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# CONTENTS

## ABSTRACT

The degenerate systems having Lagrangians depending on time derivatives of arbitrary order and on time explicitly are investigated, the highest order of the involved derivatives being assumed to be generally different for various generalized coordinates. First, the equivalent Lagrangians are discussed and it is shown how the Lagrangian formalism can be developed and the relevant Lagrangian constraints obtained for such systems. After this, the corresponding Hamiltonian formalism, based on the Dirac's theory of degenerate systems, is given.

In this manner, one obtains two types of primary constraints, only one of which takes part in the equations of motion, the Hamilton-Dirac equations and corresponding secondary constraints, and a general relation between the primary and the Lagrangian constraints. Also, one achieves the separation of all the constraints into those of the first and of the second class, by a suitable definition of these notions. On this basis the generalized Dirac bracket is introduced and the meaning of the specific first-class constraints is given. Finally, the results obtained are illustrated by a comparatively simple, but characteristic example.


## 1. INTRODUCTION AND HISTORY OF THE PROBLEM

It is known that the term "degenerate systems" is used in analytical mechanics to designate systems having the Hessian of their Lagrangian with respect to generalized velocities zero, so that the conventional transition from Lagrangian to Hamiltonian formalism is no longer possible. P. DIRAC [1–4] was the first to study such systems, what originated from his previous analysis of homogeneous dynamic variables. He showed how the Hamiltonian formalism can be formulated in this case, and how the quantization of these systems can be subsequently realized. In contrast to the standard case, this canonical formalism is characterized by the presence of certain constraints among the canonical variables and by appearance of a number of arbitrary constraint multipliers in the general equation of motion, the role of Poisson bracket being taken over by appropriately generalized, so-called Dirac bracket. S. SHANMUGADHASAN [5–6] analysed the influence of degeneracy on the Lagrange equations, proving that a certain number of their linear combinations reduce to first-order differential equations, and on this ground he formulated the corresponding Hamiltonian formalism. K. KAMIMURA [7] established the interrelation between these two approaches, i.e. between Lagrangian and Hamiltonian constraints. The structure of Dirac bracket was investigted in detail, from a modern mathematical standpoint, by E. SUDARSHAN and N. MUKUNDA [8–9]. If one applies the calculus of functionals, developed by V. VOLTERRA [10], the majority of these results can be systematically extended to classical field thery [11]. Degenerate systems and applications of this formalism to various domains of contemporary theoretical physics are dealt with in detail in some recent monographs [12–14].

On the other hand, the analytical mechanics of systems describable by Lagrangians depending on arbitrary order time derivatives of the generalized coordinates, the so-called generalized mechanics, has recently received much attention. M. OSTROGRADSKIY [15] began to study these systems from the standpoint of the calculus of variations, and he demonstrated that the Euler-Lagrange equations may always be substituted by an equivalent system of first-order differential equations. Using his invariant theory of the calculus of variations, TH. DE DONDER [16] obtained the canonical form of the equations for the extremals, generalizing it subsequently to the case of several independent variables. Shortly after that, F. BOPP and B. PODOLSKI [17–18] attempted a generalization of electrodynamics, based on a Lagrangian depending on second-order time derivatives of electromagnetic potentials. Inspired by these papers, M. BORNEAS [19–20] was the first to formulate explicitly the corresponding generalized momenta and the Hamiltonian for one and several independent variables. Independently from the above authors, J. KOESTLER and J. SMITH [21] obtained generalized Hamilton equations for such systems, as well as the associated Poisson and Lagrange brackets, and L. and P. RODRIGUES [22] formulated the corresponding canonical transformations. Simultaneously, K. THIELHEIM [23], and C. DE SOUZA and P. RODRIGUES [24] extended these results to classical field theory, including the energy, momentum and angular momentum densities. Based on this, it is possible to construct a complete theory of canonical transformations, for discrete and continuous systems in classical and covariant formulation [25–26]. In a recent monograph, concerning the generalized mechanics and field theory [27], a geometrical approach of this formalism with a contemporary mathematical point of view is presented.

However, the degenerate systems in generalized mechanics have so far received little attention and only a few papers dealing with this problem were published recently. This investigation was started by studies of equivalent Lagrangians by C. HAYES [28], C. RYAN [29] and D. ANDERSON [30]. The two last authors concluded that the Lagrangian of such a degenerate system has to be linear with respect to the highest order derivatives, and that in every class of equivalent Lagrangians it is always possible to find one which would correspond to a regular system. A critical analysis of these results given by H. TESSER [31] showed that this problem can be treated correctly only within the framework of Dirac's theory.

The first to deal with the degenerate systems in generalized mechanics was T. KIMURA [32], who showed that a correct treatment of the constraints appearing in the process leads to a consistent formulation of the Hamiltonian formalism. D. GITMAN, S. LYAKHOVICH and L. TYUTIN [33, 14] in their analysis of correspoding canonical formalism also briefly discussed this problem and extended it to physical fields. V. TAPIA [34] investigated degenerate systems with Lagrangians depending on second-order time derivatives, and applied an immediate extension of Dirac's theory of the first-order systems to this case. He thus arrieved at the relevant Hamilton-Dirac equations and Poisson bracket, and illustrated the results by two characteristic Lagrangians linear with respect to the second-order derivatives. In these cases always exists an equivalent first-order Lagrangian, which represents a standard mechanical system. C. GALVAO and N. LEMOS [35] gave a critical analysis of these papers, particularly from the point of view of the quantization of these systems. They studied a special case of degenerate systems with second-order derivatives, the Lagrangian of which can be represented as a sum of a usual Lagrangian and an additional term having the form of a total time derivative. By systematical applying of Dirac's theory of degenerate systems, they showed that this canonical formalism here yields the same results as the ones obtained with the use of the Lagrangian of the equivalent non-degenerate system of the usual type.

Independently from the above authors, V. NESTERENKO [36] analysed in more detail arbitrary systems with singular second-order Lagrangian, including the case non-reducible to standard systems. He developed the corresponding canonical formalism, investigated the fundamental characteristics of these systems, and applied the results to the action of a relativistic particle with an additional term proportional to the curvature of the particle trajectory. In a similar way, Y. SAITO, R. SUGANO, T. OHTA and T. KIMURA [37] formulated the canonical formalism for singular Lagrangians with arbitrary order derivatives and showed, inter alia, that all secondary constraints can be classified into two groups, only one of which corresponds to analogous constraints for usual singular systems. In the case when all the constraints belong to the first class, these authors gave the procedure of constructing the generator of gauge transformations with the aid of these constraints. After this, they applied the obtained results to the case of gravitation interaction with a acceleration-dependent potential appearing in the Lagrangian.

C. BATTLE, J. GOMIS, J. PONS and N. ROMAN-ROY [38–39] continued the study of these problems, primarily from the standpoint of the interrelation between Lagrangian and Hamiltonian formalism. They gave a strict proof of equivalence of the Euler-Lagrange and the Hamilton-Dirac equations in general case, and studied various sorts of constraints appearing in case of singular Lagrangians with second-order derivatives. The theory of degenerate systems obtained in this manner can be extended to classical field theory as well, which was carried out by V. TAPIA [40], J. BARCELOS-NETO and N. BRAGA [41] for the case where the Lagrangian density depends on second-order time derivatives. The former author extended the results of his previous work to physical fields, analyzing the non-standard Lagrangians linear with respect to the second-order derivatives, while the latter authors obtained similar results, which they applied subsequently to Klein-Gordon field.

Recently, several papers concerning the applications of this generalized canonical formalism were published. They are based on the assumption that the physical system studied can be better described by some singular Lagrangian depending on time derivatives of higher order. This theory was thus applied to Bopp-Podolski electromagnetic field [42], to Hilbert-Einstein gravitational field [43] and to the relativistic model of string with rigidity [44]. In all these investigations, the usual Lagrangian was complemented by adding an appropriate term depending on time derivatives of higher, usually the second order. The applicability of this method to the cited and similar physical systems was discussed in a number of papers (see for example [35]) and, although it cannot completely eliminate the built-in disadvantages of the theory, it has already given some encouraging results.

