{xx}, D_{xy}, D_{yy},

elles désignent les dérivées correspondantes prises par rapport à x et y. En vertu de la définition de l'involution de *Darboux* du troisième ordre, les équations (3) ne doivent point être résolubles par rapport aux dérivées du quatrième ordre. Il en résulte les deux conditions de l'involution sous les formes suivantes

$$
(4) \qquad \Phi_\delta^2 - f_s \Phi_\delta + f_t = 0,
$$

$$
(5) \qquad D_x \Phi + \frac{f_t}{\Phi_\delta} D_y \Phi - D_{yy} f = 0.
$$

2. Système des équations différentielles ordinaires des caractéristiques

Grâce aux équations (3) on peut former les équations suivantes

$$
\begin{aligned}
& \frac{\partial\alpha}{\partial x} + \Phi_\delta \frac{\partial\alpha}{\partial y} + \frac{f_t}{\Phi_\delta} m + D_{xx} f = 0, \\
& \frac{\partial\beta}{\partial x} + \Phi_\delta \frac{\partial\beta}{\partial y} - m + D_{yx} f = 0, \\
(3') \qquad & \frac{\partial\gamma}{\partial x} + \Phi_\delta \frac{\partial\gamma}{\partial y} + D_x \Phi = 0, \\
& \frac{\partial\delta}{\partial x} + \Phi_\delta \frac{\partial\delta}{\partial y} + D_y \Phi = 0,
\end{aligned}
$$

où l'on a posé: $\alpha \equiv z_{xxx}$, $\beta \equiv z_{xxy}$, $\gamma \equiv z_{xyy}$, $m \equiv (f_t/\Phi_\delta)\, D_x \Phi - D_{yx} f$. Complétons ces dernières équations par les égalités évidentes

$$
(6) \qquad
\begin{aligned}
&\frac{\partial z}{\partial x} + \Phi_\delta \frac{\partial z}{\partial y} - p - \Phi_\delta\, q = 0,\\
&\frac{\partial p}{\partial x} + \Phi_\delta \frac{\partial p}{\partial y} - r - \Phi_\delta\, s = 0,\\
&\frac{\partial q}{\partial x} + \Phi_\delta \frac{\partial q}{\partial y} - s - \Phi_\delta\, t = 0,
\end{aligned}
$$

$$
(6') \qquad
\begin{aligned}
&\frac{\partial r}{\partial x} + \Phi_\delta \frac{\partial r}{\partial y} - \alpha - \Phi_\delta\, \beta = 0,\\
&\frac{\partial s}{\partial x} + \Phi_\delta \frac{\partial s}{\partial y} - \beta - \Phi_\delta\, \gamma = 0,\\
&\frac{\partial t}{\partial x} + \Phi_\delta \frac{\partial t}{\partial y} - \gamma - \Phi_\delta\, \delta = 0.
\end{aligned}
$$

L'ensemble d'équations (3'), (6), (6') représente un système de la forme de *Charpit*, [1], à dix fonctions inconnues: $z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$ des variables indépendantes x et y. Par conséquent, l'intégration du système (3'), (6), (6') revient à l'intégration du système d'équations différentielles ordinaires des caractéristiques

$$
(7) \qquad
\begin{aligned}
dx = \frac{dy}{\Phi_\delta} = \frac{dz}{p+\Phi_\delta q} = \frac{dp}{r+\Phi_\delta s} = \frac{dq}{s+\Phi_\delta t} = \frac{dr}{\alpha+\Phi_\delta \beta} = \frac{ds}{\beta+\Phi_\delta \gamma} = \frac{dt}{\gamma+\Phi_\delta \delta} = \\
= -\frac{d\alpha}{D_{xx}f + mf_t/\Phi_\delta} = -\frac{d\beta}{D_{yx}f - m} = -\frac{d\gamma}{D_x \Phi} = -\frac{d\delta}{D_y \Phi}.
\end{aligned}
$$

Pour nos équations (1) et (2) on peut utiliser aussi le système des caractéristiques de la forme

$$
(8) \qquad dx = \frac{dy}{\Phi_\delta} = \frac{dz}{p+\Phi_\delta\, q} = \frac{dp}{r+\Phi_\delta\, s} = \frac{dq}{s+\Phi_\delta\, t} = \frac{ds}{\beta+\Phi_\delta\, \gamma} = \frac{dt}{\gamma+\Phi_\delta \delta} = -\frac{d\delta}{D_y \Phi},
$$

où r, γ, β doivent être exprimés par les autres variables figurant aux équations (1) et (2).

3. L'intégrale complète

Nous partons de l'équation

$$
(9) \qquad z = V(x, y, C_1, \ldots, C_6),
$$

où C_i sont les paramètres distincts et indépendantes des variables x et y. Supposons que $V \in C^4(D')$, D' désignant un domaine de $x, y, C_1 \ldots, C_6$. Formons les équations dérivées

$$(10) \qquad p = V_x(x, y, C_1, \ldots, C_6), \quad q = V_y(x, y, C_1, \ldots, C_6),$$

$$(11) \qquad r = V_{xx}(x, y, C_1, \ldots, C_6),$$

$$(12) \qquad s = V_{xy}(x, y, C_1, \ldots, C_0), \quad t = V_{yy}(x, y, C_1, \ldots, C_6),$$

$$(13) \qquad \gamma = V_{xyy}(x, y, C_1, \ldots, C_6),$$

$$(14) \qquad \delta = V_{yyy}(x, y, C_1, \ldots, C_0),$$

sous la condition suivante dans le domaine D'

$$(15) \qquad \Delta \equiv D\left(\frac{V, V_x, V_y, V_{xy}, V_{yy}, V_{yyy}}{C_1, C_2, C_3, C_4, C_5, C_6}\right) \neq 0.$$

Si le résultat de l'élimination des paramètres C_i des équations (11), (13) et (9) (10), (12), (14) ne donne que les équations (1) et (2), nous dirons dans ce cas que l'intégrale complète du système (1)—(2) est définie par l'équation (9).

On va maintenant démontrer que grâce aux conditions (4)—(5) la fonction V doit satisfaire aux conditions complémentaires.

