INSTITUT ZA MATEMATIKU

NOVI SAD

# ALGEBRAIC CONFERENCE

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# P R E F A C E

The Conference of algebraists from Belgrade, Zagreb, Skopje, Sarajevo, Titograd, Novi Sad and some other yugoslav mathematical centers took place in Novi Sad on May 30th Yuneth. There were 55 official participants with 30 reports.

Beside the offical participants, some other mathematicians attended the Conference.

It was the second conference of yugoslav algebraists. The first conference took place in Skopje in 1980.

The participants agreed that such meetings are very useful for interchanging scientific informations, fruitful collaboration in varions algebraic fields, and making contacts for other types of activities.

It was decided that the communications of the conference will be published in a separate volume.

This book of proceedings is a result of the preceding agreement. The next algebraic conference will be organized by the algebraists of the Faculty of Sciencie from Belgrade in 1982.

Novi Sad
Mart 1982.

Editor

C O N T E N T S

Algebraic conference
N O V I  S A D  1981

# POLYNOMIAL SUBALGEBRAS

G. Čupona and S. Markovski

A polynomial subalgebra of an algebra $\underline{A} = (A, \mho)$ is a subset B of the carrier of the algebra which is closed under the polynomials belonging to a set of $\mho$-polynomials. In this paper polynomial subalgebras are considered, together with a few properties and examples. A special attention is given to the polynomial subalgebras of the algebras belonging to a variety.

1. Throughout the paper $\mho$ and $\mho'$ will be two sets of operational symbols and $X = \{x_1, x_2, \dots, x_n, \dots\}$ will be the set of individual variables. By $\mho_X$ will be denoted the set of all $\mho$-polynomials, i.e. $\mho_X = \mathrm{Term}\,(\mho)$. If $p \in \mho_X$ and if each variable that occurs in p is in the set $\{x_1, \dots, x_n\}$, then we will usually write $p=p(x_1, \dots, x_n)$. Let $\wedge\colon \mho \to \mho'_X$ be a mapping such that if $f \in \mho(n)$, then $f^\wedge = f^\wedge(x_1, \dots, x_n)$. The mapping $\wedge$ induces a mapping from $\mho_X$ into $\mho'_X$ (denoted with the same symbol $\wedge$) defined by: (i) $x^\wedge = x$, for each $x \in X$ and (ii) $f \in \mho(n)$, $p = fp_1 \dots p_n \Rightarrow p^\wedge = f^\wedge(p_1^\wedge, \dots, p_n^\wedge)$.

Let $\underline{A}$ be an $\mho$-algebra, $\underline{A}'$ an $\mho'$-algebra and $\phi\colon A \to A'$ a mapping such that $\phi(f_{\underline{A}}(a_1, \dots, a_n)) = f^\wedge_{\underline{A}'}(\phi(a_1), \dots, \phi(a_n))$ for each $f \in \mho(n)$ and $a_1, \dots, a_n \in A$. The mapping $\phi$ in this case will be called a $\wedge$-homomorphism from $\underline{A}$ into $\underline{A}'$. Moreover, if $A \subseteq A'$ and if the embedding of $\underline{A}$ into $\underline{A}'$ is a $\wedge$-homomorphism, then $\underline{A}$ is said to be a $\wedge$-subalgebra of $\underline{A}'$. (We will sometimes say polynomial homomorphism (polynomial subalgebra) instead of $\wedge$-homomorphism ($\wedge$-subalgebra).)

If $\underline{A}'$ is an $\mho'$-algebra, then an $\mho$-algebra $\underline{{}^\wedge A'}$ by the same carrier $A'$ is defined by: $f_{\underline{{}^\wedge A'}}(a_1', \dots, a_n') = f^\wedge_{\underline{A}'}(a_1', \dots, a_n')$, for each $f \in \mho(n)$ and $a_1', \dots, a_n' \in A'$. We say that $\underline{{}^\wedge A'}$ is induced from $\underline{A}'$ by $\wedge$.

Let $\mathcal{C}'$ be a class of $\sigma'$-algebras and $\mathcal{C}$ be a class of $\sigma$-algebras. Then by ${}^{\wedge}\mathcal{C}'$ will be denoted the class of $\sigma$-algebras which are $\wedge$-subalgebras of $\sigma'$-algebras belonging to $\mathcal{C}'$, and by $\mathcal{C}^{\wedge}$ the class of $\sigma'$-algebras $\underline{A}'$ such that all $\wedge$-subalgebras of $\underline{A}'$ are in $\mathcal{C}$. We say that a pair $(\mathcal{C}, \mathcal{C}')$ is $\wedge$-compatible if each algebra $\underline{A} \in \mathcal{C}$ is a $\wedge$-subalgebra of an algebra $\underline{A}' \in \mathcal{C}'$ such that ${}^{\wedge}\underline{A}' \in \mathcal{C}$.

The following properties give some connections between $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}^{\wedge}$ and ${}^{\wedge}\mathcal{C}'$.

$\underline{1}^{\circ}$. (a) If $\mathcal{C}$ is a class of $\sigma$-algebras and $\mathcal{C}'$ a class of $\sigma'$-algebras, then: ${}^{\wedge}(\mathcal{C}^{\wedge}) \subseteq \mathcal{C}$, $\quad \mathcal{C}' \subseteq ({}^{\wedge}\mathcal{C}')^{\wedge}$.

(b) The equation ${}^{\wedge}(\mathcal{C}^{\wedge}) = \mathcal{C}$ holds iff each $\sigma$-algebra $\underline{A} \in \mathcal{C}$ is a $\wedge$-subalgebra of an $\sigma'$-algebra $\underline{A}'$ such that each $\wedge$-subalgebra of $\underline{A}'$ is in $\mathcal{C}$.

(c) The equation $({}^{\wedge}\mathcal{C}')^{\wedge} = \mathcal{C}'$ holds iff $\mathcal{C}'$ contains any $\sigma'$-algebra $\underline{A}'$ such that every $\wedge$-subalgebra $\underline{A}$ of $\underline{A}'$ is $\wedge$-subalgebra of $A'' \in \mathcal{C}'$.

$\underline{2}^{\circ}$. If $(\mathcal{C}, \mathcal{C}')$ is a $\wedge$-compatible, then $\mathcal{C} \subseteq {}^{\wedge}\mathcal{C}'$.

$\underline{3}^{\circ}$. If $\mathcal{C}'$ is a quasivariety of $\sigma'$-algebras, then ${}^{\wedge}\mathcal{C}'$ is also a quasivariety of $\sigma$-algebras. ([8], p. 274).

We note that there are known infinite many varieties of $\sigma'$-algebras $\mathcal{C}'$ such that ${}^{\wedge}\mathcal{C}'$ is a proper quasivariety. This suggests to look for a description of the set of varieties $\mathcal{C}'$ of $\sigma'$-algebras such that ${}^{\wedge}\mathcal{C}'$ to be also a variety of $\sigma$-algebras.

$\underline{4}^{\circ}$. Let $\mathcal{C}'$ be a variety of $\sigma'$-algebras and $\underline{A}$ be an $\sigma$-algebra. Let $\underline{F}'$ be the $\sigma'$-algebra which is freely generated by A in $\mathcal{C}'$ and let $\rho$ be the least congruence on $\underline{F}'$ such that:

$$a = f_{\underline{A}}(a_1,\ldots,a_n) \text{ in } A \Rightarrow a \;\rho\; f^{\wedge}_{\underline{F}'}(a_1,\ldots,a_n).$$

Then $A \in {}^{\wedge}\mathcal{C}'$ if the following condition is satisfied:

$$a,b \in A \Rightarrow (a\rho b \Rightarrow a=b).$$

$\underline{5}^{o}$. Let $\mathcal{C}' = \mathrm{Var}_{\mathcal{O}'}\ \Sigma'$ be a variety of $\mathcal{O}'$-algebras defined by a set of identities $\Sigma'$. Denote by $\langle\Sigma'\rangle$ the set of identities which are consequences from $\Sigma'$, i.e. which hold in all $\mathcal{O}'$-algebras belonging to $\mathcal{C}'$, and denote by $\hat{}\Sigma'$ the set of $\mathcal{O}$-identities $p \equiv q$ such that $p\hat{} \equiv q\hat{} \in \langle\Sigma'\rangle$. Then $\hat{}\mathcal{C}'$ is a variety iff $\hat{}\mathcal{C}' = \mathrm{Var}\ \hat{}\Sigma'$. And, if $\hat{}\mathcal{C}'$ is the variety of all $\mathcal{O}$-algebras then $\hat{}\Sigma'$ consists of trivial identities, i.e. the identities of the form $p \equiv p$, where $p \in \mathcal{O}_X$.

$\underline{6}^{o}$ Let $\mathcal{C} = \mathrm{Var}_{\mathcal{O}}\ \Sigma$, $\mathcal{C}' = \mathrm{Var}_{\mathcal{O}'}\ \Sigma'$ be such that $\mathcal{C} \subseteq \hat{}\mathcal{C}'$. Denote by $\Sigma''$ the following set of $\mathcal{O}'$-identities:

$$\{p\hat{} \equiv q\hat{} \mid p \equiv q \in \Sigma\} \cup \Sigma',$$

and let $\mathcal{C}'' = \mathrm{Var}_{\mathcal{O}'}\ \Sigma''$. Then the pair $(\mathcal{C}, \mathcal{C}')$ is $\wedge$-compatible iff $\mathcal{C}' \subseteq \mathcal{C}''$.

$\underline{7}^{o}$. If $\mathcal{C}'$ is an axiomatizable class of $\mathcal{O}'$-algebras, then $\hat{}\mathcal{C}'$ can be defined by a system of open formulas. ([7]).

$\underline{8}^{o}$. Let $\Sigma'$ be a class of $\mathcal{O}'$-identities satisfyng the following condition:

(**) If $u', v'$ are finite sequences on $\mathcal{O}' \cup X$, $p' \in \mathcal{O}_X$ and if there is a $q' \in \mathcal{O}_X$ such that $u'p'v' \equiv q' \in \langle\Sigma'\rangle$, then there is a $q'' \in \langle\Sigma'\rangle$ such that $u'xv' \equiv q'' \in \langle\Sigma'\rangle$, where $x$ is a variable which does not occur in $u'p'v'$.

Then $\hat{}\mathrm{Var}_{\mathcal{O}'}\ \Sigma'$ is a variety of $\mathcal{O}$-algebras ([5]).

$\underline{2}$. Now, we will state some results concerning special classes of algebras, which will throw better look on the properties $\underline{1}^{o}$-$\underline{8}^{o}$.

1) Let $\underline{Sem}$ be the variety of semigroups. If $\mathcal{O}' = \{\cdot\} = \mathcal{O}'(2)$ and if $p(x_1, \ldots, x_n) \in \mathcal{O}_X$, then by the associative law an (2) identity of the form $p \equiv x_{i_1} x_{i_2} \ldots x_{i_k}$ holds in $\underline{Sem}$, where $i_\nu \in \{1, 2, \ldots, n\}$. Thus, we can assume that if $\mathcal{C}$ is a variety of semigroups, then $\mathcal{C} = \mathrm{Var}\Sigma$, where $\Sigma$ is a set of identities of the forms $x_{i_1} \ldots x_{i_k} \equiv x_{j_1} \ldots x_{j_s}$, where $i_\nu, j_\lambda \in \{1, 2, \ldots\}$, including the identity $x_1(x_2x_3) = (x_1x_2)x_3$.

The following result is known as Cohn-Rebane's theorem ([1] page 185):

If $\underline{A}$ is an $\mathcal{O}$-algebra, then there is a semigroup $\underline{S}$ and a mapping $f \mapsto \bar{f}$ of $\mathcal{O}$ into S such that $A \subseteq S$ and $f_{\underline{A}}(a_1,\dots,a_n) = \bar{f}a_1\dots a_n$ for each $f \in \mathcal{O}(n)$ and all $a_1,\dots,a_n \in A$. Then we say that $\underline{A}$ is an $\mathcal{O}$-subalgebra of the semigroup $\underline{S}$. If $\mathcal{C}'$ is a class of semigroups, then by $\mathcal{C}'(\mathcal{O})$ will be denoted the class of $\mathcal{O}$-algebras which are $\mathcal{O}$-subalgebras of semigroups belonging to $\mathcal{C}'$. Thus, the Cohn-Rebane's theorem can be formulated as follows:

1.1) $\underline{\underline{Sem}}$ ($\mathcal{O}$) is the variety of all $\mathcal{O}$-algebras.

We will state some other results. First, we will give some definitions. If $p \in \mathcal{O}_X$ and if $b \in X \cup \mathcal{O}$, then $|p|_b$ is the number of occurences of the symbol b in p. Also, by $\underline{\underline{Absem}}$ we denote the variety of commutative semigroups, and by $\underline{C}_{r,m}$ the variety $\underline{\underline{Absem}}$ $(x^r=x^{r+m})$, where r and m are positive integers. Then we have:

1.2) $\underline{A} \in \underline{\underline{Absem}}$ ($\mathcal{O}$) if $\underline{A}$ satisfies any identity $p \equiv q$, where $p,q \in \mathcal{O}_X$ are such that $|p|_b = |q|_b$, for each $b \in \mathcal{O} \cup X$ ([10]).

1.3) $\underline{C}_{r,m}(\mathcal{O})$ is a variety iff r=1 or $\mathcal{O} = \mathcal{O}(1)$. ([6]).

We note that, if $\mathcal{O}(o) = \emptyset$, then 1.1) and 1.2) are consequences from $\underline{8}^o$. If in 1.1) or 1.2) we have $\mathcal{O}(o) = \emptyset, \mathcal{O} \setminus \mathcal{O}(o) \neq \emptyset$ (or in 1.3) $\mathcal{O} \neq \emptyset$), then the condition (**) of $\underline{8}^o$ is not satisfied.

2) Let $\mathcal{O} = \{f\} = \mathcal{O}(n)$, $\mathcal{O}' = \{\cdot\} = \mathcal{O}'(2)$ and $f^\wedge = x_1x_2\dots x_n$. If $\mathcal{C}'$ is a class of groupoids, then $^\wedge\mathcal{C}'$ is denoted by $\mathcal{C}'(n)$. Also, $\underline{\underline{Sem}}$ (xyz=xyxz, xyz=xzyz), $\underline{\underline{Sem}}$ (xyz=xyxz), $\underline{\underline{Sem}}$ $(x^r=x^{r+m})$ will be denoted respectively by: $\underline{D}$, $\underline{D}^\ell$, $\underline{P}_{r,m}$. And, $\underline{\underline{Sem}}_n$ is the class of n-semigroups, i.e. algebras with an associative n-ary operation.

2.1) $\underline{\underline{Sem}}(n) = \underline{\underline{Sem}}_n$.

2.2) $\underline{P}_{r,m}(n)$ is a variety iff r=1 or n-1 is a divisor of m.

2.3) $\underline{C}_{r,m}(n)$ is a variety for all r,m,n.

2.4) $\underline{D}(n)$ is a variety for every n.

2.5) $\underline{D}^{\ell}(n)$ is a proper quasivariety for every $n \geq 3$.

2.6) Let $\Sigma'$ be a set of semigroup identities $p \equiv q$ such that

$$|p|_i \equiv |q|_i \pmod{n-1} \qquad (***)$$

for each $i=1,2,\ldots$, where $n \geq 3$, and let $\mathcal{C}' = \underline{\underline{Sem}}(\Sigma)$. Then $\mathcal{C}'(n)$ is a variety. (We note that this result is a corrolary from $\underline{8}^{\circ}$; and, conversely, if a variety $\mathcal{C}' = \underline{\underline{Sem}}(\Sigma')$ satisfies the condition (**) of $\underline{8}^{\circ}$, then (***) is satisfied for every identity $p \equiv q \in \Sigma'$.)

The above results are proved in the papers [3], [4], [5], [9]. Some of the results in 1) and 2) suggest the following conjecture: If $\mathcal{C}'$ is a variety of semigroups such that $\mathcal{C}(\mathcal{O})$ is a variety of $\mathcal{O}$-algebras for every $\mathcal{O}$, then $\mathcal{C}'(n)$ is a variety of n-semigroups for every $n \geq 2$.

3) If $\underline{R}$ is a ring, then by 1.1) there is a semigroup $\underline{S}$ and a pair of elements $a,b \in S$ such that $x+y = axy$, $x\bullet y = bxy$ ("$\bullet$" is the multiplication in the ring $\underline{R}$). But, if $\underline{S}$ is a semigroup with at least two elements, and if the operations + and $\bullet$ defined on $\underline{S}$ by: $x+y = axy$, $x\bullet y = bxy$, where $a,b \in S$, then $(S;+,\bullet)$ is never a ring. This example shows that it can happen a pair ($\mathcal{C}$, $\mathcal{C}'$) to be not $\wedge$-compatible, although $\mathcal{C} \subseteq {}^{\wedge}\mathcal{C}'$. In [2] there are given several examples of such noncompatible pairs. We note that in each of the examples 1.1)-1.3), 2.1)-2.4) we have a compatible pair of varieties.

4) Now we will finish our considerations by an example of a variety $\mathcal{C}' = \mathrm{Var}\Sigma'$ such that ${}^{\wedge}\mathcal{C}'$ is not a variety although ${}^{\wedge}\Sigma'$ does not contain non trivial identities. Namely, let $\mathcal{O}' = \mathcal{O}'(2) = \{\cdot\}$, $\mathcal{O} = \mathcal{O}(3) = \{f\}$, and $f^{\wedge} = (x_1x_2)x_3$. If $\Sigma' = \{(((x_1x_2)x_1)x_2)x_1 = ((x_1x_1)x_1)(x_2x_2)\}$, then ${}^{\wedge}\Sigma'$ does not contain nontrivial identities, but ${}^{\wedge}\mathrm{Var}\Sigma'$ is a proper subclass of the class of ternary groupoids (i.e. algebras with a ternary operation).

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SEMIGROUPS WITH n-PROPERTIES

B. Trpenovski

Semigroups with n-property were introduced in [8] in the following way: a semigroup S posseses the n-property iff every n-subsemigroup of S is a subsemigroup, i.e. $Q \subseteq S$, $Q^{n+1} \subseteq Q \Rightarrow Q^2 \subseteq Q$. The problem of describing the structure of a semigroup with n-property is a special case of a problem formulated in [1]. Nevertheless, this special case is not easy to deal with and a structure description is given in [8] only for unipotent semigroups of that type. Using the idea of involving an (n+1)-ary operation, $n > 1$, in a semigroup, in this paper we introduce several classes of semigroups and give structure descriptions which follow the same pattern of structure description for unipotent semigroups with n-property.

First we collect some of the results from [8] in the following

Theorem 1. (i) Every semigroup with n-property is periodic;

(ii) If S is a group, then S posseses the n-property iff the order of every element of S is relatively prime with n;

(iii) Let H be a group with n-property, P a set such that $H \cap P = \emptyset$ and $\phi : P \to H$ a mapping. Extend $\phi$ to a mapping from $S^* = H \cup P$ onto H by $\phi(x) = x$ for all $x \in H$ and define an operation in $S^*$ by

$$x o y = \phi(x)\phi(y).$$

Then $S^* = S[H,P,\phi]$ will be a unipotent semigroup with n-property. Conversely, every unipotent semigroup with n-property can be obtained in that way. □

The general pattern sugested by (iii) of the above theorem is the structure set $[H,P,\phi]$. To be more precise, and for convenience, we bring out the following

Lemma 1. Let $P$ be a partial semigroup, $E$ a semigroup such that $P \cap E = \emptyset$ and $\phi: P \to E$ a homomorphism. Extend $\phi$ to a mapping $\phi^*: S = P \cup E \to E$ by $\phi^*(e) = e$ for all $e \in E$ and define an operation in $S$ by

$$x \circ y = \begin{cases} xy \text{ if } x,y \in P \text{ and } xy \text{ is defined in } P, \\ \phi^*(x)\phi^*(y) \text{ otherwise.} \end{cases}$$

Then $S(\circ)$ will be a semigroup with $E$ as an ideal and $\phi^*$-epimorphism.

Proof. This is in fact Lemma III.4.1 of [3]. Note that a mapping $\phi$ from a partial semigroup $P$ into a semigroup $E$ is a homomorphism if $\phi(xy) = \phi(x)\phi(y)$, $x,y \in P$, whenever $xy$ is defined in $P$.

We will denote the semigroup constructed in Lemma 1 by $S[P,E,\phi]$.

A subclass of the class of semigroups with n-property can be defined in the following way: a semigroup $S$ is said to be a $\lambda_o^n$-semigroup iff $Q \subseteq S$, $Q^{n+1} \subseteq Q \Rightarrow SQ \subseteq Q$. The structure of a $\lambda_o^n$-semigroup is very simple which is seen from

Lemma 2. ([7], Theorem 1). A semigroup $S$ is a $\lambda_o^n$-semigroup iff $S$ is periodic and $xy = e_y$ for all $x,y \in S$, where $e_y$ is the corresponding idempotent in $\langle y \rangle$. (Here $n > 1$).

In order to obtain more interesting classes of semigroups we can substitute the left-ideality by corresponding n-ideality and, alternatively, taking subsemigroups, beside n-subsemigroups, to posseses the ideal property. In that way we can introduce the following two classes of semigroups: a semigroup $S$ is said to be a $\lambda_1^n$-semigroup ($\lambda_2^n$-semigroup) iff $Q \subseteq S$, $Q^{n+1} \subseteq Q \Rightarrow S^nQ \subseteq Q$ ($Q \subseteq S$, $Q^2 \subseteq Q \Rightarrow S^nQ \subseteq Q$). Each of this two classes, for $n = 1$, represents the class of $\lambda$-semigroups (see, for example, [2], [4]). So, dealing with $\lambda_1^n$-or $\lambda_2^n$-semigroups we asume that $n > 1$. For any semigroup which belongs to either of this two classes we say that posseses "left-ideal n-property". Observe that

Lemma 3. (i) Every $\lambda_o^n$-semigroup is a $\lambda$-semigroup and a semigroup with n-property;

(ii) Every $\lambda$-semigroup is a $\lambda_2^n$-semigroup;

(iii) Every $\lambda_1^n$-semigroup is a $\lambda_2^n$-semigroup.

The following is almost abvious:

Lemma 4. Every subsemigroup and every homomorphic image of a $\lambda_o^n$-, $\lambda_1^n$-, $\lambda_2^n$-semigroup is a $\lambda_o^n$-, $\lambda_1^n$-, $\lambda_2^n$-semigroup, respectively.

Lemma 5. Let S be a semigroup. Then:

(i) S is a $\lambda$-semigroup iff $Sa \subseteq \langle a \rangle$ for every $a \in S$;

(ii) S is a $\lambda_1^n$-semigroup iff $S^n a \subseteq \langle a \rangle_n$ for every $a \in S$, where $\langle a \rangle_n = \{a^{kn+1} | k \in \mathbb{N}^o\}$ is the cyclic n-subsemigroup of S generated by a; if S is a $\lambda_1^n$-semigroup then $S^n a \subseteq \langle a \rangle$ for every $a \in S$;

(iii) S is a $\lambda_2^n$-semigroup iff $S^n a \subseteq \langle a \rangle$ for all $a \in S$.