## 2. BASIC IDEAS OF THE GENERALIZED MECHANICS

Consider any discrete physical system with $N$ degrees of freedom, describable by a Lagrangian of the form

$$L = L(q_i, \dot{q}_i, \ddot{q}_i, \dots, q_i^{(M_i)}, t), \qquad (i = 1, 2, \dots, N) \tag{2.1}$$

depending on arbitrary order time derivatives of the generalized coordinates. It will be assumed here that the highest order of these derivatives $M_i$ are generally different for various values of the index $i$, the maximum of these order will be designated by $M$, i.e. $M_i \leq M$. The behaviour of the system considered is determined by Hamilton's variational principle

$$\delta W = \delta \int_{t_0}^{t_1} L\, dt = 0 \tag{2.2}$$

which yields the corresponding Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_i} - \dots + (-1)^{M_i}\frac{d^{M_i}}{dt^{M_i}}\frac{\partial L}{\partial q_i^{(M_i)}} = 0\,. \tag{2.3}$$

If one introduces the notion of the functional (variational) derivative, which for the functionals of the form

$$F_{[y_i(x)]} = \int_a^b \mathcal{F}(y_i(x), y_i'(x), \dots, y_i^{(M)}(x); x)\, dx \tag{2.4}$$

is defined as

$$\frac{\delta F}{\delta y_i^{(m)}(x)} \stackrel{\text{def}}{=} \frac{\partial \mathcal{F}}{\partial y_i^{(m)}} - \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y_i^{(m+1)}} + \dots + (-1)^{M-m}\frac{d^{M-m}}{dx^{M-m}}\frac{\partial \mathcal{F}}{\partial y_i^{(M)}}\,, \tag{2.5}$$

these Lagrange equations can be compactly written as

$$\frac{\delta W}{\delta q_i(t)} = 0\,, \qquad W = \int_{t_0}^{t_1} L\, dt\,. \tag{2.6}$$

To achieve the transition to the corresponding Hamiltonian formalism, the appropriate generalized momenta have to be introduced. They are defined by

$$\begin{aligned} p_{i/m} &\stackrel{\text{def}}{=} \frac{\delta W}{\delta q_i^{(m)}(t)} = \frac{\partial L}{\partial q_i^{(m)}} - \frac{d}{dt}\frac{\partial L}{\partial q_i^{(m+1)}} + \dots + (-1)^{M_i-m}\frac{d^{M_i-m}}{dt^{M_i-m}}\frac{\partial L}{\partial q_i^{(M_i)}} = \\ &= \sum_{j=0}^{M_i-m}(-1)^j\frac{d^j}{dt^j}\frac{\partial L}{\partial q_i^{(m+j)}}\,, \qquad \begin{pmatrix} i = 1, 2, \dots, N \\ m = 1, 2, \dots, M_i \end{pmatrix}\,. \end{aligned} \tag{2.7}$$

and, for different values of the index $i$, this sum may have different number of terms. From this definition follows immediately a reccurence formula

$$p_{i/m} = \frac{\partial L}{\partial q_i^{(m)}} - \dot{p}_{i/m+1} \tag{2.8}$$

The Lagrange equations (2.3) may, then, be replaced by the following equivalent system of equations [**15–16, 21**],

$$\dot{p}_{i/m} = -\frac{\partial H}{\partial q_i^{(m-1)}}, \qquad \dot{q}_i^{(m-1)} = \frac{\partial H}{\partial p_{i/m}} \tag{2.9}$$
$$(i = 1, 2, \dots, N; \; m = 1, 2, \dots, M_i)$$

where the corresponding Hamiltonian is given by

$$H(q_i^{(m-1)}, p_{i/m}, t) = p_{i/m} q_i^{(m)} - L(q_i^{(m-1)}, q_i^{(M_i)}, t) \tag{2.10}$$

and the summation over the repeated indices is understood. These are the generalized Hamilton (canonical) equations for the case considered. They form a system of $2\sum_{i=1}^{N} M_i$ first-order differential equations for the unknown functions

$$q_i, \dot{q}_i, \dots, q_i^{(M_i-1)}; p_{i/1}, p_{i/2}, \dots, p_{i/M_i},$$

which here play the role of canonical variables. Utilizing (2.7) the Lagrange equations (2.3) can be written in an alternative form

$$\frac{\partial L}{\partial q_i} = \dot{p}_{i/1}, \qquad (i = 1, 2, \dots, N). \tag{2.11}$$

This Hamiltonian can be formed by elimination of the highest order derivatives $q_i^{(M_i)}$ only and their substitution by corresponding generalized momenta. To this aim, the subsystem (2.7), consisting of the equations pertaining to $m = M_i$ has to be solved explicitly with respect to $q_i^{(M_i)}$ and these solutions have to be inserted into (2.10). In this manner, one obtains the Hamiltonian as a function of the canonical variables and such procedure is possible only if the Jacobian

$$\Delta = \left| \frac{\partial p_{i/M_i}}{\partial q_k^{(M_k)}} \right| = \left| \frac{\partial^2 L}{\partial q_i^{(M_i)} \partial q_k^{(M_k)}} \right| \tag{2.12}$$

is not zero. If, however, this Jacobian is zero, the subsystem mentioned cannot be completely solved with respect to all the highest order derivatives, and the transition to the Hamiltonian formalism is no longer possible. The systems exhibiting this property are degenerate in Dirac's sense [**2**].

## 3. EQUIVALENT LAGRANGIANS OF HIGHER ORDER

First, let us consider the equivalent Lagrangians and investigate the possibility to reduce the degenerate systems to the regular ones. From the equivalence of the Lagrange equations and Hamilton's principle itself, it follows that all the Lagrangians of the form

$$L' = L(q_i, \dot{q}_i, \dots, q_i^{(M_i)}, t) + \frac{d}{dt} f(q_i, \dot{q}_i, \dots, q_i^{(M_i-1)}, t) \tag{3.1}$$

are equivalent to one another in the sense that they all yield the same Lagrange equations. If the last term is written out explicitly

$$L' = L(q_i, \dot{q}_i, \dots, q_i^{(M_i)}, t) + \frac{\partial f}{\partial q_i^{(m-1)}} q_i^{(m)} + \frac{\partial f}{\partial t}, \tag{3.2}$$

it can be seen that all Lagrangians of this type differ mutually by a term which is linear with respect to the highest order derivatives and are generally of the same order as $L$. Moreover, all the elements of the Hessian (2.12) are equal with both $L$ and $L'$, i.e.

$$W_{ik} \equiv \frac{\partial^2 L'}{\partial q_i^{(M_i)} \partial q_k^{(M_k)}} = \frac{\partial^2 L}{\partial q_i^{(M_i)} \partial q_k^{(M_k)}}. \tag{3.3}$$

Hence, one immediately concludes, that if the system considered is regular ($\Delta \neq 0$) or degenerate ($\Delta = 0$), the same will hold for all the systems described by any of the equivalent Lagrangians $L'$ of the same order.

However, it may happen that the terms containing the highest order derivatives in (3.2) cancel out and the equivalent Lagrangian $L'$ will then be of the order smaller by unity than $L$. For this to occur, it is necessary that the Lagrangian $L$ of the system considered be linear with respect to the highest order derivatives, i.e.

$$L(q_i, \dot{q}_i, \dots, q_i^{(M_i)}, t) = a_i q_i^{(M_i)} + b, \tag{3.4}$$

where the coefficients $a_i$ and $b$ may be functions of the remaining variables. In this case, the cancellation of all the terms containing $q_i^{(M_i)}$ requires that

$$\frac{\partial f}{\partial q_i^{(M_i-1)}} = -a_i \qquad (i = 1, 2, \dots, N). \tag{3.5}$$

If the arbitrary function $f$ in (3.1) is chosen so as to satisfy this set of requirements, the ensuing Lagrangian $L'$ will be of the lower order, i.e. $M - 1$. The system described by this Lagrangian may be either regular or degenerate, depending on the rank of its Hessian matrix (2.12), which is also of lower order.

Hence, Ryan's statement [29] that the Lagrangian of any degenerate system of higher order has to be necessarily linear with respect to the highest order derivatives does not hold in general. This provides only sufficient and not necessary condition and corresponds to the case of maximim degeneracy, in which all the elements of the Hessian are identically zero. The same is valid for Anderson's statement [30] that it is always possible to find, within the class of equivalent Lagrangians, one of them which corresponds to the regular system. This is possible only if the Lagrangian of the system is linear with respect to the highest order derivatives. For example, for the degenerate system described by the Lagrangian

$$L = \tfrac{1}{2} a(\ddot{q}_1^2 + 2\ddot{q}_1\ddot{q}_2 + \ddot{q}_2^2) \tag{3.6}$$

no equivalent Lagrangian of lower order exist, and this system is in no way reducible to some regular one.

## 4. LAGRANGIAN FORMALISM FOR DEGENERATE SYSTEMS

Let us analyse in some detail the Lagrange equations (2.3), by suitable generalisation of the procedure given by Sudarshan and Mukunda in their monograph [9]. First, let us notice that the highest order derivatives of the generalized coordinates apper in the last terms of the left-hand side in (2.3). Their form can be obtained by successive derivation of $\partial L/\partial q_i^{(M_i)}$.

$$\left.\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial L}{\partial q_i^{(M_i)}} = \dfrac{\partial^2 L}{\partial q_i^{(M_i)}\partial q_k^{(m)}} q_k^{(m+1)} + \dfrac{\partial^2 L}{\partial q_i^{(M_i)}\partial t} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d^{M_i}}{dt^{M_i}}\dfrac{\partial L}{\partial q_i^{(M_i)}} = \dfrac{\partial^2 L}{\partial q_i^{(M_i)}\partial q_k^{(M_k)}} q_k^{(M_k+M_i)} + (***), \end{array}\right\} \tag{4.1}$$

where (***) denotes all the remaining terms, containing only lower-order derivatives. Thus the Lagrange equations may be presented in a concise form as

$$W_{ik} q_k^{(M_k+M_i)} = A_i(q_k, \dot{q}_k, \dots, q_k^{(M_k+M_i-1)}, t), \qquad (i,k = 1,2,\dots,N) \tag{4.2}$$

which are linear with respect to the highest derivatives, where

$$W_{ik} = \frac{\partial^2 L}{\partial q_i^{(M_i)}\partial q_k^{(M_k)}}. \tag{4.3}$$

An equivalent form of these equations can be found by completing the equations (2.3) with terms up to the $M$-th derivatives, i.e. by replacing $M_i$ by $M$

$$W'_{ik} q_k^{(M_k+M)} = A_i(q_k, \dot{q}_k, \dots, q_k^{(M_k+M-1)}, t), \qquad (i,k = 1,2,\dots,N) \tag{4.4}$$

where

$$W'_{ik} = \frac{\partial^2 L}{\partial q_i^{(M)}\partial q_k^{(M_k)}}. \tag{4.5}$$

The corresponding new Hessian is given by

$$\Delta' = |W'_{ik}| = \left| \frac{\partial^2 L}{\partial q_i^{(M)}\partial q_k^{(M_k)}} \right| \tag{4.6}$$

and it differs from the previous one (2.12) by the fact that, for any value of the index $i$ corresponding to $M_i < M$ the correspoding $i$-th line contains only zeros, so that in general the rank of this Hessian is $R' \leq R$.

