Zbornik radova 11(19)

APPLICATIONS OF MATHEMATICS IN MECHANICS

Matematički institut SANU

Zbornik radova 11(19)

APPLICATIONS OF MATHEMATICS IN MECHANICS

Matematički institut SANU

Zbornik radova 11(19)

APPLICATIONS OF MATHEMATICS IN MECHANICS

Editor:
Bogoljub Stanković

Matematički institut SANU

Издавач: Математички институт САНУ, Београд, Кнеза Михаила 35
серија: Зборник радова, књига 11(19)
За издавача: Богољуб Станковић, главни уредник
Уредник: Богољуб Станковић
Технички уредник: Драган Благојевић
Штампа: "Академска издања", Земун
Штампање завршено новембра 2006.

CIP – Каталогизација у публикацији
Народна библиотека Србије, Београд

531.011:531.1/.3(082)
Applications of Mathematics in Mechanics / editor Bogoljub Stanković.
– Beograd : Matematički institut SANU, 2006 (Zemun : Akademska izdanja).
– 154 str. : tabele ; 24 cm.
– (Zbornik radova / Matematički institut SANU ; knj. 11(19))
Tiraž 350. – Str. 3: Preface / Bogoljub Stanković.
– Bibliografija uz svako poglavlje.

ISBN 86–80593–39-7
1. Stanković, Bogoljub
а) Механика – Математички методи – Зборници
COBISS.SR-ID 135229707

Contents

PREFACE

The aim of *Zbornik radova* is to foster further growth of pure and applied mathematics, publishing papers which contain new ideas and scopes in the mathematics. The papers have to be prepared in such a manner that they can inform readers in a favourable way, introducing them in a narrower field of mathematical theories pointing at research possibilities. It can be for the individual use or for discussions in College or University seminars.

We are open for contacts and cooperations.

Bogoljub Stanković
Editor-in-Chief

Teodor Atanacković and Bogoljub Stanković

GENERALIZED FUNCTIONS IN SOLVING LINEAR MATHEMATICAL MODELS IN MECHANICS

CONTENTS

0. Introduction

The aim of this paper is to consider the necessity of introducing the generalized functions for the construction and solving mathematical models.

Mathematical models in mechanics have been usually given by a partial differential equation with some boundary and initial conditions.

With regards to the construction of a mathematical model the following remarks are worthy of mention:

First we have to catch sight and then to select the basic elements of the situation (of the object) we wish to model. Consequently, a mathematical model is only an approximation of the object to which it corresponds.

Or to put in another, more pessimistic consideration: All models are wrong, some models are "useful" [30]. But there are several requirements that mathematical models must satisfy in order to be "useful". Structural stability of the model is probably the most important requirement. Also, because of the approximate value of a model, it is natural to expect that if we can find a family of solutions to the model equation and if there exists a subfamily which is convergent, then the limit has also to be a solution. The difficulty lies in finding a topology not overly restrictive but such that the found limit has a meaning for the treated object.

That is one of the sources of the "weak" and "generalized" solutions to mathematical models which will be used in this paper, as well.

Many authors have pointed at shortcomings of the classical analysis with regards to the solving partial differential equations. L. Hörmander [27] illustrated them by the equation of the vibrating string

$$\frac{\partial^2}{\partial x^2} v(x,t) - \frac{\partial^2}{\partial t^2} v(x,t) = 0.$$

Its classical solution has been given by $v(x,t) = f(x+t) + g(x-t)$, where f and g are arbitrary functions with continuous second derivatives. In his opinion the limits of sequences of such solutions have also to be taken as solutions (Laplace operator has just this property).

He continues with such a consideration for the nonhomogeneous equation

$$\frac{\partial^2}{\partial x^2} v(x,t) - \frac{\partial^2}{\partial t^2} v(x,t) = F(x,t),$$

where $F(x,t)$ is continuous and equals zero outside a bounded set. If F has also continuous first partial derivatives, then the cited nonhomogeneous equation has a classical solution

$$v(x,t) = -\frac{1}{2} \iint\limits_{\tau - t + |x-\xi| < 0} F(\xi,\tau)\, d\xi\, d\tau.$$

In case that $F(x,t)$ is only continuous, the found solution $v(x,t)$ has continuous first partial derivatives and has to be admitted as solution, as well. Such solutions are called "weak solutions".

Secondly, partial differential equations have been given by partial derivatives which are very restrictive operations in usual topology in $\mathbb{R}^n$ (in classical analysis) and have not to be continuous. The first systematic elaborated idea to overcome these shortcomings of the classical derivatives has been given by S. L. Sobolev (cf. [57]). He started from the space $\mathbf{L}^p(\Omega)$, $p \geqslant 1$, where Ω is an open set in $\mathbb{R}^n$. Let $\varphi, \psi \in C^m(\overline{\Omega})$, $\operatorname{supp}\psi = K$, K compact set in Ω. Then

$$\int\limits_{\Omega} \Big[\varphi(x) \frac{\partial^m \psi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} + (-1)^{m+1} \psi(x) \frac{\partial^m \varphi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}\Big]\, dx = 0, \qquad m_1 + \cdots + m_n = m.$$

If we know only that $\varphi \in \mathbf{L}^p(\Omega)$, $p \geqslant 1$, and that there exists $\omega_{m_1,\ldots,m_n} \in \mathbf{L}_{\mathrm{loc}}(\Omega)$ such that

$$\int\limits_{\Omega} \Big[\varphi(x) \frac{\partial^m \psi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} + (-1)^{m+1} \psi(x) \omega_{m_1,\ldots,m_n}(x)\Big]\, dx = 0$$

for every ψ with the cited properties, then $\omega_{m_1,\ldots,m_n}$ is defined as Sobolev's generalized derivative

$$\frac{\partial^m \varphi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \stackrel{\text{def}}{=} \omega_{m_1,\ldots,m_n}(x).$$

This is the basic idea for the theory of Sobolev's spaces which are very useful in the theory of partial differential equations.

Schwartz's distributions (cf. [56]) generalize Sobolev's idea and represent a theory which gives impressive results in the theory of partial differential equations. To every locally integrable function it corresponds in a unique way a distribution. Every distribution has all partial derivatives which are continuous operators. The space $\mathcal{D}'$ of distributions is the least extension of the space of continuous functions in which all elements have all partial derivatives. Moreover, derivatives are continuous operators. Consequently, if we have a convergent sequence or a convergent filter with the countable basis of the filter (cf. [56, I, p. 53]) as solution to a linear partial differential equation in $\mathcal{D}'$, then the limit of this sequence or of this filter is also a solution to this equation.

To this day many spaces of generalized functions have been elaborated (cf. [13], [18], [20], [24], [31], [32], [40], [47], [53], [56]) which can be useful in considering mathematical models. Not only to find a generalized solution to a model, but also to improve the classical methods for solving them. In this sense the integral transforms of generalized functions have an important role.

A very significant fact is that the spaces of generalized functions have not only been used to solve a mathematical models, but also in the construction of models.

Some elements and relations in the theoretical physics can be defined only by using generalized functions. Let us mention first of all the Dirac δ-"function". The quantum field theory is an impressive example of a theory which uses generalized functions to express some phenomena from physics (cf. [15], [16], [29], [65]).

The utility of mathematics for many problems of science and society is increasingly evident. However we can not neglect some doubt in this linking. Namely, mathematics pretends to claims of absolute certainty by means of mathematical proofs. But this certainty is paid for by logical disconnection from empirical reality. One can find cited the following Einstein sentence (cf. [12]): "As far as the properties of mathematics refer to reality, they are not certain and as far as they are certain, they do not refer to reality".

So in considerations mathematical models we have two extreme positions:

First, if a solution to the constructed mathematical model is not quite mathematically rigorous, but none the less leads to an excellent conformity with experimental observation, then one can consider such solutions valued by nature, if not by mathematics.

Second, one may choose to recognize mathematical models and their solutions if and only if the model is based on classical foundations and solutions have been obtained in absolute mathematical rigorousness.

In this paper we shall work with generalized solutions which are:

- well-defined;
- obtained in a mathematically correct way which allows to see why their introduction is necessary;
- solutions of linear mathematical models arising from mechanics and which claim can be validated by natural conditions;
- elements of spaces acceptable to the specialists working in mechanics.
- a pointer to the very abstract possibilities of the today's cutting-edge mathematics.

The paper is divided into three parts. In the first we repeat some definitions and results from spaces of generalized functions we need subsequently. In the second part we give constructions of some interesting new mathematical models in mechanics. In the third we solve the constructed models illustrating the possibilities of methods which have been offered by generalized functions in solving mathematical models in mechanics. We have not insisted on complete mathematical proofs if they were overly large and if they can be found in the published papers cited.

1. Spaces of generalized functions

In this paper we use the space of distributions $\mathcal{D}'$ with some subspaces and the space of hyperfunctions $\mathcal{B}$.

1.1. The space of distributions.

1.1.1. *Definitions and notation.* We repeat some definitions and facts that we need in our exposition. There are now a lot of books in which one can find spaces of distributions elaborated in different volumes. We cite only some, we use (cf. [24], [56], [66]). If the cited result is not well-known, then we give the proof, as well.

Let Ω denote an open subset of $\mathbb{R}^n$ (Ω can be $\mathbb{R}^n$ on the whole). The *support of a function* φ ($\operatorname{supp}\varphi$) defined on Ω is the closure in Ω of the set $\{x \in \Omega;\ \varphi(x) \neq 0\}$. The space $\mathcal{D}(\Omega)$ is the space $\{\varphi \in C^\infty(\mathbb{R}^n);\ \operatorname{supp}\varphi \subset \Omega\}$. A sequence $\{\varphi_j\} \subset \mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to zero if and only if there exists a compact set $K \subset \Omega$ such that:

1. $\operatorname{supp}\varphi_j \subset K$, $j \in \mathbb{N}$;
2. for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \equiv \mathbb{N}_0^n$, $\varphi_j^{(\alpha)} \to 0$ uniformly on K;

$$\varphi_j^{(\alpha)} = \Big(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\Big)\varphi_j.$$

$\mathcal{D}'(\Omega)$ is the space of all continuous linear functionals on $\mathcal{D}(\Omega)$. It is called *the space of distributions* on Ω. The value of a distribution f at a function $\varphi \in \mathcal{D}(\Omega)$ will be denoted by $\langle f, \varphi\rangle$.

Every locally integrable function f on Ω defines the *regular distribution* $[f]$, by $\langle [f], \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx$, $\varphi \in \mathcal{D}(\Omega)$. Two functions $f, g \in L^1_{\mathrm{loc}}(\Omega)$ define the same distribution $[f] = [g]$ on Ω if and only if $f = g$ a.e. on Ω.

Suppose that $u_x \in \mathcal{D}'(\mathbb{R}^n)$, $v_y \in \mathcal{D}'(\mathbb{R}^m)$. By

$$\langle w, \varphi\rangle = \langle u_x, \langle v_y, \varphi(x,y)\rangle\rangle = \langle v_y, \langle u_x, \varphi(x,y)\rangle\rangle$$

is defined the distribution $w \in \mathcal{D}'(\mathbb{R}^{n+m})$, where $\varphi \in \mathcal{D}(\mathbb{R}^{n+m})$ and x, y denote variables in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The distribution w is called *tensor product* of the distributions u_x and v_y; one writes $w = u_x \otimes v_y$.

Let u_x and v_y belong to $\mathcal{D}'(\mathbb{R}^n)$. If there exists a distribution $z \in \mathcal{D}'(\mathbb{R}^n)$ defined by $\langle z, \varphi\rangle = \langle u_x \otimes v_y, \varphi(x+y)\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then z is called the *convolution*of u_x and v_y and is denoted by $u_x * v_y$.

From the properties of convolution we mention only: if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$, then $\varphi * u \in C^\infty(\mathbb{R}^n)$ and $\varphi * u = u * \varphi = \langle u_x, \varphi(y-x)\rangle$.

Let $D^m u$ denote the m-th derivative in the sense of distributions (see Section 1.1.2), then $D^m\delta * u = D^m u$, $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$.

An important subspace of $\mathcal{D}'(\mathbb{R}^n)$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. Let us define it. By $\mathcal{S}(\mathbb{R}^n)$ we denote the *space of rapidly decreasing functions* φ with the property that for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, $\sup_{x\in\mathbb{R}^n}|x^\alpha\varphi^{(\beta)}(x)| < \infty$.

The space of linear continuous functional on $\mathcal{S}(\mathbb{R}^n)$ is called the *space of tempered distributions* and is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Let Γ denote the closed, convex and acute cone and $C = \operatorname{int}\Gamma$. Let K be a compact set in $\mathbb{R}^n$. By $\mathcal{S}'(\Gamma + K)$ is denoted the space of tempered distributions with supports in the closed set $\Gamma + K \subset \mathbb{R}^n$. Then $\mathcal{S}'(\Gamma+)$ is defined by

$$\mathcal{S}'(\Gamma+) = \bigcup_{K\subset\mathbb{R}^n} \mathcal{S}'(\Gamma + K).$$

The set $\mathcal{S}'(\Gamma+)$ forms an algebra that is associative and commutative if for the operation of multiplication one takes the convolution, denoted by $*$.

1.1.2. *Derivatives of a distribution.* Let D^{α_i} denote the α_i-th derivative in x_i of a distribution. It is defined as

$$\langle D^{\alpha_i} f, \varphi\rangle = \big\langle (-1)^{\alpha_i} f, \varphi^{(\alpha_i)}\big\rangle, \ \ f \in \mathcal{D}'(\Omega),\ \varphi \in \mathcal{D}(\Omega).$$

Then for $\alpha = (\alpha_1, \dots, \alpha_n)$, $D^\alpha f = D^{\alpha_1} \cdots D^{\alpha_n} f$.

We list some properties of the *derivatives of distributions*:

1. Every distribution has all derivatives D^{α_i} and $D^{\alpha_i} D^{\alpha_j} = D^{\alpha_j} D^{\alpha_i}$, $i, j = 1, \dots, n$.

2. The differentiation of distributions is a linear and continuous mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$.

3. In particular, every regular distribution has derivatives of any order. In this sense every locally integrable function has distributional derivatives. The derivative of a regular distribution has not to be regular distribution.

4. If $F \in C^\alpha(\Omega)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, then $D^\alpha[F] = [F^{(\alpha)}]$. Moreover, if $a \in C^\infty(\Omega)$, then $aD^\alpha[F] = [aF^{(\alpha)}]$.

5. If $F, G \subset C(\Omega)$ and $D_{x_i}[F] = [G]$, then there exists $F^{(1)}_{x_i}$ and $F^{(1)}_{x_i} = G$, $i \in (1, \dots, n)$.

6. Let η denote the function

$$\eta(x) = \begin{cases} 0, & |x| \geqslant 1, \\ \exp(|x|^2 - 1)^{-1}, & |x| < 1,\ |x|^2 = x_1^2 + \dots + x_n^2 \end{cases}$$

and let $c = \int_{\mathbb{R}^n} f(x)\, dx$, $\omega_k(x) = c^{-1} k^n f(kx)$, $\Omega_{1/k} = \{x \in \mathbb{R}^n,\ d(x, \Omega) \leqslant 1/k\}$, $d(x, \Omega) = \inf_{y \in \Omega} |x - y|$. If $f, g \in \mathrm{L}^p(\Omega)$, $1 \leqslant p < \infty$ and $D^\alpha[f] = [g]$, then for $\Omega_1 \subset \Omega$ and $(\Omega_1)_{1/k} \subset \Omega$, $k \geqslant k_0$, $\|(f * \delta_n)^{(\alpha)} - g\|_{\mathrm{L}^p(\Omega_1)} \to 0$, $n \to \infty$, where $\delta_n = \omega_{1/n} \subset \mathcal{D}(\Omega)$ and $*$ is the sign of convolution.

7. Some properties which can be useful in solving differential and partial differential equations.

If $\Omega \subset \mathbb{R}$, $u \in \mathcal{D}'(\Omega)$ and $D^m u(x) + f_{m-1}(x) D^{m-1} u(x) + \dots + f_0(x) u(x) = F(x)$, where $f_i \in C^\infty(\Omega)$, $i = 0, 1, \dots, m-1$, and $F \in C^p(\Omega)$, $p \in \mathbb{N}_0$, then the solution u is defined by a function belonging to $C^{m+p}(\Omega)$ and represents the classical solution.

Let $\{\delta_n\}_{n \in \mathbb{N}}$, $\operatorname{supp} \delta_n$, $n \in \mathbb{N}$, belong to the compact set $K \subset \mathbb{R}^n$ and $[\delta_n] \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$; let also L be a linear differential operator with constant coefficients. Then every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ to $L(u) = 0$ is a limit in $\mathcal{D}'(\mathbb{R}^n)$ of a sequence $\{u_j\}_{j \in \mathbb{N}}$ of classical solutions to $L(u) = 0$. The sequence $\{u_j\}$ can be $u_j = u * \delta_j$ (cf. [64]).

8. Derivatives of a regular distribution

8.1. One dimensional case. Let $f \in C^{(p)}((-\infty, b))$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and H_a be a function such that $H_a(x) = 0$, $-\infty < x < a < b$; $H_a(x) = 1$, $0 \leqslant a \leqslant x < b$. Denote by $[H_a f]$ the regular distribution defined by $H_a f$. Hence, $[H_a f] \in \mathcal{D}'((-\infty, b))$, $\operatorname{supp}[H_a f] \subset [a, b)$ or $[H_a f] \in \mathcal{D}'([a, b))$, as well as $\mathcal{D}'([a, b)) = \{T \in \mathcal{D}'(-\infty, b); \operatorname{supp} T \subset [a, b)\}$. By $[f_a^{(p)}]$, $p \in \mathbb{N}$, we denote the distribution defined by the function $f_a^{(p)}$ equals to $f^{(p)}(x)$, $x \in (a, b)$ and equals zero for $x \in (-\infty, a)$ and is not defined for $x = a$.

Since the function $(H_a f)^{(k)}$ has in general a discontinuity of the first kind in $x = a$, $k = 0, 1, \dots, p$, by the well-known formula (cf. [56])

$$\begin{aligned} D^p[H_a f] &= [f_a^{(p)}] + f^{(p-1)}(a)\delta(x-a) + \cdots + f(a)\delta^{(p-1)}(x-a) \\ &= [f_a^{(p)}] + R_{p,a}(f) = [H_a f^{(p)}] + R_{p,a}(f), \end{aligned} \tag{1.1}$$

where $D^p[H_a f]$ is the derivative of order p in the sense of distributions, and (cf. [56])

$$R_{p,a}(f) = f^{(p-1)}(a)\delta(x-a) + \cdots + f(a)\delta^{(p-1)}(x-a).$$

Definition 1.1. [60] Let α be a positive real number such that $m - 1 < \alpha < m$ for a fixed $m \in N$. The α-th fractional derivative of a function $f \in C([0,\infty))$ is defined by

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^m}{dx^m} \int_0^x f(x-t) t^{m-1-\alpha} dt, \quad x > 0,$$

if this derivative exists.

Proposition 1.1. *Let α be a real number such that $0 < \alpha < 1$ and let $f \in C((0,b))$, f bounded on $[0,\varepsilon]$, $\varepsilon > 0$, or, more generally, let $|f(x)| \leqslant Mx^{-(\beta-\alpha)}$, $0 < x < \varepsilon$, for an $\varepsilon > 0$, $0 < \alpha < \beta < 1$. Then:*

$$[f^{(\alpha)}(x)|_{(0,b)}] = \frac{1}{\Gamma(1-\alpha)} D_x \left[H_0(x) \int_0^x f(x-t) t^{-\alpha} dt \right].$$

Proof. By (1.1)

$$\begin{aligned} [f^{(\alpha)}(x)|_{(0,b)}] &= \frac{1}{\Gamma(1-\alpha)} D_x \left[H_0(x) \int_0^x f(x-t) t^{-\alpha} dt \right] \\ &\quad - \frac{1}{\Gamma(1-\alpha)} \lim_{x \to 0+} \int_0^x f(x-t) t^{-\alpha} dt \\ &= \frac{1}{\Gamma(1-\alpha)} D_x \left[H_0(x) \int_0^x f(x-t) t^{-\alpha} dt \right]. \end{aligned}$$ □

We have to prove that $\lim_{x\to 0} \int_0^x f(x-t) t^{-\alpha} dt = 0$. Let $x > 0$. Then we have

$$\begin{aligned} \int_0^x (x-t)^{-(\beta-\alpha)} t^{-\alpha} dt &= \frac{2-\beta}{(1-(\beta-\alpha))x} \int_0^x t^{-\alpha}(x-t)^{1-(\beta-\alpha)} dt \\ &= \frac{(2-\beta)x^{1-(\beta-\alpha)}}{(1-(\beta-\alpha))x} \int_0^x t^{-\alpha}\left(1 - \frac{t}{x}\right)^{1-(\beta-\alpha)} dt. \end{aligned}$$

Since $0 \leqslant t < x$, $\left|(1-t/x)^{1-(\beta-\alpha)}\right| < 1$,

$$\left|\int_0^x (x-t)^{-(\beta-\alpha)} t^{-\alpha} dt\right| \leqslant \frac{(2-\beta)x^{1-\beta}}{(1-(\beta-\alpha))(1-\alpha)} \to 0, \quad x \to 0^+.$$

We denote $\left[f^{(\alpha)}(x)|_{(0,b)}\right]$ by $D^\alpha[H_0 f]$.

8.2. The n-dimensional case. We use the following notation: $P = \prod_{i=1}^n [a_i, b_i)$, $0 \leqslant a_i < b_i$, $i = 1, \dots, n$; $\Omega = \overline{\mathbb{R}}^n_- + P$, then $P \subset \Omega$; $H_a^n(x) = H_{a_1}(x_1) \cdots H_{a_n}(x_n)$, $H_{a_i}(x_i) = 1$, $a_i \leqslant x_i < b_i$; $H_{a_i}(x_i) = 0$, $x_i < a_i$, $i = 1, \dots, n$. Let f be a function with continuous partial derivatives on Ω; $[H_a^n f]$ is the distribution, defined by $H_a^n f$, belonging to $\mathcal{D}'(\Omega)$ and to $\mathcal{D}'(P)$, as well. Finally, $\left(\partial^p f/\partial x_i^p\right)_{a_i}$ is the function equal to $\partial^p f/\partial x_i^p$ on the $\operatorname{int} P \cup \{x, x_j = a_j,\ j \neq i\}$, and equal to zero on $\Omega \setminus P$, but is not defined for $x_i = a_i$.

Proposition 1.2. *With the notation as above, we have*

$$D_{x_i}^p[H_a^n f] = \left[H_a^n \left(\frac{\partial^p}{\partial x_i^p} f\right)_{a_i}\right] + R_{p,a_i}(f), \quad p \in \mathbb{N}, \tag{1.2}$$

where

$$R_{p,a_i}(f) = \left[H_a^n \frac{\partial^{p-1}}{\partial x_i^{p-1}} f(x)|_{x_i=a_i}\right] \times \delta(x_i - a_i) + \cdots + \left[H_a^n f(x)|_{x_i=a_i}\right] \times \delta^{(p-1)}(x_i - a_i). \tag{1.3}$$

Proof. The method of the proof is just the same as for (1.1). □

Proposition 1.3. *With the notation as in Proposition 1.2, we have*

$$\begin{aligned} D_{x_j}^q D_{x_i}^p [H_a^n f] &= \left[H_a^n \frac{\partial^q}{\partial x_j^q}\left(\left(\frac{\partial^p}{\partial x_i^p} f\right)_{a_i}\right)_{a_j}\right] \\ &+ \left[H_a^n \frac{\partial^{q-1}}{\partial x_j^{q-1}}\left(\frac{\partial^p}{\partial x_i^p} f\right)_{a_i}(x)|_{x_j=a_j}\right] \times \delta(x_j - a_j) \\ &+ \left[H_a^n \left(\frac{\partial^p}{\partial x_i^p} f\right)_{a_i}(x)|_{x_j=a_j}\right] \times \delta^{(q-1)}(x_j - a_j) + D_{x_j}^q R_{p,a_i}(f). \end{aligned}$$

Proof. We have only to apply $D_{x_j}^q$ to (1.2). □

Remark. To realize $D_{x_j}^q R_{p,a_i}$ we have to use (1.3).

We illustrate the use of Proposition 1.3 by calculating the following expressions $D_{x_2} D_{x_1}[H_a^2 f]$, $D_{x_1}^2 D_{x_2}^2 [H_a^2 f]$ and $D_{x_1}^\alpha D_{x_2}^2 [H_a^2 f]$.

1) $D_{x_2} D_{x_1}[H_a^2 f]$. Let us start with the first derivatives.

$$D_{x_1}[H_a^2 f] = \left[H_a^2\left(\frac{\partial}{\partial x_1} f\right)_{a_1}\right] + \delta(x_1 - a_1) \times [H_{a_2}(x_2) f(a_1, x_2)],$$

$$D_{x_2}[H_a^2 f] = \left[H_a^2\left(\frac{\partial}{\partial x_2} f\right)_{a_2}\right] + [H_{a_1}(x_1) f(x_1, a_2)] \times \delta(x_2 - a_2),$$

$$D_{x_2}D_{x_1}[H_a^2 f] = D_{x_2}\Big[H_a^2\Big(\frac{\partial}{\partial x_1}f\Big)_{a_1}\Big] + \delta(x_1-a_1)\times D_{x_2}[H_{a_2}(x_2)f(a_1,x_2)]$$
$$= \Big[H_a^2\Big(\frac{\partial^2}{\partial x_1 \partial x_1}f\Big)_{a_1,a_2}\Big] + D_{x_1}[H_{a_1}(x_1)f(x_1,a_2)]\times\delta(x_2-a_2)$$
$$-f(a_1,a_2)\delta(x_1-a_1)\times\delta(x_2-a_2) + \delta(x_1-a_1)\times D_{x_2}[H_{a_2}(x_2)f(a_1x_2)],$$

where $\Big(\frac{\partial^2}{\partial x_2 \partial x_1}f\Big)_{a_1,a_2} = \frac{\partial^2}{\partial x_2 \partial x_1}f(x,y),\ (x,y)\in(a_1,b_1)\times(a_2,b_2)$.

Remark. a) This formula is derived by supposing that:

$$\Big(\frac{\partial f}{\partial x_1}f\Big)_{a_1}(x_1,x_2)\Big|_{x_2=a_2} = \Big(\frac{\partial}{\partial x_1}f(x_1,a_2)\Big)_{a_1}$$
$$\Big(\frac{\partial f}{\partial x_2}f\Big)_{a_2}(x_1,x_2)\Big|_{x_1=a_2} = \Big(\frac{\partial}{\partial x_2}f(a_1,x_2)\Big)_{a_2}.$$

b) It follows that $D_{x_1}D_{x_2}[H_a^2 f] = D_{x_2}D_{x_1}[H_a^2 f]$.

2) $D_{x_1}^2 D_{x_2}^2[H_a^2 f]$. By a similar procedure as in 1) we have

$$D_{x_1}^2 D_{x_2}^2[H_a^2 f] = \Big[H_a^2\Big(\frac{\partial^4}{\partial x_1^2 \partial x_2^2}f\Big)_{a_1,a_2}\Big] + D_{x_1}^2[H_{a_1}f(x_1,a_2)]\times\delta^{(1)}(x_2-a_2)$$
$$+\delta^{(1)}(x_1-a_1)\times D_{x_2}^2[H_{a_2}f(a_1,x_2)] + D_{x_1}^2\Big[H_{a_1}\frac{\partial}{\partial x_2}f(x_1,a_2)\Big]\times\delta(x_2-a_2)$$
$$+\delta(x_1-a_1)\times D_{x_2}^2\Big[H_{a_2}\frac{\partial}{\partial x_1}f(a_1,x_2)\Big] - f(a_1,a_2)\big(\delta^{(1)}(x_1-a_1)\times\delta^{(1)}(x_2-a_2)\big)$$
$$-\frac{\partial}{\partial x_2}f(a_1,a_2)\big(\delta^{(1)}(x_1-a_1)\times\delta(x_2-a_2)\big) - \frac{\partial}{\partial x_1}f(a_1,a_2)\big(\delta(x_1-a_1)\times\delta^{(1)}(x_2-a_2)\big)$$
$$-\frac{\partial^2}{\partial x_1 \partial x_2}f(a_1,a_2)\big(\delta(x_1-a_1)\times\delta(x_2-a_2)\big).$$

3) $D_{x_1}^\alpha D_{x_2}^2[H_a^2 f]$, $a_1=0$, $b_1=\infty$ and θ is Heaviside's function.

$$\Big[D_{x_1}^\alpha\Big(\frac{\partial^2}{\partial x_2^2}f\Big)_{a_2}\Big] = \frac{1}{\Gamma(1-\alpha)}D_{x_1}\Big[H_a^2\Big(\Big(\frac{\partial^2}{\partial x_2^2}f\Big)_{a_2} *_{x_1}\theta(x_1)x_1^{-\alpha}\Big)\Big]$$
$$= \frac{1}{\Gamma(1-\alpha)}D_{x_1}D_{x_2}^2\big[(H_a^2 f *_{x_1}\theta(x_1)x_1^{-\alpha})\big]$$
$$-\frac{1}{\Gamma(1-\alpha)}D_{x_1}\Big[\Big(H_a^2\frac{\partial}{\partial x_2}f(x_1,x_2)\big|_{x_2=a_2} *_{x_1}\theta(x_1)x_1^{-\alpha}\Big)\Big]\times\delta(x_2)$$
$$-\frac{1}{\Gamma(1-\alpha)}D_{x_1}\big[(H_a^2 f(x_1,x_2)\big|_{x_2=a_2} *_{x_1}\theta(x_1)x_1^{-\alpha})\big]\times\delta^{(1)}(x_2).$$

9. If $u,v\in\mathcal{D}'(\mathbb{R}^n)$ and $u*v$ exists, then for $m=(m_1,\dots,m_n)\in\mathbb{N}_0^n$ one has $D^m(u*v) = (D^m u * v) = (u * D^m v)$.

1.1.3. *The convergence of a sequence of distributions.* A sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{D}'(\Omega)$ is called convergent to $u \in \mathcal{D}'(\Omega)$ if for every $\varphi \in \mathcal{D}(\Omega)$ the limit $\lim\limits_{n\to\infty}\langle u_n, \varphi\rangle = \langle u, \varphi\rangle$ exists and is finite.

If the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{L}^1_{\mathrm{loc}}(\Omega)$ converges in $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$ to zero, then the sequence $\{[u_n]\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ converges to zero in $\mathcal{D}'(\Omega)$, as well. In particular, the space $\mathcal{C}(\Omega)$ can replace $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$ in this statement (cf. [56]).

1.1.4. *Distributional-valued functions.* Let $\Omega_x \subset \mathbb{R}^n$ and $\Omega_t \subset \mathbb{R}^m$ be open sets. We define the function w on Ω_x with values in $\mathcal{D}'(\Omega_t)$; $w : \Omega_x \ni x \to w(x) \in \mathcal{D}'(\Omega_t)$. Such a function w is called *distributional-valued function.* A distributional-valued function w defined on $\Omega_x \subset \mathbb{R}$ is of the class $\mathcal{C}^1$ if the limit

$$\lim_{\varepsilon\to 0}\Big\langle \frac{1}{\varepsilon}\big((w(x+\varepsilon\xi) - w(x)), \varphi\big)\Big\rangle$$

exists for every $\varphi \in \mathcal{D}(\Omega_t)$ where x and $x + \varepsilon\xi$ belong to Ω_x, i.e.

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(w(x+\varepsilon\xi) - w(x)\big) \text{ exists in } \mathcal{D}'(\Omega_t).$$

We put by definition that in $\mathcal{D}'(\Omega_t)$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(w(x+\varepsilon\xi) - w(x)\big) = w^{(1)}_{x_j}(x).$$

Repeating p times this procedure, we obtain the *distributional-valued function of class* $\mathcal{C}^p$ (cf. [64]).

1.1.5. *The Laplace transform of distributions.* To define the *Laplace transform* (in short LT) of distributions we start with the Laplace transform of tempered distributions. The notion and definitions we will use were given in 1.1.

If $\Gamma + K$ is convex, as it will be in our case, then the LT of $f \in \mathcal{S}'(\Gamma+)$ is defined by

$$\hat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), e^{-zt}\rangle,\ z \in C + i\mathbb{R}^n,$$

where $t = (t_1, \dots, t_n)$, $z = (z_1, \dots, z_n)$, $zt = z_1t_1 + \cdots + z_nt_n$ and $C = \operatorname{int}\Gamma$. It is one to one operation.

For the properties of so defined LT one can consult [66]. We shall cite only those used in the sequel:

1) $\mathcal{L}\Big(\dfrac{\partial^m}{\partial t_i^m}f\Big)(z) = (z_i)^m\mathcal{L}(f)(z)$.
2) If $f \in \mathcal{S}'(\Gamma+)$ and $g \in \mathcal{S}'(\Gamma+)$, then $\mathcal{L}(f \times g)(z,s) = \mathcal{L}(f)(z)\mathcal{L}(g)(s)$, $z \in C + i\mathbb{R}^n$, $s \in C + i\mathbb{R}^n$.
3) If $f, g \in \mathcal{S}'(\Gamma+)$, then $f * g \in \mathcal{S}'(\Gamma+)$ and $\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z)$, $z \in C + i\mathbb{R}^n$.
4) If $f \in L_{\mathrm{loc}}([0,\infty))$ and is bounded in a neighborhood of zero, $0 < \beta < 1$, $n = 1$, then $\mathcal{L}(f^{(\beta)})(z) = z^\beta\mathcal{L}(f)(z)$.
5) $\mathcal{L}(\delta(t - t_0))(z) = e^{-zt_0}$.
6) $\mathcal{L}(f)(z+a) = \mathcal{L}(e^{-at}f)(z)$, $\operatorname{Re} a > 0$.

7) If $f \in \mathrm{L}_{\mathrm{loc}}(\mathbb{R}^n_+)$ and $|f(x)| \leqslant Me^{qx}$, $x \geqslant x_0 > 0$, then $f(x)e^{-qx} \in \mathcal{S}'(\overline{\mathbb{R}^n_+})$ and

$$\int\limits_{\mathbb{R}^n_+} e^{-(z+q)t} f(t)\,dt = \int\limits_{\mathbb{R}^n_+} e^{-zt} e^{-qt} f(t)\,dt = \mathcal{L}(e^{-qt}f)(z).$$

Let $\mathcal{H}_a^{(\alpha,\beta)}(C)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, $a \geqslant 0$, denote the sets of holomorphic functions on $C + i\mathbb{R}^n$ which satisfy the following growth condition:

$$|f(z)| \leqslant Me^{a|x|}(1+|z|^2)^{\alpha/2}(1+d^{-\beta}(x,\partial C)), \quad z = x+iy \in C + i\mathbb{R}^n,$$

where ∂C is the boundary of C and $d(x, \partial C)$ is the distance between x and ∂C. We set

$$\mathcal{H}_a(C) = \bigcup_{\alpha,\beta \geqslant 0} \mathcal{H}_a^{(\alpha,\beta)}(C) \quad \text{and} \quad \mathcal{H}_+(C) = \bigcup_{a \geqslant 0} \mathcal{H}_a(C).$$

Proposition 1.4. [66, p. 191] *The algebras $H_+(C)$ and $S'(C^*+)$ and also their subalgebras $H_0(C)$ and $S'(C^*)$ are isomorphic. This isomorphism is accomplished via the LT.* ($C^* = \{t \in R^n;\ tx = t_1x_1 + \cdots + t_nx_n \geqslant 0,\ \forall x \in C\}$).

A property of the defined LT which can be used in a practical way is the following:

– Let P be the set $\prod_{i=1}^n [a_i, b_i)$, $0 \leqslant a_i < b_i$, $i = 1, \ldots, n$. Then $\overline{P}$ is compact. Since $\overline{\mathbb{R}^n_+}$ is a closed convex and acute cone, $\mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P})$ is well defined (see 1.1.1).

– Let $f \in \mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P})$. The LT of f, $\mathcal{L}(f)$, can be obtained by subsequent applications of the LT-s $\mathcal{L}_1(f), \ldots, \mathcal{L}_n(f)$, $\mathcal{L}(f) = \mathcal{L}_1(f) \circ \cdots \circ \mathcal{L}_n(f)$.

– If $\sigma \geqslant 0$, $f \in \mathcal{S}'(C^*+)$ and $g = e^{\sigma t}f$, then by definition $\mathcal{L}(g)(s) = \langle f(t), e^{-(s-\sigma)t}\rangle$, $\operatorname{Re} s > \sigma$.

– Let $F(s)$ be a function holomorphic for $\operatorname{Re} s > \sigma$. The function $F(\xi + \sigma)$ is holomorphic for $\operatorname{Re}\xi > 0$. If $F(\xi + \sigma) \in \mathcal{H}(\mathbb{R}_+)$, then there exists $f \in \mathcal{S}'(\overline{\mathbb{R}_+})$ such that $\mathcal{L}(e^{\sigma t}f)(s) = F(s)$.

H. Komatsu defined the Laplace transform for any hyperfunction (cf. [33]). The same idea we use to define the Laplace transform for a large class of distributions.

Let $\mathcal{A}$ be the vector space:

$$\mathcal{A} = \{T \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P});\ \operatorname{supp} T \subset \{(\overline{\mathbb{R}^n_+} + \overline{P}) \setminus P\}\},\ \omega \in \mathbb{R},$$

where $e^{\omega t} = e^{\omega t_1} \cdots e^{\omega t_n}$. $\mathcal{A}$ is a subspace of $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P})$. We can define an equivalence relation in $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P})$ by $f \sim g \Longleftrightarrow f - g \in \mathcal{A}$. Let $\mathcal{B}$ denote

$$\mathcal{B} = e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P})/\mathcal{A},\ \ b \in \mathcal{B} \Longleftrightarrow b = \operatorname{class}(T) \equiv d(T),\ \ T \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}^n_+} + \overline{P}).$$

Definition 1.2. [60] Let $\mathcal{D}'(P)$ denote the space of distributions defined on P. Then

$$\mathcal{D}'_\omega(P) = \{T \in \mathcal{D}'(P); \exists \overline{T} \in e^{\omega t}\mathcal{S}'((\overline{\mathbb{R}^n_+} + \overline{P})), \overline{T}|_P = T\},$$

where $\overline{T}|_P$ is the restriction of $\overline{T}$ on P. Since $\mathcal{D}'$ is not a flabby sheaf, $\mathcal{D}'_\omega(P) \neq \mathcal{D}'(P)$.

Proposition 1.5. *$\mathcal{D}'_\omega(P)$ is algebraically isomorphic to $\mathcal{B}$.*

Proof. If $T \in \mathcal{D}'_\omega(P)$, then there exists $\overline{T} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$ such that $\overline{T}|_P = T$. We can define the mapping $\lambda : \mathcal{D}'_\omega(P) \to \mathcal{B}$, for $T \in \mathcal{D}'_\omega(P)$, $\lambda(T) = cl(\overline{T}) \in \mathcal{B}$. The inverse mapping λ^{-1} exists and $\lambda^{-1}(cl(\overline{T})) = \overline{T}|_P = T \in \mathcal{D}'(P)$. T does not depend on the chosen element from $cl(\overline{T})$. If we take an other representative T_1 of the $cl(\overline{T})$, then $T_1 = \overline{T} + S$, $S \in \mathcal{A}$. Then $T_1|_P = \overline{T}|_P$. Now it is easily seen that λ is an algebraic isomorphism of two vector spaces. □

Definition 1.3. The LT of elements in $D'_\omega(P)$ is defined by

$$\mathcal{L}(\mathcal{D}'_\omega(P)) = \mathcal{L}(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P}))/\mathcal{L}(\mathcal{A}).$$

If $T \in D'_\omega(P)$, then $L(T) = cl(L\overline{T})$, where $\overline{T}$ is such that $\overline{T}|_P = T$.

Remark. Let H_P be the function $H_P(t) = 1$, $t \in P$, $H(t) = 0$, $t \in \mathbb{R}^n \setminus P$. Then:

a) If $f \in \mathrm{L}_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$, then the regular distribution $[H_P f]$ defined by $H_P f$ belongs to $\mathcal{D}'_\omega(P)$ and f has the LT in the sense of Definition 1.3.

b) If $f \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+ + \overline{P}_+)$ and $g \in \mathcal{A}$, then $f * g \in \mathcal{A}$, as well.

1.1.6. *Extension of a distribution.* We know that there exist distributions defined on an open set Ω which can not be extended to an open set $\Omega_1 \supset \overline{\Omega}$. This is a consequence that $\mathcal{D}'$ is not a flabby shief. There are theorems which give the conditions for the extendability. We cite one such theorem we use in the sequel:

Proposition 1.6. [64] *Let T be a distribution on a bounded open set $\Omega \subset \mathbb{R}^n$ and let $\Omega_1 \supset \overline{\Omega}$. Then T is extendable to Ω_1 if and only if there exist constants C and $k \in N_0$ satisfying $|\langle T, \varphi\rangle| \leqslant C \sum_{|\alpha| \leqslant k} \lim_{x \in \mathbb{R}^n} |\varphi^{(\alpha)}(x)|$ for $\varphi \in \mathcal{D}(\Omega)$.*

1.2. The space of hyperfunctions.

1.2.1. *Notation and definitions.* The space of hyperfunctions was introduced by M. Sato (cf. [52], [53]) in 1958. By H. Komatsu's opinion ([32]), the idea of hyperfunctions has been employed most successfully since a long time ago. He cited some examples from mathematics and physics, to prove it.

The theory of hyperfunctions in many variables calls for deep results in algebraic topology (cf. [32], [53]). But if one restricts oneself to the one dimensional case, this theory is of easier access. Fortunately we need only this theory of one variable.

Let Ω be an open set in $\mathbb{R}$ and V an open set in $\mathbb{C}$ containing Ω as a relatively closed set (Ω is a closed subset of V). Let $\mathcal{O}(V)$ denote the space of *holomorphic functions* on V. Then *hyperfunctions* on Ω are by definition the elements in the quotient space $\mathcal{B}(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$. If $F \in \mathcal{O}(V \setminus \Omega)$, then we denote by $[F]$ the class of F; F is called a *defining function* of the hyperfunction $[F]$.

The definition of $\mathcal{B}(\Omega)$ does not depend on the choice of the complex neighborhood of V.

$\mathcal{B}$ is a flabby sheaf. Consequently, if Ω_1 is an open subset of Ω, then any hyperfunction $f \in \mathcal{B}(\Omega_1)$ can be extended to an $\tilde{f} \in \mathcal{B}(\Omega)$. This is a very important property of $\mathcal{B}$. Distributions have not this property. That is the reason for Definition 1.2.

$\mathcal{B}(\Omega)$ contains $\mathcal{C}(\Omega)$, $\mathrm{L}_{\mathrm{loc}}(\Omega)$, $\mathcal{D}'(\Omega)$, the space of real analytic functions on Ω, ultradistributions on $\Omega \ldots$. One can find in [32] what conditions has to satisfy the defining function F of an hyperfunction $f = [F]$ so that f belongs to some subspaces of $\mathcal{B}(\Omega)$.

Let $\Omega = (-\infty, b)$ and $-\infty < a < b$; then the space of hyperfunctions with support in $[a, b)$ is $\mathcal{B}_{[a,b)} = \mathcal{O}(\mathbb{C}_{x<b} \setminus [a, b))/\mathcal{O}(\mathbb{C}_{x<b})$, where $\mathbb{C}_{x<b} = \{z \in \mathbb{C};\ \mathrm{Re}\, z < b\}$.

1.2.2. *The space of Laplace hyperfunctions and their Laplace transform.* Let $\mathbb{O}$ be the radial compactification of the complex plane and V an open set in $\mathbb{O}$. $\mathcal{O}^{\exp}(V)$ the space of functions F on V such that F is holomorphic on $\mathbb{C} \cap V$ and on each compact set $K \subset V$, $|F(z)| \leqslant Ce^{H|z|}$, $z \in K \cap \mathbb{C}$, with constants H and C. The space $\mathcal{B}^{\exp}_{[a,\infty]}$ of Laplace hyperfunctions with support in $[a, \infty]$ is defined by

$$\mathcal{B}^{\exp}_{[a,\infty]} = \mathcal{O}^{\exp}(\mathbb{O} \setminus [a, \infty])/\mathcal{O}^{\exp}(\mathbb{O}).$$

An $f \in \mathcal{B}^{\exp}_{[a,\infty]}$ is represented by $F \in \mathcal{O}^{\exp}(\mathbb{O} \setminus [a, \infty])$, $f = [F] = \{F + G;\ G \in \mathcal{O}^{\exp}(\mathbb{O})\}$. The *Laplace transform* $\tilde{\mathcal{L}} f(\xi)$ of an $f = [F] \in \mathcal{B}^{\exp}_{[a,\infty]}$ is defined by

$$\tilde{\mathcal{L}} f(\xi) = \int_L e^{-\xi z} F(z)\, dz \in \tilde{\mathcal{L}}\mathcal{B}^{\exp}_{[a,\infty]},$$

where L is a path composed of a ray from $e^{i\alpha}_{\infty}$ to a point $c < a$ and a ray from c to $e^{i\beta}_{\infty}$ with $-\pi/2 < \alpha < \beta < \pi/2$.

Proposition 1.7. [33] *The Laplace transformation $\tilde{\mathcal{L}}$ is an isomorphism $\mathcal{B}^{\exp}_{[a,\infty]} \to \tilde{\mathcal{L}}\mathcal{B}^{\exp}_{[a,\infty]}$, where $\tilde{\mathcal{L}}\mathcal{B}^{\exp}_{[a,\infty]}$ is the space of all holomorphic functions $\hat{f}(\xi)$ of exponential type defined on a neighborhood Ω of the semi-circle $S = \{e^{i\gamma}_{\infty};\ |\gamma| < \pi/2\}$ in $\mathbb{O}$ such that*

$$\varlimsup_{\rho \to \infty} \frac{1}{\rho} \log |\hat{f}(\rho e^{i\gamma})| \leqslant -a \cos \gamma, \quad |\gamma| < \pi/2. \tag{1.4}$$

If $\hat{f}(\xi) \in \tilde{\mathcal{L}}\mathcal{B}^{\exp}_{[a,\infty]}$, then a defining function $F(z)$ of its inverse image f is given by the integral

$$F(z) = \frac{1}{2\pi i} \int_{s_0}^{\infty} e^{\xi z} \hat{f}(\xi)\, d\xi,$$

where s_0 is a fixed point in Ω and the integral part is a convex curve in Ω.

The restriction mapping $\mathcal{O}^{\exp}(\mathbb{O} \setminus [a, \infty]) \to \mathcal{O}(\mathbb{C}_{x<b} \setminus [a, b))$ induces a natural mapping $\omega : \mathcal{B}^{\exp}_{[a,\infty]} \to \mathcal{B}_{[a,b)}$ which is surjective, but not injective. It has been proved (cf. [33]) that ω is surjective and

$$\mathcal{B}_{[a,b)} \cong \mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{B}^{\exp}_{[b,\infty]}.$$

Consequently,

$$\mathcal{L}\mathcal{B}_{[a,b)} \cong \mathcal{L}\mathcal{B}^{\exp}_{[a,b]}/\mathcal{L}\mathcal{B}^{\exp}_{[b,\infty]}. \tag{1.5}$$

If $g \in \mathcal{B}_{[a,\infty)}$, then $\tilde{\mathcal{L}} g = [\tilde{\mathcal{L}} \tilde{g}] = \{\tilde{\mathcal{L}} \tilde{g} + \tilde{\mathcal{L}} h;\ h \in \mathcal{B}^{\exp}_{[b,\infty]}\}$, $\tilde{g} \in \mathcal{B}^{\exp}_{[a,\infty]}$, $\tilde{g} \in \omega^{-1}(g)$.

Let $L_{loc}^{exp}([0,\infty))$ denote the space of locally integrable functions q on $[0,\infty)$ satisfying $|q(x)| \leqslant Ce^{H_x}$, $H \in \mathbb{R}$, $x \geqslant 0$. We write $\overline{\theta q}$ for the element in $\mathcal{B}_{[0,\infty)}$ which corresponds to q. Then the classical Laplace transform of q, $\mathcal{L}q(s) = \int_0^\infty e^{-st}q(t)\,dt$, belongs to $\tilde{\mathcal{L}}\mathcal{B}_{[0,\infty]}^{exp}$ (by Proposition 2.1) and $\widetilde{\theta q}$ may be regarded as the Laplace hyperfunction for which $\tilde{\mathcal{L}}\widetilde{\theta q}(\xi) = \mathcal{L}q(\xi)$; $\widetilde{\theta q}$ is an extension of $\overline{\theta q}$ on $[0,\infty]$. Here θ stands for the Heaviside function.

The delta distribution δ imbedded in $\mathcal{B}_{[0,\infty]}^{exp}$ is $\delta = \left[-\frac{1}{2\pi i}\frac{1}{z}\right]$ and with the notation

$$\delta^{(\alpha)}(x) = \begin{cases} D^\alpha\delta(x), & \alpha = 0,1,\ldots \\ x_+^{-\alpha-1}/\Gamma(-\alpha), & \alpha \neq 0,1,\ldots,\ \alpha \in \mathbb{R}_+ \end{cases}$$

where $x_+^{-\alpha-1}$, $\alpha > 0$ is the distribution with support in $[0,\infty)$ and $x_+^{-\alpha-1} = x^{-\alpha-1}$, $x > 0$, then $\tilde{\mathcal{L}}\delta^{(\alpha)}(\xi) = \xi^\alpha$, $\alpha \in \mathbb{R}_+ \cup \{0\}$ and $\tilde{\mathcal{L}}(\delta(x-x_0))(\xi) = e^{-x_0\xi}$, $x_0 > 0$. If $f, g \in L_{loc}([0,\infty))$, then $\overline{\theta f} * \overline{\theta g} = (\widetilde{\theta f}) * (\widetilde{\theta g})\big|_{(-\infty,\infty)} = \overline{\theta(f*g)}$. Here $f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau$ is the convolution.

1.2.3. *Final comments.* Let us remark the following facts which concern the hyperfunctions:

1. It is a very large space containing the most interesting functions and generalized function spaces.
2. The Laplace transform of hyperfunctions is defined by (1.5), for hyperfunctions having an arbitrary growth order. Specially, every locally integrable function on $[0,\infty)$ has the Laplace transform in this sense.
3. The space $\tilde{\mathcal{L}}\mathcal{B}_{[a,\infty]}^{exp}$ has been characterized by (1.4).
4. The Laplace transform $\tilde{\mathcal{L}}$ is a generalization of the classical one. If $f \in L_{loc}([0,\infty))$ and has a classical Laplace transform $\hat{f}(s)$, then it has $\tilde{\mathcal{L}}f(s)$ and $\tilde{\mathcal{L}}f(s) = \hat{f}(s)$.
5. The properties of $\tilde{\mathcal{L}}$ are the same as the properties of the Laplace transform of tempered distributions cited in Section 1.1.5.

At the end we mention that H. Komatsu extended the theory of Laplace hyperfunctions to the hyperfunctions having values in a complex Banach space (cf. [36]) and applied it to find solutions to some partial differential equations using the theory of semigroups.

2. Mathematical models of some elastic and viscoelastic rods

In this Section we shall derive equations corresponding to lateral motion of elastic and viscoelastic rods with different boundary conditions, which will be treated in Section 3 or are treated in some of our papers listed in the References.

2.1. Elastic axially loaded rod. Consider a rod shown in Figure 1. We shall consider in-plane motion of the rod. Let $\bar{x} - B - \bar{y}$ be a rectangular Cartesian coordinate system with the origin fixed at the point B of the rod.

The rod is simply supported at end B and connected to the moving support at end C. At the end C the rod is loaded by an axial *force* $\mathbf{F}$ having fixed direction and

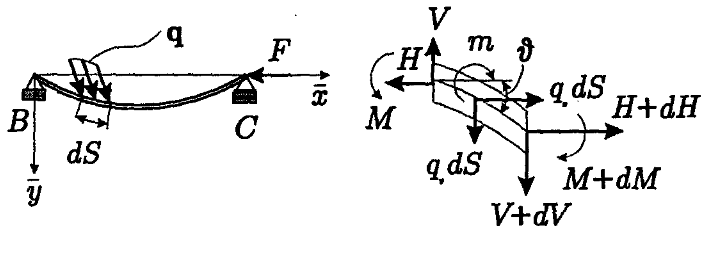

FIGURE 1. Coordinate system and load configuration

of intensity $F(t)$ that may be a function of time. Also the rod is loaded by *distributed forces* of intensity $\mathbf{q}(S,t)$ per unit length of the rod axis in the undeformed state. We further assume that the rod is initial straight and that its length is L. Let S be the arc length of the rod axis. We consider an element of the rod of length dS in the undeformed state. In the deformed state the length of this element is ds, so that

$$\varepsilon = \frac{ds - dS}{dS}, \tag{2.1}$$

is the *strain of the rod axis.* We shall use S as the independent space variable, so that $S \in [0, L]$. In an arbitrary section of the rod the *contact force* $\mathbf{Q}$ and the *contact couple* M represent the influence of the part $[0, S)$ on the part $(S, L]$ of the rod. Let $\mathbf{Q} = H\mathbf{e}_1 + V\mathbf{e}_2$ and $\mathbf{q}(S,t) = q_x\mathbf{e}_1 + q_y\mathbf{e}_2$ where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the $\bar{x}$ and $\bar{y}$ axis, respectively. Then, the *equilibrium equations*, written in the deformed configuration, for and element of the rod of the length dS in the undeformed state read

$$\frac{\partial H}{\partial S} = -q_x, \quad \frac{\partial V}{\partial S} = -q_y, \quad \frac{\partial M}{\partial S} = -V\frac{\partial x}{\partial S} + H\frac{\partial y}{\partial S} - m, \tag{2.2}$$

where m denotes the intensity of the distributed couples along the length of the rod. To equations (2.2) we adjoin the following *geometrical conditions*

$$\frac{\partial x}{\partial S} = (1+\varepsilon)\cos\vartheta, \qquad \frac{\partial y}{\partial S} = (1+\varepsilon)\sin\vartheta,$$

where ϑ is the angle between the tangent to the rod axis at an arbitrary cross section and $\bar{x}$ axis.

Next we formulate the constitutive equations. We neglect the influence of the shear stresses so that the cross section of the rod that is orthogonal to the rod axis in the undeformed state is orthogonal in the deformed state too (for more general rod theories that take into account the influence of the shear stresses see [2], for example). Then, the *strain measures* are $\partial\vartheta/\partial S$ and ε. Note that $\partial\vartheta/\partial S$ is *not* the curvature κ of the rod axis in the deformed state. Indeed, let s be the arc length of the rod axis in the deformed state. Then the curvature is $\kappa = \partial\vartheta/\partial s$, so that

$$\frac{\partial\vartheta}{\partial S} = \frac{\partial\vartheta}{\partial s}(1+\varepsilon) = \kappa(1+\varepsilon),$$

where we used (2.1). We treat materially linear rod so that the contact couple is proportional to $\partial\vartheta/\partial S$ and the strain of the rod axis ε is proportional to the component of the contact force in the direction normal to the cross section in the deformed state (i.e., in the direction of the tangent to the rod axis). Let $\mathbf{t} = \cos\vartheta \mathbf{e}_1 + \sin\vartheta \mathbf{e}_2$ be a unit tangent to the rod axis. Then the component of $\mathbf{Q}$ in the direction of $\mathbf{t}$ is $N = V\cos\vartheta + H\sin\vartheta$. With this, the constitutive equations that we use are

$$M = EI\frac{d\vartheta}{dS}; \quad \varepsilon = \frac{N}{EA} = \frac{V\cos\vartheta + H\sin\vartheta}{EA}. \tag{2.3}$$

In (2.3) E is the *modulus of elasticity* of the material of the rod, A is the cross-sectional area and $I = \int_A \eta^2 dA$ is the *second moment of inertia* of the cross section with respect to the principal axes of the cross section passing through the center of gravity. The constants EI and EA are called bending and extensional rigidity of the rod, respectively. We note that for the case of a rod with variable cross section both EI and EA become functions of the arc length S. The constitutive equations (2.3) were given by Pflüger [45]. Note however that $(2.3)_2$ does not have the important property that $N \to \infty$ as $\varepsilon \to -1$. Thus, $(2.3)_2$ is valid only for $\varepsilon > -1$. There are several generalizations of $(2.3)_2$ that satisfy the property $N \to \infty$ as $\varepsilon \to -1$. For example, in [37] the relation

$$N = EA\frac{\varepsilon^3}{1+\varepsilon},$$

was proposed, with $EA > 0$ being a constant. In [1] more complicated relation

$$N = \frac{EA}{1+\gamma}\Big(\varepsilon + 1 - \frac{1}{(\varepsilon+1)^\gamma}\Big), \tag{2.4}$$

with $\gamma > 0$ was proposed. For ε small, i.e., $|\varepsilon| \ll 1$, the normal force N obtained from (2.4) is of the form $N = EA\varepsilon + O(\varepsilon^2)$, that is (2.4) approximates the Hooke's law in the limit when $|\varepsilon| \to 0$. For further discussion on $(2.3)_2$ see [2] and [39]. We shall use $(2.3)_2$ but with the restriction

$$\varepsilon > -1. \tag{2.5}$$

Finally we define q_x, q_y and m. By using the *D'Alembert's principle* (active and inertial forces and couples are in equilibrium) we shall add to the active distributed forces and couples the inertial terms and obtain from the system (2.2)–(2.3) equations of motion of the rod. Thus, we assume that

$$q_x = -\rho\frac{\partial^2 x}{\partial t^2} + q_x^{\text{pres.}}, \quad q_x = -\rho\frac{\partial^2 y}{\partial t^2} + q_y^{\text{pres.}}, \quad m = -J\frac{\partial^2 \vartheta}{\partial t^2} + m^{\text{pres.}}, \tag{2.6}$$

where ρ is the mass density of the rod (mass of the rod per unit length of the rod axis in the undeformed state), J is the moment of inertia of the rod cross-section, $q_x^{\text{pres.}}, q_y^{\text{pres.}}$ are prescribed values of the distributed forces along the $\bar{x}$ and $\bar{y}$ axes respectively and $m^{\text{pres.}}$ is the value of the prescribed distributed couples.

With (2.6) we can write the complete system of equations describing in plane motion of an elastic rod with extensible axis

$$\frac{\partial H}{\partial S} = \rho\frac{\partial^2 x}{\partial t^2} - q_x^{\text{pres.}};$$

$$
\begin{aligned}
&\frac{\partial V}{\partial S} = -\rho\frac{\partial^2 y}{\partial t^2} + q_y^{\text{pres.}};\\
&\frac{\partial M}{\partial S} = -V\left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\cos\vartheta\\
&\qquad + H\left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\sin\vartheta + J\frac{\partial^2\vartheta}{\partial t^2} - m^{\text{pres.}};\\
&\frac{\partial x}{\partial S} = \left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\cos\vartheta;\\
&\frac{\partial y}{\partial S} = \left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\sin\vartheta;\\
&\frac{\partial\vartheta}{\partial S} = \frac{M}{EI},
\end{aligned} \tag{2.7}
$$

To the system (2.7) we must add the boundary conditions. For the rod shown in Figure 1 those conditions read

$$
\begin{aligned}
&H(L,t) = -F, \quad M(0,t) = 0, \quad M(L,t) = 0,\\
&x(0,t) = 0, \quad y(0,t) = 0, \quad y(L,t) = 0.
\end{aligned} \tag{2.8}
$$

We define as a *trivial* solution the solution of (2.7), (2.8) in which the rod axis remains straight. Suppose that $q_x^{\text{pres.}}(S,t) = q_y^{\text{pres.}}(S,t) = m^{\text{pres.}}(S,t) = 0$. It is easy to see that the trivial solution of (2.7),(2.8) is[1]

$$
\begin{aligned}
&H^0(S,t) = -F, \quad V^0(S,t) = 0, \quad M^0(S,t) = 0,\\
&x^0(S,t) = \left(1-\frac{F}{EA}\right)S, \quad y^0(S,t) = 0, \quad \vartheta^0(S,t) = 0.
\end{aligned}
$$

We study the disturbed motion of the trivial state. Thus, let us denote by $\Delta H(S,t)$, $\dots, \Delta V(S,t), \dots, \Delta\vartheta(S,t)$ the perturbations of the variables $H^0(S,t), \dots, \Delta\vartheta^0(S,t)$ describing the trivial configuration. Then for the disturbed state we have

$$
\begin{aligned}
H(S,t) &= H^0(S,t) + \Delta H(S,t),\\
V(S,t) &= V^0(S,t) + \Delta V(S,t),\\
&= \dots\dots\dots\dots\dots .\\
\vartheta(S,t) &= \vartheta^0(S,t) + \Delta\vartheta(S,t).
\end{aligned} \tag{2.9}
$$

By substituting (2.9) into (2.7) and neglecting the higher order terms in $\Delta H(S,t)$, $\dots, \Delta\vartheta(S,t)$ we obtain

$$
\begin{aligned}
&\frac{\partial \Delta H}{\partial S} = \rho\frac{\partial^2 \Delta x}{\partial t^2},\\
&\frac{\partial \Delta V}{\partial S} = -\rho\frac{\partial^2 \Delta y}{\partial t^2},\\
&\frac{\partial \Delta M}{\partial S} = -\Delta V\left(1-\frac{F}{EA}\right) - F\left(1-\frac{F}{EA}\right)\Delta\vartheta + J\frac{\partial^2\Delta\vartheta}{\partial t^2},
\end{aligned}
$$

[1]Note that (2.5) requires that $F < EA$

$$
\begin{aligned}
&\frac{\partial \Delta x}{\partial S} = \Big(1 - \frac{1}{EA}\Big),\\
&\frac{\partial \Delta y}{\partial S} = \Big(1 - \frac{F}{EA}\Big)\Delta\vartheta,\\
&\frac{\partial \Delta\vartheta}{\partial S} = \frac{\Delta M}{EI}.
\end{aligned} \tag{2.10}
$$

The system (2.10) could be simplified if we assume that we can differentiate the functions involved. Thus, by differentiating $(2.10)_2$ with respect to S and by using $(2.10)_2$ and $(2.10)_{5,6}$ we obtain

$$
EI\frac{\partial^4 \Delta y}{\partial S^4} + F\frac{\partial^2 y}{\partial S^2} - J\frac{1}{(1-F/EA)}\frac{\partial^4 \Delta y}{\partial S^2 \partial t^2} + \rho\left(1 - \frac{F}{EA}\right)\frac{\partial^2 \Delta y}{\partial t^2} = 0, \tag{2.11}
$$

subject to

$$
\Delta y(0,t) = 0, \quad \Delta y(L,t) = 0, \quad \frac{\partial^2 \Delta y}{\partial S^2}(0,t) = 0, \quad \frac{\partial^2 \Delta y}{\partial S^2}(L,t) = 0. \tag{2.12}
$$

We write next the system (2.11), (2.12) in the dimensionless form. By introducing the following quantities

$$
\begin{gathered}
\xi = \frac{S}{L}, \quad u = \frac{\Delta y}{L}, \quad i^2 = \frac{I}{A}, \quad \mu = \frac{L}{i},\\
\lambda = \frac{FL^2}{EI}, \quad \tau = t\left(\frac{EI}{\rho L^4}\right)^{1/2}, \quad \alpha = \frac{J}{\rho L^2},
\end{gathered} \tag{2.13}
$$

the system (2.11), (2.12) becomes

$$
\begin{gathered}
\frac{\partial^4 u}{\partial \xi^4} + \lambda\frac{\partial^2 u}{\partial \xi^2} - \frac{\alpha}{(1-\lambda/\mu^2)}\frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \Big(1 - \frac{\lambda}{\mu^2}\Big)\frac{\partial^2 u}{\partial \tau^2} = 0,\\
\tau > 0, \quad 0 < \xi < 1,
\end{gathered} \tag{2.14}
$$

and

$$
u(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0. \tag{2.15}
$$

Equation (2.14) reduces to several special cases well known in mathematical physics. For example, suppose that we neglect compressibility of the rod axis. Then $EA \to \infty$ and $i^2 \to 0$ (see $(2.13)_3$) so that in this case the parameter μ, called *slenderness ratio*, tends to infinity i.e., $\mu \to \infty$. By using this, from (2.14) we obtain

$$
\frac{\partial^4 u}{\partial \xi^4} + \lambda\frac{\partial^2 u}{\partial \xi^2} - \alpha\frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.16}
$$

Equation (2.16) is valid for long and thin rods. Suppose further that the rotary inertia term is small i.e., $J \to 0$. In this case $\alpha \to 0$ and the equation (2.16) becomes

$$
\frac{\partial^4 u}{\partial \xi^4} + \lambda\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.17}
$$

Note that the parameter λ could be constant or a function of time. The most interesting cases are

$$\lambda = A + B\delta(\tau - \tau_0), \quad \lambda = A + B\sin\Omega\tau, \tag{2.18}$$

where A, B, τ_0 and Ω are constants and $\delta(\tau)$ is Dirac distribution.

