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TAILS OF PROBABILITY DISTRIBUTIONS AND APPLICATIONS TO LIMIT THEOREMS

by

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A thesis Submitted to the University of Belgrade in Partial Fulfillment of the Requirements for the Degree of Master of Science

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Belgrade, 2012.

ACKNOWLEDGEMENTS

First of all, I want to express my gratitude to my supervisor Prof. Dr. Pavle Mladenović, Head of the Department of Probability and Statistics, for supporting my through all stages of this thesis and for giving me the scientific liberty to develop and pursue my own ideas. Working with him has always been a pleasure and the benefit of this collaboration invaluably contributed to the success of this work.

Additionally, I would like to thank Prof. Dr. Vesna Jevremović, for her support and help in my postgraduate studies. Also I am grateful to Prof. Dr. Slobodanka Janković and Prof. Dr. Jovan Vukmirović for their support.

I have taken courses from almost all the professors in this department. I want to thank them for teaching me and helping me develop my career.

I would also like to thank all the staff and my colleagues at the Department of Propability and Statistics, Faculty of Mathematics, Belgrade University.

I never forget the departed “Prof. Emhimed Shneina”, my oldest brother, who was the first man support me in my postgraduate study, “Rahmet Allah to him”.

Finally, and above all I want to express my deepest love and gratitude to my family, especially to my wife and my children for supporting my career both ideologically and financially.

VITA

Ehfayed Khalifa Shneina was born in Tegi-Libya in 1.1.1961. In May 1984 he graduated from AL-Fateh University and was conferred a Bachelor of Science degree majoring in Mathematics. From 1986 until 1992 he was a tutor of Mathematics in the Department of Mathematics, Faculty of Science, AL-Fateh University. And a Chief of Scientific affairs of the Faculty of Teachers in Tegi, Western Mountain University, Libya, from: 1995 until 2000. He graduated on 13th December 2003, and he got Master of Science in Statistics, from Phoenix International University, Malaysia which was attended by Minister of Education in Malaysia.

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CONTENTS

Chapter 1

EXTREME VALUE THEORY (EVT)

Chapter2

STABLE DISTRIBUTIONS ON THR REAL LINE

List of Figures

Chapter 1

EXTREME VALUE THEORY

1. INTRODUCTION

Extreme value theory is a separate branch of Statistics and Probability that deals with extreme events. This theory is based on the extremal type's theorem, also called the three types theorem, stating that there are only three types of distributions that are needed to model the maximum or minimum of the collection of random observations from the same distribution. In other words, if you generate *N* data sets from the same distribution, and create a new data set that includes the maximum values from these *N* data sets, the resulting data set can only be described by one of the three models – specifically, the Gumbel, Frećhet and Weibell distributions. These models, along with the Generalized Extreme Value (GEV) distribution, are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and many other industries dealing with extreme events. There are three fundamental mathematical results that illustrate the importance of extreme value theory (EVT) in risk management applications:

(1) Extremal types Theorem.
(2) Domain of attraction Theorem.
(3) Criterion for choosing a high Threshold.

1.1 Maxima and Minima

(i) **Maxima:** Let $X_1, X_2, \dots X_n$ be a sequence of independent identically distributed (i.i.d.) random variables with distribution function $F(x)$ and suppose

$M_n = \max\{\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_n\} = \max_{1 \le i \le n} X_i$,

$m_n = \min\{\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_n\} = \min_{1 \le i \le n} X_i$,

but the relation between max and min is,

$\min\{\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_n\} = -\max\{-X_1, -X_2, \dots, -X_n\}$,

$\min_{1 \le i \le n} X_i = -\max_{1 \le i \le n}(-X_i)$. Then the distribution function of M_n is

$P\{M_n \le x\} = \mathrm{P}\{X_1 \le x, X_2 \le x, \dots, X_n \le x\}$,

$P\{M_n \le x\} = P\{X_1 \le x\} \dots P\{X_n \le x\} = [F(x)]^n = F^n(x), \ n \ge 1 \ x \epsilon \mathcal{R}, n \epsilon N.$

(ii) **Minima:** Generally, results for minima can be deduced from corresponding results for maxima by writing $\min_{1 \le i \le n} X_i = -\max_{1 \le i \le n}(-X_i)$. In conjunction with minima, it can be useful to present results in terms of the survivor function $\bar{F} = 1 - F$.

We have $P\{m_n > x\} = (1 - F(x))^n = (\bar{F}(x))^n = \bar{F}^n(x)$.

Therefore, the distribution function of the minima is

$P\{m_n \le x\} = 1 - [1 - F(x)]^n = 1 - \bar{F}^n(x)$.

Definition 1.1.1. (Extreme Value Distributions for maxima).
The following are the standard Extreme Value distribution functions for maxima:
Type I. Gumbel**:** $\Lambda(x) = G_0(x) = \exp(-e^{-x})$, $\quad -\infty < x < +\infty$;

Type II. Frećhet**:** $\Phi_\alpha(x) = G_{1,\alpha}(x) = \begin{cases} \exp(-x^{-\alpha}), & if\ x > 0, \\ 0, & if\ x \le 0, \end{cases} \quad for\ some\ \alpha > 0;$

Type III. Weibull**:** $\Psi_\alpha(x) = G_{2,\alpha}(x) = \begin{cases} \exp(-(-x)^{\alpha}), & if\ x \le 0, \\ 1, & if\ x > 0. \end{cases} \quad for\ some\ \alpha > 0.$

Definition 1.1.2. (Extreme Value Distributions for minima).

The standard converse Extreme Value distributions for minima are defined as:

$$\Lambda^*(x) = 1 - \Lambda(-x), \qquad \Phi^*(x) = 1 - \Phi_\alpha(-x)\ and\ \Psi^*(x) = 1 - \Psi_\alpha(-x).$$

Then the following are the standard Extreme Value distribution functions for minima:

Type I. Gumbel**:** $\Lambda^*(x) = G_0^*(x) = 1 - \exp(-e^{x})$, $\quad -\infty < x < +\infty$;

Type II. Frećhet**:** $\Phi_\alpha^*(x) = G_{1,\alpha}^*(x) = \begin{cases} 1 - \exp(-(-x)^{-\alpha}), & if\ x < 0, \\ 1, & if\ x \ge 0, \end{cases} \quad for\ some\ \alpha > 0;$

Type III. Weibull $\Psi_\alpha^*(x) = G_{2,\alpha}^*(x) = \begin{cases} 1 - \exp(-x^{\alpha}), & if\ x \ge 0, \\ 0, & if\ x < 0. \end{cases} \quad for\ some\ \alpha > 0.$

Generalized Extreme value distribution (GEVD) 1.1.3.

The role of the generalized extreme value (GEV) distributions in the theory of extremes is analogous to that of the normal distribution in central limit theory for sums of random variables.
Assume $X_1, X_2, \dots$ are independent identically distributed (i.i.d.) with finite variance and writing $S_n = X_1 + X_2 + \cdots + X_n$,
for the sum of the first (n) random variables, the standard version of central limit theorem (*CLT*) says that appropriately normalized sums $\frac{S_n - a_n}{b_n}$ converge in distribution to the standard normal distribution as (n) goes to infinity. The appropriate normalization used sequence of normalizing constant (a_n) and (b_n) defined by $a_n = nE(X_1)$ and $b_n = \sqrt{n\ var(X_1)}$.
In mathematical notation we have $\lim_{n\to\infty} P[b_n^{-1}(S_n - a_n) \le x] = \Phi(x),\ x \epsilon \mathcal{R}$.
For more details see [3], [5], [13], [14] and [20].

Definition 1.1.4. (The Generalized Extreme Value (GEV) distribution)
The classical extreme value theory is based on three asymptotic extreme value distributions identified by Fisher and Tippett (1928) [8]. The generalized extreme value (GEV) distribution introduced by Jenkinson (1955) combines the three distributions into a single mathematical form with the distribution function (DF) is given by

$$F_{GEV}(x;\xi,\sigma,\mu) = \begin{cases} exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{\frac{-1}{\xi}}\right]; & \xi \neq 0 \\ exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right); \;\; \xi = 0; & \end{cases}$$

where $1+\xi\left(\frac{x-\mu}{\sigma}\right) > 0$, $\mu\epsilon\mathcal{R}$ is the location parameter, $\sigma > 0$ the scale parameter, and $\xi\epsilon\mathcal{R}$ the shape parameter.
The parameter ξ is known as the shape parameter of the GEV distribution and
$F_{GEV}\ (x;\xi,\sigma,\mu)$ defines a type of distribution:
If $\xi > 0$ then $F_{GEV}(x;\xi,\sigma,\mu)$ - Frećhet distribution.
If $\xi = 0$ then $F_{GEV}(x;0,\sigma,\mu)$ - Gumbel distribution.
If $\xi < 0$ then $F_{GEV}(x;\xi,\sigma,\mu)$ - Weibull distribution.

And all graphics are made in XTREMES program [23].

The following figure (1.1) is the distribution function for the generalized extreme value distribution.

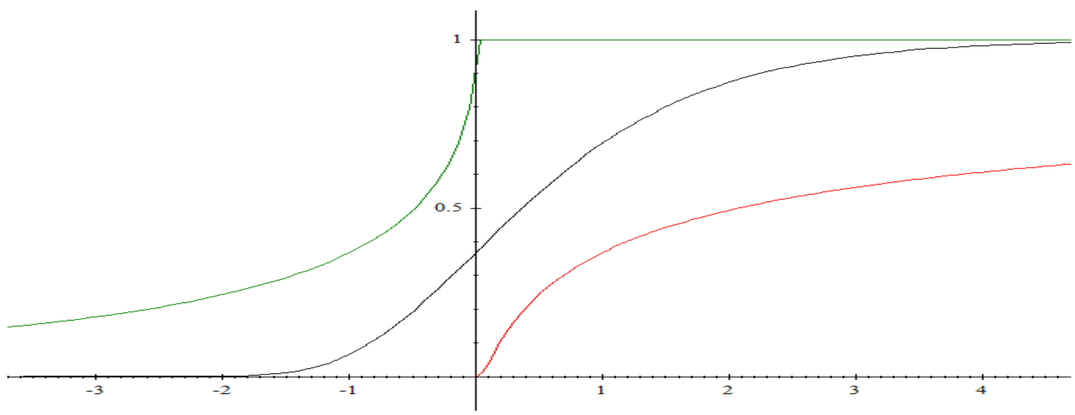

Figure (1.1): *Distribution function for the generalized extreme value distribution*.

It has the following density function

$$f_{gev}(x;\xi,\sigma,\mu)=\begin{cases}\frac{1}{\sigma}\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-\left(1+\frac{1}{\xi}\right)}\ \ exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{\frac{-1}{\xi}}\right]; & \xi\neq 0\\ \frac{1}{\sigma}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right\}; & \xi=0,\end{cases}$$

for $1+\xi\left(\frac{x-\mu}{\sigma}\right)>0.$

The following figure (1.2) is the probability density function for the generalized extreme value distribution.

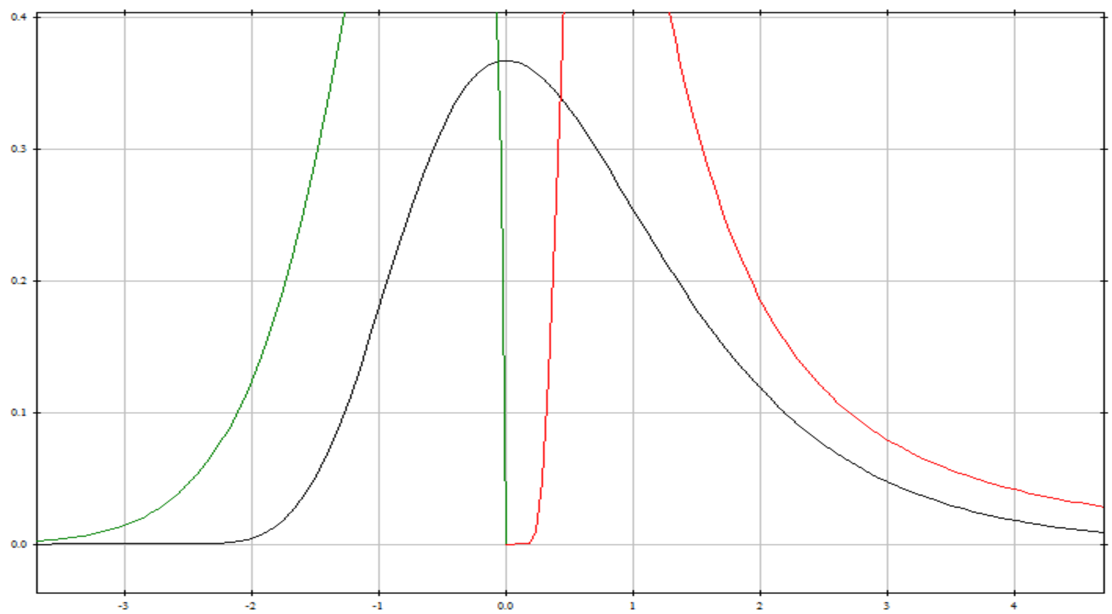

Figure (1.2): *Probability density function for the generalized extreme value distribution*.

Let M_n be random variables such that $M_n=\max\{X_1,X_2,\dots,X_n\}$ and $F^n(x)$ be distribution function of M_n.
The right endpoint of a distribution will be denote by $x_F=sup\{x\epsilon\mathcal{R}:\ \mathrm{F(x)}<1\}$,
where F is any distribution. The Gumbel and Frećhet distribution have infinite right endpoint, but the decay of the tail of the Frećhet distribution is much slower than that of the Gumbel distribution.

Suppose that block maxima M_n of independent identically distributed (i.i.d.) random variables converge in distribution under an appropriate normalization.

Recalling that $P\{M_n \leq x\} = F^n(x)$ we observe that this convergence means that there exist sequences of real constants (a_n) and (b_n), where

$a_n \geq 0,\ b_n \epsilon \mathcal{R},\ n \geq 1$ for all n, such that

$$\lim_{n\to\infty} p\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x),$$

for some non-degenerate distribution function $G(x)$. The role of the generalized extreme value distribution (GEVD) in the study of maxima is formalized by the following definition and theorem.

Extremal Types Theorem 1.1.5

The extreme type theorems play a central role of the study of extreme value theory.
In the literature, Fisher and Tippett (1928) [8], were the first who discovered the extreme type theorems and later these results were proved in complete generality by Gnedenko (1943) [9]. Galambos (1987), Leadbetter, Lindgren and Rootzen (1983) [13], Leadbetter, Lindgren and Rootzen (1986) [14] and Resnick (1987) [20], are excellent reference books on the probabilistic aspect.

Definition1.1.6 Two distribution functions F, and F^*, are called of the same type, iff there exists $a > 0,\ b\epsilon\mathcal{R}$ such that $F^*(ax + b) = F(x), for\ all\ x.$

Theorem 1.1.7. Extremal Types Theorem (Fisher and Tippett, 1928; Gnedenko, 1943).

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with distribution F, and suppose there exist normalizing constants
$a_n > 0,\ b_n\epsilon\mathcal{R},\ n \geq 1$ such that

$$P[a_n^{-1}(M_n - b_n) \leq x] = F^n(a_n x + b_n) \to G(x), \qquad (1.1)$$

where $G(x)$ is a non-degenerate limiting distribution. Then $G(x)$ belongs to the type of one of the following three distributions:

Type I. Gumbel**:** $\Lambda(x) = G_0(x) = \exp(-e^{-x}\ ) \ , \qquad -\infty < x < +\infty\,;$

Type II. Fre$\acute{c}$het**:** $\Phi_\alpha(x) = G_{1,\alpha}(x) = \begin{cases} \exp(-x^{-\alpha}), & if\ x > 0, \\ 0, & if\ \ x \leq 0, \end{cases} \qquad for\ some\ \alpha > 0;$

Type III. Weibull**:** $\Psi_\alpha(x) =\ G_{2,\alpha}(x) = \begin{cases} \exp(-(-x)^\alpha), & if\ x \leq 0, \\ \quad 1, & if\ x > 0. \end{cases} \qquad for\ some\ \alpha > 0;$

where α is a positive constant. We refer to $\Lambda = G_0 = EVo,\ \ \Phi_\alpha = G_{1,\alpha} = EV_1$ $and\ \Psi_\alpha =\ G_{2,\alpha} =\ EV_2$ as the extreme value distributions, while the constants $\boldsymbol{a_n}$ and b_n from (1.1) are called the normalizing constants.

The details of the proof can be found in Resnick (1987) [20], Proposition 0.3, pp. 9-11

Definition1.1.8. (Extreme value distribution and extremal random variable). The distribution functions Λ, Φ_α and Ψ_α as presented in theorem 1.1.7 are called standard extreme value distributions, random variables with these distributions are standard extremal random variables.

The three limiting distributions in the (GEVD) family include 1.1.9.

(i) The three types are known as the Gumbel, Frećhet and Weibull (strictly, negative Weibull), respectively.

(ii) The Frećhet (type II) is bounded below, and the negative Weibull (type III) is bounded above.

(iii) The standard Weibull is a distribution for minima.

The figure (1.3) below shows Densities for Gumbel, Frećhet and Weibull functions respectively from left to right.

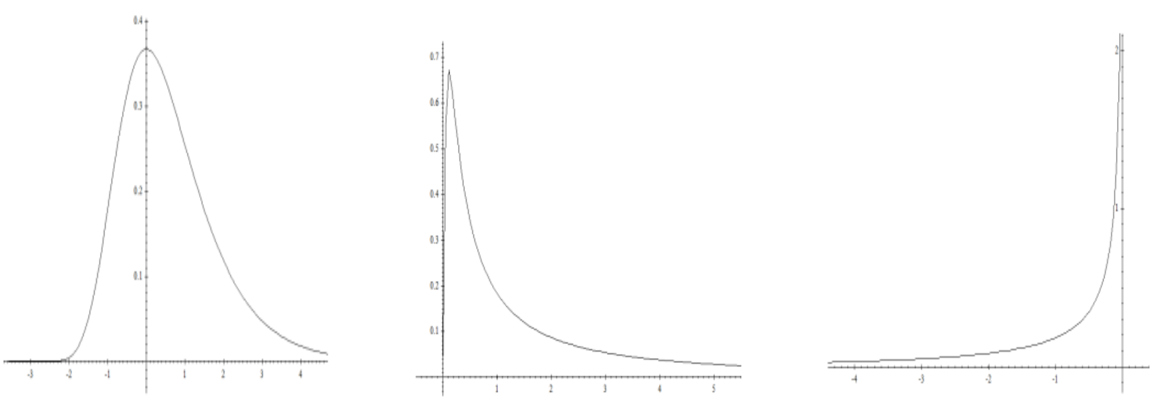

Figure (1.3): *Densities for Gumbel, Frećhet and Weibull functions respectively from left to right.*

1.2 Max-stable distribution

Definition 1.2.1. A non-degenerate distribution function G is called Max-stable if there exist real constants $a_n > 0,\ b_n \epsilon \mathcal{R},\ n \geq 1$ such that $for\ all\ x,\quad G^n(a_n x + b_n) = G(x)$. Then $G(x)$ is one of the following three forms: Type I: Gumbel, Type II: Frećhet and Type III: Weibull.