En effet, on peut démontrer aisément qu'il y a lieu les égalités suivantes

$$(16) \qquad (f_s) = -\frac{\Delta_1}{\Delta}, \quad (f_t) = -\frac{\Delta_2}{\Delta}, \quad (\Phi_\delta) = -\frac{\Delta_3}{\Delta},$$

où les parenthèses signifient le résultat de la substitution de z, p, q, s, t, δ respectivement par leurs valeurs $V, V_x, V_y, V_{xy}, V_{yy}, V_{yyy}$ et $\Delta_1, \Delta_2, \Delta_3$ les déterminants fonctionnels suivants

$$\Delta_1 \equiv D\left(\frac{V, V_x, V_y, V_{xx}, V_{yy}, V_{yyy}}{C_1, C_2, C_3, C_4, C_5, C_0}\right), \quad \Delta_2 \equiv D\left(\frac{V, V_x, V_y, V_{xy}, V_{xx}, V_{yyy}}{C_1, C_2, C_3, C_4, C_5, C_6}\right),$$

$$(17) \qquad \Delta_3 \equiv D\left(\frac{V, V_x, V_y, V_{xy}, V_{yy}, V_{xyy}}{C_1, C_2, C_3, C_4, C_5, C_6}\right).$$

La condition (4) nous donne

$$(4') \qquad (\Phi_\delta)^2 - (f_s)(\Phi_\delta) + (f_t) = 0$$

où les parenthèses ont les significations antérieurement établies. Grâce aux égalités (16) la condition (4') devient

$$\left(\frac{\Delta_3}{\Delta}\right)^2 - \frac{\Delta_1}{\Delta}\frac{\Delta_3}{\Delta} - \frac{\Delta_2}{\Delta} = 0,$$

ou

(18) $$\frac{\Delta_3}{\Delta}=\frac{\Delta_1}{\Delta}+\frac{\Delta_2}{\Delta_3}, \quad \Delta_3\neq 0.$$

On peut mettre la condition (18) sous une forme plus symétrique. En effet, en vertu des identités évidentes

$$V_{xx}+f(x, y, V, V_x, V_y, V_{xy}, V_{yyy})=0,$$
$$V_{xxy}+(D_y f)=0$$

et (16) on peut établir une rélation nouvelle

(18′) $$\Delta'=-(f_s)\,\Delta_3-(f_t)\,\Delta,$$

ou

(18″) $$\frac{\Delta'}{\Delta_3}=\frac{\Delta_1}{\Delta}+\frac{\Delta_2}{\Delta_3},$$

désignant par Δ' le déterminant fonctionnel

(19) $$\Delta'\equiv D\left(\frac{V,\, V_x,\, V_y,\, V_{xy},\, V_{yy},\, V_{xxy}}{C_1,\, C_2,\, C_3,\, C_4,\, C_5,\, C_6}\right).$$

Grâce à la relation obtenue (18″), on peut mettre la condition (18) sous la forme convenable, plus symétrique

(20) $$\frac{\Delta_3}{\Delta}=\frac{\Delta'}{\Delta_3}, \quad \Delta_3\neq 0.$$

La relation (5) ou la relation suivante

(5′) $$(D_x\,\Phi)+(D_y\,\Phi)\,(f_t/\Phi_\delta)-(D_{yy}f)=0$$

est vérifiée indentiquement, grâce à la condition (18), à savoir

$$\left(\frac{\Delta_3}{\Delta}-\frac{\Delta_2}{\Delta_3}-\frac{\Delta_1}{\Delta}\right)V_{xyyy}=0.$$

Donc, la relation (5) n'impose pas des conditions nouvelles à la fonction V.

Il est aisé à démontrer qu'il doit avoir lieu la condition nouvelle

(21) $$\delta_1=0,$$

δ_1 désignant le déterminant fonctionnel

(22) $$\delta_1\equiv D\left(\frac{V,\, V_x,\, V_y,\, V_{xx},\, V_{xy},\, V_{yy}}{C_1,\, C_2,\, C_3,\, C_4,\, C_5,\, C_6}\right),$$

car l'équation (1) ne dépend pas de la dérivée $\delta=z_{yyy}$.

On peut démontrer que les conditions (18) ou (20) et (21) sont aussi suffisantes.

Donc, l'équation (9) definit une intégrale complète du système (1)—(2) en involution de *Darboux* du troisième ordre si la fonction V admet les conditions nécessaires et suffisantes

$$\Delta \neq 0, \quad \delta_1 = 0, \quad \Delta_2/\Delta = \Delta'/\Delta_3, \tag{23}$$

Δ, Δ_3, Δ' et δ_1 désignant les déterminants fonctionnels (15), (17), (19) et (22).

4. L'intégrale générale des caractéristiques. Le théorème généralisé de Jacobi

Il est aisé à démontrer que les formules

$$z = V, \quad p = V_x, \quad q = V_y, \quad s = V_{xy}, \quad t = V_{yy}, \quad \delta = V_{yyy} \tag{24}$$

définissent les six premières intégrales distinctes du système

$$dx = \frac{dz}{p + \Phi_\delta q} = \frac{dp}{r + \Phi_\delta s} = \frac{dq}{s + \Phi_\delta t} = \frac{ds}{\beta + \Phi_\delta \gamma} = \frac{dt}{\gamma + \Phi_\delta \delta} = -\frac{d\delta}{D_y \Phi}. \tag{8'}$$

En effet, on a les équations dérivées

$$\begin{gathered} \frac{dz}{dx} = V_x + V_y \frac{dy}{dx}, \quad \frac{dp}{dx} = V_{xx} + V_{xy} \frac{dy}{dx}, \quad \frac{dq}{dx} = V_{xy} + V_{yy} \frac{dy}{dx}, \\ \frac{ds}{dx} = V_{xxy} + V_{xyy} \frac{dy}{dx}, \quad \frac{dt}{dx} = V_{xyy} + V_{yyy} \frac{dy}{dx}, \quad \frac{d\delta}{dx} = V_{xyyy} + V_{yyyy} \frac{dy}{dx} \end{gathered} \tag{25}$$

et en vertu des relations

$$(\Phi_\delta) = -\frac{\Delta_3}{\Delta}, \quad (D_y \Phi) = -V_{xyyy} + \frac{\Delta_3}{\Delta} V_{yyyy},$$

$$V_{xx} + f(x, y, V, V_x, V_y, V_{xy}, V_{yy}) = 0,$$

$$V_{xyy} + \Phi(x, y, V, V_x, V_y, V_{xy}, V_{yy}, V_{yyy}) = 0,$$

et de l'équation première du système (8)

$$\frac{dy}{dx} = (\Phi_\delta), \tag{26}$$

où les parenthèses ont les significations antérieurement établie, le résultat de l'élimination des constantes C_i, définies par les équations (24), entre les équations (25) ne donne que les équations (8').