Finally, for the case when the axial force is equal to zero, i.e., $\lambda = 0$ equation (2.17) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.19}$$

Equation (2.19) is a well known equation of lateral vibrations of an elastic rod, without the axial force.

We mention here a model, similar to (2.16) with $\lambda = 0$ recently proposed in [49] and [50]. It reads (in our notation)

$$\frac{\partial^4 u}{\partial \xi^4} - \alpha\frac{\partial^3 u}{\partial \xi^2 \partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = 0; \quad \tau > 0, \quad 0 < \xi < 1.$$

In physical terms, the model (2.19) has the damping proportional to the rate of change of the curvature of the rod. No derivation (or further physical explanation) of the term $-\alpha\frac{\partial^3 u}{\partial \xi^2 \partial \tau}$ is given in [49] and [50]. However it is stated that this new model has good mathematical properties. Some of those properties are examined in [28].

To each of the equations (2.16), (2.17), (2.19) the *boundary conditions* such as (2.15) should be adjoined. For the sake of completeness we list here other, frequently used, boundary conditions:

- Left end clamped, right end free

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau) = 0.$$

- Left and right ends simply supported

$$u(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0.$$

- Left end clamped, right end simply supported

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0.$$

- Left end clamped, right end clamped and free for axial movement

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(1,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(1,\tau) = 0.$$

- Left end clamped, right end loaded by a follower force

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau) = 0.$$

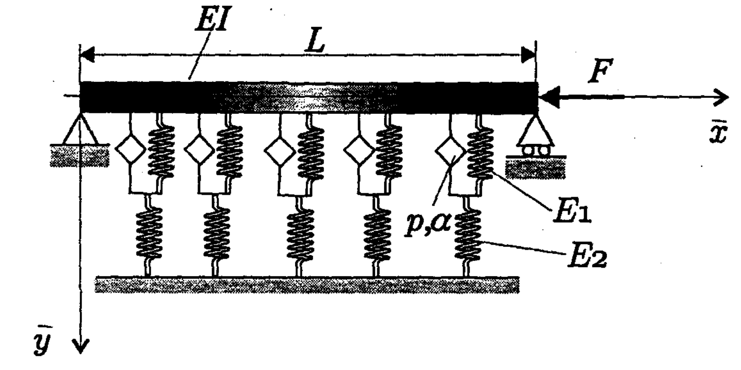

FIGURE 2. Elastic rod on a viscoleastic foundation

- Left end clamped, right end welded to a movable rigid plate (free for a transversal movement)

$$u(0,\tau)=0,\quad \frac{\partial u}{\partial \xi}(0,\tau)=0,\quad \frac{\partial u}{\partial \xi}(1,\tau)=0,\quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau)=0.$$

We note that (2.17) for the rod with λ = const and with different boundary conditions was analyzed in many publications (see [2] for references). Equation (2.16) with the boundary conditions corresponding to a simply supported rod and with $\lambda(t)$ of the form (2.18) is treated, recently, in [63].

2.2. Elastic axially loaded rod on elastic and viscoelastic foundation. We consider an elastic axially compressed rod on a special type of foundation, shown in Figure 2.

The foundation is such that it produces a distributed force f in the vertical direction, along the rod so that $q_y^{\text{pres.}} = f(S,t)$. The function $f(S,t)$ is determined by the constitutive equation of the foundation. For example, if

$$f=-cy, \tag{2.20}$$

then foundation is called *Winkler foundation*. By substituting (2.20) into (2.7) and performing the same steps as before, we obtain instead of (2.14) and (2.15) the following equation

$$\frac{\partial^4 u}{\partial \xi^4}+\lambda\frac{\partial^2 u}{\partial \xi^2}-\frac{\alpha}{(1-\lambda/\mu^2)}\frac{\partial^4 u}{\partial \xi^2\partial \tau^2}+\Big(1-\frac{\lambda}{\mu^2}\Big)\frac{\partial^2 u}{\partial \tau^2}+\beta u=0, \tag{2.21}$$
$$\tau>0,\quad 0<\xi<1,$$

subject to

$$u(0,\tau)=0,\quad \frac{\partial^2 u}{\partial \xi^2}(0,\tau)=0,\quad u(1,\tau)=0,\quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau)=0. \tag{2.22}$$

In (2.21) the constant β is given as $\beta = cL^3/EI$. In Section 4 we shall analyze the system (2.21), (2.22) for a special case when the rod is thin and long. In this case

$\alpha = J/\rho L^2 \to 0$ (see $(2.13)_7$). Also since the second moment of inertia I and the cross sectional area are connected as $I = cA^m$, where $c > 0$ and $m > 1$ we have (see $(2.13)_2$) that the radius of gyration becomes $i^2 = cA$. Thus for thin and long rods $i^2 \to 0$ and $\mu = \frac{L}{i} \to \infty$ so that (2.21) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} + \beta u = 0; \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.23}$$

Often foundation is made of viscoelastic material. In this case the functional relation between f and y is more complicated than (2.20). For example in rail track problems (see [23]) the following type of viscoelastic foundation is used

$$f + \tau_Q f^{(\alpha)} = E(y + \tau_y y^{(\alpha)}), \tag{2.24}$$

where E_p, τ_Q, τ_y and $0 < \alpha < 1$ are constants. In (2.24) we used $(\cdot)^{(\alpha)}$ to denote the α-th derivative of a function $(\cdot)$ taken in Riemann–Liouville form as (see [42], [51] and Definition 1.1 in Section 2)

$$\frac{d^\alpha}{dt^\alpha} g(t) = g^{(\alpha)} \equiv \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g(\xi)\, d\xi}{(t-\xi)^\alpha} = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g(t-\xi)\, d\xi}{\xi^\alpha}.$$

The dimension of the constants τ_y and τ_Q is $[\text{time}]^\alpha$. The constants E_p, τ_Q and τ_y in (2.24) are called of the pad and the *relaxation times*, respectively. We assume that, as a *consequence of the second law of thermodynamics*, the following inequality, is satisfied (see [11] and [3])[2]

$$E > 0, \quad \tau_Q > 0, \quad \tau_y > \tau_Q. \tag{2.25}$$

Now, by introducing new dimensionless function $F = f/EL$ the system (2.23), (2.24) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} - \frac{\alpha}{(1-\lambda/\mu^2)} \frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \left(1 - \frac{\lambda}{\mu^2}\right) \frac{\partial^2 u}{\partial \tau^2} + F = 0, \tag{2.26}$$
$$\tau > 0, \quad 0 < \xi < 1,$$

where

$$F + aF^{(\alpha)} = u + bu^{(\alpha)}, \tag{2.27}$$

subject to

$$u(0,t) = 0; \quad u(1,t) = 0; \quad \frac{\partial^2 u}{\partial \xi^2}(0,t) = 0; \quad \frac{\partial^2 u}{\partial \xi^2}(1,t) = 0,$$

and with the restriction $b > a > 0$, following from (2.25). The system (2.26),(2.27) in the special case $\alpha = 0$ was analyzed in [6].

Another important case is the case of an elastic rod on viscoelastic foundation loaded by a concentrated force at the free end (see Figure 3). The follower type concentrated force is a force having (in our case) constant intensity and the direction

[2] If one uses a rheological model shown under the rod in Figure 2, then the constants in (2.24) are given as $E = E_1E_2/(E_1+E_2)$, $\tau_Q = p/(E_1+E_2)$, $\tau_y/E = pE_2/(E_1+E_2)$ (see [54], [44]). Here E_1, E_2 are spring constants and p is the characteristic of a "springpot" an element whose stress-strin law is given as $\sigma = p\varepsilon^{(\alpha)}$.

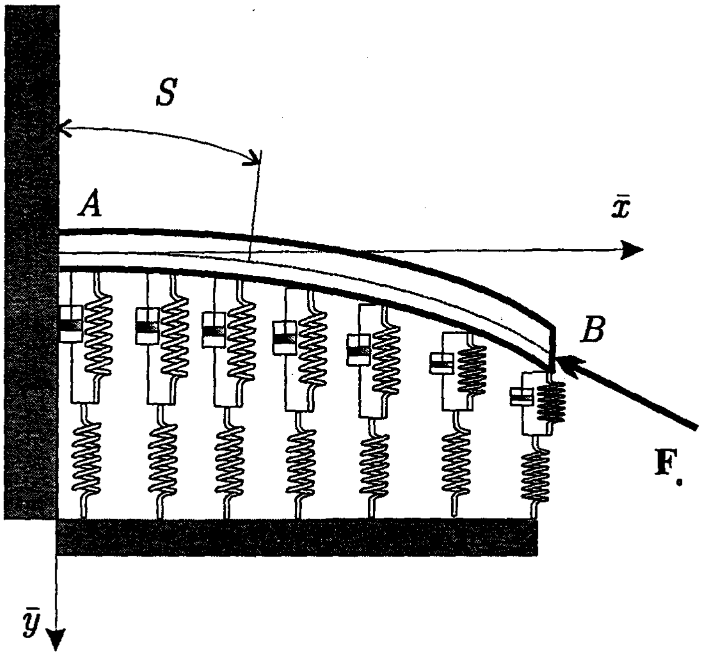

FIGURE 3. Elastic rod on viscoelastic foundation with the follower force

coinciding with the tangent to the rod axis at the point of application of force. For the case of an elastic rod with follower force and without foundation (the so called *Beck's rod*) there exists lot of results, some of them presented in [2] and [17].

The differential equations of the problem, for the rod shown in Figure 3, may be obtained by the same procedure as those used deriving (2.26) and are (see [8])

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} + \beta f = 0, \quad \tau > 0; \quad 0 < \xi < 1. \tag{2.28}$$

and

$$f + a f^{(\alpha)} = u + b u^{(\alpha)}, \tag{2.29}$$

with $0 < \alpha < 1$. The boundary conditions are

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau) = 0, \quad \tau > 0. \tag{2.30}$$

The problems of existence and stability of the solution to (2.28)–(2.30) were treated in [8]. The conclusion about stability of the system (2.28)–(2.30) i.e., the condition that guarantees that the solution $u(\xi,\tau)$ is bounded when $\tau \to \infty$ is very interesting. Namely, it is shown that the critical value λ_{cr} of the parameter λ (the rod is stable if $\lambda \leqslant \lambda_{cr}$) does not depend on parameter β. Thus, the viscoelastic foundation does not increase the stability bound! This is known to hold for elastic column with follower force on elastic foundation and constitutes the so called Herman–Smith paradox. In [8] it was shown that the same holds when elastic foundation is replaced with the viscoelastic foundation of fractional derivative type described by (2.29).

Finally we mention the problem of determining stability boundary of an elastic rod with rotary inertia positioned on viscoelastic foundation. In this case the problem is described by the system of equations (2.26), (2.27) with $\alpha \neq 0$. The stability analysis and properties of the solution are examined in [9].

2.3. Viscoelastic axially loaded rod. We consider a special type of viscoelastic rod made of material described by fractional derivatives of a strain. Suppose that the rod is made of a material whose stress-strain relation is of the form (2.24). This model is known as the *generalized Zener model* (see [11], [3], [54])

$$\sigma(t) + \tau_\sigma D_t^\beta \sigma(t) = E_0 \left[\varepsilon(t) + \tau_\varepsilon D_t^\alpha \varepsilon(t)\right], \quad t \geqslant 0, \tag{2.31}$$

where $\tau_\sigma, \beta, E_0, \tau_\varepsilon$ and α are real constants. We note that (2.31) is a special case of a stress strain relation treated in [5], [7]. By using the plane cross-section hypothesis [2] we conclude that the strain in an element of the cross-section that is on the distance z from the neutral plane is $\varepsilon_z = z/r = (\partial\vartheta/\partial S)\, z$. Thus, by multiplying (2.31) by z and integrating over the cross-section of the rod A, we obtain

$$M(t) + \tau_\sigma D_t^\beta M(t) = E_0 I \left[\frac{\partial\vartheta}{\partial S} + \tau_\varepsilon D_t^\alpha \frac{\partial\vartheta}{\partial S}\right], \tag{2.32}$$

where I is the second moment of inertia, i.e., $I = \int_A z^2 dA$. For the linearized version of the system (2.32) we can substitute $\partial\vartheta/\partial S$ with $\partial^2 y/\partial S^2$ so that (2.32) becomes

$$M(t) + \tau_\sigma D_t^\beta M(t) = E_0 I \left[\frac{\partial^2 y}{\partial S^2} + \tau_\varepsilon D_t^\alpha \frac{\partial^2 y}{\partial S^2}\right]. \tag{2.33}$$

Equation (2.33) with $\tau_\sigma = 0$ was used in [10] and in its general form (2.33) in [38] and [4].

By substituting (2.33) in (2.10) we obtain

$$\begin{gathered} \frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \\ \frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_\tau^\alpha \frac{\partial^2 u}{\partial \xi^2} - m - \mu D_\tau^\alpha m = 0, \quad \tau > 0, \quad 0 < \xi < 1, \end{gathered} \tag{2.34}$$

subject to

$$m(0,\tau) = 0, \quad m(1,\tau) = 0, \quad u(0,\tau) = 0, \quad u(1,\tau) = 0.$$

In (2.34) we used the following dimensionless quantities

$$u = \frac{\Delta y}{L}, \quad m = \frac{\Delta M L}{E_0 I}, \quad \tau = t\sqrt{\frac{E_0 I}{\rho L^4}}, \quad \lambda = \frac{F L^2}{E_0 I},$$

$$\xi = \frac{S}{L}, \quad \mu = \tau_\sigma \left(\frac{IE_0}{\rho L^4}\right)^{\alpha/2}, \quad \mu_1 = \tau_\varepsilon \left(\frac{IE_0}{\rho L^4}\right)^{\alpha/2}. \tag{2.35}$$

The second law of thermodynamics requires that $\mu_1 > \mu$.

An important generalization of (2.31) represents the so called *five-parameter model of viscoelastic body* studied in [46] and [48]. Suppose we use constitutive equation connecting the stress σ and strain ε in the form

$$\sigma_z(t) + \tau_\sigma D_t^\alpha \sigma_z(t) = E_0 \left[\varepsilon + \tau_\varepsilon D_t^\alpha \varepsilon + \tau_\gamma D^\gamma \varepsilon\right].$$

The plane cross-section hypothesis, together with the linearization of the expression for curvature, leads to

$$M(t) + \tau_\sigma D_t^\alpha M(t) = E_0 I \left[\frac{\partial^2 y}{\partial S^2} + \tau_\varepsilon D_t^\alpha \frac{\partial^2 y}{\partial S^2} + (\tau_\varepsilon)^{\gamma/\alpha} D_t^\gamma \frac{\partial^2 y}{\partial S^2}\right]. \tag{2.36}$$

The second law of thermodynamics in the case (2.36) requires that (see [11], [3], [46] and [7])

$$\gamma > \alpha; \quad \tau_\varepsilon > \tau_\sigma > 0. \tag{2.37}$$

Introducing a dimensionless quantities (2.35) and

$$\mu_2 = (\mu_1)^{\gamma/\alpha} = (\tau_\varepsilon)^{\gamma/\alpha} \left(\frac{IE_0}{\rho L^4}\right)^{\gamma/2},$$

we obtain, instead of the system (2.34), the following system of partial differential equations of integer and fractional order

$$\frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} = 0;$$

$$\frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_\tau^\alpha \frac{\partial^2 u}{\partial \xi^2} + \mu_2 D_\tau^\gamma \frac{\partial^2 u}{\partial \xi^2} - m - \mu D_\tau^\alpha m = 0; \quad \tau > 0, \quad 0 < \xi < 1,$$

with the boundary conditions $m(0,\tau) = 0$, $m(1,\tau) = 0$, $u(0,\tau) = 0$, $u(1,\tau) = 0$. The thermodynamic restrictions (2.37) become

$$\mu_1 > \mu; \quad \gamma \geqslant \alpha.$$

We note that in all cases formulated up to now, the dimensionless axial force λ can have both constant and time dependent part. For the case when an axial load is constant equal to B and additional load D is applied suddenly, at the time instant τ_0, we have $\lambda = B + C\theta(\tau - \tau_0)$. Also if we have constant axial force and at the time instant τ_0 an *impulsive force* is applied the axial force λ in this case is given as $\lambda = B + D\delta(\tau - \tau_0)$, where D is a constant.

Finally we present one more generalization of (2.31) and the corresponding constitutive equation for moments. Suppose that the stress strain relation is given in the form of so called *distributed derivative model* (see [5])

$$\int_0^1 \phi_\sigma(\gamma) \sigma^{(\gamma)} d\gamma = \int_0^1 \phi_\varepsilon(\gamma) \varepsilon^{(\gamma)} d\gamma,$$

where $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$ are known functions that are determined from experiments (constitutive functions). Then, the plane cross-section hypothesis and the procedure used in deriving (2.32) and (2.36) leads to the constitutive equation for the bending moment in the form

$$\int_0^1 \phi_\sigma(\gamma) M^{(\gamma)}(t)\,d\gamma = I\int_0^1 \phi_\varepsilon(\gamma)\Big(\frac{\partial^2 y}{\partial \xi^2}(t)\Big)^{(\gamma)} d\gamma,$$

where I is, again, the second the moment of inertia of the rod cross-section A. The restrictions that the functions $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$ must satisfy in order that the second law of thermodynamics is not violated, are derived in [7].

3. Generalized solutions to some partial differential equations

3.1. Equation in a space of generalized functions which corresponds to a partial differential equation. We denote by Ω an open set belonging to $\mathbb{R}^2$. Let

$$P(\partial)u(x,t) = f(x,t), \quad (x,t) \in \Omega, \quad f \in C(\Omega), \tag{3.1}$$

be a linear partial differential equation with coefficients belonging to $C^\infty(\Omega)$. To equation (3.1), by the property 4 of the derivative in $\mathcal{D}'(\Omega)$ (cf. Section 1, Subsection 1.1.2), it corresponds in $\mathcal{D}'(\Omega)$ the equation

$$P(D)[u(x,t)] = [f(x,t)]. \tag{3.2}$$

If there exists a solution $u(\xi,t)$ to (3.1) such that $u \in C^p(\Omega)$, where $p = (p_1, p_2)$ is the degree of the equation (3.1), this solution is called the *classical solution.* A classical solution defines a distribution (regular) which is a solution to (3.2). If the solution to (3.2) is not defined by a function from $C^p(\Omega)$ it is called *generalized solution* to (3.1). Conversely, if a solution w to (3.2) is the regular distribution $[u(x,t)]$, where $u(x,t) \in C^p(\Omega)$, then $u(x,t)$ is a solution to (3.1). In this paper we use the so defined notations of a generalized and classical solution to (3.1).

Which generalized solution can be used depends on every concrete case. We are here interested in those mathematical models which are coming from mechanics. We are also going to point at the possibility to use the classical results in construction of a generalized solution.

We will not give a general theory, but illustrate it by some special cases in which generalized functions can improve the classical results or methods. However, there is a general procedure which will be conducted in solving equations to obtain classical and generalized solutions. It is the following:

First we find the equation (3.2) in $\mathcal{D}'(\Omega)$ which corresponds to the given equation (3.1). Then we apply certain methods to solve such equation (3.2). Usually these methods in $\mathcal{D}'(\Omega)$ are less restrictive than the methods in spaces of numerical functions.

If we find a solution u to (3.2), then it can happen that it is defined by a function $u(x,t)$, $u = [u(x,t)]$. This function $u(x,t)$ can belong to $C^p(\Omega)$ and consequently be a classical solution to (3.1). Also it can belong to $C^q(\Omega)$, $0 \leqslant q < p$, or to $\mathrm{L}_{\mathrm{loc}}(\Omega)$ and then u represents a generalized solution to (3.1). But in this case we can see why $u(x,t)$ can not be a classical solution to (3.1). Sometimes having generalized solutions to (3.1) we can construct the classical ones as well. For example, let $P(D)$ in (3.2) be with constant coefficients, $\Omega = \mathbb{R}^2$ and $f = 0$. Suppose that u is a solution to such homogeneous equation (3.2). Then u is a limit in $\mathcal{D}'(\mathbb{R}^2)$ of a sequence $\{u_j\}_{j\in\mathbb{N}}$ of classical solutions to (3.1) with $f = 0$. Let $\{\delta_j\}$ be a δ-sequence, ($\delta_j \in \mathcal{D}(\mathbb{R}^2)$ and δ_j converges to δ in $\mathcal{D}'(\mathbb{R}^2)$). Now, the sequence $\{u_j\}_{j\in\mathbb{N}}$ can be $\{u * \delta_j\}_{j\in\mathbb{N}}$ (cf. [64, p. 243]).

3.2. Construction of solutions by using fundamental solutions. We have seen in Section 3.1 that to a linear partial differential equation

$$P(\partial)u(x,t) = f(x,t), \ (x,t) \subset \mathbb{R}^2, \quad f \in \mathcal{C}(\mathbb{R}^2), \tag{3.3}$$

with coefficients belonging to $\mathcal{C}^\infty(\mathbb{R}^2)$, it corresponds in $\mathcal{D}'(\mathbb{R}^2)$ the equation

$$P(D)[u(x,t)] = [f(x,t)].$$

A distribution $E \in \mathcal{D}'(\mathbb{R}^2)$ is called a *fundamental solution* of the operator $P(\partial)$, by definition, if it satisfies the equation $P(\partial)E = \delta$. If f in (3.2) is such that the convolution $E * [f(x,t)]$ exists and the operator P has constant coefficients, then $W = E * [f(x,t)]$ is a solution to (3.2). In that case W is a generalized solution to (3.3) belonging to $\mathcal{D}'(\mathbb{R}^2)$. In the mathematical literature one can find fundamental solutions for different differential operators. (cf. for example [43]).

As an illustration we consider the equation

$$\frac{\partial^4}{\partial\xi^4}u(t,\xi) + \lambda\frac{\partial^2}{\partial\xi^2}u(t,\xi) + \frac{\partial^2}{\partial t^2}u(t,\xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}, \tag{3.4}$$

which appears in mathematical models for many different phenomena subject to different boundary and initial conditions (cf. Section 2 (2.1.18)).

It is well known that a solution to (3.4) is $u(t,\xi) = Y(\xi)T(t)$, where Y and T have the analytical from:

$$Y(\xi) = C_1\cosh r_1\xi + C_2\sinh r_1\xi + C_3\cos r_2\xi + C_4\sin r_2\xi \tag{3.5}$$

$$T(t) = C_5\cos\omega t + C_6\sin\omega t, \quad \omega^2 \in \mathbb{R}_+, \tag{3.6}$$

where

$$r_1 = \sqrt{\frac{\sqrt{\lambda^2+4\omega^2}-\lambda}{2}}, \quad r_2 = \sqrt{\frac{\sqrt{\lambda^2+4\omega^2}+\lambda}{2}},$$

(cf. [2]). For ω any complex number (cf. [2], [55]).

To find generalized solutions to (3.4) belonging to $\mathcal{D}'(\mathbb{R}^2)$ we have first to find the equation in $\mathcal{D}'(\mathbb{R}^2)$ which corresponds to (3.4). In fact we seek for the corresponding equation in $\mathcal{D}'(\overline{\mathbb{R}}_+ \times \mathbb{R})$, because this space is more suitable to find a fundamental solution.

Suppose that there exists $u(t,\xi) \in C_t^{(2)}(\mathbb{R}_+,\mathbb{R})$ such that:

1. $u(t,\xi)$ is a solution to (3.4);
2. There exist $\lim_{t\to 0^+} u(t,\xi) = u_1(\xi) \in \mathcal{C}(\mathbb{R})$, $\quad \lim_{t\to 0^+} u_t^{(1)}(t,\xi) = u_2(\xi) \in \mathcal{C}(\mathbb{R})$.

Let $[\theta u]$ denote the regular distribution defined by the function $\theta(t)u(t,\xi)$, where θ is the *Heaviside* function ($\theta(t) = 0$, $t < 0$; $\theta(t) = 1$, $t \geqslant 0$). By the property of derivatives in $\mathcal{D}'$ (cf. 8 in Section 1.1.2), to (3.4) there corresponds in $\mathcal{D}'(\overline{\mathbb{R}}_+ \times \mathbb{R}) \subset \mathcal{D}'(\mathbb{R}^2)$ the following equation:

$$(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t),$$

or

$$(D_t^2 + P(D_\xi))\tilde{u} = f, \tag{3.7}$$

where $P(D_\xi) = D_\xi^4 + \lambda D_\xi^2$, $f = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t)$ and $\tilde{u} \in \mathcal{D}'(\mathbb{R}^2)$. We seek for solutions to (3.7) with the property $\operatorname{supp}\tilde{u} \subset \overline{\mathbb{R}}_+ \times \mathbb{R}$.

By the lemma in [43, p. 30], the operator $D_t^2 + P(D_\xi)$ is *quasihyperbolic* with respect to t if and only if the following condition is satisfied:

$$\exists c > 0, d \in \mathbb{R}, \forall \xi \in \mathbb{R} : \operatorname{Re} P(i\xi) - c(\operatorname{Im} P(i\xi))^2 \geqslant d.$$

In our case $P(i\xi) = \xi^4 - \lambda\xi^2$. For every $\xi \in \mathbb{R}$, $\xi^4 - \lambda\xi^2 \geqslant -\lambda^2/4$. Consequently the operator $D_t^2 + P(D_\xi)$ is quasihyperbolic.

By Proposition 5 in [43, p. 32] the unique fundamental solution E of $D_t^2 + P(D_\xi)$ with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and $E \in e^{\alpha t}\mathcal{S}'$ for an $\alpha \in \mathbb{R}$ is given by

$$E(t,\xi) = H(t)\mathcal{F}_x^{-1}\left(\frac{\sin\left(t\sqrt{P(2\pi i x)}\right)}{\sqrt{P(2\pi i x)}}\right)(t,\xi),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Using Bochner's formula (cf. [56, VII, 7, 22], or [43, p. 19])

$$E(t,|\xi|) = H(t)2\pi|\xi|^{1/2}\int_0^\infty \frac{\sin\left(t\sqrt{P(2\pi i x)}\right)}{\sqrt{P(2\pi i x)}} x^{1/2} J_{-1/2}(2\pi|\xi|x)\,dx,$$

where J_ν is the Bessel function. Since $J_{-1/2}(2\pi|\xi|x) = \dfrac{1}{\pi}\dfrac{\cos 2\pi|\xi|x}{\sqrt{|\xi|x}}$, we have

$$E(t,\xi) = 2H(t)\int_0^\infty \frac{\sin\left(t\sqrt{P(2\pi i x)}\right)}{\sqrt{P(2\pi i x)}}\frac{\cos(2\pi|\xi|x)}{\sqrt{x}}\,dx. \tag{3.8}$$

Suppose now that $u_1(\xi)$ and $u_2(\xi)$ in (3.7) have the properties that:

$$([u_2(\xi)] \otimes \delta(t)) * [E(t,\xi)], \quad ([u_1(\xi)] \otimes \delta^{(1)}(t)) * [E(t,\xi)] \tag{3.9}$$

exist, then there is a solution $\tilde{u}$ to (3.7) in $\mathcal{D}'(\mathbb{R}^2)$ with support in $\bar{\mathbb{R}}_+ \times \mathbb{R}$

$$\begin{aligned}\tilde{u} &= (([u_1(\xi)] \otimes \delta^{(1)}(t))) + ([u_2(\xi)] \otimes \delta(t))) * [E(t,\xi)] \\ &= [u_2(\xi)] * [E(t,\xi)] + [u_1(\xi)] * D_t[E(t,\xi)].\end{aligned}$$

This solution is unique in the vector space $\mathcal{G} \subset \mathcal{D}'(\mathbb{R}^2)$. $\mathcal{G}$ consists of all $q \in \mathcal{D}'(\mathbb{R}^2)$ for which there exists $E * q$ (cf. [67, Chapter III, §11.3]). We proved the following:

Theorem 3.1. [61] *Let E be given by (3.8) and let $\mathcal{G}$ be the vector space belonging to $\mathcal{D}'(\mathbb{R}^2)$ such that for every $g \in \mathcal{G}$ there exists $[E] * g$. Suppose that $u_1(\xi)$ and $u_2(\xi)$ are in $C(\mathbb{R})$ such that the convolutions (3.9) exist. Then*

$$\tilde{u} = [u_2(\xi)] * [E(t,\xi)] + [u_1(\xi)] * D_t[E(t,\xi)]$$

is a solution to $(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = 0$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R})$. But it is also the unique solution in the space $\mathcal{G} \subset D'(\mathbb{R}^2)$ satisfying the initial condition in t in the sense that

$$(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = [u_2(\xi)] \otimes \delta(t) + [u_1(\xi)] \otimes \delta^{(1)}(t).$$

Remarks. 1. If $u_1(\xi)$ and $u_2(\xi)$ also belong to $C^4(\mathbb{R})$, then by the property of convolution (cf. Section 1 Subsection 1.2, property 9)

$$D_\xi^i\tilde{u} = [u_2^{(i)}(\xi)] * [E(t,\xi)] + [u_1^{(i)}(\xi)] * D_t[E(t,\xi)], \quad i = 1, \dots, 4.$$

2. If we have two solutions $u_1(t,\xi)$ and $u_2(t,\xi)$ to (3.4) with some initial condition

$$u_1(0,\xi) = u_2(0,\xi) \text{ and } \frac{d}{dt}u_1(t,\xi)|_{t=0} = \frac{d}{dt}u_2(t,\xi)|_{t=0}, \quad \xi \in \mathbb{R},$$

then $[u_2(t,\xi)] = [u_1(t,\xi)] + h$, where $h = 0$ or $h \notin \mathcal{G}$. Let us prove it. The function $U(t,\xi) = u_2(t,\xi) - u_1(t,\xi)$ satisfies (3.4) with initial condition $U_t^{(i)}(t,\xi)|_{t=0} = 0$, $i = 0, 1$, $\xi \in \mathbb{R}$, consequently the regular distribution $[U(t,\xi)] \in \mathcal{D}'(\mathbb{R}^2)$ satisfies (3.7) with $f = 0$. Then $[U(t,\xi)] = h$, where $h = 0$ or $h \notin \mathcal{G}$. Hence $[U(t,\xi)] = [u_2(t,\xi)] - [u_1(t,\xi)] = h$.

3. The well-known solution to (3.4) $u(t,\xi) = Y(\xi)T(t)$, where Y and T are given by (3.5) and (3.6), has not the convolution with $E(t,\xi)$ in the sense of distributions, i.e., $[u(t,\xi)] * [E(t,\xi)]$ does not exist. If it were true that $[u(t,\xi)] * [E(t,\xi)]$ exists, then by 3.4 and the property of convolution:

$$\begin{aligned}[u(t,\xi)] &= [u(t,\xi)] * \delta(t,\xi) = [u(t,\xi)] * \big(D_t^2 + P(D_\xi)\big)[E(t,\xi)] \\ &= \big((D_t^2 + P(D_\xi))[u(t,\xi)]\big) * [E(t,\xi)] \\ &= \Big[\Big(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial \xi^4} + \frac{\partial^2}{\partial \xi^2}\Big)u(t,\xi)\Big] * [E(t,\xi)] = 0.\end{aligned}$$

Thus $u(t,\xi) = 0$, $t > 0$, $\xi \in \mathbb{R}$.

4. If equation (3.7) with $f = 0$ has a solution belonging to $\mathcal{D}'(\mathbb{R}^2)$, it does not belong to $\mathcal{G}$.

Proof. A solution to (3.4) in $\mathcal{D}'(\mathbb{R}^2)$ is $u(t,\xi) \equiv 0$, $(t,\xi) \in \mathbb{R}^2$. By 2 if there is a solution to (3.4) belonging to $\mathcal{D}'(\mathbb{R}^2)$ which is not identical zero, then it does not belong to $\mathcal{G}$ and the proof is complete. □

The solution $u(t,\xi) = Y(\xi)T(t)$, where Y and T have been given by (3.5) and (3.6) respectively, is in fact a solution to

$$(Y^{(4)}(\xi) + \lambda Y^{(2)}(\xi) + \omega^2 Y(\xi))T(t) + (T^{(2)}(t) - \omega^2 T(t))Y(\xi) = 0, \quad t > 0, \ \xi \in \mathbb{R},$$

for $\omega^2 \in \mathbb{R} \setminus \{0\}$. This equation can be written in the form

$$\Big(P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2\Big)Y(\xi)T(t) = 0,$$

where $P\Big(\dfrac{d}{d\xi}\Big) = \dfrac{d^4}{d\xi^4} + \lambda\dfrac{d^2}{d\xi^2} + \omega^2$. Let us suppose that $\omega^2 > 0$. Since

$$P(i\xi) = \xi^4 - \lambda\xi^2 + \omega^2 > 0, \quad \xi \in \mathbb{R}, \quad \omega^2 - \lambda^2/4 > 0,$$

by Proposition 6 in [43] there is the unique fundamental solution $E_\omega(t,\xi)$ of

$$P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2$$

with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and belonging to $e^{\alpha t}S'$ for an $\alpha \in \mathbb{R}$. It has the following representation

$$E_\omega(t,\xi) = E(t,\xi) - \omega H(t) \int_0^t \frac{\tau}{\sqrt{t^2-\tau^2}} J_1\big(\omega\sqrt{t^2-\tau^2}\big) E(\tau,\xi)\, d\tau, \tag{3.10}$$

where $E(t,\xi)$ is given by (3.8).

Theorem 3.2. *If in the Theorem* 1.1 *instead of* $E(t,\xi)$ *we take* $E_\omega(t,\xi)$, *given by* (3.10), *then we obtain an other form of solutions to*

$$\Big(P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2\Big)[u(t,\xi)] = 0$$

with $P\Big(\dfrac{d}{d\xi}\Big) = \dfrac{d^4}{d\xi^4} + \lambda\dfrac{d^2}{d\xi^2} + \omega^2$, *where* $\omega^2 - \lambda^2/4 > 0$, $\omega^2 > 0$.

3.3. Weak solutions to partial differential equation with boundary conditions.

We consider, as an illustration, the partial differential equations for the vibration rod and for lateral vibrating of an elastic rod on Winkler foundation (cf. Section 2, Subsection 2.2). To find weak (generalized) solutions we use the classical well-known results. That is the reason to consider them as a preliminary.

In this part we use some facts from the theory of linear differential operators and from Fredholm theory of integral equations. We repeat them. Let L denote a *linear differential operator* defined by the *differential expression*

$$l(u) = a_0 u^{(n)}(x) + \cdots + a_{n-1}u^{(1)}(x) + a_n u(x), \quad x_1 < x < x_2,$$

and by the homogeneous boundary condition $U_\nu(u) = 0$, $\nu = 1,\dots,n$, so to say a *differential problem* is defined. *Eigenvalues* and *eigenfunctions* of the operator L have been given by $l(u) = 0$, $U_\nu(u) = 0$, $\nu = 1,\dots,n$. *Green's function* of the operator L is the function $G(x,\xi)$ with the following properties:

(1) $G(x,\xi)$ with its $(n-2)$ derivatives in x is continuous for $x,\xi \in (x_1,x_2)$ and satisfies the prescribed boundary conditions $U_\nu(u) = 0$, $\nu = 1,\dots,n$.

(2) Except at the point $x = \xi$ the $(n-1)$-th and the n-th derivative in x are continuous for $x,\xi \in (x_1,x_2)$. At the point $x = \xi$ the $(n-1)$-th derivative in x has a jump discontinuity given by

$$\frac{\partial^{n-1}}{\partial x^{n-1}}G(\xi+0,\xi) - \frac{\partial^{n-1}}{\partial x^{n-1}}G(\xi-0,\xi) = -\frac{1}{a_0(\xi)}, \quad \xi \in (x_1,x_2).$$

(3) $G(x,\xi)$ considered as a function of x satisfies the differential equation $l(u) = 0$, $x,\xi \in (x_1, x_2)$, $x \neq \xi$.

Proposition 3.1. *If the differential problem*

$$l(u) = 0,\ U_\nu(u) = 0,\ \ \nu = 1, \dots, n$$

has only the trivial solution $u = 0$, then L has one and only one Green's function $G(x,\xi)$. This function $G(x,\xi)$ is the kernel of the integral equation

$$u(x) = \lambda \int_0^\pi G(x,\xi)u(\xi)\,d\xi + \int_0^\pi G(x,\xi)f(\xi)\,d\xi \tag{3.11}$$

which is equivalent to the differential problem

$$l(u) + \lambda u = -f,\ U_\nu(u) = 0,\ \ \nu = 1, \dots, n.$$

(cf. [19, I, p. 353]).

If a kernel $K(x,\xi)$ of the integral equation (3.11) has the property that

$$J(\varphi,\varphi) = \iint K(s,\xi)\varphi(s)\varphi(\xi)\,ds\,d\xi$$

can assume only positive or only negative values (unless φ vanishes identically) it is said to be *positive definite* or *negative definite* in both cases it is *definite*. φ is any function which is continuous or piecewise continuous in the basic domain.

Proposition 3.2. *If $K(x,\xi)$ is a continuous symmetric kernel of the integral equation* (3.11)*, then every function g of the form*

$$g(x) = \int_0^\pi K(x,\xi)h(\xi)\,d\xi,$$

where h is a piecewise continuous function on $[0,\pi]$, can be expanded in a series in the orthonormal eigenfunctions of $K(x,\xi)$

$$g(x) = \sum_{i=1}^{\infty} g_i v_i(x), \quad g_i(g,v_i) = \frac{\langle h, v_i\rangle}{\lambda_i},$$

where $\langle g, v_i\rangle \equiv \int_0^\pi g(\xi)v_i(\xi)\,d\xi$. This series converges uniformly and absolutely (cf. [19, I, p. 136]).

From the proof of this Proposition we will use the following:
For every $\varepsilon > 0$ there exists $\mathrm{N}_0(\varepsilon)$ such that:

$$\sum_{i=m}^{n} |g_i|\,|v_i(x)| < \varepsilon, \quad n,m \geqslant \mathrm{N}_0(\varepsilon),\ \ x \in [0,\pi]. \tag{3.12}$$

3.3.1. *The classical theory of a vibrating rod.* The mathematical model of the vibrating rod is (cf. Section 2, Subsection 2.1)

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)=0,\quad 0<x<\pi,\quad t>0. \tag{3.13}$$

Since for the construction of generalized solutions to (3.13) we use the classical results, we quote some of them (cf. [19, I]).

If we suppose that the solution to (3.13) has the form $u(x,t)=v(x)g(t)$, then equation (3.13) decomposes to two differential equations

$$v^{(4)}(x)-\lambda v(x)=0,\quad 0<x<\pi;\quad g^{(2)}(t)+\lambda g(t)=0,\quad t>0. \tag{3.14}$$

In [19] five various types of boundary conditions have been analyzed (see also Section 2, Subsection 2.1):

(3.15)

1. $v^{(2)}(x)=v^{(3)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., free ends
2. $v(x)=v^{(2)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., simply supported ends
3. $v(x)=v^{(1)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., clamped ends
4. $v^{(1)}(x)=v^{(3)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., moving clamped ends
5. $v(0)=v(\pi),\ v^{(1)}(0)=v^{(1)}(\pi),\ v^{(2)}(0)=v^{(2)}(\pi),\ v^{(3)}(0)=v^{(3)}(\pi),$ periodicity conditions.

In all these cases eigenvalues and eigenfunctions can be given explicitly. The next Proposition gives the properties of these eigenvalues and eigenfunctions.

Proposition 3.3. *For the differential problem* $(3.14)_1$ *and one of boundary conditions* (3.15), *there exists a denumerable infinite system of eigenvalues* $\lambda_i \geqslant 0$, $i\in N$ *and associated eigenfunctions,* v_i, $i\in N$. *Note that* $\{\lambda_i\}_{i\in\mathbb{N}}$ *is not a bounded set;* $\{v_i\}_{i\in\mathbb{N}}$ *is a complete system and arbitrary functions possessing continuous first and second and piecewise continuous third and fourth derivatives may be expanded in terms of these eigenfunctions.*

By the solutions to equations (3.14) we can construct a family of solutions to (3.15)

$$u_i(x,t)=v_i(x)(a_i\cos\nu_i t+b_i\sin\nu_i t),\quad i\in\mathbb{N}, \tag{3.16}$$

where a_i,b_i are arbitrary constants and $\nu_i=\sqrt{\lambda_i}$ ($\sqrt{\lambda_i}$ is the principal branch), $i\in\mathbb{N}$. This form of solutions contains also the initial condition in t:

$$u_i(x,0)=a_iv_i(x);\quad \frac{\partial}{\partial t}u_i(x,t)\Big|_{t=0}=b_i\nu_iv_i(x).$$

It is easily seen that every finite sum $\sum u_i(x,t)$ is a solution to (3.13), as well.

Let us go back to equation $(3.14)_1$ with the boundary condition $U_\nu(v)=0$, $\nu=1,\dots,4$, which is one of the type (3.15). In this case we have that a linear homogeneous operator L is given by $l(v)=v^{(4)}(x)=0$ and $U_\nu(v)=0$, $\nu=1,\dots,4$. From $v^{(4)}(x)=0$, it follows that $v(x)=C_1+C_2x+C_3x^2+C_4x^3$, where C_i, $i=1,\dots,4$ are arbitrary constants. For the boundary condition $U_\nu(v)=0$, $\nu=1,\dots,4$ we take for example $(3.15)_3$. Then we have to find C_i, $i=1,2,3,4$ in such a way that the chosen condition $U_\nu(v) = 0$, $\nu = 1, \dots, 4$ is satisfied. It is easily seen that all the $C_i = 0$, $i = 1, \dots, 4$. Consequently $v = 0$.

By Proposition 3.1, there exists one and only one Green's function $G(x, \xi)$ for L. This Green's function in our case is definite (cf. [19, p. 363]).

3.3.2. *Construction of generalized solutions to* (3.13), (3.14). Now, the equation (3.13) can be drowned in $\mathcal{D}'((0,\pi) \times (0,\infty))$ by the property 4 in Section 1, Subsection 1.2 of the distributional derivative. To (3.13) in $\mathcal{D}'(0,\pi) \times (0,\infty))$ it corresponds

$$D_x^4[u(x,t)] + D_t^2[u(x,t)] = 0. \tag{3.17}$$

Every solution to (3.13) defines a regular distribution, which is a solution to (3.17).

To $u(x,t) = v(x)g(t)$ corresponds in $\mathcal{D}'((0,\pi) \times (0,\infty))$ the distribution $[u(x,t)] = [v(x)] \times [g(t)]$ (*tensor product*). We know that (cf. [64, p. 120])

$$D_x^4[v(x)g(t)] = D_x^4[v(x)] \times [g(t)],$$
$$D_t^2[v(x)g(t)] = [v(x)] \times D_t^2[g(t)].$$

We proceed to find $[v(x)]$ and $[g(t)]$ in such a way that $[v(x)g(t)]$ satisfies (3.17). This equation (3.17) can be written in the form:

$$D_x^4[v(x)] \times [g(t)] - \lambda[v(x)] \times [g(t)] + [v(x)] \times D_t^2[g(t)] + \lambda[v(x)] \times [g(t)] = 0.$$

Let us find λ, $[v(x)]$ and $[g(t)]$ so that

$$D_x^4[v(x)] - \lambda[v(x)] = 0, \quad D_t^2[g(t)] + \lambda[g(t)] = 0. \tag{3.18}$$

It is well known (cf. Property 7 in Section 1, Subsection 1.2 of the distributional derivative) that these two equations (3.18) have only solutions defined by the solutions to equations (3.14). Then solutions to (3.17) have been defined by functions of the form (3.16) or by finite sums of them. Consequently we have nothing new for equation (3.17).

To find generalized solutions to (3.13), which are interesting for our differential problem (3.13), (3.15) we shall start from the classical results for the equation (3.13), we cited in Proposition 3.1.

The Green function $G(x, \xi)$ for the operator L defined on the end of the Section 3.3.1 has all the properties we need so that Proposition 3.2 can be applied.

Let $w_1(x)$ and $w_2(x)$ be continuous functions and $h_i(x)$, $i = 1, 2$, piecewise continuous functions such that

$$\omega_i(x) = \int_0^\pi G(x, \xi) h_i(\xi)\, d\xi, \quad x \in [0, \pi], \; i = 1, 2. \tag{3.19}$$

Then by Proposition 3.2 we have

$$\omega_i(x) = \sum_{j=1}^\infty \omega_{ij} v_j(x), \quad i = 1, 2, \tag{3.20}$$

where $\{v_j\}_{j \in \mathbf{N}}$ is the sequence of eigenfunctions of $G(x, \xi)$.

From (3.19) and properties of Green's function it follows by (3.20) that the functions $\omega_i(x)$, $i = 1, 2$ are not only continuous, but they have also continuous first and second order derivatives. They satisfy the boundary condition, as well. Because of the properties of eigenfunctions $v_i(x)$, $i \in \mathbb{N}$, to be continuous, to have continuous first derivative and that $v_i(0) = 0$, for every $i \in \mathbb{N}$, there exists $x_i \in (0, \pi)$, such that

$$\max_{0 \leqslant x \leqslant \pi} |v_i^{(1)}(x)| = |v_i^{(1)}(x_i)| \equiv M_i \neq 0, \quad i \in \mathbb{N}, \tag{3.21}$$

and there exists $x_i' \in (0, \pi)$, such that

$$|v_i(x_i')|/M_i < 1, \quad i \in \mathbb{N}. \tag{3.22}$$

We will also use the property of the set $\{\lambda_i\}_{i \in \mathbb{N}}$ of eigenvalues, not to be bounded. Consequently there exists $i_0 \in \mathbb{N}$ such that

$$\lambda_i^{-1} < 1, \quad i \geqslant i_0. \tag{3.23}$$

We can now construct the function $W(x,t)$

$$W(x,t) = \sum_{j=1}^{\infty} v_j(x)(a_j \cos \nu_j t + b_j \sin \nu_j t), \quad 0 \leqslant x \leqslant \pi, \ t \geqslant 0. \tag{3.24}$$

We consider two cases for constant $a_j, b_j \in \mathbb{N}$:

$$\text{(i)} \quad a_j = \frac{\omega_{1j} v_j(x_j')}{M_j}, \quad b_j = \frac{\omega_{2j} v_j(x_j')}{M_j};$$

$$\text{(ii)} \quad a_j = \frac{\omega_{1j} v_j(x_j')}{M_j \nu_j}, \quad b_j = \frac{\omega_{2j} v_j(x_j')}{M_j \nu_j},$$

where $\nu_j = \sqrt{\lambda_j}$, $\lambda_j \geqslant 0$, $j \in \mathbb{N}$.

The function $W(x,t)$ has the following properties:

1) In case (i) it is a continuous function with a continuous first derivative in x on $[0,\pi) \times [0,\infty)$. In case (ii) it has also a continuous derivative in t.

First we prove the continuity proving that the two series which constitute the function $W(x,t)$ are uniformly convergent on $[0,\pi] \times [0,\infty)$.

Case (i): By (3.12) and (3.22) we have for the first series

$$\left| \sum_{j=m}^{n} \frac{\omega_{1j} v_j(x_j')}{M_j} v_j(x) \cos v_j t \right|^2 \leqslant \left(\sum_{j=m}^{n} |\omega_{1j}| |v_j(x)| \right)^2 < \varepsilon$$

$n, m \geqslant \mathbb{N}_0$, $(x,t) \in [0,\pi] \times [0,\infty)$.

The proof for the second series is just the same.

Case (ii): We use now (3.23) in the proof of the continuity.

Let us consider the series

$$\sum_{j=1}^{\infty} v_j^{(1)}(x) \Big(\frac{\omega_{1j} v_j(x_j')}{M_j} \cos \nu_j + \frac{\omega_{2j} v_j(x_j')}{M_j} \sin \nu_j t \Big). \tag{3.25}$$

By using again (3.12) and (3.21), we have

$$\left| \sum_{j=m}^{n} v_j^{(1)}(x) \frac{\omega_{1j} v_j(x_j')}{M_j} \cos \nu_j t \right|^2 \leqslant \left(\sum_{j=m}^{n} |\omega_{1j}| |v_j(x_j')| \right)^2 < \varepsilon,$$

$n, m \geqslant \mathrm{N}(\varepsilon)$. The treatment of the second series in (3.25) is the same.

Now we can conclude that in case (i) the function $W(x,t)$ given by (3.24) has a continuous derivative in x. This derivative can be obtained by taking the derivative of every member of the series in (3.24).

The proceeding of the proof that in case (ii) we have also the derivative in t does not differ of the proof of the derivative in x.

2) In case (i) and (ii)

$$W(x,0) = \sum_{j=1}^{\infty} v_j(x) a_j,$$

and this is a continuous function with continuous derivative on $[0,\pi]$.

In case (ii) we have

$$\frac{\partial}{\partial t} W(x,t)\Big|_{t=0} = \sum_{j=1}^{\infty} v_j(x) \frac{\omega_{2j} v_j(x_j')}{M_j},$$

as well. The given series defines also a continuous function on $[0,\pi]$.

3) $W(x,t)$ satisfies the boundary condition we chose $(3.15)_3$.

4) $W(x,t)$ given by (3.24) is the limit of the sequence

$$W_n(x,t) = \sum_{j=1}^{n} v_j(x)(a_j \cos \nu_j t + b_j \sin \nu_j t), \quad n \in \mathbb{N}, \tag{3.26}$$

in $C\big([0,\pi] \times [0,\infty)\big)$. The elements of the sequence (3.26) are solutions to (3.13) (cf. (3.16)).

It is easy now to prove

Theorem 3.3. *Let us denote by:* 1) $\{\lambda_i\}_{i\in\mathbb{N}}$ *and* $\{v_i\}_{i\in\mathbb{N}}$ *the eigenvalues and eigenfunctions respectively of the differential problem*

$$\begin{aligned} &v^{(4)}(x) - \lambda v(x) = 0, \\ &v(x) = v^{(1)}(x) = 0, \ \textit{for } x = 0 \textit{ and } x = \pi. \end{aligned}$$

2) $\{\nu_i\}_{i\in\mathbb{N}}$ *the sequence defined by* $\nu_i = \sqrt{\lambda_i}$, $\lambda_i \geqslant 0$, *where* $\sqrt{\lambda_i}$ *means the principal branch,* $i \in N$.

3) $\{a_j\}_{j\in\mathbb{N}}$ *and* $\{b_j\}_{j\in\mathbb{N}}$ *the sequences*

$$\begin{aligned} &(i) \qquad a_j = \frac{\omega_{1j} v_j(x_j')}{M_j}, \quad b_j = \frac{\omega_{2j} v_j(x_j')}{M_j}, \qquad \textit{or} \\ &(ii) \qquad a_j = \frac{\omega_{1j} v_j(x_j')}{M_j \nu_j}, \quad b_j = \frac{\omega_{2j} v_j(x_j')}{M_j \nu_j}, \end{aligned}$$

where

$$M_j = \max_{0\leqslant x\leqslant \pi} \big|v_j^{(1)}(x)\big| \ \textit{and}\ x_j' \in (0,\pi),\ \big|v_j(x_j')\big|/M_j < 1,\ j \in \mathbb{N}. \tag{3.27}$$

Then the function $W(x,t) = \sum_{j=1}^{\infty} v_j(x)(a_j \cos \nu_j t + b_j \sin \nu_j t)$, $0 \leqslant x \leqslant \pi$, $t \geqslant 0$ *defines a regular distribution* $[W(x,t)] \in D'((0,\pi) \times (0,\infty))$. *This distribution is a solution to* (3.17) *and a generalized solution to* (3.13), $(3.15)_3$.

The properties of the function $W(x,t)$ are:

a) In case (i) and (ii) it is a continuous function with continuous first order partial derivative in x on $[0,\pi]\times[0,\infty)$.

b) In case (ii) it has also a continuous first order partial derivative in t on $[0,\pi]\times[0,\infty)$.

c) In case (i) and (ii) we have $W(x,0)=\sum_{j=1}^{\infty}v_j(x)a_j$, $x\in[0,\pi]$, and this is a continuous function with a continuous first order derivative on $[0,\pi]$.

d) In case (ii) we have $\frac{\partial}{\partial t}W(x,t)\Big|_{t=0}=\sum_{j=1}^{\infty}v_j(x)\nu_j b_j$, $x\in[0,\pi]$. The given series defines a continuous function on $[0,\pi]$, as well.

e) $W(x,t)$ satisfies the boundary conditions $W(x,0)=\frac{\partial}{\partial x}W(x,t)=0$, for $x=0$ and $x=\pi$, and $t\geqslant 0$.

f) In case (i) and (ii) $D_x[W(x,t)]=[\frac{\partial}{\partial x}W(x,t)]$ and in case (ii) $D_t[W(x,t)]=[\frac{\partial}{\partial t}W(x,t)]$

g) In case (i) and (ii) $W(x,t)$ and in case (ii) $\frac{\partial}{\partial t}W(x,t)$ are bounded on $[0,\pi]\times[0,\infty)$.

Proof. The function $W(x,t)$ given by (3.24) defines a distribution because of its property 1), we proved. □

If the sequence (3.26) consists of solutions to (3.13), $(3.15)_3$, then the sequence $([W_n(x,t)])_{n\in\mathbb{N}}\subset\mathcal{D}'((0,\pi)\times(0,\infty))$ is the sequence of solutions to (3.17). Since the sequence (3.26) converges in $\mathcal{C}([0,\pi]\times[0,\infty))$, the sequence $([W_n(x,t)])_{n\in\mathbb{N}}$ converges in $\mathcal{D}'((0,\pi)\times(0,\infty))$ (cf. Section 1, Subsection 1.2). Consequently, $[W(x,t)]$ as the limit of the sequence of solutions to (3.17) is also a solution to (3.17).

The other cited properties of the function $W(x,t)$ one can easily prove.

Remarks. 1) By (3.27) we have a family of functions because the sequence $\{x_j'\}_{j\in\mathbb{N}}\subset(0,\pi)$ has only to satisfy the inequality $|v_j(x_j')|/M_j<1$, $j\in\mathbb{N}$.

2) If the solution to (3.13), $(3.15)_3$ is of the form $u(x,t)=v(x)g(t)$ we have

$$u(x,0)=g(0)v(x)\ \text{and}\ \frac{\partial}{\partial t}u(x,t)\Big|_{t=0}=g'(0)v(x).$$

But in our case $W(x,t)$ given by (3.27) which defines a generalized solution to (3.13), $(3.15)_3$ satisfies a more general initial condition: in case (i) and (ii) $W(x,0)=\sum_{j=1}^{\infty}a_jv_j(x)$ and in case (ii) we have moreover

$$\frac{\partial}{\partial t}W(x,t)\Big|_{t=0}=\sum_{j=1}^{\infty}b_j\nu_jv_j(x).$$

3.3.3. *Construction of generalized solutions to equation of the lateral vibration of an elastic rod on Winkler foundation.* We consider the equation

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)+\lambda q(x)u(x,t)=0,\quad 0<x<\pi,\ t>0,\tag{3.28}$$

where $q(x)\geqslant 0$, $x\in[0,\pi]$ with boundary condition:

$$u(0,t)=\frac{\partial}{\partial x}u(x,t)\Big|_{x=0}=0;\ u(\pi,t)=\frac{\partial}{\partial x}u(x,t)\Big|_{x=\pi}=0,\ t\geqslant 0.\tag{3.29}$$

As in Section 3.3.2, we suppose that a solution of (3.28) is of the form $u(x,t) = v(x)g(t)$; then equation (3.28) becomes

$$\frac{\partial^4}{\partial x^4}v(x)g(t) + \lambda q(x)v(x)g(t) - \omega v(x)g(t) + \frac{\partial^2}{\partial x^2}v(x)g(t) + \omega v(x)g(t) = 0, \quad 0 < x < \pi,\ t > 0.$$

To find v and g we use two equations

$$v^{(4)}(x) + \lambda q(x)v(x) - \omega v(x) = 0, \quad 0 < x < \pi,$$
$$g^{(2)}(t) + \omega g(t) = 0, \quad t > 0,$$

and the boundary condition

$$v(0) = v^{(1)}(0) = 0, \quad v(\pi) = v^{(1)}(\pi) = 0. \tag{3.30}$$

Let L denote the differential expression $L(v) = v^{(4)}(x) + \lambda q(x)v(x)$. Note that L is self adjoint. To prove that L has Green's function we have to show (Proposition 3.1) that from $L(v) = 0$ and (3.30) it follows that $v = 0$. We will do it in two steps. First we consider the differential expression $l(v) = v^{(4)}(x)$ with (3.30). It is easily seen that $v^{(4)}(x) = 0$ with (3.30) gives $v = 0$. Then l has Green's function $G_l(x,\xi)$. We know that $G_l(x,\xi)$ is symmetric and definite (cf. [19, p. 363]).

Now, in the second step, we use the fact that

$$L(v) = v^{(4)}(x) + \lambda q(x)v(x) = 0, \text{ with } (2.31) \tag{3.31}$$

is equivalent to (cf. Proposition 3.1)

$$v(x) = \lambda \int_0^\pi G_l(x,\xi)q(\xi)v(\xi)\,d\xi,$$

or

$$\sqrt{q(x)}\,v(x) = \lambda \int_0^\pi G_l(x,\xi)\sqrt{q(x)q(\xi)}\,\sqrt{q(\xi)}\,v(\xi)\,d\xi.$$

The kernel $K(x,\xi) = G_l(x,\xi)\sqrt{q(x)q(\xi)}$ is also symmetric and definite. Let us denote by $y(x) = \sqrt{q(x)}\,v(x)$. Then (3.31) is equivalent to

$$y(x) = \lambda \int_0^\pi K(x,\xi)y(\xi)\,d\xi. \tag{3.32}$$

Since $K(x,\xi)$ is a continuous and symmetric kernel it possesses eigenvalues and eigenfunctions. Their number is denumerably infinite (cf. [19, p. 22]). Let λ_0 be a real number (positive) which is not an eigenvalue for the kernel $K(x,\xi)$. Then equation (3.32) and consequently equation (3.31) have only $v = 0$ as the solution. Hence we know that Green's function $G_L(x,\xi)$ exists for L with (3.30). Since L is self adjoint, $G_L(x,\xi)$ is symmetric and L has eigenvalues $\{\lambda_i\}_{i\in\mathbf{N}}$ and eigenfunctions $\{v_i(x)\}_{i\in\mathbf{N}}$. Consequently we can apply Proposition 3.2. The consequence is that we can construct generalized solutions to equation (3.28) (which depends on the chosen number λ_0) with boundary condition (3.29) processing just in the same way as in Section 3.3.2 for equation (3.13) with the same boundary condition.

We have to remark that in this case we do not know that the Green function G_L is positive definite; the eigenvalues have not to be positive. Consequently we can not assert that the function $W(x,t)$ which defines the distributional solution is bounded on $[0,\pi] \times [0,\infty)$. The stability of the solution has to be considered separately.

3.4. The Laplace transform applied to a partial differential equation. The Laplace transform is very useful in solving partial differential equations. But we have always to take into account that as a first condition for applicability of the Laplace transform on a generalized function is to have its support bounded on the left. In such a way when we have a partial differential equations with numerical functions and look for the corresponding equation in a space of generalized functions we have to use the Property 8 in Section 1, Subsection 1.2 of the derivative of a generalized function.

Working with the Laplace transform, when we find a function $F(s)$, $\operatorname{Re} s > \omega > 0$ and seek for a generalized function f, such that $\tilde{\mathcal{L}}f(s) = F(s)$, we have first to check if such f exists. For this purpose Propositions 1.4 and Proposition 2.1 in Section 1 can help. Secondly, we have to find such f. In many cases f is a numerical function. Thus, $\tilde{\mathcal{L}}^{-1}(f)$ is the regular distribution $[f]$ defined by the function f. The solution still has not to be a classical one, because the derivatives in, general, exist only in the distributional sense. An illustration how it reflects in solving a partial differential equation one can find in [61]. We consider in 3.4.1 the case when we apply the Laplace transform in one variable and in 3.4.2 in two variables to a partial differential equation.

3.4.1. *$\mathcal{M}$-valued functions as solutions to a partial differential equation.* Let $\mathcal{M}$ denote one of the following spaces: the space of L-functions (cf. [21]), $\mathcal{D}'_\omega(\overline{\mathbb{R}}_+)$ or $\mathcal{B}^{\exp}_{[0,\infty]}$. We use the Laplace transform which is defined for elements of these three spaces, consequently for elements of $\mathcal{M}$.

The partial differential equation we analyze is:

$$\frac{\partial^4}{\partial x^4}u(x,t) + \frac{\partial^2}{\partial t^2}u(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{3.33}$$

with the initial conditions

$$u(x,0) = B_0(x), \quad \frac{\partial}{\partial t}u(x,t)\Big|_{t=0} = B_1(x), \quad 0 < x < 1. \tag{3.34}$$

It is well-known that equation (3.33) has a solution of the form $u(x,t) = v(x)g(t)$ (cf. [2], [19]). In this case $B_0(x) = v(x)g(0)$ and $B_1(x) = v(x)g^{(1)}(0)$.

Let $\{[u(x,t)]\}_{0<x<1}$ denote a family of $\mathcal{M}$-valued functions of class C^4 (cf. Section 1, Subsection 1.4). For any fixed x, $[u(x,t)] \in \mathcal{M}$.

By the property 8.1 in Section 1, Subsection 1.2, to equation (3.33) it corresponds in $\mathcal{M}$ the equation

$$\frac{\partial}{\partial x^4}[u(x,t)] + D_t^2[u(x,t)] = B_1(x)\delta(t) + B_0(x)\delta^{(1)}(t), \quad 0 < x < 1. \tag{3.35}$$

Now, the solutions to (3.35) are the generalized solutions to (3.33), (3.34). Our aim is to find all the solutions to (3.35), i.e., all the generalized solutions to (3.33) with $B_0(x) = v(x)g(0)$ and $B_1(x) = v(x)g^{(1)}(0)$ which are functions with values in $\mathcal{M}$.