Remark 1.2.2. G is strictly stable iff $b_n = 0$, for all n.
We give a list of these constants when the max-stable distribution function is one of the standard extreme value (EV) distribution functions:

(i) Gumbel: $\Lambda = G_0$: $a_n = 1,\ b_n = \ln n$.

(ii) Frećhet: $\Phi_\alpha = G_{1,\alpha}$: $a_n = n^{\frac{1}{\alpha}},\ b_n = 0$.

(iii) Weibull: $\Psi_\alpha = G_{2,\alpha}$: $a_n = n^{\frac{-1}{\alpha}},\ b_n = 0$.

1.2.3. Examples:

Example (1): $G_0(x) = exp(-e^{-x}),\ a_n = 1,\ b_n = \ln n,\ x \epsilon \mathcal{R}$.

Then $\left(G_0(a_n x + b_n)\right)^n = \left(G_0(x + \ln n)\right)^n = \left(\exp\left(-e^{-(x+\ln n)}\right)\right)^n$

$= exp(-e^{-x}). {}^{1}/_{n}. n = exp(-e^{-x}) = G_0(x)$.

Example (2): $G_{1,\alpha}(x) = \exp(-x^{-\alpha}), a_n = n^{\frac{1}{\alpha}},\ b_n = 0$.

Then $\left(G_{1,\alpha}(a_n x + b_n)\right)^n = \left(G_{1,\alpha}\left(n^{\frac{1}{\alpha}} x\right)\right)^n = \left(\mathrm{e}^{-\left(n^{\frac{1}{\alpha}} x\right)^{-\alpha}}\right)^n = e^{(-x)^{-\alpha}.\frac{1}{n}.n} = exp(-(x))^{-\alpha} =$ $G_{1,\alpha}(x)$.

Example (3): $G_{2,\alpha}(x) = \exp(-(-x)^\alpha),\ a_n = n^{\frac{-1}{\alpha}}, b_n = 0$. Then
$\left(G_{2,\alpha}(a_n x + b_n)\right)^n = \left(G_{2,\alpha}\left(n^{\frac{-1}{\alpha}} x\right)\right)^n = \left(\mathrm{e}^{-\left(-n^{\frac{-1}{\alpha}} x\right)^{\alpha}}\right)^n = e^{(-(-x))^{\alpha}.\frac{1}{n}.n} = exp(-(-x))^\alpha =$ $G_{2,\alpha}(x)$.

Definition1.2.4. (Slowly Varying and Regularly Varying Function)

A positive measurable function L on $(0, \infty)$ is called

(i) Slowly varying at ∞ (write $L\epsilon\mathcal{R}V_0$) $if \quad \lim_{t\to\infty}\frac{L(tx)}{L(t)} = 1, \ \ x > 0.$

(ii) Regularly varying at ∞ with index $\rho\left(write\ L\epsilon\mathcal{R}V_\rho\right)if \lim_{t\to\infty}\frac{L(tx)}{L(t)} = x^\rho \ , x > 0.$

Further information can be found in Embrechts, P. [5], de Haan (1970) [15], Resnick [20] and many other textbooks.

Examples 1.2.5.

Example1: If $L(t) = \ln t \ \ then \ \lim_{t\to\infty}\frac{L(tx)}{L(t)} = \lim_{t\to\infty}\frac{\ln(tx)}{\ln(t)} = \lim_{t\to\infty}\frac{x/_{tx}}{1/_{t}} = \frac{1/_{t}}{1/_{t}} = 1,$

Satisfies (i) in definition (1.2.4) then $L(t) = \ln t$ is slowly varying at ∞.

Example2: If $L(t) = t^\rho \ln x \ \ then \ \lim_{t\to\infty}\frac{L(tx)}{L(t)} = \lim_{t\to\infty}\frac{(tx)^\rho \ln(tx)}{t^\rho \ln(t)} = x^\rho \lim_{t\to\infty}\frac{(t)^\rho \ln(tx)}{t^\rho \ln(t)} = x^\rho \lim_{t\to\infty}\frac{x/_{tx}}{1/_{t}} = x^\rho \frac{1/_{t}}{1/_{t}} = x^\rho.1 = x^\rho.$

Thus $L(t)$ is a regularly varying at ∞ with index ρ, and satisfies (ii) in definition (1.2.4).

Example 3: The functions: $e^x, \ \sin(x+2) \ and \ \ \exp\{\log x\}$, are not regularly varying.

Example 4: The functions: $\log x, \ \log(1+\mathrm{x}) \ , \ \log\log(e+x) \ and \exp\{(\log x)^\alpha\},$

$0 < \alpha < 1,$ are slowly varying.

1.3. Threshold Exceedances

Generalized Pareto Distribution (GPD)

The main distributional model for exceedances over thresholds is the generalized Pareto distribution.

For more information about (GPD) see [4], [5] and many other textbooks.

Remark: $X \sim Pareto\ (\alpha, k)$ the distribution function is given by $F(x) = 1 - \left(\frac{k}{k+x}\right)^{\alpha}$, $\alpha > 0, k > 0,\ x \geq 0,$ and $E(X^n) = \frac{k^n n!}{\prod_{i=1}^{n}(\alpha - i)}$, $\alpha > n$.

Definition 1.3.1. (The Generalized Pareto Distribution (GPD)).

The distribution function of the GPD is given by

$$F_{GPD}(x;\sigma,\xi) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}} & ;\ \xi \neq 0, \\ 1 - exp\left(\frac{-x}{\sigma}\right); & \xi = 0, \end{cases}$$

where $\sigma > 0, and\ x \geq 0\ when\ \xi \geq 0,$
and $0 \leq x \leq \frac{-\sigma}{\xi}\ when\ \ \xi < 0.$

The parameters ξ and σ are respectively, as the shape and scale parameters.
The following figure (1.4) is the distribution function for the generalized Pareto distribution.

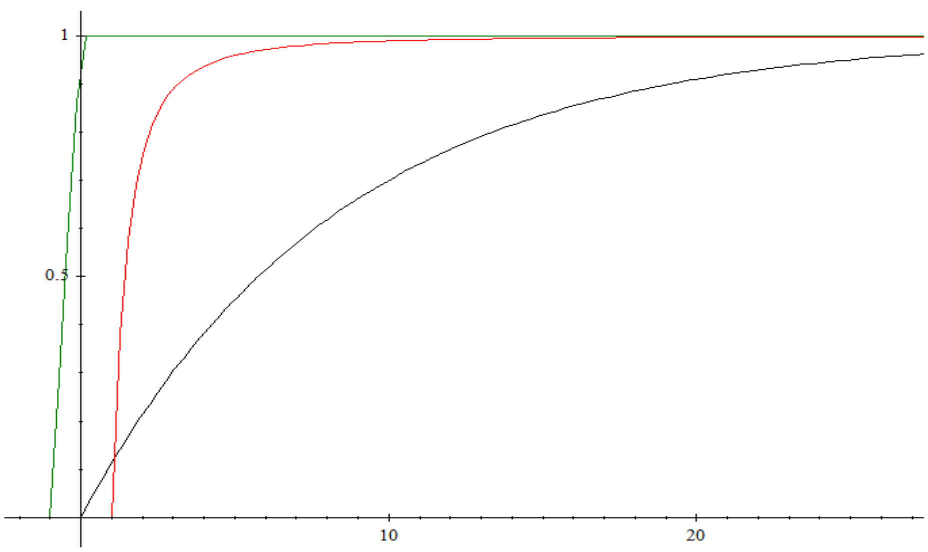

Figure (1.4): *Distribution function for the generalized Pareto distribution.*

The following analytical relationship exists between the Generalized Pareto Distribution (GPD) functions $F_{GPD}(x)$ and the generalized extreme value (GEV) distribution functions $F_{GEV}(x)$ for ξ-parameterization:

$F_{GPD}(x) = 1 + \ln(F_{GEV}(x))$, where $\alpha = \frac{1}{\xi} = \xi^{-1}$, $if \ln(F_{GEV}(x)) > -1$.

The three limiting distributions in the GPD family include the Pareto, Beta, and Standard exponential distribution functions:

(i) Exponential (GP_0): $F_{GP_0}(x) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(ii) Pareto (GP_1), $\alpha > 0$: $F_{GP_{1,\alpha}}(x) = \begin{cases} 1 - x^{-\alpha}, & x \geq 1, \\ 0, & x < 1. \end{cases}$

(iii) Beta (GP_2), $\alpha < 0$: $F_{GP_{2,\alpha}}(x) = \begin{cases} 1 - (-x)^{-\alpha}, & -1 \leq x \leq 0, \\ 0, & x < -1. \end{cases}$

The figure (1.5) below shows densities for Exponential, Pareto and Beta functions, respectively from left to right.

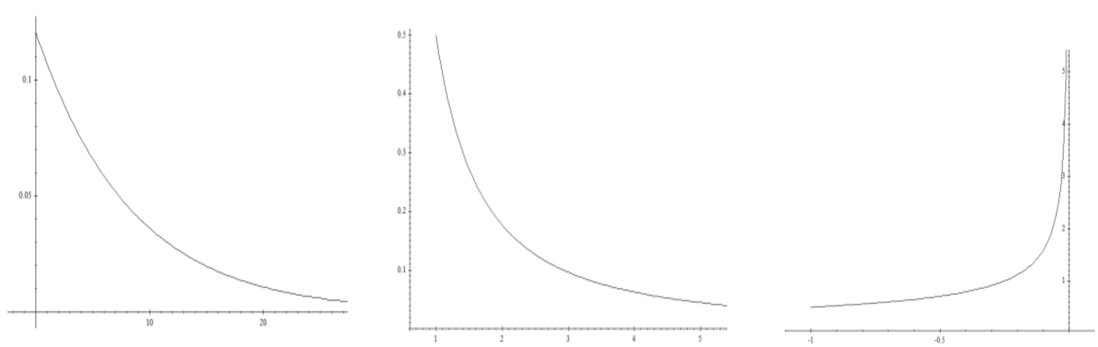

Figure (1.5): *Densities for Exponential, Pareto and Beta functions respectively from left to right.*

Relationship between Extreme Value (EV) and Generalized Pareto (GP) distributions 1.3.2.

The relationship for three different GP distributions for continuous distributions are given as follows:

(i) The exponential distribution.
The exponential distribution is denoted as GP_0.
The exponential distribution function corresponds to the Gumbel distribution as follows:
$F_{GP_0}(x) = 1 + \ln(F_{EV_0}(x)) = 1 - \exp(-x),\ x \geq 0.$

(ii) The Pareto distribution.
The Pareto (or ordinary Pareto) distribution is denoted as $GP_{1,\alpha}$. The Pareto distribution function corresponds to the Frećhet distribution as follows:

$$F_{GP_{1,\alpha}}(x) = 1 + \ln(F_{EV_{1,\alpha}}(x)) = 1 - x^{-\alpha}, \quad for\ x \geq 1, \alpha > 0\ .$$

(iii) The Beta distribution.
The Beta distribution is denoted as $GP_{2,\alpha}$. The Beta distribution function corresponds to the Weibull distribution as follows:

$$F_{GP_{2,\alpha}}(x) = 1 + \ln(F_{EV_{2,\alpha}}(x)) = 1 - (-x)^{-\alpha}, \quad for\ \ -1 \leq x \leq 0, \qquad \alpha < 0.$$

The Generalized Pareto Distribution (GPD) Properties 1.3.3:

(1) If $\xi \geq 0$, the distribution function $F_{GPD}(x; \sigma, \xi)$ is that of an ordinary Pareto distribution with

$$\alpha = \frac{1}{\xi} = \xi^{-1},\ \ k = \frac{\sigma}{\xi} = \sigma\xi^{-1}.$$

Proof: $F_{GPD}(x; \sigma, \xi) = 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}};\ \ \xi > 0,$

$$= 1 - \left(\frac{\sigma + \xi x}{\sigma}\right)^{-\xi^{-1}} = 1 - \left(\frac{\sigma}{\sigma + \xi x}\right)^{\xi^{-1}} = 1 - \left(\frac{\sigma\xi^{-1}}{\sigma\xi^{-1} + x}\right)^{\xi^{-1}}$$, that is the Pareto with $\alpha = \frac{1}{\xi} = \xi^{-1},\ \ k = \frac{\sigma}{\xi} = \sigma\xi^{-1}$.

(2) If $\xi = 0$, the distribution function $F_{GPD}(x; \sigma, 0)$ is an exponential distribution.

Proof: note that exponential distribution is given by $F(x; \lambda) = 1 - \exp(-\lambda x)$,
consider $F_{GPD}(x; \sigma, 0) = 1 - \exp\left(\frac{-x}{\sigma}\right) = 1 - \exp\left(\frac{-1}{\sigma}x\right) = F(x, \frac{1}{\sigma})$. Thus,
$F_{GPD}(x; \sigma, \xi)$ is an exponential distribution with parameter $\lambda = \frac{1}{\sigma}\ when\ \xi = 0$.

(3) If $\xi < 0$, we have a short-tailed Pareto type *II* distribution.

(4) $\lim_{\xi \to 0} F_{GPD}(x; \sigma, \xi) = F_{GPD}(x; \sigma, 0)$.

Proof: $\lim_{\xi \to 0} F_{GPD}(x; \sigma, \xi) = \lim_{\xi \to 0} 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}},$ (1)

But $\left(1+\frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}} = exp\left(\ln\left(1+\frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}}\right) = \exp\left(\frac{-1}{\xi} ln\left(\frac{\sigma+\xi x}{\sigma}\right)\right)$. (2)

Now we substitute the value of $\left(1+\frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}}$ from (2) into (1). We obtain

$$\lim_{\xi\to 0} F_{GPD}(x;\sigma,\xi) = \lim_{\xi\to 0} 1-\exp\left(\frac{-1}{\xi} ln\left(\frac{\sigma+\xi x}{\sigma}\right)\right) = 1-\lim_{\xi\to 0}\exp\left(\frac{-1}{\xi} ln\left(\frac{\sigma+\xi x}{\sigma}\right)\right),$$

$$= 1-exp\left(-\lim_{\xi\to 0}\frac{ln\left(\frac{\sigma+\xi x}{\sigma}\right)}{\xi}\right) = 1-\exp\left(-\lim_{\xi\to 0}\left(\frac{\sigma}{\sigma+\xi x}\cdot\frac{x}{\sigma}\right)\right),$$

by L'Hopitals rule $= 1-exp\left(-\lim_{\xi\to 0}\frac{x}{\sigma+\xi x}\right) = 1-\exp\left(\frac{-x}{\sigma}\right) = F_{GPD}(x;\sigma,0)$.

Thus $\lim_{\xi\to 0} F_{GPD}(x;\sigma,\xi) = F_{GPD}(x;\sigma,0)$.

(5) We have $F_{GPD}(x;\sigma,\xi)\epsilon\, DA(F_{GEV}(x;\xi,\sigma,\mu)),\ \ for\ all\ \xi\epsilon R$.

Proof: by Theorem 1.4.13.below for $\xi>0$,
$F\epsilon DA\big(F_{GEV}(x;\xi,\sigma,\mu)\big) \Leftrightarrow \bar{F}\,(x) = x^{-\xi^{-1}}L(x)$, for some function L regularly varying at ∞, $\bar{F}\,(x) = 1-F(x)$.

First, to show that $1-\ F_{GPD}(x;\sigma,\xi) = x^{-\xi^{-1}}L(x)$.

Consider $1-\ F_{GPD}(x;\sigma,\xi) = 1-\left(1-\left(1+\frac{\xi x}{\sigma}\right)^{-\xi^{-1}}\right) = \left(1+\frac{\xi x}{\sigma}\right)^{-\xi^{-1}} = \left(\frac{\sigma+\xi x}{\sigma}\right)^{-\xi^{-1}} = x^{-\xi^{-1}}\left(\frac{x^{-1}\sigma+\xi}{\sigma}\right)^{-\xi^{-1}} = x^{-\xi^{-1}}\left(\frac{1}{x}+\frac{\xi}{\sigma}\right)^{-\xi^{-1}} = x^{-\xi^{-1}}L(x),\ \ where\ L(x) = \left(\frac{1}{x}+\frac{\xi}{\sigma}\right)^{-\xi^{-1}}$.

Next, to show $L(x) = \left(\frac{1}{x}+\frac{\xi}{\sigma}\right)^{-\xi^{-1}}$ is regularly varying at ∞.

Consider

$$\lim_{t\to\infty}\frac{L(tx)}{L(t)} = \lim_{t\to\infty}\frac{\left(\frac{1}{tx}+\frac{\xi}{\sigma}\right)^{-\xi^{-1}}}{\left(\frac{1}{t}+\frac{\xi}{\sigma}\right)^{-\xi^{-1}}} = \lim_{t\to\infty}\left(\frac{(tx)^{-1}\sigma+\xi}{t^{-1}\sigma+\xi}\right)^{-\xi^{-1}} = \left(\lim_{t\to\infty}\frac{(tx)^{-1}\sigma+\xi}{t^{-1}\sigma+\xi}\right)^{-\xi^{-1}} =$$

$(\lim_{t\to\infty} x^{-1})^{-\xi^{-1}} = (x^{-1})^{-\xi^{-1}} = x^{\xi^{-1}}$. Then $L(x)$ is regularly varying at ∞ with index ξ^{-1}. Hence, $F_{GPD}(x;\sigma,\xi)\epsilon\, DA\big(F_{GEV}(x;\xi,\sigma,\mu)\big)$.

(6) $E(X^k) = \infty,\ \ for\ k \geq \frac{1}{\xi} = \xi^{-1}$.

(7) $E(X) = \frac{\sigma}{1-\xi}\ for\ \xi<1,\ \ X\sim F_{GPD}(x;\sigma,\xi)$.

Remark 1.3.4:

(1) The role of Generalized Extreme Value (GEV) in Extreme Value Theory (EVT) is a model for the block maximum distribution.

(2) The role of Generalized Pareto Distribution (GPD) in Extreme Value Theory (EVT) is a model for the excess distribution over a high threshold.

Definition 1.3.5. (Excess distribution over threshold u)

Let X be a random variable with distribution function F. The excess over the threshold u has distribution function

$$F^{(u)}(x) = P(X - u \le x | X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}, \qquad for\ 0 \le x < x_F - u,$$
$$where\ \ x_F = \sup\{x : F(x) < 1\} \le \infty,\ \ x_F = right\ endpoint\,.$$

Remark 1.3.6. The distribution function $F^{(u)}$ is called the conditional excess distribution function.