Cherchons, encore, l'intégrale de l'équation (26) ou d'une des équations équivalentes

$$\Delta_3\, dx + \Delta\, dy = 0, \tag{27}$$

$$\Delta' dx + \Delta_3\, dy = 0. \tag{28}$$

Introduisons le symbole suivant

$$(i, j, k, l, m) \equiv D\left(\frac{V, V_x, V_y, V_{xy}, V_{yy}}{C_i, C_j, C_k, C_l, C_m}\right), \tag{29}$$

i, j, k, l, m désignant les nombres entiers de 1 à 6. On peut former les identités suivantes

$$\frac{\partial}{\partial x}\left[\frac{(i,k,l,m,n)}{(i,j,k,l,m)}\right] = a_1 \Delta' + a_2 \Delta_3, \quad \frac{\partial}{\partial x}\left[\frac{(i,j,l,m,n)}{(i,j,k,l,n)}\right] = b_1 \Delta' + b_2 \Delta_3,$$

$$\frac{\partial}{\partial y}\left[\frac{(i,k,l,m,n)}{(i,j,k,l,m)}\right] = a_1 \Delta_3 + a_2 \Delta, \quad \frac{\partial}{\partial y}\left[\frac{(i,j,l,m,n)}{(i,j,k,l,n)}\right] = b_1 \Delta_3 + b_2 \Delta,$$

a_i et b_i désignant les fonctions des variables $x, y, C_1, \ldots, C_6$ bien déterminées. Grâce à la condition (20) il en résulte

$$\frac{\dfrac{\partial}{\partial x}\left[\dfrac{(i,k,l,m,n)}{(i,j,k,l,m)}\right]}{\Delta_3} = \frac{\dfrac{\partial}{\partial y}\left[\dfrac{(i,k,l,m,n)}{(i,j,k,l,m)}\right]}{\Delta} \tag{30}$$

et aussi

$$\frac{\dfrac{\partial}{\partial x}\left[\dfrac{(i,j,l,m,n)}{(i,j,k,l,n)}\right]}{\Delta'} = \frac{\dfrac{\partial}{\partial y}\left[\dfrac{(i,j,l,m,n)}{(i,j,k,l,n)}\right]}{\Delta_3}. \tag{31}$$

Donc, les équations (27) ou (28) ont les intégrales suivantes

$$I_1 \equiv \frac{(i,k,l,m,n)}{(i,j,k,l,m)} = \text{const.}, \qquad I_2 \equiv \frac{(i,j,l,m,n)}{(i,j,k,l,n)} = \text{const.}$$

$$(I_\sigma)_x \neq 0, \qquad (I_\sigma)_y \neq 0, \qquad (\sigma = 1, 2).$$

Ces intégrales ne sont pas indépendantes par rapport aux variables x et y. En effet, en vertu des relations (30) et (31), on a

$$D\left(\frac{I_1, I_2}{x, y}\right) = 0.$$

On peut formuler le théorème suivant — *théorème de Jacobi:*

L'intégrale générale du système des équations différentielles des caractéristique (8) *est définie, en vertu de l'intégrale complète* (9) *et* (23), *par les formules suivantes*

$$z=V,\ p=V_x,\ q=V_y,\ s=V_{xy},\ t=V_{yy},\ \delta=V_{yyy},$$

$$I_1\equiv D\left(\frac{V,\ V_x,\ V_y,\ V_{xy},\ V_{yy}}{C_i,\ C_k,\ C_l,\ C_m,\ C_n}\right):D\left(\frac{V,\ V_x,\ V_y,\ V_{xy},\ V_{yy}}{C_i,\ C_j,\ C_k,\ C_l,\ C_m}\right)=C_7,$$

$$(I_1)_x\neq 0,\quad (I_1)_y\neq 0.$$

5. Sur une méthode de N. Saltykow dans la théorie des équations aux dérivées partielles du second ordre

N. Saltykow avait communiqué à une des séances de l'Institut mathématique en 1959 une méthode très intéressante pour l'intégration des équations aux dérivées partielles du second ordre. Notre but est maintenant de démontrer que la méthode exposée a des liaisons avec certaines notions et certains résultats de la théorie des équations aux dérivées partielles qui sont en involution de *Darboux* du troisième ordre, [33].

Considérons une équation aux dérivées partielles du second ordre d'une fonction inconnue z de deux variables indépendantes x et y

$$f(x,\ y,\ z,\ p,\ q,\ r,\ s,\ t)=0,\qquad f_r f_t\neq 0, \tag{32}$$

où l'on a posé $p=\partial z/\partial x$, $q=\partial z/\partial y$, $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x\,\partial y$, $t=\partial^2 r/\partial y^2$, $f_r=\partial f/\partial r$, $f_s=\partial f/\partial s$. On supposera que $f\in C^2(G)$, où G est un domaine déterminé des variables $x,\ y,\ z,\ p,\ q,\ r,\ s,\ t$.

Formons les équations dérivées de l'équation (32) respectivement par rapport à x et y, à savoir

$$\begin{cases} f_r\alpha+f_s\beta+f_t\gamma+D_x f=0,\\ f_r\beta+f_s\gamma+f_t\delta+D_y f=0,\end{cases} \tag{33}$$

en posant

$$\alpha=\partial^3 z/\partial x^3,\quad \beta=\partial^3 z/\partial x^2\,\partial y,\quad \gamma=\partial^3 z/\partial x\,\partial y^2,\quad \delta=\partial^3 z/\partial y^3,\quad f_s=\partial f/\partial s,$$

$$D_x=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q}$$

$$D_y=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}$$

et ajoutons y une équation auxiliaire

$$\beta-m(x,\ y,\ z,\ p,\ q,\ r,\ s,\ t)\,c=0. \tag{34}$$

La fonction m doit satisfaire aux conditions qui seront précisées plus tard.