Suppose that we have two such solutions to (3.35) with values in $\mathcal{M}$, $w_1(x)$ and $w_2(x)$. Then $w_0(x) = w_1(x) - w_2(x)$ satisfies the homogenous equation (compare to (3.35))

$$\frac{\partial^4}{\partial x^4} w_0(x) + D_t^2 w_0(x) = 0, \quad 0 < x < 1. \tag{3.36}$$

The Laplace transform in t transforms (3.36) in

$$\frac{\partial^4}{\partial x^4} W_0(x, \hat{s}) + s^2 W_0(x, \hat{s}) = 0, \quad 0 < x < 1, \tag{3.37}$$

where $W_0(x, \hat{s}) = \tilde{\mathcal{L}}_t(w_0(x))(x, \hat{s})$. The equation (3.37) is a classical differential equation in which s, $\operatorname{Re} s > \omega > 0$, is only a parameter.

The general solution to (3.37) is of the form

$$\begin{aligned} W_0(x, \hat{s}) = C_1(s)e^{r_1 x} + C_2(s)e^{r_2 x} + C_3(s)e^{r_3 x} + C_4(s)e^{r_4 x}, \\ 0 < x < 1, \quad \operatorname{Re} s > \omega, \end{aligned} \tag{3.38}$$

where C_i, $i = 1, \dots, 4$ are functions of s and r_i, $i = 1, \dots, 4$ are solutions to equation $r^4 + s^2 = 0$.

The Propositions 1.4 and 2.1 in Section 1, give the conditions which $C_i(s)$, $i = 1, \dots, 4$, have to satisfy that $w_0(x)$ exists such that $\tilde{\mathcal{L}}(w_0(x))(x, \hat{s}) = W_0(x, \hat{s})$, $0 < x < 1$.

We known that $[v(x)g(t)]$ is solution to (3.35) with $B_0(x) = v(x)g(0)$ and $B_1(x) = v(x)g^{(1)}(0)$. Then all the solutions to (3.35) with cited values for B_0 and B_1 which are functions with values in $\mathcal{M}$ are $[v(x)g(t)] + w_0(x)$.

In such a way we proved the following theorem:

Theorem 3.4. *Let $u_1(x,t) = v(x)g(t)$ be the well known classical solution to* (3.33) *and let $\mathcal{M}$ denote one of the spaces: The space of L-functions* (cf. [19]), $\mathcal{D}'_\omega\left(\overline{\mathbb{R}}_+\right)$ *or* $\mathcal{B}^{\exp}_{[0,\infty]}$.

All the solutions to (3.35), *i.e., all the generalized solutions to* (3.33) *with initial condition*

$$u(x, 0) = v(x)g(0) \text{ and } \frac{\partial}{\partial t} u(x,t)\Big|_{t=0} = v(x)g^{(1)}(0)$$

which are functions in x with values in M are $w(x) = [v(x)g(t)] + w_0(x)$, where $\tilde{\mathcal{L}}(w_0(x))(\hat{s}) = W_0(x, \hat{s})$ and $W_0(x, \hat{s})$ is given by (3.38).

Applying the Laplace transform to (3.35) with any B_0 and B_1 we obtain a nonhomogeneous differential equation. The same procedure as for (3.37) gives us the generalized solutions to (3.33), (3.34) for any B_0, B_1.

3.4.2. *Solution of partial differential equation* (3.33) *by the Laplace transform.* We consider the equation

$$\frac{\partial^4}{\partial x^4}u(x,t) + \frac{\partial^2}{\partial t^2}u(x,t) = 0, \quad (x,t) \in \mathbb{R}^2_+, \tag{3.39}$$

with initial conditions:

$$\begin{aligned} u(0,t) &= \frac{\partial}{\partial x}u(0,t) = 0, \quad t \geqslant 0, \\ \frac{\partial^k}{\partial x^k}u(0,t) &= A_k(t), \quad k = 2,3, \quad t \geqslant 0, \\ u(x,0) = B_0(x), &\quad \frac{\partial}{\partial t}u(x,0) = B_1(x), \quad x \geqslant 0, \end{aligned} \tag{3.40}$$

where $[\theta(t)A_k(t)] \in e^{pt}\mathcal{S}'(\overline{\mathbb{R}}_+)$, $k = 2,3$, and $[\theta(x)B_i(x)] \in e^{pt}\mathcal{S}'(\overline{\mathbb{R}}_+)$, $i = 0,1$, $p > 0$. To find an equation in $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$ which corresponds to (3.40) for $x > 0$, $t > 0$, we need the relations between derivatives in the sense of distributions and the classical ones.

Let $\theta^2(x_1,x_2) = \theta(x_1)\theta(x_2)$, where θ is the Heaviside function. For a function f with continuous partial derivatives on $\mathbb{R}^2$, $[\theta^2 f]$ is the distribution, defined by $\theta^2 f$, belonging to $\mathcal{D}'(\mathbb{R}^2)$ and to $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$, as well. Let $(\partial^p f/\partial x_i^p)_0$ denote the function equal to $\partial^p f/\partial x_i^p$ on the $\mathbb{R}^2_+$ and equal zero on $\mathbb{R}^2 \setminus \overline{\mathbb{R}}^2_+$, but is not defined for $(x_1,x_2) \in \{(0,x_2) \cup (x_1,0);\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$.

With the notation as above we have (cf. 8.2 in Section 1, Subsection 1.2).

$$\begin{aligned} \frac{\partial^4}{\partial x^4}[u(x,t)] + \frac{\partial^2}{\partial t^2}[u(x,t)] &= [\theta(t)A_2(t)] \times \delta^{(1)}(x) + [\theta(t)A_3(t)] \times \delta(x) \\ &\quad + [\theta(x)B_1(x)] \times \delta(t) + [\theta(x)B_0(x)] \times \delta^{(1)}(t). \end{aligned} \tag{3.41}$$

Applying the LT we have

$$(z^4 + s^2)\mathcal{L}(u)(z,s) = \mathcal{L}(A_2)(s)z + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(z) + \mathcal{L}(B_0)(z)s,$$

or

$$\mathcal{L}(u)(z,s) = \frac{Q(z,s)}{z^4 + s^2},$$

with $Q(z,s) = \mathcal{L}(A_2)(s)z + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(z) + \mathcal{L}(B_0)(z)s$. Since

$$\frac{1}{z^4 + s^2} = \frac{1}{2is}\Big(\frac{1}{z^2 - is} - \frac{1}{z^2 + is}\Big),$$

we have

$$\frac{Q(z,s)}{z^4 + s^2} = \frac{Q(z,s)}{2is}\Big(\frac{1}{z^2 - is} - \frac{1}{z^2 + is}\Big). \tag{3.42}$$

By Proposition 1.4 in Section 1, and the property of the space $\mathcal{H}_+$, $\frac{Q(z,s)}{z^4+s^2}$ has to be holomorphic in $\{(z,s) \in \mathbb{C}^2;\ \mathrm{Re}\, z > w_1 > 0,\ \mathrm{Re}\, s > w_2 > 0\}$. Since $z^4 + s^2 = (z - z_1)(z + z_1)(z - z_2)(z + z_2)$, where $z_1 = e^{i\pi/4}\sqrt{s}$, $z_2 = e^{3i\pi/4}\sqrt{s}$, it is necessary to have

$$Q(e^{i\pi/4}\sqrt{s}, s) = 0 \text{ and } Q(-e^{3i\pi/4}\sqrt{s}, s) = 0$$

or equivalently

$$Q(e^{i\pi/4}\sqrt{s}, s) = 0 \text{ and } Q(e^{-i\pi/4}\sqrt{s}, s) = 0. \tag{3.43}$$

Let us consider the first addend in (3.42). Then $(3.43)_1$ has to be satisfied which gives

$$\mathcal{L}(A_2)(s)e^{i\pi/4}\sqrt{s} + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s}) + s\mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s}) = 0.$$

Now we can express $\mathcal{L}(A_3)(s)$,

$$\mathcal{L}(A_3)(s) = -\mathcal{L}(A_2)(s)e^{i\pi/4}\sqrt{s} - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s}) - s\mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s}).$$

With such expressed $\mathcal{L}(A_3)(s)$ the first addend in (3.42) is:

$$\begin{aligned}\frac{Q(z,s)}{2is(z^2-is)} &= \frac{\mathcal{L}(A_2)(s)(z-e^{i\pi/4}\sqrt{s})}{2is(z^2-is)} \\ &\quad + \frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s}) + s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0(e^{i\pi/4}\sqrt{s}))}{2is(z^2-is)} \\ &= \frac{\mathcal{L}(A_2)(s)}{2is(z+e^{i\pi/4}\sqrt{s})} + \left(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}} + \frac{\mathcal{L}(B_0)(z)-\mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s})}{4ie^{i\pi/4}\sqrt{s}}\right) \\ &\quad \times \left(\frac{1}{z-e^{i\pi/4}\sqrt{s}} - \frac{1}{z+e^{i\pi/4}\sqrt{s}}\right).\end{aligned} \tag{3.44}$$

By using the following formulas for the Laplace transform

$$\begin{aligned}\mathcal{L}_z^{-1}\Big(\frac{1}{z+a\sqrt{s}}\Big) &= \theta(x)e^{-ax\sqrt{s}} \\ \mathcal{L}_s^{-1}\Big(\frac{1}{\sqrt{s}}e^{-ax\sqrt{s}}\Big) &= \frac{\theta(t)}{\sqrt{\pi t}}e^{-(ax)^2/(4t)}, \ x>0, \ \mathrm{Re}\, a > 0 \\ &= \theta(t)\chi(ax,t).\end{aligned}$$

We can find the Laplace transforms in (3.44). Let us do it

$$\begin{aligned}\mathcal{L}^{-1}\Big(\frac{\mathcal{L}(A_2)(s)}{2is(z+e^{i\pi/4}\sqrt{s})}\Big) &= \mathcal{L}_s^{-1}\circ\Big(\mathcal{L}_z^{-1}\Big(\frac{1}{z+e^{i\pi/4}\sqrt{s}}\Big)\frac{\mathcal{L}(A_2)(s)}{2is}\Big) \\ &= \frac{1}{2i}\mathcal{L}_s^{-1}\Big(\frac{1}{\sqrt{s}}e^{-e^{i\pi/4}\sqrt{s}x}\Big)\frac{1}{\sqrt{s}}\mathcal{L}(A_2)(s) \\ &= \frac{\theta(x)\theta(t)}{2i\Gamma(1/2)}\chi(e^{i\pi/4}x,t) \overset{t}{*} \int_0^t (t-\tau)^{-1/2}A_2(\tau)\,d\tau.\end{aligned}$$

The second addend in (3.44) is:

$$\frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}}\Big(\frac{1}{z-e^{i\pi/4}\sqrt{s}} - \frac{1}{z+e^{i\pi/4}\sqrt{s}}\Big). \tag{3.45}$$

We shall start with:

$$\mathcal{L}^{-1}\Big(\frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\Big) \tag{3.46}$$

$$= \mathcal{L}_z^{-1} \circ \mathcal{L}_s^{-1}\bigg(\frac{\mathcal{L}(B_1)(z)}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\bigg) - \mathcal{L}_s^{-1} \circ \mathcal{L}_z^{-1}\bigg(\frac{\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\bigg).$$

The first addend in (3.46) is

$$\begin{aligned}
&\mathcal{L}_z^{-1}\bigg(B_1(z)\mathcal{L}_s^{-1}\bigg(\frac{1}{4(e^{i\pi/4}\sqrt{s})^3(e^{i\pi/4}\sqrt{s}+z)}\bigg)\bigg)\\
&= \mathcal{L}_z^{-1}\bigg(B_1(z)\mathcal{L}_s^{-1}\frac{1}{4e^{3i\pi/4}s} \overset{t}{*} \mathcal{L}_s^{-1}\frac{1}{(z+e^{i\pi/4}\sqrt{s})\sqrt{s}}\bigg)\\
&= \frac{1}{4e^{3i\pi/4}}\int_0^t \chi(e^{i\pi/4}x,\tau)\,d\tau \overset{x}{*} B_1(x).
\end{aligned} \tag{3.47}$$

For the second addend in (3.46) we have

$$\begin{aligned}
&-\mathcal{L}_s^{-1} \circ \mathcal{L}_z^{-1}\bigg(\frac{\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\bigg)\\
&= -\mathcal{L}_s^{-1}\bigg(\mathcal{L}_s(B_1)(e^{i\pi/4}\sqrt{s}) \cdot \frac{1}{4e^{3i\pi/4}s} \cdot \frac{1}{\sqrt{s}}\mathcal{L}_z^{-1}\bigg(\frac{1}{z+e^{i\pi/4}\sqrt{s}}\bigg)\bigg)\\
&= -\mathcal{L}_s^{-1}\bigg(\frac{1}{4e^{3i\pi/4}s}\frac{\theta(x)}{\sqrt{s}}e^{-e^{i\pi/4}x\sqrt{s}}\int_0^\infty e^{-e^{i\pi/4}\sqrt{s}\tau}B_1(\tau)\,d\tau\bigg)\\
&= d - \frac{1}{4e^{3i\pi/4}} \overset{t}{*} \mathcal{L}_s^{-1}\bigg(\frac{1}{\sqrt{s}}\int_0^\infty e^{-e^{i\pi/4}\sqrt{s}(x+\tau)}B_1(\tau)\,d\tau\bigg)\\
&= -\frac{1}{4e^{3i\pi/4}} \overset{t}{*} \int_0^\infty e^{-\frac{1}{4}i(x+\tau)^2/t}\frac{1}{\sqrt{\pi t}}B_1(\tau)\,d\tau\\
&= -\frac{1}{4ie^{i\pi/4}}\int_0^t du\int_0^\infty \chi(e^{i\pi/4}(x+\tau),u)B_1(\tau)\,d\tau.
\end{aligned}$$

Applying the inverse Laplace transformation the first fraction in (3.45) becomes

$$\begin{aligned}
&\mathcal{L}^{-1}\bigg(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}s\sqrt{s}(z-e^{i\pi/4}\sqrt{s})}\bigg)\\
&= \mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(z)}{4ie^{i\pi/4}s\sqrt{s}(z-e^{i\pi/4}\sqrt{s})} - \mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ie^{i\pi/4}s\sqrt{s}(z-e^{i\pi/4}\sqrt{s})}\\
&= \frac{1}{4ie^{i\pi/4}}\bigg(\mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}e^{e^{i\pi/4}x\sqrt{s}} \overset{x}{*} B_1(x) - \mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}e^{e^{i\pi/4}sx}\int_0^\infty e^{-e^{i\pi/4}\sqrt{s}u}B_1(u)\,du\bigg)\\
&= \frac{1}{4ie^{i\pi/4}}\bigg(\mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\int_0^x e^{i\frac{\pi}{4}(x-u)\sqrt{s}}B_1(u)\,du - \mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\int_0^\infty e^{-e^{i\frac{\pi}{4}(u-x)\sqrt{s}}}B_1(u)\,du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{4ie^{i\pi/4}} \mathcal{L}_s^{-1} \left(\frac{1}{s\sqrt{s}} \int_x^\infty e^{-e^{i\frac{\pi}{4}(u-x)\sqrt{s}}} B_1(u)\, du \right) \\
(3.48) \qquad & \\
&= \frac{-1}{4ie^{i\pi/4}} \int_0^t \int_x^\infty \chi(e^{i\pi/4}(u-x), \tau) B_1(u)\, du\, d\tau.
\end{aligned}
$$

If we collect all the results obtained in (3.47)–(3.48), then the inverse LT of (3.45) is a function denoted by $F(B_1, x, t, \pi/4)$,

$$
\begin{aligned}
F\Big(B_1, x, t, \frac{\pi}{4}\Big) = &-\frac{1}{4ie^{i\pi/4}} \int_0^t \int_x^\infty \chi(e^{i\pi/4}(u-x), \tau) B_1(u)\, du\, d\tau \\
&- \frac{1}{4ie^{i\pi/4}} \int_0^t \chi(e^{i\pi/4}x, \tau)\, d\tau \overset{x}{*} B_1(x) \\
&+ \frac{1}{4ie^{i\pi/4}} \int_0^t du \int_0^\infty \chi(e^{i\pi/4}(x+\tau), u) B_1(\tau)\, d\tau.
\end{aligned}
$$

To find the inverse LT of (3.44), it is yet to be find the inverse LT of

$$
\frac{s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s}))}{4e^{3i\pi/4}s\sqrt{s}} \left(\frac{1}{z - e^{i\pi/4}\sqrt{s}} - \frac{1}{z + e^{i\pi/4}\sqrt{s}} \right). \tag{3.49}
$$

If we compare (3.49) with (3.45), we can observe that in the structure of (3.49) we have additionally only a product by s. Since $F(B_0, x, 0, \frac{\pi}{4}) = 0$, the inverse LT of (3.49) is $\partial F(B_0, x, t, \pi/4)/\partial t$.

The procedure of finding the inverse Laplace transform of the second addend in (3.42) is just the same as for the first one. The details, the complete solution and the comments one can find in [61].

Remark. If in equation (3.39), $(x,t) \in (0,1) \times \mathbb{R}_+$, then we can consider the equation (3.41) in $\mathcal{D}'_\omega((0,1) \times \mathbb{R}_+)$ (cf. Section 1, Subsection 1.5).

3.5. The case in which a generalized function appears just in the model. We shall study the existence and properties of the solutions to the following system of coupled partial differential equations (cf. Section 2, (2.3.4)):

$$
\begin{aligned}
&\frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial t^2} = 0, \\
&\frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_t^\alpha \frac{\partial^2 u}{\partial \xi^2} - m - \mu D_t^\alpha m = 0, \quad t > 0, \quad 0 < \xi < 1,
\end{aligned} \tag{3.50}
$$

with boundary conditions

$$
m(0,t) = 0, \; m(1,t) = 0, \;\; u(0,t) = 0, \;\; u(1,t) = 0, \;\; t \geqslant 0. \tag{3.51}
$$

We assume solutions to (3.50), (3.51) in the form $m(\xi,t) = M(\xi)V(t)$, $u(\xi,t) = U(\xi)T(t)$. Then for every $k = \pm 1, \pm 2, \dots$ the system (3.50), (3.51) reduces to

$$M_k(\xi) = C_k \sin k\pi\xi, \quad U_k = C_k \sin k\pi\xi, \tag{3.52}$$

and

$$\begin{gathered} -(k\pi)^2 V_k(t) - \lambda(k\pi)^2 T_k(t) + T_k^{(2)}(t) = 0, \\ V_k(t) + \mu V_k^{(\alpha)}(t) + (k\pi)^2 T_k(t) + \mu_1 (k\pi)^2 T_k^{(\alpha)}(t) = 0, \quad k \in \pm\mathbb{N}, \end{gathered} \tag{3.53}$$

where C_k are arbitrary constants.

Throughout this example we shall assume that, firstly $\mu \neq 0$, $\mu_1 \neq 0$ and secondly

$$\lambda \equiv \frac{FL^2}{E_0 I} = B + A\delta(t - t_0), \quad t_0 > 0.$$

The second assumption means that the axial force is subject to an impulsive change. Consequently, in equation $(4.4)_1$ we have the product $\delta(t-t_0)T_k(t)$. Since δ can be treated as a measure, this product has a meaning for any $t_0 > 0$ if $T_k \in C([0,\infty))$. Then $\delta(t - t_0)T_k(t) = T_k(t_0)\delta(t - t_0)$ (cf. [56]). This fact one has to take into account when we construct the generalized solutions. Such solution can be only a regular generalized function defined by a continuous function $T_k(t)$.

To solve the system (3.53) we will use the Laplace transform (cf. Section 1, Subsections 1.5 and 2.2) applied on functions or generalized functions with support in $\overline{\mathbb{R}}_+$. A function and its derivatives with the support in $\overline{\mathbb{R}}_+$ can have discontinuities at zero. For this reason, when we construct the system in $\mathcal{D}'(\mathbb{R})$ which corresponds to the system (3.53), we have to take care of the property 8.1 of a derivative given in Section 1, Subsection 1.2. Let us take for short in (3.53) that $k = 1$.

So if T is bounded in $[0,\varepsilon)$, for an $\varepsilon > 0$ (an assumption which is supposed to be satisfied in this case), then

$$\begin{aligned} D_t^\alpha[\theta(t)T(t)] &= [\theta(t)D_t^\alpha T(t)], \quad 0 < \alpha < 1, \\ D_t^{(2)}[\theta(t)T(t)] &= [\theta(t)T^{(2)}(t)] + T^{(1)}(0)\delta(t) + T(0)\delta^{(1)}(t). \end{aligned}$$

Consequently, to (3.53) it corresponds in $\mathcal{D}'(\overline{\mathbb{R}}_+)$

$$\begin{gathered} D^2[\theta T] - B\pi^2[\theta T] - \pi^2[\theta V] = T(0)\delta^{(1)}(t) + T^{(1)}(0)\delta(t) + \pi^2 A T(t_0)\delta(t - t_0), \\ \mu D^\alpha[\theta V] + \mu_1\pi^2 D^\alpha[\theta T] + [\theta V] + \pi^2[\theta T] = 0. \end{gathered}$$

Applying the generalized Laplace transform (cf. 1.1.5) with the following notation: $\mathcal{L}[(\theta T)](s) = \widehat{T}(s)$, $\mathcal{L}[\theta V](s) = \widehat{V}(s)$, $T(0) = T_0$, and $T^{(1)}(0) = T_0^1$, we have

$$\begin{gathered} -\pi^2\widehat{V}(s) - (B\pi^2 - s^2)\widehat{T}(s) = T_0 s + T_0^1 + \pi^2 A T(t_0)e^{-t_0 s}, \\ (1 + \mu s^\alpha)\widehat{V}(s) + \pi^2(1 + \mu_1 s^\alpha)\widehat{T}(s) = 0. \end{gathered} \tag{3.54}$$

The solution to system (3.54) is

$$\widehat{T}(s) = \frac{s^\alpha + 1/\mu}{\Delta(s)}\,(T_0^1\mu + T_0\mu s + \pi^2 A T(t_0)\mu e^{-t_0 s}),$$

$$-\widehat{V}(s) = \frac{s^\alpha + 1/\mu_1}{\Delta(s)}\left(T_0^1\mu_1 + T_0\mu_1 s + \pi^2 AT(t_0)\mu_1 e^{-t_0 s}\right), \tag{3.55}$$

where

$$\Delta(s) = \mu s^{\alpha+2} + s^2 + (\mu_1\pi^2 - B\mu)\pi^2 s^\alpha + (\pi^2 - B)\pi^2 = \mu s^{\alpha+2} + s^2 + as^\alpha + d,$$

and

$$a = \pi^2(\mu_1\pi^2 - B\mu); \quad d = \pi^2(\pi^2 - B).$$

The next step is to find the distribution which corresponds to (3.55). The main part to the solution (3.55) is the function

$$\hat{f}(s) = \frac{s^\alpha + 1/\mu}{\Delta(s)}. \tag{3.56}$$

To the function $\hat{f}(s)$ we can apply Theorem 3 in [19, Vol 1, p. 263], as well. In fact, there exists $f \in L_{\text{loc}}[0,\infty)$ and $x_1 > 0$, such that

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ts}\hat{f}(s)\,ds, \quad x > x_1, \quad t \geqslant 0, \tag{3.57}$$

$(\mathcal{L}f)(s) = \hat{f}(s)$. Here $(\mathcal{L}f)(s)$ denotes the classical Laplace transform of f defined as $(\mathcal{L}f)(s) = \int_0^\infty e^{-st}f(t)\,dt$.

Since the integral in (3.57) converges uniformly for $0 \leqslant t_0 \leqslant t \leqslant t_1 < \infty$, $f(t)$ is a continuous function in $[0,\infty)$. Consequently, $f(t)$ is bounded in the interval $[0,\varepsilon]$, $0 < \varepsilon < \infty$. How such an integral can be calculated, see for example [25]. But we will find an analytic form for f which is, in our opinion, more suitable then integral (3.57) (cf. [59]).

Let us analyze the function f defined by (3.57). Put $c = \frac{1}{\mu}(d - a/\mu)$. Then

$$\begin{aligned}\frac{1}{\Delta(s)} &= \frac{1/\mu}{(s^2 + a/\mu)(s^\alpha + 1/\mu) + c}\\ &= \frac{1/\mu}{(s^2 + a/\mu)(s^\alpha + 1/\mu)} \times \left(1 + \sum_{\nu=1}^{\infty}(-c)^\nu\left(\frac{1}{s^2 + a/\mu}\right)^\nu\left(\frac{1}{s^\alpha + 1/\mu}\right)^\nu\right).\end{aligned}$$

First we find the function $\phi_\alpha(t)$, $t \geqslant 0$, such that

$$(\mathcal{L}\phi_a)(s) = \sum_{\nu=1}^{\infty}(-c)^\nu\left(\frac{1}{s^2 + a/\mu}\right)^\nu\left(\frac{1}{s^\alpha + 1/\mu}\right)^\nu. \tag{3.58}$$

Then,

$$\frac{1}{\Delta(s)} = \frac{1}{\mu}\frac{1}{(s^2 + a/\mu)(s^\alpha + 1/\mu)}\left(1 + (\mathcal{L}\phi_a)(s)\right).$$

We will denote by $\omega(t)$ the function

$$\omega(t) = \alpha t^{\alpha-1}E_\alpha^{(1)}(z), \tag{3.59}$$

where $z = -t^\alpha/\mu$, $t \geqslant 0$ and $E_\alpha(z)$ is Mittag-Leffler's function (see [22] and [26]). We know that $(\mathcal{L}\omega)(s) = (s^\alpha + 1/\mu)^{-1}$ (cf. [25]). In our analysis of the terms of the series (3.58), we have to distinguish three cases: $a > 0$, $a = 0$ and $a < 0$. Thus,

$$\left(\frac{1}{s^2 + a/\mu}\right)^\nu \left(\frac{1}{s^\alpha + 1/\mu}\right) = \begin{cases} \left(\sqrt{\frac{\mu}{a}}\right)^\nu \mathcal{L}\left(\left(\sin\sqrt{\frac{a}{\mu}}t * \omega(t)\right)^{*\nu}\right)(s), & a > 0 \\ \mathcal{L}((t * \omega(t))^{*\nu})(s), & a = 0 \\ \left(\sqrt{\frac{\mu}{-a}}\right)^\nu \mathcal{L}\left(\left(\sinh\sqrt{\frac{-a}{\mu}}t * \omega(t)\right)^{*\nu}\right)(s)(s), & a < 0, \end{cases}$$

where $f^{*\nu}$ means ν-fold convolution of f. We have to evaluate the obtained convolutions. First for the function ω given by (3.59) we need some properties of the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Namely, $E_\alpha(z)$ is an entire function with the properties:

$$E_\alpha(z) = \frac{-1}{\Gamma(1-\alpha)}\frac{1}{z} + O(|z|^{-2}), \quad |\arg(-z)| < (1 - \alpha/2)\pi, \quad z \to \infty,$$

$$E_\alpha^{(1)}(z) = \sum_{k=1}^{\infty} \frac{kz^{k-1}}{\Gamma(\alpha k + 1)}.$$

By [13, p. 36],

$$E_\alpha^{(1)}(z) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{z^2} + O(|z|^{-3}), \quad |\arg(-z)| < (1 - 3\alpha/4)\pi, \quad z \to \infty.$$

Consequently,

$$\omega(t) \sim \frac{\alpha}{\Gamma(1+\alpha)} t^{\alpha-1} = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \quad t \to 0;$$

$$\omega(t) \sim \frac{\alpha}{\Gamma(1-\alpha)} t^{\alpha-1} \mu^2 \frac{1}{t^{2\alpha}} = -\frac{\mu^2}{\Gamma(-\alpha)} t^{-(1+\alpha)}, \quad t \to \infty,$$

and

$$\omega(t) \sim O(t^{\alpha-1}), \quad t \to 0;$$
$$\omega(t) \sim O(t^{\alpha-1}), \quad t \to \infty.$$

Then, there exists a constant C_1 such that $|\omega(t)| \leqslant C_1 t^{\alpha-1}$, $0 < t < \infty$. Now, we can estimate the terms in the series

$$\phi_a(t) = \sum_{\nu=1}^{\infty} (-c)^\nu \left(\sqrt{\mu/a}\right)^\nu \left(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\right)^{*\nu}(t), \quad t \geqslant 0, \tag{3.60}$$

in our three cases: $a > 0$, $a = 0$, and $a < 0$. Let us start with $a > 0$. If $\nu = 1$, $a > 0$:

$$\left|\left(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\right)(t)\right| \leqslant C_1 \int_0^t \tau^{\alpha-1} d\tau = \frac{C_1}{\alpha} t^\alpha = C_2 \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad t \geqslant 0.$$

For any $\nu \in \mathbb{N}$ and $t \geqslant 0$,

$$\left|\left(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\right)^{*\nu}(t)\right| \leqslant C_2^\nu\left(\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)^{*\nu}(t)$$
$$\leqslant C_2^\nu \mathcal{L}^{-1}\frac{1}{s^{(\alpha+1)\nu}} \leqslant C_2^\nu \frac{t^{(\alpha+1)\nu-1}}{\Gamma(\nu(\alpha+1))}.$$

Let us set $F_\nu(t) = (-c)^\nu\left(\sqrt{\mu/a}\right)^\nu\left(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\right)^{*\nu}$, $\nu = 1, 2, \ldots$. Then for $t \geqslant 0$,

$$\sum_{\nu=1}^{\infty}|F_\nu(t)| \leqslant t^{-1}\sum_{\nu=1}^{\infty}\left(|C|\sqrt{\mu/a}\,t^{\alpha+1}\right)^\nu\frac{1}{\Gamma(\nu\alpha+1)} \leqslant t^{-1}\left(E_\alpha\left(|C|\sqrt{\mu/a}\,t^{\alpha+1}\right)-1\right).$$

Hence, the series (3.60) is absolute convergent for $t \geqslant 0$ and $\phi_\alpha(t)$ is bounded on every compact set $K \subset [0,\infty)$.

Now we have the following properties of F_ν, $\nu \in \mathbb{N}$:

(1) $\mathcal{L}(F_\nu)(s) = (-c)^\nu\left(\frac{1}{s^2+a/\mu}\right)^\nu\left(\frac{1}{s^\alpha+1/\mu}\right)^\nu$;

(2) $\int\limits_0^\infty e^{-x_0 t}|F_\nu(t)|dt \leqslant |c|^\nu\left(\sqrt{\mu/a}\right)^\nu C_2^\nu\frac{1}{x_0^{\nu(\alpha+1)}}$, $\nu = 1, 2, \ldots$

(3) The series $\sum\limits_{\nu=1}^{\infty}\int\limits_0^\infty e^{-x_0 t}|F_\nu(t)|dt \leqslant \sum\limits_{\nu=1}^{\infty}\left(\frac{|c|\sqrt{\mu/a}\,C_2}{x_0^{\alpha+1}}\right)^\nu$ converges, if $x_0^{\alpha+1} > |c|\sqrt{\frac{\mu}{a}}C_2$.

By Theorem 2, in [19, Vol. 1, p. 305], $\mathcal{L}(\phi_a)(x) = \widehat{\phi}_a(s)$, $a > 0$, with $\widehat{\phi}_a(s)$ given by (3.58).

In the other two cases the procedure is just the same. We have only to use the following evaluations, for $\nu \in \mathbb{N}$ and $t \geqslant 0$:

$$|(\tau * \omega(\tau))^{*\nu}(t)| \leqslant (\tau * (C_1\tau^{\alpha-1}))^{*\nu}(t) = C_2^\nu\left(\tau * \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\right)^{*\nu}(t)$$
$$\leqslant C_2^\nu\left(\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}\right)^{*\nu}(t) = C_2^\nu\frac{t^{(\alpha+2)\nu-1}}{\Gamma(\nu(\alpha+2))},$$

and

$$\left|\left(\sin h\sqrt{-a/\mu}\,\tau\right)^{*\nu}(t)\right| \leqslant \frac{t^{\nu-1}}{\Gamma(\nu)}e^{\sqrt{-a/\mu}\,t}, \quad t \geqslant 0.$$

Now, the function $\hat{f}(s)$ in (3.56) is:

$$\hat{f}(s) = \frac{s^\alpha + 1/\mu}{\Delta(s)} = \frac{1}{\mu}\frac{1}{s^2+a/\mu}(1+\widehat{\phi}_a(s)),$$

where $\widehat{\phi}_a(s)$ is given by (3.58). Consequently for $t \geqslant 0$,

$$f(t) = (\mathcal{L}^{-1}\hat{f})(t) = \frac{1}{\mu}\left[\left(\mathcal{L}^{-1}\frac{1}{s^2+a/\mu}\right)(t) + \left(\left(\mathcal{L}^{-1}\frac{1}{s^2+a/\mu}\right) * \phi_a\right)(t)\right], \tag{3.61}$$

and

(3.62)
$$f^{(1)}(t) = \left(\mathcal{L}^{-1}s\hat{f}(s)\right)(t) = \frac{1}{\mu}\left[\left(\mathcal{L}^{-1}\frac{s}{s^2+a/\mu}\right)(t) + \left(\left(\mathcal{L}^{-1}\frac{s}{s^2+a/\mu}\right) * \phi_a\right)(t)\right],$$

where

$$\phi_a(t) = \sum_{\nu=1}^{\infty}(-c)^\nu\Big(\Big(\mathcal{L}^{-1}\frac{1}{s^2+a/\mu}\Big)^{*\nu} * \omega^{*\nu}\Big)(t).$$

Note that in all three case: $a > 0$, $a = 0$ and $a < 0$, we have $f(0) = 0$. Hence, $s\hat{f}(s) = (\mathcal{L}f^{(1)})(s)$. Also

$$\frac{1}{\Delta(s)} = \frac{1}{s^\alpha + 1/\mu}\hat{f}(s) = \mathcal{L}(\omega * f)(s), \quad \frac{s}{\Delta(s)} = \mathcal{L}(\omega * f^{(1)})(s).$$

Now we can fix the form of the solution to (3.53), for $t \geqslant 0$,

$$\begin{aligned} T(t) &= T_0^1\mu f(t) = T_0\mu f^{(1)}(t) + \pi^2 AT(t_0)\mu\theta(t-t_0)f(t-t_0); \\ -V(t) &= T_0^1\mu_1\Big(f(t) + \Big(\frac{1}{\mu_1} - \frac{1}{\mu}\Big)(\omega * f)(t)\Big) \\ &\quad + T_0\mu_1\Big(f^{(1)}(t) + \Big(\frac{1}{\mu_1} - \frac{1}{\mu}\Big)(\omega * f^{(1)})(t)\Big) \\ &\quad + \pi^2 AT(t_0)\mu_1\theta(t-t_0)\Big(f(t-t_0) + \Big(\frac{1}{\mu_1} - \frac{1}{\mu}\Big)(\omega * f)(t-t_0)\Big) \end{aligned} \tag{3.63}$$

where $f(t)$ and $f^{(1)}(t)$ are given by (3.61) and (3.62). To analyze the character of the solution (3.63) we will find the first and second derivatives of f. By (3.62) $f^{(1)}(t)$, $f^{(2)}(t)$ and $f^{(3)}(t)$ belong to $C_{[0,\infty)}$.

From the properties of the generalized Laplace transform it follows that (3.63) is the unique solution in $L_{\mathrm{loc}}([0,\infty))$ such that T and V are bounded in $[0,\varepsilon]$ for $\varepsilon > 0$.

We state now the main results of this section:

Theorem 3.5. *A solution to* (3.53) *is given by* (3.63). *This solution is continuous on* $[0,\infty)$. *If* $A = 0$, *then the solution belongs to* $C^2_{[0,\infty)}$ *and is a classical one; it can be obtained by the classical Laplace transform. In the general case the functions* $T(t)$ *and* $V(t)$ *define regular distributions* $[T(t)]$ *and* $[V(t)]$ *which are solutions to* (??) *and generalized solution to* (3.53).

Remark. The continuity of T and V follows from the fact that $f(0) = 0$.

Theorem 3.6. *A family of solutions to* (3.50) *and* (3.51) *is*

$$m_k(\xi,t) = M_k(\xi)V_k(t), \quad u_k(\xi,t) = U_k(\xi)T_k(t), \quad k \in \pm\mathbb{N},$$

where M_k *and* U_k *are given by* (3.52), $k \in \mathbb{N}$, *and* V_k *and* T_k *are given by* (3.63) *when instead of* π *we take* πk.

3.6. Localization of the solution. The mathematical model of lateral vibration of a viscoelastic axially loaded rod (cf. Section 2, (2.3.12)) is

$$\begin{aligned} &\frac{\partial^2 m}{\partial \xi^2} + \lambda\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial t^2} = 0; \\ &\frac{\partial^2 u}{\partial \xi^2} + \mu_1\, D_t^\alpha\frac{\partial^2 u}{\partial \xi^2} + \mu_2 D_t^\beta\,\frac{\partial^2 u}{\partial \xi^2} = m + \mu\, D_t^\alpha m, \quad 0 < t,\ 0 < \xi < 1, \end{aligned} \tag{3.64}$$

with the boundary conditions:

$$m(0,t)=0;\ \ m(1,t)=0;\ \ u(0,t)=0,\ \ u(1,t)=0,\ \ t\geqslant 0. \tag{3.65}$$

We consider the vibrations of the rod when it is loaded by a compressive axial force F such that the intensity λ of the force F is $\lambda = B + A\theta(t-t_0)$, $t_0>0$, where θ is Heaviside's function and A, B are constants.

To stress possibilities of the Laplace transform of generalized functions (cf. Section 1, Subsections 1.5 and 1.2) we consider more general system which can appear as a model of an other situation, as well, namely:

$$\begin{aligned}&\frac{\partial^2 m}{\partial\xi^2}+\lambda\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial t^2}=g(t)\sin k\pi\xi,\ \ k\in\mathbb{N},\\ &\frac{\partial^2 u}{\partial\xi^2}+\mu_1 D_t^{\alpha}\frac{\partial^2 u}{\partial\xi^2}+\mu_2 D_t^{\beta}\frac{\partial^2 u}{\partial\xi^2}=m+\mu D_t^{\alpha}m,\end{aligned} \tag{3.66}$$

$0<t$, $0<\xi<1$, with the same boundary conditions (3.65), where $g\in\mathcal{C}([0,\infty))$ and without any growth condition. In case $g=0$ system (3.66) becomes (3.64).

Let us remark that in system (3.66) we have a coefficient which is a discontinuous function with a discontinuity in $t=t_0>0$. Since the product of a discontinuous function and a generalized function, e.g., of a distribution and a hyperfunction, is not defined, we can not to expect such a generalized solution to (3.66). So we have to localize the procedure of the construction of the solutions to (3.66). Therefore, we construct a solution for the domain $D_1=\{(\xi,t);\ 0<\xi<1,\ 0<t<t_0\}$ with boundary conditions (3.65) and initial conditions in $t=0$ and then for the domain $D_2=\{(\xi,t);\ 0<\xi<1,\ t_0<t\}$ using the Laplace transform presented in Section 1, Subsection 1.1.5. At the end we try to find a "global" solution to (3.66).

We start with the separation of variables.

Let us suppose that the solutions of the system (3.66), (3.65) have the from

$$m(\xi,t)=M(\xi)V(t),\ u(\xi,t)=U(\xi)T(t).$$

It is easily seen that for M and U, which satisfy the boundary conditions from (5.2), we have a family of solutions:

$$M_k(\xi)=C_k\sin k\pi\xi;\ \ U_k(\xi)=C_k\sin k\pi\xi,\ \ k\in\mathbb{N}.$$

In order to find the corresponding values T_k and V_k we have to solve the system:

$$\begin{aligned}&T_k^{(2)}(t)-\lambda(k\pi)^2T_k(t)-(k\pi)^2V_k(t)=g(t);\\ &V_k(t)+\mu V_k^{(\alpha)}(t)+(k\pi)^2T_k(t)+\mu_1(k\pi)^2T_k^{(\alpha)}+\mu_2(k\pi)^2T_k^{(\beta)}(t)=0,\ \ 0<t.\end{aligned} \tag{3.67}$$

We start with the domain D_1. Then we analyze system (3.67) in the interval $(0,t_0)$ with initial condition in $t=0$ and with $\lambda=B$. In this case to (3.67) it corresponds in $\mathcal{D}'_\omega([0,t_0))$ (cf. 8.1 Section 1, Subsection 1.1.2):

$$\begin{aligned}&D^2[H_0T_k]-B(k\pi)^2[H_0T_k]-(k\pi)^2[H_0V_k]=[H_0g]+T_{k0}\delta^{(1)}(t)+T_{k0}^1\delta(t),\\ &[H_0V_k]+\mu D^{\alpha}[H_0V_k]+(k\pi)^2[H_0T_k]+\mu_1(k\pi)^2D^{\alpha}[H_0T_k]+\mu_2(k\pi)^2D^{\beta}[H_0T_k]=0,\end{aligned} \tag{3.68}$$

where $T_{k0} = T_k(0)$, $T^1_{k0} = T^{(1)}_k(0)$. Applying the LT to (3.68) we get

$$(3.69)\qquad \begin{aligned} &(s^2 - B(k\pi)^2)\widehat{\widehat{T}}_k(s) - (k\pi)^2\widehat{V}_k(s) = \widehat{g}(s) + T_{k0}s + T^1_{k0} + \widehat{r}_1(s);\\ &(1+\mu s^\alpha)\widehat{V}_k(s) + (k\pi)^2(1+\mu_1 s^\alpha + \mu_2 s^\beta)\widehat{\widehat{T}}_k(s) = \widehat{r}_2(s), \end{aligned}$$

where $r_1, r_2 \in \mathcal{A}$. For simplicity we solve system (3.68) for $k = 1$. Let

$$\begin{aligned} \Delta_{10}(s) &= \begin{vmatrix} s^2 - B\pi^2 & -\pi^2 \\ \pi^2(1+\mu_1 s^\alpha + \mu_2 s^\beta) & (1+\mu s^\alpha) \end{vmatrix} \\ &= \mu s^{2+\alpha} + s^2 + \pi^2(\mu_1\pi^2 - B\mu)s^\alpha + \pi^4\mu_2 s^\beta + \pi^2(\pi^2 - B) \\ &= \mu s^{2+\alpha} + s^2 + a s^\alpha + b s^\beta + d, \end{aligned}$$

where $a = \pi^2(\mu_1\pi^2 - B\mu)$, $b = \pi^4\mu_2$, $d = \pi^2(\pi^2 - B)$,

$$\begin{aligned} \Delta_{11}(s) &= \begin{vmatrix} T_{10}s + T^1_{10} + \widehat{g}(s) + \widehat{r}_1(s) & -\pi^2 \\ \widehat{r}_2(s) & (1+\mu s^\alpha) \end{vmatrix} \\ &= \mu(s^\alpha + 1/\mu)(T_{10}s + T^1_{10} + \widehat{g}(s) + \widehat{r}_1(s)) + \pi^2\widehat{r}_2(s), \end{aligned}$$

$$\begin{aligned} \Delta_{12}(s) &= \begin{vmatrix} s^2 - B\pi^2 & T_{10}s + T^1_{10} + \widehat{g}(s) + \widehat{r}_1(s) \\ \pi^2(1+\mu_1 s^\alpha + \mu_2 s^\beta) & \widehat{r}_2 s \end{vmatrix} \\ &= -\pi^2(T_{10}\mu_2 s^{1+\beta} + T_{10}\mu_1 s^{1+\alpha} + T_{10}s + T^1_{10}\mu_1 s^\alpha + T^1_{10}\mu_2 s^\beta + T^1_{10}) \\ &\quad - \pi^4(1+\mu_1 s^\alpha + \mu_2 s^\beta)\widehat{g}(s) - \pi^2(1+\mu_1 s^\alpha + \mu_2 s^\beta)\widehat{r}_1(s) + (s^2 - B\pi^2)\widehat{r}_2(s). \end{aligned}$$

If in Δ_{10}, Δ_{11} and Δ_{12} we replace π with $k\pi$, then we have Δ_{k0}, Δ_{k1} and Δ_{k2} respectively. The solutions $\widehat{\widehat{T}}_k(s)$, $\widehat{V}_k(s)$, $k \in \mathbb{N}$ to system (3.69) are

$$\widehat{\widehat{T}}_k(s) = \frac{\Delta_{k1}(s)}{\Delta_{k0}(s)};\quad \widehat{V}_k(s) = \frac{\Delta_{k2}(s)}{\Delta_{k0}(s)}.$$

Suppose that $\Delta_{k0}(s) \neq 0$, $\operatorname{Re} s > x^0_k > 0$, $k \in \mathbb{N}$. Let us introduce the new variable $\zeta_k = s - x^0_k$ in $\Delta_{ki}(s)/\Delta_{k0}(s)$, $i = 1, 2$,

$$\frac{\Delta_{ki}(s)}{\Delta_{k0}(s)} = \frac{\Delta_{ki}(\zeta_k + x^0_k)}{\Delta_{k0}(\zeta_k + x^0_k)} \equiv Q_{ki}(\zeta_k),\ i = 1, 2,\quad k \in \mathbb{N}.$$

Now the functions $Q_{ki}(\zeta_k)$ are holomorphic on $\mathbb{R}_+ + i\mathbb{R}$ and belong to the space $\mathcal{H}(\mathbb{R}_+)$. Hence, there exist $q_{ki} \in \mathcal{S}'(\overline{\mathbb{R}}_+)$ such that

$$\langle q_{ki}(t), e^{-\zeta_k t}\rangle = Q_{ki}(\zeta_k),\ i = 1, 2,\quad k \in \mathbb{N},$$

or

$$\langle q_{ki}(t), e^{-(s - x^0_k)t}\rangle = \frac{\Delta_{ki}(s)}{\Delta_{k0}(s)},\ \operatorname{Re} s > x^0_k,\ i = 1, 2.\ k \in \mathbb{N}.$$

Hence, a solution to the system (3.68) for a fixed $k \in \mathbb{N}$ is:

$$\begin{aligned} T^0_k(t) &= e^{x^0_k t} q_{k1}(t)|_{[0,b)};\\ V^0_k(t) &= e^{x^0_k t} q_{k2}(t)|_{[0,b)}. \end{aligned}$$

Note that T_k^0 and V_k^0 belong to $\mathcal{D}'([0,b))$ for every b, $0 < b < \infty$ (cf. Section 1, Subsection 1.1.5).

By the similar method as we applied in [59] we can prove that T_k^0 and V_k^0 are regular distributions defined by T_k and V_k which have the following properties for $k \in \mathbb{N}$:

(1) $T_k \in \mathcal{C}^2((0,t_0]) \cap \mathcal{C}^1([0,t_0])$, $T_k^{(2)} \in \mathrm{L}^1([0,t_0]) \cap \mathcal{C}((0,t_0])$; $T_k^{(2)}(t)$ is not bounded in $t=0$, $\lim\limits_{t\to 0+} T_k(t) = T_{k0}$, $k \in \mathbb{N}$.

(2) $V_k \in \mathrm{L}^1([0,t_0]) \cap \mathcal{C}((0,t_0])$ and $V_k(t)$ is not bounded at $t=0$, $V_k(t) = O(t^{-(\beta-\alpha)})$, $k \in \mathbb{N}$ but it satisfies Proposition 1.1 in Section 1.

If additionally $T_k(0) = 0$, then $T_k \in \mathcal{C}^2([0,t_0])$ and $V_k \in \mathcal{C}([0,t_0])$, $V_k^{(1)} \in \mathrm{L}^1([0,t_0]) \cap \mathcal{C}((0,t_0))$.

Consequently, by our definition of the classical solution and generalized solution, in D_1 we have a classical solution to (3.66). The functions T_k and V_k, $k \in \mathbb{N}$, which satisfy (3.67) in $(0,t_0)$ are

$$\begin{aligned} T_k(t) &= T_k(0)\big(\mu F_{\alpha+1}(t) + F_1(t)\big) \\ &\quad + T_k^{(1)}(0)\big(\mu F_\alpha(t) + F_0(t)\big) + \big((\mu F_\alpha + F_0) * g\big)(t); \end{aligned} \tag{3.70}$$

and

$$\begin{aligned} V_k(t) = -(k\pi)^2 \big\{ & T_k(0)[\mu_2 F_{1+\beta}(t) + \mu_1 F_{1+\alpha}(t) + F_1(t)] \\ & + T_k^{(1)}(0)[\mu_1 F_\alpha(t) + \mu_2 F_\beta(t) + F_0(t)] \\ & + \big((\mu_1 F_\alpha + \mu_2 F_\beta + F_0) * g\big)(t)\big\}, \quad 0 < t < t_0, \end{aligned} \tag{3.71}$$

where $F_p(t) = \mathcal{L}^{-1}(s^p/\Delta_{k0}(s))(t)$, $0 \leqslant t < t_0$ (cf. [62]).

With regard to domain D_2 we have to find a solution to system (3.67) but in the interval (t_0,b) for any $b > t_0$, and $\lambda = B + A$. We proceed in the following way: First we have to localize the supposed solution to (3.67) on the interval (t_0,b). Then we suppose that there exists a solution T_k, V_k to (3.67) such that $H_{t_0}T_k \in \mathcal{C}^1([t_0,b))$, $(H_{t_0}T_k)^{(2)} \in \mathrm{L}^1([t_0,b))$; $V_k \in \mathcal{C}((t_0,b)) \cap \mathrm{L}^1([t_0,b))$.

By (1.1) and Proposition 1.1 in Section 1, to (3.67), on the interval $[t_0,b)$ it corresponds in $\mathcal{D}'_\omega([t_0,b))$

$$\begin{aligned} D^2[H_{t_0}T_k] - (A+B)(k\pi)^2[H_{t_0}T_k] - (k\pi)^2[H_{t_0}V_k] \\ = T_k(t_0)D^1\delta(t-t_0) + T_k^{(1)}(t_0)\delta(t-t_0) + [H_{t_0}g]; \end{aligned} \tag{3.72}$$

and

$$[H_{t_0}V_k] + \mu D^\alpha[H_{t_0}V_k] + (k\pi)^2[H_{t_0}T_k] + \mu_1(k\pi)^2 D^\alpha[H_{t_0}T_k] + \mu_2(k\pi)^2 D^\beta[H_{t_0}T_k] = 0.$$

Let

$$\begin{aligned} \overline{T}_k &\in e^{\omega t}\mathcal{S}'\big(\overline{\mathbb{R}}_+ + [t_0,b)\big) \text{ such that } \overline{T}_k|_{(-\infty,b)} = H_{t_0}T_k, \\ \overline{V}_k &\in e^{\omega t}\mathcal{S}'\big(\overline{\mathbb{R}}_+ + [t_0,b)\big) \text{ such that } \overline{V}_k|_{(-\infty,b)} = H_{t_0}V_k \\ \overline{g} &\in e^{\omega t}\mathcal{S}'\big(\overline{\mathbb{R}}_+ + [t_0,b)\big) \text{ such that } \overline{g}|_{(-\infty,b)} = H_{t_0}g. \end{aligned}$$

Applying to (3.72) the defined LT, we get

$$(s^2-(A+B)(k\pi)^2)\widehat{\overline{T}}_k(s)-(k\pi)^2\widehat{\overline{V}}_k(s) = T_k(t_0)se^{-t_0s}+T_k^{(1)}(t_0)e^{-t_0s}+\widehat{\overline{g}}(s)+\widehat{r}_1(s);$$

$$(1+\mu s^\alpha)\widehat{\overline{V}}_k(s)+(k\pi)^2(1+\mu_1 s^\alpha+\mu_2 s^\beta)\widehat{\overline{T}}_k(s) = \widehat{r}_2(s),$$

where r_1 and $r_2 \in \mathcal{A}$. By $\widehat{\overline{T}}_k$ is denoted the LT of $\overline{T}_k$.

When we solve this system in $\widehat{\overline{T}}_k$, $\widehat{\overline{V}}_k$ and use the inverse LT, we get

$$\begin{aligned}(H_{t_0}T_k)(t) &= T_k(t_0)\theta(t-t_0)\big(\mu G_{\alpha+1}(t-t_0)+G_1(t-t_0)\big)\\ &\quad+T_k^{(1)}(t_0)\theta(t-t_0)\big(\mu G_\alpha(t-t_0)+G_0(t-t_0)\big)\\ &\quad+\big((\mu G_\alpha+G_0)*H_{t_0}g\big)(t);\end{aligned} \tag{3.73}$$

$$\begin{aligned}&(H_{t_0}V_k)(t)\\ &=-(k\pi)^2\big\{T_k(t_0)\theta(t-t_0)[\mu_2 G_{1+\beta}(t-t_0)+\mu_1 G_{\alpha+1}(t-t_0)+G_1(t-t_0)]\\ &\qquad+T_k^{(1)}(t_0)\theta(t-t_0)[\mu_1 G_\alpha(t-t_0)+\mu_2 G_\beta(t-t_0)+G_0(t-t_0)]\\ &\qquad+[(\mu_1 G_\alpha+\mu_2 G_\beta+G_0)*(H_{t_0}g)](t)\big\},\quad t_0<t<b,\end{aligned} \tag{3.74}$$

where $G_p(t)=\mathcal{L}^{-1}(s^p/\Delta'_{k0})$, Δ'_{k0} equals Δ_{k0} in which instead of B we have $A+B$. Therefore, we can use the properties of solution (3.70), (3.71) to system (3.67) taking into account that we have $A+B$ instead of B.

We have now a solution for the domain D_1, given by (3.70), (3.71) and a solution for the domain D_2 given by (3.73), (3.74). The properties of $H_{t_0}T_k$ and $H_{t_0}V_k$ in $t=t_0$ follow by the properties of T_k and V_k in $t=0$.

Theorem 3.7. *If in the system* (3.66) *with the boundary condition* (3.65), $\lambda = B$ *and* $g \in C([0,b))$, *for any* $b>0$, *then we have the classical solutions in* $(0,1)\times(0,b)$ *for every* $b>0$. *These solutions are*

$$m_k(\xi,t)=C_k\sin k\pi\xi V_k(t),\quad u_k(\xi,t)=C_k\sin k\pi\xi V_k(t),\quad k\in\mathbf{N}, \tag{3.75}$$

where T_k, V_k *are of the form* (3.70), (3.71) *for* $0<t<b$. *In case* $\lambda=B+A\theta(t-t_0)$, $0<t_0$, $A\neq 0$, *there exist the regular distributions* $R_k, Q_k \in \mathcal{D}'((0,1)\times(0,b))$ *defined by the functions* $r_k(\xi,t)$ *and* $q_k(\xi,t)$ *respectively,* $k\in N$, *such that:*

(1) r_k *and* q_k *belong to* $C^\infty([0,1])\times L^1([0,b])$, $0<t_0<b<\infty$.
(2) *The restriction of* $r_k(\xi,t)$ *and* $q_k(\xi,t)$ *to* D_1 *are* $m_k(\xi,t)$ *and* $u_k(\xi,t)$ *given by* (3.75), *where* V_k *and* T_k *have been given by* (3.70) *and* (3.71);
(3) *The restriction of* $r_k(\xi,t)$ *and* $q_k(\xi,t)$ *to* D_2 *are the same functions* m_k *and* u_k *given by* (3.75) *in which instead of* T_k *and* V_k *we have* $H_{t_0}T_k$ *and* $H_{t_0}V_k$ *given by* (3.73), (3.74).

Proof. We have only to prove that two regular distributions defined on $D_1\cup D_2$ by $m_k(\xi,t)$ and $u_k(\xi,t)$ for a fixed $k\in\mathbf{N}$ can be extended to $(0,1)\times(0,b)$ for any b, $0<t_0<b<\infty$. By the properties of V_k and T_k, we cited it is easily seen that the condition of Proposition 1.6 in Section 1 is satisfied. Consequently such extension exists. □

Bibliography

[1] S.S. Antman and J.F. Pierce, *Global behavior of buckled states of compressible elastic rods*, SIAM J. Appl. Math. 50 (1990), 395–419.

[2] T.M. Atanacković, *Stability theory of elastic rods*, World Scientific, River Edge, N.J., 1997.

[3] T.M. Atanacković, *A modified Zener model of a viscoelastic body*, Continuum Mech. Thermodyn. 14 (2002), 137–148.

[4] T.M. Atanacković, B. Stanković, *Dynamics of a Viscoelastic rod of Fractional Derivative Type*, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM) 82 (2002), 377–386.

[5] T.M. Atanacković, *A model for the uniaxial isothermal deformation of a Viscoelastic body*, Acta Mechanica 159 (2002), 77–86.

[6] T.M. Atanacković and B. Stanković, *Stability of an Elastic rod on a Fractional derivative type of Foundation*, J. Sound Vibration (in press).

[7] T.M. Atanacković, *On a distributed derivative model of a viscoelastic body*, Comptes rendus de.l' Acadeèmie des Sciences – Mechanics 331 (2003) 687–692.

[8] T.M. Atanacković and B. Stanković, *On a system of differential equations with fractional derivatives arising in rod Theory*, J. Phys. A: Math. Gen. 37 (2004), 1241–1250.

[9] T.M. Atanacković and B. Stanković, *On the copmpressed elastic rod with rotary inertia on viscoelastic foundation*, Preprint.

[10] B.S. Bačlić and T.M. Atanacković, *Stability and creep of a fractional derivative order viscoelastic rod*, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur. 21 (2000), 115–131.

[11] R.L. Bagley and P.J. Torvik, *On the fractional calculus model of viscoelastic behavior*, Journal of Rheology 30 (1986), 133–155.

[12] H. Bass, *The Carnegie Initiative on the Doctorate: The case of Mathematics*, Notices AMS 50:7 (2003), 767–776.

[13] L. Berg, *Asymptotische Auffassung der Operatorenrechnung*, Studia Math. 21 (1962), 215–229.

[14] L. Berg, *Asymptotische Darstellungen und Entwicklungen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1968.

[15] N.N. Bogolubov, V.S. Vladimirov and A.N. Tavkelidze, *On automodel asymptotics in quantum field theory*, Proc. Steklov Inst. Math. Issue 1 (1978), 27–54.

[16] Yu. Brichkov, Yu. Shirokov, *On asymptotic behavior of the Fourier transform*, TMF 4:3 (1970), 301–309 (in Russian).

[17] J. Carr and Z.M. Malharden, *Beck's problem*, SIAM J. Appl. Math. 37:2 (1979), 261–262.

[18] J.F. Colombeau, *Elementary Introduction to New Generalized Functions*, North Holand, Amsterdam, 1985.

[19] R. Courant, D. Hilbert, *Methods of Mathematical Physics, Vol. I*, Interscience, New York, 1953.

[20] V.A. Ditkin, *On the theory of operational calculus*, Dokl. AN SSSR 123 (1958), 395–396 (in Russian).

[21] G. Doetsch, *Haudbuch der Laplace-Transformation, I and II*, Birkhäuser, Basel, 1950–1955.

[22] A. Erdélyi (Editor), *Higher Transcendental Functions*, McGraw-Hill, New York, 1955.

[23] R.A. Fenander, *A fractional derivative railpad model included in a railway track model*, J. Sound Vibration 212 (1998), 889–903.

[24] I.M. Gelfand, G.E. Shilov, *Generalized functions*, 1, 2, 3, Moscow, 1958 (in Russian).

[25] R. Gorenflo and F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, in: *Fractional Calculus in Continuum Mechanics*, (Ed–s: A. Carpinteri, F. Mainardi), Springer-Verlag, Wien, 1997, pp. 223–276.

[26] R. Gorenflo, Yu. Luchko, S. Rogozin, *Mittag-Leffler type functions: notes on growth properties and distributions of zeros*, Preprint A–04/97, Fachbereich Mathematik und Informatik, Freie Universität, Berlin (1997).

[27] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, I, II, Springer–Verlag, Berlin, 1983; (Mir, Moskva, 1986).

[28] W. T. Van Horssen and M. A. Zarubinskaya, *On and Elastic Dissipation Model for a Cantilevered Beam*, Quarterly Appl. Math. 61 (2003), 565–573.

[29] I. Imai, *Applied Hyperfunction theory*, Kluwer, Dordrecht, 1992.

[30] F. Jensen, *How to succeed in Modelling*, Qual. Reliab. Engng. Int. 15 (1999), 159.

[31] H. Komatsu, *Ultradistributions, I*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 20 (1973), 23–105.

[32] H. Komatsu, *An introduction to the theory of hyperfunctions*, Lecture Notes in Mathematics 287, Springer–Verlag, Berlin, 1973.

[33] H. Komatsu, *Laplace transforms of hyperfunctions – a new foundation of the Heviside calculus*, J. Fac. Sci., Univ. Tokyo, Sec. 1A, 34 (1987), 805–820.

[34] H. Komatsu, *Solution of differential equations, by means of Laplace hyperfunctions*, structure of solutions of differential equations, Katata/Kyoto, 1995, World Sciientific, River Edge, N.J. 1996, pp. 227–252.

[35] H. Komatsu, *Operational calculus, hyperfunctions and ultradistributions*, Algebraic Analysis 1, Academic Press, 1988, 357–372.

[36] H. Komatsu, *Operational calculus and semi–groups of operators*, Lecture Notes in Math. 1540 (1993), 213–234.

[37] F. K. Labisch, *On shearable nonlinearly elastic rings*, Int. J. Eng. Sciences 30 (1992), 101–117.

[38] G.-G. Li, Z.-Y. Zhu and C.-J. Cheng, *Dynamical stability of viscoelastic column with fractional derivative constitutive relation*, Appl. Math. Mech. 22 (2001), 294–303.

[39] A. Magnusson, M. Ristinamaa and C. Ljung, *Behavior of the extensible elastica solution*, Int. J. Solids Structures 38 (2001), 8441–8457.

[40] J. Mikusiński, *Sur les fondements du calcul opératoire*, Studia Math. 11 (1995), 41–70.

[41] J. Mikusiński, *Operational Calculus, I, II*, Pergamon Press, 1983.

[42] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York, 1974.

[43] N. Ortner, *Construction of Fundamental Solutions*, in: *Topics in Boundary Element Research*, Ed. C. A. Brebbia, 1987.

[44] K. D. Papoulia and J. M. Kelly, *Visco-Hyperelastic Model for Filled Rubbers Used in Vibration Isolation*, ASME J. Eng. Materials and Technology, 119 (1997), 292–297.

[45] A. Pflüger, *Stabilitätsprobleme der Elastistatik*, Springer-Verlag, Berlin, 1964.

[46] T. Pirtz, *Five-parameter fractional derivative model for polymeric damping materials*, J. Sound Vibration 265 (2003), 935–952.

[47] E. E. Rosinger, *Non-Linear Partial Differential Equations, An Algebraic View of Generalized Solutions*, North-Holland, Amsterdam, 1990.

[48] Yu. Rossikhin, and M. V. Shitikova, *Analysis of Dynamic Behavior of Viscoelastic rods Whose Rheological Models Contain Fractional Derivatives of Two Different Orders*, Z. Angew. Math. (ZAMM) 81 (2001), 363–376.

[49] D. L. Russell, *On the positive root of the fourth derivative operator*, Quarterly Appl. Math. 16 (1988), 751–753.

[50] D. L. Russell, *A comparison of certain elastic dissipation mechanisms via decoupling and projection techniques*, Quarterly Appl. Math. 19 (2001), 374–396.

[51] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach, Amsterdam, 1993.