Definition 1.3.7. (Mean excess function)
The mean excess function of a random variable X with finite mean is given by

$$e_{F^u} = E[X - u | X > u] = \frac{1}{1 - F(u)} \int_u^\infty (1 - F(x))\, dx\,, \quad u > 0.$$

Examples 1.3.8. (Excess distribution of exponential and GPD)

(1) If F is the distribution function of an exponential random variable then
$F^{(u)}(x) = F(x);\ \ for\ all\ x.$

Proof: since $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0. \end{cases}$ where $\lambda > 0$,

$$\Rightarrow F^{(u)}(x) = \frac{F(x+u) - F(u)}{1 - F(u)} = e^{-\lambda u}\, \frac{\left[1 - e^{-\lambda x}\right]}{e^{-\lambda u}} = 1 - e^{-\lambda x} = F(x),$$

$$\Rightarrow F^{(u)}(x) = F(x).$$

(2) If X has distribution function,
$$F = F_{GPD}(x; \sigma, \xi) \Rightarrow F^{(u)}(x) = F_{GPD}(x; \sigma(u), \xi), \quad \sigma(u) = \sigma + \xi(u),$$
where $0 \le x < \infty\ if\ \xi > 0\ and\ 0 \le x \le \frac{-\sigma}{\xi}\ if\ \xi < 0.$

Proof: by definition 1.3.5, $F^{(u)}(x) = \frac{F(x+u) - F(u)}{1 - F(u)}$ $, \Rightarrow F^{(u)}(x) = \frac{F_{GPD}\,(x+u) - F_{GPD}\,(u)}{1 - F_{GPD}\,(u)} =$

$$\frac{\left(1 - \left(1 + \frac{\xi(x+u)}{\sigma}\right)^{-\xi^{-1}}\right) - \left(1 - \left(1 + \frac{\xi u}{\sigma}\right)^{-\xi^{-1}}\right)}{1 - \left(1 - \left(1 + \frac{\xi u}{\sigma}\right)^{-\xi^{-1}}\right)} =$$

$$\frac{\left(1 + \frac{\xi u}{\sigma}\right)^{-\xi^{-1}} - \left(1 + \frac{\xi(x+u)}{\sigma}\right)^{-\xi^{-1}}}{\left(1 + \frac{\xi u}{\sigma}\right)^{-\xi^{-1}}} = 1 - \left(\frac{\sigma + \xi(x+u)}{\sigma + \xi u}\right)^{-\xi^{-1}} =$$
$$= 1 - \left(\frac{\sigma + \xi x + \sigma(u) - \sigma}{\sigma + \sigma(u) - \sigma}\right)^{-\xi^{-1}} = 1 - \left(\frac{\sigma(u) + \xi x}{\sigma(u)}\right)^{-\xi^{-1}} = 1 - \left(1 + \frac{\xi x}{\sigma(u)}\right)^{-\xi^{-1}}$$

$$= F_{GPD}(x;\sigma(u),\xi), \ \ where \ \ \sigma(u) = \sigma + \xi(u) \Rightarrow \xi(u) = \sigma(u) - \sigma.$$

(3) Then mean excess function of the GPD can be calculated by

$$E(X) = \frac{\sigma}{1-\xi}, and\ definition\ 1.3.7, \quad e_{F^u} = \frac{\sigma(u)}{1-\xi} = \frac{\sigma + \xi u}{1-\xi}, \qquad \sigma + \xi u > 0,$$

where $0 \le u < \infty\ if\ 0 \le \ \xi < 1\ and\ 0 \le u < \frac{-\sigma}{\xi}\ if\ \xi < 0.$

Theorem 1.3.9. (Pickands-Balkema-de Haan Theorem (1974)).

Suppose that $X_1, X_2, \dots, X_n$ are n independent realizations of a random variable X with a distribution function *F(x)*. Let x_F be the finite or infinite right endpoint of the distribution F. The distribution function of the excesses over certain high threshold u is given by

$$F^{(u)}(x) = P(X - u \le x | X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}, \ \ for\ 0 \le x < x_F - u.$$

If $F\epsilon\ DA\big(F_{GEV}(x;\xi,\sigma,\mu)\big)$, then there exists a positive measurable function $\sigma(u)$ such that

$$\lim_{x \to x_F} sup\left|F^{(u)}(x) - F_{GPD}(x;\sigma(u),\xi)\right| = 0.$$

1.4. Domain of attraction of the extremal type distributions.

Definition 1.4.1. (Domain of Attraction (DA)). Suppose $\{X_n,\ n \geq 1\}$ is a sequence of independent identically distributed (i.i.d.) random variables with the common distribution function *F*. The distribution *F* belongs to the domain of attraction of the extreme value distribution *G*, $F \epsilon DA(G)$ if there exist constants $a_n > 0,\ b_n \epsilon \mathcal{R},\ n \geq 1$ such that

$$F^n\,(a_n x + b_n) = P[M_n \leq a_n x + b_n] \to G(x), \text{ as } n \to \infty \text{, where } \quad M_n = \max_{1 \leq i \leq n} X_i\ .$$

The following theorem from [14] is very useful in finding the domain of attraction of *F* and gives necessary and sufficient conditions:

Theorem 1.4.2. The following conditions are necessary and sufficient for a distribution function, *F*, to belong to the domain of attraction of the three extremal types:

Type I Gumbel: There exists a strictly positive function $g(t)$ defined on the set $(-\infty, x_F)$, such that for every real number x the equality $\lim_{t \uparrow x_F} \frac{1-F(t+xg(t))}{1-F(t)} = e^{-x}$ holds true.

Type II Frećhet: $x_F = +\infty$, and $\lim_{t \to \infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha},\ for\ some\ \alpha > 0\ and\ all\ x > 0.$

Type III Weibull**:** $x_F < +\infty$, and $\lim_{h \downarrow 0} \frac{1-F(x_F - hx)}{1-F(x_F - h)} = x^{\alpha}\ , for\ some\ \alpha > 0\ and\ all\ x > 0.$

The proofs of the theorem can be found in Leadbetter et al. (1983) [13], Leadbetter et al. (1986) [14], Resnick (1987) [20], Galambos (1987), etc...

Theorem 1.4.3. (Characterization of DA (G))

The distribution function *F* belongs to the domain of attraction of the extreme value distribution *G* with norming constants $a_n > 0,\ b_n \epsilon \mathcal{R}$ iff

$$\lim_{n \to \infty} n\bar{F}\,(a_n x + b_n) = -\ln G(x),\ \ x \epsilon \mathcal{R}\text{ , when } G(x) = 0 \text{ the limit is interpreted as } \infty.$$

Where $\bar{F}(a_n x + b_n) = 1 - F(a_n x + b_n)$. More information see [5], [20].

Definition 1.4.4. (Von Mises function). Let *F* be a distribution function with right endpoint $x_F \leq \infty$. Suppose there exists some $z < x_F$ such that *F* has representation $\bar{F}(x) = c.exp\left\{-\int_z^x \frac{1}{a(t)} dt\right\},\ z < x < x_F$, where *c* is some positive constant, *a*(.) is a positive and absolutely continuous function with density $a'\ and\ \lim_{x \uparrow x_F} a'(x) = 0$. Then *F* is called a Von Mises function, the function *a*(.) is the auxiliary function of *F*.
For more details see Resnick [20], proposition 1.4. and de Haan [2].

Theorem 1.4.5. (Von Mises Condition).

(i) Let F be an absolutely continuous distribution function with density f satisfying $\lim_{x\to\infty}\frac{xf(x)}{\bar{F}(x)}=\alpha>0,\ \ then\ \ F\epsilon DA(\Phi_\alpha)$.

(ii) Let F be an absolutely continuous distribution function with density f which is positive on some finite interval (z, x_F). If $\lim_{x\uparrow x_F}\frac{(x_F-\mathrm{x})f(x)}{\bar{F}(x)}=\alpha>0,\ \ then\ \ F\epsilon DA(\Psi_\alpha)$.

The proof for details see Resnik (1987) [20], proposition 1.15 and proposition 1.16, pp. 63.

Properties of Von Mises functions 1.4.6. Every Von Mises function F is absolutely continuous on (z, x_F) with positive density f. The auxiliary function can be chosen as $a(x)=\frac{\bar{F}(x)}{f(x)}$. Moreover, the following properties hold:

(i) If $x_F=\infty$, then $\bar{F}\ \epsilon RV_{-\infty}$ and $\lim_{\mathrm{x}\to\infty}\frac{xf(x)}{\bar{F}(x)}=\infty$.

(ii) If $x_F<\infty$, then $\bar{F}(x_F-\mathrm{x}^{-1})\epsilon RV_{-\infty}$ and $\lim_{x\uparrow x_F}\frac{(x_F-\mathrm{x})f(x)}{\bar{F}(x)}=\infty$.

For more details see [5], pp.140.

Examples 1.4.7.

We give some examples of Von Mises functions. See [5], pp. 139.

Example (1): (Exponential distribution)

$\bar{F}(x)=e^{-\lambda x}, x\geq 0, \lambda>0$. F is a Von Mises function with auxiliary function $a(x)=\lambda^{-1}$.
Proof: $\bar{F}(x)=1-F(x)=e^{-\lambda x}$, $F(x)=1-\ e^{-\lambda x}$, $F'(x)=f(x)=\lambda e^{-\lambda x}$,

then the auxiliary function $a(x)=\frac{\bar{F}(x)}{f(x)}=\frac{e^{-\lambda x}}{\lambda e^{-\lambda x}}=\frac{1}{\lambda}=\lambda^{-1}$.

Example (2): (Weibull distribution)

$\bar{F}(x)=e^{-cx^\tau}, x\geq 0,\ \ c,\tau>0$. F is a Von Mises function with auxiliary function $a(x)=c^{-1}\tau^{-1}x^{1-\tau}, x>0$.

Proof: $\bar{F}(x)=1-F(x)=e^{-cx^\tau}$, $F(x)=1-\ e^{-cx^\tau}$, $F'(x)=f(x)=e^{-cx^\tau}(c\tau x^{\tau-1})$,

then the auxiliary function $a(x)=\frac{\bar{F}(x)}{f(x)}=\frac{e^{-cx^\tau}}{e^{-cx^\tau}(c\tau x^{\tau-1})}=\frac{1}{c\tau x^{\tau-1}}=c^{-1}\tau^{-1}x^{1-\tau}, x>0$.

Theorem 1.4.8. (Von Mises(1936)). F is absolutely continuous distribution function and $x_F = sup\{x: F(x) < 1\}$. If

(i) $F^{''}(x) < 0,\ for\ all\ x\epsilon(z, x_F),\ x_F \leq \infty$.

(ii) $F^{'}(x) = 0,\ for\ x \geq x_F$.

(iii) $\lim_{x\to x_F} \frac{F^{'}(x)(1-F(x))}{\left(F^{'}(x)\right)^2} = 1, \qquad then\ F\epsilon DA(\Lambda)$.

That is sufficient conditions for continuous function.

The proof for details see Resnik (1987) [20] and [5].

Example: Let $F(x) = 1 - e^{-x},\ x > 0$.

$$Then\ F^{'}(x) = f(x) = e^{-x}, \qquad x \geq 0,$$

$and\ f(x) = \frac{F^{'}(x)(1-F(x))}{\left(F^{'}(x)\right)^2} = \frac{1-F(x)}{F^{'}(x)} = \frac{e^{-x}}{e^{-x}} = 1,$

Therefore $f^{'}(x) = 0$, and $F\epsilon DA(\Lambda)$. See Resnik (1987) [20], pp. 42.

The following theorems from [14], [2] and [20] are giving necessary and sufficient conditions:

Theorem 1.4.9. (Gnedenko (1943), Mejzler (1949), De Haan (1970)). For a distribution function F set $H(x) = \frac{1}{1-F(x)}$, $\quad x_F = \sup\{t : F(t) < 1\}$, so that $H^{\leftarrow} = H^{-1}$ is defined on $(1, \infty)$.

The following are equivalent:

(i) $F\epsilon DA(\Lambda)$, if there exist constants $a_n > 0,\ b_n\epsilon\mathcal{R},\ n\epsilon N$ such that

$\lim_{n\to\infty} F^n(a_n x + b_n) = \exp(-e^{-x})$, for all x.

(ii) $H\epsilon\Gamma$, there exist g such that for every real number x: $\lim_{t\uparrow x_F} \frac{1-F(t+xg(t))}{1-F(t)} = e^{-x}$.

(iii) $H^{\leftarrow} = H^{-1}\epsilon\Pi$, there exist a such that for all $x > 0$: $\quad \lim_{t\to\infty} \frac{H^{-1}(tx)-H^{-1}(t)}{a(t)} = \ln x$.

The proofs for details see Resnik (1987) [20], proposition 0.10., pp. 28-30.

Theorem 1.4.10. (De Haan, 1970). $F\epsilon DA(\Lambda)\ \ iff\ \lim_{x\to x_{x_F}} \frac{(1-F(x))\int_x^{x_F}\int_y^{x_F}(1-F(t))\,dt\,dy}{\left(\int_x^{x_F}(1-F(t))dt\right)^2} = 1,$

in this case $\frac{1}{1-F}\ \epsilon\ \Gamma$, and the auxiliary function can be chosen $g(t) = \frac{\int_x^{x_F}\int_y^{x_F}(1-F(t))\,dt\,dy}{\int_x^{x_F}(1-F(t))dt}$, or

$g(t) = \frac{\int_x^{x_F}(1-F(t))dt}{(1-F(x))}$, and norming constants can be chosen
$a_n = g(b_n),\ \ b_n = F^{-1}\left(1 - \frac{1}{n}\right)$.
The proofs for details see Resnik (1987) [20], proposition 1.9., pp. 48-50.

Theorem 1.4.11. (De Haan, 1970). $F\epsilon DA(\Lambda)$ iff

$\lim_{x\to x_F} \frac{\int_x^{x_F}(1-F(t))^{\alpha}dt}{(1-F(x))\int_x^{x_F}(1-F(t))^{\alpha-1}dt} = \frac{\alpha-1}{\alpha}, for\ some\ \alpha > 1$. In this case is true for all $\alpha > 1$. The proof for details see Resnik (1987) [20], proposition 1.10., pp. 50-52.

Domain of attraction of $\Lambda(x) = G_0(x) = \exp(-e^{-x})\ ,\quad -\infty < x < +\infty$.

For a proof of the following Theorem we refer to Resnick [20], Corollary 1.7 and proposition 1.9.

Theorem 1.4.12. (Characterization of DA (Λ)) Gnedenko, 1943.

The distribution function *F* belongs to the domain of attraction of Λ, if and only if $x_F \le \infty$,
$\bar{F}(x) = 1 - F(x) = c(x)exp\left\{-\int_z^x \frac{g(t)}{a(t)}dt\right\},\ z < x < x_F,$
$where\ c(x) \to c > 0, g(x) \to 1, a^{'}(x) \to 0\ as\ x \uparrow x_F$. A possible choice for the function *a* is $a(x) = \int_x^{x_F} \frac{\bar{F}(t)}{\bar{F}(x)}dt,\ x < x_F$. If $F\epsilon DA(\Lambda)$, then in this case $a_n^{-1}(M_n - b_n) \xrightarrow{d} \Lambda$, where the norming constants a_n can be chosen as $a_n = a(b_n) = F^{-1}(1 - (ne)^{-1}) - F^{-1}\left(1 - \frac{1}{n}\right)$,

$b_n = F^{-1}\left(1 - \frac{1}{n}\right)$.

Domain of attraction of $\Phi_\alpha(x) = \exp(-x^{-\alpha})\ ,\quad x > 0$.

Theorem 1.4.13. (Characterization of DA (Φ_α))

The distribution function *F* belongs to the domain of attraction of Φ_α, $\alpha > 0$, if and only if

$x_F = +\infty, \bar{F}(x) = 1 - F(x) = x^{-\alpha}L(x)\epsilon RV_{-\alpha}$, for some function L slowly varying at ∞. If $F\epsilon DA(\Phi_\alpha)$, then in this case $a_n^{-1}M_n \xrightarrow{d} \Phi_\alpha$, where the norming constants a_n can be chosen as the $a_n = F^{-1}\left(1 - \frac{1}{n}\right),\ \ b_n = 0$.

The proof for details see Resnik (1987) [20], pp. 54-57.

Domain of attraction of $\Psi_\alpha(x) = \exp(-(-x)^{\alpha}), x < 0$.

Theorem 1.4.14. (Characterization of DA(Ψ_α))

The distribution function F belongs to the domain of attraction of Ψ_α, $\alpha > 0$, if and only if

$x_F < +\infty,\ and\ \bar{F}(x) = 1 - F\left(x_F - \frac{1}{x}\right)\epsilon RV_{-\alpha}$. If $F\epsilon DA(\Psi_\alpha)$, then in this case

$a_n^{-1}(M_n - x_F) \xrightarrow{d} \Psi_\alpha$, where the norming constants a_n can be chosen as

$a_n = x_F - F^{-1}\left(1 - \frac{1}{n}\right),\ b_n = x_F$.

The proofs for details see Resnik (1987) [20], pp. 59-62. And [5], pp. 135.

Definition 1.4.15. See [5]. (Quantile function).
The generalized inverse of the distribution function F,

$F^{-1}(t) = F^{\leftarrow}(t) = \inf\{\, x\epsilon\mathcal{R}: F(x) \geq t\}$, $\ 0 < t < 1$, is called the quantile function of the distribution function F. The quantity $x_t = F^{-1}(t) = F^{\leftarrow}(t)$ defines the t- quantile of F.

The following figure (1.6) below is the quantile function for the generalized extreme value distribution.

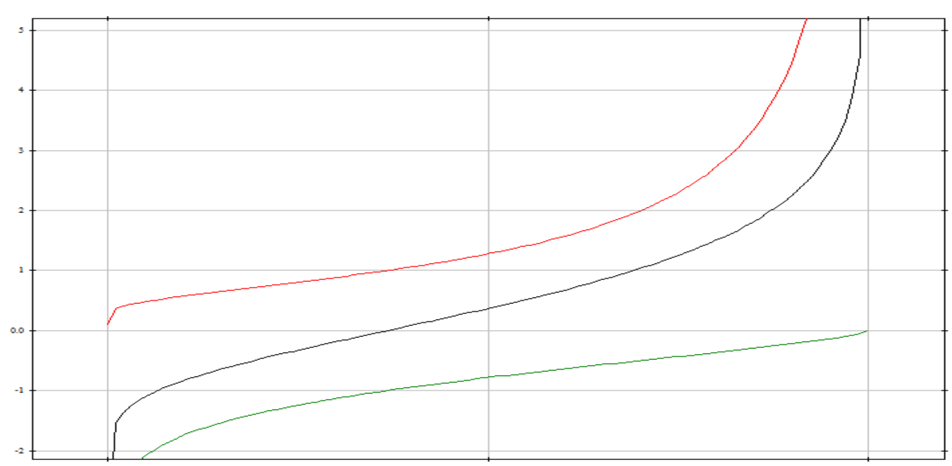

Figure (1.6): *quantile function for the generalized extreme value distribution.*

The domain of attraction of the distribution function F is determined by the asymptotic behavior of the tail $1 - F(x)$, as $x \to +\infty$.
The following theorem from [14] is important for determining of normalizing constants a_n and b_n in (1.1).

Theorem 1.4.16. Let $\{X_n\}$ be an independent identically distributed (i.i.d.) sequence random variables. Let $\tau \epsilon [0, +\infty)$, and suppose that $\{u_n\}$ is a sequence of real numbers, such that $n\big(1 - F(u_n)\big) \to \tau, \ as\ n \to \infty$, then $P\{M_n \le u_n\} \to e^{-\tau}, \ as\ n \to \infty$.