Proof. For (i) and (ii) see [2] Lemma 2 and [7] Lemma 3. From (ii) and Lemma 3 it follows that if S is a $\lambda_2^n$-semigroup then $S^n a \subseteq \langle a \rangle$ for all $a \in S$. Conversely, if Q is a subsemigroup of a $\lambda_2^n$-semigroup S and $q \in Q$, from $S^n q \subseteq \langle q \rangle \subseteq Q$ if follows that $S^n Q \subseteq Q$.

Lemma 6. Let S be a $\lambda$-, $\lambda_1^n$-, $\lambda_2^n$-semigroup. Then:

(i) S is periodic;

(ii) The set E of all idempotents of S is a right-zero subsemigroup of S and is an ideal in S;

(iii) For all $a \in S$, $m_a = 1$ where $m_a$ is the period of a and: if S is a $\lambda$-semigroup, then $|\langle a \rangle| \leq 3$, if S is a $\lambda_1^n$-semigroup then $|\langle a \rangle| \leq n + 2$, if S is a $\lambda_2^n$-semigroup then $|\langle a \rangle| \leq 2n + 1$.

Proof. For $\lambda$- and $\lambda_1^n$-semigroups see [2] and [7]. Let S be a $\lambda_2^n$-semigroup, a S and $\langle a \rangle = \{a, a^2, \dots\}$. If $\langle a \rangle$ were infinite, then $Q = \langle a^{n+1} \rangle$ would be a subsemigroup of S which does not contain $a^{2n+1}$ but, on the other hand, $a^{2n+1} = a^n a^{n+1} \in S^n Q \subseteq Q$ and this is a contradiction. So, S is periodic since $\langle a \rangle$ is finite. Let $e_a$ be the idempotent in $\langle a \rangle$ and $x \in S$. Then

$$xe_a = xe_a \dots e_a \in S^n e_a \subseteq \langle e_a \rangle = \{e_a\},$$

i.e.

$$xe_a = e_a. \tag{1}$$

From (1) it follows that E is a right-zero subsemigroup of S. If $e \in E$ and $x \in S$, again from (1) it follows that $exex = e.e.x = ex$, so $ex \in E$, i.e. $ES \subseteq E$ which proves (ii). Let $K_a$ be the periodic part of $\langle a \rangle$ and $y \in K_a$. From (1) we have that $y = ye_a = e_a$ and $K_a = \{e_a\}$. Let $\langle a \rangle = \{a, a^2, \ldots, a^s = e_a\}$ and $Q = \langle a^{n+1} \rangle$; if $s \geq 2n+1$ then $a^{2n+1} \in Q$ which is a contadiction and so we have $|\langle a \rangle| \leq 2n+1$.

In what follows S will be a semigroup of any of the classes $\lambda_o^n$, $\lambda$, $\lambda_1^n$, $\lambda_2^n$. Let us put

$$P = S \backslash E$$

where E is as before, the set of all idempotents of S. Then P will be a partial semigroup such that for every $a \in P$ there exists some $k \in \mathbb{N}$ with $a^k$ not defined in P, which is a consequence of the periodicity of S; we may call such a partial semigroup a power breaking partial semigroup. We have therefore seen that

a) P is a power breaking partial semigroup.

Let us define a mapping $\phi: S \to E$ by $\phi(x) = e_x$, $e_x$ the idempotent in $\langle x \rangle$, and let $xy = z$, $x,y,z \in S$. For some $m \in \mathbb{N}$, $m > \frac{n}{2}$, we have that $z^m = e_z$ and then, by Lemma 3 and 5,

$$e_z = xy \ldots xy \in S^n y \subseteq \langle y \rangle$$

which implies that $e_z = e_y$ and

$$\phi(xy) = e_y = e_x e_y = \phi(x)\phi(y),$$

and $\phi$ is an epimorphism from P onto E. The restriction $\psi = \phi_{|P}$ then is a homomorphism from P into E, which establishes

b) There is a homomorphism $\psi: P \to E$.

The operation in S can be, now, expressed of follows:

c) $xy = \begin{cases} xy \text{ if } x,y \in P \text{ and } xy \text{ is defined in } P \\ \phi(x)\phi(y) \text{ otherwise.} \end{cases}$

Finaly, from Lemma 2 and 6 it follows that P posseses the left-ideal property which can be introduced in the following way:

d) (i) If S is a $\lambda_o^n$-semigroup then P is just a set; (ii) if S is a $\lambda$-semigroup then $xy = y^2$ whenever $xy$ is defined in P; (iii) if S is a $\lambda_1^n$-semigroup then $x_o x_1 \ldots x_n = x_n^s$, $s < n+2$, whenever $x_o x_1 \ldots x_n$

is defined in P; (iv) if S is a $\lambda_2^n$-semigroup then as in (iii) with $s < 2n+1$.

Conversely, let E be a left-zero semigroup, P a power breaking partial semigroup, $P \cap E = \emptyset$, and $\phi: P \to E$ a homomorphism. Extend $\phi$ to a mapping $\phi^*: S = P \cup E \to E$ by $\phi(e) = e$ for all $e \in E$ and define an operation in S as in Lemma 1. According to Lemma 1 S(o) will be a semigroup and $\phi^*$ an epimorphism. It is easily seen that S is periodic. Finaly: (i) if P is a set without operation defined on it, then S(o) will be a $\lambda_o^n$-semigroup; (ii) if $xy = y^2$ whenever xy is defined in P, then $Sy \subseteq \{\phi(y), y^2\} \subseteq \langle y \rangle$ since $\phi(y)$ is the corresponding idempotent to y (if $y^s$ is not defined in P then $y^s = [\phi(y)]^s = \phi(y)$) and so, S(o) will be a $\lambda$-semigroup; for (iii) and (iv), similarly as in (ii) we can see that S(o) will be a $\lambda_1^n$-, $\lambda_2^n$-semigroup, respectively.

From the above discusion follows

Theorem 2. A semigroup S posseses the left-ideal n-property iff $S = S[P,E,\phi]$ where E is a left-zero semigroup, P a power breaking partial semigroup and: (i) S is a $\lambda_o^n$-semigroup, (ii) $\lambda$-semigroup, (iii) $\lambda_1^n$-semigroup, (iv) $\lambda_2^n$-semigroup iff (i) P is a set without operation defined on it, (ii) $xy = y^2$, $x,y \in P$, whenever xy is defined in P, (iii) $x_o x_1 \ldots x_n = x_n^s$, $s < n+2$, whenever $x_o x_1 \ldots x_n$ is defined in P and (iv) $x_o x_1 \ldots x_n = x_n^s$, $s < 2n+1$ whenever $x_o x_1 \ldots x_n$ is defined in P.

Let us observe that it is very easy to formulate right dual of left-ideal n-property and, by symetry, to translate all results. Also, we can, now, obtain structure description for semigroups with ideal n-property which can be introduced in an obvious way. For example, for the corresponding class of $\lambda_o^n$-semigroups we will come to zero semigroups while in all other classes with ideal n-property E reduces to one idempotent and some additional identities will be needed: instead of $xy = y^2$ or $x_o x_1 \ldots x_n = x_n^s$ we will have $x_y = x^2 = y^2$ or $x_o x_1 \ldots x_n = x_o^s = x_n^s$ whenever xy, respectively $x_o x_1 \ldots x_n$ is defined in P. Let us observe that the class of $\lambda_1^n$-semigroups can be interpreted as a class of n-semigroups (for a structure description see [9]).

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# IDEMPOTENT PURE CONGRUENCES ON CLIFFORD SEMIGROUPS

Branka P.Alimpić and Dragica N.Krgović

In this note we describe idempotent pure congruences on Clifford semigroups and we get an expression for the greatest idempotent pure congruence on such a semigroup (Theorem 4). This is a solution of the problem stated by M.Petrich [3].

First we briefly mention congruence pair of an inverse semigroup. All undefined terminology and notation can be found in [1] and [3].

For a congruence $\rho$ on an inverse semigroup S the <u>kernel</u> and the <u>trace</u> of $\rho$ is defined by

$$\ker \rho = \{a\in S \mid (\exists e\in E_S)\ a\rho e\},$$

$$\mathrm{tr}\,\rho = \rho|E_S$$

respectively [2]. This associates to each congruence $\rho$ on S the ordered pair $(\ker\rho, \mathrm{tr}\rho)$.

Conversely, the pair $(K,\bar{\rho})$ is a <u>congruence pair</u> for S if K is a normal subsemigroup of S, $\bar{\rho}$ is a normal congruence on $E_S$, and they satisfy

$$\text{(i)}\ ae\in K \wedge e\bar{\rho}a^{-1}a \Rightarrow a\in K \qquad (a\in S,\ e\in E_S),$$

$$\text{(ii)}\ kk^{-1}\bar{\rho}k^{-1}k \qquad (k\in K).$$

<u>Theorem 1</u>. [2] Let S be an inverse semigroup. If $(K,\bar{\rho})$ is a congruence pair for S, then $\rho_{(K,\bar{\rho})}$ defined by

$$a\rho_{(K,\bar{\rho})}b \Leftrightarrow a^{-1}a\bar{\rho}b^{-1}b \wedge ab^{-1}\in K$$

is the unique congruence $\rho$ on S for which ker $\rho$ = K and tr $\rho = \overline{\rho}$. Conversely, if $\rho$ is a congruence on S, then (ker $\rho$, tr$\rho$) is a congruence pair for S and $\rho_{(\ker\rho,\ \mathrm{tr}\rho)} = \rho$.

A congruence $\rho$ on an inverse semigroup S is <u>idempotent pure</u> if for any $a \in S$, $e \in E_S$, $a\rho e$ implies $a \in E_S$ [3].

<u>Theorem 2</u> [3]. Let $S=[Y;\ G_\alpha,\ \varphi_{\alpha,\beta}]$ be a Clifford semigroup. The pair $(K,\overline{\rho})$ is a congruence pair on S if and only if $K=[Y; K_\alpha,\ \psi_{\alpha,\beta}]$

(i) $K_\alpha$ is a normal subgroup of $G_\alpha$, $\alpha \in Y$,

(ii) $\psi_{\alpha,\beta} = \varphi_{\alpha,\beta}|K_\alpha$,

(iii) $\overline{\rho}$ is a congruence on $E_S$ such that

$$e_\alpha > e_\beta \wedge e_\alpha \overline{\rho} e_\beta \Rightarrow K_\beta \varphi^{-1}_{\alpha,\beta} \subseteq K_\alpha.$$

(iv) $e_\alpha > e_\beta \Rightarrow K_\alpha \varphi_{\alpha,\beta} \subseteq K_\beta$.

According to Theorem 2 we have

<u>Corollary 1</u>. Let $S=[Y;\ G_\alpha,\ \varphi_{\alpha,\beta}]$ be a Clifford semigroup. Then $(E,\overline{\rho})$ is a congruence pair on S if and only if $\overline{\rho}$ is a congruence on $E_S$ such that

(1) $\qquad \alpha > \beta \wedge e_\alpha \overline{\rho} e_\beta \Rightarrow \varphi_{\alpha,\beta}$ is one-one.

By Theorem 1 and Corollary 1 we get a description of an idempotent pure congruence on Clifford semigroup.

<u>Theorem 3</u>. Let $S=[Y;\ G_\alpha,\ \varphi_{\alpha,\beta}]$ be a Clifford semigroup. If $\overline{\rho}$ is a congruence on $E_S$ such that (1) holds, then the relation $\rho$ defined by

$$a_\alpha \rho\, b_\beta \overset{\text{def}}{\Longleftrightarrow} e_\alpha \overline{\rho} e_\beta \wedge a_\alpha \varphi_{\alpha,\alpha\beta} = b_\beta \varphi_{\beta,\alpha\beta}$$

is an idempotent congruence on S, Conversely, if $\rho$ is an idempotent pure congruence on S, then

$$\alpha > \beta \wedge e_\alpha \rho\, e_\beta \Rightarrow \varphi_{\alpha,\beta} \text{ is one-one},$$

$$a_\alpha \rho b_\beta \Leftrightarrow e_\alpha \rho e_\beta \wedge a_\alpha \varphi_{\alpha,\alpha\beta} = b_\beta \varphi_{\beta,\alpha\beta}.$$

If $\rho$ is a pure congruence on an inverse semigroup S then $\rho$ restricted to each subgroup of S is the equality relation. The converse is an open problem [3].

Let $\rho$ be a congruence on a Clifford semigroup S and $\rho$ restricted to each subgroup of S be the equality relation. If $a_\alpha \rho e_\beta$, then according to Theorem 1, $a_\alpha^{-1} a_\alpha \rho e_\beta$, i.e. $e_\alpha \rho e_\beta$. Hence, $a_\alpha \ \rho \ e_\alpha$, which implies $a_\alpha = e_\alpha$.

Thus, for a Clifford semigroup S, a congruence $\rho$ is pure if and only if $\rho$ restricted to each subgroup of S is the equality relation.

<u>Theorem 4</u>. Let $S=[Y;\ G_\alpha, \varphi_{\alpha,\beta}]$ be a Clifford semigroup. The relation $\tau$ on S defined by

$$a_\alpha \tau\, b_\beta \overset{\text{def}}{\iff} (\forall \gamma \in Y)(e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} = e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}) \wedge a_\alpha\varphi_{\alpha,\alpha\beta} = b_\beta\varphi_{\beta,\alpha\beta}$$

is the greatest idempotent pure congruence on S.

<u>Proof</u>. Let $\bar{\tau}$ be the relation on $E_S$ defined by

$$e_\alpha \ \bar{\tau}\ e_\beta \overset{\text{def}}{\iff} (\forall \gamma \in Y)(e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} = e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma})$$

Evidently, $\bar{\tau}$ is an equivalence relation on $E_S$. Let $e_\alpha, e_\beta, e_\delta \in E_S$. Then

$$\begin{aligned}
e_\alpha \ \bar{\tau}\ b_\beta &\iff (\forall \gamma \in Y)(e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} = e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}) \\
&\Rightarrow (\forall \gamma \in Y)(e_{\alpha(\delta\gamma)}\varphi^{-1}_{\delta\gamma,\alpha(\delta\gamma)} = e_{\beta(\delta\gamma)}\varphi^{-1}_{\delta\gamma,\beta(\delta\gamma)}) \\
&\Rightarrow (\forall \gamma \in Y)(e_{\alpha(\delta\gamma)}\varphi^{-1}_{\delta\gamma,\alpha(\delta\gamma)}\varphi^{-1}_{\gamma,\delta\gamma} = e_{\beta(\delta\gamma)}\varphi^{-1}_{\delta\gamma,\beta(\delta\gamma)}\varphi^{-1}_{\gamma,\delta\gamma}) \\
&\Rightarrow (\forall \gamma \in Y)(e_{(\alpha\delta)\gamma}\varphi^{-1}_{\gamma,(\alpha\delta)\gamma} = e_{(\beta\delta)\gamma}\varphi^{-1}_{\gamma,(\beta\delta)\gamma}) \\
&\iff e_{\alpha\delta}\ \bar{\tau}\ e_{\beta\delta} \iff e_\alpha e_\delta \ \bar{\tau}\ e_\beta e_\delta .
\end{aligned}$$

Hence, $\bar{\tau}$ is a congruence on $E_S$.

Let $\alpha>\beta$ and $e_\alpha\ \bar{\tau}\ e_\beta$, i.e.

$(\forall\gamma\in Y)(e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} = e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma})$.

Then, for $\gamma=\alpha$, we get $e_\alpha = e_\beta\varphi^{-1}_{\alpha,\beta}$. Consequently, $\varphi_{\alpha,\beta}$ is one-one. By Theorem 3, $\tau$ is idempotent pure. Let $\rho$ be any idempotent pure congruence on S. Then, by Theorem 3,

$$a_\alpha\ \rho\ b_\beta \Leftrightarrow e_\alpha\ \rho\ e_\beta \wedge a_\alpha\varphi_{\alpha,\alpha\beta} = b_\beta\varphi_{\beta,\alpha\beta},$$

and

(2) $e_\alpha\rho e_\beta \Rightarrow (\forall\gamma\in Y)(\forall X_\gamma\in G_\gamma)X_\gamma e_\alpha\rho X_\gamma e_\beta \Leftrightarrow (\forall\gamma\in Y)(\forall X_\gamma\in G_\gamma)X_\gamma\varphi_{\alpha\gamma}\rho X_\gamma\varphi_{\gamma,\beta\gamma}$.

Let $X_\gamma\in e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma}$, i.e. $X_\gamma\varphi_{\gamma,\alpha\gamma} = e_{\alpha\gamma}$. Then by (2), $e_{\alpha\gamma}\rho X_\gamma\varphi_{\gamma,\beta\gamma}$. Since $\rho$ is idempotent pure, we have $X_\gamma\varphi_{\gamma,\beta\gamma} = e_{\beta\gamma}$, so $X_\gamma\in\varphi^{-1}_{\gamma,\beta\gamma}$. Hence, $e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma}\subseteq e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}$, for all $\gamma\in Y$. Analogously, we can prove $e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}\subseteq e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma}$ and thus $e_\alpha\ \bar\tau\ e_\beta$.

Therefore $\tau$ is the greatest idempotent pure congruence on S.

<u>Corollary 2</u>. Let $\sigma$ be the least group congruence and let $\tau$ be the greatest idempotent pure congruence on a Clifford semigroup S. Then $\tau=\sigma$ if and only if the homomorphisms $\varphi_{\alpha,\beta}$ are all one-one.

<u>Proof</u>. Let the homomorphisms $\varphi_{\alpha,\beta}$ are all one-one. Then $e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} = e_\gamma = e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}$ for all $\gamma\in Y$, so $e_\alpha\ \bar\tau\ e_\beta$ for all $\alpha,\beta\in Y$. Thus, $\mathrm{tr}\tau = \omega_{E_S}$, i.e. $\tau$ is a group congruence. Since $\ker\tau=E$, $\tau$ is the least group congruence $\sigma$. Conversely, let $\tau=\sigma$. Then $\mathrm{tr}\tau=\omega_{E_S}$, so $e_\alpha\ \tau\ e_\beta$ for every $\alpha,\beta\in Y$. According to Theorem 3, $\varphi_{\alpha,\beta}$ is one-one, for every $\alpha,\beta$ such that $\alpha>\beta$.

If the homomorphisms $\varphi_{\alpha,\beta}$ are all one-one, Theorem 4 and Corollary 2 yield the following expression of the greatest idempotent pure congruence on S:

$$a_\alpha\ \tau\ b_\beta \Leftrightarrow a_\alpha\varphi_{\alpha,\alpha\beta} = b_\beta\varphi_{\beta,\alpha\beta}$$

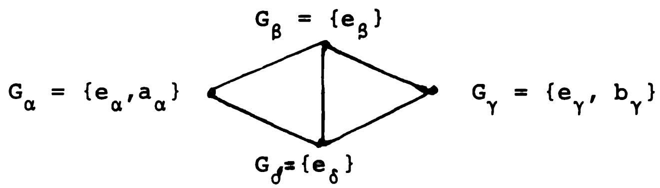
<u>Remark</u>. The corollary 2 can be obtained by the following assertions: A Clifford semigroup is E-unitary if and only if the

homomorphisms $\varphi_{\alpha,\beta}$ are all one-one [1], and an inverse semigroup is E-unitary if and only if $\sigma=\tau$ [3].

<u>Corollary 3</u>. Let none of the homomorphisms $\varphi_{\alpha,\beta}(\alpha>\beta)$ of a Clifford semigroup S be one-one. Then $\tau=\mathcal{E}$.

<u>Proof</u>. $e_\alpha \ \tau \ e_\beta \Rightarrow e_\alpha \ \tau \ e_{\alpha\beta} \Rightarrow \varphi_{\alpha,\alpha\beta}$ is one-one. Hence $\operatorname{tr}\tau=\mathcal{E}_E$. Since $\ker \tau = E$, we get $\tau=E_S$.

The following example shows that the converse is not true.




$e_\alpha \ \tau \ e_\beta \Rightarrow e_\delta \ \tau \ e_\gamma \Rightarrow \varphi_{\gamma,\delta}$ is one-one.
$e_\beta \ \tau \ e_\gamma \Rightarrow e_\alpha \ \tau \ e_\delta \Rightarrow \varphi_{\alpha,\delta}$ is one-one
$e_\beta \ \tau \ e_\delta \Rightarrow e_\alpha \ \tau \ e_\delta \Rightarrow \varphi_{\alpha,\delta}$ is one-one
Hence, $\tau=\mathcal{E}$ , and, for example, $\varphi_{\beta,\alpha}$ is one-one.

<u>Theorem 5</u>. Let $\tau$ be the greatest idempotent pure congruence on a Clifford semigroup $S=[Y;\ G_\alpha, \varphi_{\alpha,\beta}]$, and $\alpha>\beta$. The following conditions are equivalent:

(i) $e_\alpha \ \tau \ e_\beta$.

(ii) $(\forall\gamma\in Y)\varphi_{\alpha\gamma,\beta\gamma}$ is one-one.

(iii) $(\forall\gamma\in Y)\ e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma} \subseteq e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma}$.

<u>Proof</u>. (i) $\Rightarrow$ (ii). Let $\alpha,\beta\in Y$ and $\alpha>\beta$. Then $e_\alpha \ \tau \ e_\beta$ $\Rightarrow (\forall\gamma\in Y)\ e_\alpha e_\gamma \ \tau \ e_\beta e_\gamma \Rightarrow (\forall\gamma\in Y)\ e_{\alpha\gamma} \ \tau \ e_{\beta\gamma} \Rightarrow (\forall\gamma)\ \varphi_{\alpha\gamma,\beta\gamma}$ is one-one.

(ii) $\Rightarrow$ (iii). Let $x_\gamma\in e_{\beta\gamma}\varphi^{-1}_{\gamma\ \beta\gamma}$. Then

$$x\in e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma} \Leftrightarrow x_\gamma\varphi_{\gamma,\beta\gamma} = e_{\beta\gamma} \Rightarrow e_{\beta\gamma} = (x_\gamma\varphi_{\gamma,\alpha\gamma})\varphi_{\alpha\gamma,\beta\gamma}$$
$$\Rightarrow x_\gamma\varphi_{\gamma,\alpha\gamma} = e_{\alpha\gamma} \text{ (because } \varphi_{\alpha\gamma,\beta\gamma} \text{ is one-one)}$$
$$\Rightarrow x_\gamma \in e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma}.$$

(iii) $\Rightarrow$ (i). Let $X_\gamma \in e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma}$. Then

$$X_\gamma \in e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} \Leftrightarrow X_\gamma\varphi_{\gamma,\alpha\gamma} = e_{\alpha\gamma}$$

$$\Rightarrow e_{\alpha\gamma}\varphi_{\alpha\gamma,\beta\gamma} = (X_\gamma\varphi_{\gamma,\alpha\gamma})\varphi_{\alpha\gamma,\beta\gamma} = X_\gamma\varphi_{\gamma,\beta\gamma}$$

$$\Rightarrow X_\gamma\varphi_{\gamma,\beta\gamma} = e_{\beta\gamma} \Rightarrow X_\gamma \in e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}.$$

Hence, $e_{\alpha\gamma}\varphi^{-1}_{\gamma,\alpha\gamma} \subseteq e_{\beta\gamma}\varphi^{-1}_{\gamma,\beta\gamma}$. According to (iii) and Theorem 4 we have $e_\alpha \,\tau\, e_\beta$.

An inverse semigroup is <u>E-disjunctive</u> if and only if $\tau = \mathcal{E}$ [3].