Grâce à l'équation (34), on peut mettre **les équations** dérivées (33) sous la forme suivante

(33′)
$$\alpha+(mf_s+f_t)/(mf_r)\beta+D_x f/f_r=0,$$
$$\gamma+f_t/(mf_r+f_s)\delta+D_y f/(mf_r+f_s)=0.$$

Supposons que les coefficients de b et d des **équations** ci-dessus satisfont aux conditions

$$(mf_s+f_t)/mf_r=-m,\quad f_t/(mf_r+f_s)=-m;$$

on aura alors une seule condition pour la **fonction** m:

(35)
$$f_r m^2+f_s m+f_t=0.$$

Par conséquent les équations (33′) et (34) **peuvent être** écrites sous la forme

(36)
$$\frac{\partial r}{\partial x}-m\frac{\partial r}{\partial y}+\frac{l}{f_r}D_x f=0,$$
$$\frac{\partial s}{\partial x}-m\frac{\partial s}{\partial y}=0,$$
$$\frac{\partial t}{\partial x}-m\frac{\partial t}{\partial y}-\frac{m}{f_t}D_y f=0.$$

Les équations obtenues (36) et les identités **évidentes de** la forme suivante

(37)
$$\frac{\partial z}{\partial x}-m\frac{\partial z}{\partial y}=p-mq,$$
$$\frac{\partial p}{\partial x}-m\frac{\partial p}{\partial y}=r-ms,$$
$$\frac{\partial q}{\partial x}-m\frac{\partial q}{\partial y}=s-mt,$$

représentent un système de la forme de ***Charpit* à** six fonctions inconnues: z, p, q, r, s, t de deux variables **indépendantes** x **et** y, [18]. (Dans le livre de *Hilbert et Courant* [1], pour le système **de cette espèce** on emploie la dénomination „système des équations qui ont *la même partie principale*").

Le système correspondant d'équations **différentielles** ordinaires „*caractéristiques*" devient

(38)
$$dx=\frac{dy}{-m}=\frac{dz}{p-mq}=\frac{dp}{r-ms}=\frac{dq}{s-mt}=-\frac{dr}{D_x f/f_r}=\frac{ds}{0}=\frac{dt}{mD_y f/f_t}.$$

Il ne reste qu'à poser la question de l'intégration de l'équation donnée (32) à l'aide de l'intégration du système (38).

Posons maintenant le problème suivant: associons à l'équation (32) une autre équation de la forme (34) de telle manière que ces deux équations soient en involution de *Darboux* du troisième ordre.

Pour établir les conditions de la dite involution nous utiliserons, par exemple, un procédé antérieurement cité, [33].

Pour cela formons les équations dérivées des équations (32) et (34)

$$\begin{aligned} &f_r z_{xxxx} + f_s z_{xxxy} + f_t z_{xxyy} + D_{xx} f = 0, \\ &f_r z_{xxxy} + f_s z_{xxyy} + f_t z_{xyyy} + D_{xy} f = 0, \\ &f_r z_{xxyy} + f_s z_{xyyy} + f_t z_{yyyy} + D_{yy} f = 0, \\ &z_{xxxy} - m z_{xxyy} - \gamma D_x m = 0, \\ &z_{xxyy} - m z_{xyyy} - \gamma D_y m = 0. \end{aligned} \tag{39}$$

D_x, D_y, D_{xx}, D_{xy}, D_{yy} désignant respectivement les dérivées cerrespondantes des fonctions m et f prises une ou deux fois par rapport à x et y; quant à z_{xxxx}, z_{xxxy}, etc., elles désignent les dérivées partielles du quatrième ordre de la fonction z: $\partial^4 z/\partial x^4$, $\partial^4 z/\partial x^3 \partial y$, etc.

D'après les conditions de l'involution de *Darboux* du troisième ordre, les équations (39) sont insolubles par rapport aux dérivées du quatrième ordre: z_{xxxx}, z_{xxxy}, $\ldots$, z_{yyyy}. Il en résulte que tous les déterminants du cinquième ordre de la matrice des coefficients du système (39) sont égaux à zéro. On peut mettre le déterminant Δ du cinquième ordre formé des coefficients des dérivées du quatrième ordre de la fonction z — le déterminant du système (39) — sous la forme suivante

$$\Delta \equiv f_r f_t (f_r m^2 + f_s m + f_t).$$

En égalant ce déterminant à zéro et grâce à la condition $f_r f_t \neq 0$, on a

$$f_r m^2 + f_s m + f_t = 0. \tag{35}$$

Ce n'est que la condition (35) qui a été obtenue par la méthode de *N. Saltykow*. En utilisant la condition (35) et $f_r f_t \neq 0$, tous les autres déterminants du cinquième ordre seront égaux à zéro sous la condition suivante

$$\gamma f_t D_y m - m (\gamma f_r D_x m + D_{xy} f) = 0. \tag{40}$$

Donc, si les conditions (35) et (40) sont satisfaites, les équations (32) et (34) seront en involution de *Darboux* du troisième ordre.

Comme il est bien connu, [33], on peut associer aux équations (32) et (34), qui sont en involution de *Darboux* du troisième ordre, un système d'équations différentielles ordinaires, appelé *système des caractéristiques*.

Or, en ce qui concerne la formation du système mentionné d'équations différentielles ordinaires, on peut tirer d'abord du système (39), grâce aux conditions (34) et (40), les équations suivantes

$$
(41)\qquad
\begin{aligned}
&\frac{\partial\alpha}{\partial x}-m\frac{\partial\alpha}{\partial y}-\frac{\gamma f_t D_x m - m D_{xx} f}{m f_r}=0,\\
&\frac{\partial\beta}{\partial x}-m\frac{\partial\beta}{\partial y}-\gamma D_x m=0,\\
&\frac{\partial\gamma}{\partial x}-m\frac{\partial\gamma}{\partial y}-\gamma D_y m=0,\\
&\frac{\partial\delta}{\partial x}-m\frac{\partial\delta}{\partial y}-\frac{m}{f_t}(\gamma f_r D_y m + D_{yy} f)=0.
\end{aligned}
$$

Enfin y ajoutons les équations évidentes

$$
(42)\qquad
\begin{aligned}
&\frac{\partial z}{\partial x}-m\frac{\partial z}{\partial y}=p-mq,\\
&\frac{\partial p}{\partial x}-m\frac{\partial p}{\partial y}=r-ms,\\
&\frac{\partial q}{\partial x}-m\frac{\partial q}{\partial y}=s-mt,\\
&\frac{\partial r}{\partial x}-m\frac{\partial r}{\partial y}=\alpha-m\beta=-\frac{D_x f}{f_r},\\
&\frac{\partial s}{\partial x}-m\frac{\partial s}{\partial y}=\beta-m\gamma=0,\\
&\frac{\partial t}{\partial x}-m\frac{\partial t}{\partial y}=\gamma-m\delta=\frac{m}{f_t}D_y f.
\end{aligned}
$$

Les systèmes (41) et (42) font un système de *Charpit* par rapport aux fonctions inconnues: $z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$ de deux variables indépendantes x et y. Ce système de *Charpit* est équivalent au système cherché des caractéıistiques

$$
(43)\qquad
\begin{aligned}
dx&=\frac{dy}{-m}=\frac{dz}{p-mq}=\frac{dp}{r-ms}=\frac{dq}{s-mt}=\frac{dr}{D_x f/f_r}=\frac{ds}{0}=-\frac{dt}{mD_y f/f_t}=\\
&=\frac{d\alpha}{(\gamma f_t D_x m - m D_{xx} f)\, m f_r}=\frac{d\beta}{\gamma D_x m}=\frac{d\gamma}{\gamma D_y m}=\frac{d\delta}{m(\gamma f_r D_y m + D_{yy} f) f_t}.
\end{aligned}
$$

On peut utiliser les intégrales du système des caractéristiques pour former les solutions cherchées de l'équation donnée (32) (avec (34), (35) et (40)), [24].