If the system considered is degenerate, i.e. $R' < N$, the corresponding square matrix $\{W'_{ik}\}$ is singular and there exist $N - R'$ linearly independent null eigenvectors $\xi_i^{(p)}$ $(p = 1, 2, \ldots, P' = N - R')$, so that among its rows (or columns) $N - R'$ linear relations of the form

$$\xi_i^{(p)} W'_{ik} = 0 \qquad (p = 1, 2, \ldots, P' = N - R') \tag{4.7}$$

hold. Multiplying the Lagrange equations (4.4) by $\xi_i^{(p)}$ and summing over the index $i$, the left-hand side vanishes, as a consequence of (4.7), and hence

$$F_p \equiv \xi_i^{(p)} A_i(q_k, \dot{q}_k, \ldots, \overset{(M_k+M-1)}{q_k}, t) = 0 \qquad (p = 1, 2, \ldots, P' = N - R'). \tag{4.8}$$

Consequently, a certain number of linear combinations of the Lagrange equations in this case is of lower order than the maximum, and this number is equal to the degree of degeneracy $P' = N - R'$. They are the corresponding primary Lagrangian constraints and represent a generalization of those discovered by S. SHANMUGADHASAN [5] for degenerate systems with the Lagrangian of the usual form.

If the functional matrix $\{\partial F_p / \partial \overset{(M_k+M-1)}{q_k}\}$ has the rank $P' = N - R'$, the equations (4.8) contain all the variables mentioned, and the functions $F_p$ are mutually independent. If, however, the rank of this matrix is $P_1 < P'$, only $P_1$ of these equations contain the highest order derivatives, and the remaining $P' - P_1$ equations do not, i.e. they are of an order lower by unity

$$F_{p_1}(q_k, \dot{q}_k, \ldots, \overset{(M_k+M-2)}{q_k}, t) = 0 \qquad (p_1 = P_1 + 1, \ldots, P'). \tag{4.9}$$

By continuing this procedure, one successively obtains further relations with ever decreasing order of the highest derivatives, until the relations not containing them are arrived at.

All these relations have to be preserved in time and this so-called consistency condition here assumes the form

$$\frac{dF_p}{dt} = \frac{\partial F}{\partial \overset{(m-1)}{q_i}} \overset{(m)}{q_i} + \frac{\partial F_p}{\partial t} = 0. \tag{4.10}$$

By applying this condition, the following may result: a) some new differential equations of the highest order, b) new relations of lower order of the type (4.8), and c) identities, either immediate or following from the previous relations, much as in the case of usual Lagrangians [7]. Applying the consistency condition repeatedly to these newly formed relations, one can continue the procedure until its further use yields identities only. Separating from the relations obtained solely the independent ones, one may classify them into two basic types: the relations containing only the canonical variables $\overset{(m-1)}{q_i}$

$$A_{k'}(\overset{(m-1)}{q_i}, t) = 0 \qquad (k' = 1, 2, \ldots, K_1) \tag{4.11}$$

and the ones containing their $M_i$-th time derivatives as well

$$B_{k''}(\overset{(m-1)}{q_i}, \overset{(M_i+m-1)}{q_i}, t) = 0 \qquad (k'' = 1, 2, \ldots, K_2). \tag{4.12}$$

These relations, all the corressponding Lagrangian constraints for the case studied, represent a generalization of those of type A and B; introduced by SUDARSHAN and MUKUNDA [9].

## 5. TRANSITION TO HAMILTONIAN FORMALISM AND PRIMARY CONSTRAINTS

In order to formulate the corresponding Hamiltonian formalism, one should generalize Dirac's theory [2–4] of the canonical formalism for degenerate systems, as well as the results of other authors [34–37] related to systems describable by Lagrangians containing the derivatives of higher order. To this aim, let us consider the system of equations defining the generalized momenta (2.7) for $m = M_i$

$$p_{i/M_i} = \frac{\partial L}{\partial q_i^{(M_i)}} = f_i(q_k, \dot{q}_k, \dots, q_k^{(M_k)}, t) \qquad (i = 1, 2, \dots, N). \tag{5.1}$$

If the Hessian (2.12) of this system is of the rank $R < N$, it will be posible to solve only $R$ of these equations with respect to as many highest derivatives

$$q_r^{(M_r)} = \varphi_r(p_{r'/M_{r'}}, q_k^{(m-1)}, q_\rho^{(M_\rho)}, t) \qquad (r = 1, 2, \dots, R;\ \rho = R{+}1, \dots, N). \tag{5.2}$$

Substituting the variables $q_r^{(M_r)}$ by these functions into the Hamiltonian (2.10), it takes the form

$$H = H(q_i^{(m-1)}, q_\rho^{(M_\rho)}, p_{i/m}, t) \tag{5.3}$$

with $N - R$ highest derivatives $q_\rho^{(M_\rho)}$, as they cannot be eliminated by their procedure.

Since the Hamiltonian depends on the variables $q_\rho^{(M_\rho)}$, both directly and indirectly via $q_r^{(M_r)}$, one obtains

$$\frac{\partial H}{\partial q_\rho^{(M_\rho)}} = \left(p_{\rho/M_\rho} - \frac{\partial L}{\partial q_\rho^{(M_\rho)}}\right) + \frac{\partial \varphi_r}{\partial q_\rho^{(M_\rho)}}\left(p_{r/M_r} - \frac{\partial L}{\partial q_r^{(M_r)}}\right) \approx 0, \tag{5.4}$$

where the symbol $\approx$ denotes weak equality in Dirac's sense [2]. Namely, under the weak equations are understood such relations which are not valid per se, but only as a consequence of the relations defining the generalized momenta, directly or indirectly, and this is denoted by symbol $\approx$. These relations restrict the independence of the canonical variables, they are valid solely in a corresponding subspace of phase space and cannot be used before working out the Poisson brackets (see formula (6.10)), because then they would cease to be well-defined quantities. Hence, this procedure yields, as a first step, the Hamiltonian in the form of a "generalized canonical quantity" (5.3), according to terminology from [7], and just by using the definition of generalized momenta it becomes independent of the remaining highest derivatives $q_\rho^{(M_\rho)}$, i.e.

$$p_{i/M_i} = \frac{\partial L}{\partial q_i^{(M_i)}} \implies H \approx H(q_i^{(m-1)}, p_{i/m}, t). \tag{5.5}$$

As the Jacobian (2.12) of the system of equations (5.1) has the rank $R$, from the theorem of dependence of functions it follows, that there must exist $N-R$ relations among the generalized momenta $p_i/M_i$ not containing the highest derivatives $q_i^{(M_i)}$, which can be obtained in the following manner. Solving the subsystem formed by the first $R$ equations of the system (5.1) with respect to the variables $q_r^{(M_r)}$, and inserting the functions so obtained into the remaining $N-R$ equations of this system, one finds

$$p_{\rho/M_\rho} = \psi_\rho(q_k^{(m-1)}, p_{r/M_r}, t) \qquad (\rho = R+1, \dots, N). \tag{5.6}$$

These relations do not contain the highest derivatives $q_\rho^{(M_\rho)}$, since in the opposite case it would be possible to solve the system (5.1) with respect to more than $R$ of the derivatives $q_i^{(M_i)}$. They can be expressed, also, in the form of following weak equations

$$\begin{gathered}\Phi_p^{(0)}(q_i^{(m-1)}, p_{i/M_i}, t) \equiv p_{\rho/M_\rho} - \psi_\rho(q_k^{(m-1)}, p_{r/M_r}, t) \approx 0 \\ (p = \rho - R = 1, 2, \dots, P = N - R).\end{gathered} \tag{5.7}$$

In addition to these constraints, in this case other constraints among the canonical variables may also exist. For this reason, let us consider the complete set of equations (2.7), where the higher order derivatives appear in the last terms as $q_k^{(M_k+m-1)}$, so that this system is of the form

$$\begin{aligned}&p_{i/m} = f_{i,m}(q_k, \dot{q}_k, \dots, q_k^{(M_k+m-1)}, t) \\ &(i, k = 1, 2, \dots, N;\ m = 1, 2, \dots, M_i).\end{aligned} \tag{5.8}$$

The possibility of solving this set of equations with respect to the higher order derivatives depends on the character of the corresponding Jacobian

$$\widetilde{\Delta} = \left| \frac{\partial p_{i/m}}{\partial q_k^{(M_k+m-1)}} \right|. \tag{5.9}$$

If its rank is $\widetilde{R} \geq R$, one may solve any $\widetilde{R}$ of the equations (5.8) with respect to $\widetilde{R}$ variables $q_k^{(M_k+m-1)}$ for $k = 1, 2, \dots, R_0$ and insert these functions into the remaining equations. The result of this procedure is

$$p_{\rho/m} = \psi_{\rho,m}(q_k^{(m'-1)}, p_{r/m'}, t), \quad (\rho = R_0 + 1, \dots, N;\ m = 1, 2, \dots, M_i); \tag{5.10}$$

which can also be represented as

$$\begin{gathered}\Phi_p(q_i^{(m-1)}, p_{i/m}, t) \equiv p_{\rho/m} - \psi_{\rho,m}(q_k^{(m'-1)}, p_{r/m'}, t) \approx 0 \\ (p = 1, 2, \dots, \widetilde{P} = \textstyle\sum_\rho M_\rho \geq P)\end{gathered} \tag{5.11}$$

The obtained relations among the canonical variables could be called generalized primary constraints in Hamiltonian formalism, in analogy with the corresponding constraints in Dirac's theory. They include the relations (5.7), which are responsible for the direct transition to Hamiltonian formalism, and which will be further on distinguished from the others by calling them fundamental primary constraints. It could be remarked here, that the primary constraints other than the fundamental ones are not even mentioned by the majority of authors. V. TAPIA [34] treats them on equal footing with the fundamental ones, and SAITO, SUGANO, OHTA and KIMURA [37] classify them as secondary constraints.