<u>Theorem 6</u>. A Clifford semigroup $S=[Y; G_\alpha, \varphi_{\alpha,\beta}]$ is E-disjunctive if and only if $(\forall\alpha,\beta\in Y)(\alpha>\beta \Rightarrow (\exists\gamma\in Y)\varphi_{\alpha\gamma,\beta\gamma}$ is not one-one).

<u>Proof</u>. If there exist $\alpha,\beta$ Y such that $\alpha>\beta$ and the homomorphism $\varphi_{\alpha\gamma,\beta\gamma}$ is one-one for all $\gamma\in Y$, then $e_\alpha\tau e_\beta$ by Theorem 5. Hence, $\tau\neq\mathcal{E}$. Converesely, let $\tau\neq\mathcal{E}$. Then there exist $\alpha,\beta\in Y$ such that $\alpha\neq\beta$ and $e_\alpha\tau e_\beta$, from which it follows that $e_\alpha\tau e_{\alpha\beta}$. We distinguish three cases: $\alpha\beta=\alpha, \alpha\beta=\beta$ and $\alpha>\alpha\beta$. If $\alpha\beta=\alpha$ then $\beta>\alpha$. Since $e_\alpha\tau e_\beta$ it follows that $\varphi_{\beta\gamma,\alpha\gamma}$ is one-one for all $\gamma\in Y$ by Theorem 5. By analogy, $\alpha\beta=\beta$ implies $\varphi_{\alpha\gamma,\beta\gamma}$ is one-one for all $\gamma\in Y$. If $\alpha>\alpha\beta$ *then* $e_\alpha\tau e_{\alpha\beta}$ implies $\varphi_{\alpha\gamma,\alpha\beta\gamma}$ is one-one for all $\gamma\in Y$ by Theorem 5.

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# Completions of Models and Galois Thery

Žarko Mijajlović

## 1. Completion of models.

It is often considered the following problem: Given two classes $\mathfrak{M}$, $\mathfrak{M}'$ of structures , define a map $\sigma$ so that for $\underline{A} \in \mathfrak{M}$ the pair $\sigma(\underline{A}) = (\underline{B}, p)$ completes $\underline{A}$ in a certain way, where $\underline{B} \in \mathfrak{M}'$ and $p : \underline{A} \longrightarrow \underline{B}$ is a morphism. In order to make the consideration clear, we assume that $\mathfrak{M}, \mathfrak{M}'$ are classes of models respectively in languages $L$, $L'$ so that $L \subseteq L'$, and that $\mathfrak{M}, \mathfrak{M}'$ are closed under isomorphic images. For $\underline{A}, \underline{B} \in \mathfrak{M} \cup \mathfrak{M}'$ by $\mathrm{Mor}(\underline{A}, \underline{B})$ a set of homomorphisms is denoted. Instead of $p \in \mathrm{Mor}(\underline{A}, \underline{B})$ we shall write $p : \underline{A} \longrightarrow \underline{B}$. If $\underline{A}$ is a model, then $A$ stands for the domain of $\underline{A}$.

Definition 1.1. Let $\underline{A} \in \mathfrak{M}$. A pair $(\underline{B}, p)$ is a completion of $\underline{A}$ (in respect to $\mathfrak{M}'$) iff the following holds:

$1^\circ$ $\underline{B} \in \mathfrak{M}'$ , $2^\circ$ $p : \underline{A} \longrightarrow \underline{B}$,

$3^\circ$ For any $\underline{C} \in \mathfrak{M}'$ , any $q : \underline{A} \longrightarrow \underline{C}$ there is $\alpha : \underline{B} \longrightarrow \underline{C}$ such that $\alpha p = q$, i.e. the displayed diagram commutes.

$$\begin{tikzcd} \underline{B} \arrow[rr, "\alpha"] & & \underline{C} \\ & \underline{A} \arrow[ul, "p"] \arrow[ur, "q"'] & \end{tikzcd}$$

Example 1.2. $1^\circ$ $\mathfrak{M}$ is the class of all linear orderings, $\mathfrak{M}'$ is the class of all complete orderings, $\underline{B}$ is obtained from $\underline{A}$ by adjoing to $\underline{A}$ all Dedekind cuts, $p$ is inclusion, $\mathrm{Mor}(\underline{A}, \underline{B})$ is the set of all embeddings from $\underline{A}$ into $\underline{B}$.

$2^\circ$ $\mathfrak{M}$ is the class of all algebraically closed fields, $\underline{B}$ is an algebraic closure of $\underline{A}$, $\mathrm{Mor}(\underline{A}, \underline{B})$ is the set of all embeddings from $\underline{A}$ into $\underline{B}$.

$3^\circ$ $\mathfrak{M}$ is the class of all distributive lattices, $\mathfrak{M}'$ is the set of all Boolean algebras, $\underline{B}$ is a Boolean algebra generated by a lattice of sets in Stone representation theorem isomorphic to $\underline{A}$. $\mathrm{Mor}(\underline{A}, \underline{B})$ is the set of all embeddings from $\underline{A}$ into $\underline{B}$ preserving end-points 0,1 (if they exist in $\underline{A}$).

The following proposition sumarizes some simple properties of this notion:

Proposition 1.3. $1^\circ$ If $(\underline{B}, p)$ is a completion of $\underline{A}$, $\underline{B}' \leq \underline{B}$, $\underline{B}' \in \mathfrak{M}'$ and $p : \underline{A} \longrightarrow \underline{B}'$ then $(\underline{B}', p)$ is a completion of $\underline{A}$.

$2^\circ$ If $\underline{A}$ has a completion $(\underline{B}, p)$ then $\underline{A}$ has a completion $(\underline{C}, q)$ so that $q$ is an inclusion.

3° If $\underline{C} \in \mathfrak{M}'$ , $p:\underline{A} \longrightarrow \underline{C}$, and $\underline{A}$ has a completion in $\mathfrak{M}'$, then $\underline{A}$ has a completion $(\underline{B},p)$ for some $\underline{B} \subseteq \underline{C}$, $\underline{B} \in \mathfrak{M}'$ .

4° If $\underline{A}_1, \underline{A}_2 \in \mathfrak{M}$ , $\alpha:\underline{A}_1 \longrightarrow \underline{A}_2$, and $(\underline{B}_1,p)$, $(\underline{B}_2,q)$ are completions respectively of $\underline{A}_1, \underline{A}_2$, then there is $\beta:\underline{B}_1 \longrightarrow \underline{B}_2$ such that $q\alpha = \beta p$.

Proof: Properties 2° and 3° follow by the abstractness of the class $\mathfrak{M}'$ .

4°
$$\begin{tikzcd} \underline{B}_1 \arrow[r, "\beta"] & \underline{B}_2 \\ \underline{A}_1 \arrow[u, "p"] \arrow[ur, "r"] \arrow[r, "\alpha"] & \underline{A}_2 \arrow[u, "q"'] \end{tikzcd}$$

In the displayed diagram , the map $r=q\alpha$ is constructed first, then $\beta$ .

In some cases it is possible to strength the property 4° in the previous proposition.

Theorem 1.4. Assume $\mathfrak{M}'$ is closed under nonempty intersectins of decreasing chains of models in $\mathfrak{M}'$ , and suppose all homomorphisms in question are injective. Then for any completions $(\underline{B},p)$, $(\underline{C},q)$ of $\underline{A} \in \mathfrak{M}$ there is an isomorphism $\alpha:\underline{B} \xrightarrow{\sim} \underline{C}$ so that $\alpha p=q$.

Proof: Let $(\underline{B},p)$, $(\underline{C},q)$ be completions of $\underline{A}$, and let $\alpha:\underline{B} \longrightarrow \underline{C}$ $\beta:\underline{C} \longrightarrow \underline{B}$ be such that $\alpha p=q$, $\beta q=p$ which exist by the definition of completion. If one of $\alpha,\beta$ is not an isomorphism, then it follows that one of $\alpha$ , $\beta$ is not onto. Let $\underline{B}'= \beta\alpha(\underline{B})$. Thus $\underline{B}' \subseteq \underline{B}$, and as one of $\alpha,\beta$ is not onto it follows $B' \neq B$. Let $a \in A$. As $\alpha p=q$, $\beta q=p$ we have $\beta\alpha p=p$, so $pa = \beta\alpha pa$ , hence $pa \in B'$. Therefore, $p:\underline{A} \longrightarrow \underline{B}'$. Since $\beta\alpha:\underline{B} \xrightarrow{\sim} \underline{B}'$ it follows $\underline{B}' \in \mathfrak{M}'$ . Hence,

(1) $(\underline{B}',p)$ is a completion of $\underline{A}$.

Now we construct a decreasing sequence $\underline{B}=\underline{B}_0 \supsetneq \underline{B}_1 \supsetneq \cdots \supsetneq \underline{B}_\xi \supsetneq \cdots$ where $\xi < |B|^+ = k$, i.e. $k$ is the cuccessor cardinal of card(B). Let $\underline{B}_1=\underline{B}'$. Assume $(\underline{B}_\xi,p)$ has been constructed for all $\xi<\lambda$ , $\lambda<k$. Then $\underline{B}_\xi \subsetneq \underline{B}$, and $(\underline{B}_\xi,p)$ is a completion of $\underline{A}$. If $\lambda = \xi+1$ $\underline{B}_\lambda$ is constructed in the same way, as $\underline{B}'$ has been constructed from $\underline{B}$. If $\lambda<\kappa$ is a limit ordinal, let $\underline{B}_\lambda = \bigcap_{\xi<\lambda} \underline{B}_\xi$. As for all $\xi<\lambda$ $p:\underline{A} \longrightarrow \underline{B}_\xi$ it follows that $B_\lambda$ is nonempty, thus by assumption on $\mathfrak{M}'$ we have $\underline{B}_\lambda \in \mathfrak{M}'$ , and $(\underline{B}_\lambda,p)$ is a completion of $\underline{A}$. Therefore, $\underline{D}= \bigcap_{\xi<\kappa} \underline{B}_\xi$ is nonempty, and $(\underline{D},p)$ is a completion of $\underline{A}$. But for $\xi < k$ $B_\xi - B_{\xi+1} \neq \emptyset$, so $\mathrm{card}(\bigcup_{\xi<\kappa} (B_\xi - B_{\xi+1})) \geqslant k$, what contradicts to $|B_0| < k$. Hence, $\alpha,\beta$ are onto, thus they are isomorphisms. ∎

**Corollary 1.5.** Assume $\mathfrak{M}'$ is as in the previous theorem. Then all completions of $\underline{A} \in \mathfrak{M}$ are isomorphic.

**Corollary 1.6.** Assume $\mathfrak{M}'$ is as in the previous theorem, and let $\alpha : \underline{A}_1 \xrightarrow{\sim} \underline{A}_2$. If $(\underline{B}_1, p)$, $(\underline{B}_2, q)$ are completions respectively of $\underline{A}_1$, $\underline{A}_2$, then there is an isomorphism $\beta : \underline{B}_1 \xrightarrow{\sim} \underline{B}_2$ such that $q\alpha = \beta p$.

Proof: Observe that $(\underline{B}_1, p)$, $(\underline{B}_2, q\alpha)$ are completions of $\underline{A}_1$.

## 2. Embeddings of fields.

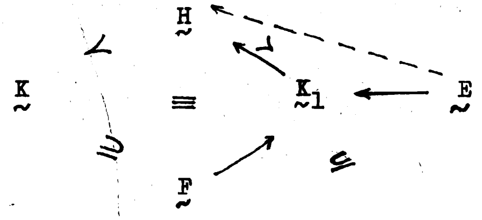
Let $\underline{F}$, $\underline{E}$, $\underline{K}$ be fields such that $\underline{F} \subseteq \underline{E}$, $\underline{F} \subseteq \underline{K}$. Then an embedding $\alpha : \underline{E} \longrightarrow \underline{K}$ is an $\underline{F}$-embedding iff $\alpha$ fixes $F$ pointwise, i.e. $\forall x \in F \ \alpha x = x$. Now we prove a theorem on $\underline{F}$-embeddings by use of model theoretic methods. Notation and all notions from model theory are as in [2] .

Theorem 2.1. Let $\underline{E}, \underline{F}$ be fields, $\underline{F} \subseteq \underline{E}$, $\underline{E}$ is algebraic over $\underline{F}$, and let $\underline{K} \supseteq \underline{F}$ be an arbitrary field. Assume for all fields $\underline{L}$, $\underline{F} \subseteq \underline{L} \subseteq \underline{E}$ such that $|\underline{L}:\underline{F}| < \infty$, there exist an $\underline{F}$-embedding of $\underline{L}$ into $\underline{K}$. Then there exist an $\underline{F}$-embedding of $\underline{E}$ into $\underline{K}$.

Proof: Let $\mathcal{L}_F = \mathcal{L} \cup \{\underline{a} : a \in F\}$, where $\mathcal{L}$ is the language of the theory of the fields, and $\underset{\sim}{K} = (\underline{K}, a)_{a \in F}$, $\underset{\sim}{E} = (\underline{E}, a)_{a \in F}$, and $\Gamma = \mathrm{Th}(\underline{K}) \cup \Delta(\underline{E})$, where $\Delta(\underline{E})$ is the diagram of $\underline{E}$. Without loss of generality we may assume that $\Delta(\underline{E})$ is closed under conjunctions of formulas.

(1) $\Gamma$ is a consistent theory.

We prove (1): Let $\varphi(\underline{b}_1, \ldots, \underline{b}_n) \in \Delta(\underline{E})$, where $\varphi(x_1, \ldots, x_n)$ belongs to $\mathcal{L}_F$, and $b_1, \ldots, b_n \in E$. Let $\underline{L} = \underline{F}(b_1, \ldots, b_n)$ be the subfield of $\underline{E}$ generated by $b_1, \ldots, b_n$ and $F$. Then $\underline{F} \subseteq \underline{L} \subseteq \underline{E}$, and since all elements $b_1, \ldots, b_n$ are algebraic over $\underline{F}$ it follows $|\underline{L}:\underline{F}| < \infty$ , thus there is an $\underline{F}$-embedding $\alpha : \underline{L} \longrightarrow \underline{K}$. As $\alpha$ is an $\underline{F}$-embedding it follows that $\alpha$ is an embedding from $\underset{\sim}{L}$ into $\underset{\sim}{K}$, also. Hence, for $\underset{\sim}{K}' = (\underset{\sim}{K}, b_1, \ldots, b_n)$, $\underset{\sim}{K}'$ is a model of $\mathrm{Th}(\underline{K}) + \varphi(\underline{b}_1, \ldots, \underline{b}_n)$. By Compactness Theorem $\Gamma$ is consistent, i.e. (1) holds.



Let $(\underset{\sim}{K}_1, k_e)_{e \in E}$ be a model of $\Gamma$. Then $\underset{\sim}{K}_1 \equiv \underset{\sim}{K}$, i.e. $\underset{\sim}{K}_1$ is elementary equivalent to $\underset{\sim}{K}$, so there is a model $\underset{\sim}{H}$ such that $\underset{\sim}{K} \prec \underset{\sim}{H}$, and $\underset{\sim}{K}_1 \xrightarrow{\prec} \underset{\sim}{H}$. Since

$(\underset{\sim}{K}_1, k_e)_{e\in E} \models \Delta(\underline{E})$, $\underline{E}$ is embedded into $\underset{\sim}{K}_1$, and therefore into $\underline{H}$ also, say by $\beta$. We prove $\beta(\underset{\sim}{E}) \subseteq \underset{\sim}{K}$. Let $e \in E$. Since $\underline{E}$ is algebraic over $\underline{F}$, there is a polynomial p over $\underline{F}$ so that $p(e)=0$. Let $d_1,\dots,d_n$ be all distinct roots of p in $\underline{H}$. Then $\underline{H} \models \Psi$, where $\Psi$ is $\exists x_1\dots x_n(p(x_1)=0 \wedge \dots \wedge p(x_n)=0 \wedge \bigwedge_{1\le i<j\le n} x_i \neq x_j)$. As $\underset{\sim}{K} \prec \underset{\sim}{H}$ it follows $\underset{\sim}{K} \models \Psi$. Hence, all roots of p in $\underline{H}$ belong to $\underline{K}$. As $p(\beta(e))=0$ it follows $\beta(e) \in K$. Observe that $\beta: \underline{E} \longrightarrow \underline{K}$ is an $\underline{F}$-embedding since $\beta: \underset{\sim}{E} \longrightarrow \underset{\sim}{K}$ is an embedding. This finishes the proof of theorem. ∎

As a consequence of the last theorem we prove that an algebraic closure of a field $\underline{F}$ is a completion of $\underline{F}$ in the sense of the first part of this paper, see also Example 1.2.2°. For convenience we repeat the definition of algebraic closure of a field.

<u>Definition 2.2.</u> Let $\underline{F}$, $\underline{E}$ be fields, and assume $\underline{F} \le \underline{E}$. Then $\underline{E}$ is an algebraic closure of $\underline{F}$ iff
1° $\underline{E}$ is algebraic over $\underline{F}$, 2° $\underline{E}$ is algebraically closed.

<u>Theorem 2.3.</u> Let $\underline{E}$ be an algebraic closure of $\underline{F}$. Then $(\underline{E},p)$ is a completion of $\underline{F}$ in respect to the class $\mathfrak{M}'$ of all algebraically closed fields, where p is the inclusion. The converse also holds, that is if $(\underline{E},p)$ is a completion of $\underline{F}$ in respect to $\mathfrak{M}'$, then $\underline{E}$ is an algebraic closure of $\underline{F}$.

Here, for $\underline{F} \in \mathfrak{M}$, $\underline{E} \in \mathfrak{M}'$, $Mor(\underline{F},\underline{E})$ is the set of all embeddings from $\underline{F}$ into $\underline{E}$.

<u>Proof</u>: Suppose $\underline{E}$ is an algebraic closure of $\underline{F}$, and let $p:\underline{F} \longrightarrow \underline{E}$ be inclusion. Obviously, parts 1° and 2° in Definition 1.1. are satisfied. So we prove 3°. Let $\underline{K}$ be an algebraically closed field and $q:\underline{F} \longrightarrow \underline{K}$ an arbitrary embedding. However, without loss of generality we may assume that q is the inclusion. To prove that $\underline{E}$ is $\underline{F}$-embedded, i.e. that there is $\alpha:\underline{E} \longrightarrow \underline{K}$ with $\alpha p=q$, into $\underline{K}$ according to the Theorem 2.1. it suffices to prove that for all intermediate fields $\underline{L}$, $\underline{F} \le \underline{L} \le \underline{E}$, $|\underline{L}:\underline{F}| < \infty$, $\underline{L}$ is $\underline{F}$-embedded into $\underline{K}$. So let $|\underline{L}:\underline{F}| < \infty$, $\underline{F} \le \underline{L} \le \underline{E}$. Then there are $a_1,\dots,a_n \in E$ algebraic over $\underline{F}$ so that $\underline{L} = \underline{F}(a_1,\dots,a_n)$

By induction on n we prove that $\underline{F}(a_1,\dots,a_n)$ is $\underline{F}$-embedded into $\underline{K}$.

Assume $\gamma:\underline{F}(a_1,\dots,a_{n-1}) \rightarrow \underline{K}$ is an $\underline{F}$-embedding. Let $p(x)$ be an irreducible polynomial for $a_n$ over $\underline{F}(a_1,\dots,a_{n-1})$. Let $b_i = \gamma(a_i)$, $i=1,\dots,n-1$. In that case we have

$\varphi : \underline{F}(a_1,\dots,a_{n-1}) \longrightarrow \underline{F}(b_1,\dots,b_{n-1})$.

Then $p'(x)$, the image of $p(x)$ under $\varphi$ is also irreducible over $\underline{F}(b_1,\dots,b_{n-1})$, as otherwise we would have for some polynomials $p_1', p_2'$ $p'=p_1'p_2'$, and so $p=p_1p_2$, a contradiction. $\underline{K}$ is closed, so there is $b_n \in K$ such that $p(b_n)=0$. Then there is $\psi \supseteq \varphi$ such that $\psi : \underline{F}(a_1,\dots,a_n) \longrightarrow \underline{F}(b_1,\dots,b_n)$, $\psi(a_i)= b_i$, $i=1,\dots,n$.

The existence of $\psi$ is provided by the following well-known

Lemma 2.4. Let $\underline{H}$, $\underline{K}_1$, $\underline{K}_2$ be fields such that $\underline{H} \subseteq \underline{K}_1,\underline{K}_2$. Let $a \in K_1$, $b \in K_2$. If $a,b$ are roots of the same irreducible polynomial $p(x)$, then there is an $\underline{H}$-isomorphism $\lambda : \underline{H}(a) \longrightarrow \underline{H}(b)$, $\lambda(a)=b$, $\underline{H}(a)$ is a subfield of $\underline{K}_1$ generated by $H \cup \{a\}$, and similarly for $\underline{H}(b)$.

Now we prove the converse of the theorem.

Let $(\underline{E},p)$ be a completion of $\underline{F}$, where p is an inclusion. Further, let $\underline{K}$ be an algebraic closure of $\underline{F}$ so that $p : \underline{F} \longrightarrow \underline{K}$. The class $\mathcal{M}'$ of all algebraically closed fields is closed under intersections of decreasing chains, thus by Corollary 1.5. $\underline{E}$ and $\underline{K}$ are $\underline{F}$-isomorphic, hence $\underline{E}$ is also an algebraic closure of $\underline{F}$.

As a consequence of the previous theorem we have thatall results in 1st section are applicable in case of fields and algebraically closed fields.

We observe, also, that the proof of Theorem 2.1. may be obtained by use of ultraproducts, or saturated models instead of Compactness Theorem.

## 3. Galois Theory, infinitary extensions.

In this section we shall present a model-theoretic approach in determining infinite algebraic extensions of fields. The only real hardle to be overcome is the existence of suitible extensions, first of all the existence of an algebraic closure. However, we shall give an outline only, providing proofs at main steps, as we do not pretend to have an original contribution in this section. Therefore, the value of this part is of methodological character only. As a guide we had in mind the first part of [4].

Theorem 3.1. Every field is a subfield of a closed field.

For the proof of this theorem the following lemma is needed.

Lemma For every field $\underline{F}$ and every nonconstant polynomial $p(x)$ over $\underline{F}$ there is a field $\underline{F}'$, $\underline{F} \subseteq \underline{F}'$ and p has a root in $\underline{F}'$.

<u>Proof of Lemma</u>: Let $q(x)\,|\,p(x)$ be a nonreducible polynomial over $\underline{F}$. Let $\underline{F}'= F[x]/(q)$, $(q)$ is the ideal of $F[x]$ generated by $q$.

<u>Proof of Theorem 3.1.</u> Let $C=\{c_p:\ p\in F[x]\}$ be a set of new constant symbols so that for $p\neq q$ $c_p$ and $c_q$ are distinct constant symbols. By above lemma the set $T=$ Theory of fields $+\ \Delta(\underline{F})\ +$ $\{p(c_p)=0:\ p\in \underline{F}[x]\}$ is finitelly consistent, so by Compactness Theorem there is a model $H=(\underline{H},c_p)_{p\in \underline{F}[x]}$ of $T$. Then $\underline{F}$ is embedded into $\underline{H}$, and every polynomial $p$ over $\underline{F}$ of degree $\geqslant 1$ has a root in $\underline{H}$. Hence, there is a sequence $\underline{F}=\underline{F}_0\subseteq \underline{F}_1\subseteq \underline{F}_2\subseteq \dots$ , $\underline{F}_1=\underline{H}$, so that every polynomial $p$ over $\underline{F}_i$ has a root in $\underline{F}_{i+1}$. Thus, $\underline{K}=\bigcup_i \underline{F}_i$ is algebraically closed, and $\underline{F}\subseteq \underline{K}$.