[52] M. Sato, *Theory of Hyperfunctions*, Sugaku 10 (1958), 1–27, (In Japanese).

[53] M. Sato, *Theory of hyperfunctions*, J. Fac. Sci. Univ. Tokyo, Sec. I 8 (1959), 139–193.

[54] A. Schmidt and L. Gaul, *Implementation von Stoffgesetzen mit fraktionalen Ableitungen in die Finite Elemente Methode*, Z. Angewandte Math. Mech. (ZAMM) 83 (2003), 26–37.

[55] A. P. Seyranian, *Bifurcations of Eigenvalues and Stability Problems in Mechanics*, in: *Modern Problems of Structural Stability*, Springer–Verlag, 2002.

[56] L. Schwartz, *Théorie des distributions*, Hermann, Paris, I, II, 1951.

[57] S. L. Soboleff, *Méthode nonvelle à resondre le problème de Cauchy pour les équations linéaires hyperbolic normales*, Mat. Sb. 1 (43) (1936), 39–72.

[58] B. Stanković, *Hyperfunctions*, Zbornik radova 7(15), Matematički Institut SANU, Beograd (1997), 71–110.

[59] B. Stanković, T. M. Atanacković, *On a model of a viscoelastic rod*, Fractional calculus and Applied Analysis 4:4 (2001), 502–522.

[60] B. Stanković, *Solution of mathematical models by localization*, Bul. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur. 27 (2002), 1–17.

[61] B. Stanković, *Solutions to a partial differential equation that appear in Mechanics*, Bull. Acad. Serbe Sci. Arts, Sci. Math. 28 (2003), 41–60.

[62] B. Stanković, T. M. Atanacković, *On a viscoelastic rod with constitutive equation containing fractional derivatives of two different orders*, Mathematics and Mechanics of solids (to appear).

[63] B. Stanković and T. M. Atanacković, *On the copmpressed elastic rod with rotary inertia and varying compressive force*, Preprint.

[64] Z. Szmydt, *Fourier transformation and linear differential equations*, PWN – Polish Scientific publishers, Warszawa, 1977.

[65] V.S. Vladimirov, B.I. Zavjalov, *Tauberian theorems in quantum field theory*, Itogi Nauki i tehniki, Sovremenie problemi matematiki 25 (1980), 95–130 (in Russian).

[66] V.S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, Moscow, 1979.

[67] V.S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow, 1988 (in Russian).

Marko Nedeljkov and Stevan Pilipović

GENERALIZED FUNCTION ALGEBRAS AND PDEs WITH SINGULARITIES. A SURVEY

CONTENTS

Introduction

The aim of this survey article is to explain basic ideas of generalized function algebras (Part I), to illustrate the analysis of second order PDEs (Part II) and first order hyperbolic systems (Part III) with singular coefficients and singular data.

Part I deals with the basic definitions of Colombeau type spaces and algebras, while Parts II and III present main statements of our investigations on PDEs of quoted types. Especially, complete proofs on solving elliptic linear equations with singular coefficients and singular data are given.

Colombeau had constructed his well-known algebras by purely algebraic methods, [9, 10]. Since then, algebras of Colombeau generalized numbers and functions became a very useful framework for linear problems with singularities and especially for nonlinear problems. Here we refer to monographs [3, 11, 29, 31, 68, 72] for the so-called Colombeau approach and for another approach, we refer to monographs [87, 88].

Many linear and nonlinear problems with irregular data or irregular coefficients, have been successfully analyzed by the mean of appropriate approximations through nets of C^∞ functions which fits into Colombeau algebra $\mathcal{G}$ of generalized functions. We extend the references of this article in order to emphasize a part of large literature related to linear and nonlinear equations in the framework of generalized function algebras.

For the general theory of Colombeau type generalized functions, beside the quoted monographs, we refer to [29, 30, 46, 48, 49, 50] for generalized functions on manifolds, to [4, 5, 17, 18, 21, 62, 83, 98] for embeddings of different function spaces and to [19, 24, 25, 63, 89, 90] for the topology in Colombeau generalized function algebras and spaces.

Linear equations and pseudodiffierential operators were studied in [23, 34, 35, 52, 66, 82], while local and microlocal analysis within Colombeau type algebras were studied in [37, 38, 55, 56, 76, 78, 91].

Concerning nonlinear equations, not mentioned in this paper, one can see also [3, 15, 20, 27, 28, 36, 47, 64, 92].

We give in Part I a general concept of extending the theory of locally convex spaces and algebras to the corresponding Colombeau type spaces and algebras. Since we are mainly interested in equations, we give several constructions which are related to Sobolev and Hölder spaces of functions. Also, for the later use of a class of stochastic equations, we recall the definition of a generalized stochastic process. In the last subsection of Part II, we introduce generalized semigroups of operators which will be used in the analysis of a semilinear heat equation with singularities.

We present in Part II, Subsection 1.1 our method of solving various classes of second order elliptic linear differential equations with strong singularities in the framework of generalized function algebras. We start with the Whitney type definitions of generalized functions on closed sets. Then, with all the details, we construct and discuss corresponding boundary value problems in relation to the classical results. The results of this subsection are not published before.

A quasilinear Dirichlet problem for uniformly elliptic equations whose coefficients have the lack of regularity assumptions and with singular boundary conditions is considered in Subsection 1.2 [84]. In our setting of a problem, we replace an equation $\operatorname{div} \mathbf{A}(Du) = 0$ with a net of equations with regular coefficients and a singular boundary condition with an appropriate regularized net of boundary conditions.

In Subsection 2.1 is considered a semilinear wave equations [60] in space dimension $n \leqslant 9$ with singular data and various types of nonlinearities. In general, a nonlinear term is regularized with respect to a small parameter ε such that it becomes globally Lipschitz for each ε. A net of solutions to a net of Cauchy problems obtained in this way determines a generalized solution. For certain growth conditions on a nonlinear term the equation is uniquely solved without regularizations. Note, in certain cases, a solution to the regularized equation is also a solution to the non-regularized one.

Subsection 2.2 deals with some classes of wave equations with stochastic perturbations as singularities.

We study in Section 3 [67] a class of heat equations with singularities extending the use of semigroups to some classes of PDE's with singular coefficients. The general idea is simple and it lies in the core of a construction of generalized functions. Regularized PDE, in fact a net of equations, is solved with an appropriate net of semigroups. The solution obtained in this way will represent a generalized function. For this reason, we will use different variants of Colombeau-like generalized function algebras. By the use of semigroups related to the Schrödinger operator $\triangle - V$ we present the results concerning the semilinear heat equation with singular Cauchy data.

Part III is devoted to solving a class of Riemann problems to one-dimensional 2×2 conservation law systems which do not always admit a classical entropy solution consisting of elementary waves. Two new solution types, delta and singular shock waves, could appear in such a situation. We shall use two solutions concepts for describing them.

Also, we shall briefly describe what might happen when such a wave interacts with the wave of the same type or with some other elementary wave.

We present in the Appendix a very general construction of generalized function Colombeau type algebras through a purely topological description of such algebras. We will show that such algebras fit very well in the general theory of the well known sequence spaces forming appropriate algebras [17].

PART I: BASIC DEFINITIONS

1. Different algebras and spaces of generalized functions

1.1. Extensions over locally convex spaces and algebras. This subsection contains special constructions of Colombeau type algebras. One can find a more abstract general approach in [17] as well as in the Appendix.

Let E be a vector space on $\mathbb{C}$ with an increasing sequence of seminorms μ_n, $n \in \mathbb{N}$. The space of moderate nets of $\mathcal{E}_M(E)$, respectively, of null nets of $\mathcal{N}(E)$, is constituted by nets $R_\varepsilon \in E^{(0,1]}$ with the properties

$$(\forall n \in \mathbb{N})(\exists a \in \mathbb{R})(\mu_n(R_\varepsilon) = \mathcal{O}(\varepsilon^a)),$$

$$\text{respectively, } (\forall n \in \mathbb{N})(\forall b \in \mathbb{R})(\mu_n(R_\varepsilon) = \mathcal{O}(\varepsilon^b)).$$

($\mathcal{O}$ is the Landau symbol.) The quotient space $\mathcal{G}(E) = \mathcal{E}_M(E)/\mathcal{N}(E)$ with elements $[F_\varepsilon], [G_\varepsilon], \ldots,$ (equivalence classes are denoted by $[\cdot]$) is called the Colombeau extension of E. Putting $v_n(R_\varepsilon) = \sup\{a;\ \mu_n(R_\varepsilon) = \mathcal{O}(\varepsilon^a)\}$ and $e_n\big((R_\varepsilon)\varepsilon, (S_\varepsilon)\varepsilon\big) = \exp\big(-v_n(R_\varepsilon - S_\varepsilon)\big)$, $n \in \mathbb{N}$, we obtain that $(e_n)_n$ is a sequence of ultra-pseudometrics defining the ultra-metric topology (sharp topology) on $\mathcal{G}(E)$.

If $E = \mathbb{C}$ (or $E = \mathbb{R}$) and the seminorms are equal to the absolute value, then the corresponding spaces are $\mathcal{E}_0$, $\mathcal{N}_0$; $\mathcal{E}_0$ is an algebra and $\mathcal{N}_0$ is an ideal and, as a quotient, one obtains Colombeau algebra of generalized complex numbers $\bar{\mathbb{C}} = \mathcal{E}_0/\mathcal{N}_0$ (or $\bar{\mathbb{R}}$). These algebras are not fields as expected at the first sight, but rings. If a set O is open in $\mathbb{R}^n$ and $E = C^\infty(O)$ is endowed with the usual sequence of seminorms (this is Schwartz space $\mathcal{E}(O)$),

$$\mu_{K_\nu,\nu}(\phi) = \sup_{x \in K_\nu, |\alpha| \leqslant \nu} |\phi^{(\alpha)}(x)|,\ \nu \in \mathbb{N}_0,$$

where $(K_\nu)_\nu$ is an increasing sequence of compact sets that exhausts O, then the above definition gives Colombeau simplified algebra $\mathcal{G}(O) = \mathcal{E}_M(O)/\mathcal{N}(O)$ [9], [72]. Its elements are called generalized functions and we keep this name for elements of any spaces or algebras constructed as extensions of some functional space E.

Then the embedding of compactly supported Schwartz distributions (elements of $\mathcal{E}'(O)$) is made through the convolution with a net of mollifiers $\phi_\varepsilon = \varepsilon^{-n}\phi(\cdot/\varepsilon)$ constructed by a rapidly decreasing function $\phi \in \mathcal{S}(\mathbb{R}^n)$ with the properties $\int \phi(t)\,dt = 1$, $\int t^m \phi(t)\,dt = 0$, $m \in \mathbb{N}^n$. The embedding is given by $f \mapsto [f * \phi_\varepsilon|_O]$. By the sheaf properties of $\mathcal{D}'(O)$ and $\mathcal{G}(O)$, this embedding is extended to $\mathcal{D}'(O)$.

Besides a cited paper, one can also look in [9] for Colombeau's original approach.

1.2. Colombeau generalized functions with uniform bounds. In general, for an open set $O \subset \mathbb{R}^n$, denote by $C_b^\infty(O)$ the algebra of smooth functions on O bounded together with all their derivatives.

We shall briefly repeat some definitions of Colombeau algebra given in [77]. Denote $\mathbb{R}^2_+ := \mathbb{R} \times (0,\infty)$, $\overline{\mathbb{R}^2_+} := \mathbb{R} \times [0,\infty)$. Let $C_{\overline{b}}^\infty(\mathbb{R}^2_+)$ be a set of all functions $u \in C^\infty(\mathbb{R}^2_+)$ satisfying $u|_{\mathbb{R}\times(0,T)} \in C_b^\infty(\mathbb{R}\times(0,T))$ for every $T>0$. Let us remark that every element of $C_b^\infty(\mathbb{R}^2_+)$ has a smooth extension up to the line $\{t=0\}$, i.e., $C_b^\infty(\mathbb{R}^2_+) = C_b^\infty(\overline{\mathbb{R}^2_+})$. This is also true for $C_{\overline{b}}^\infty(\mathbb{R}^2_+)$.

We will give explicit definitions an one can easily make the description of these spaces and algebras as in Subsection 1.1. $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is the set of all maps $G : (0,1)\times\mathbb{R}^2_+ \to \mathbb{R}$, $(\varepsilon, x, t) \mapsto G_\varepsilon(x,t)$, $G_\varepsilon \in C_{\overline{b}}^\infty(\mathbb{R}^2_+)$ for every $\varepsilon \in (0,1)$ satisfying: for every $(\alpha,\beta) \in \mathbb{N}_0^2$ and $T>0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{-N}), \text{ as } \varepsilon \to 0.$$

$\mathcal{N}_g(\mathbb{R}^2_+)$ is the set of all $G_\varepsilon \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$ satisfying: for every $(\alpha,\beta) \in \mathbb{N}_0^2$, $a \in \mathbb{R}$ and $T>0$

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{a}), \text{ as } \varepsilon \to 0.$$

Clearly, $\mathcal{N}_g(\mathbb{R}^2_+)$ is an ideal of the multiplicative differential algebra $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$. Thus one defines the multiplicative differential algebra $\mathcal{G}_g(\mathbb{R}^2_+)$ of generalized functions by $\mathcal{G}_g(\mathbb{R}^2_+) = \mathcal{E}_{M,g}(\mathbb{R}^2_+)/\mathcal{N}_g(\mathbb{R}^2_+)$. All operations in $\mathcal{G}_g(\mathbb{R}^2_+)$ are defined by the corresponding ones in $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$.

If one uses $C_b^\infty(O)$ instead of $C_b^\infty(\mathbb{R}^2_+)$, for an open connected set $O \subset \mathbb{R}^n$, one obtains $\mathcal{E}_{M,g}(O)$, $\mathcal{N}_g(O)$ and consequently, the space of generalized functions on a real line, $\mathcal{G}_g(O)$.

Additionally, if functions from $\mathcal{E}_{M,g}(\mathbb{R})$ and $\mathcal{N}_g(\mathbb{R})$ are substituted with reals, one obtains the ring $\mathcal{E}_{M,0}$ and it ideal $\mathcal{N}_0$, respectively. Thus, the ring of generalized real numbers is defined by $\overline{\mathbb{R}} = \mathcal{E}_{M,0}/\mathcal{N}_0$.

In the sequel, G denotes an element (equivalence class) in $\mathcal{G}_g(O)$ defined by $G_\varepsilon \in \mathcal{E}_{M,g}(O)$.

Since $C_{\overline{b}}^\infty(\mathbb{R}^2_+) = C_{\overline{b}}^\infty(\overline{\mathbb{R}^2_+})$, a restriction of a generalized function to $\{t=0\}$ is defined in the following way. For given $G \in \mathcal{G}_g(\mathbb{R}^2_+)$, its restriction $G|_{t=0} \in \mathcal{G}_g(\mathbb{R})$ is the class determined by a function $G_\varepsilon(x,0) \in \mathcal{E}_{M,g}(\mathbb{R})$. In the same way as above, $G(x-ct) \in \mathcal{G}_g(\mathbb{R})$ is defined by $G_\varepsilon(x-ct) \in \mathcal{E}_{M,g}(\mathbb{R})$.

If $G \in \mathcal{G}_g$ and f is a smooth function polynomially bounded together with all its derivatives, then one can easily show that the composition $f(G)$, defined by a representative $f(G_\varepsilon)$, $G \in \mathcal{G}_g$ makes sense. It means that $f(G_\varepsilon) \in \mathcal{E}_{M,g}$ if $G_\varepsilon \in \mathcal{E}_{M,g}$, and $f(G_\varepsilon) - f(H_\varepsilon) \in \mathcal{N}_g$ if $G_\varepsilon - H_\varepsilon \in \mathcal{N}_g$.

The equality in the space of the generalized functions $\mathcal{G}_g$ is not appropriate for conservation laws as one can see in [72]. A generalized function $G \in \mathcal{G}_g(O)$ is said to be associated with $u \in \mathcal{D}'(O)$, $G \approx u$, if for some (and hence every) representative G_ε of G, $G_\varepsilon \to u$ in $\mathcal{D}'(O)$ as $\varepsilon \to 0$. Two generalized functions G and H are said to be associated, $G \approx H$, if $G - H \approx 0$. One can easily verify that the association is linear and an equivalence relation.

A generalized function $G \in \mathcal{G}_g(O)$ is pointwiselly non-negative if for every $x \in O$, $G(x) \geqslant 0$, i.e., there exists $Z_\varepsilon \in \mathcal{N}_0$ such that $G_\varepsilon(x) \geqslant Z_\varepsilon$, for ε small enough.

A generalized function $G \in \mathcal{G}_g(O)$ is distributionally non-negative if for every $\psi \in C_0^\infty(O)$, $\int_O G_\varepsilon(x)\psi(x) \geqslant 0$, for ε small enough.

Let $u \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Let $\mathcal{A}_0$ be the set of all functions $\phi \in \mathcal{D}(\mathbb{R})$ satisfying $\phi(x) \geqslant 0$, $x \in \mathbb{R}$, $\int \phi(x)\,dx = 1$ and $\operatorname{supp}\phi \subset [-1,1]$. Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $x \in \mathbb{R}$. Then

$$\iota_\phi : u \mapsto \text{ class of } u * \phi_\varepsilon$$

defines a mapping of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ into $\mathcal{G}_g(\mathbb{R})$. It is clear that ι_ϕ commutes with the derivation. Also, $\iota_\phi(\delta)$ is a class defined by a delta net ϕ_ε.

1.3. Generalized function algebras over Hölder spaces. Let O be a bounded open set in $\mathbb{R}^n$ and $\alpha \in (-1,1)$. Recall [26, p. 94], a domain O and its boundary are of class $C^{k,\alpha}$, for $0 \leqslant \alpha \leqslant 1$, if at each point $x_0 \in \partial O$ there is a ball $B = B_{x_0}$ and a bijection $\psi : B \to D$ such that $\psi(B \cap O) \subset \mathbb{R}^n_+$, $\psi(B \cap \partial O) \subset \partial\mathbb{R}^n_+$, and $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$. A domain O has a boundary portion $T \in \partial O$ of class $C^{k,\alpha}$ if at each point $x_0 \in T$ there is a ball B_{x_0} in which the above conditions are satisfied and $B \cap \partial O \subset T$.

We will consider the Colombeau extensions in cases $E = C^{k,\alpha}(\bar{O})$, $k \in \mathbb{N}$ and $E = C^\infty(\bar{O})$. We will use the norms

$$|f|_{k,O} = \sup\{|f^{(p)}(x)|;\, |p| \leqslant k, x \in O\},$$
$$|f|_{k,\alpha,O} = |f|_{k,O} + [f]_{k,\alpha,O},\ k \in \mathbb{N}_0,$$

where, for $f \in C^\infty(\bar{O})$, $k \in \mathbb{N}_0$,

$$[f]_{k,\alpha,O} = \sup\Big\{\frac{|f^{(p)}(x) - f^{(p)}(y)|}{|x-y|^\alpha};\ (x,y) \in O,\ x \neq y,\ |p| = k\Big\}.$$

The completion of $C^\infty(\bar{O})$ with respect to the norm $|\cdot|_{k,\alpha,O}$ defines $E_k = C^{k,\alpha}(\bar{O})$, $k \in \mathbb{N}$. Recall, if $k + \alpha < k' + \alpha'$, then the embedding of $C^{k,\alpha}(\bar{O})$ into $C^{k',\alpha'}(\bar{O})$ is a compact linear operator.

Note that the sequences of norms $\|\cdot\|_{k,\alpha}$, $k \in \mathbb{N}$ and $\|\cdot\|_k$, $k \in \mathbb{N}$ define the same uniform structure on $C^\infty(\bar{O})$ as the usual one.

In case $E = C^\infty(\bar{O})$, we need one more construction. Let G_ε be a net in $C^{0,\alpha}(\bar{O})$ such that

$$G_\varepsilon \in C^{k,\alpha}(\bar{O}), \quad \varepsilon < \varepsilon_k, \quad k \in \mathbb{N},$$

where ε_k strictly decreases to zero.

Two such nets are in relation, $G_\varepsilon \sim R_\varepsilon$, if

$$G_\varepsilon = R_\varepsilon, \quad \varepsilon < \varepsilon_0, \text{ for some } \varepsilon_0 \in (0,1).$$

This is an equivalence relation and with the corresponding classes, elements in $C^{0,\alpha}(\bar{O})/\sim$, we define spaces $\mathcal{E}_M[E]$, $\mathcal{N}[E]$. Then we define the corresponding Colombeau type space $\mathcal{G}[E] = \mathcal{E}_M[E]/\mathcal{N}[E]$. Note that there exists a canonical isomorphism of $\mathcal{G}[E]$ onto $\mathcal{G}(E)$ if $E = C^\infty(\bar{O})$. In case $E = C^{k,\alpha}(\bar{O})$, we have $\mathcal{G}(E) = \mathcal{G}[E]$.

1.4. Colombeau–Sobolev type spaces and algebras. Although we have introduced two general concepts of constructions of generalized function algebras, still the flexibility of Colombeau main idea enable us to construct some other types of spaces and algebras useful in the analysis of problems with singularities.

Let O be an open, connected subset of $\mathbb{R}^n$ with a smooth boundary. Let $H^{r,s}(O)$ be Sobolev space of functions in $L^s(O)$ with all distributional derivatives of order $|\alpha| \leqslant r$ belonging to $L^s(O)$, equipped with the usual norm. In case $s = 2$, we simply write $H^r(O)$. One can find Colombeau type algebras $\mathcal{G}_{L^p,L^q}$ in [7] and [60]. We shall describe the special case of the last one, $\mathcal{G}_{L^2,L^2}$ space, denoted by $\mathcal{G}_{2,2}$ in the present paper.

$\mathcal{E}_{2,2}([0,T)\times\mathbb{R}^n)$ is the algebra of all $G_\varepsilon \in \mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $T>0$ and $\alpha \in \mathbb{N}_0^n$ there exists $N \in \mathbb{N}$ such that

$$\|\partial^\alpha G_\varepsilon\|_{L^2([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^2}$ is moderate or that it has a moderate bound.

$\mathcal{N}_{2,2}([0,T)\times\mathbb{R}^n)$ is the algebra of all $G_\varepsilon \in \mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $T>0$, $\alpha \in \mathbb{N}_0^n$ and $a \in \mathbb{R}$

$$\|\partial^\alpha G_\varepsilon\|_{L^2([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{a}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^2}$ is negligible.

As above, we define

$$\mathcal{G}_{2,2}([0,T)\times\mathbb{R}^n) = \mathcal{E}_{2,2}([0,T)\times\mathbb{R}^n)/\mathcal{N}_{2,2}([0,T)\times\mathbb{R}^n).$$

One can similarly define spaces $\mathcal{E}_{2,2}(\mathbb{R}^n)$, $\mathcal{N}_{2,2}(\mathbb{R}^n)$ and $\mathcal{G}_{2,2}(\mathbb{R}^n)$ but independently of time variable t.

Let Q denote $[0,T)\times O$ or O. The proof that $\mathcal{N}_{2,2}(Q)$ is an ideal of $\mathcal{E}_{2,2}(Q)$ is given in [7]. Sobolev embedding theorems give that $\mathcal{E}_{2,2}(Q) \subset \mathcal{E}_g(Q)$ and $\mathcal{N}_{2,2}(Q) \subset \mathcal{N}_g(Q)$. Thus there exists a canonical mapping $\mathcal{G}_{2,2}(Q) \to \mathcal{G}_g(Q)$. Also, this means that in $\mathcal{G}_{2,2}(Q)$ instead of L^2-norm on the strip $[0,T)\times\mathbb{R}^n$ one can use L^∞-norm on $[0,T)$ and L^2-norm on $\mathbb{R}^n$ and vice versa.

1.5. Generalized stochastic processes. At the beginning we recall some basic facts from classical stochastic analysis. Let (Ω,Σ,μ) be a probability space. A weakly measurable mapping

$$X : \Omega \to \mathcal{D}'(\mathbb{R}^d)$$

is called a generalized stochastic process on $\mathbb{R}^d$.

For each fixed function $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the mapping $\Omega \to \mathbb{R}$ defined by

$$\omega \to \langle X(\omega), \varphi\rangle$$

is a random variable.

The space of generalized stochastic processes will be denoted by $\mathcal{D}'_\Omega(\mathbb{R}^d)$. The characteristic functional of a process X is

$$C_X(\varphi) = \int e^{i\langle X(\omega),\varphi\rangle} d\mu(\omega), \ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Take probability space to be the space of tempered distributions $\Omega = \mathcal{S}'(\mathbb{R}^d)$ and Σ to be the Borel σ-algebra generated by the weak topology. Then there is a unique probability measure μ on (Ω, Σ) such that

$$\int e^{i\langle X(\omega),\varphi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R}^d)}}, \ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

It is a well known result following from the Bochner–Minlos theorem (see [39], for example). White noise process $\dot{W} : \Omega \to \mathcal{D}'(\mathbb{R}^d)$ is the identity mapping

$$\langle \dot{W}(\omega), \varphi\rangle = \langle \omega, \varphi\rangle, \ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

It is a generalized Gaussian process with mean zero and variance

$$D(\dot{W}(\varphi)) = E(\dot{W}(\varphi)^2) = \|\varphi\|^2_{L^2(\mathbb{R}^d)},$$

where E denotes expectation.

Definition 1. *$\mathcal{G}_g$-Colombeau random generalized function* on a probability space (Ω, Σ, μ) is a mapping $U : \Omega \to \mathcal{G}_g(Q)$ such that there exists a function $U : (0,1) \times Q \times \Omega \to \mathbb{R}$ with the following properties:

1) For fixed $\varepsilon \in (0,1)$, $(x,\omega) \to U(\varepsilon, x, \omega)$ is jointly measurable in $Q \times \Omega$.

2) $\varepsilon \to U(\varepsilon, \cdot, \omega)$ belongs to $\mathcal{E}_g(Q)$ almost surely in $\omega \in \Omega$, and it is a representative of $U(\omega)$.

By $\mathcal{G}_g^\Omega(Q)$ we denote the algebra of $\mathcal{G}_g$-Colombeau random generalized functions on Ω. A family of $\mathcal{G}_g$-Colombeau random generalized functions is called $\mathcal{G}_g$-Colombeau generalized stochastic process.

Definition 2. *$\mathcal{G}_{2,2}$-Colombeau random generalized function* on a probability space (Ω, Σ, μ) is a mapping $U : \Omega \to \mathcal{G}_{2,2}(Q)$ such that there exists a function $U : (0,1) \times Q \times \Omega \to \mathbb{R}$ with the following properties:

1) For fixed $\varepsilon \in (0,1)$, $(x,\omega) \to U(\varepsilon, x, \omega)$ is jointly measurable in $Q \times \Omega$.

2) $\varepsilon \to U(\varepsilon, \cdot, \omega)$ belongs to $\mathcal{E}_{2,2}(Q)$ almost surely in $\omega \in \Omega$, and it is a representative of $U(\omega)$.

By $\mathcal{G}_{2,2}^\Omega(Q)$ we denote the algebra of $\mathcal{G}_{2,2}$-Colombeau random generalized functions on Ω. A family of $\mathcal{G}_{2,2}$-Colombeau random generalized functions is called $\mathcal{G}_{2,2}$-Colombeau generalized stochastic process.

As usual, the variable ε will be written as a subindex

1.6. Vector valued Colombeau type spaces. We will make some necessary modifications to define and use generalized semigroups. The main difference comparing with the previous section is that one does not need all derivatives of a representative. This will open new possibilities for applications, but also make a work with them more complicated (spaces of such generalized functions are not algebras in general).

Definition 3. $\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ (respectively $\mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)$), $T>0$, is the vector space of nets G_ε of functions

$$G_\varepsilon \in C^0\left([0,T):H^2(\mathbb{R}^n)\right)\cap C^1\left((0,T):L^2(\mathbb{R}^n)\right),\ \varepsilon\in(0,1)$$

with the property: for every $T_1\in(0,T)$ there exists $a\in\mathbb{R}$, (respectively, for every $a\in\mathbb{R}$) such that

$$(1)\qquad \max\left\{\sup_{t\in[0,T)}\|G_\varepsilon(t)\|_{H^2},\ \sup_{t\in[T_1,T)}\|\partial_t G_\varepsilon(t)\|_{L^2}\right\}=\mathcal{O}(\varepsilon^a),\ \text{as}\varepsilon\to 0.$$

The quotient space $\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)=\dfrac{\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)}{\mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)}$ is a Colombeau type vector space.

Dropping the conditions on $\partial_t G_\varepsilon$ in (1) we obtain spaces $\mathcal{E}_{C^0,H^2}([0,T):\mathbb{R}^n)$, $\mathcal{N}_{C^0,H^2}([0,T):\mathbb{R}^n)$ and $\mathcal{G}_{C^0,H^2}([0,T):\mathbb{R}^n)$.

By Sobolev lemma we have

Lemma 4. *If $n\leqslant 3$, then $\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an algebra with the multiplication and $\mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an ideal of $\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$. Therefore, $\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an algebra with the multiplication. The same holds for $\mathcal{E}_{C^0,H^2}([0,T):\mathbb{R}^n)$, $\mathcal{N}_{C^0,H^2}([0,T):\mathbb{R}^n)$ and $\mathcal{G}_{C^0,H^2}([0,T):\mathbb{R}^n)$.*

Substituting H^2-norm with L^2-norm in (1) we obtain vector spaces

$$\mathcal{E}_{C^1,L^2}([0,T):\mathbb{R}^n),\ \mathcal{N}_{C^1,L^2}([0,T):\mathbb{R}^n)\ \text{and}\ \mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n).$$

Canonical mapping $\iota_{L^2}:\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)\to\mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n)$ is defined by $\iota_{L^2}(G)=H$, where $H=[G_\varepsilon]$ and G_ε is a representative of G.

Space $\mathcal{G}_{H^2}(\mathbb{R}^n)$ is defined in a similar way as $\mathcal{G}_{C^1,H^2}(\mathbb{R}^n)$, but with representatives independent of time variable t. This space is also an algebra in case $n\leqslant 3$. Let us give more details for space $\mathcal{G}_{H^{2,\infty}}([0,T):\mathbb{R}^n)$.

$\mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)$, (respectively, $\mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)$) is the space of nets G_ε of functions $G_\varepsilon\in H^{2,\infty}(\mathbb{R}^n)$, $\varepsilon\in(0,1)$, with the property: there exists $a\in\mathbb{R}$ (respectively, for every $a\in\mathbb{R}$) such that

$$\|G_\varepsilon\|_{H^{2,\infty}(\mathbb{R}^n)}=\mathcal{O}(\varepsilon^a),\ \text{as}\varepsilon\to 0.$$

Both spaces are algebras with the usual multiplication and $\mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)$ is an ideal. Colombeau type algebra is defined by

$$\mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)=\frac{\mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)}{\mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)}.$$

2. Generalized semigroups

The definitions and assertions are from [67].

Let $(E, \|\cdot\|)$ be a Banach space and let $\mathcal{L}(E)$ be the space of all linear continuous mappings $E \to E$.

Definition 5. $\mathcal{S}\mathcal{E}_M([0,\infty) : \mathcal{L}(E))$ is the space of nets S_ε of continuous mappings $S_\varepsilon : [0,\infty) \to \mathcal{L}(E)$, $\varepsilon \in (0,1)$, with the properties $S_\varepsilon(0) = I$, $\varepsilon \in (0,1)$, and that for every $T > 0$ there exists $a \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} \|S_\varepsilon(t)\| = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \to 0. \tag{2}$$

$\mathcal{S}\mathcal{N}([0,\infty) : \mathcal{L}(E))$ is the space of nets N_ε of continuous mappings $N_\varepsilon : [0,\infty) \to \mathcal{L}(E)$, $\varepsilon \in (0,1)$, with the properties:

(a) For every $b \in \mathbb{R}$ and $T > 0$, $\sup\limits_{t\in[0,T)} \|N_\varepsilon(t)\| = \mathcal{O}(\varepsilon^b)$, as $\varepsilon \to 0$.

(b) There exists a net H_ε in $\mathcal{L}(E)$ such that $\lim_{t\to 0} \dfrac{N_\varepsilon(t)}{t} x = H_\varepsilon x$, for every $x \in E$ and ε small enough, and for every $b > 0$, $\|H_\varepsilon\| = \mathcal{O}(\varepsilon^b)$, as $\varepsilon \to 0$.

Proposition 6. *$\mathcal{S}\mathcal{E}_M([0,\infty) : \mathcal{L}(E))$ is an algebra with respect to composition and $\mathcal{S}\mathcal{N}([0,\infty) : \mathcal{L}(E))$ is an ideal of $\mathcal{S}\mathcal{E}_M([0,\infty) : \mathcal{L}(E))$.*

Now we define Colombeau type algebra as the factor algebra

$$\mathcal{S}\mathcal{G}([0,\infty) : \mathcal{L}(E)) = \frac{\mathcal{S}\mathcal{E}_M([0,\infty) : \mathcal{L}(E))}{\mathcal{S}\mathcal{N}([0,\infty) : \mathcal{L}(E))}.$$

Elements of $\mathcal{S}\mathcal{G}([0,\infty) : \mathcal{L}(E))$ will be denoted by $S = [S_\varepsilon]$, where S_ε is a representative of the above class.

Definition 7. $S \in \mathcal{S}\mathcal{G}([0,\infty) : \mathcal{L}(E))$ is called a Colombeau C_0-semigroup if it has a representative S_ε such that, for some $\varepsilon_0 > 0$, S_ε is a C_0-semigroup, for every $\varepsilon < \varepsilon_0$.

In the sequel we will use only representatives S_ε of a Colombeau C_0-semigroup S which are C_0-semigroups, for ε small enough.

Proposition 8. *Let S_ε and $\tilde{S}_\varepsilon$ be representatives of a Colombeau C_0-semigroup S, with the infinitesimal generators A_ε, $\varepsilon < \varepsilon_0$, and $\tilde{A}_\varepsilon$, $\varepsilon < \tilde{\varepsilon}_0$, respectively, where ε_0 and $\tilde{\varepsilon}_0$ correspond (in the sense of Definition 7) to S_ε and $\tilde{S}_\varepsilon$, respectively. Then $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \bar{\varepsilon}_0 = \min\{\varepsilon_0, \tilde{\varepsilon}_0\}$ and $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to be an element of $\mathcal{L}(E)$, denoted again by $A_\varepsilon - \tilde{A}_\varepsilon$. Moreover, for every $a \in \mathbb{R}$, $\|A_\varepsilon - \tilde{A}_\varepsilon\| = \mathcal{O}(\varepsilon^a)$, as $\varepsilon \to 0$.*

Now we define the infinitesimal generator of a Colombeau C_0-semigroup S. Denote by $\mathcal{A}$ the set of pairs $(A_\varepsilon, D(A_\varepsilon))$ where A_ε is a closed linear operator on E with dense domain $D(A_\varepsilon) \subset E$, for every $\varepsilon \in (0,1)$. We introduce an equivalence relation in $\mathcal{A}$ by: $(A_\varepsilon, D(A_\varepsilon)) \sim (\tilde{A}_\varepsilon, D(\tilde{A}_\varepsilon))$ if there exists $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ there exists $C > 0$ such that, for $x \in D(A_\varepsilon)$, $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \leqslant C\varepsilon^a \|x\|$, as $\varepsilon \to 0$.

Since A_ε has a dense domain in E, $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to be an operator in $\mathcal{L}(E)$ satisfying $\|A_\varepsilon - \tilde{A}_\varepsilon\| = \mathcal{O}(\varepsilon^a)$, $\varepsilon \to 0$, for every $a \in \mathbb{R}$.

We denote by A the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to Proposition 8, the following definition makes sense.

Definition 9. $A \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau C_0-semigroup S if there exists a representative A_ε of A such that A_ε is the infinitesimal generator of S_ε, for ε small enough.

We collect some obvious properties in the following proposition (cf. [79]).

Proposition 10. *Let S be a Colombeau C_0-semigroup with the infinitesimal generator A. Then there exists $\varepsilon_0 \in (0,1)$ such that:*
(a) *Mapping $t \mapsto S_\varepsilon(t)x : [0,\infty) \to E$ is continuous for every $x \in E$ and $\varepsilon < \varepsilon_0$.*

(b)
$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} S_\varepsilon(s)x\,ds = S_\varepsilon(t)x,\ \varepsilon < \varepsilon_0,\ x \in E.$$

(c)
$$\int_0^t S_\varepsilon(s)x\,ds \in D(A_\varepsilon),\ \varepsilon < \varepsilon_0,\ x \in E.$$

(d) *For every $x \in D(A_\varepsilon)$ and $t \geqslant 0$, $S_\varepsilon(t)x \in D(A_\varepsilon)$ and*

$$\frac{d}{dt}S_\varepsilon(t)x = A_\varepsilon S_\varepsilon(t)x = S_\varepsilon(t)A_\varepsilon x,\ \varepsilon < \varepsilon_0.$$

(e) *Let S_ε and $\tilde{S}_\varepsilon$ be representatives of Colombeau C_0-semigroup S, with infinitesimal generators A_ε and $\tilde{A}_\varepsilon$, $\varepsilon < \varepsilon_0$, respectively. Then, for every $a \in \mathbb{R}$ and $t \geqslant 0$*

$$\left\|\frac{d}{dt}S_\varepsilon(t) - \tilde{A}_\varepsilon S_\varepsilon(t)\right\| = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \to 0.$$

(f) *For every $x \in D(A_\varepsilon)$ and every $t, s \geqslant 0$,*

$$S_\varepsilon(t)x - S_\varepsilon(s)x = \int_s^t S_\varepsilon(\tau)A_\varepsilon x\,d\tau = \int_s^t A_\varepsilon S_\varepsilon(\tau)x\,d\tau,\ \varepsilon < \varepsilon_0.$$

Theorem 11. *Let S and $\tilde{S}$ be two Colombeau C_0-semigroups with infinitesimal generators A and B, respectively. If $A = B$, then $S = \tilde{S}$.*

Example 12. *Semigroups of Schrödinger-type operators.* Let $V \in \mathcal{G}_{W^{2,\infty}}(\mathbb{R}^n)$ be of logarithmic type. Then differential operators $A_\varepsilon u = (\Delta - V_\varepsilon)u$, $u \in W^2(\mathbb{R}^n)$, $\varepsilon < 1$, are infinitesimal generators of C_0-semigroups S_ε, $\varepsilon < 1$, and S_ε is a representative of a generalized C_0-semigroup $S \in \mathcal{SG}\left([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n))\right)$.

Let $\varepsilon < 1$. Operator A_ε is the infinitesimal generator of the corresponding C_0-semigroup $S_\varepsilon : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n))$ defined by the Feynman–Kac formula:

$$S_\varepsilon(t)\psi(x) = \int_O \exp\left(-\int_0^t V_\varepsilon(\omega(s))\,ds\right)\psi(\omega(t))\,d\mu_x(\omega),\ t \geqslant 0,\ x \in \mathbb{R}^n,$$

for $\psi \in L^2(\mathbb{R}^n)$, where $O = \prod_{t\in[0,\infty)} \overline{\mathbb{R}^n}$ and μ_x is the Wiener measure concentrated at $x \in \mathbb{R}^n$ (cf. [86] or [95]).

The assumption on V implies that there exists $C > 0$ such that

$$|S_\varepsilon(t)\psi(x)| \leqslant \exp\Big(t \sup_{s\in\mathbb{R}^n} |V_\varepsilon(s)|\Big) \int_O |\psi(\omega(t))|d\mu_x(\omega)$$
$$= \varepsilon^{-Ct}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x-y|^2}{4t}\Big)|\psi(y)|dy,$$

for every $t > 0$, $x \in \mathbb{R}^n$ and $\varepsilon < 1$. Recall that the heat kernel is given by

$$E_n(t,x) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{x^2}{4t}\Big), \quad t > 0,\ x \in \mathbb{R}^n,$$

and its $L^1(\mathbb{R}^n)$-norm equals 1 for every $t > 0$. By the Young inequality,

$$|S_\varepsilon(t)\psi| \leqslant e^{-Ct}\|E_n(t,\cdot)\|_{L^1(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)},\ t > 0,\ \varepsilon < 1.$$

Therefore, there exists $C_0 > 0$ such that $\sup_{t\in[0,T)} \|S_\varepsilon(t)\psi\|_{L^2} \leqslant C_0\varepsilon^{-CT}\|\psi\|_{L^2}$, $\varepsilon < 1$, for every T, i.e., $S_\varepsilon(t)$, $t \in [0,T]$, satisfies relation (2) and

$$S = [S_\varepsilon] \in SG\left([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n))\right).$$

Remark 13. We refer to [70] for an approach to generalized semigroup theory based on uniformly continuous classical semigroups.

PART II: SECOND ORDER EQUATIONS

3. Elliptic PDEs

3.1. Linear elliptic PDE. We will consider elliptic boundary value problems with very singular boundary data and coefficients. Because of that the solutions are considered in a large space of generalized functions and moreover, the concept of being a solution is adequately extended.

3.1.1. Introduction. The restriction of a generalized function on $A \subset \Omega$ is defined by the restriction of a representative. Recall, the support of $G \in \mathcal{G}(\Omega)$ is the complement of the largest open subset of Ω where G is the zero generalized function. The space of all compactly supported generalized functions is denoted by $\mathcal{G}_c(\Omega)$.

In the sequel we use the notation $(A)_{-\gamma} = \{x \in A : \operatorname{dist}(x, \partial A) \geqslant \gamma\}$ and $\phi_\gamma = \gamma^{-n}\phi(\cdot/\gamma)$, $\gamma > 0$, where $\phi = \phi_1(x_1)\cdots\phi_n(x_n) \in C_0^\infty(\mathbb{R}^n)$, $\int \phi_i(x)\,dx = 1$, $i = 1, \dots, n$. Note that ϕ_γ, $\gamma > 0$ is a delta net. For the sake of simplicity, let us assume that ϕ is a radially symmetric, positive function in the open unit ball and that ϕ is supported by the closed unit ball in $\mathbb{R}^n$. Let $\psi_\varepsilon = 1_{(\Omega)_{2\varepsilon}} * \phi_\varepsilon$. We define an inclusion ι of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ in the following way. If $g \in \mathcal{D}'(\Omega)$, then $\iota(g) = G$ is represented by $G_\varepsilon = (g \cdot \psi_\varepsilon) * \phi_\varepsilon$.

Let $\mathbb{D}$ be some space of test functions. We say that $G_1, G_2 \in \mathcal{G}(\Omega)$ are equal in $\mathbb{D}$-sense, $G_1 \stackrel{\mathbb{D}}{=} G_2$, if $\langle G_1, \varphi\rangle = \langle G_2, \varphi\rangle$ in $\overline{\mathbb{C}}$ for every $\varphi \in \mathbb{D}$. Similarly, a generalized function G_1 and $g \in \mathbb{D}'$ equals in $\mathbb{D}$-sense, $G_1 \stackrel{\mathbb{D}}{=} g$, if $\langle G_1, \varphi\rangle = \langle g, \varphi\rangle$ for every $\varphi \in \mathbb{D}$. Usually, $\mathbb{D}$ denotes Sobolev space H^m or H_0^m.

The s-association $(\approx_s)$, $s \geqslant 0$ of generalized complex numbers is defined as follows. For $G \in \overline{\mathbb{C}}$, $G \approx_s 0$ means that G has a representative G_ε such that $G_\varepsilon = o(\varepsilon^s)$ as $\varepsilon \to 0$. If G_1, G_2 are in $\mathcal{G}(\Omega)$, then $G_1 \stackrel{\mathbb{D}}{\approx}_s G_2$ if $\langle G_1 - G_2, \varphi\rangle \approx_s 0$ for every $\varphi \in \mathbb{D}$. If $s = 0$, then the notation $\dots$-association is often used instead of $\dots$-0-association. As in the previous case one can define association between a generalized function in $\mathcal{G}$ and a generalized function in $\mathbb{D}'$.

We will recall the definition of the space of generalized functions on a closed set (cf. [5]) in order to give a meaning to the Dirichlet problem in $\mathcal{E}_M$ and thus in $\mathcal{G}$.

Let X be a non-void subset of $\mathbb{R}^n$ and $\{G_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\}$ be a family of mappings $G_\varepsilon^\alpha : (0,1)\times X \to \mathbb{C}$. Denote by $\mathcal{E}_{W,M}(X)$ the vector space of families $\{G_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\}$ which satisfy the following conditions:

(a) $\{G_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\}$ has a locally moderate growth when $\varepsilon \to 0$. This means that for every $\alpha \in \mathbb{N}_0^n$ and $x_0 \in X$ there exist a neighborhood V of x_0, $N \in \mathbb{R}$, $C > 0$ and $\eta > 0$ such that $|G_\varepsilon^\alpha(x)| \leqslant C\varepsilon^{-N}$, $x \in V \cap X$, $\varepsilon \in (0,\eta)$.

(b) There exists $\eta > 0$ such that the family $\{X \ni x \mapsto G_\varepsilon^\alpha(x),\ \varepsilon < \eta,\ \alpha \in \mathbb{N}_0^n\}$ satisfies requirements defining Whitney's C^∞-function on X, that is for every $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant m$ and $x_0 \in X$ there exist a neighborhood V of x_0 and $c_\varepsilon > 0$ such that

$$\left|G_\varepsilon^\alpha(x) - \sum_{|\beta|\leqslant m-|\alpha|} \frac{(x-x')^\beta G_\varepsilon^{\alpha+\beta}(x')}{\beta!}\right| \leqslant c_\varepsilon |x-x'|^{m-|\alpha|-1}, \tag{3}$$

for every $x, x' \in V$, $\varepsilon \in (0,\eta)$.

(c) Constants c_ε are locally bounded above by $c\varepsilon^{-N}$ as $\varepsilon \to 0$. More precisely, for every $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant m$ and $x_0 \in X$ there exist a neighborhood V of x_0, $N \in \mathbb{R}$, $C > 0$ and $\eta > 0$ such that (3) holds with $c_\varepsilon = C\varepsilon^{-N}$.

The ideal $\mathcal{N}_W(X)$ of $\mathcal{E}_{W,M}(X)$ is the set of those $\{G_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\}$ which satisfy: for every $\alpha \in \mathbb{N}_0^n$, and $x_0 \in X$ there exists a neighborhood V of x_0 such that for every $q > 0$ there exist $C > 0$ and $\eta > 0$ such that $|G_\varepsilon^\alpha(x)| \leqslant C\varepsilon^q$, $x \in V \cap X$, $\varepsilon \in (0,\eta)$.

Put $\mathcal{G}_W(X) = \mathcal{E}_{W,M}(X)/\mathcal{N}_W(X)$. Clearly, if $G \in \mathcal{G}(\Omega)$, where Ω is an open set containing X, then $\{D^\alpha G_\varepsilon|_X,\ \alpha \in \mathbb{N}_0^n\} \in \mathcal{E}_{W,M}$ defines the restriction $G|_X \in \mathcal{G}_W$.

Theorem 14. [5] *Let X be a closed subset of $\mathbb{R}^n$. Then the restriction map $\mathcal{G}(\mathbb{R}^n) \to \mathcal{G}_W(X)$ is surjective. In particular, for given $\{G_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\} \in \mathcal{E}_{W,M}(X)$ there exists $F_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ such that $\{D^\alpha F_\varepsilon|_X - G_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\} \in \mathcal{N}_W(X)$.*

3.1.2. Generalized Dirichlet problem. A differential operator of the form $P(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha$, where $a_\alpha \in \mathcal{G}(\mathbb{R}^n)$, is called a generalized differential operator. A representative of $P(x,D)$ is given by $P_\varepsilon(x,D) = \sum_{|\alpha|\leqslant m} a_{\alpha,\varepsilon}(x) D^\alpha$, where $a_{\alpha,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n$, is a representative of a_α, $|\alpha| \leqslant m$. Note that if $b_{\alpha,\varepsilon}$ is another representative of a_α, $|\alpha| \leqslant m$, then

$$\sum_{|\alpha|\leqslant m} a_{\alpha,\varepsilon}(x) D^\alpha G_\varepsilon - \sum_{|\alpha|\leqslant m} b_{\alpha,\varepsilon}(x) D^\alpha G_\varepsilon \in \mathcal{N}(\mathbb{R}^n),\ G_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n).$$

Let O be a bounded open set in $\mathbb{R}^n$, $H \in \mathcal{G}(\mathbb{R}^n)$ and let $F \in \mathcal{G}_W(\partial O)$ be defined by a family $\{F_\varepsilon^\alpha,\ \alpha \in \mathbb{N}_0^n\}$. Consider the following boundary value problem

$$P(x,D)G \overset{\mathbb{D}_0^m(O)}{\approx} H,\ \text{in } O,\ G|_{\partial O} = F \quad (\mathbb{D}_0^m(O) = H^m(O) \cap H_0^{m-1}(O)). \tag{4}$$

Theorem 14 implies that there exists $\tilde F \in \mathcal{G}(\mathbb{R}^n)$ such that $\tilde F|_{\partial O} = F$. Let $V = P(x,D)\tilde F$ and U be a solution to the problem

$$P(x,D)U \overset{\mathbb{D}_0^m(O)}{\approx} H - V \text{ in } O,\ U|_{\partial O} = 0.$$

Then $G = U + \tilde F$ is a solution to (4).

So, in the sequel we shall consider the following problem

$$P(x,D)G \overset{\mathbb{D}_0^m(O)}{\approx} H \text{ in } O,\ G|_{\partial O} = 0, \tag{5}$$

in terms of representatives,

$$\lim_{\varepsilon\to 0}\int (P_\varepsilon(x,D)G_\varepsilon(x)-H_\varepsilon(x))\psi(x)\,dx=0,\ \psi\in\mathbb{D}_0^m(O)$$
$$\{D^\alpha G_\varepsilon|_{\partial O},\ \alpha\in\mathbb{N}_0^n\}\in\mathcal{N}_W(\partial O).$$

Theorem 15. *With the assumptions given above, for every $s\geqslant 0$ there exists a solution $G\in\mathcal{G}(\mathbb{R}^n)$ to (5) in $\mathbb{D}_0^m(O)$-s-associated sense.*

Proof. Let $P_\varepsilon^*(x,D)=\sum_{|\alpha|\leqslant m}\tilde{a}_{\alpha,\varepsilon}(x)D^\alpha$ be the adjoint operator to $P_\varepsilon(x,D)$. Since $\tilde{a}_{\alpha,\varepsilon}\in\mathcal{E}_M(\mathbb{R}^n)$, there exist $\tilde{N}_1>0$ and $C_1>0$ such that

$$B_\varepsilon=\max\left\{|\nabla\tilde{a}_{\alpha,\varepsilon}(x)|,\ |\tilde{a}_{\alpha,\varepsilon}(x)|,\ x\in\overline{O},\ |\alpha|\leqslant m\right\}\leqslant C_1\varepsilon^{-\tilde{N}_1},$$

for ε small enough, say $\varepsilon<\eta_1$.

Let $\varepsilon\in(0,\eta_1)$ be given. Let Π_ε be a cube $\{x:|x_i|\leqslant b,\ i=1,\dots,n\}$, which contains O. Put $N_\varepsilon=B_\varepsilon b\varepsilon^{-q}/2$, where q will be determined later, and divide Π_ε by hyperplanes

$$x_i=bk/N_\varepsilon,\ i=1,\dots,n,\ k=0,\pm1,\dots,\pm(N_\varepsilon-1),\ N_\varepsilon\in\mathbb{N},$$

into $(2N_\varepsilon)^n$ cubes $\Pi_{j,\varepsilon}$, $j=1,\dots,(2N_\varepsilon)^n$. These cubes can be renumerated such that Π_j, $j=1,\dots,J_\varepsilon$ cover $\overline{O}$ and denote $O_{j,\varepsilon}=O\cap\Pi_{j,\varepsilon}$. Then $J_\varepsilon=\mathcal{O}(\varepsilon^{-n(q+\tilde{N}_1)})$ as $\varepsilon\to 0$.

Denote by $X_{j,\varepsilon}$ the center of $\Pi_{j,\varepsilon}$ and $\tilde{A}_{\alpha,j,\varepsilon}=\tilde{a}_{\alpha,\varepsilon}(X_{j,\varepsilon})$. For ε small enough, let $\{\psi_{j,\varepsilon}\}$ be a partition of the unity defined in the following way.

$$\tilde{\psi}_{j,\varepsilon}=1_{O_{j,\varepsilon}}*\phi_{\varepsilon^d},\ \psi_{j,\varepsilon}=\frac{\tilde{\psi}_{j,\varepsilon}}{\sum_{j=1}^{J_\varepsilon}\tilde{\psi}_{j,\varepsilon}},\ j=1,\dots,J_\varepsilon.$$

Note that $\psi_{j,\varepsilon}\equiv 1$ on $\tilde{K}_{j,\varepsilon}\subset O_{j,\varepsilon}$ and $\operatorname{mes}(\operatorname{supp}\psi_{j,\varepsilon}\setminus\tilde{K}_{j,\varepsilon})\leqslant C_0\varepsilon^d$, where C_0 does not depend on j and d to be chosen later. Moreover,

$$\sup_{|\alpha|\leqslant m,\ j\leqslant J_\varepsilon}\|\tilde{a}_{\alpha,\varepsilon}-\tilde{A}_{\alpha,j,\varepsilon}\|_{L^\infty(O_{j,\varepsilon})}\leqslant B_\varepsilon\frac{b}{2N_\varepsilon}=\varepsilon^q.$$

Since $H_\varepsilon\in\mathcal{E}_M(\mathbb{R}^n)$, there exist $N_2>0$ and $C_2>0$ such that

$$\|H_\varepsilon\psi_{j,\varepsilon}\|_{L^\infty(\overline{O})}\leqslant\|H_\varepsilon\|_{L^\infty(\overline{O})}\leqslant C_2\varepsilon^{-N_2},\ j=1,\dots,J_\varepsilon,$$

for ε small enough. Denote $H_{j,\varepsilon}:=H_\varepsilon\psi_{j,\varepsilon}$.

Let $\tilde{G}_{j,\varepsilon}\in\mathcal{E}_M(\mathbb{R}^n)$ be a solution to

$$P_{j,\varepsilon}(D)\tilde{G}_{j,\varepsilon}=\sum_{|\alpha|\leqslant m}A_{\alpha,j,\varepsilon}D^\alpha\tilde{G}_{j,\varepsilon}=H_{j,\varepsilon},\ j\leqslant J_\varepsilon,$$

which exists by Theorem 1 in [82], where $P_{j,\varepsilon}(D)$ is the adjoint operator for $P_{j,\varepsilon}^*(D)=\sum_{|\alpha|\leqslant m}\tilde{A}_{\alpha,j,\varepsilon}D^\alpha$. Put $G_{j,\varepsilon}(x)=\tilde{G}_{j,\varepsilon}(x)\kappa_{j,\varepsilon}(x)\xi_\varepsilon(x)$, where

$$\kappa_{j,\varepsilon}=1_{(\tilde{K}_{j,\varepsilon})_{-\varepsilon^d/2}}*\phi_{\varepsilon^d/2},\ \xi_\varepsilon=1_{(O)_{-3\varepsilon^d/4}}*\phi_{\varepsilon^d/4},\ j\leqslant J_\varepsilon.$$

Let $K_{j,\varepsilon} = \{x \in O_{j,\varepsilon} : G_{j,\varepsilon}(x) = \tilde{G}_{j,\varepsilon}(x)\}$, $j = 1, \dots, J_\varepsilon$. Obviously,

$$\sup_{j \leqslant J_\varepsilon} \operatorname{mes}(O_{j,\varepsilon} \setminus K_{j,\varepsilon}) = O(\varepsilon^{nd}) \text{ as } \varepsilon \to 0.$$

By inspecting the proof of Theorem 1 in [82] one can see that there exist C_0, $M > 0$ and $\tilde{N}_0 > 0$, which depend only on H and a_α such that

$$\sup_{j \leqslant J_\varepsilon} \|\tilde{G}_{j,\varepsilon}\|_{L^\infty(\overline{O})} \leqslant C_0 \varepsilon^{-M-\tilde{N}_0},$$

for ε small enough.

One can easily see that $\operatorname{supp}(G_{i,\varepsilon}) \cap \operatorname{supp}(G_{j,\varepsilon}) = \emptyset$, $i \neq j$ and $G_{j,\varepsilon} \in \mathcal{E}_M(\overline{O})$.

Let $G_\varepsilon = \sum_{j=1}^{J_\varepsilon} G_{j,\varepsilon}$ and let ψ be an arbitrary function in $\mathbb{D}_0^m(O) = H^m(O) \cap H_0^{m-1}(O)$. Then

$$\int_O \psi(x) P_\varepsilon(x, D) \sum_{j=1}^{J_\varepsilon} G_{j,\varepsilon}(x) - \int_O \psi(x) H_\varepsilon(x)\, dx = I_1 + I_2,$$

where

$$I_1 = \int_O \psi(x) P_\varepsilon(x, D) \sum_{j=1}^{J_\varepsilon} G_{j,\varepsilon}(x) - \int_O \psi(x) \sum_{j=1}^{J_\varepsilon} P_{j,\varepsilon}(D) G_{j,\varepsilon}(x)\, dx,$$

$$I_2 = \int_O \psi(x) \sum_{j=1}^{J_\varepsilon} P_{j,\varepsilon}(D) G_{j,\varepsilon}(x) - \int_O \psi(x) H_\varepsilon(x)\, dx.$$

By using $H_\varepsilon = \sum_{j=1}^{J_\varepsilon} H_{j,\varepsilon}$, and $P_{j,\varepsilon}(D)\tilde{G}_{j,\varepsilon} = H_{j,\varepsilon}$, $j \leqslant J_\varepsilon$, and passing to the adjoint operators, we have

$$\begin{aligned}
|I_2| &= \left| \sum_{j=1}^{J_\varepsilon} \int_O P_{j,\varepsilon}^*(D)\psi(x)(\tilde{G}_{j,\varepsilon}(x) - G_{j,\varepsilon}(x))\, dx \right| \\
&\leqslant \sum_{j=1}^{J_\varepsilon} \int_O |P_{j,\varepsilon}^*(D)\psi(x)| \cdot |\tilde{G}_{j,\varepsilon}(x)| \cdot |1 - \kappa_{j,\varepsilon}(x)\xi_\varepsilon(x)|\, dx \\
&\leqslant \sum_{j=1}^{J_\varepsilon} \sum_{|\alpha| \leqslant m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{O})} \int_O |D^\alpha \psi(x)| \cdot |\tilde{G}_{j,\varepsilon}(x)| \cdot |1 - \kappa_{j,\varepsilon}(x)\xi_\varepsilon(x)|\, dx.
\end{aligned}$$

Using Cauchy–Schwartz inequality and bounds for $a_{\alpha,\varepsilon}$, $|\alpha| \leqslant m$ it follows

$$\begin{aligned}
|I_2| &\leqslant C_1 \varepsilon^{-\tilde{N}_1} \|\psi\|_{H^m(O)} \sum_{j=1}^{J_\varepsilon} \left(\int_{O_{j,\varepsilon}} |\tilde{G}_{j,\varepsilon}(x)|^2 |1 - \kappa_{j,\varepsilon}(x)\xi_\varepsilon(x)|^2 dx \right)^{1/2} \\
&\leqslant C_1 \varepsilon^{-\tilde{N}_1} \|\psi\|_{H^m(O)} \|\tilde{G}_{j,\varepsilon}\|_{L^\infty(\overline{O})} \sum_{j=1}^{J_\varepsilon} (\operatorname{mes}(O_{j,\varepsilon} \setminus K_{j,\varepsilon}))^{1/2} \\
&= \mathcal{O}(\varepsilon^{d/2 - \tilde{N}_1 - \tilde{N}_0 - M - n(q + \tilde{N}_1)}),\ \varepsilon \to 0.
\end{aligned}$$

This implies that $|I_2| \to 0$ as $\varepsilon \to 0$ if $d > 2(\tilde{N}_1 + \tilde{N}_0 + M + n(q + \tilde{N}_1))$. Thus, the first constant d is now determined. Further on,

$$
\begin{aligned}
|I_1| &\leqslant \sum_{j=1}^{J_\varepsilon} \left| \int_{K_{j,\varepsilon}} \psi(x)(P_\varepsilon(x,D)G_{j,\varepsilon}(x) - P_{j,\varepsilon}(D)G_{j,\varepsilon}(x))\,dx \right| \\
&\quad + \sum_{j=1}^{J_\varepsilon} \left| \int_{O_{j,\varepsilon}\setminus K_{j,\varepsilon}} \psi(x)(P_\varepsilon(x,D)G_{j,\varepsilon}(x) - P_{j,\varepsilon}(D)G_{j,\varepsilon}(x))\,dx \right| \\
&\leqslant \sum_{j=1}^{J_\varepsilon} \left| \int_{K_{j,\varepsilon}} G_{j,\varepsilon}(x)(P_\varepsilon^*(x,D)\psi(x) - P_{j,\varepsilon}^*(D)\psi(x))\,dx \right| \\
&\quad + \sum_{j=1}^{J_\varepsilon} \left| \int_{O_{j,\varepsilon}\setminus K_{j,\varepsilon}} G_{j,\varepsilon}(x)(P_\varepsilon^*(x,D)\psi(x) - P_{j,\varepsilon}^*(D)\psi(x))\,dx \right| \\
&\leqslant \sum_{j=1}^{J_\varepsilon} \int_{K_{j,\varepsilon}} |G_{j,\varepsilon}(x)| \sum_{|\alpha|\leqslant m} |\tilde{a}_{\alpha,\varepsilon}(x) - A_{\alpha,j,\varepsilon}| \cdot |D^\alpha\psi(x)|\,dx \\
&\quad + \sum_{|\alpha|\leqslant m} J_\varepsilon \sup_{j\leqslant J_\varepsilon} \int_{O_{j,\varepsilon}\setminus K_{j,\varepsilon}} |G_{j,\varepsilon}(x)| \cdot \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{O})} |D^\alpha\psi(x)|\,dx \\
&\leqslant \varepsilon^q \sum_{|\alpha|\leqslant m} \int_O |G_{j,\varepsilon}(x)| \cdot |D^\alpha\psi(x)|\,dx \\
&\quad + J_\varepsilon \sum_{|\alpha|\leqslant m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{O})} \sup_{j\leqslant J_\varepsilon} \left(\int_{O_{j,\varepsilon}\setminus K_{j,\varepsilon}} |G_{j,\varepsilon}(x)|^2 \right)^{1/2} \|D^\alpha\psi\|_{L^2(O)} \\
&\leqslant \varepsilon^q \|G_\varepsilon\|_{L^2(O)} \|\psi\|_{H^m(O)} \\
&\quad + J_\varepsilon \sum_{\alpha\leqslant m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{O})} \sup_{j\leqslant J_\varepsilon} \|\psi\|_{H^m(O)} \|G_{j,\varepsilon}\|_{L^\infty(\overline{O})} (\operatorname{mes}(O_{j,\varepsilon}\setminus K_{j,\varepsilon}))^{1/2} \\
&\leqslant \varepsilon^q \|G_{j,\varepsilon}\|_{L^\infty(\overline{O})} (\operatorname{mes}(O))^{1/2} \|\psi\|_{H^m(O)} \\
&\quad + C\varepsilon^{d/2-n(q+\tilde{N}_1)} \|\psi\|_{H^m(O)} \sum_{|\alpha|\leqslant m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{O})} \sup_{j\leqslant J_\varepsilon} \|G_{j,\varepsilon}\|_{L^\infty(\overline{O})} \\
&\leqslant \mathcal{O}(\varepsilon^{q-M-\tilde{N}_0}) + \mathcal{O}(\varepsilon^{d/2-n(q+\tilde{N}_1)-\tilde{N}_1-M-\tilde{N}_0}),\ \varepsilon \to 0.
\end{aligned}
$$

This implies that $|I_1| \to 0$ as $\varepsilon \to 0$ if $d > 2(n(q+\tilde{N}_1) + \tilde{N}_1 + M + \tilde{N}_0)$ and $q > M + \tilde{N}_0$. More precisely, $|I_1|$ and $|I_2|$ are $\mathbb{D}_0^m$-s-associated with zero if one chooses $q > M + \tilde{N}_0 + s$, and then $d > 2(n(q+\tilde{N}_1) + \tilde{N}_1 + M + \tilde{N}_0) + s$. This proves the theorem. □

Remark 16. Denote by $\mathbb{D}_0^{m,\tilde{a},t}(O)$ the space of nets with elements $\Psi_\varepsilon \in C^{m,\tilde{a}}(\overline{O}) \cap H_0^{m-1}(O)$, $\varepsilon \in (0,1)$ such that $\|D^\alpha\Psi_\varepsilon\|_{L^\infty(\overline{O})} = \mathcal{O}(\varepsilon^{-t})$, $\varepsilon \to 0$, $|\alpha| \leqslant m$. Then

$$\|\Psi_\varepsilon\|_{H^m(O)} \leqslant \sum_{|\alpha|\leqslant m} (\mathrm{mes}(O))^{1/2}\|D^\alpha\Psi_\varepsilon\|_{L^\infty(\overline{O})} = \mathcal{O}(\varepsilon^{-t}),\ \varepsilon\to 0.$$

If $q > M + \tilde{N}_0 + s + t$ and $d > 2(n(q+\tilde{N}_1) + \tilde{N}_1 + M + \tilde{N}_0 + t) + s$, then

$$|\langle P_\varepsilon(x,D)G_\varepsilon, \Psi_\varepsilon\rangle - \langle H_\varepsilon, \Psi_\varepsilon\rangle| = o(\varepsilon^s),\ \text{as } \varepsilon\to 0,\ \Psi_\varepsilon \in \mathbb{D}_0^{m,t}(O),$$

i.e., for every $s > 0$ there exists a solution to

$$P(x,D)G \overset{\mathbb{D}_0^{m,t}(O)}{\approx_s} H \text{ in } O,\ G|_{\partial O} = 0.$$

This result will be used in the following two theorems. The assumption $\Psi_\varepsilon \in H_0^{m-1}(O)$ is crucial in the construction of the solution. This will restrict applications of the above theorem to strictly elliptic problem of order greater than two.