The proofs of the theorem can be found in Leadbetter et al. (1983) [13], [14], Resnick (1987) [20], Galambos (1987), etc...

Examples of Domain of Attraction 1.4.17.

Example (1) Exponential distribution (type I, Gumbel)

We consider $F(x) = 1 - e^{-x}$. We have $\frac{1-F\left(t+xg(t)\right)}{1-F(t)} = \frac{e^{-\left(t+xg(x)\right)}}{e^{-t}} = e^{-xg(t)} = e^{-x}, if\ g(t) = 1$.

Therefore, the distribution function $F(x)$, belongs to the domain of attraction of the function $G_0(x)$, and we have the type (I) of extreme value distribution, i.e. there exist constants a_n and b_n, such that the following equality holds true:

$$P\left\{M_n \le \frac{x}{a_n} + b_n\right\} \to \exp(-e^{-x}).$$

We now determine the constants a_n and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n} e^{-x}$ as $n \to \infty, i.e.$

$1 - F(x) = e^{-x}$ (1.1),

it follows from (1.1) that $1 - F(u_n) = e^{-u_n}, \ \ \tau = e^{-x}$,

then $e^{-u_n} \sim {}^{\tau}/_{n}$ $, \tau > 0$,

$\ln e^{-u_n} \sim \ln {}^{\tau}/_{n}$, $but\ \tau = e^{-x}\ then\ \ln \tau = -x$,

$-u_n \sim \ln \tau - \ln n$, $so\ \ -u_n \sim -(x + \ln n)$,

then $u_n \sim x + \ln n\, , as\ n \to \infty$,

$P\{M_n \le u_n\} \to e^{-\tau}, \ as\ n \to \infty$.

Using Theorem 1.4.16 we obtain

$P\{M_n \leq x + \ln n\} \to e^{-e^{-x}}$, (1)

but $P\left\{M_n \leq \frac{x}{a_n} + b_n\right\} \to G(x)$ (2).

Now we compare the equation (1) with the equation (2). We obtain

$a_n = 1,\ b_n = \ln n\ and\ G(x) = e^{-e^{-x}}$.

Example (2) Pareto distribution (or Pareto's law) (type II, Frećhet)

We consider $F(x) = 1 - x^{-\alpha}, \quad \alpha > 0, \quad x \geq 1.$ We have $\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = \frac{(tx)^{-\alpha}}{(t)^{-\alpha}} = x^{-\alpha}$.

In this example, the distribution function $F(x)$, belongs to the domain of attraction of the function $G_1(x)$, and we have the type (*II*) of extreme value distribution, i.e. there exist constants a_n and b_n, such that the following equality holds true:

$$P\left\{M_n \leq \frac{x}{a_n} + b_n\right\} \to \exp(-x^{-\alpha}).$$

We now determine the constants a_n and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n} x^{-\alpha}$, , $\alpha > 0$, as $n \to \infty, i.e.$

$1 - F(u_n) \sim {}^{\tau}/_{n}\,,\ \tau = x^{-\alpha}, \quad \tau > 0,$

$(u_n)^{-\alpha} \sim \frac{x^{-\alpha}}{n}, as\ n \to \infty,$

$(u_n)^{-\alpha} \sim \left(\frac{x}{n^{\frac{-1}{\alpha}}}\right)^{-\alpha}, as\ n \to \infty,$ and we obtain

$u_n \sim n^{\frac{1}{\alpha}} x,\ as\ n \to \infty.$

Using Theorem 1.4.16 we obtain

$P\left\{M_n \leq n^{\frac{1}{\alpha}} x\right\} \to e^{-x^{-\alpha}}\ , \quad as\ n \to \infty,$

we get $a_n = n^{\frac{-1}{\alpha}},\ b_n = 0\ and\ G(x) = e^{-x^{-\alpha}}$.

Example (3) Uniform distribution (type III, Weibull)

We consider $F(x) = 1 - x \quad on\ [0,1]$.

Obviously $x_F = 1$, and we have $\lim_{h\downarrow 0} \frac{1-F(x_F-hx)}{1-F(x_F-h)} = \lim_{h\downarrow 0} \frac{1-F(1-hx)}{1-F(1-h)} = \lim_{h\downarrow 0} \frac{1-hx}{1-h} = \lim_{h\downarrow 0} \frac{-x}{-1} = x$, by LHopitals Rule, $\alpha = 1$.

So $F\epsilon DA(\Psi_1)$ (of type III) Weibull distribution.

We now determine the constants a_n and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n}x$, , $\alpha > 0$, as $n \to \infty, i.e.$

$1 - F(u_n) \sim {}^{\tau}/_{n}\,,\ \tau = x, \quad \tau > 0,$

$u_n \sim \frac{x}{n}\ , \quad as\ n \to \infty,$

$P\{M_n \le u_n\} \to e^{-\tau},\ as\ n \to \infty.$

Using Theorem 1.4.16 we obtain

$P\left\{M_n \le \frac{x}{n}\right\} \to e^{-x}\ , \quad as\ n \to \infty,$

thus, $a_n = n,\ b_n = 0\ and\ \ G(x) = e^{-x}$.

Example (4): If $F(x) = \begin{cases} 1 - e^{\frac{1}{x}}, & x < 0 \\ 1, & x \ge 0. \end{cases}$

Determine the type of extreme value distribution and the normalizing constants?

We consider $F(x) = 1 - e^{\frac{1}{x}}$.

We have $\frac{1-F(t+xg(t))}{1-F(t)} = \frac{e^{\frac{1}{t+xg(x)}}}{e^{\frac{1}{t}}} = e^{\frac{-xg(t)}{t(t+xg(t))}} = e^{\frac{-xt^2}{t^2+xt^3}} = e^{\frac{-x}{1+xt}}, \quad put\ g(t) = t^2, \Rightarrow$
$\lim_{t\to 0^-} e^{\frac{-x}{1+xt}} = e^{-x}, as\ t \to 0^-, \Rightarrow$

$$\lim_{t\to 0^-} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x} \quad , as\ t \to 0^-,$$

then $F\epsilon DA(\Lambda(\mathrm{x}))$ (of type I, Gumbel distribution).

We now determine the constants a_n, b_n.

We consider $F(x) = 1 - e^{\frac{1}{x}},\ x < 0, \qquad \Rightarrow 1 - F(x) = e^{\frac{1}{x}},\ x < 0, \qquad (4.1)$

$$then\ 1 - F(u_n) \sim {}^{\tau}/_{n}, \quad \tau > 0, \quad \tau = e^{\frac{1}{x}}$$

$1 - F(u_n) \sim {}^{e^{\frac{1}{x}}}/_{n}, \quad x < 0,$ (note that $\tau = e^{\frac{1}{x}} \Rightarrow \ln\tau = \frac{1}{x}, \tau > 0$).

From (4.1)$\Rightarrow e^{\frac{1}{u_n}} \sim {}^{e^{\frac{1}{x}}}/_{n} \Rightarrow \ln e^{\frac{1}{u_n}} \sim \ln({}^{e^{\frac{1}{x}}}/_{n}) \Rightarrow \frac{1}{u_n} \sim \ln({}^{e^{\frac{1}{x}}}/_{n}) \Rightarrow u_n \sim \left(\ln {}^{e^{\frac{1}{x}}}/_{n}\right)^{-1}$,

$$u_n \sim \left(\ln e^{\frac{1}{x}} - \ln n\right)^{-1} \Rightarrow u_n \sim \left(\frac{1}{x} - \ln n\right)^{-1} \Rightarrow u_n \sim (\ln\tau - \ln n)^{-1},$$

$$P\{M_n \le u_n\} \to e^{-\tau}, \qquad as\ n \to \infty,$$

$$\tau = e^{-x}, \qquad x\epsilon R, \quad u_n = (\ln\tau - \ln n)^{-1},$$

$$u_n = (\ln e^{-x} - \ln n)^{-1} = (-x - \ln n)^{-1} = -(x + \ln n)^{-1},$$

$$\Rightarrow\ P\{M_n \le -(x + \ln n)^{-1}\} \to \exp(-e^{-x}),$$

But $(x + \ln n)^{-1} = \left(\ln n(\frac{x}{\ln n} + 1)\right)^{-1} = (\ln n)^{-1}\left(1 - \frac{x}{\ln n} + \sigma\left(\frac{1}{\ln n}\right)\right) = \frac{1}{\ln n} - \frac{x}{(\ln n)^2} + \sigma\left(\frac{1}{(\ln n)^2}\right)$,

$$\Rightarrow \{M_n \le -(x + \ln n)^{-1}\} = \left\{M_n \le -\frac{1}{\ln n} + \frac{x}{(\ln n)^2} + \sigma\left(\frac{1}{(\ln n)^2}\right)\right\}$$

$$= \left\{M_n \le \frac{-\ln n + x + \sigma(1)}{(\ln n)^2}\right\} = \{(\ln n)^2 . M_n + \ln n \le x + \sigma(1)\},$$

$= \left\{(\ln n)^2\left[M_n + \frac{1}{\ln n}\right] \le x + \sigma(1)\right\} \Rightarrow M_n \le \frac{x}{(\ln n)^2} - \frac{1}{\ln n}$,

Using Theorem 1.4.16 we obtain $P\left\{M_n \le \frac{x}{(\ln n)^2} - \frac{1}{\ln n}\right\} \to e^{-e^{\frac{1}{x}}}$.

Thus $a_n = (\ln n)^2\ \ and\ \ b_n = -\frac{1}{\ln n} = -(\ln n)^{-1}$.

Example (5) suppose $X_1, X_2, \ldots$ be financial loss, independent identically distributed (i.i.d.) with distribution function F and defined as; $F(x) = 1 - \exp(-\lambda x)\ \ where\ \lambda > 0, x > 0.$ Choose normalizing sequences

$a_n = \frac{1}{\lambda},\ b_n = \frac{\ln n}{\lambda}, \quad$ Calculate $F^n(a_n x + b_n)$?

Proof: Since $F(x) = 1 - \exp(-\lambda x)$, then $F^n(x) = 1 - exp(-\lambda x)^n$

So that $F^n(a_n x + b_n) = F^n\left(\frac{1}{\lambda}x + \frac{\ln n}{\lambda}\right) = \left[1 - exp\left(-\lambda\left(\frac{1}{\lambda}x + \frac{\ln n}{\lambda}\right)\right)\right]^n =$

$$= [1 - \exp(-x - \ln n)]^n = [1 - \exp(-x) . \exp(\ln \mathrm{n}^{-1})]^n =$$

$\left(1 - \frac{\exp(-x)}{n}\right)^n = \left(1 - \frac{1}{n}\exp(-x)\right)^n$, then $G(x) = \lim_{n\to\infty} F^n(a_n x + b_n) =$

$$= \lim_{n\to\infty}\left(1 - \frac{1}{n}\exp(-x)\right)^n = exp(-e^{-x}) = G_0(x) = \Lambda(x).$$

Thus $F \epsilon DA(\Lambda(\mathrm{x}))$.

1.5. Tailes

Definition 1.5.1. (Fat –tailed distribution)

The distribution of random variable X is said to have a fat tail if

$P[X > x] = \bar{F}(x) = 1 - F(x) \sim x^{-\alpha},\ as\ x \to \infty, \alpha > 0.$

Remark 1.5.2 Cauchy distributions are examples of fat-tail distributions.

Definition 1.5.3. (Heavy-tailed distribution)

The distribution of a random variable X with distribution function F is said to have a heavy right tail if $\lim_{x\to\infty} e^{\lambda x}\bar{F}(x) = \infty,\ for\ all\ \lambda > 0,\ \ \bar{F}(x) = 1 - F(x),\ \bar{F}(x\ ) =\ \ P[X > x].$

Definition 1.5.4. (Long-tailed distribution)

The distribution of a random variable X with distribution function F is said to have a long right tail if $\lim_{x\to\infty} P[X > x + t : X > x] = 1, for\ all\ t > 0$, or equivalently

$$\bar{F}(x + t) \sim \bar{F}(x), as\ x \to \infty.$$

Remark 1.5.5. Extreme Value Theory (EVT): Three types of distributions

Type I (Gumbel): Medium tail.

Type II (Frećhet): Heavy tail.

Type III (Weibull): Short tail.

1.6. Tail equivalence

Definition 1.6.1. See [5], [20] (Tail equivalence)

Two distribution functions *F(x)* and *G(x)* are called tail equivalent if they have the same right endpoint, i.e. if $x_0^F = x_0^G = x_0$ and $\lim_{x \to x_0^F} \frac{1-F(x)}{1-G(x)} = A,\ for\ some\ A > 0\ and$

$x_0 = inf\{x: F(x) = 1\}$.

Definition 1.6.2. (Tail equivalence)

Two distribution functions *F(x)* and *G(x)* are right tail equivalent iff

$$x_0^F = x_0^G = x_0, \qquad 1 - F(x) \sim 1 - G(x)\ as\ x \to x_0-;\ and\ \lim_{x \to x_0-} \frac{1 - F(x)}{1 - G(x)} = 1.$$

The following Theorems and Results from [7], [9], [19] and [20].

Definition 1.6.3. Two distribution functions *U(x)* and *V(x)* are of the same type if for some

$A > 0, B \epsilon R, \quad V(x) = U(Ax + B),$ for all *x*.

Theorem 1.6.4. Suppose *U(x)* and *V(x)* are two non-degenerate distribution functions. If for a sequence $F_n(x)$is a distribution functions and constants $a_n > 0, b_n \epsilon R,\ n \geq 1\ and$

$\alpha_n > 0, \beta_n \epsilon R, \ \ F_n(a_n\, x + b_n) \to^c U(x), \quad F_n(\alpha_n x + \beta_n) \to^c V(x),$

$$\Longrightarrow \frac{\alpha_n}{a_n} \to A > 0, \quad \frac{\beta_n - b_n}{a_n} \to B \epsilon R\ \ and\ V(x) = U(Ax + B).$$

Remark 1.6.5. The set of normalizing constants $a_n > 0, b_n \epsilon R,\ n \geq 1$ is asymptotically equivalent to the set of normalizing constants $\alpha_n > 0,\ \beta_n \epsilon R, n \geq 1\ iff\ \frac{\alpha_n}{a_n} \to 1,\ \frac{\beta_n - b_n}{a_n} \to 0.$

Domain of attraction results:

Theorem 1.6.6. *F(x)* and *G(x)* are distribution functions such that $\lim_{x\to x_0-}\frac{1-F(x)}{1-G(x)}=\alpha,$

$0<\alpha<\infty$. If there exist normalizing constants $a_n>0, b_n\epsilon R,\ n\geq 1$ such that

$$F^n(a_nx+b_n)\to\Phi(x) \qquad (1.6.1),$$

$\Phi(x)$ non-degenerate, then $G^n(a_nx+b_n)\to\Phi^{\alpha^{-1}}(x)$.

Proof: Suppose first *F(x)* and *G(x)* are tail equivalent and (1.6.1) holds.

Since $\frac{1-F(x)}{1-G(x)}\to\alpha, as\ x\to x_0-,\ \ 1-G(x)\sim\alpha^{-1}\big(1-F(x)\big),$

An equivalent formulation of (1.6.1) is $n\big(1-F(a_nx+b_n)\big)\to-\ln\Phi(x),$

for $x\ such\ that\ \Phi(x)>0$. For such $x,\ a_nx+b_n\to x_0-$, and hence from tail equivalence

$$n\big(1-G(a_nx+b_n)\big)\sim n\alpha^{-1}\big(1-F(a_nx+b_n)\big)\to-\alpha^{-1}\ln\Phi(x),$$

Then $G^n(a_nx+b_n)\to\Phi^{\alpha^{-1}}(x)$.

Remark 1.6.7. From three extreme value we have for all *x* and $\gamma>0$:

(i) $\Lambda(x)^\gamma=\Lambda(x-\ln\gamma),$

(ii) $\Phi_\alpha(x)^\gamma=\Phi_\alpha\big(\gamma^{-\alpha^{-1}}x\big),$

(iii) $\Psi_\alpha(x)^\gamma=\Psi_\alpha\big(\gamma^{\alpha^{-1}}x\big).$

Result 1.6.8. *F(x)* and *G(x)* are distribution functions and $\Phi_\alpha(x)$ is an extreme value distribution. Suppose $F^n(a_nx+b_n)\to\Phi_\alpha(x), for\ \ a_n>0, b_n\epsilon R,\ n\geq 1.$

Then $G^n(a_nx+b_n)\to\ \Phi_\alpha(Ax+B)\ and\ A>0\ \ iff\ B=0\ and\ \lim_{x\to\infty}\frac{1-F(x)}{1-G(x)}=A^\alpha.$

Proof: if $\lim_{x\to\infty}\frac{1-F(x)}{1-G(x)}=A^\alpha,$ then by Theorem 1.6.6 we have that $G^n(a_nx+b_n)\to$ $\{\Phi_\alpha(x)\}^{A^{-\alpha}}=\Phi_\alpha(Ax),$ and from remark 1.6.7. (ii) Then $\{\Phi_\alpha(x)\}^{A^{-\alpha}}=\Phi_\alpha\big((A^{-\alpha})^{-\alpha^{-1}}x\big)=$ $\Phi_\alpha(Ax)$.

For the converse, we can let $a_n=F^{-1}\big(1-\frac{1}{n}\big)=\mu_n^F,\ \ b_n=0$, so that we are given that

$G^n\left(\frac{a_n}{A}x+b_n-a_n\frac{B}{A}\right)\to\Phi_\alpha(x)\ as\ n\to\infty$, then

$G^n\left(\frac{\mu_n^F}{A}x - \frac{\mu_n^F B}{A}\right) \to \Phi_\alpha(x)\ as\ n \to \infty,$ (1.6.2)

Then from (1.6.2) we obtain $a_n = \frac{\mu_n^F}{A},\ \ b_n = -\frac{\mu_n^F B}{A}$.

But since $G(x)\epsilon\Phi_\alpha(x)$, we have that $G^n(\mu_n^G x) \to \Phi_\alpha(x)\ \ as\ n \to \infty,$ (1.6.3)

then from (1.6.3) we obtain $\alpha_n = \mu_n^G,\ \ \beta_n = 0$, and therefore by Theorem 1.6.4

$\frac{\alpha_n}{a_n} \to 1 \Longrightarrow \frac{\mu_n^G}{\frac{\mu_n^F}{A}} = \frac{\mu_n^G}{\mu_n^F} \longrightarrow A^{-1}$ (1.6.4) and

$$\frac{b_n - \beta_n}{\alpha_n} \to 0 \Longrightarrow \frac{-\dfrac{\mu_n^F B}{A} - 0}{\mu_n^G} = \frac{-\mu_n^F B A^{-1}}{\mu_n^G} \longrightarrow 0 \qquad (1.6.5) \qquad as\ n \to \infty.$$

Since $A > 0$, (1.6.4) and (1.6.5) can both hold iff $B = 0$.