<u>Theorem 3.2.</u> Every field $\underline{F}$ has an algebraic closure.

<u>Proof</u>: Let $\underline{K}$ be an algebraically closed field, $\underline{F}\subseteq\underline{K}$ (such field exists by Theorem 3.1.). Then the subfield $\underline{E}$ of $\underline{K}$ with domain $E=\{a\in K:\ a$ is algebraic over $\underline{F}\}$ is an algebraic closure of $\underline{F}$.

<u>Theorem 3.3.</u> If $\underline{E}$ is an algebraic closure of $\underline{F}$, then $\underline{E}$ is a minimal algebraically closed field extending $\underline{F}$.

<u>Proof</u>: Let $F\subseteq A\subseteq E$, $A$ is an algebraically closed subfield of $\underline{E}$. Let $a\in A$ and $p\in F[x]$ be irreducible polynomial such that $p(a)=0$. As $\underline{A}$ is closed, $p$ splits into linear factors, $p(x)=m(x-a_0)\dots(x-a_n)$, $m\in F$, $a_i\in A$. But $a_i$ are roots of $p(x)$ in $\underline{E}$ also, so by the uniqueness of factorisation of polynimials ($F[x]$ is a unique factorisation domain) it follows that $a$ is one of $a_0,\dots,a_{n-1}$, i.e. $a\in A$.

<u>Theorem 3.4</u>. Let $\underline{E}$ be an algebraically closed field and $\underline{F}$ $\underline{E}$. Then $\underline{E}$ is an algebraic closure of $\underline{F}$ iff $\underline{E}$ is a minimal algebraically closed extension of $\underline{F}$.

<u>Proof</u>: By theorems 2.3, 3.3.

<u>Theorem 3.5.</u> Let $\underline{E}$ be an algebraic closure of $\underline{F}$. Then for every algebraically closed field $\underline{H}$ such that $\underline{F}\subseteq\underline{H}$ there is an $F$-embedding $f:\ \underline{E}\longrightarrow\underline{H}$.

<u>Proof</u>: by Theorem 2.3.

<u>Theorem 3.6</u>. $1^o$ Let $\underline{F}'$, $\underline{G}'$, be algebraic closures of fields $\underline{F}$, $\underline{G}$ respectively. Then every embedding $f:\underline{F}\longrightarrow\underline{G}$ can be extended to an embedding $g:\ \underline{F}'\longrightarrow\underline{G}'$.

$2^o$ Every two algebraic closures $\underline{E}$, $\underline{K}$ of a field $\underline{H}$ are isomorphic.

<u>Proof</u>: Follows by Theorem 2.3, Proposition 1.3, and Theorem 1.4.

<u>Theorem 3.7.</u> Let $\underline{E}$ be a field. Then $\underline{E}$ is not a finite union of it's proper subfields.

<u>Proof</u>: Assume $E = K_1 \cup K_2 \cup \ldots \cup K_n$, $K_i$ are subfields of $\underline{E}$.
Case $1^o$ Suppose E is finite. Then the multiplicative group of $\underline{E}$ is cyclic, say generated by a. Then for some i $a \in K_i$, so $E \leq K_i$, a contradiction.
Case $2^o$ Suppose E is infinite. First we prove the following
<u>Claim</u>: Let $\underline{G}$ be a group and $G_1,\ldots,G_n$ it's subgroups. If G is finite union of some cosets of $G_1,\ldots,G_n$ then for some $k \leq n$ $|G:G_k| < \infty$.
<u>Proof of Claim</u>: Suppose for all $k \leq n$ $|G:G_k| = \infty$, and let $G = C_1^1 \cup \ldots \cup C_{r_1}^1 \cup C_1^2 \cup \ldots \cup C_{r_n}^n$ where $C_j^i$ is a coset og $G_i$. As $|G:G_1| = \infty$, there is a coset C of $G_1$ so that $C \neq C_j^1$, $j \leq r_1$. Further, $C \cap C_j^1 = \emptyset$, $C \leq G$, hence $C \leq C_1^2 \cup \ldots \cup C_{r_n}^n$, thus for some $y \in G$ $C = G_1 y$, and $G_1 \leq C_1^2 y^{-1} \cup \ldots \cup C_{r_n}^n y^{-1}$. For some $x_j$ $C_j^1 = G_1 x_j$, therefore $G = S_1^2 \cup \ldots \cup S_{r_n}^n$ where $S_j^i$ is a coset $G_i$. Thus G is a union of cosets of subgroups $G_2,\ldots, G_n$. Repeating this process we obtain cosets $T_1^n,\ldots,T_m^n$ of $G_n$ so that $G = T_1^n \cup \ldots \cup T_m^n$. Thus $|G:G_n| < \infty$, contradicting our assumption.

Now we return to the proof of the theorem. By Claim, for some $i \leq n$ $|E':K_i'| < \infty$, where $E'$ denotes the additive part of the field $\underline{E}$. Therefore, $K_i'$ is also infinite. Let $a \in E - K_i$. If $k_1, k_2 \in K_i$, $k_1 \neq k_2$, then $K + k_1 a \neq K + k_2 a$, so $|K':K_i'| = \infty$, a contradiction. Thus, $E - K_i = \emptyset$, i.e. $E = K_i$. ∎

The following theorem is the main step in this approach, and it is due to M.Isaacs. It's proof is strongly based on the Theorem 2.1.

<u>Theorem 3.8</u>. Let $\underline{F} \subseteq \underline{E}_1, \underline{E}_2$ be fields, $\underline{E}_1$, $\underline{E}_2$ are algebraic over $\underline{F}$. Let $P_i = \{p \in \underline{F}[x]: \text{p has a root in } \underline{E}_i\}$. Then $\underline{E}_1 \cong \underline{E}_2$ iff $P_1 = P_2$.

<u>Proof</u>: First we prove
<u>Claim</u>: Let $\underline{E}$ be an algebraic extension of $\underline{F}$ so that for all $a \in E$ the minimal polynimial $m_a$ of a over $\underline{F}$ has a root in a field $\underline{K}$. Then there is an $\underline{F}$-isomorphism $f:\underline{E} \longrightarrow \underline{K}$.
<u>Proof of Claim</u>: By Theorem 2.1. it suffices to prove the claim in case $|\underline{E}:\underline{F}| < \infty$. So assume $|\underline{E}:\underline{F}| < \infty$, and let $\underline{K}'$ be an algebraic closure of $\underline{K}$. Let J be the set of all $\underline{F}$-isomorphisms of $\underline{E}$ into $K'$. Then $|J| < \infty$ since $|\underline{E}:\underline{F}| < \infty$. For $f \in J$ we have $F \leq f^{-1}(K) \subseteq E$. Further,

$$(1) \qquad E = \bigcup_{f \in J} f^{-1}(K).$$

Really, if $a \in E$ let $m_a$ be a minimal polynomial of a over $\underline{F}$. By assumption $m_a$ has a root b in $\underline{K}$. There is an $\underline{F}$-isomorphism $h:\underline{F}[a] \longrightarrow \underline{F}[b]$, and therefore there is an $\underline{F}$-embedding $g \supseteq h$, $g:\underline{E} \rightarrow \underline{K}$. So $g \in J$, and $a \in g^{-1}(K)$, i.e. (1) holds.

Now, by Theorem 3.7. for some $f \in J$ $E = f^{-1}(K)$ i.e. $f(E) \subseteq K$. This is the end of the proof of Claim.

Now we continue the proof of the theorem.

We have that for each $a \in E_1$ $m_a$ has a root in $\underline{E}_1$, thus it has a root in $\underline{E}_2$. By Claim there is an $\underline{F}$-isomorphism $f:\underline{E}_1 \longrightarrow \underline{E}_2$.

Similarly, there is an $\underline{F}$-isomorphism $g:\underline{E}_2 \longrightarrow \underline{E}_1$. Let $L = gf(E_1)$. Then $F \subseteq L \subseteq E_1$ and L is $\underline{F}$-isomorphic (via gf) to $\underline{E}_1$. We show $L=E_1$. Let $a \in E_1$. $m_a$ has exactly (say) r roots in $\underline{E}_1$, so, as L is $\underline{F}$-isomorphic to $\underline{E}_1$, $m_a$ has exactly r roots in L. Thus, $a \in L$, i.e. $E_1=L$. Hence, g is onto so g is an $\underline{F}$-isomorphism from $\underline{E}_2$ to $\underline{E}_1$.

<u>Corollary</u>: If $\underline{E}$ is an algebraic extension of $\underline{F}$ and every nonconstant $p \in \underline{F}[x]$ has a root in $\underline{E}$, then $\underline{E}$ is an algebraic closure of $\underline{F}$.

<u>Proof</u>: Let $\underline{E}_1=\underline{E}$ and let $\underline{E}_2$ be an algebraic closure of $\underline{F}$. Using the same notation as in the previous theorem we have $P_1=P_2$, so by the same theorem $\underline{E}_1$ and $\underline{E}_2$ are $\underline{F}$-isomorphic.

<u>Corollary</u>: Let $\underline{F} \leq \underline{E}_1, \underline{E}_2$ be fields, $\underline{E}_1$, $\underline{E}_2$ are algebraic over $\underline{F}$. If $\underline{E}_1$, $\underline{E}_2$ are elementary equivalent over $\underline{F}$, i.e. $(\underline{E}_1 a)_{a \in F} \equiv (E_2, a)_{a \in F}$, then $\underline{E}_1$, $\underline{E}_2$ are $\underline{F}$-isomorphic.

<u>Proof:</u> Observe that $(\underline{E}_1, a)_{a \in F} \equiv (\underline{E}_2, a)_{a \in F}$ implies $P_1=P_2$.

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# THE MAXIMAL SEMILATTICE DECOMPOSITION OF AN n-SEMIGROUP

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The purpose of this paper is to generalize the notion of the maximal semilattice decomposition of a semigroup to n-ary case.

1. Some definitions. Let S be an n-semigroup i.e. an algebra S whit an associative n-operation

$$(x_1,x_2,\dots,x_n) \to x_1x_2\dots x_n$$

S is called an n-semilattice if S is commutative, idempotent and satisfies the following identity

$$x_1^{i_1}x_2^{i_2}\dots x_k^{i_k} = x_1^{j_1}x_2^{j_2}\dots x_k^{j_k}$$

where $i_1+i_2+\dots+i_k = j_1+j_2+\dots+j_k = n$, $i_\nu, j_\nu > 0$.

A congruence $\alpha$ on an n-semigroup S is called a semilattice congruence if $S/\alpha$ is an n-semilattice.

An ideal I of S is said to be completely simple iff

$$x_1x_2\dots x_n \in I \Leftrightarrow x_1 \in I \quad \text{or} \quad x_2 \in I \text{ or } \dots \text{ or } x_n \in I.$$

A subset F of S is a filter in S iff $I = S\backslash F$ is a completely simple ideal.

2. Characterisation of semilattice congruences whit completely simple ideals.

2.1. Let $\Sigma$ be the set of all completely simple ideals in S. Then the relation $\alpha$ defined by

$$x\alpha y \Longleftrightarrow (\forall I \in \Sigma)(x,y \in I \text{ or } x,y \notin I)$$

is a semilattice congruence.

Proof. Since the elements of $\Sigma$ are completely simple ideals, one easily obtains that $\alpha$ is a congruence on S; so it remains to show that $\alpha$ is a semilattice congruence. Let $I \in \Sigma$ and $x_1,x_2,\dots,x_n \in S$. Since I is a completely simple ideal we have that

$$x^n \in I \Leftrightarrow x \in I; \; x_1 x_2 \ldots x_n \in I \Leftrightarrow x_{i_1} x_{i_2} \ldots x_{i_n} \in I$$

where $\nu \to i_\nu$, is a permutation of $\{1,2,\ldots,n\}$;

$$x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} \in I \Leftrightarrow x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k} \in I,$$

$$i_1 + i_2 + \ldots + i_k = j_1 + j_2 + \ldots + j_k = n,$$

which implies

$$x^n \alpha x; \quad x_1 x_2 \ldots x_n x \alpha x_{i_1} x_{i_2} \ldots x_{i_n};$$

$$x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} \, \alpha \, x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k}$$

i.e. $\alpha$ is a semilattice congruence. □

Let us denote the congruence $\alpha$ of 1.1. by $\alpha_\Sigma$. We shall show now that the converse of 1.1. is also true:

2.2. If $\alpha$ is a semilattice congruence on S, then there is a family $\Sigma$ of completely simple ideals in S such that $\alpha = \alpha_\Sigma$.

Proof. Let $\alpha$ be a semilattice congruence on S and let us associate to each element $x \in S$ the subset $F_x$ of S defined by

$$F_x = \{y \in S \mid x \alpha x^{n-1} y\}.$$

The set $F_x$ is nonempty and a filtre in S. Namelly it is clear that $x \in F_x$. If $u_1, u_2, \ldots, u_n \in F_x$, then we have that

$$x \alpha x^{n-1} u_n \alpha x^{n-2} (x^{n-1} u_{n-1}) u_n$$

so we get

$x \alpha x^{(n-1)(n-1)} u_1 u_2 \ldots u_n$, so $u_1 u_2 \ldots u_n \in F_x$. Conversely let $u_1 u_2 \ldots u_n \in F_x$. Then

$$x \alpha x^{n-1} u_1 u_2 \ldots u_n \alpha x^{n-1} u_1 u_2 \ldots u_n u_n^{n-1} \alpha x u_n^{n-1} \alpha x^{n-1} u_n,$$

i.e. $u_n \in F_x$. Since $x^{n-1} u_1 u_2 \ldots u_n \alpha x^{n-1} u_{i_1} u_{i_2} \ldots u_{i_n}$ where $\nu \to i_\nu$, is a permutation of $\{1,2,\ldots,n\}$ we get $u_1, u_2, \ldots, u_n \in F_x$, i.e. $F_x$ is a filtre.

Put $I_x = S \setminus F_x$ and let $\Sigma_\alpha = \{I_x \mid x \in S\}$. So $\Sigma_\alpha$ is a set of completely simple ideals in S. We shall show that $\alpha = \alpha_{\Sigma_\alpha}$.

Let $y \alpha z$, $I \in \Sigma_\alpha$ and $y \notin I_x$. Therefore $y \in F_x$ i.e. $x \alpha x^{n-1} y$. Since $x^{n-1} y \alpha x^{n-1} z$ we have that $z \in F_x$, i.e. $z \notin I_x$. We have thus shown that $\alpha \subseteq \alpha_{\Sigma_\alpha}$. Conversely, let $x \alpha_{\Sigma_\alpha} y$; then $x \in F_x$

implies $y \in F_x$, i.e. $x \alpha x^{n-1}y$. For the same reason $y \in F_y$ implies $y \alpha y^{n-1}x$. But since $\alpha$ is a semilattice congruence, we have

$$x^{n-1}y \,\alpha\, y^{n-1}x \quad \text{and} \quad x \,\alpha\, y. \square$$

Let us note that:

2.3. If $\Sigma_1$ and $\Sigma_2$ are sets of completely simple ideals and $S \notin \Sigma_1$, $S \notin \Sigma_2$, then $\alpha_{\Sigma_1} = \alpha_{\Sigma_2}$ if and only if $\Sigma_1 = \Sigma_2$.$\square$

3. The least semilattice congruence.

It is clear that the intersection $\eta$ of all semilattice congruences is a semilattice congruence. So:

3.1. $x \eta y$ iff for every completely simple ideal I in S $x,y \in I$ or $x,y \notin I$. $\square$

Now we shall give another description of $\eta$. Let us denote by N(x) the minimal filtre in S containing x, i.e. N(x) is the filtre generated by x.

A direct consequence of 3.1. and the definition of N(x) is

3.2. $x \eta y \Longleftrightarrow N(x) = N(y)$.$\square$

The classes of the congruence $\eta$ are called $\eta$-classes. If $x \in S$, then the $\eta$-class which contains x is denoteed by $N_x$. With this notations we have that:

3.3. I) $N_{x_1 x_2 \dots x_n} = N_{x_{i_1} x_{i_2} \dots x_{i_n}}$, where $i_1, i_2, \dots, i_n$ is a permutation of $\{1,2,\dots,n\}$.

II) $N_x n = N_x$.

III) $N_{x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}} = N_{x_1^{j_1} x_2^{j_2} \dots x_k^{j_k}}$, where $i_1 + i_2 + \dots + i_k = j_1 + j_2 + \dots + j_k = n$.

IV) $N_x$ is a subsemigroup of S.$\square$

As in the binary case S is said to be $\eta$-simple iff S has no proper completely simple ideals.

For the n-ary case, and in a similar way as in the binary case, we can prove some analogous properties for the semilattice decomposition, a part of wich is formulated below.

A constructive way for abtaining $N(x)$, which has an inductive nature, is given with the following statement:

3.4. Let $x$ be an element in $S$. Let $N_1(x) = \{x, x^n, x^{2(n-1)+1}, \dots, x^{k(n-1)+1} \dots\}$ and let $N_{n+1}(x)$ be the n-semigroup generated by all elements $y$ in $S$ such that $N_n(x) \cap J(y) \neq \emptyset$, where $J(y) = y \cup S^{n-1}y \cup \cup S^{n-2}yS \cup \dots \cup yS^{n-1} \cup S^{n-1}yS^{n-1}$. Then $N(x) = \bigcup_{n=1}^{\infty} N_n(x)$.□

3.5. If $I$ is an ideal of some $\eta$-class of an n-semigroup $S$, then $I$ has no proper completely simple ideals.

Proof. Let $S$ be an n-semigroup, $z \in S$ and $I$ an ideal of $N_z$. It will suffice to prove that $I$ is the only filtre of $I$. Let $F$ be a filtre of $I$, $a \in F$ and let

$$T = \{x \in S \mid a^{2n-2}x \in F\}.$$

We shall show that $T$ is a filtre of $S$. Let $x_1, x_2, \dots, x_n \in T$; then $a^{2n-2}x_i \in F$ for $i = 1,2,\dots,n$. By the inclusion $F \subseteq I \subseteq N_z$ we have that $N_{a^{2n-2}x_i} = N_{a^{n-1}x_i} = N_{x_i a^{n-1}} = N_z$ which implies $a^{2n-2}x_i,\ x_i a^{2n-2} \in I$. Since $a^{2n-2}x_i,\ a \in F$ it follows that $(a^{2n-2}x_i)a^{2n-2} = a^{2n-2}(x_i a^{2n-2}) \in F$ and $x_i a^{2n-2} \in F$. $N_z$ is an n-subsemigroup of $S$, so $(a^{n-1}x_1x_2a^{n-1})a^{n-2} \in N_z$ which implies

$$N_{a^{n-1}x_1x_2a^{n-1}a^{n-2}} = N_{a^{n-1}x_1x_2} = N_z,$$

and finally $a^{2n-3}x_1x_2 \in I$. Since $F$ is a filtre, then

$$a(a^{2n-3}x_1x_2)a^{3n-n} = (a^{2n-2}x_1)(x_2a^{2n-2})a^{n-2} \in F$$

implies $a^{2n-3}x_1x_2 \in F$. By induction, if follows that $T$ is an n-subsemigroup.

Let $x_1x_2\dots x_n \in T$. By the inclusion $F \subseteq I \subseteq N_z$ we have that $a, a^{2n-2}x_1x_2\dots x_n \in N_z$ and $N(a) \subseteq N(a^{n-1}x_1) \subseteq N(a^{n-2}x_1x_2) \subseteq \dots \subseteq N(ax_1x_2\dots \dots x_{n-1}) \subseteq N(a^{n-2}x_1x_2\dots x_n) = N(a) = N(z)$.

So we have shown that

$$a^{n-1}x_i,\ x_ia^{n-1},\ a^{n-2}x_1x_2,\dots,ax_1x_2\dots x_{n-1}\in N_z.$$

Since J is an ideal it follows that

$$a^{2n-2}x_i,\ x_ia^{2n-2},\ a^{2n-3}x_1x_2,\dots,a^nx_1x_2\dots x_{n-1}\in I.$$

We have that $a^{2n-2}x_1x_2\dots x_n$, $a\in F$, so

$$(a^{2n-2}x_1x_2\dots x_n)a^{2n-2} = a^{n-2}(a^nx_1x_2\dots x_{n-1})(x_na^{2n-2})\in F$$

which implies $a^nx_1x_2\dots x_{n-1}$, $x_na^{2n-2}\in F$. But then

$$a^{2n-2}(x_na^{2n-2}) = (a^{2n-2}x_n)a^{2n-2}\in F \text{ and so } a^{2n-2}x_n\in F,$$

i.e. $x_n\in T$. By reapeting this procedure with $a, a^nx_1x_2\dots x_{n-2}x_{n-1}\in F$ we get $x_{n-1}\in T$. Thus T is a filtre.

Ie is clear that $F\subseteq T\subseteq I$. Let $x\in T\cap I$. Then $a^{2n-2}x\in F$. Since F is a filtre, it follows that $x\in F$. But from $a\in N_z\cap T$ it follows that $N_z\subseteq T$. So $T\cap I = I$ and finally $F = I$.$\square$

As a consequence of 3.5 we conclude that

<u>3.6</u>. Every n-semigroup is an n-semilattice of $\eta$-simple n-semigroups.$\square$

<u>3.7</u>. If I is a completely simple ideal of an n-semigroup S and if $I\cap N_x = \emptyset$, then $I\cap N_x$ is completely simple.$\square$

The following is a consequence of 3.7.

<u>3.8</u>. Every completely simple ideal of an n-semigroup S is a union of $\eta$-classes.$\square$

If $Y_s$ denotes the set of all $\eta$-classes of an n-semigroup S, then the following holds:

<u>3.9</u>. If I is a completely simple ideal of an n-semigroup S, then $J = \{N_x\in Y_s \mid x\in I\}$ is a completely simple ideal in $Y_s$. Conversely, if J is a completely simple ideal in $Y_s$, then $I = \{x\in S \mid N_x\in J\}$ is a completely simple ideal in S.$\square$

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# The dual space of reductive groups over local fields

M. Tadić

Let $G$ be the group of rational points of a connected reductive group defined over nonarchimedean local field $k$ .Then $G$ is a totally disconnected locally compact group.

If $H$ is a Hilbert space and $\pi$ a homomorphism from $G$ into the group of all unitary operators on $H$ such that the mapping $g \to \pi(g)v$ is continuous for all $v \in H$ , then $(\pi,H)$ is called the unitary representation of $G$ . If $H$ has no non-trivial closed $G$-invariant subspaces then we say that $(\pi,H)$ is irreducible unitary representation of $G$ .

The Hecke algebra of $G$ is denoted by $H(G)$ . It is the algebra of all compactely supported locally constant functions on $G$ under the convolution. Let

$$\pi(f) = \int_G f(g)\pi(g)dg$$

for $f \in H(G)$ . Then $H$ is $H(G)$-modul and $H^{\infty} = \pi(H(G))H$ is $H(G)$-submodul. If $(\pi,H)$ is irreducible then $V$ is a simple $H(G)$-modul. In this way the problem of classification of irreducible unitary representations of $G$ can be reduced to the purely algebraic problem of classifying certain classes of simple $H(G)$-modules.