3.1.3. Applications. Let

(6)
$$L = \sum_{i,j=1}^n a_{ij}(x)D_iD_j + \sum_{i=1}^n b_i(x)D_i + c(x)$$

be a differential operator with real coefficients such that

(7)
$$a_{ij}, b_i, c \in C^1(\overline{\Omega}),\ c\leqslant 0,\ a_{ij} = a_{ji},\ i,j = 1,\dots,n.$$

Assume that

(8)
$$\Omega \text{ is bounded and } \partial\Omega \text{ is of } C^\infty \text{ class.}$$

Remark 17. For a method used in this and the following section, the regularization of coefficients of a differential operator and a function h is needed. Since $C^\infty(\overline{\Omega})$ is not dense in $C^{\tilde{\alpha}}(\overline{\Omega}) = C^{0,\tilde{\alpha}}(\overline{\Omega})$, $\tilde{\alpha}\in(0,1)$, (cf. [97, Remark 2 in 4.5.1]) we suppose that the coefficients and h are in $C^1(\overline{\Omega})$.

Assume that there exist $\lambda > 0$ and $\Lambda > 0$ such that

(9)
$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2,\ x\in\Omega,\ \xi\in\mathbb{R}^n.$$

By Theorem 6.14 and (6.42) in [26], we have

(10) If $h\in C^{\tilde{\alpha}}(\overline{\Omega})$, there exists a unique solution $g\in C^{2,\tilde{\alpha}}(\overline{\Omega})$ to Dirichlet problem $Lg = h$ in Ω, $g = 0$ on $\partial\Omega$ and

(11)
$$\|g\|_{C^{2,\tilde{\alpha}}(\overline{\Omega})} \leqslant C\|h\|_{C^{\tilde{\alpha}}(\overline{\Omega})},\ \text{for some } C > 0.$$

Let $h\in C^1(\overline{\Omega})$, $h = 0$ outside of $\overline{\Omega}$ and $H_\varepsilon = h * \check{\phi}_\varepsilon$ (ϕ is a radially symmetric function in $C_0^\infty(\mathbb{R}^n)$ $\int\phi(x)\,dx = 1$, $\check{\phi}_\varepsilon = \varepsilon^{-n}\phi(-\cdot/\varepsilon)$.) Consider the Dirichlet problems

(12)
$$Lg = h \text{ in } \Omega,\ g|_{\partial\Omega} = 0,$$

(13)
$$LZ_\varepsilon = H_\varepsilon \text{ in } \Omega,\ Z_\varepsilon|_{\partial\Omega} = 0,\ \text{for fixed } \varepsilon\in(0,1),$$

(14)
$$LG \overset{H^2(\Omega)\cap H_0^1(\Omega)}{\approx} H \text{ in } \Omega,\ G|_{\partial\Omega} = 0.$$

The last equation means that

(15) $$\lim_{\varepsilon\to 0}\int (LG_\varepsilon - H_\varepsilon)\psi\, dx = 0, \text{ for every } \psi \in H^2(\Omega)\cap H_0^1(\Omega), G|_{\partial\Omega} \in \mathcal{N}_W(\partial\Omega),$$

in terms of representatives. Let g, Z_ε and G_ε be solutions to (12), (13) and (15), respectively, where solutions g and Z_ε exist by (10) and G_ε is the solution given in Theorem 15.

Theorem 18. *Let L and Ω satisfy assumptions given above. Moreover, assume that L has coefficients which are smooth on $\overline{\Omega}$. The generalized solution G to* (14) *constructed in Theorem* 15 *is $C^{\tilde\alpha}(\overline{\Omega})\cap H_0^1(\Omega)$-associated, $\tilde\alpha > 0$, with the classical solution g to* (12).

Proof. By [97], Theorem 1 in 4.5.2, $C^{\tilde\alpha}(\overline{\Omega}) = [C^0(\overline{\Omega}), C^1(\overline{\Omega})]_{\tilde\alpha}$. ($[\cdot,\cdot]_{\tilde\alpha}$ denotes an interpolation space.) This implies [97, (7) in 4.5.2],

(16) $$\|f\|_{C^{\tilde\alpha}(\overline{\Omega})} \leqslant \|f\|^{1-\tilde\alpha}_{C^0(\overline{\Omega})}\|f\|^{\tilde\alpha}_{C^1(\overline{\Omega})},\ f\in C^1(\overline{\Omega}).$$

and in the special case,

$$\|h - H_\varepsilon\|_{C^{\tilde\alpha}(\overline{\Omega})} \leqslant \|h - H_\varepsilon\|^{1-\tilde\alpha}_{C^0(\overline{\Omega})}\|h - H_\varepsilon\|^{\tilde\alpha}_{C^1(\overline{\Omega})} \to 0,\ \varepsilon\to 0,$$

where h and H_ε are the functions from (12) and (15) respectively.

The above inequality, boundedness of Ω and (11) imply $\|g - Z_\varepsilon\|_{H^1(\Omega)} \to 0$, as $\varepsilon \to 0$ and

$$\int (g(x) - Z_\varepsilon(x))\theta(x)\, dx \to 0, \quad \text{as } \varepsilon\to 0,\ \theta\in C^{\tilde\alpha}(\overline{\Omega})\cap H_0^1(\Omega).$$

We have to prove that

$$\int_\Omega (G_\varepsilon(x) - Z_\varepsilon(x))\theta(x)\, dx \to 0, \text{ as } \varepsilon\to 0.$$

The boundary value problem $L^*\psi = \theta$ in Ω, $\psi|_{\partial\Omega} = 0$ has a solution in $C^{2,\tilde\alpha}(\overline{\Omega})\cap H_0^1(\Omega)$. This follows from (10) since the adjoint operator L^* satisfies the assumptions given in this paragraph. We have

$$\int_\Omega G_\varepsilon(x)\theta(x)\, dx = \int_\Omega G_\varepsilon(x)L^*\psi(x)\, dx = \int_\Omega LG_\varepsilon(x)\psi(x)\, dx,$$
$$\int_\Omega Z_\varepsilon(x)\theta(x)\, dx = \int_\Omega Z_\varepsilon(x)L^*\psi(x)\, dx,\ \psi\in C^{2,\tilde\alpha}(\overline{\Omega})\cap H_0^1(\Omega).$$

By Theorem 15

$$\int_\Omega (G_\varepsilon(x) - Z_\varepsilon(x))\theta(x)\, dx = \int_\Omega (LG_\varepsilon(x) - H_\varepsilon(x))\psi(x)\, dx \to 0, \text{ as } \varepsilon\to 0.$$

This proves the theorem. □

We continue to consider L of the form (6) which satisfies (7), (8) and (9). Let $\overline{\Omega}$ satisfies the cone property [97, Definition in 4.2.3] in addition. Let

$$\tilde L_\varepsilon = \sum_{i,j=1}^n a_{ij,\varepsilon}(x)D_iD_j + \sum_{i=1}^n b_{i,\varepsilon}(x)D_i + c_\varepsilon(x)$$

be regularized operator for L, where $(1_\Omega a_{ij}) * \check{\phi}_\varepsilon(x) = a_{ij,\varepsilon}(x)$, $(1_\Omega b_i) * \check{\phi}_\varepsilon(x) = b_{i,\varepsilon}(x)$, $(1_\Omega c) * \check{\phi}_\varepsilon(x) = c_\varepsilon(x)$, $x \in \mathbb{R}$, $\check{\phi}_\varepsilon(x) = \phi(-x)$. Clearly, $c_\varepsilon(y) \leqslant 0$, $y \in \Omega$. Since $a_{ij,\varepsilon} \to a_{ij}$, $b_{i,\varepsilon} \to b_i$ and $c_\varepsilon \to c$ in $C^0(\overline{\Omega})$ and $C^1(\overline{\Omega})$ as $\varepsilon \to 0$, one gets convergence in $C^{\tilde{\alpha}}(\overline{\Omega})$, too.

Multiplying a_{ij} in (9) by 1_Ω, then multiplying all the members of (9) by $\phi_\varepsilon(y-x)$ and by integrating over $\mathbb{R}^n$, we obtain

$$\lambda|\xi|^2 \int_\Omega \phi_\varepsilon(y-x)\,dy \leqslant \sum_{i,j=1}^n a_{ij,\varepsilon}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2 \int_\Omega \phi_\varepsilon(y-x)\,dy \leqslant \Lambda|\xi|^2,$$

where $y \in \Omega$, $\xi \in \mathbb{R}^n$, $\varepsilon < \varepsilon_0$. By using radial symmetry and non-negativity of the function ϕ and the cone property of $\overline{\Omega}$,

$$\int_\Omega \phi_\varepsilon(y-x)\,dy \geqslant \int_{\Gamma_{x,h}} \phi_\varepsilon(y-x)\,dy \geqslant \tilde{D} > 0,$$

where $\Gamma_{x,h}$ is a cone with the vertex at zero and height h, and $y \in x + \Gamma_{x,h} \subset \Omega$. The constant $\tilde{D}$ does not depend on ε and we have

$$\tilde{D}\lambda|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij,\varepsilon}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2,\ x \in \Omega,\ \xi \in \mathbb{R}^n, \varepsilon \leqslant \varepsilon_0.$$

Theorem 19. *Let L satisfy given assumptions. Then the generalized solution G to*

$$\tilde{L}G \overset{H^2(\Omega)\cap H_0^1(\Omega)}{\approx} H \text{ in } \Omega,\ G|_{\partial\Omega} = 0,$$

constructed in Theorem 15, *is $C^{\tilde{\alpha}}(\overline{\Omega}) \cap H_0^1(\Omega)$-associated to the classical solution $g \in H^{2,\tilde{\alpha}}(\overline{\Omega})$ to* (12).

Proof. Let $\theta \in C^{\tilde{\alpha}}(\overline{\Omega}) \cap H_0^1(\Omega)$ be given. Then (10) and (11) applied to L_ε imply that for every $\varepsilon \in (0,1)$ the solution Ψ_ε to $\tilde{L}_\varepsilon^*\Psi_\varepsilon = \theta$ in Ω, $\Psi_\varepsilon|_{\partial\Omega} = 0$ belongs to $C^{2,\tilde{\alpha}}(\overline{\Omega}) \cap H_0^1(\Omega)$ and

$$\sup_{|\beta|\leqslant 2, x\in\Omega} |D^\beta\Psi_\varepsilon(x)| \leqslant C\Big(\sup_{x\in\Omega}|\theta(x)| + \sup_{x,y\in\overline{\Omega}} \frac{|\theta(x)-\theta(y)|}{|x-y|^{\tilde{\alpha}}}\Big) = \|\theta\|_{C^{\tilde{\alpha}}(\overline{\Omega})},\ \varepsilon \in (0,1),$$

where $C > 0$ depends only on λ, Λ and the diameter of the set Ω. By the remark after Theorem 15 we have that for every $s > 0$ there exists G_ε such that

$$\langle \tilde{L}_\varepsilon G_\varepsilon, \Psi_\varepsilon\rangle - \langle H_\varepsilon, \Psi_\varepsilon\rangle = o(\varepsilon^s),\ \varepsilon \to 0.$$

Denote by Z_ε a smooth solution to $\tilde{L}_\varepsilon Z_\varepsilon = H_\varepsilon$ in Ω, $Z_\varepsilon|_{\partial\Omega} = 0$, $\varepsilon \in (0,1)$ is fixed. Then

$$\begin{aligned}(17)\qquad \int_\Omega (G_\varepsilon(x) - Z_\varepsilon(x))\theta(x)\,dx &= \int_\Omega (G_\varepsilon(x) - Z_\varepsilon(x)) L_\varepsilon^*\Psi_\varepsilon(x)\,dx \\ &= \int_\Omega (\tilde{L}_\varepsilon G_\varepsilon(x) - \tilde{L}_\varepsilon Z_\varepsilon(x))\Psi_\varepsilon(x)\,dx = \langle \tilde{L}_\varepsilon G_\varepsilon - H_\varepsilon, \Psi_\varepsilon\rangle = o(\varepsilon^s),\ \varepsilon \to 0.\end{aligned}$$

Let $\tilde{H}_\varepsilon = \tilde{L}_\varepsilon g$, where $g \in C^{2,\tilde{\alpha}}(\overline{\Omega})$ is the solution to (12). Inequality (11) and (16) imply

$$\|(\tilde{L}_\varepsilon - L)g\|_{C^{\tilde{\alpha}}(\overline{\Omega})} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \|a_{ij,\varepsilon} - a_{ij}\|_{C^{\tilde{\alpha}}(\overline{\Omega})} \|D_iD_jg\|_{L^\infty(\Omega)}$$
$$+ \sum_{i=1}^{n} \|b_{i,\varepsilon} - b_i\|_{C^{\tilde{\alpha}}(\overline{\Omega})} \|D_ig\|_{L^\infty(\Omega)} + \|c_\varepsilon - c\|_{C^{\tilde{\alpha}}(\overline{\Omega})} \|g\|_{L^\infty(\Omega)} \to 0,$$

as $\varepsilon \to 0$. This implies that $\tilde{H}_\varepsilon - h \xrightarrow{C^{\tilde{\alpha}}(\overline{\Omega})} 0$ as $\varepsilon \to 0$, i.e., $\tilde{H}_\varepsilon - H_\varepsilon \xrightarrow{C^{\tilde{\alpha}}(\overline{\Omega})} 0$ (since $h \in C^1(\Omega)$. Finally, (10) implies $\|Z_\varepsilon - g\|_{C^{2,\tilde{\alpha}}(\overline{\Omega})} \leqslant C\|\tilde{H}_\varepsilon - H_\varepsilon\|_{C^{\tilde{\alpha}}(\overline{\Omega})} \to 0$. This and (17) prove the theorem. □

3.1.4. A class of generalized elliptic differential operators of order $2m$. Consider a family of equations

$$(18)\qquad \begin{aligned} &P_\varepsilon(x, D)G_\varepsilon = \sum_{|\alpha| \leqslant 2m} a_{\alpha,\varepsilon}(x) D^\alpha G_\varepsilon(x) = H_\varepsilon(x),\ x \in \Omega, \\ &\{D^\alpha G_\varepsilon|_{\partial\Omega},\ \alpha \in \mathbb{N}_0^n\} \in \mathcal{N}_W(\partial\Omega), \end{aligned}$$

where:
1. Ω is an open bounded set with a smooth boundary $\partial\Omega$.
2. $a_{\alpha,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$ is complex valued, $|\alpha| \leqslant 2m$, $H_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$.
3. For every $\varepsilon \in (0, \varepsilon_0)$, $P_\varepsilon(x, D)$ is uniformly and strongly elliptic (cf. [96, Ch. 36, (36.3)]) and moreover, there exist $C_0 > 0$ and $p_0 \geqslant 0$ such that

$$C_0\varepsilon^{p_0}\|u\|^2_{H^m(\Omega)} \leqslant \mathrm{Re}\langle P_\varepsilon(x, D)u, u\rangle_{L^2(\Omega)},\ u \in C_0^\infty,\ \varepsilon < \varepsilon_0.$$

Then, for every fixed $\varepsilon < \varepsilon_0$,

$$(19)\qquad \begin{aligned} &P_\varepsilon(x, D) : H_0^m(\Omega) \to H^{-m}(\Omega) \text{ is a surjective isomorphism} \\ &\text{and the solution to (18) satisfies } \|G_\varepsilon\|_{H^m} \leqslant C_0^{-1}\varepsilon^{-p_0}\|H_\varepsilon\|_{H^{-m}}(\Omega). \end{aligned}$$

(cf. [96, Theorem 36.2, Lemma 23.1])

The second assumption means that there exist $\nu_1, \nu_2 > 0$ such that

$$(20)\qquad \sup_{|\alpha| \leqslant m} \|a_{\alpha,\varepsilon}\|_{L^\infty(\overline{\Omega})} = \mathcal{O}(\varepsilon^{-\nu_1}),$$

$$(21)\qquad \|H_\varepsilon\|_{L^\infty(\overline{\Omega})} = \mathcal{O}(\varepsilon^{-\nu_2}),\ \text{as } \varepsilon \to 0.$$

Note that (21) implies $\|H_\varepsilon\|_{H^{-m}(\Omega)} = \mathcal{O}(\varepsilon^{-\nu_2})$, as $\varepsilon \to 0$.

Theorem 20. *Let $P_\varepsilon(x, D)$, H_ε and Ω satisfy the conditions given above. Then:*

(a) *For every $s \geqslant 0$ there exists a solution $G_{s,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$ to (18) in the $H_0^{2m}(\Omega)$-s-associated sense, i.e.,*

$$\langle P_\varepsilon(x, D)G_{s,\varepsilon} - H_\varepsilon, \psi\rangle = o(\varepsilon^s),\ \varepsilon \to 0,\ \psi \in H_0^{2m}(\Omega)$$

and $\{D^\alpha G_{s,\varepsilon}|_{\partial\Omega} = 0,\ \alpha \in \mathbb{N}_0^n\} \in \mathcal{N}_W(\partial\Omega)$. The solution constructed in the proof will be called s-solution.

(b) *Let $T_{s,\varepsilon}$ be an s-solution to$P_\varepsilon(x,D)T_{s,\varepsilon} = R_\varepsilon$, where R_ε satisfies* (21). *Moreover, assume that*

$$\varepsilon^{-(p_0+s_0m)}\|R_\varepsilon - H_\varepsilon\|_{H^{-m}(\Omega)} \to 0 \text{ as } \varepsilon \to 0,$$

where $s_0 > \nu_1+\nu_2+p_0+s$. Then $T_{s,\varepsilon}$ and $G_{s,\varepsilon}$ are $H^{-m}(\Omega)$-0-associated. Specially, if R_ε and H_ε are in the same class of $\mathcal{G}(\Omega)$, then the appropriate solutions are $H^{-m}(\Omega)$-$\tilde{s}$-associated for arbitrary $\tilde{s} > 0$. This is a kind of the uniqueness of solution to (18).

(c) *Let $P(x,D)$ have $C^\infty(\mathbb{R}^n)$ coefficients which satisfy condition* 3 *with $p_0 = 0$ and let C_0 do not depend on ε. Assume that $H \in H^{-m}(\Omega)$ is of the form $H = \sum_{|\alpha|\leqslant m} D^\alpha f_\alpha$, where $f_\alpha \in L^2(\Omega)$, $f_\alpha(x) = 0$ for $x \notin \Omega$, $|\alpha| \leqslant m$. Let*

(i) *$U \in H_0^m(\Omega)$ be the solution to $P(x,D)U = H$, $U|_{\partial\Omega} = 0$ (which exists by the Lax–Milgram Lemma) and*

(ii) *$G_{s,\varepsilon}$ be the s-solution to*

$$P(x,D)G_{s,\varepsilon} = H_\varepsilon \text{ in } \Omega,\ G_\varepsilon|_{\partial\Omega} = 0, \tag{22}$$

in H_0^{2m}-s-associated sense, where $H_\varepsilon = H * \phi_{\varepsilon^d}$ for an appropriate $d = d(s) > 0$. Then $U \overset{H^{-m}(\Omega)}{\approx} G_0$.

(d) *Denote by G_ε the solution in $\mathbb{D}_0^m(\Omega)$-0-associated sense to* (22) *constructed in the proof of Theorem* 15 *and by $G_{0,\varepsilon}$ the solution to* (22) *in $H_0^{2m}(\Omega)$-0-associated sense. Then $G_\varepsilon \overset{\mathbb{D}_P(\Omega)}{\approx} G_{0,\varepsilon}$, where $\mathbb{D}_P(\Omega)$ is the set of all nets Ψ_ε in $H_0^0(\Omega)$ such that there exist a function $\psi \in H_0^{2m}(\Omega)$ and $\eta > 0$ such that $\Psi_\varepsilon = P_\varepsilon^*\psi$, for every $\varepsilon < \eta$.*

Proof. (a) Recall, (19) implies that for every fixed $\varepsilon < \varepsilon_0$ there exists a solution $g_\varepsilon \in H_0^m(\Omega)$ to equation $P_\varepsilon(x,D)g_\varepsilon = H_\varepsilon$, in Ω, that is

$$\langle P_\varepsilon(x,D)g_\varepsilon, \psi\rangle = \langle H_\varepsilon, \psi\rangle,\ \psi \in H_0^m(\Omega). \tag{23}$$

By ellipticity of $P_\varepsilon(x,D)$ for every fixed ε, the solution g_ε to (23) is in $C^\infty(\Omega)$. Let us prove that $g_\varepsilon \in \mathcal{E}_M(\Omega)$. Let D_i be an arbitrary derivative of the first order. Then

$$D_i(P_\varepsilon(x,D)g_\varepsilon(x)) = P_\varepsilon(x,D)D_ig_\varepsilon(x) + \tilde{P}_\varepsilon(x,D)g_\varepsilon(x),$$

where $\tilde{P}_\varepsilon(x,D) = \sum_{|\alpha|\leqslant 2m} D_ia_{\alpha,\varepsilon}(x)D^\alpha g_\varepsilon(x)$. Integration by parts implies

$$\begin{aligned}\|\tilde{P}_\varepsilon(\cdot,D)g_\varepsilon\|_{H^{-m}(\Omega)} &= \sup_{\|\phi\|_{H^m(\Omega)}\leqslant 1}\left|\int_\Omega \tilde{P}_\varepsilon(x,D)g_\varepsilon(x)\phi(x)\,dx\right| \\ &\leqslant \sup_{\|\phi\|_{H^m(\Omega)}\leqslant 1}\int_\Omega\left|\sum_{|\alpha|\leqslant m} b_{\alpha,\varepsilon}(x)D^\alpha g_\varepsilon(x)\right|\cdot\left|\sum_{|\beta|\leqslant m} c_{\beta,\varepsilon}D^\beta\phi(x)\right|dx,\end{aligned}$$

where $b_{\alpha,\varepsilon}, c_{\beta,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$, $|\alpha| \leqslant m$, $|\beta| \leqslant m$. Since (19) and (23) imply

$$\|g_\varepsilon\|_{H^m(\Omega)} = \mathcal{O}(\varepsilon^{-(p_0+\nu_2)}),\ \varepsilon \to 0,$$

then

$$\|\tilde{P}_\varepsilon(\cdot,D)g_\varepsilon\|_{H^{-m}(\Omega)} = \mathcal{O}(\varepsilon^{-\nu_3}),\ \varepsilon \to 0,$$

and

$$P_\varepsilon(x,D)D_i g_\varepsilon(x) = D_i H_\varepsilon(x) - \tilde{P}_\varepsilon(x,D)g_\varepsilon(x) \in H^{-m} \cap \mathcal{E}_M(\mathbb{R}^n)$$

for a suitable $\nu_3 > 0$. Thus, (19) implies $\|D_i g_\varepsilon\|_{H^m(\Omega)} = \mathcal{O}(\varepsilon^{-\nu_4})$, $\varepsilon \to 0$, for some $\nu_4 > 0$. By induction with the respect to the orders of derivatives, it follows that $g_\varepsilon \in \mathcal{E}_M(\Omega)$. Put

$$\kappa_\varepsilon = \mathbf{1}_{(\Omega)_{-3\varepsilon^{s_0}/4}} * \phi_{\varepsilon^{s_0}/4}, \tag{24}$$

where s_0 will be determined later. Note that $\operatorname{mes}(\Omega \setminus (\Omega)_{-\varepsilon^{s_0}}) = \mathcal{O}(\varepsilon^{s_0})$ as $\varepsilon \to 0$. Define $G_{s,\varepsilon} = g_\varepsilon \kappa_\varepsilon$. Then $G_{s,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$ since $g_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$.

For arbitrary $\psi \in H_0^{2m}$,

$$\begin{aligned}
|I_\varepsilon| &= |\langle P_\varepsilon(x,D)G_{s,\varepsilon} - H_\varepsilon, \psi\rangle| = |\langle G_{s,\varepsilon} - H_\varepsilon, P_\varepsilon^*(x,D)\psi\rangle| \\
&\leqslant \int_\Omega |g_\varepsilon(x)(1-\kappa_\varepsilon(x))| \cdot |P_\varepsilon^*(x,D)\psi(x)|\,dx \\
&\leqslant 2 \sup_{|\alpha|\leqslant m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{\Omega})} \|\psi\|_{H^{2m}(\Omega)} \|g_\varepsilon\|_{L^2(\Omega)} (\operatorname{mes}(\Omega \setminus (\Omega)_{-\varepsilon^{s_0}}))^{1/2} \\
&= \mathcal{O}(\varepsilon^{s_0 - (\nu_1+\nu_2+p_0)}),\ \varepsilon \to 0.
\end{aligned}$$

The assertion follows by choosing $s_0 > \nu_1 + \nu_2 + p_0 + s$.

(b) Let t_ε be the solution to (23) when H_ε is replaced by R_ε and $T_\varepsilon = t_\varepsilon \kappa_\varepsilon$, where κ_ε is given by (24). Then, (19) implies

$$\begin{aligned}
\|G_{s,\varepsilon} - T_{s,\varepsilon}\|_{H^m(\Omega)} &= \|(g_{s,\varepsilon} - t_{s,\varepsilon})\kappa_\varepsilon\|_{H^m(\Omega)} \\
&\leqslant \sup_{|\alpha|\leqslant m} \|g_{s,\varepsilon} - t_{s,\varepsilon}\|_{H^m(\Omega)} \|D^\alpha \kappa_\varepsilon\|_{L^\infty(\Omega)} \\
&\leqslant C_0^{-1} \varepsilon^{-p_0} \varepsilon^{-s_0 m} \|F_\varepsilon - R_\varepsilon\|_{H^{-m}(\Omega)}.
\end{aligned}$$

Now, for $\psi \in H^{-m}(\Omega)$

$$|\langle G_{s,\varepsilon} - T_{s,\varepsilon}, \psi\rangle| \leqslant C\|G_{s,\varepsilon} - T_{s,\varepsilon}\|_{H^m(\Omega)} \|\psi\|_{H^{-m}(\Omega)} \leqslant C_1 \varepsilon^{-p_0 - s_0 m} \|F_\varepsilon - R_\varepsilon\|_{H^{-m}(\Omega)}$$

and the proof follows.

(c) The assertion is a direct consequence of (b).

(d) Let $\Psi_\varepsilon \in \mathbb{D}_P(\Omega)$ and G_ε be the solution constructed in Theorem 15. By the definition of $\mathbb{D}_P(\Omega)$ there exists $\psi_1 \in H_0^{2m}(\Omega)$ such that $P_\varepsilon^*(x,D)\psi_1 = \Psi_\varepsilon$ for every $\varepsilon < \eta$. Then

$$\begin{aligned}
\int_\Omega (G_\varepsilon - G_{0,\varepsilon})(x)\Psi_\varepsilon(x)\,dx &= \int_\Omega (G_\varepsilon - G_{0,\varepsilon})(x) P_\varepsilon^*(x,D)\psi_1(x)\,dx \\
&= \int_\Omega P_\varepsilon(x,D)(G_\varepsilon - G_{0,\varepsilon})(x)\psi_1(x)\,dx \to 0,\ \varepsilon \to 0.
\end{aligned}$$

This proves (d). □

3.2. Quasilinear elliptic PDE. First we give a simple example in order to illustrate our approach to a class of quasilinear elliptic PDE.

Example 21. Let $\delta_{(0,1)}(x_1,x_2)=\delta(x_1)\delta(x_2-1)$, $(x_1,x_2)\in\mathbb{R}^2$ be the delta distribution concentrated at $(0,1)$ and B_1 be the ball with the radius 1 and center $(0,0)$. Define $\delta_{(0,1)}|_{\partial B_1}$ by $\langle\delta_{(0,1)}|_{\partial B_1},\phi\rangle=\phi(0,1)$, where ϕ is a smooth function on the circle ∂B_1. Consider a Dirichlet problem formally written as

$$\Delta u(x)=0,\ x\in B_1\subset\mathbb{R}^2,\quad u|_{\partial B_1}=\delta_{(0,1)}|_{\partial B_1}.$$

We approximate $\delta(x_1)\delta(x_2-1)$ by a net $\frac{1}{\varepsilon^2}\phi\Big(\frac{x_1}{\varepsilon}\Big)\phi\Big(\frac{x_2-1}{\varepsilon}\Big)$, $(x_1,x_2)\in\mathbb{R}^2$, $\varepsilon\in(0,1)$, with the properties $\phi\in C_0^\infty(\mathbb{R}),\int\phi=1$ and $\operatorname{supp}\phi\in[-1,1]$ (the net of mollifiers). Then, we replace the given problem with the family of problems

$$\Delta u_\varepsilon(x)=0,\quad x\in B_1\subset\mathbb{R}^2,$$

$$u_\varepsilon\big|_{\partial B_1}=\frac{1}{\varepsilon^2}\phi\Big(\frac{x_1}{\varepsilon}\Big)\phi\Big(\frac{\sqrt{1-x_1^2}-1}{\varepsilon}\Big),\ |x_1|<\varepsilon$$

and zero on the rest of the boundary.

Using the Poisson formula we obtain a family of solutions U_ε of corresponding classical solutions.

Assume that $\phi(0)=0$. Then, in the sense of the weak convergence in $\mathcal{D}'(B_1)$,

$$U_\varepsilon(x)\to f(x)=C_\phi\frac{|x|^2-1}{x_1^2+(x_2-1)^2}\quad\text{as }\varepsilon\to 0,\text{where } C_\phi=-\frac{1}{4\pi}\phi'(0)\int_{-1}^{1}u^2\phi(u)\,du.$$

Note that f is a solution to $\Delta u=0$ in B_1. Moreover, for any point $(x_{10},x_{20})\in\partial B_1\setminus(0,1)$

$$\lim_{(x_1,x_2)\to(x_{10},x_{20})}\frac{x_1^2+x_2^2-1}{x_1^2+x_2^2-2x_2+1}=0\quad((x_1,x_2)\in B_1)$$

and, for $\theta=\pi/2$, $\lim\limits_{r\to1-}\frac{r^2-1}{r^2-2r+1}=-\infty$. This shows the "blow up" of a solution at $(0,1)$.

Let Q_ε be a net of elliptic nonlinear operators of divergent type of the form

$$Q_\varepsilon(u)=\operatorname{div}\mathbf{A}_\varepsilon(Du)=a_\varepsilon^{i,j}(Du)D_{i,j}u,\varepsilon<1,\tag{25}$$

where $a_\varepsilon^{i,j}(p)=D_{p_i}A_\varepsilon^j(p)$, or, in case $n=2$, let Q_ε be a net of elliptic nonlinear operators of the form

$$Q_\varepsilon(u)=a_\varepsilon^{i,j}(x,u,Du)D_{i,j}u,\ u\in C^\infty(\bar{O}).\tag{26}$$

We assume that $a_\varepsilon^{i,j}$, $\varepsilon\in(0,1)$ are smooth functions on O. If λ_ε and Λ_ε denote respectively the minimum and maximum eigenvalues, then we have

$$0<\lambda_\varepsilon(x,t,p)|\xi|^2\leqslant a_\varepsilon^{i,j}(x,t,p)\xi_i\xi_j\leqslant\Lambda_\varepsilon(x,t,p)|\xi|^2,$$
$$p\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n\setminus\{0\},\ x\in O,\ t\in\mathbb{R},\ \varepsilon<\varepsilon_0.$$

Assume additionally:

(27) $(\forall d \in \mathbb{N}_0^n)(\exists l_d \in \mathbb{R})(\exists a_d \in \mathbb{R})$

$$\sup\left\{\frac{|\partial_x^d a_\varepsilon^{i,j}(x,t,p)|}{(1+|t|+|p|)^{a_d}};\ x \in \bar{O},\ t \in \mathbb{R},\ p \in \mathbb{R}^n\right\} = \mathcal{O}(\varepsilon^{l_d}).$$

(28) $(\exists C > 0)(\exists \mu > 0)(\exists b \in \mathbb{R})$

$$\frac{\varepsilon^\mu}{C}(1+|t|+|p|)^b \leqslant \lambda_\varepsilon(x,t,p)) \leqslant \Lambda_\varepsilon(x,t,p) \leqslant \frac{C}{\varepsilon^\mu}(1+|t|+|p|)^b,$$
$$p \in \mathbb{R}^n,\ x \in \bar{O},\ t \in \mathbb{R}, \varepsilon < \varepsilon_0.$$

In the case when the net $\mathcal{Q}_\varepsilon$ is of the form (26) then $n = 2$, and if it is of the divergent form (25), then we exclude variables x and t in the conditions given above. Note that condition (28) implies $\Lambda_\varepsilon/\lambda_\varepsilon \leqslant C^2/\varepsilon^{2\mu}$, $\varepsilon < \varepsilon_0$. We will consider this net in the framework of $\mathcal{G}_{C^{k,\alpha}}$. In this case we will use the notation $\mathcal{F} = \mathcal{E}_{C^{k,\alpha}}$. With the given properties $\mathcal{Q}_\varepsilon$ is called the net of uniformly elliptic moderate continuous operators.

Example 22. (i) All the examples given in [26, pp. 260–262], for $n = 2$ can serve as examples in our framework but now with singular boundary conditions.

(ii) Consider in $\mathbb{R}^3$ the operator

$$Q(x,u,Du) = \Big(1 + \sum_{i=1}^{3} \delta(D_i)\Big)\Delta u \quad (\delta \text{ is the delta distribution}).$$

With the regularization of $\delta = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon})$, we have

$$Q_\varepsilon(x,u,Du) = \Big(\frac{1}{\varepsilon}\phi\Big(\frac{D_1 u}{\varepsilon}\Big) + \frac{1}{\varepsilon}\phi\Big(\frac{D_2 u}{\varepsilon}\Big)\frac{1}{\varepsilon}\phi\Big(\frac{D_3 u}{\varepsilon}\Big) + 1\Big)\Delta u.$$

(ϕ is a compactly supported smooth function with the integral equals 1.) Then

$$\lambda_\varepsilon = 1 \text{ and } \Lambda_\varepsilon = \Big(\frac{1}{\varepsilon}\phi\Big(\frac{p_1}{\varepsilon}\Big) + \frac{1}{\varepsilon}\phi\Big(\frac{p_2}{\varepsilon}\Big)\frac{1}{\varepsilon}\phi\Big(\frac{p_3}{\varepsilon}\Big) + 1\Big).$$

This operator is of the form (25) for which all the assumptions given above hold.

We need a "slope condition" adapted to the setting of Colombeau theory.

Definition 23. Let $E = C^{k,\alpha}(\bar{O})$ for some $k \in \mathbb{N}$ (cf. 1.1 and $\mathcal{G}_{C^{k,\alpha}}$), $\psi_\varepsilon \in \mathcal{F} = \mathcal{E}_{C^{k,\alpha}}$ and $\Gamma_\varepsilon = \{(x, z_\varepsilon),\ x \in \partial O,\ z_\varepsilon = \psi_\varepsilon(x)\}$. The boundary ∂O satisfies a moderate slope condition if for any $P_\varepsilon \in \Gamma_\varepsilon$ there exist hyperplanes $\pi^+_{\varepsilon,P_\varepsilon}$ and $\pi^-_{\varepsilon,P_\varepsilon}$ defined by $z_\varepsilon = \pi^+_{\varepsilon,P_\varepsilon}(x)$ and $z_\varepsilon = \pi^-_{\varepsilon,P_\varepsilon}(x)$ such that

$$\pi^-_{\varepsilon,P_\varepsilon}(x) \leqslant \psi_\varepsilon(x) \leqslant \pi^+_{\varepsilon,P_\varepsilon}(x),\ x \in \partial O,\ \varepsilon < \varepsilon_0$$

and such that for some $K > 0$ and some $m \in \mathbb{R}$,

$$\sup\{|D\pi^+_{\varepsilon,P_\varepsilon}(x)|, |D\pi^-_{\varepsilon,P_\varepsilon}(x)|; x \in \partial O, P_\varepsilon \in \Gamma_\varepsilon\} \leqslant K\varepsilon^m, \varepsilon < \varepsilon_0.$$

Proposition 24. *Let $\mathcal{Q}_\varepsilon$ be a net of uniformly elliptic operators of the form* (25) *or* (26) *with $a_\varepsilon^{i,j} \in C^{k+1}(\bar{O})$ ($k \in \mathbb{N}$) satisfying* (27) *with $d \leqslant k+1$ and* (28).

Let $E = C^{k+2,\alpha}(\bar{O})$, $\psi_\varepsilon \in \mathcal{E}_{C^{k+2,\alpha}}$, where ∂O is of $C^{k+2,\alpha}$ class and it satisfies a moderate slope condition with ψ_ε. Then, there exists $U_\varepsilon \in \mathcal{E}_{C^{k+2,\alpha}}$ such that

$$\mathcal{Q}_\varepsilon(U_\varepsilon) = 0,\ U_\varepsilon\big|_{\partial O} = \psi_\varepsilon,\ \varepsilon < 1.$$

This theorem implies the solvability in $\mathcal{G}_{C^{k,\alpha}}$. The process of regularization of equation $\operatorname{div} \mathbf{A}(Du) = 0$, $u_{|\partial O} = \psi$ with singular coefficients and singular data leads to the approximated net of solutions by the mean of previous theorem.

4. Hyperbolic PDEs

4.1. Semilinear wave equation. In our approach we connect two areas: the L^2-theory for the nonlinear wave equation

(29) $$\partial_t^2 u - \Delta u + g(u) = 0,\ g(0) = 0,\ u = u(x,t),\ x \in \mathbb{R}^n,\ t \geqslant 0,$$

with the initial data

$$u(x,0) = a(x),\ u_t(x,) = b(x),\ x \in \mathbb{R}^n,$$

involving energy estimates and the theory of generalized functions where nonlinear operations makes sense for a large collection of singular objects.

Concerning g, if it is not globally Lipschitz, then it is substituted by a net of globally Lipschitz functions $g_\varepsilon(u)$. Then the obtained net of equations, called regularized equation, is solved for each fixed ε.

In some cases g is not regularized and the growth conditions on g are involved for the existence and uniqueness of a solution similarly as in the classical theory.

We use here the algebra $\mathcal{G}_{\infty,L^2}([0,T) \times \mathbb{R}^n)$. Also we use the notation $\mathcal{F} = \mathcal{E}_{L^2}([0,T) \times \mathbb{R}^n)$. Consider a family of equations in $\mathcal{E}_{\infty,L^2}([0,T) \times \mathbb{R}^n)$

(30) $$(\partial_t^2 - \Delta)G_\varepsilon = -g(G_\varepsilon),\ G_\varepsilon|_{t=0} = A_\varepsilon,\ \partial_t G_\varepsilon|_{t=0} = B_\varepsilon,\ \varepsilon \in (0,1),$$

where $A_\varepsilon, B_\varepsilon \in \mathcal{E}_{\infty,L^2}(\mathbb{R}^n)$ and $g : \mathbb{R}^n \to \mathbb{R}$ is smooth, polynomially bounded together with all its derivatives and $g(0) = 0$.

Equation (30), with the regularization g_ε instead of g is called the regularized equation for (29).

Proposition 25. a) *Let $n \leqslant 5$. Then there exists a regularized net g_ε such that for every $T > 0$ there exists a unique solution to* (30) *in $\mathcal{G}_{\infty,L^2}([0,T) \times \mathbb{R}^n)$.*

b) *Let $n = 6$ and let $\|A_\varepsilon\|_{H^{3,2}}$ and $\|B_\varepsilon\|_{H^{2,2}}$ be bounded by $(\log(\log(\varepsilon^{-1})))^s$, as $\varepsilon \to 0$, where $s < 1$. Then there exists a regularized net g_ε such that for every $T > 0$ there exists a unique solution to* (30) *in $\mathcal{G}_{\infty,L^2}([0,T) \times \mathbb{R}^n)$.*

Remark 26. Let $n = 7$. In order to obtain the existence of a unique solution with the moderate growth of all its derivatives, we need that $H^{3,2}$-norms of initial data are bounded by $\underbrace{\log(\log\ldots(\log}_{q}\varepsilon^{-1})\ldots)^s$ with respect to ε for some s and q. This follows from [80, Theorem 4.8]. Cases $n = 8, 9$ can be handled out using the procedure and Lemmas 2.1–2.20 in the same paper as well as a composition of the logarithmic function sufficiently many times.

The proof of quoted theorem for $n = 3$ implies the next corollary.

Corollary 27. *Let $n = 3$, $g(y)$ be globally Lipschitz and its first derivative be polynomially bounded. Then for every $T > 0$ there exists a solution to (30) in $\mathcal{G}_{\infty,L^2}([0,T)\times\mathbb{R}^n)$.*

Remark 28. If $g(y)$ is globally Lipschitz, for $n = 4,5,6$, we need to assume appropriate conditions for the first and second derivatives of g. If $n = 7,8,9$, then the assumptions of corollary are more complicated.

For a non-regularized wave equation, the following is true.

Proposition 29. *Equation*

$$(\partial_t^2 - \Delta)G = -G^3,\ G|_{t=0} = A,\ \partial_t G|_{t=0} = B,$$

where $A, B \in \mathcal{G}_{\infty,L^2(\mathbb{R}^3)}$, has a unique solution in $\mathcal{G}_{\infty,L^2}([0,T)\times\mathbb{R}^3)$ for every $T > 0$ if there exist representatives of initial data such that

$$\|(\nabla^2 A_\varepsilon, \nabla B_\varepsilon)\|_{L^2} = o((\log\varepsilon^{-1})^{1/2}).$$

The presented results are from [60]. One can look also in [10, 74] for some other results concerning the wave equation.

4.2. Stochastic wave equations. We will present the main results from the paper [69] concerning the different stochastic wave equations. Before that, we have to define Colombeau generalized processes like in [75].

Consider the problem

$$(\partial_t^2 - \partial_x^2)U + F(U)\cdot S = 0, \tag{31}$$

$$U|_{t=0} = A,\ \partial_t U|_{t=0} = B, \tag{32}$$

where A and B are $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes on $\mathbb{R}$, that is, $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R})$, and $S \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ is $\mathcal{G}_{2,2}$-Colombeau generalized stochastic process on $\mathbb{R}^2$ with compact support. We suppose that the function F is smooth, polynomially bounded together with all its derivatives and that $F(0) = 0$. We look for a solution $U \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$. We substitute F by a family of smooth functions F_ε, $\varepsilon \in (0,1)$, which is called the regularization of F. This is done in the following way. We choose the smooth function F_ε with the property that there exists a net a_ε such that for every $\alpha \in \mathbb{N}_0$ there exist $\varepsilon_0 \in (0,1)$ and $m^\alpha \in \mathbb{N}$ such that

$$F_\varepsilon(y) = F(y), \text{for}|y| \leqslant a_\varepsilon,\ \varepsilon < \varepsilon_0$$
$$\|D^\alpha F_\varepsilon(y)\|_{L^\infty} = \mathcal{O}(a_\varepsilon^{m_\alpha}).$$

In the sequel we shall denote $m = \sup_{|\alpha|\leqslant 1} m^\alpha$.

Denote by $\tilde{F} = [F_\varepsilon]$, where $F_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ has properties as above. Then, instead of non-regularized equation (31)–(32), we consider the regularized one

$$(\partial_t^2 - \partial_x^2)U + \tilde{F}(U)\cdot S = 0, \tag{33}$$

$$U|_{\{t=0\}} = A,\ \ \partial_t U|_{\{t=0\}} = B, \tag{34}$$

where $S = [S_\varepsilon] \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ and $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R})$.

Note that for $U_\varepsilon, V_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$ such that $U_\varepsilon - V_\varepsilon \in \mathcal{N}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$, we have that $\tilde{F}(U_\varepsilon) - \tilde{F}(V_\varepsilon) \in \mathcal{N}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$.

First, one can see the connection of regularized and non-regularized one dimensional wave equation in the following theorem.

Theorem 30. *Let G, a primitive function of F, be nonnegative and $G(0) = 0$. Let Colombeau generalized stochastic process $S \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R})$ be nonnegative and depend only on the variable x, i.e., there exists a representative S_ε of S such that $S_\varepsilon(x) \geqslant 0$, for all ε small enough and $x \in \mathbb{R}$. Suppose that $\|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2} = o(a_\varepsilon)$, as $\varepsilon \to 0$, where a_ε is the corresponding net used in regularization of function F. Then, for every $T > 0$, the solution to the regularized equation (33)–(34) is also the solution to the non-regularized equation (31)–(32).*

4.2.1. Cubic wave equation with nonnegative stochastic process.

We consider the problem

$$(\partial_t^2 - \triangle)U + U^3 \cdot S = 0, \tag{35}$$

$$U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B, \tag{36}$$

where we suppose that $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$ are $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes such that

$$\|(B_\varepsilon, \nabla A_\varepsilon)\|_{L^2} = o\big((\log \varepsilon^{-1})^{1/4}\big), \tag{37}$$

and $S \in \mathcal{G}^{\Omega}_g(\mathbb{R}^3)$ is nonnegative $\mathcal{G}_g$-Colombeau generalized stochastic process which depends only on variable x and such that

$$\|S_\varepsilon\|_{L^\infty} = o\big((\log \varepsilon^{-1})^{1/2}\big). \tag{38}$$

Theorem 31. *Let stochastic processes $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$ have representatives which satisfy condition (37) and $S \in \mathcal{G}^{\Omega}_g(\mathbb{R}^3)$ be nonnegative stochastic process that depends only on variable x and has a representative which satisfies (38). Then, for every $T > 0$, problem (35),(36) has a unique solution almost surely in $\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$.*

Remark 32. If a Colombeau stochastic generalized process S is a image of a generalized stochastic process, then one can use a regularization which ensures estimate (38). This remark could be added after each further assertion when we need estimates on a stochastic term.

4.2.2. Cubic wave equation with multiplicative stochastic process.

We consider the problem

$$(\partial_t^2 - \triangle)U + U \cdot S + U^3 = 0, \tag{39}$$

$$U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B, \tag{40}$$

where stochastic processes $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$ are such that

$$\|(B_\varepsilon, \nabla A_\varepsilon)\|_{L^2} = o\big((\log \varepsilon^{-1})^{1/2}\big), \tag{41}$$

and $S \in \mathcal{G}_g^{\Omega}([0,T) \times \mathbb{R}^3)$ is such that

$$\|S_\varepsilon\|_{L^\infty} = o\big(\log(\log \varepsilon^{-1})^{1/2T}\big). \tag{42}$$

Theorem 33. *Let $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes $A, B \in \mathcal{G}_{2,2}^{\Omega}(\mathbb{R}^3)$ satisfy condition* (41) *and $\mathcal{G}_g$-Colombeau stochastic process $S \in \mathcal{G}_g^{\Omega}([0,T) \times \mathbb{R}^3)$ satisfy* (42). *Then, for every $T > 0$, problem* (39)–(40) *has a unique solution almost surely in $\mathcal{G}_{2,2}^{\Omega}([0,T) \times \mathbb{R}^3)$.*

4.2.3. Cubic Klein–Gordon equation with additive stochastic process. We consider the problem

$$(\partial_t^2 - \Delta)U + U + U^3 + S = 0, \tag{43}$$

$$U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B, \tag{44}$$

where stochastic processes $A, B \in \mathcal{G}_{2,2}^{\Omega}(\mathbb{R}^3)$ satisfy

$$\|(B_\varepsilon, \nabla A_\varepsilon)\|_{L^2} = o\big((\log \varepsilon^{-1})^{1/2}\big), \tag{45}$$

and $S \in \mathcal{G}_{2,2}^{\Omega}([0,T) \times \mathbb{R}^3)$ is such that

$$\|S_\varepsilon\|_{L^\infty} = o\left(\left(\log \varepsilon^{-1}\right)^{1/2}\right) \tag{46}$$

$$S_\varepsilon \text{ has a compact support.} \tag{47}$$

Theorem 34. *Let $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes $A, B \in \mathcal{G}_{2,2}^{\Omega}(\mathbb{R}^3)$ and $S \in \mathcal{G}_{2,2}^{\Omega}([0,T) \times \mathbb{R}^3)$ satisfy conditions* (45) *and* (46)–(47), *respectively. Then, for $T > 0$, the problem* (43)–(44) *has a unique solution almost surely in $\mathcal{G}_{2,2}^{\Omega}([0,T) \times \mathbb{R}^3)$.*

The literature concerning the stochastic wave equation and generalized processes is quite rich. One can look in papers [1] or [75], for example.

5. Semilinear parabolic PDE

Two types of equations in generalized functions algebra, $\mathcal{G}_{C^1,H^2}([0,T) : \mathbb{R}^n)$ (given below). The first one is a Cauchy problem

$$(\partial_t - \Delta)U + VU = 0, \; U(0,x) = U_0(x),$$

where potential V is a singular distribution, for example the delta distribution or a linear combination of its derivatives. It will be presented here.

The second type is a nonlinear Cauchy problem

$$(\partial_t - \Delta)U + VU = f(t,U), \; U(0,x) = U_0(x),$$

where f satisfies certain conditions.

In both types of equations U_0 is an element of Colombeau-type space, $\mathcal{G}_{H^2}(\mathbb{R}^n)$. This involves singular data, embedded singular distributions, for example of the form $U_0 = \sum_{i=0}^{2} f_i^{(i)}$, $f_i \in L^2$, $i = 0,1,2$, again the important standpoint of our approach.

We will present the use of generalized C_0-semigroups in solving a class of heat equations with singular potentials and singular data. First note that the multiplication of elements $G \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ and $H \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ gives an element in $\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$. Indeed, if $G_\varepsilon \in \mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)$ and $H_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ then $G_\varepsilon H_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$.

Similarly, if $G_\varepsilon \in \mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)$ or $H_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)$, then $G_\varepsilon H_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)$. Thus, multiplication of potential $V \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ and a function $U \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ which is expected to be a solution to equation

$$\partial_t U = (\Delta - V)U,\ U(0,x) = U_0(x),$$

makes sense.

Definition 35. Let A be represented by a net A_ε, $\varepsilon \in (0,1)$, of linear operators with the common domain $H^2(\mathbb{R}^n)$ and with ranges in $L^2(\mathbb{R}^n)$. A generalized function $G \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$, $T>0$, is said to be a solution to equation $\partial_t G = AG$ if

$$\sup_{t\in[0,T)} \|\partial_t G_\varepsilon(t,\cdot) - A_\varepsilon G_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \text{for every } a \in \mathbb{R}.$$

5.0.4. General potential. We will consider in this subsection singular potentials, elements of $\mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$. Especially when the potential is a power of the delta generalized function.

Theorem 36. *Let $V \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ be of logarithmic type, $U_0 = [U_{0\varepsilon}] \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ and $[S_\varepsilon]$ be defined as in Example 12. Let $T>0$. Then $U = SU_0 \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ $(U_\varepsilon(t,x) = S_\varepsilon(t)U_{0\varepsilon}(x),\ \varepsilon<1)$ is the unique solution to equation*

$$\partial_t U(t,x) - \Delta U(t,x) + V(x)U(t,x) = 0,\ U(0,x) = U_0(x). \tag{48}$$

in the sense of Definition 35.

Note that in our construction of a solution to (48) the perturbations with elements in $\mathcal{N}_{H^{2,\infty}}$ null-nets do not effect the solution. More precisely, if V_ε is substituted by $V_\varepsilon + R_\varepsilon$, $R_\varepsilon \in \mathcal{N}_{H^{2,\infty}}$, in (48), we have the same generalized solution.

5.0.5. Powers of the generalized delta function as a potential. Let ϕ_ε be a net of mollifiers

$$\phi_\varepsilon = \varepsilon^{-n}\phi(\cdot/\varepsilon),\ \varepsilon \in (0,1), \tag{49}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int \phi(x)dx = 1$ and $\phi(x) \geqslant 0$, $x \in \mathbb{R}^n$. It represents the generalized delta function $\delta = [\phi_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$.

Different ϕ_ε's (with the prescribed properties on ϕ) define different infinitesimal generators. Let us show this. Put $A_\varepsilon = \Delta - \phi_\varepsilon$ and $\tilde{A}_\varepsilon = \Delta - \tilde{\phi}_\varepsilon$, $\varepsilon < 1$. The equality of infinitesimal generators would imply that

$$\|(A_\varepsilon - \tilde{A}_\varepsilon)u\|_{L^2}^2 = \varepsilon^{-2n}\int_{\mathbb{R}^n} |\phi(y) - \tilde{\phi}(y)|^2 |u(\varepsilon y)|^2 dt \leqslant C_a \varepsilon^a \|u\|_{L^2}^2,\ \varepsilon<1$$

for every $a>0$ (and corresponding $C_a>0$). Thus, it follows that $\phi = \tilde{\phi}$.

Let $m \in \mathbb{N}$. We will use the $\delta^m = [\phi_\varepsilon^m]_{m\in\mathbb{N}}$ as the definition of m-th power of $\delta \in \mathcal{G}(\mathbb{R}^n)$. Let $A_{\varepsilon,m}u = (\Delta - \phi_\varepsilon^m)u$, $u \in H^2(\mathbb{R}^n)$, $\varepsilon < 1$. $A_{\varepsilon,m}$ is the infinitesimal generator of the semigroup

$$S_{\varepsilon,m} : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n)),\ S_\varepsilon(t) = \exp((\Delta - \phi_\varepsilon^m)t),\ t \geqslant 0 \text{ (cf. [79])}.$$

It follows that $S_{\varepsilon,m}$ is a representative of a generalized C_0-semigroup $S \in \mathcal{LG}([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n))$.

We know that $S_{\varepsilon,m}\psi$, $\varepsilon < 1$ and $\psi \in L^2(\mathbb{R}^n)$, given by

$$S_\varepsilon(t)\psi(x) = \int_O \exp\left(-\int_0^t \phi_\varepsilon^m(\omega(s))ds\right)\psi(\omega(t))d\mu_x(\omega),\ x \in \mathbb{R}^n,\ t \geqslant 0.$$

Since $\phi_\varepsilon(x) \geqslant 0$, $x \in \mathbb{R}^n$, $\varepsilon < 1$, it follows that the set $\{S_{\varepsilon,m} :\ \varepsilon \in (0,1),\ t \geqslant 0\}$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ (not only moderate). Thus (31) holds for $S_{\varepsilon,m}$.

Our goal is to prove the following theorem, where the assumption $n \geqslant 2$ is crucial.

Theorem 37. *Let $n \geqslant 2$, $m \in \mathbb{N}$, $T > 0$ and $U_0 \in H^2(\mathbb{R}^n)$. Then*

$$U_{\varepsilon,m}(t,x) = \int_O \exp\left(-\int_0^t \varepsilon^{-mn}\phi^m\left(\frac{\omega(s)}{\varepsilon}\right)ds\right)U_0(\omega(t))\,d\mu_x(\omega), \qquad x \in \mathbb{R}^n,\ t \geqslant 0,\ \varepsilon < 1 \tag{50}$$

defines a representative of a solution $U \in \mathcal{G}_{C^1,H^2}([0,T) : \mathbb{R}^n)$ to the equation

$$\partial_t U(t,x) - \Delta U(t,x) + \delta^m(x)U(t,x) = 0,\ U(0,x) = U_0(x).$$

The solution is unique in the sense of Definition 35.

Moreover net (50) *converges to $\tilde{U}(t,\cdot) = e^{-\Delta t}U_0(\cdot)$ uniformly on compact sets of $\mathbb{R}^n$ of $H^2(\mathbb{R}^n)$, for every $t \geqslant 0$.*

We need several notions and properties of n-dimensional Brownian motions, $n \geqslant 2$. Recall that the hitting time τ_A of a subset A of $\mathbb{R}^n$ is defined by $\tau_A = \inf\{t > 0 :\ \omega(t) \in A\}$ ($t_A = \infty$ if $\omega(t) \notin A$ for all $t > 0$). We refer to [85, Ch. 1 Sec. 2], for the elementary properties of hitting times. Recall, a Borel set A is said to be polar if $\mu_x(\{\omega \in O :\ \omega(t) \in A \text{ for some } t < \infty\}) = 0$. We will use the fact that every one-point set is polar for $n \geqslant 2$. This is not true for $n = 1$ and that is the essential reason for different results in the cases $n \geqslant 2$ and $n = 1$.

Let $B_\varepsilon = \{x \in \mathbb{R}^n :\ \|x\| \leqslant \varepsilon\}$, $B = B_1$. Take $\varepsilon \in (0,1)$, $t > 0$ and define

$$W_{B_\varepsilon}(t) = \{\tau_{B_\varepsilon} < t\} = \{\omega :\ \text{there exists } 0 < s < t,\ \omega(s) \in B_\varepsilon\},$$
$$W_{B_\varepsilon} = \bigcup_{t>0} W_{B_\varepsilon}(t).$$

Clearly, $W_{B_\varepsilon}(t) \subset W_{B_\varepsilon}(s)$, $0 < t \leqslant s$. Note that

$$W_{B_\varepsilon}(s) \setminus W_{B_\varepsilon}(t) = \{t \leqslant \tau_{B_\varepsilon} < s\},\ 0 < t < s$$

and

$$W_{B_\varepsilon}(s) \setminus W_{B_\varepsilon}(t) \subset W_{B_\varepsilon} \setminus W_{B_\varepsilon}(t) \subset \{t - 1 < \tau_{B_\varepsilon}\}, \tag{51}$$

for $s > t > 1$.

Choose an increasing sequence $(t_m)_m$ such that $t_{m+1} > t_m + 1$, and (51) holds for every $m \in \mathbb{N}$.

Lemma 38. 1) *For every compact subset K of $\mathbb{R}^n$ and $\varepsilon < 1$, there exists a constant $C_\varepsilon > 0$ such that $\mu_x(W_{B_\varepsilon}) \leqslant C_\varepsilon$.* 2) $\lim_{\varepsilon \to 0} \sup_{x \in K} \mu_x(W_{B_\varepsilon}) = 0$.

Powers of the generalized delta function, δ^α, $\alpha \in (0,1)$, are defined in this paper by

$$\delta^\alpha = [(\phi_\varepsilon)^\alpha * \phi_\varepsilon], \ \varepsilon \in (0,1). \tag{52}$$

The reason for introducing (52) is simple: When $\alpha \in (0,1)$, the function ϕ_ε^α, $\varepsilon < 1$ is not smooth, in general. Note, generalized function $[\phi_\varepsilon * \phi_\varepsilon]$ is only associated with the generalized delta function $\delta = [\phi_\varepsilon]$.

Since one-point sets are not polar for $n = 1$, we could not use the same arguments as in the case $n \geqslant 2$. Note that functions in $H^2(\mathbb{R})$ are continuous and bounded.

Proposition 39. *Let $\alpha \in (0,1)$, $T > 0$ and $U_0 \in H^2(\mathbb{R})$. Then by*

$$U_\varepsilon(t,x) = \int_O \exp\left(-\int_0^t (\phi_\varepsilon)^\alpha * \phi_\varepsilon(\omega(s))\,ds\right) U_0(\omega(t))\,d\mu_x(\omega) \qquad t > 0,\ x \in \mathbb{R},\ \varepsilon < 1, \tag{53}$$

is defined a representative of a solution $U(t,x) \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R})$ to equation

$$\partial_t U(t,x) - \Delta U(t,x) + \delta^\alpha(x) U(t,x) = 0,\ U(0,x) = U_0(x).$$

The solution is unique in the sense of Definition 35.

Net (53) has a subsequence $(U_{\varepsilon_\nu,\alpha}(t,x))_{\nu \in \mathbb{N}}$, converging to $\tilde{U}(t,x) = e^{-\Delta t} U_0(x)$, $t \geqslant 0$, $x \in \mathbb{R}$ in the weak topology of $L^2([0,T) \times \mathbb{R})$.

Example 40. Assume $n \geqslant 2$, $T > 0$, $V \in H^{1,\infty}(\mathbb{R}^n)$, and $f \in C^1([0,\infty) \times \mathbb{R}^n)$ satisfies $f(s,0) = 0$, $s \in \mathbb{R}$ and $|f(s,y_1) - f(s,y_2)| \leqslant C|y_1 - y_2|$.

Let $U_0(x) = \delta(x)$, $x \in \mathbb{R}^n$, i.e., $U_{0\varepsilon} = \phi_\varepsilon$, $\varepsilon < 1$ (cf. (49)). Then for fixed $\varepsilon < 1$,

$$\partial_t U_\varepsilon(t,x) = (\Delta_x - V(x)) U_\varepsilon(t,x) + f(t, U_\varepsilon(t,x)),\ U_\varepsilon(0,x) = \phi_\varepsilon,$$

has a unique classical solution U_ε in $C^0([0,T), L^1(\mathbb{R}^n)) \cap C^1((0,T), L^1(\mathbb{R}^n))$ and $U_\varepsilon(t,x) \in H^{2,1}(\mathbb{R}^n))$ for every $t > 0$. Again we have $U_\varepsilon(t,x) \in C^0((0,T) : H^2(\mathbb{R}^n))$, $\varepsilon < 1$. We will show that there exists a sequence $(U_{\varepsilon_\nu})_{\nu \in \mathbb{N}}$ converging to some $U \in L^q_{\text{loc}}((0,T), \mathbb{R}^n)$, $1 \leqslant q < n/(n-1)$, in $L^q_{\text{loc}}((0,T), \mathbb{R}^n)$ such that $\partial_t U = (\Delta - V)U$ in $\mathcal{D}'((0,T), \mathbb{R}^n)$.