Given any $\varepsilon > 0$, there exists because of (1.6.4) an integer N_ε such that for $n > N_\varepsilon$, we have

$$\left|\frac{\mu_n^G}{\mu_n^F} - A^{-1}\right| < \varepsilon, \quad -\varepsilon < \frac{\mu_n^G}{\mu_n^F} - A^{-1} < \varepsilon,$$

$$i.e., \quad \mu_n^F(A^{-1} - \varepsilon) < \mu_n^G < (A^{-1} + \varepsilon)\mu_n^F.$$

Since $\mu_n^G < \mu_{n+1}^G \to \infty$, we have that for every x sufficiently large there exists an integer $n = n(x)$ such that $x\varepsilon[\mu_n^G, \mu_{n+1}^G]$. Then $\frac{1-F(x)}{1-G(x)} \le \frac{1-F(\mu_n^G)}{1-G(\mu_{n+1}^G)} \le$

$$\left[1 - F\left(\mu_n^F(A^{-1} - \varepsilon)\right)\right](n+1) = \frac{1 - F\left(\mu_n^F(A^{-1} - \varepsilon)\right)}{1 - F(\mu_n^F)}\left[(n+1)\left(1 - F(\mu_n^F)\right)\right] \to (A^{-1} - \varepsilon)^{-\alpha},$$

$as\ n \to \infty$, by Theorem 1.4.2 (type *II*).

Therefore $\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} \le (A^{-1} - \varepsilon)^{-\alpha}$, similarly $\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} \ge (A^{-1} + \varepsilon)^{-\alpha}$, and since ε is arbitrary $\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} = A^\alpha$.

Result 1.6.9. *F(x)* and *G(x)* are distribution functions and $\Psi_\alpha(x)$ is an extreme value distribution. Suppose $F^n(a_n x + b_n) \to \Psi_\alpha(x), for\ \ a_n > 0, b_n\epsilon R,\ n \ge 1$.

Then $G^n(a_n x + b_n) \to\ \Psi_\alpha(Ax + B)\ and\ A > 0\ \ iff\ B = 0, x_0^F = x_0^G = x_0\ and$

$$\lim_{x\to x_0^-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}.$$

Proof: if $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = A^{-\alpha}$, then by Theorem 1.6.6 we have $G^n(a_n x + b_n) \to \{\Psi_\alpha(x)\}^{A^\alpha} = \Psi_\alpha(Ax)$, and from remark 1.6.7. (iii) Then $\{\Psi_\alpha(x)\}^{A^\alpha} = \Psi_\alpha\left((A^\alpha)^{\alpha^{-1}} x\right) = \Psi_\alpha(Ax)$.

For the converse, we can suppose $a_n = x_0^F - \mu_n^F$, $b_n = x_0^F$, so that we are given that

$G^n\left(\frac{a_n}{A} x + b_n - a_n \frac{B}{A}\right) \to \Psi_\alpha(x)\ as\ n \to \infty$, then

$$G^n\left((x_0^F - \mu_n^F)A^{-1}x + x_0^F - (x_0^F - \mu_n^F)A^{-1}B\right) \longrightarrow \Psi_\alpha(x), \tag{1.6.6}$$

Then from (1.6.6) we obtain $a_n = (x_0^F - \mu_n^F)A^{-1}$, $b_n = x_0^F - (x_0^F - \mu_n^F)A^{-1}B$.

This means that $G(x)\epsilon\Psi_\alpha(x)$. Therefore $x_0^G < \infty\ and$

$$G^n\left((x_0^G - \mu_n^G)x + x_0^G\right) \to \Psi_\alpha(x)\ \ as\ n \to \infty, \tag{1.6.7}$$

Then from (1.6.7) we obtain $\alpha_n = x_0^G - \mu_n^G$, $\beta_n = x_0^G$, and therefore by Theorem 1.6.4

$$\frac{\alpha_n}{a_n} \to 1 \Longrightarrow \frac{x_0^G - \mu_n^G}{(x_0^F - \mu_n^F)A^{-1}} \to 1 \Longrightarrow \frac{x_0^F - \mu_n^F}{x_0^G - \mu_n^G} \longrightarrow A, \tag{1.6.8}$$

and

$$\frac{\beta_n - b_n}{\alpha_n} \to 0 \Longrightarrow \frac{x_0^G - (x_0^F - (x_0^F - \mu_n^F)A^{-1}B)}{x_0^G - \mu_n^G} \longrightarrow 0, \qquad (1.6.9) \qquad as\ n \to \infty.$$

Combining (1.6.8) and (1.6.9) we have that $\frac{x_0^G - \left(x_0^F - A(x_0^G - \mu_n^G)A^{-1}B\right)}{x_0^G - \mu_n^G} \longrightarrow 0$,
then $\frac{x_0^G - x_0^F}{x_0^G - \mu_n^G} + B \longrightarrow 0$, and since $x_0^G - \mu_n^G \longrightarrow 0$, we have $x_0^G = x_0^F = x_0$ and $B = 0$.

From (1.6.8) for any $\varepsilon > 0$, there exists N_ε such that for $n > N_\varepsilon$, we have

$$\left|\frac{x_0 - \mu_n^G}{x_0 - \mu_n^F} - A^{-1}\right| < \varepsilon, \quad -\varepsilon < \frac{x_0 - \mu_n^G}{x_0 - \mu_n^F} - A^{-1} < \varepsilon,$$

$$i.e., x_0 - (x_0 - \mu_n^F)(A^{-1} + \varepsilon) < \mu_n^G < x_0 - (A^{-1} - \varepsilon)(x_0 - \mu_n^F).$$

For any $x < x_0$ but sufficiently close to x_0, there exists an integer $n = n(x)$ such that
$x\ \epsilon\ [\mu_n^G, \mu_{n+1}^G]$. Then $\frac{1-F(x)}{1-G(x)} \leq \frac{1-F(\mu_n^G)}{1-G(\mu_{n+1}^G)} \leq$

$$\begin{aligned}&\left[1 - F\left(x_0 - \ (x_0 - \mu_n^F)\ (A^{-1} + \varepsilon)\right)\right](n+1)\\ &\qquad\qquad = \frac{1 - F\left(x_0 - \ (x_0 - \mu_n^F)\ (A^{-1} + \varepsilon)\right)}{1 - F(x_0 - \ (x_0 - \mu_n^F)\ )}\left[(n+1)\left(1 - F(x_0 - \ (x_0 - \mu_n^F)\ )\right)\right]\\ &\qquad\qquad \to (A^{-1} + \varepsilon)^{\alpha},\end{aligned}$$

$as\ n \to \infty,$ by Theorem 1.4.2 (type *III*).

Therefore $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} \le (A^{-1} + \varepsilon)^{\alpha}$. Similarly $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} \ge (A^{-1} - \varepsilon)^{\alpha}$, and since ε is arbitrary $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} = A^{-\alpha}$.

Corollary 1.6.10. Let *F(x)* and *G(x)* are distribution functions.

(i) If $F^n(\mu_n^F x) \to \Phi_\alpha(x)\ and\ \frac{\mu_n^G}{\mu_n^F} \to A^{-1}$ then $\lim_{x \to \infty} \frac{1-F(x)}{1-G(x)} = A^{\alpha}$.

(ii) If $x_0^F = x_0^G < \infty, F^n\left((x_0 - \mu_n^F)x + x_0\right) \to \Psi_\alpha(x),\ and\ \lim_{n \to \infty} \frac{x_0 - \mu_n^G}{x_0 - \mu_n^F} = A^{-1}$,

$$then\ \ \lim_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}.$$

Result 1.6.11. *F(x)* and *G(x)* are distribution functions. Suppose $F^n(a_n x + b_n) \to \Phi_\alpha(x), for\ \ a_n > 0, b_n \epsilon R,\ n \ge 1.$

If $G^n(a_n x + b_n) \to \Phi(x), \Phi(x)$ non-degenerate, then $\Phi(x) = \Phi_\alpha(Ax)$ for some $A > 0$ and

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^{\alpha}.$$

Proof: To show $\beta = \alpha$: we have $G^n(A^{-1}\mu_n^F x) \to \Phi_\beta(x)$ and $G^n(\mu_n^G x) \to \Phi_\beta(x)$ so that by Theorem 1.6.4, $\frac{\mu_n^G}{\mu_n^F} \to A^{-1}$. By corollary 1.6.10. (i), $\lim_{x \to \infty} \frac{1-F(x)}{1-G(x)} = A^{\alpha}$,

But $n\{1 - F(\mu_n^F x)\} \to -\ln \Phi_\alpha(x) = x^{-\alpha}$, (1.6.10)

and $n\{1 - G(\mu_n^F x)\} \to -\ln \Phi_\beta(Ax) = (Ax)^{-\beta}$, (1.6.11)

$as\ n \to \infty$. Dividing gives

$$\lim_{n \to \infty} \frac{1 - F(\mu_n^F x)}{1 - G(\mu_n^F x)} = \lim_{n \to \infty} \frac{x^{-\alpha}}{(Ax)^{-\beta}} = A^{\beta}.x^{\beta - \alpha}.$$

Since $\lim_{x \to \infty} \frac{1-F(x)}{1-G(x)} = \lim_{x \to \infty} A^{\beta}.x^{\beta - \alpha} = A^{\alpha}\ and\ \mu_n^F \to \infty$, suppose $\beta = \alpha$.

Result 1.6.12. *F(x)* and *G(x)* are distribution functions. Suppose
$F^n(a_n x + b_n) \to \Psi_\alpha(x), for \;\; a_n > 0, b_n \epsilon R,\; n \geq 1.$

If $\;G^n(a_n x + b_n) \to \;\Phi(x), \Phi(x)$ non-degenerate, then $\Phi(x) = \Psi_\alpha(Ax)$ for some $A > 0$,

$x_0^F = x_0^G = x_0,$ and $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} = A^{-\alpha}$.

Proof: To show $\beta = \alpha$: we have $G^n\big((x_0 - \mu_n^F)A^{-1}x + x_0\big) \to \;\Psi_\beta(x),$

$G^n\big((x_0 - \mu_n^G)x + x_0\big) \to \;\Psi_\beta(x),$ so that by Theorem 1.6.4, $\lim_{n\to\infty} \frac{x_0 - \mu_n^G}{x_0 - \mu_n^F} = A^{-1}$.

By corollary 1.6.10. (ii), $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} = A^{-\alpha}$.

Also, $n\{1 - F((x_0 - \mu_n^F)x + x_0)\} \to -\ln \Psi_\alpha(x) = (-x)^\alpha$, (1.6.12)

and $n\{1 - G((x_0 - \mu_n^F)x + x_0)\} \to -\ln \Psi_\beta(Ax) = (-Ax)^\beta$, (1.6.13)

$as\; n \to \infty$. Dividing gives

$$\lim_{n\to\infty} \frac{1 - F((x_0 - \mu_n^F)x + x_0)}{1 - G((x_0 - \mu_n^F)x + x_0)} = \lim_{n\to\infty} \frac{(-x)^\alpha}{(-Ax)^\beta} = A^{-\beta}.x^{\alpha-\beta}.$$

Since $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} = \lim_{x \to x_0^-} A^{-\beta}.x^{\alpha-\beta} = A^{-\alpha}$, for all $x < 0 \;\; and\; \mu_n^F \to \infty$, assume $\beta = \alpha$.

Corollary 1.6.13. Let *F(x)* and *G(x)* are distribution functions. Suppose there exist

$a_n > 0, b_n \epsilon R,\; n \geq 1$, such that $F^n(a_n x + b_n) \to \Lambda(x)$. If $G^n(a_n x + b_n) \to \;\Phi(x),$

$\Phi(x)$ non-degenerate, then $\Phi(x) = \Lambda(Ax + B)$, for some $A > 0, B$.

Result 1.6.14. *F(x)* and *G(x)* are distribution functions. Suppose
$F^n(a_n x + b_n) \to \Lambda(x), for \;\; a_n > 0, b_n \epsilon R,\; n \geq 1$. Then $G^n(a_n x + b_n) \to \;\Lambda(Ax + B),$

and $A > 0$ iff $A = 1, x_0^F = x_0^G = x_0,$ and $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)} = e^B$.

Theorem 1.6.15. Let *F(x)* and *G(x)* are distribution functions and let $\Phi(x)$ be an extreme value distribution. Suppose *F(x)* ϵ $\Phi(x)$ and that $F^n(a_n x + b_n) \to \;\Phi(x)$, for normalizing constants

$a_n > 0, b_n \epsilon R,\ n \geq 1$. Then $G^n(a_n x + b_n) \to \Phi'(\mathrm{x}),\ \Phi'(\mathrm{x})$ non-degenerate, iff for some $A > 0, B$: $\Phi'(\mathrm{x}) = \Phi(Ax + B)$, $x_0^F = x_0^G = x_0$, $\lim_{x \to x_0^-} \frac{1-F(x)}{1-G(x)}$ exists, and if

(i) $\Phi(x) = \Phi_\alpha(x)$, then $B = 0$ and $\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} = A^\alpha$;

(ii) $\Phi(x) = \Psi_\alpha(x)$, then $B = 0$ and $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = A^{-\alpha}$;

(iii) $\Phi(x) = \Lambda(x)$, then $A = 1$ and $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = e^B$.

Theorem 1.6.16. *F(x)* and *G(x)* are distribution functions and $\Phi(x)$ be an extreme value distribution. Suppose $F^n(a_n x + b_n) \to \Phi(x)$, for normalizing constants

$a_n > 0, b_n \epsilon R,\ n \geq 1$. Then $(FG)^n(a_n x + b_n) = F^n(a_n x + b_n) G^n(a_n x + b_n) \to \Phi(Ax + B)$,

Iff

(i) $\Phi(x) = \Phi_\alpha(x)$: $B = 0, 0 < A \leq 1$, and $\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} = (A^{-\alpha} - 1)^{-1}$.

(ii) $\Phi(x) = \Psi_\alpha(x)$: $B = 0,\ 1 \leq A < \infty$, and $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = (A^\alpha - 1)^{-1}$.

(iii) $\Phi(x) = \Lambda(x)$: $A = 1, B < 0$ and, $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = \left(e^{-\beta} - 1\right)^{-1}$.

Proof (i) by Theorem 1.6.15. Replacing $G(x)\ by\ FG(x)$ we have that *B=0* and

$\lim_{x\to\infty} \frac{1-F(x)}{1-FG(x)} = A^\alpha$. For $x > 0$,

$$F^n(a_n x + b_n).G^n(a_n x + b_n) \to \Phi_\alpha(Ax), \qquad (1.6.14)$$

and $$F^n(a_n x + b_n) \to \Phi_\alpha(x)\,, \qquad (1.6.15),$$

so that, since $(FG)^n(a_n x + b_n) \leq F^n(a_n x + b_n)$, we have $\Phi_\alpha(Ax) \leq \Phi_\alpha(x)$. Therefore, $Ax \leq x$ and $A \leq 1$. Also for $x > 0$,

Dividing gives $\frac{F^n(a_n x+b_n).G^n(a_n x+b_n)}{F^n(a_n x+b_n)} \to \frac{\Phi_\alpha(Ax)}{\Phi_\alpha(x)}$ $then\ G^n(a_n x + b_n) \to \frac{\Phi_\alpha(Ax)}{\Phi_\alpha(x)} =$

$\Phi_\alpha\left((A^{-\alpha} - 1)^{-\alpha^{-1}} x\right)$, and by Theorem 1.6.15. (i) we have $\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} = (A^{-\alpha} - 1)^{-1}$.

Proof (ii) by Theorem 1.6.15. Replacing $G(x)\ by\ FG(x)$ we have that *B=0* and

$\lim_{x\to x_0^-} \frac{1-F(x)}{1-FG(x)} = A^{-\alpha}$. For $x > 0$,

$$F^n(a_n x + b_n).G^n(a_n x + b_n) \to \Psi_\alpha(Ax), \qquad (1.6.16)$$

and $\qquad F^n(a_n x + b_n) \to \Psi_\alpha(x),$ (1.6.17),

so that, since $(FG)^n(a_n x + b_n) \le F^n(a_n x + b_n)$, so that $\Psi_\alpha(Ax) \le \Psi_\alpha(x)$ and for $x < 0$, $Ax \le x$ so that $A \ge 1$.

Dividing gives $\frac{F^n(a_n x+b_n).G^n(a_n x+b_n)}{F^n(a_n x+b_n)} \to \frac{\Psi_\alpha(Ax)}{\Psi_\alpha(x)}$ $then\ G^n(a_n x + b_n) \to \frac{\Psi_\alpha(Ax)}{\Psi_\alpha(x)} =$

$\Psi_\alpha\left((A^\alpha - 1)^{\alpha^{-1}} x\right)$, and by Theorem 1.6.15.(ii) we have $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = (A^\alpha - 1)^{-1}$.

Proof (iii) by Theorem 1.6.15. Replacing $G(x)\ by\ FG(x)$ we have that *A=1* and

$\lim_{x\to x_0^-} \frac{1-F(x)}{1-FG(x)} = e^B$. For $x > 0$,

$$F^n(a_n x + b_n). G^n(a_n x + b_n) \to \Lambda(x + B), \qquad (1.6.18)$$

and $\qquad F^n(a_n x + b_n) \to \Lambda(x)$, (1.6.19),

so that, since $(FG)^n(a_n x + b_n) \le F^n(a_n x + b_n)$, we have $\Lambda(x + B) \le \Lambda(x)$. Therefore $x + B \le x$ and $B \le 0$.

Dividing gives $\frac{F^n(a_n x+b_n).G^n(a_n x+b_n)}{F^n(a_n x+b_n)} \to \frac{\Lambda(x+B)}{\Lambda(x)}$ $then\ G^n(a_n x + b_n) \to \frac{\Lambda(x+B)}{\Lambda(x)} =$

$\Lambda(x - \ln(e^{-B} - 1))$, and by Theorem 1.6.15. (iii) we have $\lim_{x\to x_0^-} \frac{1-F(x)}{1-G(x)} = \left(e^{-\beta} - 1\right)^{-1}$.

Chapter2

STABLE DISTRIBUTIONS ON THE REAL LINE

2. INTRODUCTION

Stable distributions are a rich class of probability distributions that allow skewness and heavy tails and have many mathematical properties. In probability theory, a random variable is said to be stable distributed if it has the property that a linear combination of two independent copies of the variable has the same distribution. The stable distribution family is also sometimes referred to as the Levy alpha-stable distribution. The general stable distribution requires four parameters for complete description: $S_\alpha(\sigma, \beta, \mu)$, where $\alpha \epsilon (0,2]$ is an index of stability and also called the tail index, tail exponent or characteristic exponent, a skewness parameter $\beta \epsilon [-1,1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \epsilon R$.
And all graphics are made in XTREMES program [23].

The figure (2.1) below shows Probability Density Function when

$(\alpha = 2,1.5,1,0.5, \beta = 0, \sigma = 1, \mu = 0)$.