Let $\hat{G}$ denote the dual space of $G$ . It is the set of all unitary equivalence classes of irreducible unitary representations of $G$ . Let $L^1(G)$ be the Banach algebra of all integrable functions on $G$ . For $f \in L^1(G)$ set

$$||f|| = \sup\{||\pi(f)||_{\pi} , (\pi,H_{\pi}) \in \hat{G}\}$$

Then $||\ \ ||$ is a new norm on $L^1(G)$ and completion of $L^1(G)$ with respect to this norm is denoted by $C^*(G)$ . Then $C^*(G)$ is called a $C^*$-algebra of $G$ . Denote by $C^*(G)^{\wedge}$ the set of all equivalence classes of irreducible $*$ -representations of $C^*(G)$ .

Then there is a natural bijection from $C^*(G)^\wedge$ onto $\hat{G}$. Let

$$C\ell(T) = \{\sigma \in C^*(G) \quad ; \quad \bigcap_{\pi \in T} \ker \pi \subseteq \ker \sigma\}$$

for $T \subseteq C^*(G)^\wedge$. Operator $C\ell$ defines so-called Jacobson topology on $C^*(G)^\wedge$. By using the natural bijection $\hat{G}$ becomes a topological space.

Note that we can describe the topology of $\hat{G}$ by algebraic tools, namely in terms of characters.

Now we shall describe the topology of dual space of $SL(2,k)$. In $[0,1] \times \{0,1\}$ we identify $(t,0)$ and $(t,1)$ for $0 < t < 1$ and let $J$ denote the space obtained by this identification. Then $SL(2,k)^\wedge$ is homeomorphic to the disjoint of two copies of $J$, countably many tori $\{z \in C \ ; \ |z|=1\}$ and countably many points. Tori and two copies of $J$ correspond to irreducible subrepresentations of the unitary principal series and the complementary series, points correspond to absolutely cuspidal irreducible representations. Note that $SL(2,k)^\wedge$ is not a Hansdorff space.


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# ON SOME FINITE GRUPOIDS WHOSE EQUATIONAL THEORIES ARE NOT FINITELY BASED

Valentina Harizanov

For an algebra $\mathfrak{A}$, $L(\mathfrak{A})$ denotes its language. By the equational theory of an algebra $\mathfrak{A}$ we mean the set of all identities of the form $t_1=t_2$, where $t_1$ and $t_2$ are terms of the language $L(\mathfrak{A})$, which are satisfied in $\mathfrak{A}$. Equational theory of $\mathfrak{A}$ will be denoted by $Id(\mathfrak{A})$. A basis of the equational theory $Id(\mathfrak{A})$ is every subset of it from which in equational logic every identity from $Id(\mathfrak{A})$ can be derived. R. C. Lyndon proved that the equational theory of every two-element algebra has a finite basis ([2]), and that there exists a finite algebra of finite type whose equational theory is not finitely based. He gave the example of seven-element algebra of type $\langle 0,2\rangle$ with such property ([3]). Later V. V. Vişin ([5]) found four-element grupoid and V. L. Murskiĭ ([4]) found three-element grupoid, both having the mentioned property.

We shall give an example of n-element grupoid, where n is an arbitrary natural number greater than 3, with equational theory which is not finitely based.

<u>Theorem 1</u>. Let a binary operation on the set $A_n=\{a_0,a_1,\dots,a_{n-1}\}$ be defined by

$$a_i\cdot a_j=\begin{cases} a_{j+1}, & \text{if } i>j\geqslant 1;\\ a_0, & \text{otherwise.}\end{cases}$$

If $n\geqslant 4$, then the equational theory of the grupoid $\mathfrak{A}_n=(A_n,\cdot)$ is not finitely based.

Proof. We shall not give a complete proof, because some parts of it can be proved as in [5].

By a left-associated term we mean a term whose all left parentheses are at the very beginning; in this case it is of the form $(\ldots((x_{i_1}\cdot x_{i_2})\cdot x_{i_3})\cdot\ldots)\cdot x_{i_s}$. The value of such term in the algebra $\mathfrak{A}_n$ can be easily determined, i. e.

$$(\ldots(a_{i_1}\cdot a_{i_2})\cdot\ldots)\cdot a_{i_s}=\begin{cases} a_{i_s+1}, & \text{if } i_1 > i_2 \geqslant\ldots\geqslant i_{s-1}\geqslant i_s\geqslant 1;\\ a_0, & \text{otherwise;}\end{cases}$$

where $a_{i_1},\ldots,a_{i_s}\in A_n$. Therefore the following identities belong to $\mathrm{Id}(\mathfrak{A}_n)$:

(I) $(x\cdot y)\cdot y=x\cdot y$,

($I_k^p$) $((\ldots((x\cdot y_1)\cdot y_2)\cdot\ldots)\cdot y_k)\cdot y_1=$

$((\ldots((x\cdot y_{p(1)})\cdot y_{p(2)})\cdot\ldots)\cdot y_{p(n)})\cdot y_{p(1)}$, where $k\geqslant 2$ and $p\in S_k$ ($S_k$ is the set of all nonidentical permutations of the set $\{1,2,\ldots,k\}$ ).

If the identity $t_1=t_2$ belongs to $\mathrm{Id}(\mathfrak{A}_n)$, then at least one of the following conditions is fulfilled:

(a) $t_1$ and $t_2$ have in $\mathfrak{A}_n$ a constant value $a_0$,

(b) $t_1$ and $t_2$ are left-associated terms,

(c) None of the terms $t_1$ and $t_2$ is left-associated.

Denote by $\mathrm{Id}'(\mathfrak{A}_n)$ the set of all identities $t_1=t_2$ from $\mathrm{Id}(\mathfrak{A}_n)$ which satisfy (b) and do not satisfy (a). Similarly as in [5] we can prove that every set of identities from $\mathrm{Id}'(\mathfrak{A}_n)$ whose terms on both left and right sides have at most m occurences of variables can be derived from the set $I_m=\{I\}\cup\bigcup_{2\leqslant i<m}\{I_i^p\colon p\in S_i\}$ ( $\bigcup_{i\in\phi}S_i\overset{\mathrm{def}}{=}\phi$).

Assume that the equational theory $\mathrm{Id}(\mathfrak{A}_n)$ has a finite basis $\mathfrak{B}$. Then from the set $\mathfrak{B}'=\mathfrak{B}\cap\mathrm{Id}'(\mathfrak{A}_n)$ all identities from $\mathrm{Id}'(\mathfrak{A}_n)$ can be derived. Since $\mathfrak{B}'$ is a finite set, the number of occurences of variables on each side of the terms from $\mathfrak{B}'$ is less than some fixed natural number m. Thus, from Im we can derive all identities from $Id'(\mathfrak{K}_n)$. However for $p \in S_m$ the identity $(I_m^p)$ belongs to $Id'(\mathfrak{K}_n)$ but it is not derivable from Im. From the obtained contradiction it follows that $Id(\mathfrak{K}_n)$ is not finitely based. □

Remark. Grupoid $\mathfrak{K}_4$ from the previous theorem is isomorphic to the grupoid

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 3 | 1 | 0 |

from [5]. The function $f=\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & 2 & 1 & 3 \end{pmatrix}$ is corresponding isomorphism.

The following theorem is immediate consequence of Theorem 1. and [4].

Theorem 2. For each natural number $n \geq 3$ there exists n-element grupoid such that its equational theory is not finitely based.

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# ON CAUCHY SEQUENCES OF A VALUED FIELD*

Jusuf Alajbegović

## 1. Introduction

The purpose of this note is to prove several statements what will show thw relationship between the notion of the Cauchy sequences and the notion of the distinguished pseudo-Cauchy family in a valued field.

The idea for this occured in clearing out some difficulties appearing in the proof of Ribenboim's characterization of the valued fields which are complete by stages.

## 2. Basic notions

We first list the necessary definitions and background results from [5] .

Definition 1. An additive abelian group $\Gamma$ is called totally ordered if there is a total ordering „$\leqslant$" on $\Gamma$ such that :

$$(\forall\alpha,\beta,\gamma\in\Gamma)\ \alpha\leqslant\beta \Rightarrow \alpha+\gamma\leqslant\beta+\gamma \quad .$$

A subgroup $\Delta$ of a totally ordered abelian group $\Gamma$ is called isolated if: $(\forall\gamma\in\Gamma)(\forall\delta\in\Delta)\ 0\leqslant\gamma\leqslant\delta\Rightarrow\gamma\in\Delta$ .

The rank of a totally ordered additive abelian group is the ordinal type of its set of nonzero isolated subgroups ordered under $\supseteq$ .

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<u>Definition 2.</u> A valued field is a pair $(K,v)$ where $K$ is a field and $v$ is mapping from $K$ onto a set $\Gamma \cup \{\infty\}$, where $\Gamma$ is a totally ordered additive abelian group and $\infty \notin \Gamma$ is a symbol such that $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$ and $\gamma < \infty$ for each $\gamma \in \Gamma$, for which the following conditions are satisfied:

i) $v(K\setminus\{0\}) = \Gamma$ , $v(0) = \infty$ ;

ii) $v(xy) = v(x) + v(y)$ for all $x,y$ in $K$ ;

iii) $v(x+y) \geq \min\{v(x),v(y)\}$ for all $x,y$ in $K$ .

Group $\Gamma = \Gamma_v = v(K\setminus\{0\})$ is called the value group of the valuation $v$ , and the rank of the group $\Gamma_v$ is called the rank of $v$ . The set $A_v = \{x \in K : v(x) \geq 0\}$ is a subring of $K$ and $K$ is the field of fractions of $A_v$ . It is well known that there is a bijective correspondence between the set of prime ideals $P$ of the ring $A_v$ and the set of isolated subgroups $\Delta$ of totally ordered group $\Gamma_v$ given by

$$P \mapsto v(A_P\setminus P) \quad \text{or by} \quad \Delta \mapsto v^{-1}(\{\gamma \in \Gamma_v : \delta\in\Delta \Rightarrow \delta < \gamma\}) \cup \{0\}$$

such that $P \subsetneq P'$ implies $\Delta \supsetneq \Delta'$ where $\Delta$ correspond to $P$ and $\Delta'$ to $P'$ . Also, if $P \subsetneq P'$ are prime ideals of $A_v$ and $\Delta \supsetneq \Delta'$ are the corresponding isolated subgroups of $\Gamma_v$ , then $(A_P/P, v_{P'}/P)$ is a valued field if $v_{P'}/P$ is defined by:

$$(\forall x \in A_P\setminus P)\ v_{P'}/P\,(x+P) = \theta(v(x)) ,$$

where $\theta : \Delta \to \Delta/\Delta'$ is the canonical mapping.

Avalued topology $\mathcal{T}_v$ on a valued field $(K,v)$ is the topology with the open neighbourhoods base for $0$ consisting of all the sets $V_\gamma(0) = \{x \in K : v(x) > \gamma\}$ , $\gamma \in \Gamma_0$ , where $\Gamma_0$ is cofinal in $\Gamma_v$ . A valued field $(K,v)$ is called complete if the field $K$ is complete in the topology $\mathcal{T}_v$ .

<u>Definition 3.</u> A family $\langle a_\tau \rangle_{\tau \in T}$ of elements from a valued field (K,v) is called pseudo-Cauchy (or P.C.F.) if the set T is well ordered, without the last element, and :

$(\exists \tau_0 \in T)(\forall \tau, \tau', \tau'' \in T)\ \tau_0 \leq \tau < \tau' < \tau'' \Rightarrow v(a_\tau - a_{\tau'}) \quad v(a_{\tau'} - a_{\tau''})$

If the family $\langle a_\tau \rangle_{\tau \in T}$ satisfies also the following two conditions:

i) $(\forall \tau \in T)\ a_\tau \in A_v$ ;

ii) $\{ x \in K : \tau_0 \leq \tau \Rightarrow \gamma_\tau \leq v(x) \}$ is a prime ideal of $A_v$, distinct from $M_v = \{ x \in K : v(x) > 0 \}$, where $\gamma_\tau = v(a_\tau - a_{\tau'})$ for $\tau_0 \leq \tau < \tau'$, then $\langle a_\tau \rangle_{\tau \in T}$ is called the distinguished pseudo-Cauchy family (or D.P.F.) of a valued field (K,v) .

It is not difficult to show that a P.C.F. $\langle a_\tau \rangle_{\tau \in T}$ with elements in $A_v$ is distinguished if and only if the set $\langle \gamma_\tau \rangle_{\tau \geq \tau_0}$ is cofinal in a nonzero isolated subgroup of $\Gamma_v$ .

If T is a well ordered set without the last element and $\langle a_\tau \rangle_{\tau \in T}$ a family of elements from a valued field (K,v), then an element $a \in K$ is called pseudo-limit of the family $\langle a_\tau \rangle_{\tau \in T}$ if there is an element $\tau_0 \in T : \tau_0 \leq \tau < \tau' \Rightarrow v(a - a_\tau) < v(a - a_{\tau'})$. In that case the family $\langle a_\tau \rangle_{\tau \in T}$ is called pseudo-convergent to a , and we write $a \in \underset{\tau \in T, (K,v)}{\mathrm{Ps.lim}}\, a_\tau$ .

<u>Definition 4.</u> A valued field (K,v) is complete by stages if the following two conditions are satisfied :

i) $(\forall P \subsetneq P'$ : succesive prime ideals in $A_v$ )
$(A_P/P, v_{P'}/P)$ is complete valued field ;

ii) If T is a well ordered set, $\langle P_\tau \rangle_{\tau \in T}$ a family of prime ideals of $A_v$ and $\langle a_\tau \rangle_{\tau \in T}$ a family of elements in $A_v$ such that : $(\forall \tau, \tau' \in T)\ \tau \leq \tau' \Rightarrow P_{\tau'} \subseteq P_\tau \wedge a_\tau - a_{\tau'} \in P_\tau$ , then there is an element $a \in A_v : (\forall \tau \in T)\ a - a_\tau \in P_\tau$ .

3. Some remarks concerning the proof of Ribenboim's theorem

In [4] and [5] Ribenboim stated and proved the following theorem :

Theorem A A valued field (K,v) is complete by stages if and only if every D.P.C.F. in (K,v) has a pseudo-limit in K .

The end of the proof of that theorem in [5] is incorect, as it was shown in [1] , and an earlier proof of the author in [4] is incomplete . Proposition 3.1. and its proof clears out this situation. We first give a part of the statement contained in the Remark 2.p.48.[6] .

Lema 3.1. Let $T = W(\lambda)$ denote the set of ordinal numbers less than a limit ordinal $\lambda$ , $\langle M_\tau \rangle_{\tau \in T}$ a family of ideals of a ring A , and $\langle a_\tau \rangle_{\tau \in T}$ a family of elements from A such that

$$(\forall \tau, \tau' \in T) \ \tau < \tau' \Rightarrow M_{\tau'} \subseteq M_\tau \wedge a_\tau - a_{\tau'} \in M_\tau \ .$$

Then there is an element $\tau_0 \in T$ such that: $(\forall \tau \in T) \ a_\tau - a_{\tau_0} \in M_\tau$ or $((\exists T' \text{cofinal in } T)(\forall \tau, \tau' \in T') \ \tau < \tau' \Rightarrow a_\tau - a_{\tau'} \in M_\tau \setminus M_{\tau'}$ .

Proposition 3.1. Let (K,v) be a valued field such that every distinguished pseudo-Cauchy family in (K,v) has a pseudo-limit in (K,v). If T is a well ordered set,without the last element, $\langle P_\tau \rangle_{\tau \in T}$ a family of prime ideals of $A_v$ and $\langle a_\tau \rangle_{\tau \in T}$ a family of elements in $A_v$ such that :

$$(\forall \tau, \tau' \in T) \ \tau \leq \tau' \Rightarrow P_{\tau'} \subseteq P_\tau \wedge a_\tau - a_{\tau'} \in P_\tau \ ,$$

then there exists an element $a \in A_v$ such that

$$a - a_\tau \in P_\tau \ \text{for all} \ \tau \in T \ .$$

## 4. Cauchy-sequences and D.P.C. families in a valued field

<u>Theorem 4.1.</u> Let $(K,v)$ be a valued field, $\mathrm{rank}(v) = 1$, and $\langle a_n \rangle_{n\in\omega}$ Cauchy-sequence in $(K,v)$ with $a_n \in A_v$ for all $n \in \omega$. Then the following holds:

$$((\exists n_0 \in \omega)\ a_{n_0} = \lim_{n\in\omega,(K,v)} a_n) \vee ((\exists N - \text{cofinal in } \omega)\ \langle a_n \rangle_{n\in N} \text{ is D.P.C.F. in } (K,v)) .$$

<u>Proof</u> Since by assumption $\mathrm{rank}(v) = 1$, we can choose an increasing sequence $\langle \gamma_n \rangle_{n\in\omega}$ of positive elements in $\Gamma_v$ such that $\langle \gamma_n \rangle_{n\in\omega}$ is cofinal in $\Gamma_v$. Thus, if we denote $M_n = \{x \in K : v(x) > \gamma_n\}$ for all $n \in \omega$, $\langle M_n \rangle_{n\in\omega}$ is the basis of open neighbourhoods for the element 0 in the valued topology on $(K,v)$. The sequence $\langle a_n \rangle_{n\in\omega}$ is Cauchy in $(K,v)$, hence: $(\forall n\in\omega)(\exists i_n \in \omega : n < i_n)(\forall m \in \omega)\ i_n < m \Rightarrow a_{i_n} - a_m \in M_n$.

Thus we can choose an increasing sequence $\langle i_n \rangle_{n\in\omega}$ and a sequence $\langle b_n \rangle_{n\in\omega}$, with $b_n = a_{i_n}$ for all $n \in \omega$, such that:

$$(\forall n,n' \in \omega)\ n < n' \Rightarrow b_n - a_{i_{n'}} \in M_{n'}$$

In particular $\langle \gamma_{i_n} \rangle_{n\in\omega}$ is cofinal in $\Gamma_v$. Lema 2.1. implies that either there exists some $n_1 \in \omega$ with $b_{n_1} - b_n \in M_n$ for all $n \in \omega$, or there exists a set $N$ cofinal in $\omega$ such that $b_n - b_{n'} \in M_n \setminus M_{n'}$ for all $n < n'$ in $N$. In the first case we take $n_0 = i_{n_1}$, hence $a_{n_0} = \lim_{n\in\omega} a_n$, since $v(b_{n_1} - b_n) > \gamma_n$ for all $n \in \omega$ and $\langle \gamma_n \rangle_{n\in\omega}$ is cofinal in $\Gamma_v$.

In the second case, $\langle b_n \rangle_{n\in N}$ is obviously P.C.F. in $(K,v)$. Furthermore, the family $\langle b_n \rangle_{n\in N}$ is distinguished since:

$$(\forall n \in \omega)\ n < n' \Rightarrow n < i_n < i_{n'} \Rightarrow \gamma_n < \gamma_{i_{n'}} \Rightarrow a_{i_n} - a_{i_{n'}} \in M_{i_{n'}} \Rightarrow$$
$$\Rightarrow v(a_{i_n} - a_{i_{n'}}) > \gamma_{i_{n'}} > \gamma_n \text{ , i.e. } v(b_n - b_{n'}) > \gamma_n .$$

Thus , $\langle b_n \rangle_{n\in N}$ is cofinal in $\Gamma_v$ , and the theorem is established .

<u>Theorem 4.2.</u> Let (K,v) be a valued field with the valued topology satisfying the first axiom of separability .
If $\langle a_n \rangle_{n\in\omega}$ is a Cauchy sequence of (K,v) with elements in $A_v$ , then there exists a set N cofinal in $\omega$ such that $\langle a_n \rangle_{n\in N}$ is D.P.C.F. in (K,v) if and only if the set $\{a_n : n\in\omega\}$ is infinite .

<u>Proof</u> If $\langle a_n \rangle_{n\in N}$ is D.P.C.F. in (K,v) , then the set $\{a_n : n\in\omega\}$ is infinite , since :

$(\exists n_0\in\omega)(\forall n,n',n''\in\omega)\ n_0 \leqslant n < n' < n'' \Rightarrow v(a_n - a_{n'}) < v(a_{n'} - a_{n''}) \Rightarrow$
$\Rightarrow a_n - a_{n'} \neq 0$ .

Conversely , let $\{a_n : n\in\omega\}$ be an infinite set and $\langle a_n \rangle_{n\in\omega}$ Cauchy sequence in (K,v) , with $a_n \in A_v$ for all $n\in\omega$ .
In the proof of the previous theorem the assumption that rank(v) = 1 is used to show the existence of a sequence $\langle \gamma_n \rangle_{n\in\omega}$ cofinal in $\Gamma_v$ and $0 < \gamma_n < \gamma_{n'}$ , for all $n < n'$ in $\omega$ . Such a sequence exists in this case becouse the valued topology on (K,v) satisfais the first axiom of separability . Hence we can apply the conclusions of Theorem 4.1. for the sequence $\langle a_n \rangle_{n\in\omega}$. So , it can happen that there exists some $n_0\in\omega$ such that $a_{n_0} = \lim\limits_{n\in\omega,(K,v)} a_n$ . In that case we can take $N = \{n\in\omega : a_n \neq a_{n_0}\}$ and it is obvious that $a_{n_1} \neq \lim\limits_{n\in N,(K,v)} a_n$ for all $n_1$ in N , and $\langle a_n \rangle_{n\in N}$ is Cauchy sequence in (K,v).
Thus, Theorem 4.1. allows us to conclude that there exists a set N′ cofinal in N such that $\langle a_n \rangle_{n\in N'}$ is D.P.C.F. in (K,v) and of course , N′ is cofinal in $\omega$ .

<u>Example</u> According to [2] ( Exercise 2 , p.p.452-453 ) the field $K=k((X))^{\Gamma}$ of formal power series with coefficients in a given field k and exponents in a given totally ordered abelian group $\Gamma$ , with respect to the natural valuation $\vartheta$ on K , is a valued field . Furthermore , $\Gamma_{\vartheta}=\Gamma$ .

Let $\Gamma = \underset{i\in I}{H}\Gamma_i$ - the Hahn product of the groups $\Gamma_i = R$ ( R - the additive group of real numbers ) for all $i \in I$ . Then $\Gamma_{\vartheta}$ is its own principal isolated subgroup in the case that I=N (N - the set of natural numbers ) , and is equal to the union of the sequence of isolated subgroups of $\Gamma_{\vartheta}$ distinct from $\Gamma_{\vartheta}$ in the case that I=Z (Z-the set of integers). Obviously , in both cases , rank($\Gamma_{\vartheta}$) is infinite . Furthermore , the valued topology on $(K,\vartheta)$ satisfais the first axiom of separability as it was shown in [3] .

<u>Remark</u> Theorems 4.1. and 4.2. clear out the situation when a Cauchy sequence in a valued has a subsequence which is a distinguished pseudo-Cauchy family in (K,v) .

In the case of discrete , rank one , valuation v on a field K it is not hard to show (see [5] ) that if $\langle a_{\tau}\rangle_{\tau\in T}$ is P.C.F. in (K,v) , then there exists an increasing sequence $\langle \tau_n\rangle_{n\in\omega}$ which makes $\langle a_{\tau_n}\rangle_{n\in\omega}$ a Cauchy sequence in (K,v) .