Remark 41. The classical theory of semigroups is used here as a tool for finding generalized solutions to a nonlinear heat equations. One can find different approach in [12], [33] or [100].

PART III: HYPERBOLIC SYSTEMS

6. Semilinear hyperbolic systems

Let

$$(\partial_t + \Lambda(x,t)\partial_x)y(x,t) = F(x,t,y(x,t)),\ y(x,0) = A(x) \tag{54}$$

be a semilinear hyperbolic system, where Λ is a real diagonal matrix and a mapping $y \mapsto F(x,t,y)$ is in $\mathcal{O}_M(\mathbb{C}^n)$ with uniform bounds for $(x,t) \in K \subset\subset \mathbb{R}^2$. In [72] a generalized solution to (54) is constructed when A is an arbitrary generalized function and F has a bounded gradient with respect to y for $(x,t) \in K \subset\subset \mathbb{R}^2$.

Here, F is substituted by $F_{h(\varepsilon)}$ which has a bounded gradient with respect to y for every fixed ε and converges pointwisely to F as $\varepsilon \to 0$. Our aim is to find a generalized solution to

$$(\partial_t + \Lambda(x,t)\partial_x)Y(x,t) = F_{h(\varepsilon)}(x,t,Y(x,t)),\ Y(x,0) = A(x). \tag{55}$$

We fix a decreasing function $h:(0,1) \to (0,\infty)$ such that $h(\varepsilon) = \mathcal{O}((\log\varepsilon^{-1})^{1/2})$, $h(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Denote by B_r the cube $|x| \leqslant r$, $|t| \leqslant r$, $|y| \leqslant r$, where $y = (u_1, v_1, ..., u_n, v_n)$. Let ε_i be a decreasing sequence of positive numbers such that $h(\varepsilon_{i+1}) = i$, $i \in \mathbb{N}$. This implies that $h(\varepsilon) \geqslant i-1$ if $\varepsilon < \varepsilon_i$. Let

$$\begin{aligned} S_i = B_i \cap \{(x,t,u,v),\ &|F(x,t,u,v)| \leqslant i-1\} \\ \cap \{(x,t,u,v),\ &|\nabla_{u,v}F(x,t,u,v)| \leqslant i-1\},\ i \in \mathbb{N}. \end{aligned}$$

Let κ_i be the characteristic function of S_i, $i \in \mathbb{N}$. Put

$$\begin{aligned} \kappa_{h(\varepsilon)} &= (\kappa_i * \phi_{1/h(\varepsilon)}),\ \varepsilon \in [\varepsilon_{i+1}, \varepsilon_i),\ i \in \mathbb{N}, \\ F^k_{h(\varepsilon)} &= F^k \kappa_{h(\varepsilon)},\ \varepsilon \in (0,\varepsilon_1),\ k \in \{1,\dots,n\}. \end{aligned}$$

Then there exists a constant $C = C(C_0) > 0$ and $\varepsilon_1 > 0$ such that

$$\begin{aligned} \|F_{h(\varepsilon)}\|_{L^\infty(\mathbb{R}^{2+2n})} &\leqslant Ch(\varepsilon) \\ \|\nabla_{u,v}F_{h(\varepsilon)}\|_{L^\infty(\mathbb{R}^{2+2n})} &\leqslant Ch(\varepsilon)^2,\ \varepsilon \in (0,\varepsilon_1). \end{aligned}$$

Definition 42. $G = (G_1,\dots,G_n) \in (\mathcal{G}(\mathbb{R}^2))^n$ is a solution to (55) if any of its representative G_ε satisfies the system

$$\begin{aligned} (\partial_t + \Lambda(x,t)\partial_x)G_\varepsilon(x,t) &= F_{h(\varepsilon)}(x,t,G_\varepsilon(x,t)) + d_{1,\varepsilon}(x,t), \\ G_\varepsilon(x,0) &= A_\varepsilon(x) + d_{2,\varepsilon}(x), \end{aligned} \tag{56}$$

where $A_\varepsilon \in (\mathcal{E}_M(\mathbb{R}))^n$ is a representative for some $d_{2,\varepsilon} \in (\mathcal{N}(\mathbb{R}))^n$, and $d_{1,\varepsilon} \in (\mathcal{N}(\mathbb{R}^2))^n$. We call (55) and (56) the h-regularized system.

Theorem 43. *Assume that every component of the mapping $y \mapsto F(x,t,y)$ belongs to $\mathcal{O}_M(\mathbb{C}^n)$ and has uniform bounds for $(x,t) \in K \Subset \mathbb{R}^2$. Then the h-regularized system* (55) *has a unique solution in $(\mathcal{G}(\mathbb{R}^2))^n$ whenever the initial data is in $(\mathcal{G}(\mathbb{R}))^n$.*

The proof follows by using the method of characteristics and the fixed point theorem.

Theorem 44. *Let the initial data A in* (54) *belong to $(C(\mathbb{R}))^n$.*

(a) *The solution G_h to the regularized system* (55) *is L^∞-associated with the continuous local solution g to* (54) *in K_{T_0}, for some $T_0 > 0$.*

(b) *Assume that* (54) *is globally well posed. Then the solution G_h to* (55) *is L^∞-associated with the continuous solution g to* (54) *on each K_T.*

Remark 45. If for every compact set $K \subset \mathbb{R}^2$ there exists $C > 0$ such that

$$\sup_{(x,t)\in K, y\in\mathbb{C}^n} |F(x,t,y)| \leqslant C \quad \text{or} \quad \sup_{(x,t)\in K, y\in\mathbb{C}^n} |\nabla_y F(x,t,y)| \leqslant C,$$

then system (54) is globally well posed.

The presented assertions are from [65]. One can look also in [71] or [2]. Multidimensional case is done in [52], initial-boundary problem is a topic of paper [43], while some nice results in special cases can be found in papers [14] and [73].

7. Systems of conservation laws

7.1. Introduction. For an $n \times n$ hyperbolic system (n real eigenvalues) in one space dimension

$$U_t + f(U)_x = 0,\ U : O \subset \mathbb{R}^2 \to \mathbb{R}^n,\ f : \mathbb{R}^n \to \mathbb{R} \tag{57}$$

with Riemann initial data

$$U|_{t=0} = \begin{cases} U_0, & x < 0, \\ U_1, & x > 0, \end{cases} \quad U_0, U_1 \text{ are real vectors,} \tag{58}$$

there exists a unique entropy solution, provided $\|U_1 - U_0\|_{L^\infty}$ small enough (Lax in 50's (see [53], for example)). The classical solution to the above Riemann problem consists of shock, rarefaction waves and contact discontinuities. (If n real eigenvalues are all different, then system (57) is called strictly hyperbolic.) Also, methods for solving an arbitrary Cauchy problem (Glimm scheme, wave front tracking algorithm (Di Perna for 2×2 system (70's), Bressan et all (90's) for $n \times n$ system), see [8] are based on the fact that the total variation of the initial data is small enough.

So, the first reason for introducing solutions containing Dirac δ distribution is a possible managing of the system with "large" initial total variation. The second reason for doing this is that some systems of conservation laws, perturbed by a "viscosity" matrix, $U_t + f(U)_x = \varepsilon A(U)U_{xx}$ have solutions which limit contains terms with δ distribution, as $\varepsilon \to 0$ ([72] and [41]). The third reason is that if one perturb Riemann data by smooth functions

$$U|_{t=0} = U_{0,\varepsilon} \to \begin{cases} U_0, & x < 0 \\ U_1, & x > 0 \end{cases}, \text{ as } \varepsilon \to 0,$$

local smooth solution has not only gradient catastrophe (the case of shock waves), but also L^∞ catastrophe (the L^∞ norm of the smooth solution goes to infinity in a finite time).

The aim of this section is threefold:

1. We shall give two solution concepts where δ function appears (delta and singular shock waves), see [57], [58] or [61].

2. Describe when it is possible to find such solutions (delta and delta singular locus, see [58] for these notions).

3. Give some results of interaction of delta and singular shock waves with other types of elementary waves in some special cases (see [59] or [61]).

At the end, we shall present some of numerous open problems concerning the above topics.

7.2. Some examples. Dirac δ distribution, as a part of a viscosity limit for solutions to some systems of conservation laws was numerically observed in [51]. In papers [72] and [41] is proved that the viscosity limit of some Riemann data for the system

$$\begin{aligned} u_t + (u^2/2)_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}$$

contain δ distribution. In [42], the arbitrary Riemann problem for the system

$$\begin{aligned} u_t + (u^2 - v)_x &= 0 \\ v_t + (u^3/3 - u)_x &= 0 \end{aligned} \tag{59}$$

which is a modified model of spreading ion acoustic waves is solved. For some initial data (u_1 is in the area Q_7 at Figure 1), the solution (in approximating sense) is given by

$$\begin{aligned} u_\varepsilon(x,t) &= G_\varepsilon(x - ct) + a\sqrt{\frac{t}{\varepsilon}}\,\rho\Big(x - c\frac{t}{\varepsilon}\Big), \\ v_\varepsilon(x,t) &= H_\varepsilon(x - ct) + a^2\frac{t}{\varepsilon}\,\rho^2\Big(x - c\frac{t}{\varepsilon}\Big), \end{aligned}$$

where G_ε and H_ε converge to appropriate step functions defined by the Riemann initial data, $\rho_\varepsilon^2(\cdot) := \varepsilon^{-1}\rho^2(\cdot/\varepsilon)$, where $\rho \in C_0^\infty$, $\int \rho^2 = 1$, converges to the delta distribution and ρ_ε^i converges to zero in $\mathcal{D}'$ as $\varepsilon \to 0$, $i = 1, 3$.

Pressureless gas dynamics model

$$\begin{aligned} u_t + (uv)_x &= 0 \\ (uv)_t + (uv^2)_x &= 0 \end{aligned} \tag{60}$$

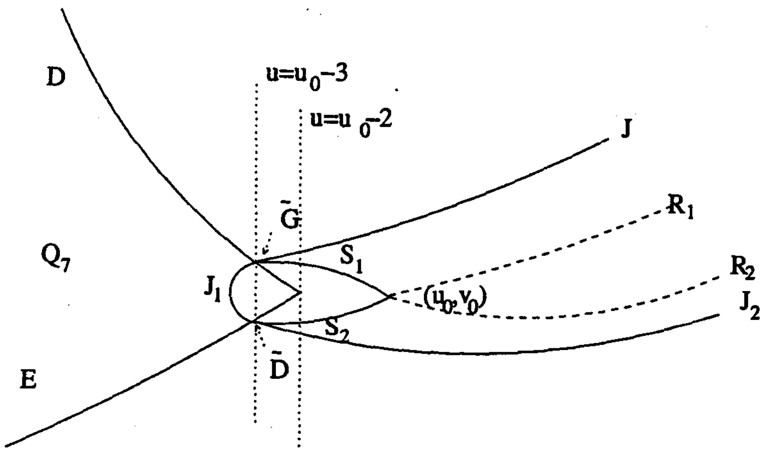

FIGURE 1. Singular shock wave area

can be transformed (after the elimination of the variable u_t from the second equation) into

$$\begin{aligned} u_t + (u^2)_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}$$

for which the Riemann problem is solved in [94]. Some of the solutions contains δ distribution as a term. The authors used Dafermos–Di Perna viscosity limit method (viscous term is given by $\varepsilon t u_{xx}$, so the viscous approximation allows self-similar solutions), and the results are justified by assuming that $u \in L^\infty$, v is a Borel measure, and u has an appropriate value at line of discontinuity.

The same ideas are used in [99] for the systems of the form

$$\begin{aligned} u_t + (f(u)v)_x &= 0 \\ (uv)_t + (f(u)v^2)_x &= 0. \end{aligned}$$

In [40], a measure theoretic solution to the above system is given.

In [32], a solution to an arbitrary Riemann problem for the system

$$\begin{aligned} u_t + (u^2/2)_x &= 0 \\ v_t + ((u-1)v)_x &= 0 \end{aligned} \tag{61}$$

which is a very simplified MHD-model, by using Vol'pert idea of multiplication of functions with bounded total variation and distributions. In some cases, the solution contains a term with δ distribution.

Finally, we shall mention the paper [22], where the author found viscosity limits to the Riemann problem

$$\begin{aligned} u_t + (f(u))_x &= 0 \\ (uv)_t + (g(u)v)_x &= 0 \end{aligned}$$

(under mild assumptions on functions f and g). Again, some limits contain δ distribution as a term.

7.3. Solution concepts. In the sequel we shall restrict ourselves to 2×2 systems, i.e., system (57) we shall write in the following form

$$\begin{aligned} u_t + (f_1(u)v + f_2(u))_x &= 0 \\ v_t + (g_1(u)v + g_2(u))_x &= 0, \end{aligned} \tag{62}$$

where f_1, f_2, g_1, g_2 are smooth functions, polynomially bounded with all its derivatives. One can see that the above system can be substituted by more general one

$$\begin{aligned} (\tilde{f}_1(u)v + \tilde{f}_2(u))_t + (f_1(u)v + f_2(u))_x &= 0 \\ (\tilde{g}_1(u)v + \tilde{g}_2(u))_t + (g_1(u)v + g_2(u))_x &= 0, \end{aligned}$$

without any major change in statements and concepts given here.

As it was written in the introduction, we shall present two solution concepts which are suitable for multiplication of distributions (in fact, δ distribution with a discontinuous function). The first one is based on the Colombeau generalized function space introduced in [77]. The second one is based on splitting δ distribution into two parts, which are divided by a discontinuity line.

7.3.1. First solution concept. We shall use Colombeau space $\mathcal{G}_g$ in this section. Let us start with a simple lemma often used in the rest of it.

Lemma 46. *The generalized function defined by the representative $\phi_\varepsilon(x-ct) \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$, $\phi \in \mathcal{A}_0$, $c \in \mathbb{R}$, is associated with $\delta(x-ct) \in \mathcal{D}'(\mathbb{R}^2_+)$.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^\infty_+)$ and

$$I_\varepsilon := \iint \varepsilon^{-1}\phi((x-ct)/\varepsilon)\psi(x,t)\,dx\,dt.$$

Changing the variables $(x-ct)/\varepsilon \mapsto y$, $t \mapsto s$, using the Lebesgue dominated convergence theorem and the properties of the functions from $\mathcal{A}_0$ gives

$$\begin{aligned} I_\varepsilon &= \iint \phi(y)\psi(\varepsilon y + cs, s)\,dy\,ds \\ &\to \int\Big(\int \phi(y)\,dy\Big)\psi(cs,s)\,ds = \int \psi(cs,s)\,ds, \text{ as } \varepsilon \to 0. \quad \Box \end{aligned}$$

The step functions, mapped by ι into $\mathcal{G}_g(\mathbb{R})$, belong to the following important class of generalized functions. $G \in \mathcal{G}_g(O)$ is said to be of a bounded type if

$$\sup_{x \in O} |G_\varepsilon(x)| = \mathcal{O}(1) \text{ as } \varepsilon \to 0,$$

for every $T > 0$.

Definition 47. (a) $G \in \mathcal{G}(\mathbb{R})$ is said to be a generalized step function with value (y_0, y_1) if it is of bounded type and

$$G_\varepsilon(y) = \begin{cases} y_0, & y < -\varepsilon \\ y_1, & y > \varepsilon \end{cases}$$

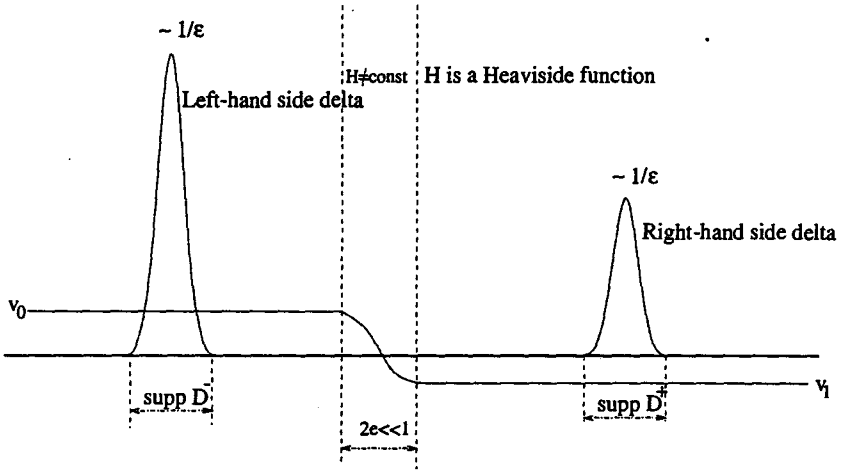

FIGURE 2. Delta shock wave

Denote $[G] := y_1 - y_0$.

(b) $D \in \mathcal{G}_g(\mathbb{R})$ is said to be generalized slitted delta function ($S\delta$-function for short) with value (α_0, α_1) if $D = \alpha_0 D^- + \alpha_1 D^+$, where $\alpha_0 + \alpha_1 = 1$ and $D^\pm \in \mathcal{G}_g(\mathbb{R})$ are associated to delta distribution and $D^- G \approx y_0\delta$ and $D^+ G \approx y_1\delta$, for any generalized step function G with value (y_0, y_1).

Remark 48. One can give fixed representatives for a generalized split delta function in the following way

$$D_\varepsilon^\pm(y) := \frac{1}{\varepsilon}\phi\Big(\frac{y - (\pm 2\varepsilon)}{\varepsilon}\Big), \ \phi \in \mathcal{A}_0.$$

Note that $D_\varepsilon^\pm$ are in fact shifted model delta nets.

Lemma 49. *If G is a generalized step function with value (y_0, y_1) and D is an $S\delta$-function with value (α_0, α_1), then the following hold.*

(i) *$f(G)$ is a generalized step function with value $(f(y_0), f(y_1))$, where f is a smooth function.*

(ii)
$$G \cdot D \approx (y_0\alpha_0 + y_1\alpha_1)\delta.$$

Proof. The proof is a straightforward consequence of the definitions. □

Remark 50. The support property of $S\delta$-function ensures the uniqueness in the association sense of its product with a generalized step function. This was done in order to deal with conservation law systems given is a general form. Of course that some other choices can be more efficient in specific cases (see [11] and literature there, for example).

The generalized initial data for our system are now generalized step functions G and H with values (u_0, u_1) and (v_0, v_1), respectively. One can see that the inclusion by ι_ϕ of a classical step function gives a generalized step function in the sense of Definition 47 (a) for every $\phi \in \mathcal{A}_0$.

Definition 51. $(U,V) \in (\mathcal{G}(\mathbb{R}^2_+))^2$ is called delta shock wave solution to Riemann problem (62, 58) if (a) and (b) hold:

(a)
$$\begin{aligned} U_t + (f_1(U)V + f_2(U,V))_x &\approx 0 \\ V_t + (g_1(U)V + g_2(U,V))_x &\approx 0. \\ U|_{t=0} = G,\ V|_{t=0} &= H. \end{aligned}$$

(b) $U(x,t) = G(x-ct)$, and $V(x,t) = H(x-ct)+s(t)(\beta_0 D^-(x-ct)+\beta_1 D^+(x-ct))$.

Here, G and H are generalized step functions, f_i, g_i, $i = 1,2$, are smooth functions polynomially bounded together with all their derivatives, f_2 and g_2 are also sublinearly bounded with respect to V, $c \in \mathbb{R}$ is a speed of the shock wave, $s \in C^1([0,\infty))$, $s(0) = 0$, $\beta_0 + \beta_1 = 1$, and $D = \beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)$ is an Sδ-function.

The function $s(t)\beta_0$ is called the left-handed strength of the wave, and $s(t)\beta_1$ is called the right-handed strength of the wave. Its sum ($s(t)$) is called the strength of delta or singular shock wave.

Definition 52. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be m-singular delta function (mSD-function for short) with value (β_0, β_1) if $d = \beta_0 d^- + \beta_1 d^+$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $(d^\pm)^i \approx 0$, $i \in \{1,\dots,m-1\}$, $(d^\pm)^m \approx \delta$, $(d^-)^m G \approx y_0\delta$ and $(d^+)^m G \approx y_1\delta$, for each generalized step function G with value (y_0, y_1).

Let m be an odd positive integer. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be m'-singular delta function (m'SD-function for short) with value (β_0,β_1) if $d = \beta_0 d^- + \beta_1 d^+$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $(d^\pm)^i \approx 0$, $i \in \{1,\dots,m-2,m\}$, $(d^\pm)^{m-1} \approx \delta$, $(d^-)^{m-1}G \approx y_0\delta$ and $(d^+)^{m-1}G \approx y_1\delta$, for each generalized step function G with value (y_0,y_1). (That implies $\beta_0^m + \beta_1^m = 1$.)

An Sδ-function D and an mSD-function (or an m'SD-function) d are said to be compatible if $d^m D \approx 0$ (or $d^{m-1}D \approx 0$).

Remark 53. One can construct such functions in a similar way as an Sδ-function, with $\operatorname{supp} d_\varepsilon^- \subset (-\infty,\varepsilon)$, $\operatorname{supp} d_\varepsilon^+ \subset (\varepsilon,\infty)$. Compatibility conditions can be achieved by demanding that D and d have disjoint supports for ε small enough, for example.

The definition of m'SD-function d implies $Gd^m \approx 0$ if G is a generalized step function.

Now we shall give the definition of singular shock wave and a useful lemma.

Definition 54. $(U,V) \in (\mathcal{G}(\mathbb{R}^2_+))^2$ is called singular shock wave solution to Riemann problem (1–3) if (a) and (b) hold:

(a)
$$\begin{aligned} U_t + (f_1(U)V + f_2(U,V))_x &\approx 0 \\ V_t + (g_1(U)V + g_2(U,V))_x &\approx 0. \\ U|_{t=0} = G,\ V|_{t=0} &= H. \end{aligned}$$

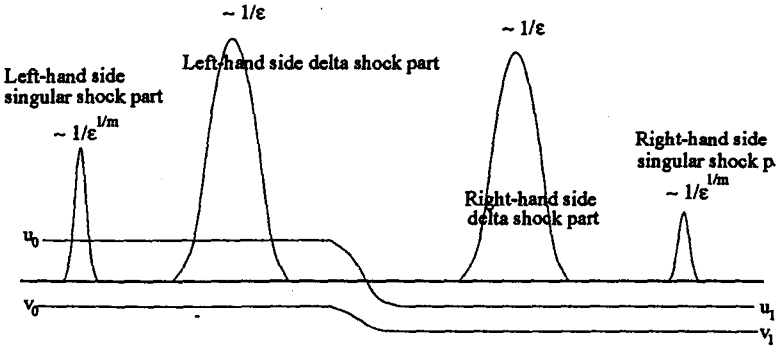

FIGURE 3. m singular shock wave

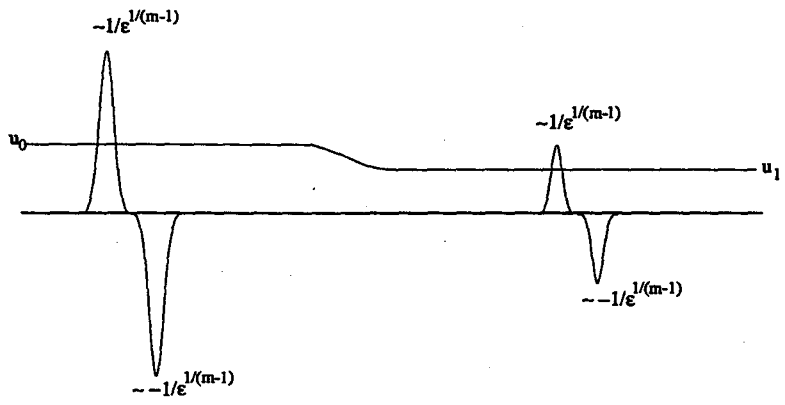

FIGURE 4. m' part of a singular shock wave

(b)
$$\begin{aligned} U(x,t) &= G(x-ct) + s_1(t)(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)), \\ V(x,t) &= H(x-ct) + s_2(t)(\beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)) \\ &\quad + s_3(t)(\gamma_0 d^-(x-ct) + \gamma_1 d^+(x-ct)). \end{aligned}$$

Here G and H are generalized step functions, f_i, g_i, $i = 1, 2$, are polynomials of the degree at most m $c \in \mathbb{R}$ is a speed of the shock, $s, s_1, s_2 \in C^1([0,\infty))$, $s_1(0) = s_2(0) = s_3(0) = 0$, D is an $S\delta$-function, as before, and d_j are mSD or m'SD-function, $j = 1, 2$.

Here, the strength of a singular shock wave is $s_2(t)$, and the left-and right-hand sided strengths are defined as in the case of delta shock wave.

From Definition 52 it follows that $\alpha_0^k + \alpha_1^k = \beta_0^k + \beta_1^k = \gamma_0 + \gamma_1 = 1$, where $k = m$ in the case of mSD-and $k = m - 1$ in the case of m'SD-functions.

Lemma 55. a) *Let* $d \in \mathcal{G}_g(\mathbb{R})$ *be an mSD-function with value* (β_0, β_1), $\beta_0^m + \beta_1^m = 1$, $G \in \mathcal{G}_g(\mathbb{R})$ *generalized step function with value* (y_0, y_1), $s \in C^1(\mathbb{R}_+)$, $s(0) = 0$, *and* $\Gamma(y) = \sum_{i=0}^m a_i y^i$ *be a real valued polynomial. Then*

$$\begin{aligned}&\Gamma(G(x-ct)) + s(t)d(x-ct))\\ &\approx \Gamma(G(x-ct)) + a_m s^m(t)\big(\beta_0^m (d^-)^m (x-ct) + \beta_1^m (d^+)^m (x-ct)\big)\\ &\approx \Gamma(G(x-ct)) + a_m s^m(t)\delta(x-ct).\end{aligned}$$

b) *Let* $d \in \mathcal{G}_g(\mathbb{R})$ *be an* m'*SD-function with value* (β_0, β_1), $\beta_0^{m-1} + \beta_1^{m-1} = 1$, *while* G, s *and* Γ *are as above. Then*

$$\begin{aligned}&\Gamma(G(x-ct) + s(t)d(x-ct))\\ &\approx \Gamma(G(x-ct)) + a_{m-1}s^{m-1}(t)\big(\beta_0^{m-1}(d^-)^{m-1}(x-ct) + \beta_1^{m-1}(d^+)^{m-1}(x-ct)\big)\\ &\qquad + m a_m s^{m-1}(t)\big(\beta_0^{m-1} y_0 (d^-)^{m-1}(x-ct) + \beta_1^{m-1} y_1 (d^+)^{m-1}(x-ct)\big)\\ &\approx \Gamma(G(x-ct)) + a_{m-1}s^{m-1}(t)\delta(x-ct)\\ &\qquad + m a_m s^{m-1}(t)(\beta_0^{m-1} y_0 + \beta_1^{m-1} y_1)\delta(x-ct).\end{aligned}$$

7.3.2. Second solution concept. We shall now briefly describe the second solution concept we are using. Suppose $\overline{R_+^2}$ is divided into finitely disjoint open sets $O_i \neq \emptyset$, $i = 1, \dots, n$ with piecewise smooth boundary curves Γ_i, $i = 1, \dots, m$, that is $O_i \cap O_j = \emptyset$, $\bigcup_{i=1}^n \overline{O}_i = \overline{R_+^2}$ where $\overline{O}_i$ denotes the closure of O_i. Let $C(\overline{O}_i)$ be the space of bounded and continuous real-valued functions on $\overline{O}_i$, equipped with the L^∞-norm. Let $\mathcal{M}(\overline{O}_i)$, be the space of measures on $\overline{O}_i$.

We consider the spaces $C_\Gamma = \prod_{i=1}^n C(\overline{O}_i)$, $\mathcal{M}_\Gamma = \prod_{i=1}^n \mathcal{M}(\overline{O}_i)$. The product of an element $G = (G_1, \dots, G_n) \in C_\Gamma$ and $D = (D_1, \dots, D_n) \in \mathcal{M}_\Gamma$ is defined as an element $D \cdot G = (D_1 G_1, \dots, D_n G_n) \in \mathcal{M}_\Gamma$, where each component is defined as the usual product of a continuous function and a measure.

Every measure on $\overline{O}_i$ can be viewed as a measure on $\overline{\mathbb{R}_+^2}$ with support in $\overline{O}_i$. This way we obtain a mapping

$$\begin{aligned}&m : \mathcal{M}_\Gamma \to \mathcal{M}(\overline{\mathbb{R}_+^2})\\ &m : D \mapsto D_1 + D_2 + \cdots + D_n.\end{aligned}$$

A typical example is obtained when $\overline{\mathbb{R}_+^2}$ is divided into two regions O_1, O_2 by a piecewise smooth curve $x = \gamma(t)$. The delta function $\delta(x - \gamma(t)) \in \mathcal{M}(\overline{\mathbb{R}_+^2})$ along the line $x = \gamma(t)$ can be split in a non unique way into a left-hand side $D^- \in \mathcal{M}(\overline{O}_1)$ and the right-hand component $D^+ \in \mathcal{M}(\overline{O}_2)$ such that

$$\delta(x - \gamma(t)) = \alpha_0(t)D^- + \alpha_1(t)D^+ = m(\alpha_0(t)D^- + \alpha_1(t)D^+)$$

with $\alpha_0(t) + \alpha_1(t) = 1$. A solution of the form

$$\begin{aligned}&u(x,t) = G(x-ct)\\ &v(x,t) = H(x-ct) + s(t)(\alpha_0 D^- + \alpha_1 D^+)\end{aligned} \tag{63}$$

is called delta shock wave.

The solution concept which allows to incorporate such two sided delta functions as well as shock waves is modeled along the lines of the classical weak solution concept and proceeds as follows:
Step 1: Perform all nonlinear operations of functions in the space C_Γ.
Step 2: Perform multiplications with measures in the space $\mathcal{M}_\Gamma$.
Step 3: Map the space $\mathcal{M}_\Gamma$ into $\mathcal{M}(\overline{\mathbb{R}^2_+})$ by means of the map m and embed it into the space of distributions.
Step 4: Perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

Note that in the case of absence of a measure part (Step 2), this is the precisely the concept of a weak solution to equations in divergence form.

7.4. Existence theorems. Delta locus is the set of all points $U_1 = (u_1, v_1) \in \mathbb{R}^2$ such that there exists a delta shock solution to system (62, 58). Singular delta locus is the set of all points $U_1 \in \mathbb{R}^2$ such that there exists a singular shock solution to system (62), (58). In this case f_1, f_2, g_1 and g_2 have to be polynomials.

As one can see, these definitions are just a simple analogue to the definition of Hugoniot locus.

7.4.1. Existence theorems using the first solution concept. We shall present (without proofs) two theorems about delta locus and singular delta locus from [58].

Theorem 56. a) *Let* $f_1 \not\equiv \text{const}$. *Then a delta shock wave solution to* (62), (58) *exists if* $u_0 \neq u_1$, $f_1(u_0) \neq f_1(u_1)$ *and*

$$\begin{aligned} c &= \frac{f_1(u_1)v_1 + f_2(u_1, v_1) - f_1(u_0)v_0 - f_2(u_0, v_0)}{u_1 - u_0} \\ &= \frac{g_1(u_0)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1) - f_1(u_0)}, \end{aligned} \tag{64}$$

where c *is the velocity of the delta shock. The set of all points* (u_1, v_1) *such that* (64) *holds is the delta locus of the system (for the point* (u_0, v_0)*).*

b) *If* $f_1(u_0) = f_1(u_1) = 0$ *(specially, if* $f_1 \equiv 0$*) and* $g_1 \not\equiv \text{const}$, *then the delta locus is the set of all points* (u_1, v_1) *such that*$g_1(u_0) \neq g_1(u_1)$.

c) *If* $f_1 \equiv 0$ *and* $g_1 \equiv b \in \mathbb{R}$, *then the delta locus is the set of all points* (u_1, v_1) *such that* $b(u_1 - u_0) = f_2(u_1) - f_2(u_0)$.

The main point of the proof is to express c from the first equation of the system and then substitute this value of c into the second one. Then, one has to find appropriate $s(t) = \sigma t$, and other coefficients to "compensate" so called Rankine–Hugoniot deficit in the second equation and eliminate terms associated to δ'. The exact proof contains a lot of details, so we shall omit it. One has to use the definitions and lemmas above.

For the second theorem, we have to make same assumption and give notation. Suppose that the maximal degree of all polynomials in the fluxes equals m. Let

$$f_1(y) = \sum_{i=0}^m a_{1,i}y^i,\ f_2(y) = \sum_{i=0}^m a_{2,i}y^i,\ g_1(y) = \sum_{i=0}^m b_{1,i}y^i,\ g_2(y) = \sum_{i=0}^m b_{2,i}y^i.$$

The following lemma is obvious consequence of the given definitions.

Lemma 57. a) *Let $d \in \mathcal{G}_g(\mathbb{R})$ be an mSD-function with value (β_0, β_1), $\beta_0^m + \beta_1^m = 1$, $G \in \mathcal{G}_g(\mathbb{R})$ generalized step function with value (y_0, y_1), $s \in C^1(\mathbb{R}_+)$, $s(0) = 0$, and let $\Gamma(y) = \sum_{i=0}^m a_i y^i$ be a real valued polynomial. Then*

$$\begin{aligned}
&\Gamma(G(x-ct)) + s(t)d(x-ct)) \\
&\approx \Gamma(G(x-ct)) + a_m s^m(t)\big(\beta_0^m (d^-)^m(x-ct) + \beta_1^m (d^+)^m(x-ct)\big) \\
&\approx \Gamma(G(x-ct)) + a_m s^m(t)\delta(x-ct).
\end{aligned}$$

b) *Let $d \in \mathcal{G}_g(\mathbb{R})$ be an $m'SD$-function with value (β_0, β_1), $\beta_0^{m-1} + \beta_1^{m-1} = 1$, while G, s and Γ are as above. Then*

$$\begin{aligned}
&\Gamma(G(x-ct) + s(t)d(x-ct)) \\
&\approx \Gamma(G(x-ct)) + a_{m-1}s^{m-1}(t)\big(\beta_0^{m-1}(d^-)^{m-1}(x-ct) + \beta_1^{m-1}(d^+)^{m-1}(x-ct)) \\
&\quad + m a_m s^{m-1}(t)\big(\beta_0^{m-1} y_0 (d^-)^{m-1}(x-ct) + \beta_1^{m-1} y_1 (d^+)^{m-1}(x-ct)) \\
&\approx \Gamma(G(x-ct)) + a_{m-1}s^{m-1}(t)\delta(x-ct) \\
&\quad + m a_m s^{m-1}(t)(\beta_0^{m-1} y_0 + \beta_1^{m-1} y_1)\delta(x-ct).
\end{aligned}$$

Now, we are in position to state the main theorem in this subsection.

Theorem 58. *Let $G(x-ct)$ and $H(x-ct)$ be generalized step functions with the speed c and values (u_0, u_1) and (v_0, v_1), respectively. Let*

$$\alpha = c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)) = c[H] - [g_1(G)H + g_2(G)]$$

be Rankine–Hugoniot deficit.

A singular shock wave solution exists if one of the following two assertions are true:

(i) *There exists a solution $(\alpha_0, y_0) \in \mathbb{R}^2$ to the system*

$$\begin{aligned}
\alpha[f_1(G)]\alpha_0 + \sigma[H]a_{1,m}y_0 &= \sigma(v_1 a_{1,m} + a_{2,m}) + \alpha f_1(u_1) \\
\alpha[g_1(G)]\alpha_0 + \sigma[H]b_{1,m}y_0 &= \sigma(v_1 b_{1,m} + b_{2,m}) + \alpha(g_1(u_1) - c),
\end{aligned}$$

for some $\sigma \in \mathbb{R} \setminus \{0\}$. If m is an even number, then σ also has to be positive and $y_0 \in [0,1]$.

(ii) *m is an odd number and there exists a solution $(\alpha_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$ to the system*

$$\begin{aligned}
&\alpha[f_1(G)]\alpha_0 + \sigma(a_{1,m-1}[G] + m a_{1,m}[GH] + m a_{2,m}[G])y_0 \\
&\qquad = \sigma(a_{1,m-1}v_1 + m a_{1,m}u_1 v_1 + m a_{2,m}u_1 + a_{2,m-1}) + \alpha f_1(u_1) \\
&\alpha[g_1(G)]\alpha_0 + \sigma(b_{1,m-1}[G] + m b_{1,m}[GH] + m b_{2,m}[G])y_0 \\
&\qquad = \sigma(b_{1,m-1}v_1 + m b_{1,m}u_1 v_1 + m b_{2,m}u_1 + b_{2,m-1}) + \alpha(g_1(u_1) - c)
\end{aligned}$$

for some $\sigma \in \mathbb{R}_+$.

The speed of the singular shock waves is always given by

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} = \frac{[f_1(G)H + f_2(G)]}{[G]}$$

Again, we shall omit the proof because it is even longer than the proof of the previous theorem, and the idea is almost the same. The main difference is that now $d^{\pm}$ plays significant role in constructing a solution, even they are zeros in distributional sense.

Remark 59. By using the first solution concept, all results from the introduction are recovered, except (60). System (60) does not have delta shock wave solution, but singular shock wave solution, where the density of the gas is distributionally greater or equal to zero. This singular shock solution gives the same distribution limit as the original one, obtained by using the measure theoretical and vanishing viscosity in method given in [99] and [40].

Remark 60. Let us note that the delta locus is just a curve in $\mathbb{R}^2_+$ and the delta singular locus is an area, in general. This has deep consequence: it is not easy to solve arbitrary Riemann problem without using a singular delta shock locus. The usual delta shock wave can not be followed or can not follow any of the elementary waves, if the system (57) is strictly hyperbolic. But the combination of 1-rarefaction and singular shock wave or singular shock and 2-rarefaction wave is quite possible, even in this case.

7.4.2. Existence theorems using the second solution concept. In the paper [57], the following theorem is proved.

Theorem 61. *A point (u_1, v_1) is in the delta locus of a point (u_0, v_0) for the Riemann problem* (62, 58) *if f_2 and g_2 do not depend on v and the following holds:*

$$\text{(a)}\quad g_1(u_0) \neq g_1(u_1), \qquad \text{(b)}\quad f_1(u_0)\frac{k_1(g_1(u_1)-c)}{g_1(u_1)-g_1(u_0)} = f_1(u_1)\frac{k_1(g_1(u_0)-c)}{g_1(u_1)-g_1(u_0)},$$

where $k_1 = c[G] - [f_1(G)H + f_2(G)]$, and c is a speed of the delta shock wave.

We shall prove the theorem, just to show the simplicity of this solution concept.

Proof. Let us denote by $s_0(t) = s(t)\beta_0$ and $s_1(t) = s(t)\beta_1$. The substitution of functions in (63) into (62)–(58) and the use of the Rankine–Hugoniot conditions gives the following equation

$$\begin{aligned}&(-c[G]+[f_1(G)H+f_2(G)])\delta(x-ct)+(f_1(s_0(t)u_0)\delta^-(x-ct)+f_1(s_1(t)u_1)\delta^+(x-ct))_x\\&= (-c[G]+[f_1(G)H+f_2(G)])\delta(x-ct)+(f_1(s_0(t)u_0)+f_1(s_1(t)u_1))\delta'(x-ct) = 0.\end{aligned}$$

Suppose that $u_0 \neq u_1$. From the above equation, one obtains the value of the speed c and the coupling equations for s_0 and s_1:

$$\begin{aligned} c &= [f_1(G)H + f_2(G)]/[G] \\ s_0(t)f_1(u_0) &+ s_1(t)f_1(u_1) = 0. \end{aligned} \tag{65}$$

Doing the same for the second equation, one obtains

$$\begin{aligned}&-c[H] + (s_0(t)+s_1(t))'\delta(x-ct) - c(s_0(t)+s_1)\delta'(x-ct)\\&+ [g_1(G)H + g_2(G)]\delta(x-ct) + (s_0(t)g_1(u_0)+s_1(t)g_1(u_1))\delta'(x-ct) = 0.\end{aligned} \tag{66}$$

Since c is already determined,

$$(s_0(t) + s_1(t))' = c[H] - [g_1(G)H + g_2(G)], \text{ i.e., } s_0(t) + s_1(t) = k_1 t,$$

and k_1 is called Rankine–Hugoniot deficit [42]. Now, one obtains the following system of equations for s_0 and s_1:

$$\begin{aligned}(g_1(u_0) - c)s_0(t) + (g_1(u_1) - c)s_1(t) &= 0\\ s_0(t) + s_1(t) &= k_1 t.\end{aligned}$$

If $g_1(u_0) = g_1(u_1)$, then $k_1 = 0$, i.e., there is no delta shock wave solution. Otherwise,

$$s_0(t) = \frac{k_1(g_1(u_1) - c)}{g_1(u_1) - g_1(u_0)}, \quad s_1(t) = \frac{k_1(c - g_1(u_0))}{g_1(u_1) - g_1(u_0)}$$

are determined. Using these values and the second equation in (65), one gets the assertion of the theorem.

Now, let $u_0 = u_1$. Then, from the above equations, one can see that $k_1 = 0$ and there is no delta shock wave solution to (62)–(58). □

Remark 62. Again, the solutions admitting delta shock wave, mentioned in introduction has also the same solution in this sense, except (60).

7.4.3. Admissibility conditions. A delta or singular shock wave is said to be admissible wave (entropy one) if it is overcompressive, i.e. $\lambda_i(u_0) \geqslant c \geqslant \lambda_i(u_1)$, $i = 1, \dots, n$, where λ_i is i-th characteristics for the system $u_t + \nabla f(u)u_x = 0$, which is equivalent to (62) for smooth solutions, and c is a speed of the delta or singular shock wave.

If the i-th characteristic field is linearly degenerate, i.e., $\nabla\lambda_i \cdot r_i = 0$, where r_i is i-th eigenvector for the matrix ∇f, then the delta or singular shock wave with singular support on the i-th characteristics is admissible and called delta or singular delta discontinuity (one can see [61] for a precise definition).

7.5. Intersection of delta or singular shock waves with themselves and other elementary waves. The main idea is simple: if delta or singular shock wave interacts with some (delta or singular) shock wave at the point (x_0, t_0), one has to solve the new initial data problem

$$(67) \qquad u|_{t=t_0} = \begin{cases} u_0, & x < x_0,\\ u_2, & x < x_0;\end{cases} \qquad v|_{t=t_0} = \begin{cases} v_0, & x < x_0,\\ v_2, & x < x_0,\end{cases} + \gamma\delta_{(x_0,t_0)}.$$

Before considering interaction of delta or singular shock wave with a rarefaction wave, in order to see what result is expected one can decompose the rarefaction wave into a large family of approximate, but non-entropy physical waves.

In the case of interaction of delta or singular shock wave with rarefaction or shock wave, γ is a strength of an initial delta or singular shock wave. If delta or singular shock waves interacts mutually, then γ is sum of strengths of these waves.

Definition 63. The set of points (u_2, v_2) for which there exists a solution to (62), (67) in a form of delta (singular) shock wave is called γ-second delta (singular delta) locus.

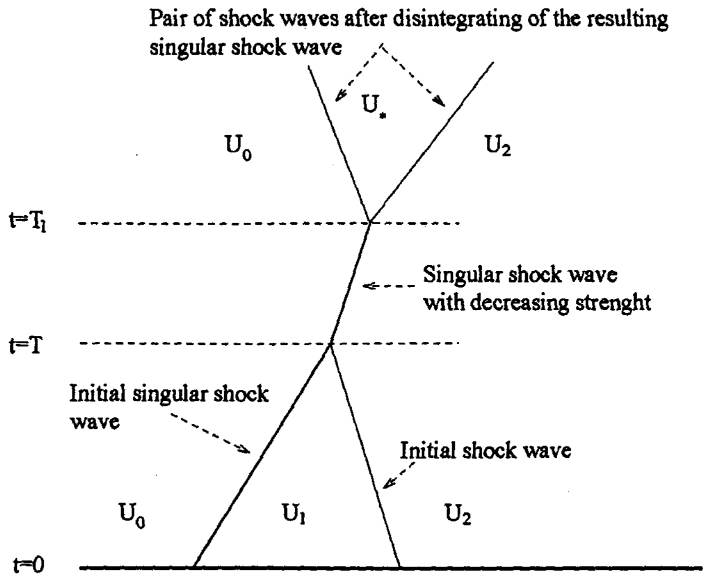

FIGURE 5. Splitting of singular shock wave

7.5.1. Some results for the first solution concept. A second singular locus is not easy to write in a simple form for general case (62), and we shall skip it here. Let us just mention that a γ-second delta (singular delta) locus contains as its subset delta (singular delta) locus for $\gamma > 0$, as it was shown in [59].

The major problem is intersection of a delta or singular shock wave and rarefaction wave due to continuous integration of delta function and some continuous function (rarefaction fan). One has to solve an ordinary differential equation and consider the admissibility condition afterward. Due to this fact, the singular support of singular shock wave is a curve, not a straight line as before. In special cases it can be done more easily. For example, for system (59) a complete analysis of interaction for singular shock waves and any other elementary wave or another singular shock wave is done (see [59]).

We shall present one phenomena obtained in the cited paper. For an usual Riemann problem, a strength of singular shock wave increases (linearly) with time. During a interaction with rarefaction wave it can decrease with time. If the strength reaches zero, then the singular shock wave decouples into two ordinary shock waves (see Figure 5, where "initial shock wave" is non-admissible one–it is a result of rarefaction wave approximation with a fan of such shock waves)

7.5.2. Some results for the second solution concept. In contrast to previous case, there exist a theorem describing the second delta locus in a simple form.

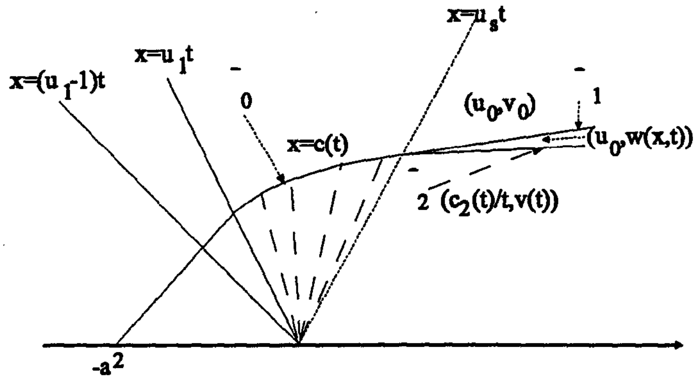

FIGURE 6. Delta contact discontinuity and unbounded part of solution

Theorem 64. *A point (u_2, v_2) is in the second delta locus of the point (u_0, v_0) if one of the following is true.*

(a) $f_1 \not\equiv \text{const}$ *and*

$$\frac{f_1(u_2)v_2 + f_1(u_2) - f_1(u_0)v_0 - f_1(u_0)}{u_2 - u_0} = \frac{g_1(u_0)f_1(u_2) - g_1(u_2)f_1(u_0)}{f_1(u_2) - f_1(u_0)},$$

(b) $f_1 \equiv 0$ *and* $g_1(u_0) \neq g_1(u_2)$

(c) *If* (u_2, v_2) *is in a Hugoniot locus of the point* (u_0, v_0).

But, complete analysis is done only for system (61) so far (up to our knowledge). Again, we obtained few new interesting things. The first one is an existence of delta contact discontinuity (which is possible only if a given system is not genuinely nonlinear). And the second one is that we start with piecewise constant function, the solution can be unbounded in a region with Lebesgue measure greater than zero. That is, a part of solution (after intersection of a delta shock wave and rarefaction wave) is L^∞_{loc} function (going to infinity as $1/\sqrt{y}$ as $y \to 0$). One can see illustration in Figure 6: the function w is unbounded, the line1 denotes the delta contact discontinuity curve, while the line 2 denotes a shock wave curve. We shall demonstrate how a delta shock curve $x = c(t)$ can be found during the intersection of delta shock wave and rarefaction wave in the case of system (61).

The function $x = c(t)$ has to satisfy the following ordinary differential equation:

$$-c'(t)\Big(\frac{c(t)}{t} - u_0\Big) + \frac{1}{2}\Big(\Big(\frac{c(t)}{t}\Big)^2 - u_0^2\Big) = 0, \quad c(t_0) = x_0, \tag{68}$$

which has the unique solution $c(t) = u_0 t - a\sqrt{2(u_0 - u_1)t}$, $t \geqslant t_0$. This example also shows why the intersection problem depends highly on a system in question: The equation (68) should be explicitly solved, which is not always possible.

But the main problems and interesting phenomena of this intersection appears when the delta shock wave is no longer overcompressive during the interaction. One can found complete results in [61].

7.6. Numerical verification. Let us give an example for a possible approach in numerical verification of a delta or singular shock wave. Using the first solution method in [58] one can find a singular shock wave solution to (60) for some Riemann data which converges to the measure valued solution described in [40] or [99]. After a "natural" change of variables $uv \mapsto w$ one gets the following system in evolution form

$$\begin{aligned} u_t + w_x &= 0 \\ w_t + (w^2/u)_x &= 0 \end{aligned} \tag{69}$$

which makes sense because u is the density, and there is no vacuum state in this case. Transformed system do not permit measure theoretical results for some initial data, since square of w appears in the flux function. But, using Colombeau generalized functions, it has the same (up to association relation) solutions for all initial data as the original one.

For the system (69) after mollifying the initial data in a usual way (convolution with a delta model net), one can try to use finite volume scheme (modified Godunov scheme, see [54]) together with moving mesh method [93]. This was done in [16]. Obtained solution resembles the solution given in [58] (obtained by the first solution method).

Remark 65. The word "resembles" in the above context means that the numerical speed of a singular shock wave is arbitrarily near the theoretical one, and the masses delta function part of singular shock wave are linearly growing with respect to time, as expected.

Conservation law systems and generalized functions are subject of a large number of papers. We shall refer to book [11], where one can find a further reference and many nice examples for Colombeau generalized function approach.

7.7. Open problems. As it was announced in the beginning of this part, now we shall present some of numerous open problems. More dimensional cases are totally excluded from the list bellow, because the number and form of problems in this case is quite large and vogue.

(i) Uniqueness in some sense (*No results so far*).
(ii) Avoiding not wanted delta or singular delta shock waves (*Overcompressibility condition is not enough*).
(iii) Overcome linearity in one variable (*There are some results using Colombeau generalized functions*).
(iv) General interactions of these new singularities (*Probably, the solution highly depends on particular systems*).
(v) How delta shock waves can be followed (or follow) rarefaction wave, as singular shock waves do (*No results so far*).

8. Appendix

8.1. Algebras of weighted sequence spaces. In this appendix we give another approach to the Colombeau type algebras which is related to the topological structure of certain exponentially weighted sequence spaces. All these classes of algebras are simply determined by the (locally convex) space E, and a sequence of weights $r : \mathbb{N} \to \mathbb{R}_+$ (or sequence of sequences) which serves to construct an ultrametric on the sequence space $E^{\mathbb{N}}$. The sequence $r = (r_n)_n$ is assumed to be decreasing to zero. This implies that sequence spaces under consideration ($\subset E^{\mathbb{N}}$) contain as a subspace $E \sim *\mathrm{diag}\, E^{\mathbb{N}}$ and that they induce the discrete topology on E. This is well-known for the sharp topology for Colombeau type algebras. But our analysis implies that if one has a Colombeau type algebra containing the Dirac delta distribution δ as an embedded Colombeau generalized function, then the topology induced on the basic space must be discrete. This is an analogous result to the Schwartz's "impossibility result" concerning the product of distributions.construction of Colombeau type algebras.

In order to simplify the construction, we will consider sequences $(f_n)_{n\in\mathbb{N}}$ instead of nets f_ε. The passage from one to another concept is simple: with $\varepsilon = 1/n$ and reversely. Consider a semi-normed algebra (E, p) such that $p(a\,b) \leqslant p(a)\,p(b)$, $a, b \in E$ and a sequence $r \in \mathbb{R}_+^{\mathbb{N}}$ decreasing to zero. Define for $f \in E^{\mathbb{N}}$

$$\| f \|_{p,r} := \limsup_{n\to\infty} p(f_n)^{r_n}.$$

This is well defined for any $f \in E^{\mathbb{N}}$, with values in $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. With this definition, let $\mathcal{F}_{p,r} = \{f \in E^{\mathbb{N}} : \| f \|_{p,r} < \infty\}$, $\mathcal{K}_{p,r} = \{f \in E^{\mathbb{N}} : \| f \|_{p,r} = 0\}$. Then the following holds:

Proposition 66. (a) *The function $d_{p,r} : \mathcal{F}_{p,r} \times \mathcal{F}_{p,r} \to \mathbb{R}_+$, $(f, g) \mapsto \| f - g \|_{p,r}$, is an ultrapseudometric on $\mathcal{F}_{p,r}$.*

(b) *$\mathcal{F}_{p,r}$ is a subalgebra of $E^{\mathbb{N}}$, and $\mathcal{K}_{p,r}$ is an ideal of $\mathcal{F}_{p,r}$; thus $\mathcal{G}_{p,r} := \mathcal{F}_{p,r}/\mathcal{K}_{p,r}$ is an algebra.*

(c) *$\tilde{d}_{p,r} : \mathcal{G}_{p,r} \times \mathcal{G}_{p,r} \to \mathbb{R}_+$, $(F, G) \mapsto d_{p,r}(f, g)$, is an ultrametric on $\mathcal{G}_{p,r}$, where $f \in F$, $g \in G$ are any representatives of the classes $F = f + \mathcal{K}_{p,r}$ resp. $G = g + \mathcal{K}_{p,r}$.*

(d) *$\mathcal{G}_{p,r} = \mathcal{F}_{p,r}/\mathcal{K}_{p,r}$ is a topological algebra, the quotient topology being the same than the topology induced by the ultrametric $\tilde{d}_{p,r}$.*

We give the construction of generalized constants. For this, E will be the underlying field $\mathbb{R}$ or $\mathbb{C}$, and $p = |\cdot|$ the absolute value. For $r = 1/\log$, we get the ring of Colombeau's numbers $\overline{\mathbb{C}}$ (and $\overline{\mathbb{R}}$). Let $r_n = \frac{1}{\log n}$, $n \geqslant 2$.

Colombeau's algebras of generalized constants represented by sequences with polynomial growth modulo sequences of more than polynomial decrease, because

$$\begin{aligned}
\limsup |x_n|^{1/\log n} < \infty &\iff \exists C : \limsup |x_n|^{1/\log n} = C \\
&\iff \exists B, \exists n_0, \forall n > n_0 : |x_n| \leqslant B^{\log n} = n^{\log B} \\
&\iff \exists \gamma : |x_n| = o(n^\gamma).
\end{aligned}$$

If we put, $\limsup = 0$ (for the ideal) then the corresponding C above equals zero and thus $\forall B > 0$ resp. $\forall\gamma$ we have $|x_n| = o(n^\gamma)$.

Consider now Hölder type spaces $E = C^{k,\alpha}(\bar{O})$ (cf. [26]), $\alpha \in (0,1]$ and $k \in \mathbb{N}_0$ (with $|\cdot|_{k,\alpha}$-norm It is a Banach space and we can apply the same construction with $p = \|\cdot\|_{k,\alpha}$. The corresponding Colombeau type algebra is defined by $\mathcal{G}_{C^{k,\alpha}} :\equiv \mathcal{F}/\mathcal{K}$, where

$$\mathcal{F} := \{u \in (C^{k,\alpha}(\bar{O}))^{\mathbf{N}} \mid \limsup \|u_n\|_{s,\infty}^{\frac{1}{\log n}} < \infty\},$$
$$\mathcal{K} := \{u \in (C^{k,\alpha}(\bar{O}))^{\mathbf{N}} \mid \limsup \|u_n\|_{s,\infty}^{\frac{1}{\log n}} = 0\}.$$

This algebra will be used for the analysis of elliptic equation in Part II.

8.1.1. Constructions with locally convex vector spaces. Consider now an algebra E which is a locally convex vector space on $\mathbb{C}$, equipped with an arbitrary set of seminorms $p \in \mathcal{P}$ determining its locally convex structure. Assume that

$$\forall p \in \mathcal{P},\ \exists \bar{p} \in \mathcal{P}, C \in \mathbb{R}_+ : \forall x, y \in E : p(xy) \leqslant C\bar{p}(x)\bar{p}(y).$$

Let

$$\mathcal{F}_{\mathcal{P},r} = \{ f \in E^{\mathbf{N}} \mid \forall p \in \mathcal{P} : \| f \|_{p,r} < \infty \},$$
$$\mathcal{K}_{\mathcal{P},r} = \{ f \in E^{\mathbf{N}} \mid \forall p \in \mathcal{P} : \| f \|_{p,r} = 0 \}.$$

Then the following holds:

Proposition 67. (a) *For every $p \in \mathcal{P}$, $d_{p,r} :\ E^{\mathbf{N}} \times E^{\mathbf{N}} \to \overline{\mathbb{R}}_+$, $(f,g) \mapsto \| f - g \|_{p,r}$, is an ultrapseudometric on $\mathcal{F}_{\mathcal{P},r}$.*

(b) *$\mathcal{F}_{\mathcal{P},r}$ is a (sub-)algebra of $E^{\mathbf{N}}$, and $\mathcal{K}_{\mathcal{P},r}$ is an ideal of $\mathcal{F}_{\mathcal{P},r}$.*

(c) *$\mathcal{G}_{\mathcal{P},r} := \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ is an algebra.*

(d) *For every $p \in \mathcal{P}$, $\tilde{d}_{p,r} : \mathcal{G}_{\mathcal{P},r} \times \mathcal{G}_{\mathcal{P},r} \to \overline{\mathbb{R}}_+$, $(F,G) \mapsto d_{p,r}(f,g)$ is an ultrametric on $\mathcal{G}_{\mathcal{P},r}$, where f, g are any representatives of the classes $F = f + \mathcal{K}_{\mathcal{P},r}$ resp. $G = g + \mathcal{K}_{\mathcal{P},r}$.*

(e) *$\mathcal{G}_{\mathcal{P},r} := \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ is a topological algebra, the quotient topology being the same than the topology induced by the family of ultrametrics $\{\tilde{d}_{p,r}\}_{p\in\mathcal{P}}$.*

Example 68. Let $E = \mathcal{C}^\infty(O)$, $\mathcal{P} = \{p_\nu\}_{\nu\in\mathbf{N}}$ with $p_\nu(f) := \sup\limits_{|\alpha|\leqslant\nu, |x|\leqslant\nu} |f^{(\alpha)}(x)|$, and $r = 1/\log$. Then, $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ with

$$\mathcal{F}_{\mathcal{P},r} = \Big\{(f_n)_n \in \mathcal{C}^\infty(O)^{\mathbf{N}} \mid \forall\nu \in \mathbf{N} : \limsup_{n\to\infty} p_\nu(f_n)^{1/\log n} < \infty\Big\},$$
$$\mathcal{K}_{\mathcal{P},r} = \Big\{(f_n)_n \in \mathcal{C}^\infty(O)^{\mathbf{N}} \mid \forall\nu \in \mathbf{N} : \limsup_{n\to\infty} p_\nu(f_n)^{1/\log n} = 0\Big\}.$$

we obtain the simplified Colombeau algebra $\mathcal{G}_s$.

So called full Colombeau algebra $\mathcal{G}$ is related to a more delicate procedure and it is omitted. We only note that the embedding of Schwartz distributions and of smooth functions into $\mathcal{G}$ is well known. Also it is well known that the multiplication of smooth function embedded into $\mathcal{G}$ is the usual multiplication.

Example 69. The following example is also of interest. Take $E = \mathcal{D}_{L^p}(O)$, $p > 1$, $\mathcal{P} = \{p_\nu\}_{\nu\in\mathbb{N}}$ with $p_\nu(f) := \sup_{|\alpha|\leqslant\nu} \|f^{(\alpha)}\|_{L^p}$, and $r = 1/\log$. Then, $\mathcal{G}_{L^p} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ with

$$\mathcal{F}_{\mathcal{P},r} = \left\{(f_n)_n \in \mathcal{D}_{L^p}(O)^{\mathbb{N}} \mid \forall\nu\in\mathbb{N} : \limsup_{n\to\infty} p_\nu(f_n)^{1/\log n} < \infty\right\},$$
$$\mathcal{K}_{\mathcal{P},r} = \left\{(f_n)_n \in \mathcal{D}_{L^p}(O)^{\mathbb{N}} \mid \forall\nu\in\mathbb{N} : \limsup_{n\to\infty} p_\nu(f_n)^{1/\log n} = 0\right\}.$$

is Colombeau type algebra used for the investigations of wave and heat equation.

8.1.2. Projective and inductive limits. *Projective limit.* The construction that follows leads to algebras of generalized ultradistributions of Beurling and Roumieu type. We will give only the general concepts of the construction.

Let $(E^\mu_\nu, p^\mu_\nu)_{\mu,\nu\in\mathbb{N}}$ be a family of semi-normed algebras over $\mathbb{C}$, such that

$$\forall\mu,\nu\in\mathbb{N} : E^\mu_{\nu+1} \hookrightarrow E^\mu_\nu,\ E^{\mu+1}_\nu \hookrightarrow E^\mu_\nu,$$

where $\hookrightarrow$ means continuously embedded. This implies that there exist constants C^μ_ν, $\tilde{C}^\mu_\nu \in \mathbb{R}_+$ such that

$$\forall\mu,\nu\in\mathbb{N} : p^\mu_\nu \leqslant C^\mu_\nu\, p^\mu_{\nu+1},\ p^\mu_\nu \leqslant \tilde{C}^\mu_\nu\, p^{\mu+1}_\nu,$$

but without loss of generality one can take C^μ_ν, $\tilde{C}^\mu_\nu = 1$, $\forall\mu,\nu\in\mathbb{N}$. Then let

$$\overleftarrow{E} := \operatorname*{proj\,lim}_{\mu\to\infty} \overleftarrow{E}^\mu = \operatorname*{proj\,lim}_{\mu\to\infty}\operatorname*{proj\,lim}_{\nu\to\infty} E^\mu_\nu = \operatorname*{proj\,lim}_{\nu\to\infty} E^\nu_\nu.$$

Define

$$\overleftarrow{\mathcal{F}}_{p,r} = \left\{f \in \overleftarrow{E}^{\mathbb{N}} \mid \forall\mu,\nu\in\mathbb{N} : \| f \|_{p^\mu_\nu, r} < \infty\right\},$$
$$\overleftarrow{\mathcal{K}}_{p,r} = \left\{f \in \overleftarrow{E}^{\mathbb{N}} \mid \forall\mu,\nu\in\mathbb{N} : \| f \|_{p^\mu_\nu, r} = 0\right\}.$$

(Here $p \equiv ((p^\mu_\nu)_\nu)^\mu$ stands (on the l.h.s.) for the whole family of seminorms.) Then Proposition 67 holds, with the slight changes of notations introduced above.