3. D'après la théorie exposée ci-haut, on doit — si l'on associe à l'équation (32), avec la condition $f_r f_t \neq 0$, une autre équation (34) avec la condition $m \neq 0$—compléter la condition (35), obtenue par la méthode de *N. Saltykow* et aussi en égalant le déterminant Δ du système (39) à zéro, avec la condition (40), en s'assurant que tous les déterminants du cinquième ordre de la matrice du système (39) sont égaux à zéro, c'est-à-dire affirment que les équations (32) et (34) sont en involution de *Darboux* du troisième ordre. Dans ce cas, on doit compléter le système des équations différentielles (38) avec les équations nouvelles et utiliser le système des caractéristiques (43).

En ce qui concerne les équations (32) et (34), on peut résoudre deux problèmes de *Jacobi* et faire la liaison entre l'intégrale du système des caractéristiques et l'intégrale complète des équations aux dérivées partielles. On peut aussi poser le problème de la formation d'une intégrale de *Cauchy* à l'aide d'une intégrale complète donnée, [33].

6. Sur le problème de Cauchy des systèmes en involution de Darboux du troisième ordre

Dans la Note, [34], en suivant l'idée de *Courant*, [1], nous avons fait une application d'un système correspondant de *Charpit* pour obtenir l'intégrale de *Cauchy* des systèmes des équations aux dérivées partielles du second ordre en involution de *Darboux-Lie*. Maintenant, nous allons traiter d'une manière analogue le problème de *Cauchy* concernant un système en involution de *Darboux* du troisième ordre.

Le problème initial pour le système de Charpit. Considérons un système des équations aux dérivées partielles en involution de *Darboux* du troisième ordre d'une fonction inconnue z de deux variables indépendantes x et y

(44) $$r+f(x, y, z, p, q, s, t)=0, \quad z_{xyy}+\Phi(x, y, z, p, q, s, t, z_{yyy})=0,$$

sous les conditions

$$\Phi_\delta^2 - f_s \Phi_\delta + f_t = 0,$$
$$D_x \Phi + (f_t/\Phi_\delta) D_y \Phi - D_{yy} f = 0,$$

avec: $p=\partial z/\partial x$, $q=\partial z/\partial y$, $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x \partial y$, $t=\partial^2 z/\partial y^2$, $z_{xyy}=\partial^3 z/\partial x \partial y^2$, $\delta=z_{yyy}=\partial^3 z/\partial y^3$, $f_s=\partial f/\partial s$, $f_t=\partial f/\partial t$, $\Phi_\delta=\partial\Phi/\partial\delta$, quant à D_x, D_y, D_{yy}, elles désignent les dérivées correspondantes prises par rapport à x et y.

On peut associer, [33], au système (44) un système de *Charpit* de la forme suivante

$$(45)\qquad \begin{aligned}
&\frac{\partial z}{\partial x}+\Phi_\delta\frac{\partial z}{\partial y}=p+\Phi_\delta q, &&\frac{\partial s}{\partial x}+\Phi_\delta\frac{\partial s}{\partial y}=\beta+\Phi_\delta\gamma,\\
&\frac{\partial p}{\partial x}+\Phi_\delta\frac{\partial p}{\partial y}=r+\Phi_\delta s, &&\frac{\partial t}{\partial x}+\Phi_\delta\frac{\partial t}{\partial y}=\gamma+\Phi_\delta\delta,\\
&\frac{\partial q}{\partial x}+\Phi_\delta\frac{\partial q}{\partial y}=s+\Phi_\delta t, &&\frac{\partial \gamma}{\partial x}+\Phi_\delta\frac{\partial \gamma}{\partial y}+D_x\Phi=0,\\
&\frac{\partial r}{\partial x}+\Phi_\delta\frac{\partial r}{\partial y}=\alpha+\Phi_\delta\beta, &&\frac{\partial \delta}{\partial x}+\Phi_\delta\frac{\partial \delta}{\partial y}+D_y\Phi=0,
\end{aligned}$$

où l'on a $\alpha=z_{xxx}$, $\beta=z_{xxy}$, $\gamma=z_{xyy}$, avec les fonctions inconnues z, p, q, r, s, t, γ, δ de deux variables indépendantes x et y.

Nous supposons d'abord que les fonctions f et Φ soient telles qu'ils existent les intégrales premières distinctes suivantes

$$(46)\qquad \begin{aligned}
&f_i(x, y, z, p, q, s, t, z_{yyy}), \qquad (i=1, 2, \ldots, 7)\\
&f_8\equiv r+f(x, y, z, p, q, s, t)\\
&f_9\equiv\gamma+\Phi(x, y, z, p, q, s, t, \delta)
\end{aligned}$$

sous la condition

$$(46')\qquad \mathcal{D}\left(\frac{f_1, f_2, f_3, f_4, f_5, f_6, f_7}{y,\ z,\ p,\ q,\ s,\ t,\ \delta}\right)\neq 0.$$

On peut obtenir ces intégrales par l'intégration du système d'équations différentielles ordinaires suivantes

$$dx=\frac{dy}{\Phi_\delta}=\frac{dz}{p+\Phi_\delta p}=\frac{dp}{r+\Phi_\delta s}=\frac{dq}{s+\Phi_\delta t}=\frac{dr}{\alpha+\Phi_\delta\beta}=$$
$$=\frac{ds}{\beta+\Phi_\delta\gamma}=\frac{dt}{\gamma+\Phi_\delta\delta}=-\frac{d\gamma}{D_x\Phi}=-\frac{d\delta}{D_y\Phi},$$

où α et β doivent être exprimés par les autres variables figurant dans les équations (44).

Grâce aux intégrales (46) l'intégrale générale du système de *Charpit* (45) est déterminée par les relations suivantes

$$(47)\qquad f_{i+1}=\prod\nolimits_i(f_1), \qquad (i=1, 2, \ldots, 8)$$

$\prod_i$ étant des fonctions arbitraires.