In particular , if (K,v) is topologicaly complete , then there exists an element $a\in K$ such that $a=\lim\limits_{n\in\omega} a_{\tau_n}$ , hence one can show that $a \in \operatorname{Ps.lim}\limits_{\tau\in T,(K,v)} a_{\tau}$ .

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# SOME REMARKS ON BOOLEAN EQUATIONS

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In this article we discuss mainly systems of Boolean equations i.e. equations of the form $AX = B$ and $XA = B$, where $A$, $B$, $X$ are matrixes whose elements are im Boolean algebra $B$ (theorem 3). The theorems 1. and 2. that follow, we shall use in the proof of the theorem 3.

Theorem 1 (Vanght). If $S$ is a Horn sentence of the language of Boolean algebras and $B_2 \models S$, then $S$ holds on all Boolean algebras.

Proof.: See, for example, |1| or |2|.

Theorem 2. Let $f : B^n \times B^m \to B$ and $g : B^n \times B^m \to B$ be Boolean functions and let for each $Y \in B^m$ the equation, in $X$, $f(X,Y) = 0$ be consistent. The following conditions

(a) $(\forall X)(f(X,Y) = 0 \Rightarrow g(X,Y) = 0)$

(b) $(\forall X)(g(X,Y) \leq f(X,Y))$

are equivalent.

We can write this theorem in the form

$$(\forall Y)(\exists X) f(X,Y) = 0 \Rightarrow (\forall Y)\left[(\forall X)(f(X,Y) = 0 \Rightarrow g(X,Y) = 0)\right.$$
$$\left.\Longleftrightarrow (\forall X)(g(X,Y) \leq f(X,Y))\right].$$

Proof.: Since for any $Y \in B$ there is $X \in B$ such that $f(X,Y) = 0$, let $X_Y$ be the element that is related to $Y$ such that $f(X_Y,Y) = 0$. Assume that (a) holds. Then for each $Y$

$$f(X_Y,Y) = 0 \Rightarrow g(X_Y,Y) = 0$$

i.e. $X_Y$ is the solution of the equation $g(X,Y) = 0$. According to Löwenheim's theorem (|3|, theorem 2.11), $f((X_Y,Y)f(X,Y) \cup (X,Y)f'(X,Y)) = 0$. Since (a) holds, we have

$$g((X_Y,Y)f(X,Y) \cup (X,Y)f'(X,Y)) = 0 .$$

Using the well known equality

$$h(tV \cup t'W) = hf(V) \cup t'f(W)$$

where $V = (V_1,\ldots,V_n)$, $W = (W_1,\ldots,W_n)$, $h : B^n \to B$, we have

$$g(X_Y,Y)f(X,Y) \cup g(X,Y)f'(X,Y) = 0 \quad \text{i.e.} \quad g(X,Y)f'(X,Y) = 0 .$$

We can write the last equality as $g(X,Y) \leq f(X,Y)$.
The proof for (b) $\Longrightarrow$ (a) is obvious.

Definition 1. Let $Q = ||q_{ij}||$ $i = 1,\ldots, m;\ j=1,\ldots,n$ and $R = ||r_{jk}||$ $j = 1,\ldots,n;\ k = 1,\ldots,p$ be two matrixes with elements belonging to Boolean algebra B. Then the product QR is the mxp matrix such that

$$(QR)_{ik} = \bigcup_{j=1}^{n} q_{ij}r_{jk} \qquad (i=1,\ldots,m;\quad k=1,\ldots,p)$$

Matrixes $Q^T$ and $Q'$ are introduced as:

$$(Q^T)_{ij} = q_{ji}$$
$$\qquad\qquad (i=1,\ldots,n;\quad j=1,\ldots,n)$$
$$(Q')_{ij} = q'_{ij}$$

and matrix I is defined by

$$I = ||\delta_{ij}|| ,$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$ $(i \neq j)$.

Definition 2. A vector $(x_1,\ldots,x_n) \in B^n$ is said to be normal if

$$\bigcup_{i=1}^{n} x_i = 1$$

and orthogonal if

$$x_ix_j = 0 \quad (i,j = 1,\ldots,n \ \text{and}\ i \neq j).$$

A vector that is normal and orthogonal is said to be orthonormal.

<u>Theorem 3.</u> Let $A = ||a_{ij}||$ and $I = ||\delta_{ij}||$ be square matrixes of order n. The following properties are equivalent:

(a) A has a right inverse, i.e. the system $AX = I$ is consistent,

(b) A has a left inverse, i.e. the system $XA = I$ is consistent,

(c) for any square matrix C of order n, the system $AX = C$ is consistent,

(d) $(\forall X)(\forall Y)(AX = AY \Longrightarrow X = Y)$,

(e) $A^T A = I$,

(f) $(A^T I')' = (I'A^T)' = A^T$,

(g) $\bigcup_{j=1}^{n} a_j^i = 1 \qquad (i = 1,\dots,n)$

where $a_j^i = a_{ij} \prod_{\substack{h=1 \\ h\neq i}}^{n} a'_{hj}$,

(h) each row and each column of the matrix A is orthonormal.

The proof of this theorem is given in "Boolean functions and equations" by S. Rudeanu using a few lemmas. Ž. Mijajlović proved (c)$\Longleftrightarrow$(e) by means of Vanght's theorem, his proof being much shorter i.e. it is reduced on the proof in Boolean algebra $B_2$. In this way we mainly prove the ofher equivalences.

<u>Proof:</u>

(a) $\Longrightarrow$ (c): Right multiplying $A.A^{-1} = I$ by C we have $A.A^{-1}.C = C$, i.e. the matrix $A^{-1}.C$ satisfies the equation $AX = B$ for any C.

(c) $\Longrightarrow$ (a): If $(\forall C)(\exists X)(AX = C)$ holds, then for $C = I$ we have $(\exists X)(AX = I)$.

(a) $\Longrightarrow$ (h): Let us now write the sentence "Each row and each column of the matrix A is orthogonal" in the more favourable way. The orthogonality of the i-th row is defined as

$$a_{ij}a_{ik} = 0 \quad (j,k = 1,\dots,n \text{ and } j \neq k),$$

namely, we have the conjuction of $n^2-n$ equalities.
The normality of i-th row is defined by $\bigcup_{j=1}^{n} x_{ij} = 1$. It means that orthonormality of a row of the matrix is defined by the conjuction of $n^2-n+1$ equalities and orthonormality of all rows

is defined by the conjuction of $n(n^2-n+1)$ equalities. Let $p = n(n^2-n+1)2$. Similarly, the orthonormality of all columns can be written as $\frac{p}{2}$ equalities. Namely, the sentence (h) can be expressed as the conjuction of p equalities. If we write all these equalities in the form $m = 0$ (i.e. 0 on the right side) and if we denote all the left sides of these equalities by $m_1, m_2, \ldots, m_p$, then we can write the sentence (h) in the form

$$\bigcup_{i=1}^{p} m_i = 0.$$

Then (a) $\Longrightarrow$ (h) can be written as

$$(\exists X)(AX = I) \Longrightarrow \bigcup_{i=1}^{p} m_i = 0$$

i.e. $$(\forall X)(AX = I \Longrightarrow \bigcup_{i=1}^{p} m_i = 0).$$

This means that (a) $\Longrightarrow$ (h) is Horn sentence and it is sufficient to prove this sentence in $B_2$. Let, in $B_2$, $(\exists X)AX = I$ holds. Then for some X

$$\begin{array}{ll} & a_{11}x_{11} \cup a_{12}x_{21} \cup \ldots \cup a_{1n}x_{n1} = 1 \\ & a_{21}x_{11} \cup a_{22}x_{21} \cup \ldots \cup a_{2n}x_{n1} = 0 \\ (1) & \vdots \\ & a_{n1}x_{11} \cup a_{n2}x_{21} \cup \ldots \cup a_{nn}x_{n1} = 0 . \end{array}$$

At least one element of the left side of the first equality is 1. Let, for example, $a_{1i}x_{i1}$ be such an element, i.e. $a_{1i} = 1$ and $x_{i1} = 1$. Since $x_{i1} = 1$, according to (1), the elements $a_{2i}, a_{3i}, \ldots, a_{ni}$ must be 0, i.e. the matrix A has in the i-th column exactly one 1. Similarly, from

$$\begin{array}{l} a_{11}x_{12} \cup a_{12}x_{22} \cup \ldots \cup a_{1n}x_{n2} = 0 \\ a_{21}x_{12} \cup a_{22}x_{22} \cup \ldots \cup a_{2n}x_{n2} = 1 \\ \vdots \\ a_{n1}x_{12} \cup a_{n2}x_{22} \cup \ldots \cup a_{2n}x_{n2} = 0 \end{array}$$

we have that some $a_{2k}x_{k2} = 1$, i.e. $a_{2k} = 1$ and $x_{k2} = 1$. Here is $k \neq i$, because in the i-th column of the matrix A all elements are, except $a_{1i}$, equal to 0. The other members of the k-th column of the matrix A are 0.

The proof for other columns is similar. Thus each column of the matrix A has exactly one 1 and all these 1's are in the different rows, i.e. each row and each column is orthonormal.

(b) $\Longrightarrow$ (h): in a similar way.

(h) $\Longrightarrow$ (a): This seutence can be written in the form

$$\bigcup_{i=1}^{p} m_i = 0 \Longrightarrow (\exists X) AX = I$$

i.e. $$(\exists X)(\bigcup_{i=1}^{p} m_i = 0 \Longrightarrow AX = I)$$

This means that the sentence (h) $\Longrightarrow$ (a) is Horn's and we prove it in $B_2$. Let (h) hold in $b_2$, i.e. each column and each row contains exactly one 1. Then for $X = A^T$ the equality $AX = I$ is satisfied.

(h) $\Longrightarrow$ (b): The some as (h) $\Longrightarrow$ (a).

(h) $\Longrightarrow$ (e): The sentence (h) $\Longrightarrow$ (e), i.e.

$$\bigcup_{i=1}^{p} m_i = 0 \Longrightarrow A^T A = I$$

is Horn's

We suppose that, in $B_2$, each row and each column contains exactly one 1. This gives $A^T A = I$.

(e) $\Longrightarrow$ (b): $A^T A = I$ means that $(\exists X) XA = I$.

(h) $\Longrightarrow$ (g): The condition $\bigcup_{j=1}^{n} a_j^i = 1 \quad (i = 1,\ldots,n)$

can be written in the form

$$\bigcap_{i=1}^{n} (\bigcup_{j=1}^{n} a_j^i) = 1$$

so we write (h) $\Longrightarrow$ (g) as

$$\bigcup_{i=1}^{p} m_i = 0 \Longrightarrow \bigcap_{i=1}^{n} (\bigcup_{j=1}^{n} a_j^i) = 1.$$

The last sentence is Horn's and we are going to prove it in $B_2$. Let (h) holds in $B_2$. For example, the i-th row contains in the j-th place (in the j-th column) 1. Then $a_j^i = 1.0'.0'\ldots 0' = 1$, thus $\bigcup_{j=1}^{n} a_j^i = 1$.

(g) $\Longrightarrow$ (h): We can write the sentence (g) $\Longrightarrow$ (h) in the form

$$\bigcap_{i=1}^{n} (\bigcup_{j=1}^{n} a_j^i) = 1 \Longrightarrow \bigcup_{i=1}^{p} m_i = 0$$

which is Horn's and it is sufficient to test it in $B_2$. If $\bigcup_{j=1}^{n} a_j^i = 1$, then in $B_2$ one element of this union is 1. Let $a_k^i = 1$, i.e. $a_{ik}\, a'_{1k}\, a'_{2k} \ldots a'_{i-1,k}\, a'_{i+1,k} \ldots a'_{nk} = 1$. This means that $a_{ik} = 1$, $a_{1k} = 0$, $a_{2k} = 0$, $a_{i-1,k} = 0$, $a_{i+1,k} = 0,\ldots, a_{nk} = 0$, namely at least one element of the i-th row of the matrix A is 1 and all other elements of the column to wich this element belongs, are 0's. Assume now that some row contains two or more 1's. Then (bearing in mind that each row contains at least one 1) there are more than n 1's in the matrix A and this means that some column contains more than one 1. Thus we get the contradiction. Hence, each row and each column has exactly one 1 and in $B_2$ this means that all rows and all columns are orthonormal.

(h) $\Longrightarrow$ (f): $(A^TI')' = (I'A^T)' = A^T$ can be written as

$$(A^TI')' = A^T \wedge (I'A^T)' = A^T$$

Each of the last two equalities contains $n^2$ equalities. If we write all these equalities (there are $2n^2$ equalities) in the form $s_1 = 0 \wedge s_2 = 0 \wedge \ldots \wedge s_q = 0$, where $q = 2n^2$, then the last conjuction becomes $\bigcup_{m=1}^{q} s_m = 0$. Therefore the sentence (h) $\Longrightarrow$ (f) is Horn's and it is sufficient to prove it in $B_2$. Let (2) hold in $B_2$, i.e. each row and each column of the matrix A contains exactly one 1. This also holds for the matrix $A^T$. If $(A^T)_{ij} = 1$, then $(A^TI')_{ij}=0$, because we multiply $(A^T)_{ij}$ by $(I')_{ii}$, i.e. by 0 and other members of the i-th row of the matrix $A^T$ are 0. Using operation ' we have $(A^TI')_{ij} = 1$. If $(A^T)_{ij} = 0$, then some other element of the i-th row of the matrix $A^T$ is 1. Multiplying the i-th row of $A^T$ and the j-th column of I' we get 1, i.e.
$(A^TI')_{ij} = 1$, hence $(A^TI')'_{ij} = 0$. This means $(A^TI')' = A^T$. Similarly we obtain $(I'A^T)' = A^T$.

(f) $\Longrightarrow$ (h): The formula (f) $\Longrightarrow$ (h) is Horn's, because it can be written as

$$\bigcup_{m=1}^{q} s_m = 0 \Longrightarrow \bigcup_{i=1}^{p} m_i = 0$$

so me prove it in $B_2$. Since

$(A^TI')' = A^T$

$\Longleftrightarrow ((A^TI')')^T = A$ (because $(A^T)^T = A$)

$\Longleftrightarrow ((A^TI')^T)' = A$ (because for evary matrix S with elements in B $(S')^T = (S^T)')$ holds)

$\Longleftrightarrow ((I')^TA)' = A$ (because for matrices $R_{kxn}$ and $Q_{nxe}$ with elements in B $(RQ)^T = Q^TR^T$ holds)

$\Longleftrightarrow I'A = A$,

the conditions $(A^TI')' = A^T$ and $I'A = A'$ are equivalent. Let $(A^TI')' = A^T$ in $B_2$, i.e. $I'A = A'$. If $a_{ij} = 0$, then at least one element of the j-th column of the matrix A is 1, because

$$(I'A)_{ij} = a_{1j} \cup a_{2j} \cup \ldots \cup a_{j-1,j} \cup 0 \cup a_{j+1,j} \cup \ldots \cup a_{nj} = 1$$

But if $a_{ij} = 1$, then all the elements of the j-th column are 0, because

$$(I'A)_{ij} = a_{1j} \cup \dots \cup a_{j-1,j} \cup 0 \cup a_{j+1,j} \cup \dots \cup a_{nj} = 0.$$

This means that each column of the matrix A contains exactly one 1.

Similarly, starting with $(I'A^T)' = A'$, i.e. $AI' = A'$, we can prove that each row of the matrix A contains exactly one 1.

(d) $\Longrightarrow$ (h): The equalities

$$AX = AY \quad \text{and} \quad X=Y$$

can be written as

$$(2) \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} (AX)_{ij} = (AY)_{ij}), \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} ((X)_{ij} = (Y)_{ij})$$

i.e. in the form

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} ((AX)_{ij} + (AY)_{ij} = 0), \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} ((X)_{ij} + (Y)_{ij} = 0$$

where we denote with $\bigwedge_{i=1}^{n}$ the conjuction of n members.

If we denote all members of the first conjuction in (2) by $p_1, \dots, p_{n^2}$ and of the second one by $r_1, \dots, r_{n^2}$ we get instead of (2)

$$\bigcup_{k=1}^{n^2} p_k = 0 \ , \quad \bigcup_{m=1}^{n^2} r_m = 0.$$

Then the implication $(\forall X)(\forall Y)(AX=AY \Longrightarrow X=Y)$ becomes

$$(3) \qquad (\forall X)(\forall Y)(\bigcup_{k=1}^{n^2} p_k = 0 \Longrightarrow \bigcup_{m=1}^{n^2} r_m = 0).$$

Since the elements $p_k$ $(k=1,\dots,n^2)$ depend upon the elements of the matrixes X,Y and A and the elements $r_m (m=1,\dots,n^2)$ depend upon the matrixes X and Y, we can write (3) in the form

$$(4) \qquad (\forall Z)(f(Z,M) = 0 \Longrightarrow g(Z,M) = 0)$$

where $Z = (x_{11},\dots,x_{nn}, y_{11},\dots,y_{nn})$ and $M = (a_{11},\dots,a_{nn})$. Note that function g does not actualy depend upon M. According to theorem 2, formula (4), bearing in mind the condition $(\forall M)(\exists Z) f(Z,M) = 0$, is equivalent to

$$(\forall Z)(g(Z,M) \le f(Z,M)).$$

The condition $(\forall M)(\exists))f(Z,M) = 0$ is satisfied, because it is equivalent to $(\forall A)(\exists X,Y)(AX = AY)$. If is sufficient to take X=Y=I.

Now we use the following: if $C \Longrightarrow (A \Longleftrightarrow B)$ then $C \Longrightarrow ((A \Longrightarrow D) \Longleftrightarrow (B \Longrightarrow D))$. Therefore the formular (d)$\Longrightarrow$(h), i.e.

$$(\forall Z)(f(Z,M) = 0 \Longrightarrow g(Z,M) = 0) \Longrightarrow b(M) = 0,$$

where $b(M) = \bigcup_{i=1}^{p} m_i$, and

$$(5) \qquad (\forall Z)(g(Z,M) \le f(Z,M)) \Longrightarrow b(M) = 0$$

are equivalent. The formula (5) and

$$(\exists Z)(g(Z,M) \leq f(Z,M) \Longrightarrow b(M) = 0)$$

are also equivalent. This means that the formula (d) $\Longrightarrow$(h) is Horn's and it is sufficient to prove it in $B_2$. Assume that all rows and columns of the matrix A are not orthonormal i.e. there exists a row or a column, that does not contain exactly one 1. If, for example, the i-th row contains all 0's, then the equalities

$$(AX)_{i1} = (AY)_{i1}, \dots, (AX)_{in} = (AY)_{in}$$

also hold in the cases when the corresponding elements of the matrixes X and Y are not equal. The other $n^2$-n equalitias do not imply the equality of the corresponding elements of the i-th rows of the matrixes X and Y, because thay are not in these matrixes. When a row contains two 1's, for example the j-th row has 1 in the k-th and h-th places, then

(6) $\qquad x_{hi} \cup x_{ki} = y_{hi} \cup y_{ki} \qquad (i=1,\dots,n)$

The last equalities do not imply

$$x_{hi} = y_{hi} \text{ and } x_{ki} = y_{ki} \quad (i = 1,\dots,n)$$

because it is sufficient to be $x_{hi} = y_{hi} = 1$ and then (6) holds, while the elements $x_{ki}$ and $y_{ki}$ can be arbitrary. The other $n^2$-n equalities (that result from equaliting the other elements of the matrixes AX and AY) do not contain the elements of the i-th row of the matrix A. When the k-th column of the matrix A contains only 0's, then the elements of the k-th row of the matrixes X and Y are not in

(7) $\qquad (AX)_{kj} = (AY)_{kj} \qquad (j=1,\dots,n)$

i.e. the equalities (7) hold , olthough the corresponding elements $x_{ki}$ and $y_{ki}$ $(i=1,\dots,n)$ can be different.

If a column of the matrix A contains at least two 1's, for example the k-th column has 1 in the i-th and j-th plases, then the elements $x_{kh}$ acn $y_{kh}$ $(h=1,\dots,n)$ are in the equalities

(8) $\qquad (AX)_{ij} = (AY)_{ih}$ and $(AX)_{jh} = (AY)_{jh} \qquad (h = 1,\dots,n)$

Since there are not two 1's in a row, the equalities (8) are of the form $\quad x_{kh} = y_{kh}$ and $x_{kh} = y_{kh} \qquad (h = 1,\dots,n)$

i.e. we have the same equalities. This means that there are elements $x_{rs}$ and $y_{rs}$ that to not take part in the equalities AX=AY,i.e. the elements $x_{rs}$ and $y_{rs}$ can be different.

(e) $\quad$ (h): Left multiplying AX=AY by $A^T$ we get X=Y.

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# ASSOCIATIVE SYSTEMS OF TOPOLOGICAL n-QUASIGROUPS

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In this paper we introduce and begin a preliminary study of a associative systems of a topological n-quasigroups. We can consider that arbitrary topological n-quasigroup which belongs to an associative topologocal system is completely regular topological space.

First, let us quote some of the results from [4], [5] and [6]

Theorem 1. [4] If n-quasigroups $Q(A_i)$ $i\epsilon\{1,2,\ldots,2n\}=N_{2n}$ are conected by a general associative law

$$A_1(A_2(a_1^{j-1},a_j^n),a_{n+1}^{j+n-1},a_{j+n}^{2n-1})=A_{2j-1}(a_1^{j-1},A_{2j}(a_j^{j+n-1}),a_{j+n}^{2n-1}) \quad (1)$$

for each $j\epsilon\{2,3,\ldots,n\}$ then

1. Each $Q(A_i)$ $i\epsilon N_{2n}$ isotope to one and only one n-group $Q(A)$ with a unit.

2. There is a binary group $Q(B)$ so that it is $A(a_1^n) = B(B(\ldots(B(a_1,a_2),a_3)\ldots),a_{n-1}),a_n)$.

For n=2 this theorem is proved by Belousov [2] and it is known, as n-analogous of Belousov's theorem about four quasigroups.

Quasigroup isotopy characterization $Q(A_i)$ in relation to n-group $Q(A)$ from the previous theorem that is the following one is proved

Theorem 2. [5] If n-quasigroups $Q(A_i)$ $i\epsilon N_{2n}$ satisfy (1) then the following equalities are valid:

$$A_{2j-1}(a_1^n)=A(T_1^1T_2^1a_1,\ldots,T_{2j-3}^{j-1}T_{2j-2}^1a_{j-1},T_{2j-1}^ja_j,T_{2j+1}^{j+1}T_{2j+2}^na_{j+1},$$
$$\ldots,T_{2n-1}^nT_{2n}^na_n) \quad\ldots\ldots\ldots\ldots(2)$$

$$A_{2j}(a_1^n)={T_{2j-1}^j}^{-1}A(T_{2j-1}^jT_{2j}^1a_1,\ldots,T_{2n-1}^nT_{2n}^1a_{n-j+1},T_3^2T_4^na_{n-j+2},\ldots,$$
$$T_{2j-1}^nT_{2j}^na_n)\quad\ldots\ldots\ldots\ldots(3)$$

for each $j\in N_n$. $(T_i^tx=A_i(\overset{t-1}{k},x,\overset{n-t}{k}))$

<u>Definition 1.</u> [6] Let it be $\Sigma\subseteq\Omega$ , where $\Omega$ is a set of all n-quasigroup operations defined on Q. The system $\Sigma$ is called $\overline{iA}$-system, if for each $A_m,A_{m+1}\in\Sigma$ where m is a fixed number ($m=2i-1$, $i\in N_n$), there are $A_t,A_{t+1}\in\Omega$ for each $t\in\{2s-1\mid s\neq i, s\in N_n\}$ it satisfed (1).