Inductive limit Consider now a family $(E^\mu_\nu, p^\mu_\nu)_{\mu,\nu\in\mathbb{N}}$ of semi-normed spaces over $\mathbb{C}$, such that

(70) $$\forall\mu,\nu\in\mathbb{N} : E^\mu_\nu \hookrightarrow E^\mu_{\nu+1},\ E^{\mu+1}_\nu \hookrightarrow E^\mu_\nu.$$

This implies that there exist constants C^μ_ν, $\tilde{C}^\mu_\nu \in \mathbb{R}_+$ such that

$$\forall\mu,\nu\in\mathbb{N} : p^\mu_{\nu+1} \leqslant C^\mu_\nu\, p^\mu_\nu,\ p^\mu_\nu \leqslant \tilde{C}^\mu_\nu\, p^{\mu+1}_\nu,$$

but again one can assume C^μ_ν, $\tilde{C}^\mu_\nu = 1$, $\forall\mu,\nu\in\mathbb{N}$. Now let

$$\forall\mu\in\mathbb{N} : \overrightarrow{E}^\mu = {}^*\operatorname*{ind\,lim}_{\nu\to\infty} E^\mu_\nu.$$

Assume that for every $\mu, \nu', \nu'' \in \mathbb{N}$ there exist $\nu\in\mathbb{N}$ and $C > 0$ such that

$$p^\mu_\nu(fg) \leqslant C p^\mu_{\nu'}(f) p^\mu_{\nu''}(g),\ f\in E^\mu_{\nu'},\ g\in E^\mu_{\nu''}.$$

Note that (70) implies that $\forall\mu\in\mathbb{N} : \overrightarrow{E}^{\mu+1} \hookrightarrow \overrightarrow{E}^\mu$. Now let

$$\overrightarrow{E} := \operatorname*{proj\,lim}_{\mu\to\infty} \overrightarrow{E}^\mu = \operatorname*{proj\,lim}_{\mu\to\infty} {}^*\operatorname*{ind\,lim}_{\nu\to\infty} E^\mu_\nu,$$

and define

$$\overrightarrow{\mathcal{F}}_{p,r} := \left\{ f \in \overrightarrow{E}^{\mathbb{N}} \mid \forall \mu \in \mathbb{N}, \exists \nu \in \mathbb{N} : f \in (E^{\mu}_{\nu})^{\mathbb{N}} \wedge \| f \|_{p^{\mu}_{\nu},r} < \infty \right\},$$

$$\overrightarrow{\mathcal{K}}_{p,r} := \left\{ f \in \overrightarrow{E}^{\mathbb{N}} \mid \forall \mu \in \mathbb{N}, \exists \nu \in \mathbb{N} : f \in (E^{\mu}_{\nu})^{\mathbb{N}} \wedge \| f \|_{p^{\mu}_{\nu},r} = 0 \right\}.$$

Proposition 70. (i) *Writing* $\overleftrightarrow{\cdot}$ *for both,* $\overleftarrow{\cdot}$ *or* $\overrightarrow{\cdot}$, *we have that* $\overleftrightarrow{\mathcal{F}}_{p,r}$ *is an algebra and* $\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an ideal of* $\overleftrightarrow{\mathcal{F}}_{p,r}$; *thus,* $\overleftrightarrow{\mathcal{G}}_{p,r} := \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an algebra.*

(ii) *For every* $\mu, \nu \in \mathbb{N}$, $d_{p^{\mu}_{\nu}} : (E^{\mu}_{\nu})^{\mathbb{N}} \times (E^{\mu}_{\nu})^{\mathbb{N}} \to \overline{\mathbb{R}}_{+}$ *defined by* $d_{p^{\mu}_{\nu}}(f,g) = \| f - g \|_{p^{\mu}_{\nu},r}$ *is an ultrapseudometric on* $(E^{\mu}_{\nu})^{\mathbb{N}}$. *Moreover* $(d_{p^{\mu}_{\nu}})_{\mu,\nu}$ *induces a topological algebra structure on* $\overleftarrow{\mathcal{F}}_{p,r}$ *(since* $d_{p^{\mu}_{\nu}}(0, f\cdot g) \leqslant d_{p^{\mu}_{\nu}}(0,f)d_{p^{\mu}_{\nu}}(0,g)$*) such that the intersection of neighborhoods of zero equals* $\overleftarrow{\mathcal{K}}_{p,r}$.

(iii) *From* (ii), $\overleftarrow{\mathcal{G}}_{p,r} = \overleftarrow{\mathcal{F}}_{p,r}/\overleftarrow{\mathcal{K}}_{p,r}$ *becomes a topological algebra which topology can be defined by the family of ultrametrics* $(\tilde{d}_{p^{\mu}_{\nu}})_{\mu,\nu}$, *where* $\tilde{d}_{p^{\mu}_{\nu}}([f],[g]) = d_{p^{\mu}_{\nu}}(f,g)$, $[h]$ *stands for the class of* h.

(iv) *If* τ_{μ} *denotes the inductive limit topology on* $\mathcal{F}^{\mu}_{p,r} = \bigcup_{\nu\in\mathbb{N}}((\tilde{E^{\mu}_{\nu}})^{\mathbb{N}}, d_{\mu,\nu})$, $\mu \in \mathbb{N}$, *then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is a topological algebra for the projective limit topology of the family* $(\mathcal{F}^{\mu}_{p,r}, \tau_{\mu})_{\mu}$.

$(\tilde{E^{\mu}_{\nu}})^{\mathbb{N}}$, consists of elements $f \in (E^{\mu}_{\nu})^{\mathbb{N}}$ with finite $d_{\mu,\nu}(f)$.
Without assuming completeness of E, it holds:

Proposition 71. (i) $\overleftarrow{\mathcal{F}}_{p,r}$ *is complete.*

(ii) *If for all* $\mu \in \mathbb{N}$, *a subset of* $\overrightarrow{\mathcal{F}}^{\mu}_{p,r}$ *is bounded if and only if it is a bounded subset of* $(E^{\mu}_{\nu})^{\mathbb{N}}$ *for some* $\nu \in \mathbb{N}$, *then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is sequentially complete.*

8.1.3. Comments on the Schwartz' impossibility result. In the definition of sequence spaces $\overrightarrow{\mathcal{F}}_{p,r}$ resp. $\overleftarrow{\mathcal{F}}_{p,r}$, we assumed $r_n \searrow 0$ as $n \to \infty$. Clearly, one could consider sequence spaces of the same type with r_n only bounded, or even $r_n \to \infty$. In the former case (r_n bounded), the space $\overleftrightarrow{\mathcal{F}}_{p,r}$ (where $\overleftrightarrow{\cdot}$ stands for $\overleftarrow{\cdot}$ or $\overrightarrow{\cdot}$) contains $*\mathrm{diag}\, E^{\mathbb{N}}$ topology, via the embedding $\overleftrightarrow{E} \ni f \mapsto (f)_n \in \overleftrightarrow{E}^{\mathbb{N}}$. In the second case (when $r_n \to \infty$), this embedding is not possible.

In the case we consider ($r_n \to 0$), the induced topology on $\overleftrightarrow{E}$ is a discrete topology. But this is necessarily so, since we want to include "divergent" sequences in $\overleftrightarrow{\mathcal{F}}_{p,r}$.

In order to have an appropriate topological algebra containing "δ", we must have that our generalized topological algebra induces a discrete topology on the original algebra $\overleftrightarrow{E}$. This conclusion is in analogy to Schwartz' impossibility statement for multiplication of distributions.

8.1.4. Sequences of scales. We can consider a sequence $(r^m)_m$ of positive sequences $(r^m_n)_n$ such that

$$\forall m, n \in \mathbb{N} : r^m_{n+1} \leqslant r^m_n; \ \lim_{n\to\infty} r^m_n = 0.$$

In addition to this, we request either of the following conditions:

$$\forall m, n \in \mathbb{N} : r_n^{m+1} \geqslant r_n^m \text{ or } \forall m, n \in \mathbb{N} : r_n^{m+1} \leqslant r_n^m.$$

Then let, in the first resp. second case :

$$\overleftrightarrow{\mathcal{F}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}, \ \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}$$

$$\text{resp. } \overleftrightarrow{\mathcal{F}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}, \ \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}$$

(where again $p = (p_\nu^\mu)_{\nu,\mu}$). Then again, $\overleftrightarrow{\mathcal{G}}_{p,r} := \overleftrightarrow{\mathcal{F}}_{p,r} / \overleftrightarrow{\mathcal{K}}_{p,r}$ will be an algebra.

The following example cower so called Egorov's type algebra. $r_n^m = \begin{cases} 1, & \text{if } n \leqslant m \\ 0, & \text{if } n > m \end{cases}$ gives the Egorov-type algebras, where the "subalgebra" contains everything and the ideal contains only stationary null sequences.

8.1.5. General remarks on embeddings of duals. Under mild assumptions on $\overleftrightarrow{E}$, we can show that our algebras of (classes of) sequences contains elements of the strong dual space $\overleftrightarrow{E}'$. Let $C^0(\mathbb{R}^s)$ be the space of continuous functions with projective topology given by sup norms on the balls of radius $\nu \in \mathbb{N}^*$, $p_\nu(f) = \sup\{|f(x)|; |x| \leqslant \nu\}$. We shall assume in the sequel that $\overleftrightarrow{E}$ is a dense subspace of $C^0(\mathbb{R}^s)$ and the inclusion mapping $\overleftrightarrow{E} \to C^0(\mathbb{R}^s)$ is continuous. Then, we have the following

Proposition 72. (i) $\delta : \overleftrightarrow{E} \to \mathbb{C}$, $\delta(\phi) := \phi(0)$ *is an element of* $\overleftrightarrow{E}'$.
(ii) *Let* $\overleftrightarrow{E}$ *be sequentially weakly dense in* $\overleftrightarrow{E}'$. *Then, a sequence* $(\delta_n)_n \in E \cap (C^0)'$ *with the property* $\exists \eta, \theta > 0 : \forall n \in \mathbb{N} : \sup_{|x|>\theta} |\delta_n(x)| < \eta$, *converging weakly to* δ, *cannot be bounded in* $\overleftrightarrow{E}$.

Thus, the appropriate choice of the sequence r appeared to be important to have at least δ embedded into the corresponding algebra. It can be chosen such that:

In $\overleftarrow{E}$ case, for every $\mu, \nu \in \mathbb{N}$

$$\limsup_{n\to\infty} p_\nu^\mu(\delta_n)^{r_n} = A_\nu^\mu \text{ and } \exists \mu_0, \nu_0 : A_{\nu_0}^{\mu_0} \neq 0.$$

In $\overrightarrow{E}$ case, for every $\mu \in \mathbb{N}$ exists $\nu \in \mathbb{N}$ such that the above limit holds.

8.2. Association. The notion of a weak limit or of a weak solutions is transferred to generalized function algebras to various notions of associations. Thus their importance is underlined through the applications to nonlinear equations or linear one with singularities.
General concept: $\mathcal{J} - X$-association. The $\mathcal{J} - X$-association of elements $F, G \in \mathcal{G} = \mathcal{F}/\mathcal{K}$ is defined in terms of an additive subgroup $\mathcal{J}$ of $\mathcal{F}$ containing the ideal $\mathcal{K}$, and a set X of generalized numbers, by

$$F \underset{\mathcal{J},X}{\approx} G \iff \forall x \in X : x \cdot (F - G) \in \mathcal{J}/\mathcal{K}.$$

As $\mathcal{J}$ is not an ideal, the association is not compatible with the multiplication in $\mathcal{F}$ (not even by generalized numbers, only by elements of E). However, in the case of differential algebras, $\mathcal{J}$ is usually chosen such that $\underset{\mathcal{J}}{\approx}$ is stable under differentiation. If the set X contains only number 1, then we simply write $F \underset{\mathcal{J}}{\approx} G \iff F - G \in \mathcal{J}/\mathcal{K}$.

For example, consider $N = \{x \in \mathbb{C}^{\mathbf{N}} \mid \lim x_n = 0\}$, the set of null sequences. This gives usual association of generalized numbers,

$$[x] \sim [y] \iff [x] \underset{N}{\approx} [y] \iff x_n - y_n \to 0$$

which is well defined because all elements of the ideal tend to zero.
Strong s-association. is defined for $s \in \mathbb{R}_+$ by

$$F \overset{s}{\simeq} G \iff F \underset{\mathcal{J}^{(s)}_{\mathcal{P}}}{\approx} G \text{ with } \mathcal{J}^{(s)}_{\mathcal{P},r} = \{f \in \mathcal{F} \mid \forall p \in \mathcal{P} : \| f \|_{p,r} < e^{-s}\}.$$

For $s = 0$, we write $F \overset{\mathcal{P}}{\simeq} G$ and simply call them strongly associated. On the other hand, $F \overset{s}{\simeq} G$ for all $s \geqslant 0$ implies $F = G$.
Weak associations. The following types of associations are defined in terms of a duality product[1] $\langle \cdot, \cdot \rangle : \overleftrightarrow{E} \times D \to \mathbb{C}$, and

$$\mathcal{J} = \mathcal{J}_M = \{f \in \overleftrightarrow{E}^{\mathbf{N}} \mid \forall \psi \in D : (\langle f_n, \psi \rangle)_n \in M\}.$$

where M is some additive subgroup of $\mathbb{C}^{\mathbf{N}}$.

$s - D'$-association is defined by $F \overset{s}{\approx} G \iff F \underset{\mathcal{J}_N, X_s}{\approx} G$ with $X_s = \{[(e^{s/r_n})_n]\}$ for $s \in \mathbb{R}$.

Example 73. In the case of Colombeau's algebra this has already been considered (with $D = \mathcal{D}$): For $s = 0$ we get the so-called weak association $[f] \approx [g] \iff f_n - g_n \to 0$ in $\mathcal{D}'$. For $s \neq 0$, $[f] \overset{s}{\approx} [g] \iff n^s(f_n - g_n) \to 0$ in $\mathcal{D}'$. In the case of ultradistributions, we take $D = \mathcal{D}^{(m)}$ and $e^{s/r_n} = \exp[s\, n^{\frac{1}{m'-1}}]$ for Beurling case, and analogous definitions in the Roumieu case.

Weak s-association is defined by $F \overset{s}{\approx} G \iff F \underset{\mathcal{J}_I}{\approx} G$ where $I = \mathcal{J}_{|\cdot|,r,s}$ for any $s \in \mathbb{R}$. For $s = 0$, we write $F \overset{\mathrm{sw}}{\approx} G$ and call F and G *strong-weak associated.*

Remark 74. Weak s-association implies $s - D'$-association, but conversely $s - D'$-association only implies weak s'-association with $s' < s$.

---

[1] D stands for a test function space such that $E \hookrightarrow D'$.

References

[1] S. Albeverio, Z. Haba, F. Russo, *Trivial solutions for a non-linear two-space-dimensional wave equation perturbed by space-time white noise*, Stochastics Stochastics Rep. 56:1-2 (1996), 127–160.

[2] H. Biagioni, *The Cauchy problem for semilinear hyperbolic systems with generalized functions as initial conditions*, Results Math. 14 (1988), 231–241.

[3] H. A. Biagioni, *A Nonlinear Theory of Generalized Functions*, Springer-Verlag, Berlin–Heidelberg–New York, 1990.

[4] H. A. Biagioni, J. F. Colombeau, *New Generalized Functions and C^∞ Functions with Values in Generalized Complex Numbers*, J. London Math. Soc. (2) 33 (1986), 169–179.

[5] H. A. Biagioni, J. F. Colombeau, *Whitney's extension theorem for generalized functions*, J. Math. Anal. Appl. 14 (1986), 574–583.

[6] H. A. Biagioni, M. Oberguggenberger, *Generalized Solutions to Burgers's Equation*, J. Differential Equations 97 (1992), 263–287.

[7] H. A. Biagioni, M. Oberguggenberger, *Generalized solutions to the Korteweg-de Vries and the regularized long-wave equations*, SIAM J. Math. Anal. 23 (1992), 923–940.

[8] A. Bressan, *Hyperbolic Systems of Conservation Laws*, Preprins S.I.S.S.A., Trieste, Italy.

[9] J. F. Colombeau, *New Generalized Functions and Multiplication of the Distributions*, North Holland, 1983.

[10] J. F. Colombeau, *Elementary Introduction in New Generalized Functions*, North Holland, 1985.

[11] J. F. Colombeau, *Multiplication of Distributions*, Lecture Notes Math. 1532, Springer-Verlag, Berlin, 1992.

[12] J. F. Colombeau, M. Langlais, *Existence and uniqueness of solutions of nonlinear parabolic equations with Cauchy data distributions*, J. Math. Anal. Appl. 145(1990), 186–196.

[13] J. F. Colombeau, A. Y. Le Roux, *Multiplications of distributions in elasticity and hydrodynamics*, J. Math. Phys. 29 (1988), 315–319.

[14] J. F. Colombeau, M. Oberguggenberger, *On a hyperbolic system with a compatible quadratic term: generalized solutions, delta waves, and multiplication of distributions*, Comm. Part. Diff. Eq. 15 (1990), 905–938.

[15] J. F. Colombeau, A. Heibig, M. Oberguggenberger, *Le probleme de Cauchy dans un espace de fonctions généralisées*, I. C. R. Acad. Sci. Paris 317 (1993), 851–855.

[16] N. Krejić, T. Krunić, M. Nedeljkov, *Numerical verification of delta shock waves in conservation laws*, Submitted.

[17] A. Delcroix, M. Hasler, S. Pilipović, V. Valmorin, *Generalized function algebras as sequence space algebras*, Proc. Amer. Math. Soc. 132 (2004), 2031–2038.

[18] A. Delcroix, M. Hasler, S. Pilipović, V. Valmorin, *Embeddings of ultradistributions and periodic hyperfunctions in Colombeau type algebras through sequence spaces*, Math. Proc. Cambridge Philos. Soc. 137 (2004), 697–708.

[19] A. Delcroix, D. Scarpalezos, *Sharp topologies on asymptotic algebras*, Mh. Math. 129 (2000), 1–14.

[20] N. Djapić, M. Kunzinger, S. Pilipović, *Symmetry group analysis of weak solutions*, Proc. London Math. Soc. (3) 84 (2002), 686–710.

[21] N. Djapić, S. Pilipović, D. Scarpalezos, *Intrinsic Microlocal Characterization of Colombeau's Generalized Functions*, J. Anal. Math. 75 (1998), 51–66.

[22] G. Ercole, *Delta shock waves as self-similar viscosity limits*, Quart. Appl. Math. 58,1 (2000), 177–199.

[23] C. Garetto, *Pseudo-differential operators in algebras of generalized functions and global hypoellipticity*, Acta Appl. Math. 80 (2004), 123–174.

[24] C. Garetto, *Topological structures in Colombeau algebras: investigation of the duals of* $Gc(\Omega)$, $G(\Omega)$ *and* $GS(\mathbb{R}^n)$, Monatshefte Math., to appear

[25] C. Garetto, G. Hörmann, *Duality theory and pseudodifferential techniques for Colombeau algebras: generalized kernels and microlocal analysis*, Bull. Cl. Sci. Math. Nat. Sci. Math., to appear.

[26] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983.

[27] T. Gramchev, *Nonlinear maps in spaces of distributions*, Math. Zeitschr. 209 (1992), 101–114.

[28] T. Gramchev, *Entropy solutions to conservation laws and singular initial data*, Nonlin. Anal. TMA 24(5) (1995), 721–733.

[29] M. Grosser, E. Farkas, M. Kunzinger, R. Steinbauer, *On the foundations of nonlinear generalized functions I and II*, Mem. Amer. Math. Soc. 153 (2001), no. 729.

[30] M. Grosser, M. Kunzinger, R. Steinbauer, J. A. Vickers, *A global theory of algebras of generalized functions*, Adv. Math. 166 (2002), 50–72

[31] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer, *Geometric theory of generalized functions with applications to general relativity*, Mathematics and its Applications 537, Kluwer, Dordrecht, 2001.

[32] B. T. Hayes, P. G. Le Floch, *Measure solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity 9 (1996), 1547–1563.

[33] A. Heibig, M. Moussaoui, *Generalized and classical solutions of nonlinear parabolic equations*, Nonlinear Anal. 24 (1995), 789–794.

[34] G. Hörmann, *First-order hyperbolic pseudodifferential equations with generalized symbols*, J. Math. Anal. Appl. 293 (2004), 40–56.

[35] G. Hörmann, M. Oberguggenberger, *Elliptic regularity and solvability for partial differential equations with Colombeau coefficients*, Electron. J. Differential Equations 14 (2004), pp. 30.

[36] G. Hörmann, M. De Hoop, *Detection of wave front set perturbations via correlation: foundation for wave-equation tomography*, Appl. Anal. 81 (2002), 1443–1465.

[37] G. Höermann, M. Kunzinger, *Microlocal properties of basic operations in Colombeau algebras*, J. Math. Anal. Appl. 261 (2001), 254–270.

[38] G. Höermann, M. Oberguggenberger, S. Pilipović, *Microlocal hypoellipticity of linear differential operators with generalized functions as coefficients*, to appear in Trans. AMS.

[39] H. Holden, B. Oxendal, J. Uboe, T. S. Zhang, *Stochastic Partail Differential Equations*, Birkhäuser, Basel, 1996.

[40] F. Huang, *Weak solution to pressureless type system*, Comm. PDEs (2005).

[41] K. Joseph, *A Riemann problem whose viscosity solutions contain δ-measures*, Asymptotic Anal. 7 (1993), 105–120.

[42] B. L. Keyfitz, H. C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Diff. Eq. 118 (1995), 420–451.

[43] I. Kmit, G. Hörmann, *Semilinear hyperbolic systems with singular non-local boundary conditions: reflection of singularities and delta waves*, Z. Anal. Anwendungen 20 (2001), 637–659.

[44] H. Komatsu, *Ultradistributions, I–III*, J. Fac. Sci. Univ. Tokyo, Sect. IA 20 (1973), 25–105; 24 (1977), 607–628; 29 (1982), 653–717.

[45] H. Komatsu, *Microlocal Analysis in Gevrey Classes and in Convex Domains*, Lecture Notes Math. 1726, Springer-Verlag, (1989), 426–493.

[46] M. Kunzinger, *Nonsmooth differential geometry and algebras of generalized functions*, J. Math. Anal. Appl. (Special issue dedicated to John Horváth) 297 (2004), 456–471.

[47] M. Kunzinger, M. Oberguggenberger, R. Steinbauer, J. A. Vickers, *Generalized flows and singular ODEs on differentiable manifolds*, Acta Appl. Math. 80 (2004), 221–241.

[48] M. Kunzinger, R. Steinbauer, J. A. Vickers, *Intrinsic characterization of manifold-valued generalized functions*, Proc. London Math. Soc. (3) 87 (2003), 451–470.

[49] M. Kunzinger, *Generalized functions valued in a smooth manifold*, Monatsh. Math. 137 (2002), 31–49.

[50] M. Kunzinger, R. Steinbauer, *Generalized pseudo-Riemannian geometry*, Trans. Amer. Math. Soc. 354 (2002), 4179–4199.

[51] D. J. Korchinski, *Solution of a Riemann Problem for a 2×2 System of Conservation Laws Possesing No Classical Weak Solution*, PhD Thesis, Adelphi University, Garden City, New York, 1977.

[52] F. Lafon, M. Oberguggenberger, *Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case*, J. Math. Anal. Appl. 160 (1991), 93–106.

[53] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia, 1973.

[54] R. J. LeVeque, *Wave-propagation algorithms for multi-dimensional hyperbolic systems*, J. Comput. Phys. 131 (1997), 327–353.

[55] J.-A. Marti, *Fundamental structures and asymptotic microlocalization in sheaves of generalized functions*, Integral Transforms and Special Functions 6 (1998), 223–228.

[56] J.-A. Marti, *$(\mathcal{C}, \mathcal{E}, \mathcal{P})$-sheaf structures and applications*, in: Michael Grosser et al., *Nonlinear Theory of Generalized Functions*, Research Notes in Mathematics, Chapman & Hall/Crc., 1999, 175–186.

[57] M. Nedeljkov, *Unbounded solutions to some systems of conservation laws – split delta shocks waves*, Mat. Vesnik 54 (2002), 145–149.

[58] M. Nedeljkov, *Delta and singular delta locus for one dimensional systems of conservation laws*, Math. Meth. Appl. Sci. 27 (2004), 931–955.

[59] M. Nedeljkov, *Singular shock waves in interactions*, Q. Appl. Math, to appear.

[60] M. Nedeljkov, M. Oberguggenberger, S. Pilipović, *Generalized solution to a semilinear wave equation*, Nonlinear Analysis 61 (2005), 461–475.

[61] M. Nedeljkov, M. Oberguggenberger, *Delta shock wave and interactions in a simple model case*, *Preprint*.

[62] M. Nedeljkov, S. Pilipović, *Convolution in Colombeau's spaces of generalized functions. parts I and II*, Publ. Inst. Math. (Beograd) (N.S.) 52(66) (1992), 95–104 and 105–112.

[63] M. Nedeljkov, S. Pilipović, *Convergence in new generalized function spaces*, Acta Sci. Math. (Szeged) 62:1-2 (1996), 231–241.

[64] M. Nedeljkov, S. Pilipović, *A note on a semilinear hyperbolic system with generalized functions as coefficients*, Nonlinear Anal., Theory Methods Appl. 30:1 (1997), 41–46.

[65] M. Nedeljkov, S. Pilipović, *Generalized solution to a semilinear hyperbolic system with a non-Lipshitz nonlinearity*, Mh. Math. 125 (1998), 255–261.

[66] M. Nedeljkov, S. Pilipović, *Hypoelliptic differential operators with generalized constant coefficients*, Proc. Edinb. Math. Soc. 41(1998), 47–60.

[67] M. Nedeljkov, S. Pilipović, D. Rajter, *Heat equation with singular potential and singular data*, P. Roy. Soc. Edinb. 135 A (2005), 863–886.

[68] M. Nedeljkov, S. Pilipović, D. Scarpalezos, *The Linear Theory of Colombeau Generalized Functions*, Pitman Res. Not. Math. 385, Longman Sci. Techn., Essex, 1998.

[69] M. Nedeljkov, D. Rajter, *Nonlinear stochastic wave equation with Colombeau stochastic generalized processes*, Math. Mod. Met. Appl. Sci. 12 (2002), 665–688.

[70] M. Nedeljkov, D. Rajter-Ćirić, *Semigroups and PDEs with perturbations*, in: A. Delcroix, M. Hasler, J.-A. Marti, V. Valmorin, *Nonlinear Algebraic Analysis and Applications*, Proc. of ICGF 2000 (2004), 219–227.

[71] M. Oberguggenberger, *Generalized solutions to semilinear hyperbolic systems*, Monatsh. Math. 103 (1987), 133–144.

[72] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Res. Not. Math. 259, Longman Sci. Techn., Essex, 1992.

[73] M. Oberguggenberger, *Case study of a nonlinear, nonconservative, non-strictly hyperbolic system*, Nonlinear Anal. Theory, Methods Appl. 19 (1992), 53–79.

[74] M. Oberguggenberger, *Generalized solutions to nonlinear wave equations*, Mat. Contemp. 27 (2004), 169–187.

[75] M. Oberguggenberger, F. Russo, *Nonlinear stochastics wave equations*, Inegral Transforms and Special Functions 6 (1998), 71–83.

[76] M. Oberguggenberger, M. Kunzinger, *Characterization of Colombeau generalized functions by their pointvalues*, Math. Nachr. 203 (1999), 147–157.

[77] M. Oberguggenberger, Y.-G. Wang, *Generalized solutions to conservation laws*, Zeitschr. Anal. Anw. 13 (1994), 7–18.

[78] M. Oberguggenberger, S. Pilipovic, D. Scarpalezos, *Local properties of Colombeau generalized functions*, Math. Nachr. 256 (2003), 88–99.

[79] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[80] H. Pecher, *Ein nichtlinearer Interpolationssatz und seine Anwendung auf nichtlineare Wellengleichungen*, Math. Z. 161 (1978), 9–40.

[81] S. Pilipović, *Colombeau's Generalized Functions and Pseudodifferential Operators*, University of Tokio, Lecture Notes Series, 1994.

[82] S. Pilipović, D. Scarpalezos, *Differential operators with generalized constant coefficients in Colombeau algebra*, Portugal. Math. 53 (1996), 305–324.

[83] S. Pilipović, D. Scarpalezos, *Colombeau ultradistributions*, Math Proc. Camb. Phil. Soc. 130 (2001), 541–553.

[84] S. Pilipović, D. Scarpalezos, *Divergebt types quasilinear Dirichlet problem with singularities*, Acta Appl. Math. to appear in 2006.

[85] S. C. Port, C. J. Stone, *Brownian Motion and Classical Potential Theory*, Academic Press, New York–San Francisco–London, 1978.

[86] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. 1 and Vol 2*, Acaddemic Press, New York–London–Toronto–Sydney–San Franceisco, 1972 and 1975.

[87] E. Rosinger, *Generalized solutions of nonlinear partial differential equations*, North Holland Mathematics Studies 146, Notas de Matemática [Mathematical Notes], 119. North Holland, Amsterdam, 1987.

[88] E. Rosinger, *Nonlinear partial differential equations. An algebraic view of generalized solutions*, North Holland Mathematics Studies 164, North Holland, Amsterdam, 1990.

[89] D. Scarpalezos, *Some Remarks on Functoriality of Colombeau's Construction: Topological and Microlocal Aspects and Applications*, Integral Transforms and Special Functions 6 (1998), 295–307.

[90] D. Scarpalezos, *Colombeau's generalized functions: topological structures; microlocal properties. A simplified point of view, I*, Bull. Cl. Sci. Math. Nat. Sci. Math. 25 (2000), 89–114.

[91] D. Scarpalezos, *Colombeau's generalized functions: topological structures; microlocal properties, A simplified point of view II*, Publ. Inst. Math. (Beograd) (N.S.) 76(90) (2004), 111–125

[92] R. Steinbauer, *Distributional description of impulsive gravitational waves*, Nonlinear theory of generalized functions, Vienna, 1997, 267–274, Chapman& Hall/CRC Res. Notes Math. 401, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[93] J. M. Stockie, J. A. Mackenzie, R. D. Russell, *A moving mesh method for one-dimensional hyperbolic conservation laws*, SIAM J. Sci. Comput. 22:5 (2001), 1791–1813.

[94] D. Tan, T. Zhang, Y. Zheng,, *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, J. Diff. Eq. 112 (1994), 1–32.

[95] M. Taylor, *Partial Differential Equations, Vol. 2*, Springer-Verlag, 1996.

[96] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, New York–San Francisco–London, 1975.

[97] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Beutscher Verlag der Wissenschaften, Berlin, 1978.

[98] V. Valmorin, *Periodic generalized functions and applications*, Dissertationes Math. (Rozprawy Mat.) 361 (1997), 75 pp.

[99] H. Yang, *Riemann problems for a class of coupled hyperbolic systems of conservation laws*, J. Diff. Eq. 159 (1999), 447–484.

[100] Y.-G. Wang, M. Oberguggenberger, *Nonlinear parabolic equations with regularized derivatives*, J. Math. Anal. Appl. 233:2 (1999), 644–658.

Vladimir Dragović

ALGEBRO-GEOMETRIC INTEGRATION IN CLASSICAL AND STATISTICAL MECHANICS

CONTENTS

Introduction

This article is an enlarged version of the talk given by the author on the Meeting on Mathematical Methods in Models of Mechanics, organized by Serbian Academy of Sciences and Arts in its Novi Sad Branch in October 2003.

We devoted the talk to the 10th anniversary of the Seminar on Mathematical Methods in Mechanics, which is being hold in the Mathematical Institute SANU. Main scientific topics in the focus of the Seminar in that period (1993–2003) are geometry of integrable dynamical systems, connections with complex algebro-geometric and finite-zone integration methods, applications in models of classical, quantum and statistical mechanics etc. Here, we are going to give a brief review of classical results and modern research streams in these areas, as well as the original results.

The article is organized as follows. The next two sections contain necessary notions and statements from algebraic geometry and integrable dynamical systems — in Section 1 we list basic definitions related to the theory of integrable systems, while Section 2 is a brief introduction to the theory of Riemann surfaces. In order to keep the presentation reasonably short, we intensively assume two references published in last few years in Belgrade [50, 26], and we refer readers to them for details and clarifications regarding algebraic geometry and Poisson structures. Let us emphasize that these mathematical techniques are the main tools for the research performed in the framework of the Seminar on Mathematical Methods in Mechanics.

In Section 3 we give a concise review of classical and modern results concerning the motion of the rigid body about the fixed point. In Section 4, the original results concerning a generalization of the classical Hess–Appel'rot rigid body system and its integration in both classical and algebro-geometric ways are presented [22, 23].

In Section 5 we return again to classical subjects, presenting Poncelet theorem on closed polygonal lines inscribed in one and circumscribed about another conic in the plane and Cayley's condition that describe analytically such polygons. In Section 6, billiards as an important class of dynamical systems are introduced. In Section 7 , we present the original results – the generalization of the Cayley's condition related to elliptical billiards in the space of arbitrary finite dimension [27, 28]. Section 8 is aimed to present the author's results on separable potential perturbations of integrable billiard systems [16, 17]. The last Section 9 is devoted to exactly solvable models in Statistical Mechanics and problems of algebro-geometric classification of the solutions of the Quantum Yang–Baxter equation. Some of the author's results, obtained in the general framework of Krichever's approach based on vacuum curvrs and vectors (see [46, 18, 19, 20]), are presented.

1. Poisson structures and completely integrable systems

The algebra $C^\infty(M)$ of smooth functions on a symplectic manifold (M, ω) admits a binary operation $\{f, g\} := \omega(X_f, X_g)$, where X_f and X_g are Hamiltonian vector fields defined by Hamiltonians f and g. Its basic properties are

- bilinearity;
- antisymmetricity: $\{f, g\} = -\{g, f\}$.
- the Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.
- the Leibnitz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

A more general class of manifolds are Poisson manifolds.

Definition 1. *Poisson algebra* is a commutative algebra with an antisymmetric bilinear operation $\{\cdot, \cdot\}$ satisfying the Jacobi identity and the Leibnitz rule. A manifold M is a Poisson manifold if there is an operation $\{\cdot, \cdot\}$ giving to $C^\infty(M)$ a structure of a Poisson algebra.

Let H be a smooth function on a Poisson manifold M. Then the dynamical system $\dot{x} = \{x, H\}$ is a Hamiltonian system with the Hamiltonian function H. A function F which is constant along the trajectories of the system is called a *first integral*. For a Hamiltonian system with the Hamiltonian function H, a function F is a first integral if and only if $\{H, F\} = 0$.

Let us recall that for functions F, H for which $\{H, F\} \equiv 0$ we say that they are *in involution*. Specially, since $\{H, H\} = 0$, the Hamiltonian function itself is a first integral for the Hamiltonian system. The following fundamental theorem describes the topological structure of flows of an important class of Hamiltonian systems.

Theorem 1 (Liouville–Arnol'd). *Let M be a symplectic manifold and assume $n = \frac{1}{2}\dim(M)$ functions in involution $F_1, \dots, F_n : M \to \mathbb{R}$ are given.*

Denote $c := (c_1, \dots, c_n) \in \mathbb{R}^n$ and $M_c = \{x \in M \mid F_k(x) = c_k\}$. If the functions $F_1, \dots, F_n$ are independent on M_c, then:

1. *M_c is a smooth manifold, invariant with respect to the Hamiltonian diffeomorphism generated by functions F_k.*
2. *If the manifold M_c is compact and connected, then it is diffeomorphic to a torus $\mathbb{T}^n = (\mathbb{S}^1)^n$.*
3. *There exist coordinates $(\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n$ such that the Hamiltonian equations with the Hamiltonian F_1 have the form $\dot{\varphi}_1 = \varpi_{1,c}, \dots, \dot{\varphi}_n = \varpi_{n,c}$ linearizing the flow.*

Definition 2. A Hamiltonian system (M^{2n}, ω, H) that has n independent first integrals in involution is *completely integrable in Liouville sense.*

All Hamiltonian systems with one degree of freedom are obvious examples of completely integrable systems. Starting with two degrees of freedom, the situation is not simple at all any more.

Example 1. The problem of geodesics on the surfaces of revolution in $\mathbb{R}^3$ is completely integrable.

Example 2. The problem of geodesics on ellipsoid in $\mathbf{E}^n$ is completely integrable, as a consequence of the Jacobi–Chasles theorem.

Completely integrable systems have, according to Theorem 1, very regular dynamics. However, they are very rare. Although, for any such a system, there exist action-angle coordinates where this system could be explicitly integrated, the construction of those coordinates is not explicit. Thus, in the theory of completely integrable systems two basic and usually difficult questions exist:

- *For a given system to show that it is completely integrable*;
- *For a given completely integrable system to perform explicit integration.*

For the systems given in the first two examples, integration is done by methods of separation of variables of Hamilton–Jacobi equation. After 1967 and discovery of infinite-dimensional completely integrable systems, such as Korteweg - de Vries equation, new techniques of solving such problems were found. These techniques are based on the inverse scattering methods, and some additional analytical, algebraic or algebro-geometric theories are used.

2. Riemann surfaces, a brief introduction

Theorem 2. *The next three definitions of genus are equivalent:*

- $g = \frac{1}{2}\dim H^1_{DR}(\Sigma)$
- $g = \dim\Omega^1(\Sigma) = \dim \check{H}^0(\Sigma;\Omega^1)$
- $g = \dim \check{H}^1(\Sigma;\mathcal{O})$.

Thus, on a Riemann surface of a genus g, there exist exactly g linearly independent holomorphic differentials. Let as consider now a case of elliptic and hyper-elliptic curves.

Example 3. On a hyper-elliptic curve of genus g given by the equation

$$y^2 = P_{2g+1}(x),$$

one basis of holomorphic differentials consists of $\omega_i = \dfrac{z^{i-1}}{P_{2g+1}(z)}dz$, $i = 1,\dots,g$. (For $g = 1$ the case of elliptic curves is included.)

Theorem 3 (Riemann–Roch). *Let D be a divisor on compact Riemann surface Σ of genus g. Then $\check{H}^0(\Sigma;\mathcal{O}_D)$ and $\check{H}^1(\Sigma;\mathcal{O}_D)$ are finitely dimensional vector spaces and*

$$\dim \check{H}^0(\Sigma;\mathcal{O}_D) - \dim \check{H}^1(\Sigma;\mathcal{O}_D) = 1 - g + \deg(D).$$

Definition 3. A divisor D satisfying $l(K-D) = 0$ is called *nonspecial.* Otherwise, a divisor D is *special,* and the number $l(K-D)$ is called *the index of speciality.*

From Poincaré–Hopf theorem, it follows

Proposition 1. *On a curve Γ of genus g, the degree of canonical divisor K_Γ is* $\deg K_\Gamma = 2g-2$. *Any divisor D of degree greater than $2g-2$ is nonspecial.*

Definition 4. A Riemann surface Γ of genus greater then 1 is *hyper-elliptic* if there exists a holomorphic two-sheeted covering $\pi : \Gamma \to \mathbb{CP}^1$.

Example 4. If a Riemann surface is hyper-elliptic in the sense of the last definition, then it represents a normalization of a curve given by the equation

$$y^2 = P_{2g+2}(x), \quad g \geq 2.$$

Two-sheeted covering π induces an involution σ on the hyper-elliptic curve Γ. If the curve is defined by the last equation, then the involution is given by the formula $\sigma(x,y) = (x,-y)$, and the set B of fixed points of the involution is in one to one correspondence with the set $B' = \{x_1, \dots, x_{2g+2}\}$ of zeroes of the polynomial $P_{2g+2} = a\prod(x - x_i)$.

Exercise 1. Let P, Q be two arbitrary points on the hyper-elliptic curve Γ. Prove that the divisors $P + \sigma(P)$ and $Q + \sigma(Q)$ are equivalent.

The class of divisors $P + \sigma(P)$ we denote by L. It does not depend, according to the last Exercise, on the choice of the point P. Let $T \subset B$ be a subset with even cardinality. We use the following notation:

$$e_T = \sum_{P_i \in T} P_i - \frac{|T|}{2} L.$$

Exercise 2. Prove:

- $2e_T = 0$.
- $e_{T_1} + e_{T_2} = e_{T_1 \Delta T_2}$, where Δ denotes the symmetric set difference.
- $e_{T_1} = e_{T_2}$ if and only if $T_1 = T_2$ or $T_1 = B \setminus T_2$.
- On hyper-elliptic curve Γ of genus g, it holds $K_\Gamma = (g-1)L$.

2.1. Matrix of periods of a Riemann surface. Suppose Γ is a given, compact, nonsingular Riemann surface of genus g. Denote by $(a_1, \dots, a_g, b_1, \dots, b_g)$ a basis of homologies $H_1(\Gamma, \mathbb{Z})$, which is *canonical*, i.e., such that

$$a_i \circ a_j = b_i \circ b_j = 0, \; a_i \circ b_j = \delta_{ij}, \quad i, j = 1, \dots, g.$$

Denote by $\tilde{\Gamma}$ the fundamental $4g$-angle, with edges $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. The surface Γ can be realized by gluing the edges of $\tilde{\Gamma}$.

Let ω, ω' be closed differential on Γ, and let

$$A_i = \int_{a_i} \omega, \quad B_i = \int_{b_i} \omega, \quad A_i' = \int_{a_i} \omega', \quad B_i' = \int_{b_i} \omega',$$

for $i = 1, \dots, g$ be their periods on canonical basis of cycles. Then

$$\iint_\Gamma \omega \wedge \omega' = \sum_{i=1}^{g} (A_i B_i' - A_i' B_i).$$

Let us fix a basis of holomorphic differentials $[\omega_1, \dots, \omega_g]$ such that

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}, \quad j, k = 1, \dots, g.$$

For a basis normalized in that way, denote by B_{jk} the matrix of b-periods:

$$B_{jk} = \int_{b_j} \omega_k, \quad j,k = 1,\dots,g.$$

Definition 5. The matrix B_{jk} is called period matrix of a Riemann surface Γ.

Proposition 2 (Riemann bilinear relations). *For the period matrix B_{jk} of a Riemann surface, it holds:*

- *The matrix B is symmetric.*
- *The matrix B has a negatively defined real part.*

Definition 6. A matrix B is called Riemannian matrix, if it satisfies properties of the last proposition. The set of such $g \times g$ matrices is called the Sigel half-plane and is denoted by $\mathcal{H}_g$.

Thus, every period matrix of a Riemann surface is a Riemannian matrix. The converse question is highly nontrivial: *Which Riemannian matrices are period matrices of some Riemann surface?* This classical and very important problem of XIX century algebraic geometry is known as *the Riemann–Shöttke problem* and it was open for more than a century. It was solved quite recently, in the middle of 1980's, using the techniques of *the soliton theory*, Japanese mathematician Shiota proved the so-called *Novikov's conjecture*. We will tell something more about this at the and of this Section.

2.2. Jacobian of a Riemann surface. The Abel map. Denote the standard basis of $\mathbb{C}^g$ by $e = [e_1,\dots,e_g]$, $(e_i)_k = \delta_{ik}$.

Exercise 3. Let B be a Riemannian matrix. Then $2g$ vectors $e_1,\dots,e_g, Be_1,\dots,Be_g$ are linearly independent over $\mathbb{R}$.

Let us consider an integer-valued lattice Λ_B in $\mathbb{C}^g$ generated by the vectors $2\pi i e_j$, Be_k, $k,j = 1,\dots,g$:

$$\Lambda_B: \quad 2\pi i M + BN, \quad M,N \in \mathbb{Z}^g.$$

Then $2g$-dimensional torus $\mathbb{T}^{2g} = \mathbb{T}(B) = \mathbb{C}^g/\Lambda_B$ defines a g-dimensional *Abel variety*, a g-dimensional complex torus.

Definition 7. If a matrix B is a period matrix of some Riemann surface Γ of genus g, then $\mathbb{T}(B)$ is called the Jacobian variety of a surface Γ, denoted by $\mathbb{T}(B) = \mathrm{Jac}(\Gamma)$.

Let a compact, smooth Riemann surface Γ of genus g be given with some canonical basis of homologies (a,b) and with corresponding normalized basis of holomorphic differentials $[\omega_1,\dots,\omega_g]$. Choosing an arbitrary point P_0 on Γ, let us consider g *Abel integrals*

$$u_i(P) = \int_{P_0}^{P} \omega_i, \quad i = 1,\dots,g,$$

assuming one and the same integration path every time.

Together with holomorphic differentials, known also as *Abel differentials of the first kind*, meromorphic differentials play important role as well.

Definition 8. The Abel differentials of the second kind $\omega_P^{(n)}$ are meromorphic differentials with a unique pole at a point P of order $n+1$, locally represented by

$$\omega_P^{(n)} = \frac{dz}{z^{n+1}} + \cdots .$$

The Abel differentials of the third kind ω_{PQ} are determined by unique simple poles P, Q with residua $+1, -1$.

These differentials are uniquely determined by the conditions:

$$\int_{a_i} \omega_P^{(n)} = 0, \quad \int_{a_i} \omega_{PQ} = 0, \quad i = 1, \dots, g.$$

Exercise 4. Prove the following formulae

$$\int_{b_i} \omega_P^{(n)} = \frac{1}{n!} \frac{d^{n-1} f_i(Q)}{dz^{n-1}}, \quad i = 1, \dots, g,\ n \in \mathbb{N}, \tag{1}$$

$$\int_{b_i} \omega_{PQ} = \int_Q^P \omega_i, \quad i = 1, \dots, g, \tag{2}$$

where $\omega_i = f_i(z)\, dz$ locally represents basic holomorphic differential around a point Q.

Exercise 5. Given four arbitrary points on a Riemann surface, prove:

$$\int_{Q_1}^{Q_2} \omega_{Q_3 Q_4} = \int_{Q_3}^{Q_4} \omega_{Q_1 Q_2}.$$

Exercise 6. Prove that the formula

$$\mathcal{A}(P) = (u_1(P), \dots, u_g(P)) \tag{3}$$

defines a mapping $\mathcal{A} : \Gamma \to \mathrm{Jac}(\Gamma)$.

Definition 9. The mapping $\mathcal{A} : \Gamma \to \mathrm{Jac}(\Gamma)$ defined by formula (3) is called the Abel mapping.

The natural question is *wether given points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ represent a divisor of zeroes and poles of some meromorphic function on a surface Γ*. The answer is given in the following

Theorem 4 (Abel). *Given points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ form divisors of zeroes and poles of a meromorphic function on a Riemann surface Γ if and only if the relation*

$$\sum_{i=1}^{n} \mathcal{A}(P_i) = \sum_{i=1}^{n} \mathcal{A}(Q_i)$$

takes place on the Jacobian $\mathrm{Jac}(\Gamma)$.

2.3. Riemann theta-function. An important tool is introduced by the following

Definition 10. Given an arbitrary $g \times g$ Riemann matrix B, $B \in \mathcal{H}_g$. The Riemann theta-function $\theta(z, B)$ is defined by the series:

$$\theta(z,B) = \sum_{n \in \mathbf{Z}^g} \exp\big((Bn, n) + (n, z)\big). \tag{4}$$

Proposition 3. *The series* (4) *converges uniformly and absolutely on every compact subset of* $\mathbb{C} \times \mathcal{H}_g$ *and it defines a holomorphic function.*

Proposition 4. *The following periodic relations are valid:*

$$\theta(z + 2\pi i e_k, B) = \theta(z, B), \quad k = 1, \dots, g,$$
$$\theta(z + Be_k, B) = \exp(-B_{kk}/2 - z_k)\,\theta(z, B), \quad k = 1, \dots, g.$$

Similarly, Riemann theta-functions with characteristics can be introduced for arbitrary real vectors $a, b \in \mathbb{R}^g$:

$$\theta[2a, 2b](z) = \exp\Big\{\frac{1}{2}(Ba, a) + (z + 2\pi i b, a)\Big\}\theta(z + 2\pi i b + Ba).$$

2.4. The Jacobi inversion problem and the Riemann theorem about zeroes of a theta-function. Starting from the case of genus 2 Riemann surfaces, there is no sense to invert fixed Abel integral. The following system

$$\zeta_1 = \int_{P_0}^{P_1} \frac{dz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{dz}{\sqrt{P_5(z)}},$$
$$\zeta_2 = \int_{P_0}^{P_1} \frac{z\,dz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{z\,dz}{\sqrt{P_5(z)}},$$

we are going to consider in the next section, in connection with the Kowalevski case of rigid body motion. The problem is to determine points P_1, P_2 as functions of given values ζ_1, ζ_2. Observing symmetric appearance of points P_1 and P_2 in the above formulae, the problem can be reduced to find expressions of symmetric functions of P_1, P_2, through ζ_1, ζ_2. Historically, it was Jacobi who solved this problem in genus two case.

For an arbitrary genus, corresponding general Jacobi problem of inversion was formulated and solved by Riemann.

Given an arbitrary, smooth Riemann surface Γ of genus g, with a fixed canonical basis of homology cycles and corresponding basis of holomorphic differentials. By using the Abel mapping, we define

$$\mathcal{A}^n : S^n(\Gamma) \to \mathrm{Jac}(\Gamma), \quad \mathcal{A}^n(P_1, \dots, P_n) = \sum_{i=1}^{n} \mathcal{A}(P_i),$$

where $S^n(X)$ denotes symmetric n-th degree of a set X.

Proposition 5. *Let a nonspecial divisor* $D = P_1 + \dots + P_g$ *be given; then in a neighborhood of the point* $\mathcal{A}^g(P_1, \dots, P_g) \in \mathrm{Jac}(\Gamma)$ *the mapping* $\mathcal{A}^g$ *is invertible.*

In the general case, the divisor $D = P_1 + \cdots + P_g$ is nonspecial. Thus, the inverse of the mapping $\mathcal{A}^g$ is defined almost everywhere. To find explicitly the inverse, Riemann essentially used theta-functions. Let us present some of their basic properties, necessary for the solution of the Jacobi inversion problem.

Suppose a vector $f \in \mathbb{C}^g$ be given. Consider the function $F(P) = \theta(\mathcal{A}(P) - f)$, where $\theta(z) = \theta(z, B)$ is the theta-function of the surface Γ. Function F is well defined and analytic on the fundamental $4g$-angle $\tilde{\Gamma}$, and for almost all f it is not identically equal zero.

Proposition 6. *If the function F is not identically zero, then it has exactly g zeroes in $\tilde{\Gamma}$.*

Definition 11. A vector $\mathcal{K} = (K_1, \dots, K_g)$, where

$$K_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \left(\int_{a_l} \omega_l(P) \int_{P_0}^{P} \omega_j \right), \quad j = 1, \dots, g,$$

is called the vector of Riemann constants.

Proposition 7. *If a function F is not identically zero and if $P_1, \dots, P_g$ are its zeroes, then $\mathcal{A}^g(P_1, \dots, P_g) = f - \mathcal{K}$.*

Theorem 5 (Riemann). *Given a vector f such that $F(P) = \theta(\mathcal{A}(P) - \mathcal{K} - f)$ is not identically zero. Then:*

- *the function F has exactly g zeroes $P_1, \dots, P_g$, giving the solution of the Jacobi problem $u_i(P_1) + \cdots + u_i(P_g) = f_i$, $i = 1, \dots, g$.*
- *The divisor $P_1 + \cdots + P_g$ is nonspecial.*

The set of zeroes of the theta-function defined on the Jacobian of the Riemann surface Γ is called the theta divisor or the Θ-divisor of the Riemann surface, denoted also by Θ_Γ.

2.5. The Baker–Akhiezer function. In the theory of integrable systems an important role plays the notion of the Baker–Akhiezer function.

Definition 12. Given n points $P_1, \dots, P_n$ on a Riemann surface of genus g, with local parameters k_i^{-1}, $i = 1, \dots, n$, $k_i^{-1}(P_i) = 0$, n polynomials $q_i(k)$ and a nonspecial divisor D, then n-point Baker–Akhiezer function ψ corresponding to the data, is

- meromorphic on $\Gamma \setminus \{P_1, \dots, P_n\}$;
- for its divisor it holds $(\psi) + D \geq 0$;
- when P tends to P_i, the function $\psi(P)\exp(-q_i(k_i(P))$ is analytical.

Theorem 6. [33] *Given a nonspecial divisor D of degree N. Then the dimension of the space of Baker–Akhiezer function is $N - g + 1$.*

Example 5. If $N = g$, then the Baker–Akhiezer function ψ is determind uniquely up to a scalar factor. It is given by the formula

$$\psi(P) = a \exp\left(\sum_{j=1}^{n} \int_Q^P \Omega_{q_j}\right) \frac{\theta\left(\mathcal{A}(P) + \sum_{j=1}^{n} U^{(q_j)} - \mathcal{A}(D) - \mathcal{K}\right)}{\theta\left(\mathcal{A}(P) - \mathcal{A}(D) - \mathcal{K}\right)},$$

where Ω_{q_j} are Abel differentials of the second order, with a principle part around P_j of the form $dq_j(k_j(P))$ normalized by the condition of annulation of the a periods; $2\pi i U^{(q_j)}$ are the vectors of their b-periods.

2.6. Riemann–Shöttke problem and Novikov's conjecture. We saw that every period matrix of a Riemann surface is a Riemannian matrix. The converse question *which Riemannian matrices are period matrices of some Riemann surface* is classical and very important problem of XIX century algebraic geometry known as *the Riemann–Shöttke problem.* It was solved quite recently, in the middle of 1980's, using the techniques of the Baker–Akhiezer functions and the soliton theory, through the so-called *Novikov's conjecture.* (see [32])

It was known after Krichever (see [33] and references therein) that there exist certain theta-function formulae associated with period matrices which give solutions of the *Kadomtsev–Petviashvili (KP)* equation from the soliton theory

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0.$$

The Novikov conjecture is a converse statement that a Riemannian matrix is a period matrix *only if it gives a solution of KP equation through the Krichever formulae.*

In a weak form the Novikov conjecture has been proven by Dubrovin in 1981 [32]. The complete solution of Novikov's conjecture and the Riemann–Shöttke problem was done by Shiota in 1986 [60, 57]. The highlight of Shiota's proof was use of a notion of *tau-function* introduced by Sato school few years before, giving opportunity to involve simultaneously the whole hierarchy of integrable systems associated with the KP equation.

3. Rotations of a heavy rigid body about a fixed point

Let us consider rotations of a rigid body about a fixed point O, under the gravitational field. Motion of the rigid body is represented in two coordinate systems: *the fixed* $Oxyz$, and *the moving frame* $OXYZ$, which is attached to the body.

Traditionally, vectors in the fixed frame are denoted by small letters, and in the moving frame by capital letters. The vector $\Omega(t) = (p, q, r)$ will denote *angular velocity in the moving frame* and velocity V of a point Q is $V = \Omega \times Q$. Now the kinetic momentum G becomes $G = \iiint_\sigma Q \times (\Omega \times Q)\, dm = J(\Omega)$, where the operator J is symmetric and called inertia tensor of a rigid body.

The operator J defines quadratic form which gives *the ellipsoid of inertia* of the body $(JX, X) = 1$. The ellipsoid describes the mass distribution in the body. Choosing the basis $e = [i, j, k]$ where the operator J is diagonal, we get $[J]_e =$

$I = \mathrm{diag}(A, B, C)$. These three numbers A, B, C, the principal momenta of inertia, which describe the mass distribution, together with the coordinates of the mass center $\chi = (x_0, y_0, z_0)$, give complete description of the dynamical properties of the rigid body. (Instead of A, B, C we will also use I_1, I_2, I_3 as a notation for the principal momenta.)

In the same basis the vector of kinetic momentum becomes

$$G = Api + Bqj + Crk.$$

Denote by $\Gamma = (\gamma, \gamma', \gamma'')$ coordinates of the vertical orth in the moving frame. Gravitational force acts in direction of Γ, and assuming $mg = 1$, we get $L = \chi \times \Gamma$, where L is the principal momentum of forces. From the equation $\dot{G} = L$, the first group of the Euler–Poisson equations follow:

$$\dot{M} = M \times \Omega + \chi \times \Gamma, \tag{5}$$

where $M = I\Omega$.

The second group of Euler–Poisson equations follow from the fact that the vector Γ is fixed in the space:

$$\dot{\Gamma} = \Gamma \times \Omega. \tag{6}$$

The equations (5) and (6) are six differential equations of motion on Ω and Γ as functions of time.

3.1. The first integrals of motion. Integrable cases.

The Euler–Poisson equations always have three first integrals of motion:

$$F_1 = \frac{1}{2}\langle I\Omega, \Omega\rangle + \langle \Gamma, \chi\rangle \quad \text{(energy integral)},$$
$$F_2 = \langle \Gamma, \Gamma\rangle \ (= 1), \quad F_3 = \langle I\Omega, \Gamma\rangle.$$

The Euler case (1751). It is defined by the condition $\chi = 0$. The additional first integral is $F_4 = \langle M, M\rangle$.

The Lagrange case (1788). This case is defined by the conditions $A = B$ and $\chi = (0, 0, z_0)$. So, the ellipsoid of inertia is symmetric, and mass-center is placed on the symmetry axis. Additional first integral, linear in impulses, is $F_4 = M_3$.

The Kowalevski case. It is well known that Kowalevski, in her celebrated 1889 paper [45], starting with a careful analysis of the solutions of the Euler and the Lagrange case of rigid-body motion, formulated a problem *to describe the parameters* (A, B, C, x_0, y_0, z_0), *for which the Euler–Poisson equations have a general solution in the form of uniform functions with only moving poles as singularities.* Here $I = \mathrm{diag}(A, B, C)$ represents the inertia operator, and $\chi = (x_0, y_0, z_0)$ is the center of mass of the rigid body.

Then, in §1 of [45], some necessary conditions were formulated and a new case was discovered, now known as Kowalevski case, as a unique possible beside the cases of Euler and Lagrange: $A = B = 2C$, $\chi = (x_0, 0, 0)$. Additional first integral found by Kowalevski is of the fourth degree in impulses

$$F_4 = \left(\Omega_1^2 - \Omega_2^2 + \frac{x_0}{I_3}\Gamma_1\right)^2 + \left(2\Omega_1\Omega_2 + \frac{x_0}{I_3}\Gamma_2\right)^2.$$

The integration of the Kowalevski case. The problem of Kowalevski of a motion of a rigid body about a fixed point, can be reduced to the solution of the system

$$\dot{s}_1 = \frac{\sqrt{P_5(s_1)}}{(s_1 - s_2)}, \quad \dot{s}_2 = \frac{\sqrt{P_5(s_2)}}{(s_2 - s_1)}, \tag{7}$$

where s_i are so-called *Kowalevski variables.*

However, considering the situation where all momenta of inertia are different, Kowalevski came to the relation analogue to the following (see [39]):

$$x_0\sqrt{A(B-C)} + y_0\sqrt{B(C-A)} + z_0\sqrt{C(A-B)} = 0.$$

And she concluded that it should be $x_0 = y_0 = z_0$ in such a case, giving the Euler case.

But, it was Appel'rot who noticed in the beginning of 1890's, that the last relation admits one more case, not mentioned by Kowalevski:

$$x_0\sqrt{A(B-C)} + z_0\sqrt{C(A-B)} = 0, \quad y_0 = 0,$$

under the assumption $A > B > C$. Such systems were considered also by Hess, even before Appel'rot, in 1890. But such intriguing position corresponding to the Kowalevski paper, made the Hess–Appel'rot systems very attractive for leading Russian mathematicians from the end of XIX century. After few years, Nekrasov and Lyapunov provided new arguments and they demonstrated that the Hess–Appel'rot systems did not satisfy the condition investigated by Kowalevski, which means that conclusion of §1 of [45] *was correct.*

A few years ago, we constructed a Lax representation for it (see [22]). We provided the Lax representation for all new systems, generalizing the Lax pair from [22]. It appeared that new systems belong to the class of *isoholomorphic systems.* This class of systems was introduced and studied in [23], in connection with the Lagrange bitop.

Such systems have specific distribution of zeroes in Lax matrices. Therefore standard integration technics of [31, 1] cannot be applied directly. Its integration requires more detailed analysis of geometry of the Prym varieties and it is based on Mumford's relation on theta-divisors of unramified double coverings.

The L operator, a quadratic polynomial in λ of the form $\lambda^2 C + \lambda M + \Gamma$, in the case $n = 4$ satisfies the condition $L_{12} = L_{21} = L_{34} = L_{43} = 0$. Such a situation, explicitly excluded by Adler–van Moerbeke (see [1, Theorem 1]) and implicitly by Dubrovin (see [31, Lemma 5 and Corollary]) has been studied for the first time in [23].

Study of the spectral curve and the Baker–Akhiezer function for the four-dimensional Hess–Appel'rot systems (see [24, 25]) shows that, similarly to [23], dynamics of the system is related to certain Prym variety Π. It is connected to the evolution of divisors of some meromorphic differentials Ω^i_j. From the condition on zeroes of the Lax matrix, it follows that differentials Ω^1_2, Ω^2_1, Ω^3_4, Ω^4_3 are *holomorphic* during the whole evolution. Compatibility of this requirement with dynamics is based on Mumford's relation $\Pi^- \subset \Theta$, (see [23]), where Π^- is a translation of the Prym variety Π.

Classical Hess–Appel'rot system. Let $J_1 < J_2 < J_3$ and $\chi = (x_0, y_0, z_0)$. Hess in [42] and Appel'rot in [4] found that if the inertia momenta and the radius vector of center of masses satisfy the conditions

$$y_0 = 0, \quad x_0\sqrt{J_2 - J_1} + z_0\sqrt{J_3 - J_2} = 0 \tag{8}$$

then, the surface

$$F_4 = M_1 x_0 + M_3 z_0 = 0 \tag{9}$$

is invariant. Integration of such system, using classical techniques can be found in [39]. In [22], an L-A pair for the Hess–Appel'rot system is constructed:

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$
$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \quad C = \frac{J_1 + J_3}{J_1 J_3}\chi,$$

where the skew-symmetric matrices represent the vectors denoted by the same letter. Also, the basic steps in algebro-geometric integration procedure are given.

The Zhukovskiĭ geometric interpretation of the conditions (8), (9) (see [67, 49]) Let us consider the ellipsoid

$$\frac{M_1^2}{J_1} + \frac{M_2^2}{J_2} + \frac{M_3^2}{J_3} = 1,$$

and the plane containing the middle axis and intersecting the ellipsoid through a circle. Denote by l corresponding normal to the plane, which passes through the fixed point O. Then the conditions (8), (9) mean that the center of masses lies on the line l.

Having this interpretation in mind, we choose the basis of moving frame such that the third axis is l, the second one is directed as the middle axis of ellipsoid, and the first one is chosen according to the orientation of the orthogonal frame. In this basis (see [9]), particular integral (9) becomes $F_4 = M_3 = 0$, the matrix J obtains the form:

$$J = \begin{pmatrix} J_1 & 0 & J_{13} \\ 0 & J_1 & 0 \\ J_{13} & 0 & J_3 \end{pmatrix},$$

and $\chi = (0, 0, z_0)$. This will serve us as a motivation for the definition of the four-dimensional Hess–Appel'rot system.