## 6. HAMILTON-DIRAC EQUATIONS AND SECONDARY CONSTRAINTS

Let us formulate now the Hamilton-Dirac equations, pertaining to the case studied, by generalization of the procedure given in [9]. To this aim, we replace $q_r^{(M_r)}$ by (5.2) in the Hamiltonian (2.10) and, according to (5.7), put $p_{\rho/M_\rho} = \psi_\rho + \Phi_p^{(0)}$ $(p = \rho - R)$. By separating the terms not containing the constraint functions $\Phi_p^{(0)}$, the Hamiltonian can be written as

$$H(q_i^{(m-1)}, p_{i/m}, t) = H_c(q_i^{(m-1)}, p_{i/m}, t) + q_{R+p}^{(M_{R+p})}\Phi_p^{(0)}(q_i^{(m-1)}, p_{i/m}, t) \qquad (6.1)$$

with

$$\begin{aligned} H_c(q_i^{(m-1)}, p_{i/m}, t) = p_{i/m} q_i^{(m)} + p_{r/M_r}\varphi_r(p_{r'/M_{r'}}, q_k^{(m-1)}, q_\rho^{(M_\rho)}, t) + \\ + \psi_\rho(q_r^{(m-1)}, p_{r/M_r}, t) q_\rho^{(M_\rho)} - L(q_i^{(m-1)}, q_r^{(M_r)}, q_\rho^{(M_\rho)}, t). \end{aligned} \qquad (6.2)$$

The differentiation of the function $H_c$ with respect to canonical variables $q_i^{(m-1)}$ and $p_{i/m}$, bearing in mind that the dependence on these variables is both explicit and implicit, via $\varphi_r$ and $\psi_p$, yields the following result

$$\begin{aligned} \frac{\partial H_c}{\partial q_i^{(m-1)}} &= p_{i/m-1} + q_\rho^{(M_\rho)} \frac{\partial \psi_\rho}{\partial q_i^{(m-1)}} - \frac{\partial L}{\partial q_i^{(m-1)}} \\ \frac{\partial H_c}{\partial p_{i/m}} &= q_i^{(m)} + q_\rho^{(M_\rho)} \frac{\partial \psi_\rho}{\partial p_{i/m}}. \end{aligned} \qquad (6.3)$$

Substituting here $\partial L/\partial q_i^{(m-1)}$ for $m = 1$ from Lagrange equations (2.11) and for $m > 1$ from recurence formula (2.8), and using the constraint functions (5.7) instead of $\psi_\rho$, one obtains

$$\dot{p}_{i/m} = -\frac{\partial H_c}{\partial q_i^{(m-1)}} - u_p \frac{\partial \Phi_p^{(0)}}{\partial q_i^{(m-1)}}, \qquad \dot{q}_i^{(m-1)} = \frac{\partial H_c}{\partial p_{i/m}} + u_p \frac{\partial \Phi_p^{(0)}}{\partial p_{i/m}}, \qquad (6.4)$$
$$(i = 1, 2, \ldots, N;\ m = 1, 2, \ldots, M_i)$$

where

$$u_p = -q_\rho^{(M_\rho)} = -q_{R+p}^{(M_{R+p})} \qquad (p = 1, 2, \ldots, P). \qquad (6.5)$$

These equations can be written, more compactly, in the form of weak equations

$$\dot{p}_{i/m} \approx -\frac{\partial H_p}{\partial q_i^{(m-1)}}, \qquad \dot{q}_i^{(m-1)} \approx \frac{\partial H_p}{\partial p_{i/m}} \qquad (6.6)$$

with the extended (total) Hamiltonian

$$H_p = H_c + u_p \Phi_p^{(0)}. \tag{6.7}$$

These are the Hamilton-Dirac equations for this type of the degenerate systems. They represent a generalization of the corresponding equations for systems with the Lagrangian of the first [2, 3] and the second order [34, 36]. These equations are also in agreement with the results of the work [37], obtained for the Lagrangians of arbitrary order, but given without any proof. Here, one has a total number of $2\sum_{i=1}^{N} M_i$ equations containing the same number of unknown functions $q_i^{(m-1)}$ and $p_{i/m}$. The second group of these equations reduces to identities for $i = \rho > R$ and $m = M_\rho$, as $H_c$ does not depend on $p_{\rho/M_\rho}$ and $\partial\Phi_{p'}^{(0)}/\partial p_{\rho/M_\rho} = \delta_{\rho\rho'}$, and each of these identities corresponds to one of the quantities $p_{\rho/M_\rho}$ missing in the Hamiltonian.

For any functions $F$ of canonical variables $q_i^{(m-1)}$, $p_{i/m}$ and the time $t$, its total time derivative is

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i^{(m-1)}} \dot{q}_i^{(m-1)} + \frac{\partial F}{\partial p_{i/m}} \dot{p}_{i/m} + \frac{\partial F}{\partial t}. \tag{6.8}$$

By substituting here $\dot{q}_i^{(m-1)}$ and $\dot{p}_{i/m}$ from the corresponding Hamilton-Dirac equations (6.4), one obtains

$$\frac{dF}{dt} = [F, H_c] + u_p [F, \Phi_p^{(0)}] + \frac{\partial F}{\partial t}, \tag{6.9}$$

where the generalized Poisson bracket is introduced by

$$[F, G] \stackrel{\text{def}}{=} \frac{\partial F}{\partial q_i^{(m-1)}} \frac{\partial G}{\partial p_{i/m}} - \frac{\partial F}{\partial p_{i/m}} \frac{\partial G}{\partial q_i^{(m-1)}} \tag{6.10}$$

and a more compact form of this equation is

$$\frac{dF}{dt} \approx [F, H_p] + \frac{\partial F}{\partial t}, \qquad H_p = H_c + u_p \Phi_p^{(0)}. \tag{6.11}$$

This is the corresponding general equation of motion for any dynamic variable, which includes the Hamilton-Dirac equations, in agreement with the results of [38].

All the primary constraints (5.11), both fundamental and others, have to be preserved in time, according to the so-called consistency condition. This condition can be expresed, with the aid of the equation (6.9), in the form

$$\frac{d\Phi_{p'}}{dt} = [\Phi_{p'}, H_c] + u_p [\Phi_{p'}, \Phi_p^{(0)}] + \frac{\partial \Phi_{p'}}{\partial t} = 0, \qquad (p' = 1, 2, \ldots, \widetilde{P}). \tag{6.12}$$

This is a system of $\widetilde{P}$ equations, linear with respect to the constraint multipliers $u_p$, although it may happen that some, or even all, of the coefficients $[\Phi_{p'}, \Phi_p^{(0)}]$ are equal to zero. Consequently, if one excludes the possible inconsistent relations, this procedure may result in: a) relations determining some of the constraint multipliers,

b) new relations among the canonical variables, and c) identities, either directly or based on the previous relations, much as with the Lagrangians of the usual type [3, 7].

The resulting new relations among the canonical variables, denoted by

$$\chi^{(1)}_{s'}(q_i^{(m-1)}, p_{i/m}, t) \approx 0 \qquad (s' = 1, 2, \ldots, S_1) \tag{6.13}$$

can be called secondary constraints of the first order. Since they have to be preserved in time as well, i.e. to be subject also to the consistency condition, this can give, among others, secondary constraints of the second order $\chi^{(2)}_{s''} \approx 0$ $(s'' = 1, 2, \ldots, S_2)$. If this procedure of applying the consistency condition is continued until, eventually, only identities are obtained, one may determine all the secondary constraints in Hamiltonian formalism, which also have the character of weak equations

$$\chi_s(q_i^{(m-1)}, p_{i/m}, t) \approx 0, \qquad \left(s = 1, 2, \ldots, \widetilde{S} = \sum S_i\right). \tag{6.14}$$

The secondary constraints obtained in this manner are somewhat more general than the ones derived by other authors, as they originate from applying the consistency condition to all (not only fundamental) primary constraints.