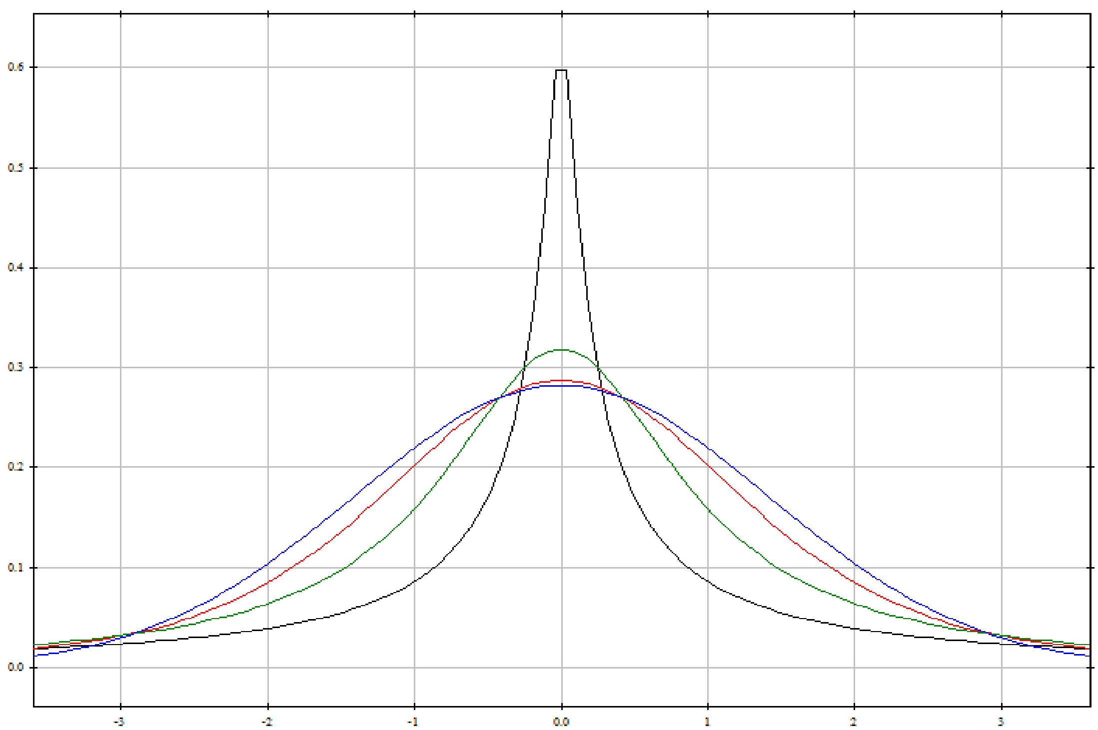

Figure (2.1): *Probability Density Function when* $(\alpha = 2,1.5,1,0.5, \beta = 0, \sigma = 1, \mu = 0)$.

The figure below shows Distribution Function when $(\alpha = 2,1.5,1,0.5, \beta = 0, \sigma = 1, \mu = 0)$.

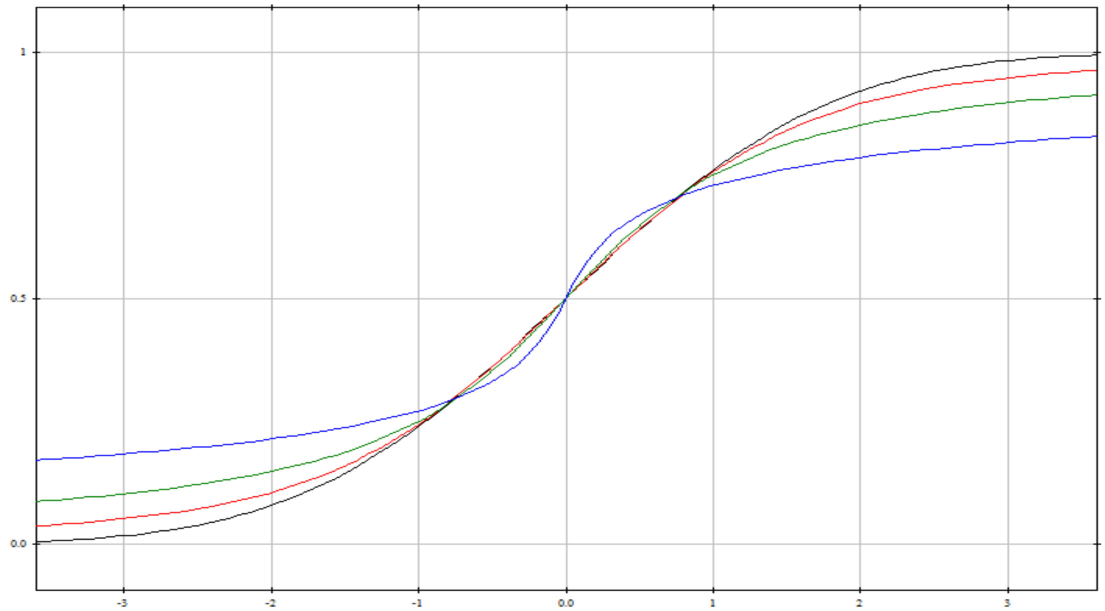

Figure (2.2): *Distribution Function when* $(\alpha = 2,1.5,1,0.5, \beta = 0, \sigma = 1, \mu = 0)$.

The following Theorems and Definitions from [22], [10].

2.1 Definitions

Here we give four equivalent definitions of a stable distribution.

The first two definitions explain why these distributions are called stable, and the third definition related it with the central limit theorem, the fourth definition specifies the characteristic function of a stable random variable.

Definition 2.1.1 A random variable *X* is called a stable distribution if for any positive numbers *A* and *B*, there is a positive number *C* and a real number *D* such that

$$AX_1 + BX_2 =^d CX + D, \qquad (2.1.1),$$

where X_1 and X_2 are independent copies of *X*, and where " $=^d$ " denotes equality in distribution.

Remark (i) If equation (2.1.1) holds for *D=0*, then it is called strictly stable.

(ii) If $X =^d - X$, then it is called symmetric stable.

Theorem 2.1.2. For any stable random variable *X*, there is a number $\alpha \epsilon (0,2]$ such that the number *C* in (2.1.1) satisfies

$$C^{\alpha} = A^{\alpha} + B^{\alpha}, \qquad (2.1.2)$$

where α is called the index of stability or characteristic exponent.

Example 2.1.3. if *X* is a Gaussian random variable with mean μ and variance σ^2

$(X \sim N(\mu, \sigma^2))$, then *X* is stable with $\alpha = 2$, because

$AX_1 + BX_2 \sim N\big((A+B)\mu, (A^2+B^2)\,\sigma^2\big)$, i.e., (1.1.1) holds with

$$C = \sqrt{A^2 + B^2} \; and \; D = (A + B - C)\mu\,.$$

Definition 2.1.4 (equivalent to definition 2.1.1). A random variable *X* is called a stable distribution if for any $n \geq 2$, there is a positive number C_n and a real number D_n such that

$$X_1 + X_2 + \cdots + X_n =^d C_n X + D_n, \qquad (2.1.3)$$

where X_i are independent copies of X..

Remark (i) The first definition displays continuous combinations of two independent identically distributed random variables, while the second definition displays the sum of any number of independent identically distributed random variables.
(ii) If equation (2.1.3) holds, then $C_n = n^{\frac{1}{\alpha}}$, for some $\alpha \epsilon (0,2]$.

Definition 2.1.5 (Equivalent to definitions 2.1.1and 2.1.4). A random variable *X* is called a stable distribution if it has a domain of attraction, i.e., if there exists a sequence of independent identically distributed random variables $Y_1, Y_2, \ldots,$ and sequences of positive numbers d_n and real numbers a_n such that

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{d_n} + a_n \to^d X, \qquad (2.1.4)$$

where $\to^d$ denotes convergence in distribution.

Remark (i) If X is Gaussian, and Y_i are independent identically distributed (i.i.d.) with finite variance, then equation (2.1.4) is just the central limit theorem.
(ii) When $d_n = n^{\frac{1}{\alpha}}$, Y is said to belong to the "normal" domain of attraction X.
Generally, $d_n = n^{\frac{1}{\alpha}} L(n)$, where $L(x)$, $x > 0$, is a slowly varying function at infinity, that is, $\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1,\ for\ all\ x > 0$. The function $L(x) = \ln x$, for example, is slowly varying at infinity (see examples 1.2.5, p. 18 in the first chapter).

Definition 2.1.6 (equivalent to definitions 2.1.1, 2.1.4 and 2.1.5). A random variable X is called a stable distribution if there exists, $0 < \alpha \leq 2,\ \ \sigma \geq 0,\ -1 \leq \beta \leq 1,\ \mu$ is a real number such that the characteristic function of stable distribution has the following form:

$$Eexp(i\theta X) = \begin{cases} \exp\{-\sigma^\alpha |\theta|^\alpha (1 - i\beta(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu\theta\} & if\ \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu\theta\} & if\ \alpha = 1, \end{cases} \tag{2.1.5}$$

and

$$sign\ \theta = \begin{cases} 1 & if\ \ \theta > 0, \\ 0 & if\ \ \theta = 0, \\ -1 & if\ \theta < 0. \end{cases}$$

Remark 2.1.7. Since (2.1.5) is characterized by four parameters, $\alpha\epsilon(0,2]$, $\sigma \geq 0, \beta\epsilon[-1,1], \mu\epsilon R$, we will denote stable distributions by $S_\alpha(\sigma, \beta, \mu)$ and write $X \sim S_\alpha(\sigma, \beta, \mu)$.

Remark 2.1.8. When $\alpha = 2$, the characteristic function (2.1.5) becomes $Eexp(i\theta X) = exp(i\mu\theta - \sigma^2\theta^2)$. This is the characteristic function of a Gaussian random variable with mean μ and variance $2\sigma^2$.

Remark 2.1.9. There are only three special cases in which a closed form expression is known for a stable distributions probability density function. These are the Gaussian case ($\alpha = 2, \beta = 0$), Cauchy case ($\alpha = 1, \beta = 0$), and Levy case ($\alpha = 0.5, \beta = \pm 1$) with the following densities:

(1) The Gaussian distribution $S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$, whose density is

$$f(x) = \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(x-\mu)^2}{4\sigma^2}}, \quad -\infty < x < \infty.$$

The distribution function, for which there is no closed form expression, is

$F(x) = P(X \le x) = \Phi\left(\frac{(x-\mu)}{\sigma}\right)$, where $\Phi(Z)$ = Probability that a standard normal random variable is less than or equal Z.

(2) The Cauchy distribution $S_1(\sigma, 0, \mu)$, whose density is

$$f(x) = \frac{\sigma}{\pi((x-\mu)^2 + \sigma^2)}, \quad -\infty < x < \infty.$$

(3) The Levy distribution $S_{0.5}(\sigma, 1, \mu)$, whose density is

$$f(x) = \frac{\sqrt{\sigma}}{\sqrt{2\pi}(x-\mu)^{\frac{3}{2}}} e^{-\frac{\sigma}{2(x-\mu)}}, \quad \mu < x < \infty.$$

The figure below shows graphics of Probability density functions for Gaussian when ($\alpha = 2, \beta = 0$), Cauchy when ($\alpha = 1, \beta = 0$), and Levy when ($\alpha = 0.5, \beta = \pm 1$), respectively from left to right.

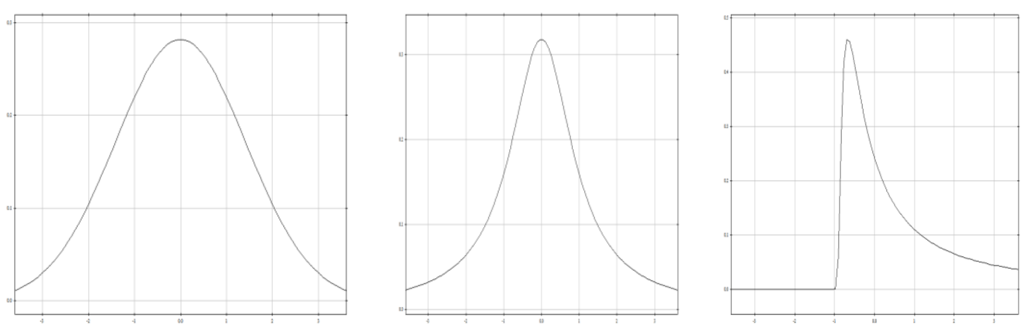

Figure (2.3): *Probability density functions for Gaussian when* ($\alpha = 2, \beta = 0$), *Cauchy when* ($\alpha = 1, \beta = 0$), *and Levy when* ($\alpha = 0.5, \beta = \pm 1$), *respectively from left to right.*

The figure below shows graphics of Probability density functions for Gaussian when ($\alpha = 2, \beta = 0$) (black line), Cauchy when ($\alpha = 1, \beta = 0$) (red line), Levy when ($\alpha = 0.5, \beta = \pm 1$) (green line).

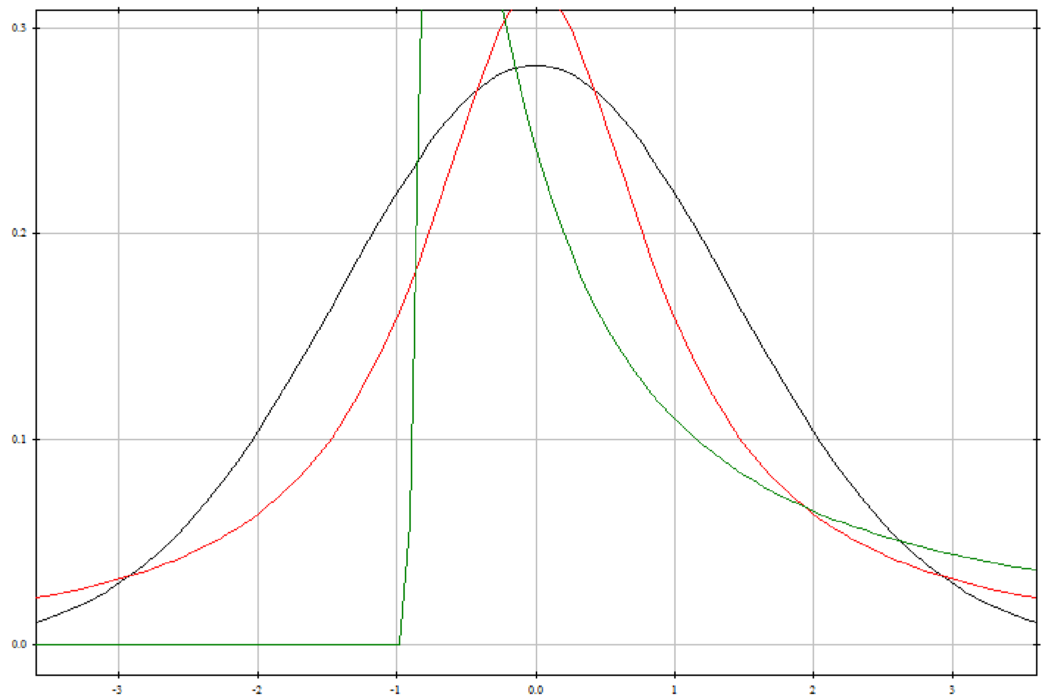

Figure (2.4): *Probability density functions for Gaussian when* ($\alpha = 2, \beta = 0$) *(black line), Cauchy when* ($\alpha = 1, \beta = 0$) *(red line), Levy when* ($\alpha = 0.5, \beta = \pm 1$) *(green line).*

The figure below shows Stable densities in the $S_\alpha(1,0,0)$, parameterization,($\alpha = 1, 1.5, 1.8, 1.95, 2$).

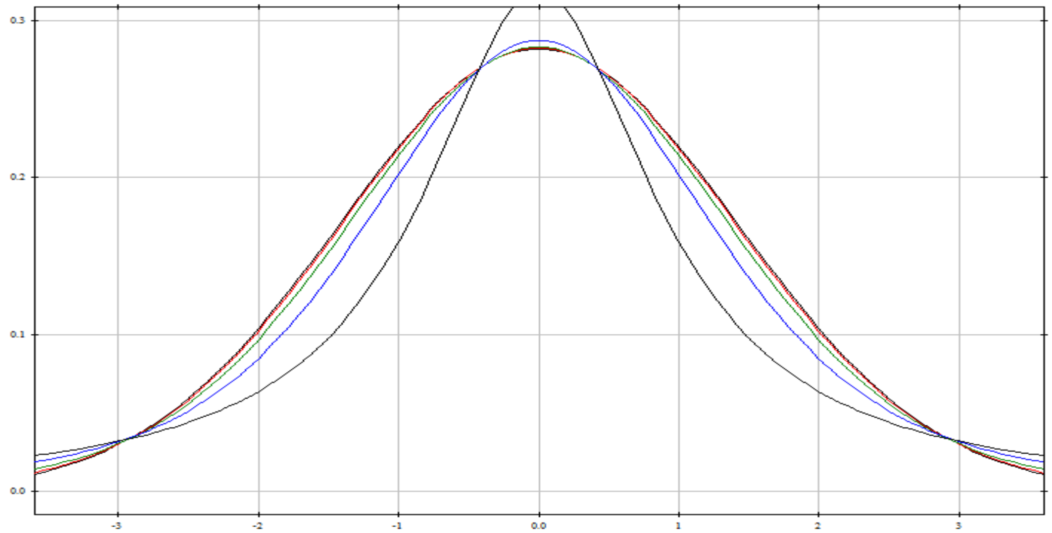

Figure (2.5): *Stable densities in the* $S_\alpha(1,0,0)$, *parameterization,* ($\alpha = 1, 1.5, 1.8, 1.95, 2$).

The figure below shows Stable densities in the $S_{0.8}(1, \beta, 0)$, parameterization,

$(\beta = -1, -0.8, -0.5, 0, 0.5, 0.8,\ and\ 1)$.

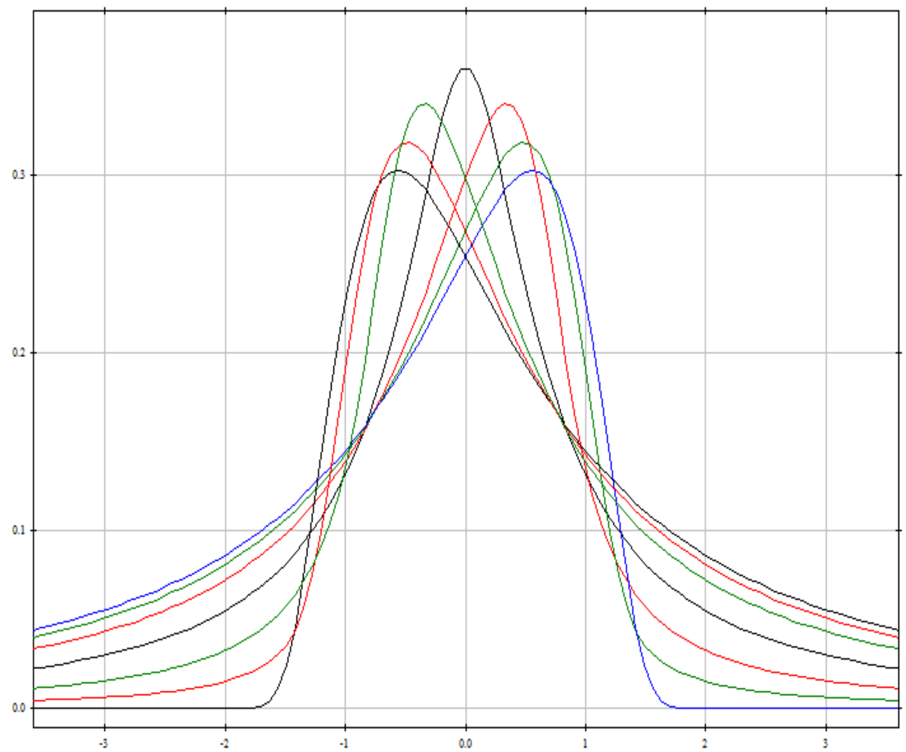

Figure (2.6): *Stable densities in the* $S_{0.8}(1, \beta, 0)$, *parameterization*,

$(\ \beta = -1, -0.8, -0.5, 0, 0.5, 0.8,\ and\ 1)$.