4. The definition of Lagrange bitop and its basic properties

The equations of motion of a heavy n-dimensional rigid body fixed at a point in the moving frame are:

$$\dot{M} = [M, \Omega] + [\Gamma, \chi], \quad \dot{\Gamma} = [\Gamma, \Omega], \tag{10}$$

where the moving frame is such that the matrix I is diagonal in it, $\mathrm{diag}(I_1, \dots, I_n)$. Here $M_{ij} = (I_i + I_j)\Omega_{ij} \in \mathrm{so}(n)$ is the kinetic momentum, $\Omega \in \mathrm{so}(n)$ is the angular velocity, $\chi \in \mathrm{so}(n)$ is a given constant matrix (describing a generalized center of the mass), $\Gamma \in \mathrm{so}(n)$. Then $I_i + I_j$ are the principal inertia momenta. These equations are on the semidirect product $\mathrm{so}(n) \times \mathrm{so}(n)$ and they were introduced in [59].

We are going to consider a four-dimensional case of these equations defined by

$$\begin{matrix} I_1 = I_2 = a \\ I_3 = I_4 = b \end{matrix} \quad \text{and} \quad \chi = \begin{pmatrix} 0 & \chi_{12} & 0 & 0 \\ -\chi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_{34} \\ 0 & 0 & -\chi_{34} & 0 \end{pmatrix} \tag{11}$$

with the conditions $a \neq b$, $\chi_{12}, \chi_{34} \neq 0$, $|\chi_{12}| \neq |\chi_{34}|$. We will call this system *the Lagrange bitop.*

Proposition 8. [22] *The equations of motion* (10) *under the conditions* (11) *have an L-A pair representation* $\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$, *where*

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \tag{12}$$

and $C = (a+b)\chi$.

One can observe that both leading terms in the operators L and A (matrices C and χ) are skewsymmetric, while in [31, 32, 34, 51, 8] one is always symmetric and another one is skewsymmetric.

Before analyzing the spectral properties of the matrices $L(\lambda)$, we will change the coordinates in order to diagonalize the matrix C. In this new basis the matrices $L(\lambda)$ have the form $\tilde{L}(\lambda) = U^{-1} L(\lambda) U$,

$$\tilde{L}(\lambda) = \begin{pmatrix} -i\Delta_{34} & 0 & -\beta_3^* - i\beta_4^* & i\beta_3 - \beta_4 \\ 0 & i\Delta_{34} & -i\beta_3^* - \beta_4^* & -\beta_3 + i\beta_4 \\ \beta_3 - i\beta_4 & -i\beta_3 + \beta_4 & -i\Delta_{12} & 0 \\ i\beta_3^* + \beta_4^* & \beta_3^* + i\beta_4^* & 0 & i\Delta_{12} \end{pmatrix},$$

where $\Delta_{12} = \lambda^2 C_{12} + \lambda M_{12} + \Gamma_{12}$, $\Delta_{34} = \lambda^2 C_{34} + \lambda M_{34} + \Gamma_{34}$, and

$$\begin{aligned} \beta_3 &= x_3 + \lambda y_3, & x_3 &= \frac{1}{2}(\Gamma_{13} + i\Gamma_{23}), \\ \beta_4 &= x_4 + \lambda y_4, & x_4 &= \frac{1}{2}(\Gamma_{14} + i\Gamma_{24}), \\ \beta_3^* &= \bar{x}_3 + \lambda \bar{y}_3, & y_3 &= \frac{1}{2}(M_{13} + iM_{23}), \\ \beta_4^* &= \bar{x}_4 + \lambda \bar{y}_4, & y_4 &= \frac{1}{2}(M_{14} + iM_{24}). \end{aligned}$$

The spectral polynomial $p(\lambda,\mu)=\det(\tilde{L}(\lambda)-\mu\cdot 1)$ has the form

$$p(\lambda,\mu)=\mu^4+P(\lambda)\mu^2+[Q(\lambda)]^2,$$

where

$$P(\lambda)=\Delta_{12}^2+\Delta_{34}^2+4\beta_3\beta_3^*+4\beta_4\beta_4^*,\quad Q(\lambda)=\Delta_{12}\Delta_{34}+2i(\beta_3^*\beta_4-\beta_3\beta_4^*).$$

We can rewrite it in terms of M_{ij} and Γ_{ij}:

$$P(\lambda)=A\lambda^4+B\lambda^3+D\lambda^2+E\lambda+F,\quad Q(\lambda)=G\lambda^4+H\lambda^3+I\lambda^2+J\lambda+K.$$

Their coefficients

$$\begin{aligned}
&A=C_{12}^2+C_{34}^2=\langle C_+,C_+\rangle+\langle C_-,C_-\rangle,\\
&B=2C_{34}M_{34}+2C_{12}M_{12}=2\left(\langle C_+,M_+\rangle+\langle C_-,M_-\rangle\right),\\
&D=M_{13}^2+M_{14}^2+M_{23}^2+M_{12}^2+M_{34}^2+2C_{12}\Gamma_{12}+2C_{34}\Gamma_{34}\\
&\quad=\langle M_+,M_+\rangle+\langle M_-,M_-\rangle+2\left(\langle C_+,\Gamma_+\rangle+\langle C_-,\Gamma_-\rangle\right),\\
&E=2\Gamma_{12}M_{12}+2\Gamma_{13}M_{13}+2\Gamma_{14}M_{14}+2\Gamma_{23}M_{23}+2\Gamma_{24}M_{24}+2\Gamma_{34}M_{34}\\
&\quad=2\left(\langle\Gamma_+,M_+\rangle+\langle\Gamma_-,M_-\rangle\right),\\
&F=\Gamma_{12}^2+\Gamma_{13}^2+\Gamma_{14}^2+\Gamma_{23}^2+\Gamma_{24}^2+\Gamma_{34}^2=\langle\Gamma_+,\Gamma_+\rangle+\langle\Gamma_-,\Gamma_-\rangle,\\
&G=C_{12}C_{34}=\langle C_+,C_-\rangle,\\
&H=C_{34}M_{12}+C_{12}M_{34}=\langle C_+,M_-\rangle+\langle C_-,M_+\rangle,\\
&I=C_{34}\Gamma_{12}+\Gamma_{34}C_{12}+M_{12}M_{34}+M_{23}M_{14}-M_{13}M_{24}\\
&\quad=\langle C_+,\Gamma_-\rangle+\langle C_-,\Gamma_+\rangle+\langle M_+,M_-\rangle,\\
&J=M_{34}\Gamma_{12}+M_{12}\Gamma_{34}+M_{14}\Gamma_{23}+M_{23}\Gamma_{14}-\Gamma_{13}M_{24}-\Gamma_{24}M_{13}\\
&\quad=\langle M_+,\Gamma_-\rangle+\langle M_-,\Gamma_+\rangle,
\end{aligned}$$

$$K=\Gamma_{34}\Gamma_{12}+\Gamma_{23}\Gamma_{14}-\Gamma_{13}\Gamma_{24}=\langle\Gamma_+,\Gamma_-\rangle. \tag{13}$$

are integrals of motion of the system (10), (11). We used two vectors $M_+,M_-\in R^3$ which correspond to $M_{ij}\in\mathrm{so}(4)$ according to

$$(M_+,M_-)\to\begin{pmatrix}0&-M_+^3&M_+^2&-M_-^1\\ M_+^3&0&-M_+^1&-M_-^2\\ -M_+^2&M_+^1&0&-M_-^3\\ M_-^1&M_-^2&M_-^3&0\end{pmatrix}$$

Here M_+^j are the j-th coordinates of the vector M_+. The system (10), (11) is Hamiltonian with the Hamiltonian function

$$\mathcal{H}=\frac{1}{2}(M_{13}\Omega_{13}+M_{14}\Omega_{14}+M_{23}\Omega_{23}+M_{12}\Omega_{12}+M_{34}\Omega_{34})+\chi_{12}\Gamma_{12}+\chi_{34}\Gamma_{34}$$

The algebra $\mathrm{so}(4)\times\mathrm{so}(4)$ is 12 dimensional. The general orbits of the coadjoint action are 8 dimensional. According to [59], the Casimir functions are coefficients of $\lambda^0,\lambda,\lambda^4$ in the polynomials $[\det\tilde{L}(\lambda)]^{1/2}$ and $-\frac{1}{2}\operatorname{Tr}(\tilde{L}(\lambda))^2$.

Since

$$[\det\tilde{L}(\lambda)]^{1/2}=G\lambda^4+H\lambda^3+I\lambda^2+J\lambda+K,\quad -\frac{1}{2}\operatorname{Tr}(\tilde{L}(\lambda))^2=A\lambda^4+E\lambda+F,$$

the Casimir functions are J, K, E, F. Nontrivial integrals of motion are B, D, H, I. They are in involution. Nontrivial integrals of motion are B, D, H, I are independent in the case $\chi_{12} \neq \pm\chi_{34}$. When $|\chi_{12}| = |\chi_{34}|$, then $2H = B$ or $2H = -B$ and there are only 3 independent integrals in involution. So we have

Proposition 9. [22] *For $|\chi_{12}| \neq |\chi_{34}|$, the system* (10), (11) *is completely integrable in the Liouville sense.*

There are two families of integrable Euler–Poisson equations introduced by Ratiu in [59]. *The generalized symmetric case* is defined by the conditions

$$I_1 = \cdots = I_n, \quad \chi \text{ arbitrary};$$

and *the generalized Lagrange case* which is defined by

$$I_1 = I_2 = a, \quad I_3 = \cdots = I_n = b, \quad \chi_{ij} = 0 \text{ if } (i,j) \notin \{(1,2),(2,1)\}.$$

The system (10), (11) does not fall in any of those families and together with them it makes the complete list of systems with the L operator of the form

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma.$$

Proposition 10. [22] *If $\chi_{12} \neq 0$, then the Euler-Poisson equations* (10) *could be written in the form* (12) *(with arbitrary C) if and only if the equations* (10) *describe the generalized symmetric case, the generalized Lagrange case or the Lagrange bitop, including the case $\chi_{12} = \pm\chi_{34}$.*

One can compare this with [63, Theorem 15, ch. 53]. The proofs of the Propositions 8–10 can be found in [22].

The $L(\lambda)$ matrix is a quadratic polynomial in the spectral parameter λ with matrix coefficients. The general theories describing the isospectral deformations for polynomials with matrix coefficients were developed by Dubrovin [31, 32] in the middle of 70's and by Adler, van Moerbeke [1] few years later. Dubrovin's approach was based on the Baker–Akhiezer function. Both approaches were applied in rigid body problems (see [51, 1] respectively).

But, as it was shown in [23], non of these two theories can be directly applied in cases like this. Necessary modifications were suggested in [23], where the procedure of algebro-geometric integration was presented. It is based on some nontrivial facts from the theory of Prym varieties, such as the Mumford relation on theta divisors of unramified double coverings and the Mumford–Dalalyan theory (see [23, 55, 56, 13, 61, 62]).

Here we are going to follow closely the procedure from [23], with necessary changes, calculations and comments. As usual, we start with the spectral curve $\Gamma : \det(\tilde{L}(\lambda) - \mu \cdot 1) = 0$. So, we have

$$\Gamma : \mu^4 + \mu^2(\Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3\beta_3^* + 4\beta_4\beta_4^*) + [\Delta_{12}\Delta_{34} + 2i(\beta_3^*\beta_4 - \beta_3\beta_4^*)]^2 = 0.$$

There is an involution $\sigma : (\lambda, \mu) \to (\lambda, -\mu)$ on the curve Γ, which corresponds to the skew-symmetry of the matrix $L(\lambda)$. Denote the factor-curve by $\Gamma_1 = \Gamma/\sigma$.

Lemma 1. *The curve Γ_1 is a smooth hyperelliptic curve of the genus $g(\Gamma_1) = 3$. The arithmetic genus of the curve Γ is $g_a(\Gamma) = 9$.*

Proof. The curve: $\Gamma_1 : u^2 + P(\lambda)u + [Q(\lambda)]^2 = 0$, is hyperelliptic, and its equation in the canonical forme is $u_1^2 = [P(\lambda)]^2/4 - [Q(\lambda)]^2$, where $u_1 = u + P(\lambda)/2$. Since $[P(\lambda)]^2/4 - [Q(\lambda)]^2$ is a polynomial of the degree 8, the genus of the curve Γ_1 is $g(\Gamma_1) = 3$. The curve Γ is a double covering of Γ_1, and the ramification divisor is of the degree 8. According to the Riemann–Hurwitz formula, the arithmetic genus of Γ is $g_a(\Gamma) = 9$.

Lemma 2. *The spectral curve Γ has four ordinary double points S_i, $i = 1, \dots, 4$. The genus of its normalization $\tilde{\Gamma}$ is five.*

Lemma 3. *The singular points S_i of the curve Γ are fixed points of the involution σ. The involution σ exchanges the two branches of Γ at S_i.*

Together with the curve Γ_1, one can consider curves C_1 and C_2 defined by the equations

$$C_1 : v^2 = P(\lambda)/2 + Q(\lambda), \quad C_2 : v^2 = P(\lambda)/2 - Q(\lambda).$$

Since the curve Γ_1 is hyperelliptic, in a study of the Prym variety Π the Mumford –Dalalyan theory can be applied (see [28, 24, 10]). Thus, using the previous Lemma, we come to

Theorem 7. a) *The Prymian Π is isomorphic to the product of the curves E_i:*

$$\Pi = \mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2).$$

b) *The curve $\tilde{\Gamma}$ is the desingularization of $\Gamma_1 \times_{\mathbf{P}^1} C_2$ and $C_1 \times_{\mathbf{P}^1} \Gamma_1$.*
c) *The canonical polarization divisor Ξ of Π satisfies*

$$\Xi = E_1 \times \Theta_2 + \Theta_1 \times E_2,$$

where Θ_i is the theta-divisor of E_i.

4.1. Equally splitting double hyperelliptic coverings.

According to the Mumford –Dalalyan theory (see [56, 13, 61]), double unramified coverings over a hyperelliptic curve $y^2 = P_{2g+2}(x)$ of genus g are in the correspondence with the divisions of the set of the zeroes of the polynomial P_{2g+2} on two disjoint nonempty subsets with even number of elements. We will consider those coverings which correspond to the divisions on subsets with *equal number of elements* and we can call them *equallysplitting*, since the Prym variety splits then as a sum of two varieties of equal dimension.

Now, let us consider with the fixed operator A from (12) the whole hierarchy of systems defined by the Lax equations

$$\dot{L}_B^{(N)} = [L_B^{(N)}, A], \quad L_B^{(N)}(\lambda) = \lambda^N B + \lambda^{N-1} M_1 + \dots + M_N.$$

So $L_B^{(N)}(\lambda)$ is a polynomial in λ of degree $N \geq 2$, and the matrix B is proportional to the matrix χ: $B = d\chi$.

Generalizing the situation from the subection above, we see that the spectral curve Γ_N is a singular curve of the form

$$p_N(\lambda, \mu) = \mu^4 + P_N(\lambda)\mu^2 + [Q_N(\lambda)]^2 = 0,$$

where the polynomials P_N, Q_N have degree $\deg P_N = \deg Q_N = 2N$. So, its normalization is a double covering over the hyperelliptic curve

$$\mu_1^2 = P_N^2(\lambda)/4 - Q_N^2(\lambda)$$

of genus $g_N = 2N - 1$. This covering corresponds to the division of the set of zeroes on subsets of zeroes of the polynomials $P_N/2 - Q_N$ and $P_N/2 + Q_N$. This is an equallysplitting covering under the assumption $|\chi_{12}| \neq |\chi_{34}|$ we fixed at the beginning. It is easy to see that all equallysplitting coverings can be realized in such a way. So we have

Theorem 8. *The Lagrange bitop hierarchy realizes all equallysplitting coverings over the hyperelliptic curves of odd genus.*

5. The Poncelet theorem and Cayley's type conditions

The following integrable mechanical system is well known: motion of a free particle within an ellipsoid in the Euclidean space of any dimension d. On the boundary, the particle obeys the billiard law. Integrability of the system is related to classical geometrical properties of elliptical billiards: the Chasles, Poncelet and Cayley theorems. According to the Chasles theorem [5] every line in this space is tangent to $d-1$ quadrics confocal to the outer ellipsoid. Even more, all segments of the particle's trajectory are tangent to the same $d-1$ quadrics [5, 53]. The Poncelet theorem [58, 6, 47, 12, 11] put some light on closed billiard trajectories: *there exists a closed trajectory with $d-1$ given confocal caustics if and only if infinitely many such trajectories exist, and all of them have the same period.* Since the periodicity of a billiard trajectory depends only on its caustic surfaces, it is a natural question to find an analytical connection between them and corresponding period.

Cayley found [10] an analytical condition for caustic conics in the Euclidean plane case. Algebro-geometric proof of Cayley's theorem from Griffiths and Harris paper [41] is going to be presented now.

Given two ellipses $C(x) = 0$ and $D(x) = 0$ in the plane. From a given point a of the first ellipse, there exist two tangents t_1, t_2 on the second conic. These tangents intersect the first one, beside the point a, also at the points b_1, b_2 respectively. *The Chasles correspondence* relates the points b_1 and b_2 to the point a.

Theorem 9 (Poncelet). *Given a polygon inscribed in one of the conics and subscribed around the another. Then there exist infinitely many such polygons; every point of the first ellipse is a vertex of one of them. All those polygons have the same number of edges.*

Next question is to find an analytical condition to determine weather for two given conics there exist n-tangle inscribed in one of them and subscribed around the another one. Such a condition was established by Cayley.

Theorem 10 (Cayley). *There exist an n-tangle inscribed in D and subscribed around C if and only if*

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{p+1} \\ C_4 & C_5 & \dots & C_{p+2} \\ \dots & \dots & \dots & \dots \\ C_{p+1} & C_{p+2} & \dots & C_{2p-1} \end{vmatrix} = 0; \quad \begin{vmatrix} C_2 & C_3 & \dots & C_{p+1} \\ C_3 & C_4 & \dots & C_{p+2} \\ \dots & \dots & \dots & \dots \\ C_{p+1} & C_{p+2} & \dots & C_{2p} \end{vmatrix} = 0,$$

where in the first case $n = 2p$, and $n = 2p+1$ in the second. The matrix elements are determined from the development: $\sqrt{C + \lambda D} = A + B\lambda + C_2\lambda^2 + C_3\lambda^3 + \cdots$.

A mechanical interpretation of these theorems will be done in the next Section.

There are several proofs of these theorems. All of them are based on the theory of elliptic curves and functions.

Let C and D be two conics in $\mathbb{CP}^2$, intersecting at four points x_0, x_1, x_2, x_3. The dual conic D^* consists of tangents on D. Let us consider a configuration

$$E = \{(x, \xi) \mid x \in \xi\} \subset C \times D^*.$$

Then, E is a Riemann surface with two involutions $i(x, \xi) = (x', \xi)$, $i'(x', \xi) = (x', \xi')$. Their composition $j = i' \circ i$ is given by $j(x, \xi) = (x', \xi')$.

Thus, the Poncelet construction, starting with $p = (x, \xi)$ gives a polygon with n edges if and only if $j^n(p) = p$.

A mapping $E \to \mathbb{C} : (x, \xi) \mapsto x$ is two-sheeted covering of a Riemann sphere $\mathbb{CP}^1$, with four ramification points x_0, x_1, x_2, x_3. Applying the Hurvitz formula, we get $\chi(E) = 2\chi(P^1) - 4 = 0$, i.e., E is an elliptic curve.

One can chose (x_0, ξ_0) as neutral element of the group of the elliptic curve E, and denote $p = (\bar{x}, \bar{\xi}) = j(x_0, \xi_0)$.

To prove the Poncelet theorem, one has to show that:

the condition $j^n(p) = p$ does not depend on choice of the point p.

It follows from the next theorem.

Theorem 11. *The Poncelet construction with arbitrary initial condition $q = (x, \xi)$ leads to a closed n-tangle if and only if $np = 0$ on the elliptic curve E.*

Suppose a pencil of conics containing the points x_0, x_1, x_2, x_3 is done by D_t : $tC(x) + D(x) = 0$. The determinant $\det(tC + D)$ is a polynomial of third degree in t, with roots t_1, t_2, t_3 different from zero. For $t \neq t_i$, we construct a tangent on D_t which contains x_0. Let $x(t)$ be the second intersecting point of this tangent with the conic C. The values $t = t_i$ are mapped to x_i, and $t = \infty$ to x_0, since $D_\infty = C$. In this way, we have proved the following

Proposition 11. *The elliptic curve E is birationally equivalent to the Riemann surface of an algebraic function $\sqrt{\det(tC + D)}$ with the origin corresponding to the point $t = \infty$ and with the point $p = (\bar{x}, \bar{\xi})$ corresponding to one of two points over $t = 0$.*

Now, the Cayley condition can be derived from the previous results, by using the following

Proposition 12. *Given an elliptic curve $E : y^2 = (x - a)(x - b)(x - c)$, with a, b, c mutually different, not equal to zero. Suppose the point corresponding to $x = \infty$ is* *chosen to be neutral on E and suppose p is one of two points which correspond to $x = 0$. Then, p is of a finite order n if and only if*

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{m+1} \\ C_4 & C_5 & \dots & C_{m+2} \\ \dots & \dots & \dots & \dots \\ C_{m+1} & C_{m+2} & \dots & C_{2m-1} \end{vmatrix} = 0, \qquad \begin{vmatrix} C_2 & C_3 & \dots & C_{m+1} \\ C_3 & C_4 & \dots & C_{m+2} \\ \dots & \dots & \dots & \dots \\ C_{m+1} & C_{m+2} & \dots & C_{2m} \end{vmatrix} = 0,$$

for $n = 2m$ in the first, and for $n = 2m + 1$ in the second case, where matrix elements are defined by

$$\sqrt{(x-a)(x-b)(x-c)} = A + Bx + C_2x^2 + C_3x^3 + \cdots .$$

The generalization of Cayley's theorem for arbitrary finite dimension is established by Dragović and Radnović [27, 28, 29, 30]. This generalization was done in [27, 28] by use of the Veselov–Moser discrete quadratic $L - A$ pair for the classical Heisenberg magnetic model [54].

The integrability of elliptical billiard systems in the Lobachevsky space was proved by Veselov in [64]. There, Veselov used discrete linear $L - A$ pair, which is quite different from the one used in the Euclidean case.

6. Basic notions on billiard systems

Let (Q, g) be a d-dimensional Riemannian manifold and let $D \subset Q$ be a domain with a smooth boundary Γ. Let $\pi : T^*Q \to Q$ be a natural projection and let g^{-1} be the contravariant metric on the cotangent bundle, in coordinates

$$|p| = \sqrt{g^{-1}(p,p)} = \sqrt{g^{ij}p_ip_j}, \quad p \in T_x^*Q.$$

Consider the *reflection mapping* $r : \pi^{-1}\Gamma \to \pi^{-1}\Gamma$, $p_- \mapsto p_+$, which associates the covector $p_+ \in T_x^*Q$, $x \in \Gamma$ to a covector $p_- \in T_x^*Q$ such that the following conditions hold:

$$|p_+| = |p_-|, \quad p_+ - p_- \perp \Gamma.$$

A *billiard* in D is a dynamical system with the phase space $M = T^*D$ whose trajectories are geodesics given by the Hamiltonian equations

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}, \quad H(p,x) = \frac{1}{2}g_x^{-1}(p,p),$$

reflected at points $x \in \Gamma$ according to the billiard law: $r(p_-) = p_+$. Here p_- and p_+ denote the momenta before and after the reflection. If some potential force field $V(x)$ is added than the system is described with the same reflection law and Hamiltonian equations with the Hamiltonian $H(p,x) = \frac{1}{2}g_x^{-1}(p,p) + V(x)$.

A function $f : T^*Q \to \mathbb{R}$ is an *integral* of the billiard system if it commutes with the Hamiltonian ($\{f, H\} = 0$) and does not change under the reflection ($f(x,p) = f(x, r(p))$, $x \in \Gamma$). The billiard is *completely integrable in the sense of Birkhoff* if it has d integrals polynomial in the momenta, which are in involution, and almost everywhere independent (see [47]).

The classical integrable examples, with smooth boundary, are billiards inside ellipsoids on the Euclidean and hyperbolic spaces and spheres, with integrals quadratic in the velocities [47]. These systems can be also considered as discrete integrable systems [54]. The explicit integrations in terms of theta-functions are performed by Veselov, Moser and Fedorov (see [54], [36]).

7. Periodical trajectories of elliptical billiards in $\mathbb{R}^d$

In this section, first, we are going to list the main steps of algebro-geometric integration of the elliptic billiard, following [54]. Then, the connection between periodic billiard trajectories and points of finite order on the corresponding hyperelliptic curve will be established and the Cayley-type conditions will be derived, as they were obtained in [27, 28].

7.1. XYZ Model and Isospectral Curves.

Following [54], the billiard system will be considered as a system with the discrete time. Using its integration procedure, the connection between periodic billiard trajectories and points of finite order on the corresponding hyperelliptic curve will be established.

Elliptical Billiard as a Mechanical System with the Discrete Time. Let the ellipsoid in $\mathbb{R}^d$ be given by $(Ax, x) = 1$. We can assume that A is a diagonal matrix, with different eigenvalues. The billiard motion within the ellipsoid is determined by the following equations:

$$x_{k+1} - x_k = \mu_k y_{k+1}$$
$$y_{k+1} - y_k = \nu_k A x_k,$$

where

$$\mu_k = -\frac{2(Ay_{k+1}, x_k)}{(Ay_{k+1}, y_{k+1})}, \quad \nu_k = -\frac{2(Ax_k, y_k)}{(Ax_k, Ax_k)}.$$

Here x_k is a sequence of points of billiard bounces, while $y_k = \dfrac{x_k - x_{k-1}}{|x_k - x_{k-1}|}$ are the momenta.

Connection between Billiard and XYZ Model. To the billiard system with the discrete time, Heisenberg XYZ model can be joined, in the way described by Veselov and Moser in [54] and which is going to be presented here.

Consider the mapping $\varphi : (x, y) \mapsto (x', y')$ given by

$$x'_k = Jy_{k+1} = J(y_k + \nu_k A x_k), \quad y'_k = -J^{-1} x_k, \quad J = A^{-\frac{1}{2}}.$$

Notice that the dynamics of φ contains the billiard dynamics:

$$x''_k = Jy'_{k+1} = -x_{k+1}, \quad y''_k = -J^{-1} x'_k = -y_{k+1},$$

and define the sequence $(\bar{x}_k, \bar{y}_k)$:

$$(\bar{x}_0, \bar{y}_0) := (x_0, y_0), \quad (\bar{x}_{k+1}, \bar{y}_{k+1}) := \varphi(\bar{x}_k, \bar{y}_k),$$

which obeys the following relations:

$$\bar{x}_{k+1} = J\bar{y}_k + \nu_k J^{-1} \bar{x}_k, \quad \bar{y}_{k+1} = -J^{-1} \bar{x}_k,$$

where the parameter ν_k is such that $|\bar{y}_k| = 1$, $(A\bar{x}_k, \bar{x}_x) = 1$. This can be rewritten in the following way:

$$\bar{x}_{k+1} + \bar{x}_{k-1} = \nu_k J^{-1}\bar{x}_k.$$

Now, for the sequence $q_k := J^{-1}\bar{x}_k$, we have:

$$q_{k+1} + q_{k-1} = \nu_k J^{-1} q_k, \quad |q_k| = 1.$$

These equations represent the equations of the discrete Heisenberg XYZ system.

Theorem 12. [54] *Let $(\bar{x}_k, \bar{y}_k)$ be the sequence connected with elliptical billiard in the described way. Then $q_k = J^{-1}\bar{x}_k$ is a solution of the discrete Heisenberg system.*

Conversely, if q_k is a solution to the Heisenberg system, then the sequence $x_k = (-1)^k Jq_{2k}$ is a trajectory of the discrete billiard within an ellipsoid.

Integration of the Discrete Heisenberg XYZ System. Usual scheme of algebro-geometric integration contains the following [54]. First, the sequence $L_k(\lambda)$ of matrix polynomials has to be determined, together with a factorization

$$L(\lambda) = B(\lambda)C(\lambda) \mapsto C(\lambda)B(\lambda) = B'(\lambda)C'(\lambda) = L'(\lambda),$$

such that the dynamics $L \mapsto L'$ corresponds to the dynamics of the system q_k. For each problem, finding this sequence of matrices requires a separate search and a mathematician with the excellent intuition. All matrices L_k are mutually similar, and they determine the same *isospectral curve* $\Gamma : \det(L(\lambda) - \mu I) = 0$. The factorization $L_k = B_k C_k$ gives splitting of spectrum of L_k. Denote by ψ_k the corresponding eigenvectors. Consider these vectors as meromorphic functions on Γ and denote their pole divisors by D_k.

The sequence of divisors is linear on the Jacobian of the isospectral curve, and this enables us to find, conversely, eigenfunctions ψ_k, then matrices L_k, and, finally, the sequence (q_k).

Now, integration of the discrete XYZ system by this method will be shortly presented. Details of the procedure can be found in [54].

The equations of discrete XYZ model are equivalent to the isospectral deformation:

$$L_{k+1}(\lambda) = A_k(\lambda)L_k(\lambda)A_k^{-1}(\lambda),$$

where

$$L_k(\lambda) = J^2 + \lambda q_{k-1} \wedge Jq_k - \lambda^2 q_{k-1} \otimes q_{k-1}, \quad A_k(\lambda) = J - \lambda q_k \otimes q_{k-1}.$$

The equation of the isospectral curve $\Gamma : \det(L(\lambda) - \mu I) = 0$ can be written in the following form:

$$\nu^2 = \prod_{i=1}^{d-1}(\mu - \mu_i)\prod_{j=1}^{d}(\mu - J_j^2), \tag{14}$$

where $\nu = \lambda\prod_{i=1}^{d-1}(\mu - \mu_i)$ and $\mu_1, \dots, \mu_{d-1}$ are zeroes of the function:

$$\phi_\mu(x, Jy) = \sum_{i=1}^{d}\frac{F_i(x, y)}{\mu - J_i^2},$$

$$F_i = x_i^2 + \sum_{j\neq i} \frac{(x \wedge Jy)_{ij}^2}{J_i^2 - J_j^2}, \quad x = q_{k-1}, \quad y = q_k.$$

It can be proved that $\mu_1, \dots, \mu_{d-1}$ are parameters of the caustics corresponding to the billiard trajectory [53]. Another way for obtaining the same conclusion is to calculate them directly by taking the first segment of the billiard trajectory to be parallel to a coordinate axe.

If eigenvectors ψ_k of matrices $L_k(\lambda)$ are known, it is possible to determine uniquely members of the sequence q_k. Let D_k be the divisor of poles of function ψ_k on curve Γ. Then [54]:

$$D_{k+1} = D_k + P_\infty - P_0,$$

where P_∞ is the point corresponding to the value $\mu = \infty$ and P_0 to $\mu = 0$, $\lambda = (q_k, J^{-1}q_{k+1})^{-1}$.

7.2. Characterization of Periodical Billiard Trajectories.

In the next lemmae, we establish a connection between periodic billiard sequences q_k and periodic divisors D_k.

Lemma 4. [27] *Sequence of divisors D_k is n-periodic if and only if the sequence q_k is also periodic with the period n or $q_{k+n} = -q_k$ for all k.*

Lemma 5. [27] *The billiard is, up to the central symmetry, periodic with the period n if and only if the divisor sequence D_k joined to the corresponding Heisenberg XYZ system is also periodic, with the period $2n$.*

Applying the previous lemma, we obtain the main statement of this section:

Theorem 13. [28] *A condition on a billiard trajectory inside ellipsoid $\mathcal{Q}_0$ in $\mathbb{R}^d$, with non-degenerate caustics $\mathcal{Q}_{\mu_1}, \dots, \mathcal{Q}_{\mu_{d-1}}$, to be periodic, up to the central symmetry, with the period $n \geq d$ is:*

$$\operatorname{rank}\begin{pmatrix} B_{n+1} & B_n & \dots & B_{d+1} \\ B_{n+2} & B_{n+1} & \dots & B_{d+2} \\ \dots & \dots & \dots & \dots \\ B_{2n-1} & B_{2n-2} & \dots & B_{n+d-1} \end{pmatrix} < n - d + 1,$$

where $\sqrt{(x-\mu_1)\cdots(x-\mu_{d-1})(x-a_1)\cdots(x-a_d)} = B_0 + B_1 x + B_2 x^2 + \cdots$.

Cases of Singular Isospectral Curve. When all $a_1, \dots, a_d, \mu_1, \dots, \mu_{d-1}$ are mutually different, then the isospectral curve has no singularities in the affine part. However, singularities appear in the following three cases and their combinations:

(i) $a_i = \mu_j$ for some i, j. The isospectral curve (14) decomposes into a rational and a hyperelliptic curve. Geometrically, this means that the caustic corresponding to μ_i degenerates into the hyperplane $x_i = 0$. The billiard trajectory can be asymptotically tending to that hyperplane (and therefore cannot be periodic), or completely placed in this hyperplane. Therefore, closed trajectories appear when they are placed in a coordinate hyperplane. Such a motion can be discussed like in the case of dimension $d-1$.

(ii) $a_i = a_j$ for some $i \neq j$. The boundary $\mathcal{Q}_0$ is symmetric.

(iii) $\mu_i = \mu_j$ for some $i \neq j$. The billiard trajectory is placed on the corresponding confocal quadric hyper-surface.

In the cases (ii) and (iii) the isospectral curve Γ is a hyperelliptic curve with singularities. In spite of their different geometrical nature, they both need the same analysis of the condition $2nP_0 \sim 2nE$ for the singular curve (14).

Aa a consequence of the Theorem 13, it can be applied not only for the case of the regular isospectral curve, but in the cases (ii) and (iii), too. Therefore, the following interesting property holds.

Theorem 14. *If the billiard trajectory within an ellipsoid in d-dimensional Eucledean space is periodic, up to the central symmetry, with the period $n < d$, then it is placed in one of the n-dimensional planes of symmetry of the ellipsoid.*

Proof. This follows immediately from Theorem 13 and the fact that the section of a confocal family of quadrics with a coordinate hyperplane is again a confocal family. □

This property can be seen easily for $d = 3$.

Example 6. Consider the billiard motion in an ellipsoid in the 3-dimensional space, with $\mu_1 = \mu_2$, when the segments of the trajectory are placed on generatrices of the corresponding one-folded hyperboloid, confocal to the ellipsoid. If there existed a periodic trajectory with period $n = d = 3$, the three bounces would have been coplanar, and the intersection of that plane and the quadric would have consisted of three lines, which is impossible. It is obvious that any periodic trajectory with period $n = 2$ is placed along one of the axes of the ellipsoid. So, there is no periodic trajectories contained in a confocal quadric surface, with period less or equal to 3.

8. Separable perturbations of integrable billiards

Appell introduced four families of hypergeometric functions of two variables in 1880's. Soon, he applied them in a solution of the Tisserand problem in the celestial mechanics. The Appell functions have several other applications, for example in the theory of algebraic equations, algebraic surfaces... The aim of this paper is to point out the relationship between the Appell functions F_4 and another subject from classical mechanics-separability of variables in the Hamilton-Jacobi equations.

The equation

$$\lambda V_{xy} + 3\,(yV_x - xV_y) + (y^2 - x^2)V_{xy} + xy\,(V_{xx} - V_{yy}) = 0, \tag{15}$$

appeared in Kozlov's paper [44] as a condition on the function $V = V(x, y)$ to be an integrable perturbation of certain type for billiard systems inside an ellipse

$$\frac{x^2}{A} + \frac{y^2}{B} = 1, \quad \lambda = A - B. \tag{16}$$

This equation is a special case of the Bertrand–Darboux equation [7, 14, 66]

$$(V_{yy}-V_{xx})(-2axy-b'y-bx+c_1)+2V_{xy}(ay^2-ax^2+by-b'x+c-c') \\ +V_x(6ay+3b)+V_y(-6ax-3b')=0.$$

It corresponds to the choice $a=-1/2$, $b=b'=c_1=0$, $c-c'=-\lambda/2$. The Bertrand–Darboux equation represents the necessary and sufficient condition for a natural mechanical system with two degrees of freedom

$$H=\frac{1}{2}(p_x^2+p_y^2)+V(x,y)$$

to be separable in elliptical coordinates or some of their degenerations.

Solutions of the equation (15) in the form of the Laurent polynomials in x, y were described in [16, 17]. The starting observation of this paper, that such solutions are simply related to the well-known hypergeometric functions of the Appell type is presented. Such a relation automatically gives a wider class of solutions of the equation (15)– new potentials are obtained for non-integer parameters. But what is more important, it shows the existence of a connection between separability of classical systems on one hand, and the theory of hypergeometric functions on the other one. Basic references for the Appell functions are [2, 3, 65]. Further, in section 3, similar formulae for potential perturbations for the Jacobi problem for geodesics on an ellipsoid from [16].

In the case of more than two degrees of freedom, the natural generalization for the equation (15) is the system:

The system

$$(a_i-a_r)^{-1}\left(x_i^2V_{rs}-x_ix_rV_{is}\right)=(a_i-a_s)^{-1}\left(x_i^2V_{rs}-x_ix_sV_{ir}\right)\quad i\neq r\neq s\neq i;$$

$$\begin{aligned}(a_i-a_r)^{-1}x_ix_r\left(V_{ii}-V_{rr}\right)&-\sum_{j\neq i,r}(a_i-a_j)^{-1}x_ix_jV_{jr}\\ &+V_{ir}\left[\sum_{j\neq i,r}(a_i-a_j)^{-1}x_j^2+(a_r-a_i)^{-1}(x_i^2-x_r^2)\right]\\ &+V_{ir}+3(a_i-a_r)^{-1}\left(x_rV_i-x_iV_r\right)=0,\quad i\neq r,\end{aligned}$$

where $V_i=\partial V/\partial x_i$, of $(n-1)\binom{n}{2}$ equations was formulated in [52] for arbitrary number of degrees of freedom n. In [52] the generalization of the Bertrand-Darboux theorem is proved. According to that theorem, the solutions of the system are potentials separable in generalized elliptic coordinates.

Some deeper explanation of the connection between the separability in elliptic coordinates and the Appell hypergeometric functions is not known yet.

8.1. Basic notations. The function F_4 is one of the four hypergeometric functions in two variables introduced by Appell [2, 3] and defined as a series:

$$F_4(a,b,c,d;x,y)=\sum\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

where $(a)_n$ is the standard Pochhämmer symbol:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1), \quad (a)_0 = 1,$$

(For example $m! = (1)_m$.)

The series F_4 is convergent for $\sqrt{x}+\sqrt{y} \leq 1$. The functions F_4 can be analytically continued to the solutions of the equations:

$$x(1-x)\frac{\partial^2 F}{\partial x^2} - y^2\frac{\partial^2 F}{\partial y^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + [c-(a+b+1)x]\frac{\partial F}{\partial x} - (a+b+1)y\frac{\partial F}{\partial y} - abF = 0,$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} - x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + [c'-(a+b+1)y]\frac{\partial F}{\partial y} - (a+b+1)x\frac{\partial F}{\partial x} - abF = 0,$$

8.2. Billiard inside an ellipse and its separable perturbations. Following [44, 15, 16] we will start with a billiard system which describes a particle moving freely within an ellipse (2). At the boundary we assume elastic reflections with equal impact and reflection angles. This system is completely integrable and it has an additional integral

$$K_1 = \frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} - \frac{(\dot{x}y - \dot{y}x)^2}{AB}.$$

We are interested in a potential perturbations $V = V(x, y)$ such that the perturbed system has an integral $\tilde{K}_1$ of the form $\tilde{K}_1 = K_1 + k_1(x, y)$, where $k_1 = k_1(x, y)$ depends only on coordinates. This specific condition leads to the equation (15) on V (see [44]).

In [15, 16] the Laurent polynomial solutions of the equation (15) were given. The basic set of solutions consists of the functions

$$V_k = \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x, y, \lambda) + y^{-2k}, \quad k \in N,$$

$$W_k = \sum_{i=0}^{k-2}\sum_{s=1}^{k-i-1} (-1)^s U_{kis}(y, x, \lambda) + x^{-2k}, \quad k \in N,$$

where

$$U_{kis} = \binom{s+i-1}{i} \frac{[1-(k-i)][2-(k-i)]\dots[s-(k-i)]}{\lambda^{s+i}s!} x^{2s} y^{-2k+2i}.$$

Now, we are going to rewrite the above formulae:

$$V_k = \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x, y, \lambda) + y^{-2k}, \quad k \in N$$

$$= \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} \frac{\Gamma(s+i)\Gamma(s+i-k+1)}{\Gamma(i+1)\Gamma(s)\Gamma(i-k+1)\Gamma(s+1)} \frac{x^{2s}y^{2(i-k)}}{\lambda^{s+i}} + y^{-2k}$$

$$= \frac{1}{y^{2k}}\left((1-k)\sum_{i=0}^{k-2}\sum_{s=1}^{k-i-1} \frac{(1)_{s+i-1}(2-k)_{s+i-1}}{i!(1)_{s-1}s!(1-k)_i}\frac{x^{2s}}{\lambda^s}\frac{(-y^2)^i}{\lambda^i} + 1\right)$$

$$= \frac{1}{y^{2k}}\left((1-k)\frac{x^2}{\lambda}\sum_{i=0}^{k-2}\sum_{s=0}^{k-i-2} \frac{(1)_{s+i}(2-k)_{s+i}}{(2)_s(1-k)_i}\frac{(x^2)^s}{s!\lambda^s}\frac{(-y^2)^i}{i!\lambda^i} + 1\right)$$

$$= \frac{1}{\tilde{y}^k\lambda^k}\big((1-k)\tilde{x}F_4(1;\ 2-k;\ 2,\ 1-k,\ \tilde{x},\ -\tilde{y}) + 1\big),$$

where $\tilde{x} = x^2/\lambda$, $\tilde{y} = -y^2/\lambda$, and F_4 is the Appell function. We have just obtained a simple formula which expresses the potentials V_k, from [16], for $k \in N$ through the Appell functions. (The scalar coefficient λ^{-k} is not essential and we will not write it any more). We can use this formula to spread the family of solutions of the equation (15) out of the set of the Laurent polynomials. We obtain new solutions of the equation (15) if we let the parameter k in the last formula to be arbitrary, not only a natural number.

Let $V(x,y) = \sum a_{nm}x^n y^m$. Then the equation (15) reduces to

$$\lambda n m a_{n,m} = (n+m)(m a_{n-2,m} - n a_{n,m-2}).$$

If one of the indices, for example the first one, belongs to Z, then V does not have essential singularities. Put $a_{0,-2\gamma} = 1$, where γ is not necessary an integer.

Let us define

$$a_{\underbrace{2s+2}_{n},\underbrace{2i-2\gamma}_{m}} = \frac{(-1)^i(1)_{s+i}(2-\gamma)_{s+i}}{(2)_s(1-\gamma)_i s! i! \lambda^{s+i}}.$$

and denote

$$V_\gamma = \tilde{y}^{-\gamma}\left((1-\gamma)\tilde{x}F_4(1, 2-\gamma, 2, 1-\gamma, \tilde{x}, \tilde{y}) + 1\right). \tag{17}$$

Then we have

Theorem 15. *Every function V_γ given with* (17) *and $\gamma \in C$ is a solution of the equation* (15)).

The theorem gives new potentials for noninteger γ.

Mechanical interpretation. With $\gamma \in R^-$ and the coefficient multiplying V_γ positive, we have potential barrier along x-axis. We can consider billiard motion in upper half plane. Then we can assume that a cut is done along negative part of y-axis, in order to get unique-valued real function as a potential.

8.3. The Jacobi problem for geodesics on an ellipsoid.

The Jacobi problem for the geodesics on an ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1$$

has an additional integral

$$K_1 = \Big(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\Big)\Big(\frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} + \frac{\dot{z}^2}{C}\Big).$$

Potential perturbations $V = V(x, y, z)$ such that perturbed systems have integrals of the form $\tilde{K}_1 = K_1 + k(x, y, z)$ satisfy the following system (see [16])

$$\begin{aligned}
&\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right) V_{xy}\frac{A-B}{AB} - 3\frac{y}{B^2}\frac{V_x}{A} + 3\frac{x}{A^2}\frac{V_y}{B} + \left(\frac{x^2}{A^3} - \frac{y^2}{B^3}\right)V_{xy}\\
&\qquad + \frac{xy}{AB}\left(\frac{V_{yy}}{A} - \frac{V_{xx}}{B}\right) + \frac{zx}{CA^2}V_{zy} - \frac{zy}{CB^2}V_{zx} = 0\\
&\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right) V_{yz}\frac{B-C}{BC} - 3\frac{z}{C^2}\frac{V_y}{B} + 3\frac{y}{B^2}\frac{V_z}{C} + \left(\frac{y^2}{B^3} - \frac{z^2}{C^3}\right)V_{yz}\\
&\qquad + \frac{yz}{BC}\left(\frac{V_{zz}}{B} - \frac{V_{yy}}{C}\right) + \frac{xy}{AB^2}V_{xz} - \frac{xz}{AC^2}V_{xy} = 0\\
&\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right) V_{zx}\frac{C-A}{AC} - 3\frac{x}{A^2}\frac{V_z}{C} + 3\frac{z}{C^2}\frac{V_x}{A} + \left(\frac{z^2}{C^3} - \frac{x^2}{A^3}\right)V_{zx}\\
&\qquad + \frac{xz}{AC}\left(\frac{V_{xx}}{C} - \frac{V_{zz}}{A}\right) + \frac{zy}{BC^2}V_{xy} - \frac{yx}{BA^2}V_{yz} = 0
\end{aligned} \tag{18}$$

The last system (18) replaces the equation (15) in this problem. Solutions of the system in the Laurent polynomial form were found in [16]. We can transform them in the following way.

$$\begin{aligned}
V_{l_0}(x, y, z) &= \sum_{0\le k\le s, k+c\le l_0} (-1)^s \binom{s+k-1}{k} (x^2)^{-l_0+k}(y^2)^s(z^2)^{l_0-(k+s)-1}\\
&\quad\times \frac{C^{s+k}(C-A)^s(C-B)^k 2^{k+s}(-l_0+1)\dots(-l_0+(k+s))}{B^k A^s (B-A)^{k+s}2^s 2^k s!(-l_0+1)\dots(-l_0+k)}(z^2)^{l_0-(k+s)-1}\\
&= \sum \frac{(s+k-1)!(-l_0+1)(-l_0+2)_{s+k-1}(z^2)^{l_0}}{k!(s-1)!s!(-l_0+1)_k(x^2)^{l_0}}\left[\frac{x^2C(A-C)}{z^2(B-A)A}\right]^s\left[\frac{y^2C(C-B)}{z^2(B-A)B}\right]^k\\
&= (-l_0+1)\left(\frac{z^2}{x^2}\right)^{l_0}\sum\frac{(1)_{s+k-1}(-l_0+2)_{s+k-1}}{(2)_{s-1}(-l_0+1)_k}\hat{x}^s\hat{y}^k\\
&= (-l_0+1)\left(\frac{z^2}{x^2}\right)^{l_0} F_4(1; -l_0+2; 2, -l_0+1, \hat{x}, \hat{y}),
\end{aligned}$$

where

$$\frac{x^2C(A-C)}{z^2(B-A)A} = \hat{x}, \quad \frac{y^2C(C-B)}{z^2(B-A)B} = \hat{y}$$

In the above formulae l_0 is an integer. We have the straightforward generalization:

Theorem 16. *For every $\gamma \in C$ the function*

$$V_\gamma = (-\gamma+1)\left(\frac{z^2}{x^2}\right)^\gamma F_4(1; -\gamma+2; 2, -\gamma+1, \hat{x}, \hat{y}),$$

is a solution of the system (18).

9. Algebro-geometric approach to the quantum Yang–Baxter equation

One of the central objects in mathematical physics in last 25 years is the R matrix, or the solution $R(t,h)$ of the quantum Yang–Baxter equation

$$R^{12}(t_1-t_2,h)R^{13}(t_1,h)R'^{23}(t_2,h) = R^{23}(t_2,h)(R^{13}(t_1,h)R^{12}(t_1-t_2,h).$$

Here t is so called *spectral parameter* and h is *Planck constant.* If the h dependence satisfies the quasi-classical property $R = I + hr + O(h^2)$ the classical r-matrix r satisfies the classical Yang–Baxter equation. Classification of the solutions of the classical Yang–Baxter equation was done by Belavin and Drinfeld in 1982. The problem of classification of the quantum R matrices is still open. Some results have been obtained in the basic 4×4 case (see [46, 18, 19, 20]).

Krichever in [46] applied the idea of "finite-gap" integration to the theory of the Yang equation

$$R^{12}L^{13}L'^{23} = L'^{23}L^{13}R^{12}.$$

The principal objects that are considered are $2n\times 2n$ matrices L, understood as 2×2 matrices whose elements are $n\times n$ matrices; $L = l^{i\alpha}_{j\beta}$ is considered as a linear operator in the tensor product $C^n\otimes C^2$. The theorem from [46] uniquely characterizes them by the following spectral data:

(1) the vacuum vectors, i.e., vectors of the form $X\otimes U$, which L maps to vectors of the same form $Y\otimes V$, where $X, Y\in C^n$ and $U, V\in C^2$;
(2) the vacuum curve $\Gamma : P(u,v) = \det L = 0$, where $L^i_j = V^\beta L^{i\alpha}_{j\beta}U_\alpha$, $(V^\beta) = (1,-v)$, $X_n = Y_n = U_2 = V_2 = 1$; $U_1 = u$, $V_1 = v$;
(3) the divisors of the vector-valued functions $X(u,v)$, $Y(u,v)$, $U(u,v)$, $V(u,v)$, which are meromorphic on the curve Γ. But the Krichever method used in [46, 18, 19, 20, 21] works with even-dimensional matrices. Here we want to discus the case of odd-dimensional matrices considering the case of 9×9 matrices. We introduce the notion of *vacuum locus* as an analogue of the vacuum curve. We also show that a vacuum locus could be a finite set for some of the solutions of the quantum Yang–Baxter equation.

Now, the matrices $L = l^{i\alpha}_{j\beta}$ are considered as a linear operator in the tensor product $C^3\otimes C^3$. The same is for matrices R. As before, we want to parametrize the vacuum vectors, i.e., vectors of the form $X\otimes U$, which L maps to vectors of the same form $Y\otimes V$, where $X, Y, U, V\in C^3$. Assume the notation:

$$U^t = (u_1,u_2,1),\ V^t = (v_1,v_2,1),\ \tilde V_1 = (1,0,-v_1),\ \tilde V_2 = (0,1,-v_2).$$

The vacuum locus is the set which parametrizes the vacuum vectors.

Lemma-Definition *The affine part of the vacuum locus is the set of* $(u_1,u_2,v_1,v_2)\in C^4$ *such that*

$$P(u_1,u_2,v_1,v_2) := \det L(\lambda) = 0$$

identically in λ, *where* $L^i_j(\lambda) = (\tilde V_1 + \lambda\tilde V_2)^\beta L^{i\alpha}_{j\beta}U_\alpha$.

The lemma follows from the fact that if two regular matrix binomials of the first degree are equivalent then they are strictly equivalent (see [37]). The condition

$\det L(\lambda) = 0$ identically in λ gives four equations in C^4 since $\det L(\lambda)$ is a polynomial of the third degree in λ. So, for the general matrix L, the set $P(u_1, u_2, v_1, v_2)$ is a finite subset of C^4. The working hypothesis among the specialists was that in a case of the solutions of the quantum Yang–Baxter equation which depend on spectral parameter, there should be an algebraic curve which parametrizes some of the vacuum vectors. However, even in the case of the solutions of the Yang–Baxter equation it is possible that the vacuum locus is a finite set. This can be proved for the famous Izergin–Korepin 9×9 R-matrix (see [43]).

Proposition 13. *The vacuum locus for the Izergin–Korepin R-matrix is a finite set.*

The structure of this set is still not clear. In order to apply some of the Krichever ideas such set should have a subset which satisfies two conditions:

- it is closed for the composition of relations properly defined;
- it is big enough to give a possibility to reconstruct matrices R, L, L' and their products.

This could lead to the construction of the solutions of the Yang-Baxter equation in which spectral parameter belongs to some discrete group.

Acknowledgement. The author has a great pleasure to use this opportunity to thank his younger colleagues Borislav Gajić, Božidar Jovanović and Milena Radnović for years of fruitful scientific collaboration. The author is grateful to academicians Bogoljub Stanković, Vojislav Marić and Stevan Pilipović for the kind invitation to Novi Sad Meeting on Mathematical Methods in Mechanical Models, October 2003 and their hospitality. The research was partially supported by the Serbian Ministry of Science and Technology Project Geometry and Topology of Manifolds and Integrable Dynamical Systems.

References

[1] M. Adler and P. van Moerbeke, *Linearization of Hamiltonian Systems, Jacobi Varieties and Representation Theory*, Advances in Math. **38** (1980), 318–379.

[2] P. Appell, *Sur les fonctions hypergeometriques de deux variables et sur des equations lineaires aux derivées partielles*, C. R. Acad. Sci. Paris **90** (1880), p. 296.

[3] P. Appell and J. Kampe de Feriet, *Fonctions hypergeometriques et hyperspheriques*, in: *Polynomes d'Hermite*, Gauthier Villars, Paris, 1926.

[4] G. G. Appel'rot, *The problem of motion of a rigid body about a fixed point*, Uchenye Zap. Mosk. Univ. Otdel. Fiz. Mat. Nauk 11 (1894), 1–112.

[5] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Nauka, Moscow, 1989 [in Russian, 3-rd edition].

[6] M. Berger, *Geometry*, Springer-Verlag, Berlin, 1987.

[7] J. Bertrand, J. Math. **17** (1852), p. 121.

[8] O. I. Bogoyavlensky *Integrable Euler equations on Li algebras arising in physical problems*, Soviet Acad. Izvestya **48** (1984), 883–938 (in Russian).

[9] A. V. Borisov, I. S. Mamaev: *Dynamics of rigid body*, RCHD, Moskva, Izhevsk, 2001, (in Russian).

[10] A. Cayley, *Developments on the porism of the in-and-circumscribed polygon*, Philosophical magazine **7** (1854), 339–345.

[11] S.-J. Chang, B. Crespi B, K.-J. Shi, *Elliptical billiard systems and the full Poncelet's theorem in n dimensions*, J. Math. Phys. **34** (1993), no. 6, 2242–2256.

[12] S.-J. Chang, K.-J. Shi, *Billiard systems on quadric surfaces and the Poncelet theorem*, J. Math. Phys. **30**:4 (1989), 798–804.

[13] S. G. Dalalyan, *Prym varieties of unramified double coverings of the hyperelliptic curves*, Uspekhi Math. Naukh **29** (1974), 165–166

[14] G. Darboux, Archives Neerlandaises (2), **6** (1901), p. 371.

[15] V. Dragović, *Integrable perturbations of a Birkhoff billiard inside an ellipse*, Prikl. Mat. Mekh. **62**:1 (1998), 166–169; English translation: J. Appl. Math. Mech. **62**:1 (1998), 159–162.

[16] V. Dragović, *On integrable potential perturbations of the Jacobi problem for the geodesics on the ellipsoid*, J. Phys. A: Math. Gen. **29**:13 (1996), L317–L321.

[17] V. Dragović, *The Appell hypergeometric functions and classical separable mechanical systems*, J. Phys A: Math. Gen. **35**:9 (2002), 2213–2221.

[18] V. I. Dragovich, *Solutions to the Yang equation with rational irreducible spectrual curves*, (in Russian), Izv. Ross. Akad. Nauk, Ser. Mat, **57**:1 (1993), 59–75; English translation: Russ. Acad. Sci., Izv., Math. **42**:1 (1994), 51–65

[19] V. I. Dragovich, *Baxter reduction of rational Yang solutions*, Vestn. Mosk. Univ., Ser. I (1992), no. 5, 84–86; English translation: Mosc. Univ. Math. Bull. **47**:5 (1992), 59–60

[20] V. I. Dragovich, *Solutions to the Yang equation with rational spectral curves*, Algebra i Analiz, Sankt-Petersburg, **4**:5 (1992), 104–116; English translation: St. Petersb. Math. J. **4**:5 (1993), 921–931

[21] V. I. Dragovich, *Solutions to the Yang equation and algebraic curves of genus greater than 1*, Funkts. Anal. Appl. **31**:2 (1997), 70–73 (in Russian).

[22] V. Dragović, B. Gajić, *An L-A pair for the Hess–Appel'rot system and a new integrable case for the Euler–Poisson equations on* so(4)×so(4), Roy. Soc. of Edinburgh: Proc A **131** (2001), 845–855

[23] V. Dragović, B. Gajić, *The Lagrange bitop on* so(4) × so(4) *and geometry of Prym varieties*, math-ph/0201036, Preprint SISSA, 79/FM 2001, American J. Math. **126** (2004), 981–1004

[24] V. Dragović, B. Gajić, *Systems of Hess–Appel'rot type*, Comm. Math. Phys. **265**:2 (2006), 397–435

[25] V. Dragović, B. Gajić, *Matrix Lax polynomials, geometry of Prym varieties and systems of Hess–Appel'rot type*, to appear

[26] V. Dragović, D. Milinković, *Analysis on manifolds*, Mathematical faculty, Belgrade, 2003, 573pp

[27] V. Dragović, M. Radnović, *Conditions of Cayley's type for ellipsoidal billiard*, J. Math. Phys. **39**: (1998), 355–362.

[28] V. Dragović, M. Radnović, *On periodical trajectories of the billiard systems within an ellipsoid in* $\mathbf{R}^d$ *and generalized Cayley's condition*, J. Math. Phys. **39**:11 (1998), 5866–5869.

[29] V. Dragović, M. Radnović, *Cayley-type conditions for billiards within k quadrics in* $\mathbf{R}^d$, J. Phys. A, Math. Gen. **37** (2004), 1269–1276.

[30] V. Dragović, M. Radnović, *Geometry of integrable billiards and pencils of quadrics*, J. Math. Pures Appl. (Liouville's Journal) **85**:6 (2006), 758–790

[31] B. A. Dubrovin, *Vpolne integriruemye gamil'tonovy sistemy svyazanye s matrichnymi operatorami i Abelevy mnogoobraziya*, Funkts. Anal. Prilozh. **11** (1977), 28–41, (in Russian)

[32] B. A. Dubrovin, *Theta-functions and nonlinear equations*, Uspekhi Math. Nauk. **36** (1981), 11–80 (in Russian)

[33] B. A. Dubrovin, I. M. Krichever and S. P. Novikov, *Integrable systems I*; in *Dynamical systems IV*, Springer-Verlag, erlin, 173–280.

[34] B. A. Dubrovin, V. B. Matveev, S. P. Novikov, *Nonlinear equations of Kortever–de Fries type, finite zone linear operators and Abelian varieties*, Uspekhi Math. Nauk **31** (1976), 55–136 (in Russian).

[35] J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics 352, Springer-Verlag, 1973

[36] Yu. Fedorov, *Classical integrable systems and billiards related to generalized Jacobians*, Acta Appl. Math. **55**, 251–301

[37] F. R. Gantmakher, *Theory of Matrices*, 4th ed., compl. Ed. by V. B. Lidskij, Moskva: Nauka, 1988, 549 p., (in Russian).

[38] L. Gavrilov, A. Zhivkov, *The complex geometry of Lagrange top*, Enseign. Math., II Sér. **44** (1998), 133–170

[39] V. V. Golubev, *Lectures on integration of the equations of motion of a rigid body about a fixed point*, Moskow: Gostenhizdat, 1953; English translation: Philadelphia: Coronet Books, 1953.

[40] P. Griffiths, J. Harris, *A Poncelet theorem in space*, Comment. Math. Helv. **52**:2 (1977), 145–160.

[41] P. Griffiths, J. Harris, *On Cayley's explicit solution to Poncelet's porism*, Enseign. Math. **24**:1–2 (1978), 31–40.

[42] W. Hess, *Ueber die Euler'schen Bewegungsgleichungen und über eine neue particuläre Lösung des Problems der Bewegung eines starren Körpers um einen festen Punkt*, Math. Ann. **37** (1890), 178–180.

[43] A. Izergin , V. Korepin, *The inverse scattering method approach to the quantuum Shabat–Mikhailov Model*, Comm. Math. Phys. **79** (1981), 303–316.

[44] V. V, Kozlov, *Some integrable generalizations of the Jacobi problem for the geodesics on the ellipsoid*, Prikl. Mat. Mekh. **59** (1995), 3–9, (in Russian).

[45] S. Kowalevski, *Sur le problème de la rotation d'un corp solide autoar d'un point fixe*, Acta Math. **12** (1889), 177–232.

[46] I. M. Krichever, *Baxter's equation and algebraic geometry*, Func. Anal. Appl. **15** (1981), 92–103 (in Russian).

[47] V. V. Kozlov, D. V. Treshchëv, *Billiards*, Amer. Math. Soc., Providence RI, 1991.

[48] H. Lebesgue, *Les coniques*, Gauthier-Villars, Paris, 1942.

[49] E. Leimanis, *The general problem of the motion of coupled rigid bodies about a fixed point*, Springer-Verlag, Berlin, Heidelberg, New York 1965.

[50] A. Lipkovski, *Algebraic geometry (Selected Topics)*, in: Bogoljub Stanković(ed.), *Three topics from contemporary mathematics*, Zb. Rad. **7(15)** (1997), 5–70, Beograd: Matematicki Institut SANU.

[51] S. V. Manakov. *Remarks on the integrals of the Euler equations of the n-dimensional heavy top*, Funkc. Anal. Appl. **10** (1976 (in Russian)), 93–94.

[52] I. Marshall and S. Wojciechowski, *When is a Hamiltonian system separable?*, J. Math. Phys. **29** (1988), 1338–1346.

[53] J. Moser, *Geometry of quadrics and spectral theory*, The Chern Symposium, Springer-Verlag, New York–Berlin, 1980, pp. 147–188.

[54] J. Moser, A. P. Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Comm. Math. Phys. **139**:2 (1991), 217–243.

[55] D. Mumford, *Theta characteristics of an algebraic curve*, Ann. scient. Ec. Norm. Sup. 4 serie, **4**(1971), 181–192.

[56] D. Mumford, *Prym varieties 1*, *A collection of papers dedicated to Lipman Bers*, Academic Press, New York, 1974, pp. 325–350.

[57] D. Mumford, *Tata Lectures on Theta*, Birkauser, Boston, 1983, 1984, Russian translation: Moscow, Mir, 1988

[58] J. V. Poncelet, *Traité des propriétés projectives des figures*, Mett-Paris, 1822.

[59] T. Ratiu, *Euler–Poisson equation on Lie algebras and the N-dimensional heavy rigid body*, Amer. J. Math. **104** (1982), 409–448.

[60] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, Invent. Math. **83** (1986), 333–382

[61] V. V. Shokurov, *Algebraic curves and their Jacobians*: in *Algebraic Geometry III*, Springer-Verlag, Berlin, 1998, 219–261.

[62] V. V. Shokurov, *Distinguishing Prymians from Jacobians*, Invent. Math. **65** (1981), 209–219

[63] V. V. Trofimov and A. T. Fomenko. *Algebra and geometry of integrable Hamiltonian differential equations*, Faktorial, Moscow, 1995 (in Russian).

[64] A. P. Veselov, *Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space*, J. Geom. Phys. **7**:1 (1990), 81–107.

[65] N. Ya. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions, Recent Advances*, Kluwer, Dordrecht, 1995 (see in particular p. 208).

[66] Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge University Press, 1952, 456 pp.

[67] N. E. Zhukovski, *Geometrische interpretation des Hess'schen falles der bewegung eines schweren starren korpers um einen festen Punkt*, Jahresber. Dtsch. Math.-Ver. **3** (1894), 62–70.