## 7. CONSISTENCY CONDITION AND LAGRANGIAN CONSTRAINTS

By applying the consistency condition to the primary constraints, a relation between them and the Lagrangian constraints can be established. Therefore, let us find the first derivative with respect to time of any of the fundamental primary constraints (5.7)

$$\frac{d\Phi_p^{(0)}}{dt} = \frac{\partial \Phi_p^{(0)}}{\partial q_i^{(m-1)}} \dot{q}_i^{(m-1)} + \frac{\partial \Phi_p^{(0)}}{\partial p_{k/M_k}} \dot{p}_{k/M_k} + \frac{\partial \Phi_p^{(0)}}{\partial t}. \tag{7.1}$$

If these constraints are differentiated with respect to $q_i^{(m-1)}$ and the resulting expressions for $\partial \Phi_p^{(0)}/\partial q_i^{(m-1)}$ are inserted into the above equation, one obtains

$$\frac{d\Phi_p^{(0)}}{dt} = -\frac{\partial \Phi_p^{(0)}}{\partial p_{k/M_k}} \left( \frac{\partial p_{k/M_k}}{\partial q_i^{(m-1)}} \dot{q}_i^{(m-1)} - \dot{p}_{k/M_k} \right) + \frac{\partial \Phi_p^{(0)}}{\partial t}. \tag{7.2}$$

Substituting here $p_{k/M_k}$ from (2.7) and applying the relation (2.8) for $m = M_k - 1$, bearing also in mind that $\partial L/\partial q_k^{(M_k)}$ depends on time by way of all the variables $q_i^{(m-1)}$, one finds

$$\frac{d\Phi_p^{(0)}}{dt} = \frac{\partial \Phi_p^{(0)}}{\partial p_{k/M_k}} \left( \frac{\partial L}{\partial q_k^{(M_k-1)}} - \frac{d}{dt}\frac{\partial L}{\partial q_k^{(M_k)}} - p_{k/M_k-1} \right) + \frac{\partial \Phi_p^{(0)}}{\partial t}. \tag{7.3}$$

To elucidate the meaning of the coefficients appearing before the brackets, one may differentiate the fundamental primary constraints with respect to $q_i^{(M)}$

$$\frac{\partial\Phi_p^{(0)}}{\partial p_{k/M_k}}\frac{\partial p_{k/M_k}}{\partial q_i^{(M)}}=\frac{\partial\Phi_p^{(0)}}{\partial p_{k/M_k}}W'_{ik}=0. \tag{7.4}$$

By comparing with the relations (4.7), one concludes that

$$\xi_k^{(p)}=\frac{\partial\Phi_p^{(0)}}{\partial p_{k/M_k}}\qquad(p=1,2,\dots,P) \tag{7.5}$$

and this establishes a relation between the coefficients considered and the constraint functions. The derivative $d\Phi_p^{(0)}/dt$ thus becomes

$$\frac{d\Phi_p^{(0)}}{dt}=\xi_k^{(p)}\left(\frac{\partial L}{\partial q_k^{(M_k-1)}}-\frac{d}{dt}\frac{\partial L}{\partial q_k^{(M_k)}}-p_{k/M_k-1}\right)+\frac{\partial\Phi_p^{(0)}}{\partial t}\approx\frac{\partial\Phi_p^{(0)}}{\partial t}, \tag{7.6}$$

as the expression in brackets is zero, according to the definition (2.7) of the generalized momenta. Furthermore, from the consistency condition $d\Phi_p^{(0)}/dt=0$, it follows directly that no such constraint can depend explicitly on time.

If this relation is again differentiated with respect to time, one finds

$$\frac{d^2\Phi_p^{(0)}}{dt^2}=-\xi_k^{(p)}\left(\frac{\partial L}{\partial q_k^{(M_k-2)}}-\frac{d}{dt}\frac{\partial L}{\partial q_k^{(M_k-1)}}+\frac{d^2}{dt^2}\frac{\partial L}{\partial q_k^{(M_k)}}-p_{k/M_k-2}\right)\approx 0. \tag{7.7}$$

The generalization of these results leads to the conclusion that the $n$-th derivative of the fundamental primary constraint is of the form

$$\begin{aligned}\frac{d^n\Phi_p^{(0)}}{dt^n}=(-1)^{n+1}\xi_k^{(p)}\bigg(&\frac{\partial L}{\partial q_k^{(M_k-n)}}-\frac{d}{dt}\frac{\partial L}{\partial q_k^{(M_k-n+1)}}+\cdots+\\ &+(-1)^n\frac{d^n}{dt^n}\frac{\partial L}{\partial q_k^{(M_k)}}-p_{k/M_k-n}\bigg)\approx 0.\qquad(n<(M_k)_{\min}\equiv M_0).\end{aligned} \tag{7.8}$$

Those among these relations in which the highest derivatives $q_k^{(M_k+m)}$ for $m\geq 0$ are missing, after the consistency condition is applied, can give certain relations between the canonical variables. This can also be interpreted as an alternative definition of the non-fundamental primary constraints, which in this approach result here from the consistency condition, and coincide with the ones called in [37] secondary constraints contained in the definition of generalized momenta.

Let us apply now this formula (7.8) to $n=M_i-1$, dividing it into the sums witk $k=i$ and $k\neq i$, respectively. If the ensuing expression is differentiated with respect to time, using again (2.8) and the Lagrange equations in the form (2.11), one obtains

$$\frac{d^{M_i}\Phi_p^{(0)}}{dt^{M_i}}\approx(-1)^{M_i}\xi_i^{(p)}\left(\frac{\partial L}{\partial q_i}-\dot p_{i/1}\right). \tag{7.9}$$

According to (2.11), the right-hand side of (7.9) represents a linear combination of the Lagrange equations, with multipliers (7.5). Since we had demonstrated that this procedure leads to the primary Lagrangian constraints (4.8), the preceding relation reduces to

$$\frac{d^{M_i}\Phi_p^{(0)}}{dt^{M_i}} \approx (-1)^{M_i+1}\xi_i^{(p)} A_i(q_k, \dot{q}_k, \ldots, q_k^{(M_k+M-1)}, t). \tag{7.10}$$

From this result one can drawn the following conclusion: using the consistency condition of the fundamental primary constraints in Hamiltonian formalism in the form $d^{M_i}\Phi_p^{(0)}/dt^{M_i} = 0$, and substituting the generalized momenta by their expressions, one obtains the primary Lagrangian constraints, which are equivalent to the corresponding secondary constraints in Hamiltonian formalism. This is a generalization of the previously established relation between primary and Lagrangian constraints in the cases where the Lagrangian is of the usual form [7, 13] or contains only second-order derivatives [36].

Finally, let us give a remark concerning the application of the consistency condition to other, non-fundamental primary constraints. Since they do not appear in the Hamilton-Dirac equations (6.4), which are equivalent to the Lagrange equations, the consistency condition applied to these constraints, in the general case, will not yield the corresponding Lagrangian constraints.

## 8. FIRST AND SECOND CLASS CONSTRAINTS

Let us now show how one may apply here Dirac's theory, based on the separation of all the constraints into those of the first and the second class [2, 3]. Here we shall extend the corresponding notions to the case where the Lagrangian depends on derivatives of arbitrary order and on time explicitly. The separation of constraints into primary and secondary ones is not essential for this analysis, so that all the constraints (5.11) and (6.14) will be assembled and denoted by

$$\Theta_\mu(q_i^{(m-1)}, p_{i/m}, t) \approx 0 \qquad (\mu = 1, 2, \ldots, T = \widetilde{P} + \widetilde{S}). \tag{8.1}$$

For the Lagrangians of the usual type, not depending on time explicitly, Dirac defined the functions of the first class as the quantities satisfying, at least weakly, the conditions

$$[F, \Theta_\mu] \approx 0, \quad (\mu = 1, 2, \ldots, T); \qquad [F, H_c] \approx 0. \tag{8.2}$$

The generalization of this definition to the case studied here most conveniently can be achieved by transition to the so-called homogeneous formalism [45], in which the time $t$ is treated as a supplementary generalized coordinate $q_{N+1} = t$, and a new independent variable $\tau$ is introduced putting $t = f(\tau)$. In this approach, homogeneous Lagrangian and Hamiltonian of the system are given by

$$L^+ = t' L(q_i, q_i'/t', t), \qquad H^+ = t'(H + p_{k+1}) \approx 0, \tag{8.3}$$

where the superscript + indicates the use of the formalism mentioned, and the prime denotes differentiation with respect to the variable $\tau$.

In the case studied presently, the functional form of the Lagrangian $L^+$ also includes the higher-order derivatives. However, it can be seen that all the above relations here remain valid, provided that $p_{N+1}$ is replaced by $p_{N+1/1}$. Then, in analogy with (8.2), a quantity $F$ will be said to be a function of the first class if, within the homogeneous formalism, it satisfies, at least weakly, the conditions

$$[F, \Theta_\mu]^+ \approx 0, \quad (\mu = 1, 2, \ldots, T); \qquad [F, H_c^+]^+ \approx 0. \tag{8.4}$$

In the opposite case it will be of the second class. Afterwards, one may return to the standard, non-homogeneous formalism, by splitting the Poisson bracket $[\ ]^+$ into the usual one and a supplementary term, bearing in mind that $\partial/\partial p_{N+1/1} = \partial/\partial H_c$. In this manner, these conditions may be written as

$$[F, \Theta_\mu] + \frac{\partial(F, \Theta_\mu)}{\partial(t, H_c)} \approx 0, \qquad [F, H_c] + \frac{\partial(F, H_c)}{\partial(t, H_c)} \approx 0, \tag{8.5}$$

in accordance with the previously established result [46] for the Lagrangians of the usual type. Hence, as the constraint functions do not depend on the Hamiltonian $H_c$ explicitly, a constraint $\Theta_\mu \approx 0$ will pertain to the first class if

$$[\Theta_\mu, \Theta_{\mu'}] \approx 0, \qquad [\Theta_\mu, H_c] + \frac{\partial \Theta_\mu}{\partial t} \approx 0. \tag{8.6}$$

For this, it is sufficient that its generalized Poisson brackets with all the other constraints are zero, the latter of the conditions is then identically satisfied owing to the consistency condition $d\Theta_\mu/dt = 0$.