Pour le système (45) on peut résoudre:

Problème A. Déterminer les solutions

(48) $z(x, y),\ p(x, y),\ q(x, y),\ r(x, y),\ s(x, y),\ t(x, y),\ \gamma(x, y),\ \delta(x, y)$

du système Charpit (45) *contenant la courbe donnée non caractéristique*

(C) $x = x_0,\quad y = \tau,\quad z = z(\tau)$

de telle manière que l'on ait le long de la courbe (C) *les conditions suivantes*

$$r + f = 0,\quad \gamma + \Phi = 0,$$

(49) $$\begin{cases} dz = p\,dx + q\,dy, & dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy, \\ dr = \alpha\,dx + \beta\,dy, & ds = \beta\,dx + \gamma\,dy, \quad dt = \gamma\,dx + \delta\,dy, \end{cases}$$

et

(50) $p = a,\quad s = b\quad pour\quad x = x_0,\quad y = y_0$

où a et b sont des constantes données, mais $(x_0, y_0) \in C$.

Grâce aux conditions (C), (49), (50), on peut d'abord déterminer les valeurs initiales des variables $p, q, r, s, t, \gamma, \delta$, c'est-à-dire les fonctions suivantes

(51) $p(\tau),\ q(\tau),\ r(\tau),\ s(\tau),\ t(\tau)\ \gamma(\tau),\ \delta(\tau).$

En vertu des conditions (49) on a pour $x = x_0$

$$z'(\tau) = q(\tau),\quad p'(\tau) = s(\tau),\quad q'(\tau) = t(\tau),$$

$$r'(\tau) = \beta(\tau),\quad s'(\tau) = \gamma(\tau),\quad t'(\tau) = \delta(\tau),$$

et aussi

(51') $q(\tau) = z'(\tau),\quad t(\tau) = z''(\tau),\quad \delta(\tau) = z'''(\tau),$

(51'') $s(\tau) = p'(\tau),\quad \gamma(\tau) = p''(\tau).$

Donc, la fonction $p(\tau)$ se détermine comme une solution de *Cauchy* de l'équation différentielle ordinaire du second ordre

(52) $p''(\tau) + \Phi[x_0, \tau, z(\tau), p(\tau), z'(\tau), p'(\tau), z''(\tau), z'''(\tau)] = 0$

satisfaisant aux conditions

(52') $\tau = y_0,\ p(y_0) = a,\ s(y_0) = b,$

où l'on suppose l'unicité de la solution du problème de *Cauchy* (52)—(52').

Grâce à la solution obtenue $p(\tau)$ du problème (52)—(52') on peut déterminer les fonctions $s(\tau)$ et $\gamma(\tau)$, (51'). Quant à la fonction $r(\tau)$, on a la relation suivante

$$r(\tau) + f[x_0, \tau, z(\tau), p(\tau), z'(\tau), p'(\tau), q'(\tau)] = 0.$$

En utilisant les fonctions déterminées (51), on peut définir les fonctions nouvelles $\lambda_i(\tau)$ et les paramètres auxiliaires u_i par les relations suivantes

$$\lambda_i(\tau) \equiv f_i[x_0, \tau, z(\tau), p(\tau), q(\tau), r(\tau), s(\tau), t(\tau), \gamma(\tau)],$$

(53) $$\lambda_i(\tau) = u_i, \qquad (i=1, 2, \dots, 9).$$

En éliminant le paramètre τ entre les relations (53), on obtient les relations bien déterminées des paramètres u_i

$$u_{i+1} = \pi_i(u_1), \qquad (i=1, 2, \dots, 8).$$

Les fonctions arbitaires $\prod_i$ dans l'intégrale générale (47) doivent avoir les formes π_i. Donc, sous les conditions (*C*), (49) et (50) les solutions (48) du problème initial pour le système de *Charpit* (2) sont déterminées par les formules

(49′) $$f_{i+1} = \pi_i(f_1), \qquad (i=1, 2, \dots, 8).$$

Alors, le procédé indiqué résoud *le problème A*.

Le problème initial pour le système en involution. Pour le système (44) en involution de *Darboux* du troisième ordre on peut résoudre

Problème B. Les solutions (48) *du système de Charpit* (45) *déterminent l'intégrale de Cauchy du système en involution de Darboux* (44) *sous les conditions* (*C*) *et* (50).

Pour cela, il suffit de démontrer que les fonctions (48) remplissent identiquement sur la surface $z = z(x, y)$ les conditions suivantes

$$r(x, y) + f[x, y, z(x, y), p(x, y), q(x, y), s(x, y), t(x, y)] \equiv 0,$$

$$\gamma(x, y) + \Phi[x, y, z(x, y), p(x, y), q(x, y), s(x, y), t(x, y), \delta(x, y)] \equiv 0,$$

$$p(x,y) - \frac{\partial z(x, y)}{\partial x} \equiv 0, \qquad q(x, y) - \frac{\partial z(x, y)}{\partial v} \equiv 0,$$

$$r(x, y) - \frac{\partial p(x, y)}{\partial x} \equiv 0, \qquad s(x, y) - \frac{\partial p(x, y)}{\partial y} \equiv s(x, y) - \frac{\partial q(x, y)}{\partial x} \equiv 0,$$

$$t(x, y) - \frac{\partial q(x, y)}{\partial y} \equiv 0, \qquad \gamma(x, y) - \frac{\partial s(x, y)}{\partial y} \equiv \gamma(x, y) - \frac{\partial t(x, y)}{\partial x} \equiv 0,$$

$$\delta(x, y) - \frac{\partial t(x, y)}{\partial y} \equiv 0.$$

La démonstration du *problème B* se peut achever d'une manière analogue comme dans le cas du système en involution de *Darboux-Lie*, [34], mais cette fois à l'aide du système de *Charpit* (45).