<u>Definition 2.</u> [6] Let it be $\Sigma\subseteq\Omega$ , where $\Omega$ is a set of all n-quasigroups defined on Q. The system $\Sigma$ is called iA-system if for each $A_m,A_{m+1}\in\Sigma$ where m is a fixed number ($m=2i-1$, $i\in N_n$) there are such $A_t,A_{t+1}\in\Sigma$, that for each $t\in\{2s-1\mid s\neq i, s\in N_n\}$ we have equality (1).

<u>Definition 3.</u> [6] Let it be $\Sigma\subseteq\Omega$ , where $\Omega$ is a set of all n-quasigroups defined on Q. $\Sigma$ is called A-system if it iA-system for each $i\in N_n$.

<u>Theorem 3.</u> [6] Let iA-system $\Sigma$ of n-quasigroups be given on Q. Then we can define group B on Q so that each operation $A\in\Sigma$ has a shape

$$A(x_1^n)=\overset{n-1}{B}(\alpha_1x_1,\ldots,\alpha_nx_n)$$

where $\alpha_i$ is automorphism of the group B and $\alpha_t$, $t\in N_n\setminus\{i\}$ some permutations of the set Q.

We remark that arbitrary permutation $\alpha_n$ from above theorem we can get as a composition of some translations of quasigroups from the system $\Sigma$, according to the theorem 2.

<u>Theorem 4.</u> [6] Let to Q be given iA-system $\Sigma$ of n-quasigroups, then we can define group B on Q so that each operation

$C \in \Sigma$ has the shape

$$C(x_1^n) = B(\overset{n-1}{B}(\Phi_1 x_1, \ldots, \Phi_n x_n), k)$$

where $\Phi_i$ are automorphisms of the group B, and k is a some element from Q.

We remark that the above theorem can be read as Hosszú-Gluskin's theorem ( see [8] ) when $\Sigma = \{A\}$ naimely $\Sigma$ is a one-element set from one n-group.

Before topological associative systems definitions we shall proove the next theorem

Theorem 5. If semitopological n-quasigroups $Q(A_i)$, $i \in N_{2n}$, out of which at least one is topological, satisfy general associative law (1) then:

1. n-Group Q(A) isotope to n-quasigroups $Q(A_i)$ is topological.
2. Group Q(B) where $A(x_1^n) = \overset{n-1}{B}(x_1^n)$ is topological.
3. All quasigroups are topological.

Proof: 1. Let $A_i$ be topological quasigroup and let it be $i=2j-1$ then from (2) we find that

$$A(x_1^n) = A_{2j-1}(T_2^{1^{-1}} T_1^{1^{-1}} x_1, \ldots, T_{2n}^{n^{-1}} T_{2n-1}^{j^{-1}} x_n)$$

so that A is topological n-quasigroup since the translations $T_i^t$ are homeomorphisms of space according to lemma 5. [9] and it is n-group at the same time so that, according to lemma 4 [9] it is topological n-group.

In the same way we proove $i=2j$ using the relation (3).

2. Let e be unit of group Q(A), then group Q(B) can be shown as a retract of n-group Q(A), that is

$$A(x,y,e,e,\ldots,e) = B(x,y)$$

so that group Q(B) is topological according to lemma 1. [9].

3. Having in mind relations (2) and (3) and the fact that Q(A) is topological n-group we can conclude that, since the translations $T_i^t$ are homeomorphisms of space according to

lemma 5. [9] that all n-quasigroups $Q(A_i)$ $i\varepsilon N_{2n}$ are topological.

Definition 4. Let Q be topological space and $\Omega$ set of all semitopological n-quasigroups defined on the Q. $\overline{iA}$-system $\Sigma$ in the set $\Omega$ we call it topological if $\Sigma$ consists of at least one topological n-quasigroup.

Definition 5. Let Q be topological space and $\Omega$ set of all semitopological n-quasigroups defined on Q. iA-system $\Sigma$ in the set $\Omega$ is called topological iA-system if $\Sigma$ consists of at least one topological n-quasigroup.

Definition 6. Let Q be topological space and $\Omega$ set of all semitopological n-quasigroups defined on the Q. A-system $\Sigma$ in the set $\Omega$ is called topological A-system if $\Sigma$ consists of at least one topological n-quasigroup.

Using the theorem 5. we shall proove the next:

Theorem 6. Let Q be topological space and $\Sigma$ topological $\overline{iA}$-system of n-quasigroups defined on Q, then we have:

1. Each of n-quasigroups from $\Sigma$ is topological.
2. All quasigroups from $\Sigma$ are isotopy to topological n-group and isotopyes are homeomorphisms. Even n-quasigroups from $\Omega$ which take part in building iA-system are topological.

Proof. It is enough to take for example that n-quasigroup $B\varepsilon\Sigma$ is topological then we proove that each one is topological taking them in pairs and joining them 2n-2 semitopological n-quasigroups from $\Omega$ so that all satisfy general associative law (1) then according to theorem 5. it follows that all are topological and that n-group is one where isotopyes are topological.

In the same way we can proove the theorem about iA-systems of topological n-quasigroups.

Theorem 7. Let Q be topological space, $\Sigma$ topological iA-system of n-quasigroups defined on the set Q, then we have:

1. All n-quasigroups from $\Sigma$ are topological.

2. All n-quasigroups are isotope to topological n-group and those isotopes are automorphisms of topological group $Q(B)$ (theorem4.).

The above theorem can be read as a generalisation of topological analogy of Hosszú-Gluskin's theorem ( [8]):

Theorem 7. If $\Sigma = \{A\}$ is topological iA-system of one topological n-group then there is topological group $Q(B)$ such that it is

$$A(x_1^n)=B(\overset{n-1}{B}(x_1, \alpha x_2, \alpha^2 x_3, \ldots , \alpha^{n-1} x_n), c)$$

where $\alpha$ is automorphism of topological group $B$, and $c$ is some element of the set $Q$, when the conditions are fulfilled

$$\alpha^{n-1}x=B(c,B(x,c^{-1})) \quad , \quad \alpha c=c \ .$$

Remark Arbitrary topological quasigroup which belongs to an associative topological system is completely regular topological space.

In connection with associative systems it is natural to ask the question: Is arbitrary quasigroup an element of an associative system?

The answer is negative what can be easily concluded on the basis of the next consideration: If each quasigroup is an element of an associative system then the arbitrary loop is a member of an associative system so that it is isotope to some group, and from this on the based of Albert's theorem it would follow that the arbitrary loop is isomorph to some group which is not true.

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# TOPOLOGICAL MEDIAL n-QUASIGROUPS

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In this note it is showed how an arbitrary medial topological n-quasigroup can be presented as a topological group, with conclusion that all medial topological n-quasigroups are completely regular topological spaces.

We denote with $a_1^n$ $(a_1,\ldots,a_n)$ and with $\overset{n}{a}$ $(\underbrace{a,\ldots,a}_{n})$

n-groupoid Q is called n-quasigroup if the equation

$$(a_1^{i-1},x,a_{i+1}^n)=b$$

solves for each $i\in\{1,2,\ldots,n\}$ and for arbitrary $a_1,\ldots,a_{i-1}$, $a_{i+1},\ldots,a_n,b\in Q$.

n-quasigroup Q( ) is called medial if it satisfies the medial law

$$((x_{11}^n),(x_{21}^n),\ldots,(x_{n1}^n))=((x_{11}^n),\ldots,(x_{1n}^n)) \quad \ldots\ldots(1)$$

for arbitrary $x_{11},\ldots,x_{nn}\in Q$.

Medial n-quasigroups are satisfied by the following theorem

Theorem 1. 1 If Q is a medial n-quasigroup then there is Abel's group Q(+) from Q such that it is

$$(x_1^n) = \sum_{i=1}^{n}\alpha_i x_i+b \quad \ldots\ldots\ldots(2)$$

where b is definite fixed element from Q and $\alpha_i$ $(i=1,\ldots,n)$ are mutually commutative automorphisms of that group.

Definition 1. Topological space Q, which is n-quasigroup as well, is called semitopological n-quasigroup if a quasigroup operation is continious with all variables and if all translations are homeomorphisms of a topological space.

Definition 2. Semitopological n-quasigroup is called topological n-quasigroup if all inverse operations are continious with all variables together.

According to the definition it immediately follows:

Lemma 1. Topological (semitopological) n-quasigroup retract is a topological (semitopological) quasigroup.

Lemma 2. Superposition of semitopological n-quasigroup and semitopological m-quasigroup is semitopological (m+n-1)-quasigroup.

Proof: Continuity of new quasigroup operation can be seen by definition, and that arbitrary translation is homeomorphism of space can be seen from the following relation:

$$A \overset{i}{+} B(a_1^{j-1},x,a_j^{m+n-2}) =$$

$$=\begin{cases} A(a_1^{i-1},B(a_i^{j-1},x,a_j^{m+i-2}),a_{m+i-1}^{m+n-2})=T^j_{Aa_1^{i-1}a_{m+i-1}^{m+n-2}}T_k^{j-1}x & i\leq j\leq m+i-1 \\ A(a_1^{j-1},x,a_j^{i-1},B(a_i^{m+i-1}),a_{m+i}^{m+n-2})=T^j_{Aa_1^{i-1}ka_{m+i}^{m+n-2}}x & i\leq j\leq i-1 \\ A(a_1^{i-1},B(a_i^{m+i-1}),a_{m+i}^{j-1},x,a_j^{m+n-2})=T^j_{Aa_1^{i-1}ka_{m+i}^{j-1}a_j^{m+n-2}}x & \\ \quad m+i-1\leq j\leq m+n-1 & \\ \quad (k=B(a_i^{m+i-1})) & \end{cases}$$

where $T^i_{Aa_1^{n-1}}x= A(a_1^{i-1},x,a_i^{n-1})$

Lemma 3. Superposition of topological n-quasigroup and topological m-quasigroup is topological (m+n-1)-quasigroup.

Proof: Continuity of arbitrary inverse operation ${}^{\pi j}C$ comes from the following relations

$$^{\pi_j}C(x_1^{m+n-1}) = x_{n+m}$$

$$^{\pi_j}(A \overset{i}{+} B)(x_1^{m+n-1})=x_{n+m}$$

$$(A \overset{i}{+} B)(x_1^{j-1},x_{m+n},x_{j+1}^{m+n-1})=x_j \quad \text{naimely}$$

$$x_j = \begin{array}{ll} A(x_1^{i-1},B(x_1^{j-1},x_{m+n},x_{j+1}^{m+i-1}),x_{m+i}^{m+n-1}) & i \quad j \quad m+i-1 \\ A(x_1^{j-1},x_{m+n},x_{j+1}^{i-1},B(x_i^{m+i-1}),x_{m+i}^{m+n-1}) & 1 \quad j \quad i-1 \\ A(x_1^{i-1},B(x_i^{m+i-1}),x_{m+i}^{j-1},x_{m+n},x_{j+1}^{m+n-1}) & m+i \quad j \quad m+n-1 \end{array}$$

Lemma 4. Topological n-quasigroup which is n-group, it is topological n-group.

Proof: It is necessary to proove the continuity of operation $x \to \bar{x}$. As it is $A(x,x,\dots,x,\bar{x})=x$ we have $^{\pi_n}A(x,\dots,x)=\bar{x}$ and it follows $U(\bar{x})=U(^{\pi_n}A(x,\dots,x)) \subseteq {}^{\pi_n}A(V(x),\dots,V(x))=\overline{V(x)}$.

Lemma 5. If topological n-quasigroup Q(A) is isotopy n-quasigroup Q(B) but isotopyes are homeomorphisms of space then Q(B) is topological n-quasigroup.

Proof: Let them $\{\alpha_i\}_{i=1}^{n+1}$ be isotopy of topological n-quasigroup Q(A) in n-quasigroup Q(B) homeomorphisms of space. Let U be neighbourhood of $B(x_1^n)$ than

$$U(B(x_1^n))=U(\alpha_{n+1}^{-1}A(\{\alpha_i\, x_i\}_1^n) \subseteq \alpha_{n+1}U'(A\{\alpha_i^{-1}\, x_i\}_1^n) \subseteq$$
$$\subseteq \alpha_{n+1}A(U_i(\alpha_i^{-1}x_i)_1^n) \subseteq \alpha_{n+1}\, A(\{\alpha_i^{-1}(V_i(x_i))\}_1^n)=B(W_1^n)$$

so that the operation B is continuous. Operation continuity $^{\pi_i}B$ is is proved by fact that $^{\pi_i}B$ are isotopyes with $^{\pi_i}A$ which are continuous, and isotopyes are homeomorphisms of space.

Definition 3. Topological n-quasigroup which satisfied the medial law, is called topological medial n-quasigroup.

Theorem 2. If Q() is medial topological n-quasigroup, then on Q there is topological Abel's group Q(+) such that it is

$$(x_1^n) = \sum_{i=1}^{n} \alpha_i x_i + b$$

where $\alpha_i$ $(i=1,\ldots,n)$ are mutually commutative automorphisms of the topological Abel's group $Q(+)$ and $b$ determined element from $Q$.

<u>Proof</u>: A theorem is proved by aryty induction:

For $n=2$, let $Q$ be binary medial topological quasigroup then there is such Abel's topological group that

$$A(x,y) = \upsilon(x) + \chi(y) + c$$

where $\upsilon$ and $\chi$ commutative automorphisms of the topological group $Q(+)$ and $c$ determined element from $Q$. The main isotope $(+)$ of topological quasigroup $Q(A)$ defined in the following way

$$x+y = A(R_a^{-1}x, L_b^{-1})$$

is evidently a topological quasigroup (taking into consideration that the translations $R_a$ and $L_b$ are homeomorphisms of the topological quasigroup), and at the same time it is Abel's group as well. To say that and are automorphisms of the topological group can be seen immediately from the fact that

$$\upsilon(x) = R_a x + (-k) \quad \text{and} \quad \chi(x) = L_b(x) + (-h).$$

We assume that the theorem relates to each natural member less than $n$.

From the assumption, that the quasigroup $Q()$ is topological and from lemma 1. it follows

$$A(u,v) = (b,u,v,\overset{n-3}{b}) \qquad \text{and}$$

$$B(x_2^n) = (a, x_2^n)$$

that they are binary namely $(n-1)$-medial topological quasigroup which we get by putting it in the medial law

$$y_i = (\overset{n}{a}) = b \text{ for } i \neq 2,3\text{ , and } y_2 = (x_1, \overset{n-1}{a}) = \alpha x_1,$$

$$y_3 = (a, x_2^n) \text{ and}$$

$$z_1 = (a, x_1, \overset{n-1}{a}) = \beta x_1 \text{ and}$$

$$z_i = (\overset{2}{a}, x_i, \overset{n-3}{a}) = \gamma x_i \quad \text{for } i \neq 1.$$

(From the quated definitions it is clear that $\alpha$, $\beta$ and $\gamma$ are space homeomorphisms.)

Having in mind the inductive assumption we have

$$A(u,v)=\gamma\, u \hat{+} \delta\, v \hat{+} d$$

$$B(x_2^n)=\lambda_2 x_2 \acute{+} \dots \acute{+} \lambda_n x_n \acute{+} c$$

where $\hat{+}$ and $\acute{+}$ are topological Abel's group and $\lambda_i$, $\gamma$ and $\delta$ automorphisms of the correspondent topological groups, c and d determined elements from Q and $\lambda_i\lambda_j=\lambda_j\lambda_i$

On the basis of medial law it follows that

$$A(\alpha x_1, B(x_2^n))=(\beta x_1, \{\gamma x_i\}_{i=2}^n) \quad \text{naimely}$$

$$\gamma\alpha\, x_1 \hat{+} \delta(\lambda_2 x_2 \acute{+} \dots \acute{+} \lambda_n x_n \acute{+} c) \hat{+} d = (\beta x_1, \{\gamma x_i\}_2^n).$$

Group $\acute{+}$ is replaced by group + which is isomorph with it since $(\acute{+})^\delta=(+)$ and $\delta$ is space homeomorphism. From preliminary relation we get

$$(x_1^n)=\mu_1 x_1 \hat{+} (\mu_2 x_2 + \dots + \mu_n x_n + \delta^{-1} c) \hat{+} d \qquad \dots\dots \qquad (3)$$

where $\mu_i=\delta^{-1}\lambda_i\gamma^{-1}$ for $i\neq 1$ and $\mu_1=\gamma\alpha\beta^{-1}$ are spaces homeomorphisms.

Medial topological quasigroup retract $(x_1^{n-1},a)$ is medial topological (n-1)-quasigroup and according to inductive supposition there is Abel's topological group $Q(\ddot{+})$ so that

$$(x_1^{n-1},a)=\nu_1 x_1 \ddot{+} \nu_2 x_2 \ddot{+} \dots \ddot{+} \nu_{n-1} x_{n-1} \ddot{+} h$$

where $\nu_i$ are automorphisms of the topological group $Q(\ddot{+})$ and h is a fixed element from Q.

Putting that $x_n=a$ in (3) we come to the equality

$$\mu_1' x_1 \hat{+} (\mu_2 x_2 + \dots + \mu_{n-1}' x_{n-1}) = \nu_1 x_1 \ddot{+} \nu_2 x_2 \ddot{+} \dots \ddot{+} \nu_{n-1} x_{n-1} \ddot{+} h$$

where $\mu_1' x=\mu_1 x \hat{+} d$ and $\mu_{n-1}' x=\mu_{n-1} x+\mu_n a+\delta^{-1} c$ and so $\mu_1'$ and $\mu_{n-1}'$ are space homeomorphisms and having in mind the fact that group $(Q,\hat{+})$, $(Q,\ddot{+})$ and $(Q,+)$ are main isotope we get from (3) using the relatoin

$$u\hat{+}v=u\ddot{+}v\ddot{+}g$$

and $$u\ddot{+}v=u+v+f$$

(f and g are fixed element from Q), that it is

$$(x_1^n)=\mu_1 x_1+\mu_2 x_2+\ldots+\mu_n x_n+b.$$

Proof that $\mu_i$ $i=1,2,\ldots,n$ are mutually commutative automorphisms of topological group Q(+) is similar to the algebraic case [1].

The question of complete regularity of topological quasigroups is not quite solved [3]. On the basis of this result we have that the medial topological quasigroup is completely regular topological space.

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# GROUPOIDS OF PSEUDOALGEBRAS

F. Ferenci

Let $A^*$ denote the set of all finite strings on the nonvoid set A and let $\lambda$ be the empty string. If V is a nonempty set disjoint from A and $\alpha$ is a mapping of V into $A^{(A^*)}$, then the system $\mathcal{A} = \alpha\langle A,V\rangle$ is a *pseudoalgebra* [3] where A is the base set, V the set of operational symbols and $\alpha(v)$ for any $v \in V$ is an operation. The set $A^{(A^*)}$ is the base set of a groupoid $G_A$ whose multiplication is defined by $(u_1 \cdot u_2)(p) = u_1(u_2(\lambda)p)$ for arbitrary $u_1, u_2 \in A^{(A^*)}$ and $p \in A^*$ [2]. The right ideal $G(\mathcal{A})$ of $G_A$ generated by $\alpha(V) = \{\alpha(v) \mid v \in V\}$ is the *groupoid of the pseudoalgebra* $\mathcal{A}$. The operations of the pseudoalgebra $\mathcal{B} = \beta\langle G(\mathcal{A}),V\rangle$ are defined by $\beta(v)(b_1 b_2 \ldots b_n) = (\ldots(\alpha(v)\cdot b_1)\cdot b_2)\cdot\ldots)\cdot b_n$ for arbitrary $v \in V$ and $b_1, b_2, \ldots, b_n \in G(\mathcal{A})$ (the multiplications on the right side of this equality are performed in $G(\mathcal{A})$ and ind the special case $n = 0$ we have $\beta(v)(\lambda) = \alpha(v)$). For these pseudoalgebras the following assertion is valid:there exists a mapping of $G(\mathcal{A})$ into A which is a homomorphism of $\mathcal{B}$ into $\mathcal{A}$ (see [1]).

There is an analogy between the previous considerations and the following fact from automata theory. If $\mathcal{A}' = \alpha'\langle A,V\rangle$ is a unary universal algebra (the operations $\alpha'(v)$ are transformations of the set A, i.e. mappings of A into itself) then let $M(\mathcal{A}')$ denote that submonoid of the full transformation monoid on A which is generated by $\alpha'(V) = \{\alpha'(v) \mid v \in V\}$. The operations of the unary universal algebra $\mathcal{B}' = \beta'\langle M(\mathcal{A}'),V\rangle$ are defined by

$\beta'(v)(b) = \alpha'(v)\cdot b$ (the multiplication is in $M(A')$). For these universal algebras the following assertion holds: there exists a mapping of $M(A')$ into A which is a homomorphism of $B'$ into $A'$ (in the primitive class generated by $A'$ , $B'$ is a free algebra whose free generating set is the unit of $M(A')$).

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Vladimir Volenec: Kvadratne kvazigrupe

(sažetak)

Kvadratna kvazigrupa je svaka kvazigrupa $(Q,\cdot)$, u kojoj vrijedi identitet

$$ca\cdot bc = ab\cdot a. \tag{1}$$

Npr. ako je u polju $(C,+,\cdot)$ kompleksnih brojeva definirana operacija $\circ$ formulom

$$a\circ b = \frac{1-i}{2}\cdot a + \frac{1+i}{2}\cdot b, \tag{2}$$

tada je $(C,\circ)$ kvadratna kvazigrupa. Skup C može se shvatiti kao euklidska ravnina. Za bilo koje dvije točke a,b te ravnine, identitet (2) može se pisati i u obliku

$$\frac{(a\circ b)-a}{b-a} = \frac{1+i}{2}.$$

To znači da su točke a, b, $a\circ b$ vrhovi trokuta direktno sličnog trokutu s vrhovima 0, 1, $\frac{1+i}{2}$, tj. $a\circ b$ je središte pozitivno orijentiranog kvadrata sa dva susjedna vrha a, b, što opravdava naziv promatranih kvazigrupa. Svaki identitet u kvadratnoj kvazigrupi $(C,\circ)$ interpretira neki geometrijski teorem. S druge strane, ovaj geometrijski model $(C,\circ)$ kvadratnih kvazigrupa daje motivaciju za ispitivanje kvadratnih kvazigrupa.

U kvadratnoj kvazigrupi $(Q,\cdot)$ vrijede i identiteti

$$aa = a \text{ (idempotentnost)}, \tag{3}$$

$$a\cdot ba = ab\cdot a \text{ (elastičnost)}, \tag{4}$$

$$ab\cdot a = ba\cdot b, \tag{5}$$

$$ba\cdot ab = a, \tag{6}$$

$$a\cdot bc = ab\cdot ac \text{ (lijeva distributivnost)}, \tag{7}$$

$$bc\cdot a = ba\cdot ca \text{ (desna distributivnost)}, \tag{8}$$

$$ab\cdot cd = ac\cdot bd \text{ (medijalnost)}. \tag{9}$$

Pomoću kvadratne kvazigrupe mogu se dobiti i neke druge kvazigrupe. Za svako $a,b\in Q$ element

$$a*b = ab\cdot a \tag{10}$$

zove se polovište para elemenata a, b, a $(Q,*)$ je komutativna idempotentna medijalna kvazigrupa.