Using these notions, one may transform the general equation of motion (6.9) to a more convenient form, in which only the functions of the first class will appear. To show this, let us write out the consistency condition for all the constraints

$$\frac{d\Theta_\mu}{dt} = [\Theta_\mu, H_c] + u_p[\Theta_\mu, \Phi_p^{(0)}] + \frac{\partial \Theta_\mu}{\partial t} = 0, \qquad (\mu = 1, 2, \ldots, T = \widetilde{P} + \widetilde{S}) \tag{8.7}$$

which represents a system of equations determining the constraint multipliers $u_p$. Designating $U_p$ to be a particular solution of these equations, and letting $V_{ap}$ ($a = 1, 2, \ldots, A$) to be the complete set of independent particular solutions of the corresponding homogeneous equations, the general solution will be

$$u_p = U_p + v_a V_{ap} \tag{8.8}$$

with arbitrary coefficients $v_a$. If this expression is inserted into the general equation of motion, it will yield

$$\frac{dF}{dt} = [F, H'] + v_a[F, \Phi_a^{(0)}] + \frac{\partial F}{\partial t}, \tag{8.9}$$

where

$$H' = H_c + U_p \Phi_p^{(0)}, \qquad \Phi_a^{(0)} = V_{ap} \Phi_p^{(0)}, \tag{8.10}$$

much as in the case of Lagrangians of the usual type [2].

To investigate the nature of these quantities, let us put $\Theta_\mu = \Phi_p^{(0)}$ and $u_p = U_p$ in the equation (8.7), multiply this relation subsequently by $U_p$ or $V_{ap}$, and carry out the summation over the index $p$, which gives

$$U_p\left([\Phi_p^{(0)}, H_c] + \frac{\partial\Phi_p^{(0)}}{\partial t}\right) = 0, \qquad V_{ap}\left([\Phi_p^{(0)}, H_c] + \frac{\partial\Phi_p^{(0)}}{\partial t}\right) = 0. \tag{8.11}$$

Using these auxiliary relations, it can be shown that the quantities $\Phi_a^{(0)}$ ($a = 1, 2, \dots, A_0'$) and $H'$ are the functions of the first class, i.e. that they satisfy the conditions (8.5). For example, if one forms the expression appearing in the left-hand side of the second condition for the function $H'$, one obtains

$$[H', H_c] + \frac{\partial(H', H_c)}{\partial(t, H_c)} \approx U_p\left([\Phi_p^{(0)}, H_c] + \frac{\partial\Phi_p^{(0)}}{\partial t}\right) \approx 0, \tag{8.12}$$

where the first auxiliary relation (8.11) was taken into account.

## 9. DETERMINATION OF THE CONSTRAINT MULTIPLIERS

All the constraints may be replaced by an equivalent set separable into the ones of the first and second class and on this basis it is possible to transform the corresponding equation of motion. If one takes the maximum number of linear combinations of the primary constraints $\Phi_{\alpha_1} = V_{\alpha,p}\Phi_p \approx 0$, belonging to the first class, the remaining constraints, denoted by $\Phi_{\beta_1} \approx 0$ will be of the second class. In a similar manner, by taking linear combinations of all primary and secondary constraints, they can also be grouped into those of the first and second class, viz. $\chi_{\alpha_2} \approx 0$ and $\chi_{\beta_2} \approx 0$ respectively. This can be schematically represented by

$$\begin{gathered} \Theta_\alpha = \{\Phi_{\alpha_1}, \chi_{\alpha_2}\} \approx 0, \qquad \Theta_\beta = \{\Phi_{\beta_1}, \chi_{\beta_2}\} \approx 0 \\ (\alpha = 1, 2, \dots, A;\ \beta = 1, 2, \dots, B = T - A). \end{gathered} \tag{9.1}$$

Let us replace now the fundamental primary constraints by an equivalent set $\Phi_a^{(0)} \approx 0$ ($a = 1, 2, \dots, A_0'$) and $\Phi_b^{(0)} \approx 0$ ($b = 1, 2, \dots, P - A_0'$), belonging to the first and second class. Putting

$$u_p\Phi_p^{(0)} = v_a\Phi_a^{(0)} + v_b\Phi_b^{(0)} \tag{9.2}$$

into the general equation of motion (6.9), one obtains

$$\frac{dF}{dt} = [F, H_c] + v_a[F, \Phi_a^{(0)}] + v_b[F, \Phi_b^{(0)}] + \frac{\partial F}{\partial t}. \tag{9.3}$$

Then, the consistency condition, applied to so formulated first and second class constraints, assume the form

$$\begin{aligned} &\frac{d\Theta_\alpha}{dt} = [\Theta_\alpha, H_c] + \frac{\partial\Theta_\alpha}{\partial t} = 0 \qquad (\alpha = 1, 2, \dots, A) \\ &\frac{d\Theta_\beta}{dt} = [\Theta_\beta, H_c] + v_b[\Theta_\beta, \Phi_b^{(0)}] + \frac{\partial\Theta_\beta}{\partial t} = 0 \qquad (\beta = 1, 2, \dots, B) \end{aligned} \tag{9.4}$$

and this system is to be used to determine the constraint multipliers. All the multipliers $v_a$ related to the fundamental primary constraints $\Phi_a^{(0)} \approx 0$ disappeared here, which means that they remain completely undetermined. Thus, the number of the undetermined constraint multipliers coincides with that of the fundamental primary constraints of the first class.

To determine the constraint multiplires $v_b$, let us form the matrix

$$D = \begin{pmatrix} [\Phi_{\beta_1}, \Phi_{\beta_1'}] & [\Phi_{\beta_1}, \chi_{\beta_2}] \\ [\chi_{\beta_2}, \Phi_{\beta_1}] & [\chi_{\beta_2}, \chi_{\beta_2'}] \end{pmatrix} , \tag{9.5}$$

the elements of which are the Poisson brackets of all the constraint functions of the second class and will be denoted by

$$\Delta_{\beta\beta'} = [\Theta_\beta, \Theta_{\beta'}] \qquad (\beta = 1, 2, \dots, B). \tag{9.6}$$

We can also introduce its inverse matrix $D^{-1}$ by

$$D \cdot D^{-1} = I \iff \Delta_{\beta\beta'} \Delta^{-1}_{\beta'\beta''} = \delta_{\beta\beta''} , \tag{9.7}$$

which will exist only if the number of constraints of the second class is even, i.e. of the form $B = 2B_0$, since otherwise the matrix $D$ is singular. Multiplying the second group of equations (9.4) by $\Delta^{-1}_{\beta'\beta}$ and summing them up subsequently with respect to the index $\beta$, it follows (for $\beta' = b \leq P - A_0'$)

$$v_b = -\Delta^{-1}_{b\beta} \left( [\Theta_\beta, H_c] + \frac{\partial \Theta_\beta}{\partial t} \right) \qquad (b = 1, 2, \dots, P - A_0'). \tag{9.8}$$

The remaining equations, those for $\beta' > P - A_0'$, as well as the ones obtained by multiplying the first group (9.4) with $\Delta^{-1}_{\beta'\beta}$, in a similar manner yield

$$\Delta^{-1}_{\beta'\beta} \left( [\Theta_\beta, H_c] + \frac{\partial \Theta_\beta}{\partial t} \right) = 0 \qquad (P - A_0' < \beta' \leq B). \tag{9.9}$$

If we introduce the expression (9.8) for $v_b$ in the general equation of motion (9.3), the result will be

$$\frac{dF}{dt} = [F, H_c] + v_a [F, \Phi_a^{(0)}] - \Delta^{-1}_{b\beta} \left( [\Theta_\beta, H_c] + \frac{\partial \Theta_\beta}{\partial t} \right) [F \Phi_b^{(0)}] + \frac{\partial F}{\partial t}. \tag{9.10}$$

This equation can be written also in an alternative form, if the relation (9.9) is multiplied by $[F, \Theta_{\beta'}]$, summed with respect to $\beta'$ for $\beta' > P - A_0'$, and the ensuing expression is added to the right-hand side of (9.10)

$$\frac{dF}{dt} = [F, H_c] + v_a [F \Phi_a^{(0)}] - \Delta^{-1}_{\beta'\beta} \left( [\Theta_\beta, H_c] + \frac{\partial \Theta_\beta}{\partial t} \right) [F, \Theta_{\beta'}] + \frac{\partial F}{\partial t}. \tag{9.11}$$

The general equation of motion, obtained by this procedure, has the same form as in the case of the usual Lagrangian [13]. However, only fundamental (and not all) primary constraints of the first class appear here, while all the constraints of the second class are present.