The figure below shows Stable densities in the $S_\alpha(1,0.5,0)$, parameterization,

$(\alpha = 0.5, 0.75, 1, 1.25, 1.5)$

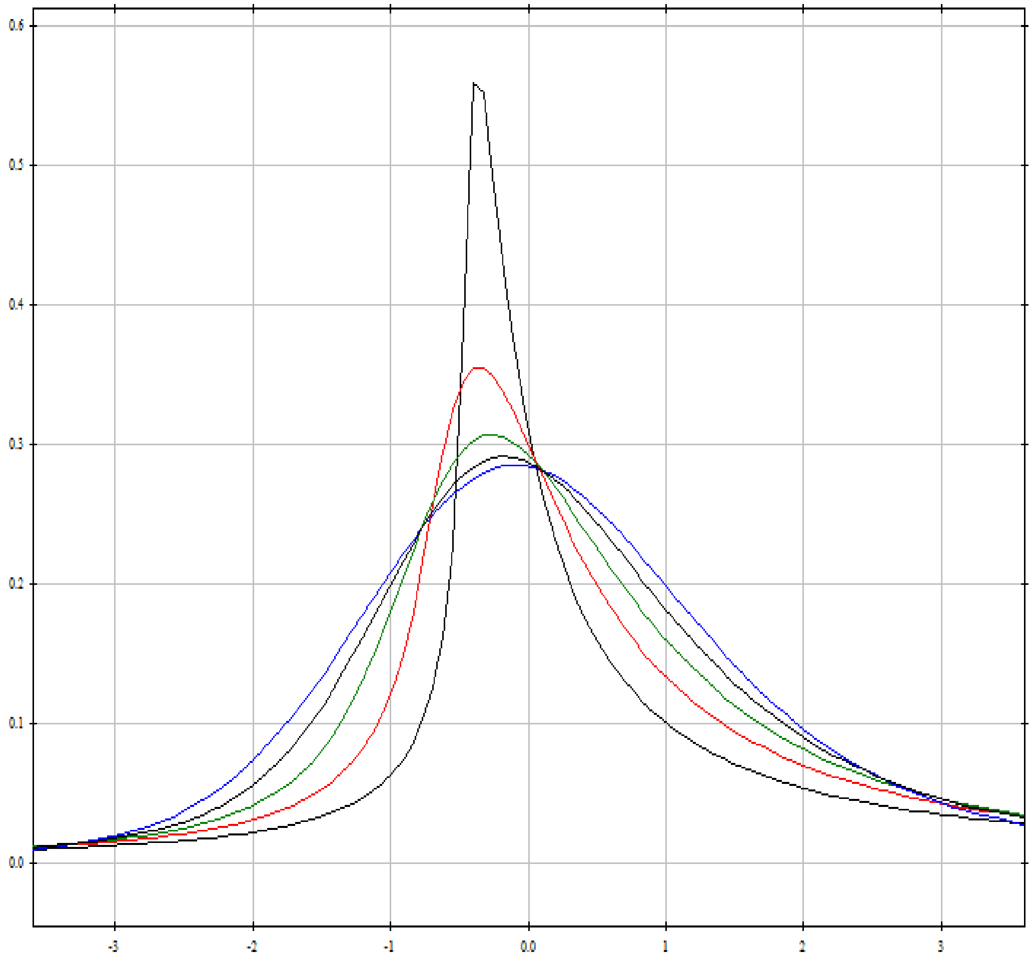

Figure (2.7): *Stable densities in the* $S_\alpha(1,0.5,0)$, *parameterization,*

$(\alpha = 0.5, 0.75, 1, 1.25, 1.5)$.

The following properties from [22].

2.2 Properties of stable random variables:

Property 2.2.1. Let $X_1 and\ X_2$ be independent random variables with $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$,

$$i = 1,2.\ then\ X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu), with$$

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{\frac{1}{\alpha}}, \qquad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \qquad \mu = \mu_1 + \mu_2.$$

Proof: Use equation (2.1.5) and first we verify this for $\alpha \neq 1$. By independence,

$$\ln Eexpi\theta(X_1 + X_2) = \ln(Eexpi\theta X_1) + \ln(Eexpi\theta X_2),$$

$$\ln(Eexpi\theta X_1) = -\sigma_1^\alpha |\theta|^\alpha (1 - i\beta_1(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu_1\theta, \qquad (2.2.1)$$

$$\ln(Eexpi\theta X_2) = -\sigma_2^\alpha |\theta|^\alpha (1 - i\beta_2(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu_2\theta, \qquad (2.2.2)$$

equation (2.2.1) + (2.2.2), then we get

$$= -(\sigma_1^\alpha + \sigma_2^\alpha)\,|\theta|^\alpha + i|\theta|^\alpha(\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha)\ sign\theta\tan\frac{\pi\alpha}{2} + i\theta(\mu_1 + \mu_2),$$

$$= -(\sigma_1^\alpha + \sigma_2^\alpha)\,|\theta|^\alpha \left\{1 - i\frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} sign\theta\tan\frac{\pi\alpha}{2}\right\} + i\theta(\mu_1 + \mu_2),$$

then, $\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{\frac{1}{\alpha}}$, $\beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}$, $\mu = \mu_1 + \mu_2$.

Second we verify for $\alpha = 1$. By independence,

$$\ln(Eexpi\theta X_1) = -\sigma_1|\theta|(1 + i\beta_1\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu_1\theta, \qquad (2.2.3)$$

$$\ln(Eexpi\theta X_2) = -\sigma_2|\theta|(1 + i\beta_2\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu_2\theta, \qquad (2.2.4)$$

then (2.2.3) + (2.2.4), we get

$$= -(\sigma_1 + \sigma_2)|\theta| \left\{1 + i\frac{\beta_1\sigma_1 + \beta_2\sigma_2}{\sigma_1 + \sigma_2}\frac{2}{\pi}(sign\theta)\ln|\theta|\right\} + i\theta(\mu_1 + \mu_2),$$

then, $\sigma = \sigma_1 + \sigma_2$, $\beta = \frac{\beta_1\sigma_1 + \beta_2\sigma_2}{\sigma_1 + \sigma_2}$, $\mu = \mu_1 + \mu_2$.

Property 2.2.2. Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and let (a) be a real constant. Then

$$X + a \sim S_\alpha(\sigma, \beta, \mu + a).$$

Proof: (i) If $\alpha \neq 1$, then

$$\ln Eexpi\theta(X + a) = \ln(Eexpi\theta X) + \ln(Eexpi\theta a),$$

But $\ln(Eexpi\theta X) = -\sigma^\alpha|\theta|^\alpha(1 - i\beta(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu\theta$, (2.2.5) and $\ln(Eexpi\theta a) = ia\theta$, (2.2.6)

because $Eexpi\theta a = E\left(e^{i\theta a}\right) = \sum_n P_n e^{i\theta a} = e^{i\theta a} \sum_n P_n = e^{i\theta a}.1 = e^{i\theta a}$.

Then (2.2.5) + (2.2.6), we get

$$\ln Eexpi\theta(X+a) = -\sigma^{\alpha}|\theta|^{\alpha}(1 - i\beta(sign\theta)\tan\frac{\pi\alpha}{2}) + i(\mu+a)\theta\ , \qquad if\ \ \alpha \neq 1.$$

(ii) If $\alpha = 1$, then

$$\ln Eexpi\theta(X+a) = -\sigma|\theta|(1 + i\beta\tfrac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu\theta + ia\theta,$$

$$\ln Eexpi\theta(X+a) = -\sigma|\theta|(1 + i\beta\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\theta(\mu+a), \qquad if\ \alpha = 1.$$

Then, $X + a \sim S_{\alpha}(\sigma, \beta, \mu + a)$.

Property 2.2.3. Let $X \sim S_{\alpha}(\sigma, \beta, \mu)$ and let (a) be a non-zero real constant. Then

$$aX \sim S_{\alpha}(|a|\sigma, sign(a)\beta, a\mu), \qquad if\ \alpha \neq 1\ ,$$

$$aX \sim S_1\left(|a|\sigma, sign(a)\beta, a\mu - \frac{2}{\pi}a(\ln|a|)\sigma\beta\right), \qquad if\ \alpha = 1.$$

Proof: (i) if $\alpha \neq 1$, then

$$\ln Eexpi\theta(aX) = -\sigma^{\alpha}|\theta a|^{\alpha}(1 - i\beta(sign(a\theta))\tan\frac{\pi\alpha}{2}) + i\mu(a\theta)\ ,$$

$$\ln Eexpi\theta(aX) = -(\sigma|a|)^{\alpha}|\theta|^{\alpha}(1 - i\beta(sign(a)sign(\theta))\tan\frac{\pi\alpha}{2}) + i(\mu a)\theta\ ,$$

then

$$aX \sim S_{\alpha}(|a|\sigma, sign(a)\beta, a\mu), \qquad if\ \alpha \neq 1\ .$$

(ii) If $\alpha = 1$, then

$$\ln Eexpi\theta(aX) = -\sigma|\theta a|(1 + i\beta\frac{2}{\pi}(sign(a\theta))\ln|a\theta|) + i\mu(a\theta),$$

$$\ln Eexpi\theta(aX) = -|\mathrm{a}|\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign(a)sign(\theta)\{\ln|a| + \ln|\theta|\}) + i\mu(a\theta),$$

$$= -|\mathrm{a}|\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign(a)sign(\theta)\ln|\theta|) + i\left(\mu a - \beta\frac{2}{\pi}|\mathrm{a}||\theta|\sigma.sign(a).\ln|a|\,sign(\theta)\right)\theta\ ,$$

then

$$aX \sim S_1\left(|a|\sigma, sign(a)\beta, a\mu - \frac{2}{\pi}|a|(\ln|a|)\sigma\beta . sign(a)\right), \qquad if\ \alpha = 1.$$

Property 2.2.4. For any $0 < \alpha < 2$,

$X \sim S_\alpha(\sigma, \beta, 0) \Leftrightarrow -X \sim S_\alpha(\sigma, -\beta, 0)$.

Proof :(i)

$$\ln EexpiθX = -\sigma^\alpha|\theta|^\alpha(1 - i\beta(sign(\theta))\tan\frac{\pi\alpha}{2}) + i\mu\theta ,$$

but $S_\alpha(\sigma, \beta, 0) = -\sigma^\alpha|\theta|^\alpha(1 - i\beta(sign(\theta))\tan\frac{\pi\alpha}{2})$, if $\alpha \neq 1$,

and $S_\alpha(\sigma, \beta, 0) = -\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign\theta \ln|\theta|)$, if $\alpha = 1$,

then $X \sim S_\alpha(\sigma, \beta, 0)$.

(ii) $S_\alpha(\sigma, -\beta, 0) = -\sigma^\alpha|\theta|^\alpha(1 + i\beta(sign\theta))\tan\frac{\pi\alpha}{2})$, if $\alpha \neq 1$,

$$= -\left\{\sigma^\alpha|\theta|^\alpha(1 + i\beta(sign\theta))\tan\frac{\pi\alpha}{2})\right\},$$

and $S_\alpha(\sigma, -\beta, 0) = -\sigma|\theta|(1 - i\beta\frac{2}{\pi}sign\theta \ln|\theta|)$, if $\alpha = 1$,

$$= -\left\{\sigma|\theta|(1 - i\beta\frac{2}{\pi}sign\theta \ln|\theta|)\right\},$$

then $-X \sim S_\alpha(\sigma, -\beta, 0)$,

from (i) and (ii) then we get $X \sim S_\alpha(\sigma, \beta, 0) \Leftrightarrow -X \sim S_\alpha(\sigma, -\beta, 0)$.

Remark in property 2.2.4. The distribution $S_\alpha(\sigma, \beta, 0)$ is said to be skewed to the right if $\beta > 0$ and to the left if $\beta < 0$. It is said to be totally skewed to the right if $\beta = 1$ and totally skewed to the left if $\beta = -1$.

Property 2.2.5. $X \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric if and only if $\beta = 0\ and\ \mu = 0$. It is symmetric about μ if and only if $\beta = 0$.

Proof: For a random variable to be symmetric, it is necessary and sufficient that its characteristic function be real. By (2.1.5), when $\beta = 0, \mu = 0$ then

$\ln Eexpi\theta X = -\sigma^\alpha|\theta|^\alpha$, if $\alpha \neq 1$,

$\ln Eexpi\theta X = -\sigma|\theta|$, if $\alpha = 1$.

Remark 2.2.6. Asymmetric stable random variable is strictly stable, but a strictly stable random variable is not necessarily symmetric.

The figure below shows Symmetric stable densities and distribution functions for $Z \sim S_\alpha(1,0,0)$, $\alpha = (0.7, 1.3, 1.9)$.

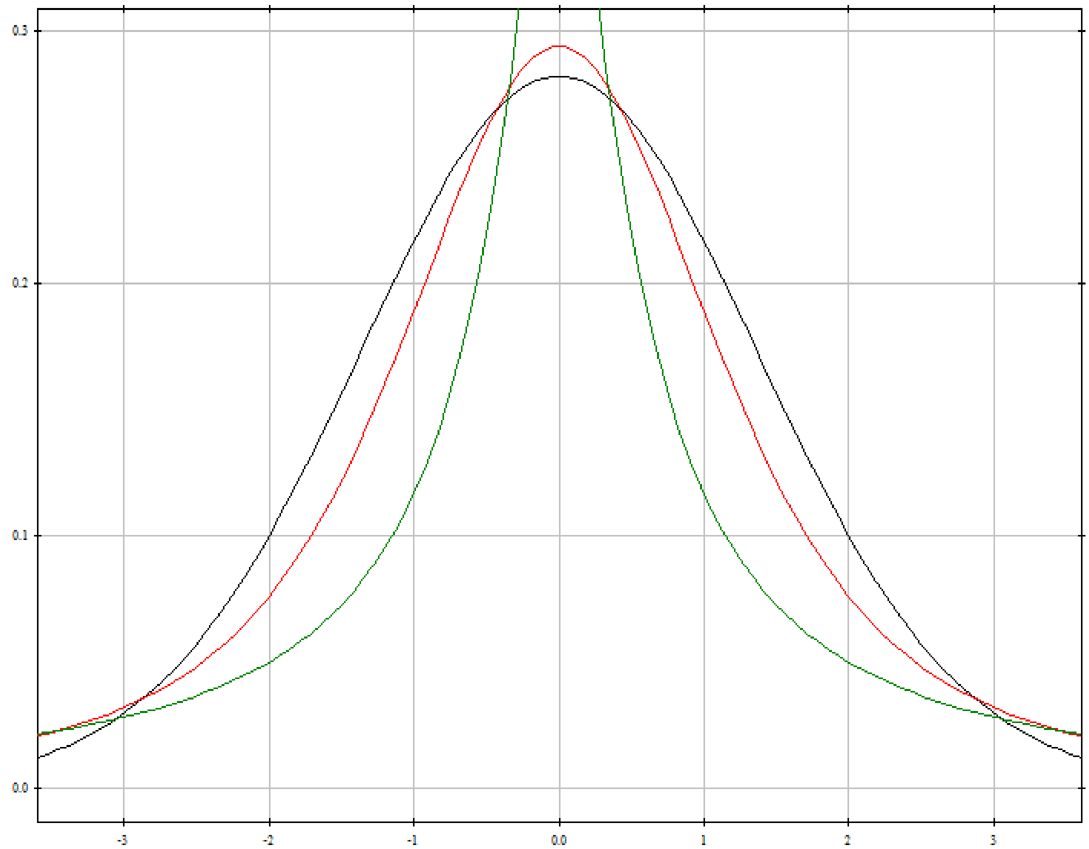

Figure (2.8): *Symmetric stable densities* for $Z \sim S_\alpha(1,0,0)$, $\alpha = (0.7, 1.3, 1.9)$.

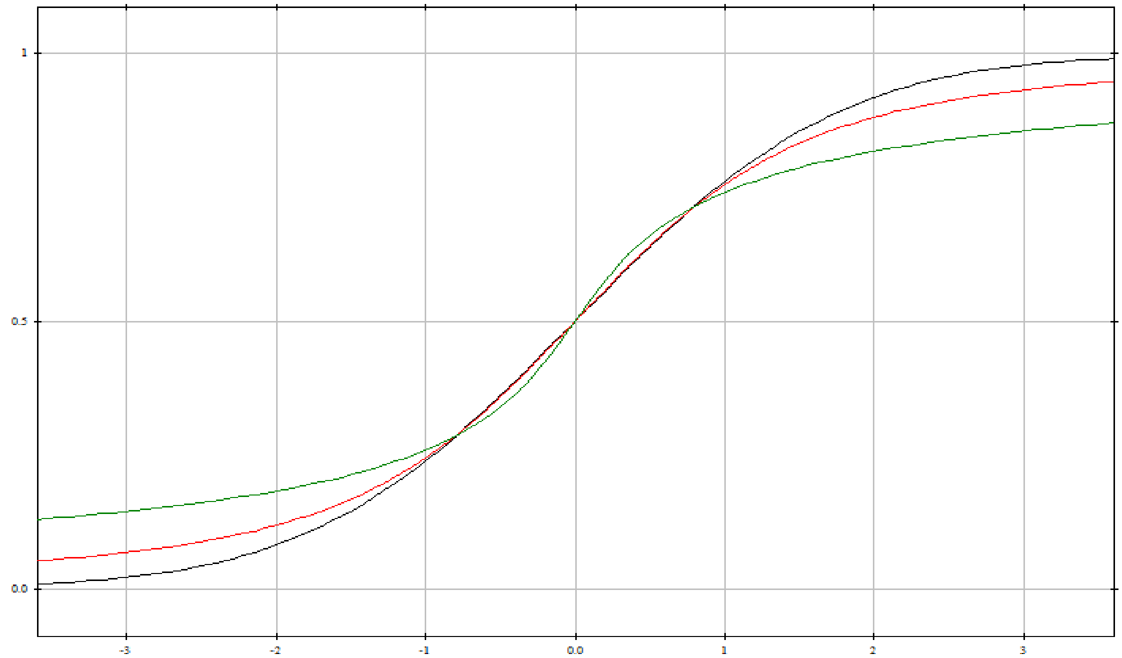

Figure (2.9): *Symmetric stable distribution functions* for $Z \sim S_\alpha(1,0,0)$, $\alpha = (0.7, 1.3, 1.9)$.