Exemple. Considérons le système

$$r - t - \frac{4}{x} p = 0,$$

(44)

$$\gamma + \delta + \frac{3}{x} s + \frac{1}{x} t + \frac{3}{x^2} p - \frac{1}{4} x^2 (x + y) = 0$$

en involution du troisième ordre (pour obtenir la seconde équation du trosième ordre en sachant la première équation du second ordre, voir le procédé [35]) et cherchons l'intégrale de *Cauchy* sous les conditions

(*C*) $$x = 1, \quad y = \tau, \quad z = \frac{1}{3} \tau^3,$$

(44') $$p = 0, \quad s = -\frac{7}{12}, \quad \text{pour} \quad x_0 = 1, \quad y_0 = 0$$

Dans ce cas les intégrales (48) sont

$$f_1 \equiv x - y,$$

$$f_2 \equiv r + 2s + t - \frac{1}{6} x^4 - \frac{1}{3} x^3 y,$$

$$f_3 \equiv p + q - xr - 2xs - xt + \frac{3}{20} x^5 + \frac{1}{4} x^4 y,$$

$$f_4 \equiv z - x(p + q) + \frac{1}{2} x^2 (r + 2s + t) - \frac{1}{15} x^6 - \frac{1}{10} x^5 y,$$

$$f_5 \equiv \frac{1}{x^2} t + \frac{3}{x^3} p - \frac{1}{4} xy,$$

$$f_6 \equiv \frac{1}{4} y - \frac{3}{x^4} p - \frac{1}{x^3} t - \frac{1}{x^2} \gamma,$$

$$f_7 \equiv \frac{1}{x^2} \left[\frac{1}{12} x^5 + \frac{1}{4} x^4 y - p - \frac{1}{2} x (r + 2s + t) \right],$$

$$f_8 \equiv r - t - \frac{4}{x} p$$

$$f_9 \equiv \gamma + \delta + \frac{3}{x} s + \frac{1}{x} t + \frac{3}{x^2} p - \frac{1}{4} x^2 (x + y)$$

et les fonctions (51) sont déterminées par les formules

$$p(\tau) = -\frac{7}{12} \tau, \quad q(\tau) = \tau^2, \quad r(\tau) = -\frac{1}{3} \tau, \quad s(\tau) = -\frac{7}{12}, \quad t(\tau) = 2\tau$$

$$\gamma(\tau) = 0, \quad \delta(\tau) = 2.$$

Les solutions du *problème A* s'obtiennent sous la forme

$$r+2s+t-\frac{1}{6}x^4-\frac{1}{3}x^3y=-\frac{4}{3}(x-y),$$

$$p+q-x(r+2s+t)+\frac{3}{20}x^5+\frac{1}{4}x^4y=(x-y)^2+\frac{19}{20},$$

$$z-x(p+q)+\frac{1}{2}x^2(r+2s+t)-\frac{1}{15}x^6-\frac{1}{10}x^5y=-\frac{1}{3}(x-y)^3-\frac{19}{60}(x-y),$$

$$\frac{1}{4}y-\frac{3}{x^4}y-\frac{1}{x^3}t-\frac{1}{x^2}\gamma=0,$$

$$\frac{1}{x^2}\left[\frac{1}{12}x^5+\frac{1}{4}x^4y-p-\frac{1}{2}x(r+2s+t)\right]=\frac{2}{3},$$

$$r-t-\frac{4}{x}p=0,$$

$$\gamma+\delta+\frac{3}{x}s+\frac{1}{x}t+\frac{3}{x^2}p-\frac{1}{4}x^2(x+y)=0,$$

ou

$$z(x,y)=\frac{1}{60}x^5y-\frac{1}{3}x^2y+\frac{1}{3}y^3+\frac{19}{60}y,$$

$$p(x,y)=\frac{1}{12}x^4y-\frac{2}{3}xy,$$

$$q(x,y)=\frac{1}{60}x^5-\frac{1}{3}x^2+y^2+\frac{19}{60},$$

$$r(x,y)=\frac{1}{3}x^3y-\frac{2}{3}y,\qquad s(x,y)=\frac{1}{12}x^4-\frac{2}{3}x,$$

$$t(x,y)=2y,\qquad \gamma(x,y)=0,\quad \delta(x,y)=2$$

et la surface $z=z(x,y)$ est la solution du système (54), (C), (54′).

7. Sur les intégrales des systèmes en involution de Darboux du troisième ordre

Il s'agit, dans ce paragraphe, d'établir la théorie de *Lagrange* des intégrales dans le cas des systèmes en involution de *Darboux* du troisième ordre, [36].

En étudiant les propriétés nouvelles pour les systèmes en involution de *Darboux* du troisième ordre nous allons utiliser la méthode de *Lagrange* de la variation des constantes dans l'intégrale complète.

Considérons un système en involution de *Darboux*

(55) $$r+f(x,y,z,p,q,s,t)=0,\quad z_{xyy}+\Phi(x,y,z,p,q,s,t,z_{yyy})=0,$$

pour lequel on a à la fois identiquement

$$\Phi_\delta^2 - f_s \Phi_\delta + f_t = 0, \quad D_x \Phi + \frac{f_t}{\Phi_\delta} D_y \Phi - D_{yy} f = 0,$$

en désignant respectivement par $p, q, r, s, t, z_{xyy}, z_{yyy}, f_s, f_t, \Phi_\delta$ les dérivées partielles: $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x \partial y$, $\partial^2 z/\partial y^2$, $\partial^3 z/\partial x \partial y^2$, $\delta = \partial^3 z/\partial y^3$, $\partial f/\partial s$, $\partial f/\partial t$, $\partial \Phi/\partial \delta$. Quant à D_x, D_y, D_{yy} elles désignent respectivement les dérivées correspondantes prises par rapport à x et y. On supposera que $f \in C^2(G)$ et $\Phi \in C^1(G_1)$, où le G et G_1, $G \subset G_1$, sont les domaines respectifs des variables x, y, z, p, q, s, t et des variables $x, y, z, p, q, s, z_{yyy}$.

Si l'on considère dans l'intégrale complète

(9) $$z = V(x, y, C_1, \ldots, C_6)$$

avec les conditions (20), les C_i, selon la méthode de la variation des constantes, comme des fonctions $C_i(x, y)$ des variables x et y, l'équation (9) définit une intégrale du système (55) seulement sous les conditions suivantes

(56) $$\nabla V \cdot C_{\xi_j} = 0, \quad \nabla V_x \cdot C_{\xi_j} = 0, \quad \nabla V_y \cdot C_{\xi_j} = 0$$
$$\nabla V_{xy} \cdot C_{\xi_j} = 0, \quad \nabla V_{yy} \cdot C_{\xi_j} = 0, \quad (j = 1, 2),$$

où l'on a

$$\nabla V = \left\{\frac{\partial V}{\partial C_i}, \ldots, \frac{\partial V}{\partial C_6}\right\}, \quad \nabla V_x = \left\{\frac{\partial^2 V}{\partial x \partial C_1}, \ldots, \frac{\partial^2 V}{\partial x \partial C_6}\right\}, \ldots,$$

$$C_{\xi_1} \equiv C_x, \quad \xi_{\xi_2} \equiv C_y', \quad C_x = \left\{\frac{\partial C_1}{\partial x}, \ldots, \frac{\partial C_6}{\partial x}\right\}, \quad C_y = \left\{\frac{\partial C_1}{\partial y}, \ldots, \frac{\partial C_6}{\partial y}\right\}.$$