Let us remark, that this asymmetry can be removed, as in the usual case [9], if one introduces all the constraints, primary and secondary, into the extended (total) Hamiltonian. In this case the number of determined constraint multipliers would remain the same, but these Hamilton-Dirac equations would not be equivalent to the Lagrange equations, in agreement with the results of the authors of [47, 48] on their non-equivalence already being in the case of effective Hamiltonian.

## 10. GENERALIZED DIRAC BRACKET

The form of the transformed general equation of motion (9.11) suggests the introduction of the following expression

$$[F,G]^* = [F,G] - [F,\Theta_\beta]\Delta^{-1}_{\beta\beta'}[\Theta_{\beta'},G]\,, \tag{10.1}$$

in analogy with the generalized Poisson bracket introduced by Dirac [2, 3] for the usual degenerate systems. The above expression may, hence, be called the generalized Dirac bracket for the quantities $F$ and $G$. Using this notion, the general equation of motion takes a compact form

$$\frac{dF}{dt} = [F,H_c]^* + v_a[F,\Phi_a^{(0)}] - [F,\Theta_\beta]\Delta^{-1}_{\beta\beta'}\frac{\partial\Theta_{\beta'}}{\partial t}, \tag{10.2}$$

where the influence of the constraints of the second class is expressed here by way of this bracket.

From the structure of both generalized Poisson and generalized Dirac bracket it follows that they are analogous to the corresponding brackets in the usual case, provided that the canonical variables $q_i$ and $p_i$ are replaced by $q_i^{(m-1)}$ and $p_{i/m}$, and the summation over $i$ is extended to embrace both indices $i$ and $m$. Hence, it can be inferred that the generalized Dirac bracket has the same properties as the Poisson and Dirac bracket. The structure of the Dirac bracket was studied in detail by SUDARSHAN and MUKUNDA [8, 9], and all the results obtained by these authors remain here valid.

In the case of Lagrangians of the usual form, it is known that the Dirac bracket is equivalent to the "truncated" Poisson bracket, in which the summation with respect to the index $i$ is taken from 1 to $N_0 = N - B_0$, where $B_0$ is one half of the number of the second class constraints. This property can be extended to the generalized Dirac bracket, utilizing the quoted correspondence, i.e. explicitly

$$[F,G]^* = \sum_{i=1}^{N-B_0}\sum_{m=1}^{M_i}\left(\frac{\partial F}{\partial q_i^{(m-1)}}\frac{\partial G}{\partial p_{i/m}} - \frac{\partial F}{\partial p_{i/m}}\frac{\partial G}{\partial q_i^{(m-1)}}\right). \tag{10.3}$$

This is equivalent to the replacement of the complete set of canonical variables by the "truncated" one, non containing the canonical variables which can be eliminated by the constraints of the second class.

It is easily verified, by direct evaluation, that certain relations expressible with the aid of the Dirac brackets here remain valid, in the same form as with the usual

Lagrangians. Thus, for example, for any function $F$ of the canonical variables and any constraint function of the second class one has

$$[F, \Theta_{\beta''}]^* = [F, \Theta_{\beta''}] - [F, \Theta_\beta]\Delta^{-1}_{\beta\beta'}\Delta_{\beta'\beta''} = 0. \tag{10.4}$$

An analogous result is arrived at if the generalized Poisson bracket is taken instead of the Dirac's one used above, provided that $F$ is replaced by

$$F^* = F - [F, \Theta_\beta]\Delta^{-1}_{\beta\beta'}\Theta_{\beta'}\,, \tag{10.5}$$

whereby the following weak equation is obtained

$$[F^*, \Theta_{\beta''}] \approx [F, \Theta_{\beta''}] - [F, \Theta_\beta]\Delta^{-1}_{\beta\beta'}\Delta_{\beta'\beta''} \approx 0. \tag{10.6}$$

## 11. MEANING OF CONSTRAINTS OF THE FIRST CLASS

Let us show that the constraints of the first class here have the same meaning as in the case of Lagrangians of the usual type [4]. To this aim, let us consider a infinitesimal canonical transformation, the generating function of which is [25]

$$G_2(q_i^{(m-1)}, \overline{p_{i/m}}, t) = q_i^{(m-1)}\overline{p_{i/m}} + \varepsilon G_0(q_i^{(m-1)}, \overline{p_{i/m}}, t)\,, \tag{11.1}$$

where $G_0$ represents the so-called generator of this transformation, and the transformation itself is determined by the relations

$$\delta q_i^{(m-1)} = \varepsilon \frac{\partial G_0}{\partial p_{i/m}}, \qquad \delta p_{i/m} = -\varepsilon \frac{\partial G_0}{\partial q_i^{(m-1)}}\,. \tag{11.2}$$

Then, the variation of any fuction of the canonical variables will be given in the first approximation by

$$\delta F \stackrel{\text{def}}{=} \overline{F} - F = \varepsilon[F, G_0]\,. \tag{11.3}$$

To determine the elementary change of any function $F$ in the transition from arbitrary coefficients $v_a$ to other ones $v_a'$, let us find the value of this function for $t = \delta t$. Expanding it into the Taylor series and substituting $dF/dt$ from the general equation of motion (9.11), it follows

$$\begin{aligned} F_{v_a}(\delta t) = F_{v_a}(0) + \delta t \Bigg\{ &[F, H_c] + v_a[F, \Phi_a^{(0)}] - \\ &- [F, \Theta_{\beta'}]\Delta^{-1}_{\beta'\beta}\left([\Theta_\beta, H_c] + \frac{\partial \Theta_\beta}{\partial t}\right) + \frac{\partial F}{\partial t}\Bigg\}_{t=0}\,. \end{aligned} \tag{11.4}$$

If the same relation is written out for some other set $v_a'$ of these arbitrary coefficients, the substraction of these two relations gives

$$\delta F = \varepsilon_a[F, \Phi_a^{(0)}], \tag{11.5}$$

where

$$\varepsilon_a = \delta t\big(v_a(0) - v_a'(0)\big). \tag{11.6}$$

From this we can conclude that, in the generalized mechanics, one obtains formally the same results as in Dirac's theory with the Lagrangians of the usual type. Hence, the above result can also be interpreted in the same manner, with the aim to preserve the validity of the causality principle. Accordingly, it will be taken that the values of $F$ corresponding to various values of the arbitrary coefficients $v_a$ ($a = 1, 2, \ldots, A'_0$) pertain to the same physical state of the system. By comparison with (11.3) one concludes, then, that the first-class functions $\Phi_a^{(0)}$ of the fundamental primary constraints are the generators of the infinitesimal canonical transformations which leave the physical state of the system unchanged.

## 12. AN EXAMPLE

To illustrate the results obtained, let us consider a comparatively simple example with the Lagrangian

$$L = \tfrac{1}{2}\ddot{q}_1^2 + q_2\ddot{q}_1 + (1-\alpha)q_1\ddot{q}_2 + \tfrac{1}{2}\beta(q_1 - q_2)^2, \tag{12.1}$$

which differs from the one studied in the SUNDERMEYER's monograph [13] only by the fact that both first derivatives are replaced by the second ones. Although this Lagrangian does not have direct physical sense, same like the example in the cited monograph, because all the known physical systems of this type belong to the classical fields, by this example we would like to point the similarities and distinctions in comparison with the case of degenerate systems of habitual form. The Hessian matrix and the Hessian itself are now given by

$$W = W' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Delta = \Delta' = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{12.2}$$

and the system considered is degenerate with the rank $R = R' = 1$. Among the columns of this matrix there is exactly one linear dependence of the type (4.7), from where one gets the coefficients $\xi_i^{(p)}$

$$0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \implies \xi_i^{(p)} = \{0, 1\}. \tag{12.3}$$

The corresponding Lagrange equations (2.3) for the generalized coordinates $q_1$ and $q_2$ in this case have the form

$$\begin{aligned} \ddddot{q}_1 + (2-\alpha)\ddot{q}_2 + \beta(q_1 - q_2) &= 0 \\ (2-\alpha)\ddot{q}_1 - \beta(q_1 - q_2) &= 0\,. \end{aligned} \tag{12.4}$$

Since $P = 1$, there exists only one linear combination (4.8) of the Lagrange equations which is of lower order and this Lagrangian constraint coincides here with the second of these Lagrange equations.