Property 2.2.7. Let $X \sim S_\alpha(\sigma, \beta, \mu)$, with $\alpha \neq 1$. Then X is strictly stable if and only if $\mu = 0$.
Proof: let X_1, X_2 be independent copies of X and let A and B be arbitrary positive constants.
By properties (2.2.1) and (2.2.3),
$AX_1 + BX_2 \sim S_\alpha\left(\sigma(A^\alpha + B^\alpha)^{\frac{1}{\alpha}}, \beta, \mu(A + B)\right)$. We must set $C = (A^\alpha + B^\alpha)^{\frac{1}{\alpha}}$ in (2.1.1) by properties (2.2.2) and (2.2.3),

$$CX + D \sim S_\alpha\left(\sigma(A^\alpha + B^\alpha)^{\frac{1}{\alpha}}, \beta, \mu(A^\alpha + B^\alpha)^{\frac{1}{\alpha}} + D\right),$$

and therefore, we have $AX_1 + BX_2 =^d CX + D \; with \; D = 0 \; iff \; \mu = 0$.

Corollary 2.2.8. Let $X \sim S_\alpha(\sigma, \beta, \mu)$, with $\alpha \neq 1$. Then $X - \mu$ is strictly stable.
Proof: use properties 2.2.2 and 2.2.7.

Remark 2.2.9. Thus, any alpha stable random variable with $\alpha \neq 1$ can be made strictly stable by shifting. This is not true when $\alpha = 1$.

Property 2.2.10. Let $X \sim S_\alpha(\sigma, \beta, \mu)$, with $\alpha = 1$. Then X is strictly stable if and only if $\beta = 0$.
Proof: let X_1, X_2 be independent copies of X and let $A > 0, B > 0$.
And use properties 2.2.3 and 2.2.1.

Corollary 2.2.11. If $X_1, X_2, \dots, X_n$ are independent identically distributed $S_\alpha(\sigma, \beta, \mu)$, then

$$X_1 + X_2 + \cdots + X_n =^d n^{\frac{1}{\alpha}} X_1 + \mu\left(n - n^{\frac{1}{\alpha}}\right), \qquad if\ \alpha \neq 1,$$

and

$$X_1 + X_2 + \cdots + X_n =^d nX_1 + \tfrac{2}{\pi}\sigma\beta, \qquad if\ \alpha = 1.$$

Remark 2.2.12. The random variable $X \sim S_\alpha(\sigma, 1{,}0)$ with $0 < \alpha < 1$ is called a stable subordinator.

Proposition 2.2.13. The "Laplace transform" $Ee^{-\gamma x}, \gamma \geq 0$, of the random variable $X \sim S_\alpha(\sigma, 1{,}0), 0 < \alpha \leq 2, \sigma \geq 0$, equals

$$Ee^{-\gamma x} = exp\left\{-\frac{\sigma^\alpha}{\cos\frac{\pi\alpha}{2}} . \gamma^\alpha\right\} \qquad if\ \ \alpha \neq 1,$$

and

$$Ee^{-\gamma x} = exp\left\{\sigma . \frac{2}{\pi}\gamma \ln \gamma\right\} \qquad if\ \ \alpha = 1.$$

Remark 2.2.14. The constant $-\sigma^\alpha\left(\cos\frac{\pi\alpha}{2}\right)^{-1}$ is negative if $0 < \alpha < 1$, and is positive if $, 1 < \alpha \leq 2$. It equals σ^2 when $\alpha = 2$.

Property 2.2.15. Let X have distribution $S_\alpha(\sigma, \beta, 0)$ with $\alpha < 2$. Then there exist two independent identically distributed (i.i.d.) random variables $Y_1\ and\ Y_2$ with common distribution $S_\alpha(\sigma, 1{,}0)$ such that

$$X =^d \left(\frac{1+\beta}{2}\right)^{\frac{1}{\alpha}} Y_1 - \left(\frac{1-\beta}{2}\right)^{\frac{1}{\alpha}} Y_2\,, \qquad if\ \alpha \neq 1,$$

and

$$X =^d \left(\frac{1+\beta}{2}\right) Y_1 - \left(\frac{1-\beta}{2}\right) Y_2 + \sigma\left(\frac{1+\beta}{\pi} \ln\frac{1+\beta}{2} - \frac{1-\beta}{\pi} \ln\frac{1-\beta}{2}\right), \qquad if\ \ \alpha = 1.$$

Proof use properties 2.2.1, 2.2.2 and 2.2.3. in [22].

Property 2.2.16. Stable distributions are infinitely divisible.

2.3 Overview in infinitely divisible:

Stable distributions have a long history in the subject of probability. They form a subset of the class of so-called "infinitely-divisible" distributions, a class of characteristic functions at the heart of general central limit theory.

The following definitions and Theorems from [11].

Definition 2.3.1. A distribution function $F(x)$ and the corresponding characteristic function $f(t)$ are said to be infinitely divisible if for every positive integer n there exist a characteristic function $f_n(t)$ such that $f(t) = (f_n(t))^n$ *then* $f_n(t) = \sqrt[n]{f(t)}$, (2.3.1)

Examples 2.3.2. Of infinitely divisible distributions include:

(i) The normal distribution with parameters (μ, σ^2) is infinitely divisible, because the characteristic function of the normal distribution has the form $f(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$, so that then for every positive integer *n* there exist a characteristic function $f_n(t)$ such that $f_n(t) = e^{i\frac{\mu}{n}t - \frac{1}{2}\left(\frac{\sigma}{\sqrt{n}}\right)^2 t^2}$ is the characteristic function of the normal distribution with parameters $\left(\frac{\mu}{n}, \frac{\sigma}{\sqrt{n}}\right)$.

(ii) The Poisson distribution with parameters (x, λ) is infinitely divisible, because the characteristic function of the Poisson distribution has the form $f(t) = e^{\lambda(e^{itx} - 1)}$, so that then for every positive integer *n* there exist a characteristic function $f_n(t)$ such that $f_n(t) = e^{\frac{\lambda}{n}(e^{itx} - 1)}$ is the characteristic function of the Poisson distribution with parameters $\left(x, \frac{\lambda}{n}\right)$.

(iii) Cauchy distribution and the "chi-squared" distribution.

Theorem 2.3.3. The characteristic function of an infinitely divisible distribution never vanishes.

Proof:

Give example: The discrete random variables taking the values -1, 0, 1, with probability $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$, its characteristic function?

$$f(t) = E\left(e^{itx}\right) = \sum_n P_n \, e^{itx_n} = \frac{3 + \cos t}{4},$$

where $e^{it} = \cos t + i \sin t$ *and* $e^{-it} = \cos t - i \sin t$,

then the result is positive and therefore does not vanish.

Before the construction of the general theory two basic elementary types of such random functions were known:

(i) The normal type then the characteristic function $f_n(t)$ is given by the formula

$$\log f_n(t) = \mathrm{n}\left(i\mu t - \frac{\sigma^2 t^2}{2}\right), \qquad (2.3.2)$$

(ii) The Poisson type then the characteristic function $f_n(t)$ is given by the formula

$$\log f_n(t) = \mathrm{n}\lambda\left(e^{itx} - 1\right), \qquad (2.3.3),$$

By combining (2.3.2) and (2.3.3) then we get the formula is

$$\log f_n(t) = \mathrm{n}\left\{i\mu t - \frac{\sigma^2 t^2}{2} + \lambda \int_{-\infty}^{+\infty}\left(e^{itx} - 1\right) dF(x)\right\}, \qquad (2.3.4),$$

$$\log f_n(t) = \mathrm{n}\left\{i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1\right) dM(x) + \int_{0}^{+\infty}\left(e^{itx} - 1\right) dN(x)\right\}, \qquad (2.3.5),$$

where $\int_{-\infty}^{0}\left(e^{itx} - 1\right) dM(x) = \lim_{a\to 0}\int_{-\infty}^{a}\left(e^{itx} - 1\right) dM(x),\ \ a < 0,$ and

$$\int_{0}^{\infty}\left(e^{itx} - 1\right) dN(x) = \lim_{a\to 0}\int_{a}^{\infty}\left(e^{itx} - 1\right) dN(x),\ \ a > 0,$$

then $$\log f_n(t) = \mathrm{n}\left\{\begin{matrix} i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1 - itx\right) dM(x) \\ + \int_{0}^{+\infty}\left(e^{itx} - 1 - itx\right) dN(x)\end{matrix}\right\}, \qquad (2.3.6),$$

$$\log f_n(t) = \mathrm{n}\left\{i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + \int_{0}^{+\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dN(x)\right\}$$

and

$$\log f(t) = \left\{i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + \int_{0}^{+\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dN(x)\right\}.$$

Can be illustrated the formulas by the following theorems:

Theorem 2.3.4. The Levy-Khinchine canonical representation:

The function $f(t)$ is the characteristic function of an infinitely divisible distribution if and only if it can be written in the form:

$\log\ f(t) = i\mu t + \int_{-\infty}^{\infty}\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]\frac{1+x^2}{x^2}\,dG(x),$ where μ is a real constant, $G(x)$ is a non-decreasing and bounded function, such that $G(-\infty) = 0$ and the integral at $x = 0$ is equal $\frac{-t^2}{2}$, i.e., $\left\{\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]\frac{1+x^2}{x^2}\right\}_{x=0} = \frac{-t^2}{2}$.

Theorem 2.3.5. The Levy canonical representation:

The function $f(t)$ is the characteristic function of an infinitely divisible distribution if and only if it can be written in the form:

$$\log\ f(t) = i\mu t - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0}\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]dM(x) + \int_{0}^{\infty}\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]dN(x),$$

where μ is a real constant, σ^2 is a real and non negative constant and the functions $M(x), N(x)$satisfy the following conditions:

(i) $M(x)\ and\ N(x)$ are non-decreasing in $(-\infty, 0)\ and\ (0, +\infty)$.
(ii) $M(-\infty) = N(+\infty) = 0$.
(iii) The integrals $\int_{-\varepsilon}^{0} x^2\, dM(x) + \int_{0}^{\varepsilon} x^2\, dN(x)$ are finite for every $\varepsilon > 0$.

Theorem 2.3.6. The Kolmogorov canonical representation:

The function $f(t)$ is the characteristic function of an infinitely divisible distribution with finite second moment iff it can be written in the form:

$\log\ f(t) = i\mu t + \int_{-\infty}^{\infty}\left[e^{itx} - 1 - itx\right]\frac{dK(x)}{x^2},$ where μ is a real constant, $K(x)$ is a non-decreasing and bounded function, such that $K(-\infty) = 0$ and the integral at $x = 0$ is equal $\frac{-t^2}{2}$, i.e.,$\left\{\left[e^{itx} - 1 - itx\right]\frac{1}{x^2}\right\}_{x=0} = \frac{-t^2}{2}$.

2.4.Tail probabilities

Theorem 2.4.1. below concerns the asymptotic behavior of the tail probabilities $P\{X > x\}$ and $P\{X < -x\}$ as $x \to \infty$. In the Gaussian case $\alpha = 2$,

$P\{X < -x\} = P\{X > x\} \sim \frac{1}{2\sigma x\sqrt{\pi}} e^{\frac{-x^2}{4\sigma^2}}$, $\quad as \;\; x \to \infty$, see (Feller 1966) [7].

When $\alpha < 2$, however, the tail probabilities behave like $x^{-\alpha}$.

The statement $h(x) \sim g(x)\; as\; x \to \infty$, will mean $\lim_{x\to\infty} \frac{h(x)}{g(x)} = 1$.

Theorem 2.4.1. Tail behavior: if $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2, -1 \leq \beta \leq 1$, then there exists a non zero constant $C_\alpha \neq 0$, such that,

$$\lim_{x\to\infty} x^\alpha P\{X > x\} = \frac{C_\alpha(1+\beta)\sigma^\alpha}{2},$$

$$\lim_{x\to\infty} x^\alpha P\{X < -x\} = \frac{C_\alpha(1-\beta)\sigma^\alpha}{2},$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\pi\alpha/2\right)} & if\ \alpha \neq 1, \\ 2/\pi & if\ \alpha = 1. \end{cases}$

(Property 1.2.15, Samorodnitsky and Taqqu (1994)) in [22].

For all $\alpha < 2$ and $-1 < \beta < 1$, both tail probabilities and densities are asymptotically power laws.

When $\beta = -1$, the right tail of the distribution is not asymptotically a power law,

Likewise, when $\beta = 1$, the left tail of the distribution is not asymptotically a power law.

2.5. Mixed distributions:

Let X_1 and X_2 be random variables with distribution functions $F_1(x)$and $F_2(x)$, respectively, and

$X = \begin{cases} X_1 & with\ probability\ p, \\ X_2 & with\ probability\ q, \end{cases}$ where $p + q = 1$.

The distribution function of the random variable X is given by
$F(x) = P\{X \leq x\} = pP\{X_1 \leq x\} + qP\{X_2 \leq x\} = pF_1(x) + qF_2(x)$,

and is called the mixture of the distributions determined by the functions $F_1\ and\ F_2$.

We now proceed to study the distribution of extreme values of sequences of independent identically distributed random variables with the distribution which is the mixture of stable distributions.

2.5.1. Mixture of stable distributions

In paper [16], (Mladenović, P., Extreme values of the sequences of independent random variables with mixed distributions, MATEMATIČKI VESNIK, 51 (1999), 29- 37.) extreme values of mixture of normal distributions and mixture of Cauchy distributions were studied, and the following was proved:

(1) Normalizing constants for maximum in the case of mixture of normal distributions depend on only one of the components in this mixture.

(2) Normalizing constants for maximum in the case of mixture of Cauchy distributions depend on both components in this mixture.

Here we consider the mixtures of stable distributions. See above Theorem 2.4.1 tail behavior, will be useful in our proofs.

As a special case, if $X \sim S_\alpha(\sigma, 0,0), then\ as\ x \to \infty,$

$$P(X > x) \sim \sigma^\alpha \frac{C_\alpha}{2} x^{-\alpha}.$$

Suppose now $X \sim S_\alpha(\sigma, -1,0)$. Since $\beta = -1$, Theorem 2.4.1 gives $\lim_{x\to\infty} x^\alpha P(X > x) = 0, i.e.,$ $P(X > x)$ tends to zero faster than $x^{-\alpha}$ as $x \to \infty$.

When $\alpha > 1,\ as\ x \to \infty,$

$$P(X > x) \sim \frac{1}{\sqrt{2\pi\alpha\,(\alpha-1)}} \left(\frac{x}{\alpha\widehat{\sigma_\alpha}}\right)^{\frac{-\alpha}{2(\alpha-1)}} exp\left(-(\alpha-1)\left(\frac{x}{\alpha\widehat{\sigma_\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right),$$

where $\widehat{\sigma_\alpha} = \sigma\left(\cos\frac{\pi}{2}(2-\alpha)\right)^{\frac{-1}{\alpha}}$.

When $\alpha = 1$,

$$P(X > x) \sim \frac{1}{\sqrt{2\pi}} exp\left(-\frac{\left(\pi/2\sigma\right)x - 1}{2} - e^{\left(\pi/2\sigma\right)x-1}\right).$$

Theorem 2.5.2. Let (X_n) be a sequence of independent random variables such that

$$X_n \sim \begin{cases} S_\alpha(\sigma_1, 0{,}0), & with\ probability\ p, \\ S_\alpha(\sigma_2, 0{,}0), & with\ probability\ q, \end{cases} \quad for\ all\ n$$

where $p, q > 0\ \ and\ \ X \sim\ S_\alpha(\sigma, 0{,}0)$ denotes the stable distribution with

$P(X > x) \sim \sigma^\alpha \frac{C_\alpha}{2} x^{-\alpha}$ and $0 < \alpha < 2,\ \sigma_1 \neq \sigma_2$.

Let $M_n = \max_{1 \le j \le n} X_j$. Then, the limiting distribution of M_n is given by

$P\{a_n(M_n - b_n) \le x\} \to \exp(-x^{-\alpha})\ ,\ n \to \infty,$

where the normalizing constants a_n and b_n are given by

$a_n = (nC)^{\frac{-1}{\alpha}}$ and $b_n = 0$,

with $C = (pA + qB),\ A = \frac{C_\alpha \sigma_1^\alpha}{2}$ and $B = \frac{C_\alpha \sigma_2^\alpha}{2}$.

Proof of Theorem 2.5.2: The distribution function of the random variable X is given by $F(x) = pF_1(x) + qF_2(x)$ where $X_1 \sim S_\alpha(\sigma_1, 0{,}0)$ and $X_2 \sim S_\alpha(\sigma_2, 0{,}0)$.

Then $1 - F_1(x) = P\{X_1 > x\} \sim Ax^{-\alpha}$ and $1 - F_2(x) = P\{X_2 > x\} \sim Bx^{-\alpha}$,

where $A = \frac{C_\alpha \sigma_1^\alpha}{2}$ and $B = \frac{C_\alpha \sigma_2^\alpha}{2}$.

For the function $F(x) = pF_1(x) + qF_2(x)$, we obtain

$1 - F(x) \sim\ Cx^{-\alpha}$, where $C = (pA + qB),\ \ x \to \infty$.

We now consider the asymptotic behavior of the tail $1 - F(x)$, as $x \to \infty$. $For\ x > 0$, we have

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t\to\infty} \frac{C(tx)^{-\alpha}}{C(t)^{-\alpha}} = \lim_{t\to\infty} x^{-\alpha} = x^{-\alpha}.$$

Hence, the distribution function $F(x)$, belongs to the domain of attraction of the function $G_1(x)$, and we have the type **(*II*)** of extreme value distribution, i.e. there exist constants a_n and b_n, such that the following equality holds true:

$$P\left\{M_n \le \frac{x}{a_n} + b_n\right\} \to \exp(-x^{-\alpha}).$$

We now determine the constants a_n and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim\ \frac{1}{n} x^{-\alpha}$ as $n \to \infty, i.e.$

$1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n} x^{-\alpha}$ as $n \to \infty$. That means

$C(u_n)^{-\alpha} \sim \frac{1}{n} x^{-\alpha}, n \to \infty,$

$(u_n)^{-\alpha} \sim \frac{x^{-\alpha}}{Cn}, \quad n \to \infty,$

$(u_n)^{-\alpha} \sim \left(\frac{x}{(Cn)^{\frac{-1}{\alpha}}} \right)^{-\alpha}, \quad n \to \infty,$ and we obtain

$u_n \sim (nC)^{1/\alpha} x$, as $n \to \infty$.

Using Theorem 1.4.16. in chapter 1, we obtain

$$P\left\{M_n \leq (nC)^{1/\alpha} x\right\} \to \exp(-x^{-\alpha}), \quad \text{as } n \to \infty \qquad (2.5.1),$$

but $P\left\{M_n \leq \frac{x}{a_n} + b_n\right\} \to G(x), \qquad (2.5.2)$.

Now we compare the equation (2.5.1) with the equation (2.5.2). We obtain

$a_n = (nC)^{-1/\alpha}$ and $b_n = 0$

where $C = (pA + qB), A = \frac{C_\alpha \sigma_1{}^\alpha}{2}, B = \frac{C_\alpha \sigma_2{}^\alpha}{2}$ $and\ G(x) = \exp(-x^{-\alpha})$. ■

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