Introduisons le symbole

(57) $$(i, j, k, l, m) \equiv \mathcal{D}\left(\frac{V, V_x, V_y, V_{xy}, V_{yy}}{C_i, C_j, C_k, C_l, C_m}\right),$$

où i, j, k, l, m désignent les nombres entiers de 1 à 6. Grâce à la condition

$$\Delta \neq 0, \quad \delta = 0$$

on peut mettre les conditions (56) sous la forme plus commode

(56′) $$\begin{cases} (1, 3, 4, 5, 6)\dfrac{\partial C_1}{\partial \xi_j} + (2, 3, 4, 5, 6)\dfrac{\partial C_2}{\partial \xi_j} = 0, \\ \qquad (1, 2, 4, 5, 6)\dfrac{\partial C_1}{\partial \xi_j} - (2, 3, 4, 5, 6)\dfrac{\partial C_3}{\partial \xi_j} = 0, \\ (1, 2, 3, 5, 6)\dfrac{\partial C_1}{\partial \xi_j} + (2, 3, 4, 5, 6)\dfrac{\partial C_4}{\partial \xi_j} = 0, \\ \qquad (1, 2, 3, 4, 6)\dfrac{\partial C_1}{\partial \xi_j} - (2, 3, 4, 5, 6)\dfrac{\partial C_5}{\partial \xi_j} = 0, \\ (1, 2, 3, 4, 5)\dfrac{\partial C_1}{\partial \xi_i} + (2, 3, 4, 5, 6)\dfrac{\partial C_5}{\partial \xi_i}, \qquad (\xi_1 \equiv x, \ \xi_2 = y, \ j = 1, 2). \end{cases}$$

Si les conditions

(58) $$(C_1, C_{k+1}) \equiv \mathfrak{D}\left(\frac{C_1, C_{k+1}}{x, y}\right) = 0, \quad \text{où } C_{k+1} = \varphi_k(C_1) \quad (k = 1, 2, \ldots, 5)$$

sont satisfaites, φ_k étant les fonctions arbitraires, les équations (56') admettent un système des solutions non triviales par rapport à (i, j, k, l, m). Mais dans ce cas grâce à $\partial C_1/\partial \xi_j \neq 0$ les relations (56) nous donnent

(59) $$\begin{cases} \nabla V \cdot \varphi' = 0, & \nabla V_x \cdot \varphi' = 0, \quad \nabla V_y \cdot \varphi' = 0 \\ \nabla V_{xy} \cdot \varphi' = 0, & \nabla V_{yy} \cdot \varphi' = 0, \quad \varphi' \{1, \varphi_1, \ldots, \varphi_5\} \end{cases}$$

ou

(59') $$\begin{cases} \varphi_1' = -\dfrac{(1, 3, 4, 5, 6)}{(2, 3, 4, 5, 6)}, & \varphi_2' = \dfrac{(1, 2, 4, 5, 6)}{(2, 3, 4, 5, 6)}, \\ \varphi_3' = -\dfrac{(1, 2, 3, 5, 6)}{(2, 3, 4, 5, 6)}, & \varphi_4' = -\dfrac{(1, 2, 3, 4, 6)}{(2, 3, 4, 5, 6)}, \\ \varphi_5' = -\dfrac{(1, 2, 3, 4, 5)}{(2, 3, 4, 5, 6)}. \end{cases}$$

En vertu de la condition

(20) $$\frac{\Delta_1}{\Delta} = \frac{\Delta'}{\Delta_1} \qquad (i = 1, 2, 3, 4),$$

les quotients dans les relations (59') ne sont pas indépendants par rapport aux variables x et y, et il ne reste que quatre relations bien déterminées pour définir les fonctions φ_i

(60) $$\varphi_i'(C_1) = F_i(C_1, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_5'), \quad (i = 1, 2, 3, 4).$$

A cause de ces quatre relations, il restera une fonction arbitraire. L'intégrale du système (55) avec une fonction arbitraire est déterminée par l'équation

$$z = V[x, y, C_1, \varphi_1(C_1), \ldots, \varphi_5(C_1)]$$

et par les équations (60) et $\nabla V \cdot \varphi' = 0$. Par analogie avec la terminologie de la théorie de *Lagrange* des intégrales dans le cas des équations aux dérivées partielles du premier ordre et des équations en involution de *Darboux-Lie*, [20], on l'appellera l'intégrale générale du système (55).

Si l'un des déterminants (C_1, C_{k+1}) est différent du zéro le système (59') n'a que les solutions triviales

(61) $$\begin{aligned} &(1, 2, 3, 4, 5) = 0, \quad (2, 3, 4, 5, 6) = 0, \quad (1, 3, 4, 5, 6) = 0, \\ &(1, 2, 4, 5, 6) = 0, \quad (1, 2, 3, 5, 6) = 0, \quad (1, 2, 3, 4, 6) = 0. \end{aligned}$$

Si l'on peut à l'aide des équations (61) déterminer les fonctions $C_i(x, y)$, alors les équations (9) et (61) nous donnent une intégrale du système (55). On l'appellera l'intégrale singulière de (55).

De cette manière, est établie la théorie de *Lagrange* des intégrales dans le cas des systèmes (55).

On peut donner la généralisation du théorème de *Jacobi* pour la théorie dite de *Lagrange* dans le cas des systèmes (55):

Chaque solution du système (55) dans un domaine bien déterminé peut être déduite de l'intégrale complète, ou de l'intégrale générale, ou bien elle est l'intégrale singulière.

On peut résoudre pour le système (55) le problème suivant de *Lagrange*, [14], [22]:

Former l'intégrale générale du système (55) à l'aide de l'intégrale générale du système correspondant des caractéristiques.

Il est bien connu qu'à chaque système en involution de *Darboux-Lie* correspond un système d'équations de *Monge* à quatre variables, [14]. Nous avons vu maintenant qu'à chaque système (55) correspond un système de la forme (60) d'équations de *Monge* à six variables. Cette remarque nous conduit à étudier comment on peut utiliser dans la théorie de l'intégration d'un système (55) les résultats jusqu'ici connus pour les équations de *Monge*

$$F_i(x_1, x_2, x_3, x_4, x_5, x_6, dx_2/dx_1, dx_3/dx_1, dx_4/dx_1, dx_5/dx_1, dx_6/dx_1)=0,$$
$$(i=1, 2, \ldots, 4).$$

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