The corresponding generalized momenta (2.7) for $m = 1$ and $M = 2$ are

$$p_{i/1} = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_i} = \begin{cases} -\dddot{q}_1 - \dot{q}_2 & \text{for } i = 1 \\ (\alpha - 1)\dot{q}_1 & \text{for } i = 2 \end{cases} \tag{12.5}$$

and for $m = 2$

$$p_{i/2} = \frac{\partial L}{\partial \ddot{q}_i} = \begin{cases} \ddot{q}_1 + q_2 & \text{for } i = 1 \\ (1-\alpha)\dot{q}_1 & \text{for } i = 2. \end{cases} \tag{12.6}$$

This system of equations cannot be solved completely with respect to all higher derivatives $\ddot{q}_k$ and $\dddot{q}_k$ $(k = 1, 2)$, what originates from the last subsystem for $m = 2$. Consequently, from (12.6) immediately follows one fundamental primary constraint

$$\Phi_1^{(0)}(q_i^{(m-1)}, p_{i/M_i}, t) \equiv p_{2/2} - (1-\alpha)q_1 \approx 0 \tag{12.7}$$

and (12.5) gives another primary constraint

$$\Phi_2(q_i^{(m-1)}, p_{i/m}, t) \equiv p_{2/1} - (\alpha - 1)q_1 \approx 0. \tag{12.8}$$

The corresponding Hamiltonian (2.10) can be found, eliminating $\ddot{q}_1$ from the first of equations (12.5) and applying the definition of $p_{2/2}$, whereby $\ddot{q}_2$ disappears too. In this manner one obtains

$$\begin{aligned} &H_c(q_i^{(m-1)}, p_{i/m}, t) = p_{i/m} q_i^{(m)} - L \approx \\ &\approx p_{1/1}\dot{q}_1 + \tfrac{1}{2}(p_{1/2} - q_2)^2 + p_{2/1}\dot{q}_2 - \tfrac{1}{2}\beta(q_1 - q_2)^2 \end{aligned} \tag{12.9}$$

and the "total" Hamiltonian (6.7) will be

$$H_p = H_c + u_1[p_{2/2} - (1-\alpha)q_1]. \tag{12.10}$$

Let us apply now the consistency condition to fundamental primary constraint (12.7) and to its time derivative. The former of these conditions gives

$$\frac{d\Phi_1^{(0)}}{dt} = [\Phi_1^{(0)}, H_p] = (\alpha - 1)\dot{q}_1 - p_{2/1} = 0 \tag{12.11}$$

and this equation coincides with the non-fundamental primary constraint (12.8), in accordance with [37]. Using this fact, the letter of the consistency conditions becomes

$$\frac{d^2\Phi_1^{(0)}}{dt^2} = [\dot{\Phi}_1^{(0)}, H_p] = (\alpha - 2)(p_{1/2} - q_2) + \beta(q_1 - q_2) = 0, \tag{12.12}$$

whence one obtains the corresponding secondary constraint

$$\chi_1(q_i^{(m-1)}, p_{i/m}, t) \equiv (\alpha - 2)(p_{1/2} - q_2) + \beta(q_1 - q_2) \approx 0. \tag{12.13}$$

This result is in full agreement with our conclusion from (7.10), so that the consistency condition $d^2\Phi_1^{(0)}/dt^2 = 0$, after putting $p_{1/2} = \ddot{q}_1 + q_2$, yields the Lagrangian constraint (12.4b), which is equivalent to the secondary constraint (12.13).

The consistency condition applied to the other primary constraint (12.8) here also results in so obtained secondary constraint

$$\frac{d\Phi_2}{dt} \approx [\Phi_2, H_p] = (2 - \alpha)(p_{1/2} - q_2) - \beta(q_1 - q_2) = 0, \tag{12.14}$$

which can be explained in the following manner. In this case, by application of this condition, one obtained a linear combination of the Lagrange equations with the coefficients equal to $\xi_i^{(p)}$ (which is not valid in the general case) and therefore it coincides with the corresponding secondary constraint. Applying the consistency condition to the secondary constraint, it follows

$$\frac{d\chi_1}{dt} \approx [\chi_1, H_p] = -\beta \dot{q}_1 + (\alpha + \beta - 2)\dot{q}_2 - (2-\alpha)p_{1/1} = 0 \tag{12.15}$$

and that is a new secondary constraint

$$\chi_2(q_i^{(m-1)}, p_{i/m}, t) \equiv -\beta \dot{q}_1 + (\alpha + \beta - 2)\dot{q}_2 - (2-\alpha)p_{1/1} \approx 0, \tag{12.16}$$

which, subjected to the consistency condition, in its turn gives

$$\frac{d\chi_2}{dt} \approx [\chi_2, H_p] = -(\alpha-2)\chi_1 - (p_{1/2}-q_2)[\beta+(\alpha-2)^2] + u_1[\beta-(\alpha-2)^2] = 0. \tag{12.17}$$

Specially, if $\beta = (\alpha-2)^2$, the last term cancels out and, bearing in mind that $\chi_1 \approx 0$, from here follows a new secondary constraint

$$\chi_3(q_i^{(m-1)}, p_{i/m}, t) \equiv p_{1/2} - q_2 \approx 0. \tag{12.18}$$

To examine the character of these constraints, i.e. to establish whether they pertain to the first or to the second class, let us form their Poisson brackets. For the primary and the first two of the secondary constraints one thus obtains

$$[\Phi_1^{(0)}, \Phi_2] = 0, \qquad [\Phi_1^{(0)}, \chi_1] = 0, \qquad [\Phi_2, \chi_1] = (\alpha-2)^2 - \beta \tag{12.19}$$

as well as

$$[\Phi_1^{(0)}, \chi_2] = (\alpha-2)^2 - \beta, \qquad [\Phi_2, \chi_2] = 0, \qquad [\chi_1, \chi_2] = -2\beta(\alpha-2). \tag{12.20}$$

If the constraint $\chi_3 \approx 0$ is also present, its Poisson brackets with the others will be

$$[\Phi_1^{(0)}, \chi_3] = 0, \qquad [\Phi_2, \chi_3] = 2-\alpha, \qquad [\chi_1, \chi_3] = 0, \qquad [\chi_2, \chi_3] = -\beta. \tag{12.21}$$

Summarizing all these results, three cases are to be distinguished here, according to the values of the parameters $\alpha$ and $\beta$. For $\alpha = 2$ and $\beta = 0$ one has $d^2\Phi_1^{(0)}/dt^2 \equiv 0$, so that no secondary constraints are obtained, i.e. there exist only two primary constraints, and both pertain to the first class. For $\beta = (\alpha-2)^2$ all the five constraints are present, with only $\Phi_1^{(0)} \approx 0$ belonging to the first class, and all the others pertaining to the second class. For $\beta \neq (\alpha-2)^2$ only the first four constraints are present, and they all belong to the second class. The constraint multipliers pertaining to the fundamental primary constraints of the first class remain completely arbitrary functions of time (the first two cases), whereas those corresponding to the constraints of the second class are fully determined.

Finally, let us discuss what would be changed in these results by including not only the fundamental, but all the primary constraints into the total Hamiltonian (as in the papers of V. TAPIA [34] and of BARCELOT-NETO and BRAGA [41].

Repeating the same procedure as above, with $H_T = H_c + u_1\Phi_1^{(0)} + u_2\Phi_2$ instead of $H_p$, one infers that the corresponding Hamilton-Dirac equations in this case are not any more equivalent to the Lagrange equations, except for a trivial value $u_2 = 0$, and the number as well as the character of the constraints are partially altered. Namely, in the first two cases all the quoted conclusions remain unaltered, but in the case $\beta \neq (\alpha - 2)^2$ the second and the third of the secondary constraints will be absent, and within the remaining constraints the fundamental primary one $\Phi_1^{(0)} \approx 0$ now will be of the first class.

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## SUBJECT INDEX

## RÉSUMÉ

Dans cet article on a étudié les systèmes dégénerés avec une lagrangienne qui dépend de dérivées d'ordre quelconque par rapport au temps et explicitement de temps lui-même. Ici on a supposé qu'en général l'ordre le plus haut de cettes dérivées est différent pour les diverses coordonnées généralisées. D'abord, les équivalentes lagrangiennes dans cette mécanique généralisée sont discutées et on a montré comment pour tels systèmes on peut développer le formalisme de Lagrange et obtenir les liaisons lagrangiennes associées. Ensuite, on a donné le formalisme d'Hamilton correspondant, basé sur la théorie de Dirac pour les systèmes dégénerés de la forme habituelle.

De cette façon, on a obtenu deux types de liaisons primaires, dont seulement un type participe dans les équations du mouvement, les équations généralisées d'Hamilton-Dirac et les liaisons secondaires correspondantes, et une relation générale entre les liaisons primaires et les liaisons lagrangiennes. De même, on a réalisé la séparation de toutes cettes liaisons aux liaisons de première et de seconde classe, en généralisant la définition de Dirac de cettes notions. A cette base, on a introduit les crochets généralisés de Dirac et on a donné le sens de liaisons spécifiques de première classe. A la fin, les résultats obtenus sont illustrés par un exemple simple, mais caractéristique.



## REZIME

U ovom radu proučavani su degenerisani sistemi sa lagranžijanom koji zavisi od vremenskih izvoda proizvoljnog reda i eksplicitno od vremena. Pri tome je pretpostavljeno da je red ovih najviših izvoda u opštem slučaju različit za razne generalisane koordinate. Prvo su analizirani ekvivalentni lagranžijani u ovoj generalisanoj mehanici i pokazano je kako se za ovakve sisteme može razviti Lagranžev formalizam i iz toga dobiti pridružene Lagranževe veze. Potom je dat odgovarajući Hamiltonov formalizam, zasnovan na Dirakovoj teoriji za degenerisane sisteme uobičajenog vida.

Na taj način, dobijena su dva tipa primarnih veza, od kojih samo jedan učestvuje u jednačinama kretanja, uopštene Hamilton-Dirakove jednačine i odgovarajuće sekundarne veze i jedna opšta relacija između primarnih i Lagranževih veza. Sem toga, postignuto je i razdvajanje svih ovakvih veza na veze prve i druge klase, uopštavajući Dirakovu definiciju ovih pojmova. Na toj osnovi uvedene su i generalisane Dirakove zagrade i dat je smisao specifičnih veza prve klase. Na kraju, dobijeni rezultati su ilustrovani na jednom prostom, ali karakterističnom primeru.

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