Neka je na skupu Q definirana operacija $\square$ ekvivalencijom

$$a \square b = c \Longleftrightarrow bc = a. \tag{11}$$

Tada je $(Q, \square)$ tzv. rot-kvazigrupa J. Dupláka, tj. kvazigrupa, u kojoj vrijedi identitet

$$x \square (x \square y) = z \square [(x \square z) \square y]. \tag{12}$$

Operacija $\square$ može se definirati i eksplicitno formulom

$$a \square b = (a \cdot ab) \cdot (a \cdot ab)(ab \cdot b). \tag{13}$$

Neka je operacija $\Delta$ definirana ekvivalencijom

$$a \Delta b = c \Longleftrightarrow a * c = b, \tag{14}$$

gdje je $*$ operacija definirana formulom (10). Tada je $(Q, \Delta)$ idempotentna kvazigrupa, u kojoj vrijedi još i identitet

$$[(a \Delta b) \Delta c] \Delta d = [(a \Delta d) \Delta c] \Delta b. \tag{15}$$

Operacija $\Delta$ može se definirati eksplicitno formulom

$$a \Delta b = [(a \cdot ab)(ab \cdot b) \cdot (ab \cdot b)][(b \cdot ba) \cdot (b \cdot ba)(ba \cdot a)]. \tag{16}$$

Vladimir Volenec, Quadratic quasigroups

(Summary)

A quadratic quasigroup is a quasigroup $(Q, \cdot)$ with the identity (1). It satisfies the identities (3)-(9). If the operations $*, \square, \Delta$ are defined on the set Q by (10), (11) resp. (14), then $(Q, *)$ is a commutative idempotent medial quasigroup, $(Q, \square)$ is a rot-quasigroup of J. Duplák, i.e. it holds the identity (12), and $(Q, \Delta)$ is an idempotent quasigroup with the identity (15). The operations $\square, \Delta$ can be defined directly by (13) and (16).

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# On enumerability of interpolants

P. Ecsedi-Tóth* and L. Turi**

## 0. Introduction

Our motivation for determining all interpolants of the arbitrarily given first order sentences $\varphi$ and $\psi$ by an effective procedure comes from computer science, namely from the theory of program verification. There, according to the well-known method of Floyd-Hoare, a program (or more precisely a program schema) must be associated by so called assertions [1,4]. This association can partially be mechanized; the difficulty arises in associating assertions to loops. If $\varphi$ is the assertion immediately before the loop and $\psi$ is the one immediately after it, then the assertion associated to the loop can be looked for among the interpolants of $\varphi$ and $\psi$ as was pointed out in [2]. Thus, by providing an effective method to generate the interpolants of $\varphi$ and $\psi$, we can completely automatize the Floyd-Hoare verification process. This, of course, represents but a little interest from a practical point of view, since the Floyd-Hoare method is object to several impediments (nevertheless, it seems to be the only general approach which has practical applications). In the same time, however, automated Floyd-Hoare process can serve as a basis for further research. In this paper we define an algorithm which enumerates the set of interpolants for arbitrarily given $\varphi$ and $\psi$. (The existence of such an enumerating algorithm easily follows from the Completeness

Theorem; in this paper, however, we avoid any use of that theorem and exhibit explicitely an enumerating algorithm.)

## 1. Zero order interpolants

Let $\sigma$ be an arbitrary zero order sentence and let $\tau\sigma$ be the set of sentence symbols (i.e. prime sentences) occuring in $\sigma$. It is well-known that every function $h:\tau\sigma\to 2$ (where $2=\{0,1\}$) can be extended uniquely to a function $\hat{h}$ over the set of sentences of the language $\tau\sigma$ according to the following recursion:

$$\hat{h}(\varphi\wedge\psi) = \min\ (\hat{h}(\varphi),\ \hat{h}(\psi))$$
$$\hat{h}(\varphi\vee\psi) = \max\ (\hat{h}(\varphi),\ \hat{h}(\psi))$$
$$\hat{h}(\neg\varphi) = 1-\hat{h}(\varphi)$$
$$\hat{h}(\varphi\to\psi) = \max\ (\hat{h}(\psi), 1-h(\varphi)).$$

Since $\tau\sigma$ is finite and hence $\tau\sigma_2$ is finite as well (where $\tau\sigma_2 = \{h \mid h:\tau\sigma\to 2\}$), we can compute $\hat{h}(\sigma)$ for every $h\in{}^{\tau\sigma}2$. Let

$$\text{Cal}\ (\sigma) = \min\{\hat{h}(\sigma) \mid h\in{}^{\tau\sigma}2\}.$$

It is clear, that Cal $(\sigma)$ is computable for every zero order sentence $\sigma$ and that the following lemma holds:

<u>Lemma 1 Let $\sigma$ be a zero order sentence. Then, Cal $(\sigma)$ = 1 if and only if $\sigma$ is a tautology.</u>

Let Con $(\sigma_1,\sigma_2)$ = Cal $(\sigma_1\to\sigma_2)$. The following assertion is easily obtained from Lemma 2 by Deduction Theorem:

<u>Lemma 2 Let $\sigma_1,\sigma_2$ be zero order sentences. Then, Con $(\sigma_1,\sigma_2)$ = 1 if and only if $\sigma_1\models\sigma_2$.</u>

Let $\varphi$ and $\psi$ be zero order sentences and set

$I_{\varphi,\psi} = \{\chi \mid \chi$ is a zero order sentence such that $\tau\chi\subseteq\tau\varphi\cap\tau\psi$ and $\varphi\models\chi$ and $\chi\models\psi\}$.

<u>Theorem 3 Let $\varphi$ and $\psi$ be zero order sentences. Then $I_{\varphi,\psi}$ is decidable; i.e. there exists an algorithm $Int_{\varphi,\psi}$ such that $Int_{\varphi,\psi}(\chi) = 1$ if and only if $\chi \in I_{\varphi,\psi}$, otherwise $Int_{\varphi,\psi}(\chi) = 0$ for arbitrary zero order sentence .</u>

<u>Proof</u> Let $Int_{\varphi,\psi}(\chi) = \min\{Con(\varphi,\chi), Con(\chi,\psi)$, $Tau(\varphi,\psi,\chi)$ where $Tau(\varphi,\psi,\chi) = 1$ if and only if $\tau\chi \subseteq \tau\varphi \cap \tau\psi$ and otherwise it is 0. It is immediate from Lemma 3 and from the finiteness of $\tau\varphi$, $\tau\psi$ and $\tau\chi$ that $Int_{\varphi,\psi}(\chi)$ is computable for every zero order sentence $\chi$ and that $Int_{\varphi,\psi}(\chi) = 1$ if and only if $\chi \in I_{\varphi,\psi}$ and otherwise it is 0.

## 2. First order interpolants

Turning to the more involved question of the first order case we recall and refine some well-known facts.

A first order formula $\varphi$ is in prenex normal form if and only if it has the form $\varphi = Q_1x_1 \ldots Q_nx_n\psi$ where $Q_i$ $(i=1,\ldots,n)$ is either the existential ($\exists$) or the universal ($\forall$) quantification symbol and no quantifier occurs in $\psi$.

<u>Lemma 4 (Prenex Normal Form Theorem)</u>

<u>There exists an algorithm Pren such that for every $\varphi$, Pren $(\varphi)$ is in prenex normal form and $\varphi$ is logically equivalent to Pren $(\varphi)$.</u>

<u>Proof</u> Trivial and can be found in any textbook on logic. We note, however, that the rigorouos definition of Pren is rather tedious (and hence is omitted here).

A first order formula in prenex normal form is said to be in standard form if and only if $\varphi = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$. The following assertion is immediate.

Lemma 5 (Standard Form Theorem) There exists an algorithm Stand such that for every $\varphi$ in prenex normal form, $\varphi$ is logically equivalent to Stand ($\varphi$) and Stand ($\varphi$) is in standard form.

From now on, we shall always assume, that for every arity we have an infinite set of function symbols.

Lemma 6 (Existential Skolem Normal Form Theorem) There exists an algorithm $\mathrm{Skol}_\exists$ such that for every $\varphi = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$ in standard form we have $\models \varphi$ if and only if $\models \mathrm{Skol}_\exists(\varphi)$ and $\mathrm{Skol}_\exists(\varphi) = \exists x_1 \ldots \exists x_n \psi(f_1(x_1)/y_1, \ldots, f_n(x_1, \ldots, x_n)/y_n)$ where none of the function symbols $f_1, f_2, \ldots, f_n$ occurs in $\varphi$ and $f_i(x_1, \ldots, x_i)/y_i$ denotes the substitution of the term $f_i(x_1, \ldots, x_i)$ into the variable $y_i$ for all $i=1, \ldots, n$.

Proof. of this lemma is quite elementary and can be found e.g. in [5].

Let Cterm be an algorithm which enumerates the set of closed terms and for every $k \in \omega$, let $\mathrm{Enum}_k$ enumerate the set $\omega^k$. The algorithm which computes the i-th component of an ordered n-tuple a is denoted by $(a)_i$. For $\varphi = \exists x_1 \ldots x_k \psi$ and $l \in \omega$ we define

$$\mathrm{Cf}\ (\varphi, l) = \psi[\mathrm{Cterm}\ ((\mathrm{Enum}_k\ (l))_1)/x_1, \ldots, \mathrm{Cterm}\ ((\mathrm{Enum}_k\ (l))_k)/x_k].$$

<u>Lemma 7 (Special Semantical Form of Herbrand's Theorem) For every existential sentence $\varphi$, $\models\varphi$ if and only if there exists an $l\in\omega$ such that</u> $\models \bigvee_{i\leq l} \mathrm{Cf}\ (\varphi,i)$.

<u>Proof.</u> This assertion follows immediately from the well-known Herbrand's Theorem (see [5]) and from the definition of Cf.

We can summerize the facts claimed in Lemmata 4,5,6,7 as follows

<u>Lemma 8 For every first order sentence $\varphi$, $\models\varphi$ if and only if there exists an $l\in\omega$ such that</u> $\models \bigvee_{i\leq l}$ <u>Cf (Skol$_{\exists}$ (Stand (Pren ($\varphi$))),i).</u>

Let us define the algorithm Calf for every first order sentence as follows.

Step 0. Let $l=0$.

Step 1. Compute Cal ( $\bigvee_{i\leq l}$ Cf (Skol$_{\exists}$(Stand (Pren ($\varphi$))),i).
If the value is 1 then Calf ($\varphi$) = 1 else go to Step 2.

Step 2. Increase $l$ by 1 and go to Step 1.

By Lemma 8 Calf ($\varphi$) stops and gives value 1 if and only if $\models\varphi$, otherwise Calf ($\varphi$) does not halt.

Let Conf ($\varphi,\psi$) = Calf ($\varphi\rightarrow\psi$).

<u>Lemma 9 For every two first order sentences $\varphi$ and $\psi$, Conf ($\varphi,\psi$) = 1 if and only if $\varphi\models\psi$ and undefined (i.e. does not halt) otherwise.</u>

<u>Proof.</u> Immediate by definitions.

Now we can provide an algorithm such that it enumerates all consequences of a given $\varphi$. Let Form be an algorithm which enumerates the set of sentences. Let Conseq ($\varphi$,n) = Form (n) if and only if Conf ($\varphi$, Form(n)) = 1.

Let Tauf ($\varphi,\psi,\chi$) = 1 if and only if $\tau\chi \subseteq \tau\varphi \cap \tau\psi$ (recall that, for a first order formula $\sigma$, $\tau\sigma$ denotes the set of non-logical symbols occuring in $\sigma$), and let Tauf ($\varphi,\psi,\chi$) = 0 otherwise.

We define the algorithm Interp ($\varphi,\psi,n$) for arbitrary first order sentences $\varphi$, $\psi$ and for $n \in \omega$ as follows.

Interp ($\varphi,\psi$,n) = Conseq ($\varphi$, $(\mathrm{Enum}_2(n))_2$) if and only if Conseq (Conseq ($\varphi$, $(\mathrm{Enum}_2(n))_2$), $(\mathrm{Enum}_2(n))_1$) = $\psi$ and Tauf ($\varphi,\psi$, Conseq ($\varphi$, $(\mathrm{Enum}_2(n))_2$) = 1.

Let $\varphi$ and $\psi$ be first order sentences such that $\varphi \models \psi$. We put

$$I_{\varphi,\psi} = \{\chi \mid \chi \text{ is a first order sentence}, \tau\chi \subseteq \tau\varphi \cap \tau\psi \text{ and } \varphi \models \chi,\ \chi \models \psi\}$$

Thus we have by Lemma 9 and by definition the following

Theorem 10 For any fixed first order sentences $\varphi,\psi,\chi$, there exists an $n \in \omega$ such that Interp ($\varphi,\psi$,n) = $\chi$ if and only if $\chi \in I_{\varphi,\psi}$; i.e. the set $I_{\varphi,\psi}$ is enumerable.

## 3. Outlook

The algorithm presented in section 2 is of high complexity. The reduction of this complexity is of great importance but the present work does not intend to deal with such questions. We provide, however, some remarks concerning this reduction.

In case of zero order logic there are faster algorithms to generate the set of interpolants. These are based on the isomorphism between the Lindenbaum-Tarski algebra of zero order sentences and the Boolean algebra of finite functions associated to the equivalence classes of zero order sentences.

In the Boolean algebra of functions we are able to find those functions which correspond to interpolants, and then to generate the appropriate representative sentences (for example in full disjunctive normal form), relatively quickly.

Analyzing the above isomorphism we have showed that, in case of zero order logic, the interpolants for any given sentences $\varphi$ and $\psi$ may be classified in a finite set of classes, and we have estimated the cardinality of this finite set. These results will appear in [3].

In the first order case further research is required.

The considerations of this paper can be generalized in several ways. For example, any application to the program verification problem needs similar algorithms for formulae with free variables instead of sentences. This will be investigated elsewhere.

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# ON AFFINE PLANES OVER $A_n^K$-QUASIGRUPS (SUMMARY)

Juraj Šiftar, Zagreb

An $A_n^k$-algebra is defined as an idempotent groupoid of order n in which every pair of distinct elements generates a subgroupoid of the same order k. Such algebras were introduced and investigated by Szamkolowicz (1962) and Puharev (1965). Each $A_n^k$-algebra Q can be used for the construction of an incidence structure $P_\ell^k$ in a natural way: the elements of Q are points, the subgroupoids of order k are lines and the incidence relation is "$\varepsilon$". Here $\ell = (n-1)/(k-1)$. For $\ell \geqslant k \geqslant 2$ $P_\ell^k$ is a regular plane, which means that each pair of distinct points lies on a unique line, that each point lies on exactly $\ell$ lines and that each line contains exactly k points. We shall consider the case when $n=k^2$ so that we have an affine plane. Also, we confine our attention to $A_n^k$-algebras which are quasigroups, in which case it easily follows that the subgroupoids of order k are subquasigroups.

Puharev proved that an affine plane over an $A_n^k$-quasigroup $(Q,\cdot)$ is desarguesian if Q is medial, i.e. if the identity

$$(a\cdot b)\cdot(c\cdot d) = (a\cdot c)\cdot(b\cdot d)$$

holds for all $a,b,c,d \in Q$. It is our aim to find some sufficient conditions on $(Q,\cdot)$ such that the corresponding affine plane $\mathfrak{a}$ belongs to the other outstanding classes of planes: translation planes, dual translation planes, semifield planes and nearfield planes.

We are able to prove the following results:

Theorem 1.

If Q has the right (or left) distributive property, then $\mathfrak{A}$ is a translation plane.

Theorem 2.

If there is in Q a subquasigroup K of order k which has the right distributive property and satisfies the condition that $(k_1 \cdot a)\cdot(k_2 \cdot b)=(k_1 \cdot k_2)\cdot(a \cdot b)$ for all $k_1, k_2 \in K$ and all $a, b \in Q \setminus K$, then $\mathfrak{A}$ is a dual translation plane. If there is another subquasigroup L of order k which intersects K and satisfies the right distributivity law, then $\mathfrak{A}$ is a semifield plane.

Theorem 3.

If Q contains two intersecting subquasigroups K and L of order k such that K is medial, L has the right distributive property and $(k_1 \cdot k_2)\cdot(\ell_1 \cdot \ell_2)=(k_1 \cdot \ell_1)\cdot(k_2 \cdot \ell_2)$ for all $k_1, k_2 \in K$ and $\ell_1, \ell_2 \in L$, then $\mathfrak{A}$ is a nearfield plane.


Sažetak

O AFINIM RAVNINAMA NAD $A_n^k$-KVAZIGRUPAMA

Idempotentnu kvazigrupu Q reda n nazivamo $A_n^k$-kvazigrupom ako svaka dva njezina različita elementa generiraju potkvazigrupu jednog te istog reda k. Ovakve se kvazigrupe na prirodan način povezuju sa strukturama incidencije koje su, u posebnom slučaju $n=k^2$, afine ravnine. Puharev je pokazao da je medijalnost $A_n^k$-kvazigrupe Q dovoljan uvjet da pripadna afina ravnina $\mathfrak{A}$ bude Desarguesove. Ovdje nalazimo neke dovoljne uvjete na kvazigrupu da bi $\mathfrak{A}$ bila ravnina translacije, dualna ravnina translacije, ravnina polupolja i ravnina skoropolja, respektivno.

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# AN ALGORITHM FOR THE CONSTRUCTION OF NON-SIMPLE MATROIDS

Dragan Acketa, Novi Sad

Abstract. We describe a simple algorithm for the construction of all non-isomorphic non-simple matroids on n+1 elements, provided all non-isomorphic matroids on n elements are known.

## Introduction

Matroid M on a finite set (carrier) S is an ordered pair (S,F), where F is a family of subsets of S, satisfying the following three axioms: I. $S \in F$ II. if $A,B \in F$, then $A \cap B \in F$

III. if $A \in F$ and $a,b \in S \setminus A$, then $\underline{b}$ is a member of all sets of F containing $A \cup \langle a \rangle$ if and only if $\underline{a}$ is a member of all sets of F containing $A \cup \langle b \rangle$

The elements of F are flats of M. All flats of a matroid M, ordered by inclusion, constitute a semimodular lattice L, each element of which (except the minimal) is a join of atoms (immediate followers of the minimal element-zero). The elements of the zero of L are loops of M, while atoms of L are also atoms of M.

Matroid is simple if all its atoms are singletons, otherwise it is non-simple .

The addition of a new element e to a flat X of a matroid M on S ($e \notin S$) is the replacement of all flats Z of M, which contain X, by $Z \cup \langle e \rangle$.

## The algorithm

Step 1: Addition of one new loop to (i.e. one new element to the zero flat of) each matroid on n elements.

Step 2: Construction of k (possibly not all distinct) loop-less non-simple matroids on n+1 elements ,which correspond to an arbitrary loop-less matroid M on n elements with k atoms, by addition of one new element to an atom of M.

Step 3: An isomorphism check of the matroids constructed in Step 2. and the elimination of the isomorphic copies.

Some explanations and comments

Step 1. uses (and establishes) an 1-1 correspodence between all non-isomorphic matroids on n elements and all non-isomorphic matroids with loops on n+1 elements.

Any loopless non-simple matroid on n+1 elements can be constructed by the procedure of Step 2. ,for there is not an element of the carrier of a matroid,which is not included in an atom of it.

Step 3. (by far the most tedious one) may be shortened if we primarily eliminate the isomorphic matroids constructed from the same matroid on n elements and observe that the loop-less matroids on n+1 elements,obtained by addition of the same p-tuple of numbers of new elements to the ordered atoms of a rank p simple matroid are isomorphic.

A modification of this algorithm is used in [1] for the construction of non-simple matroids on 8 elements. All non-isomorphic simple matroids on 8 elements are also known ([2]),and we suggest this algorithm for the construction of non-simple matroids on 9 elements.There is a very little hope for its use on larger sets,for even the construction of all non-isomorphic simple matroids on 9 elements seems to be unreachable.

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# НЕКОТОРЫЕ ХАРАКТЕРИСТИКИ ТАБЛИЦ БУЛЕВЫХ ФУНКЦИЙ И ПОСТРОЕНИЯ ВЫРАЖЕНИЙ В ОПЕРАТОРАХ ШЕФФЕРА

П. Хотомски

Функция $f: L_2^n \to L_2$, $L_2 = \{0,1\}$, представленна таблицей. Понимаем $f^j$ как $j$-ую координату вектора $f$ и с помощью конкатенации будем писать $f = \mathop{K}\limits_{j=1}^{2^n} f^j = f^1 f^2 \dots f^{2^n}$.

| $x_n$ | $\dots$ | $x_2$ | $x_1$ | $f$ |
|---|---|---|---|---|
| 0 | $\dots$ | 0 | 0 | $f^1$ |
| 0 | $\dots$ | 0 | 1 | $f^2$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| 1 | $\dots$ | 1 | 1 | $f^{2^n}$ |

Не трудно усмотреть следующие характеристики связанные с столбцами таблицы.

1. Каждый вектор $x_i$, $i=\overline{1,n}$ порождается выражением
$x_i = \mathop{K}\limits_{j=0}^{2^{n-i}-1}\Big(\mathop{K}\limits_{r=1}^{2^{i-1}} 0^{j2^i+r} \mathop{K}\limits_{r=2^{i-1}+1}^{2^i} 1^{j2^i+r}\Big)$ или короче $x_i = \mathop{K}\limits_{j=0}^{2^{n-i}-1} \mathop{K}\limits_{r=1}^{2^i} [r/(2^{i-1}+1)]^{j2^i+r}$
где [ ] – целая часть.

2. Выражение $\mathop{K}\limits_{i=1}^{n} \mathop{K}\limits_{j=0}^{2^{n-i}-1} \mathop{K}\limits_{r=1}^{2^i} [r/(2^{i-1}+1)]^{j2^i+r}$ репрезентирует аргументную часть таблицы.

3. Пусть $N_i$ десятичный эквивалент бинарной записи вектора $x_i$. Обозначим $I(n) = 2^{2^n} - 1$, тогда $N_i = \dfrac{I(n)}{I(i-1)+2} = \dfrac{I(n)\cdot I(i-1)}{I(i)}$, $i=\overline{1,n}$. Таким образом значения $N_i$ отыскиваются непосредственно и независимо друг от друга, т.е. проще чем предложено в [3].

4. Число $I(n)$ соответствует функции константе $f=1$ и удовлетворяет равенству $\prod\limits_{i=1}^{n} N_i = (I(n))^{n-1}$.

5. Посредством вектора $x_i$, $1 \le i \le n$, вектор $f$ можно представить в форме $f = \mathop{K}\limits_{j=0}^{2^{n-i}-1}\Big(\mathop{K}\limits_{r=1}^{2^{i-1}} f_0^{j2^i+r} \mathop{K}\limits_{r=2^{i-1}+1}^{2^i} f_1^{j2^i+r}\Big)$, где $f_0, f_1$ координаты вектора $f$ на местах которых в $x_i$ стоят $0, 1$, соответственно.

Используем приведенную нотацию для описания метода построения особых выражений в операторах Шеффера. Вопросы представления и минимизации в обобщенных операторах $\mathop{\uparrow}\limits_{i=1}^{m} x_i = \overline{x_1 \wedge \cdots x_m}$ рассмотрены в [1] и [2]. Там эти операторы в выражениях имеют переменную длину. Мы будем рассматривать выражения которые, исходя от таблицы строятся исключительно на бинарном операторе $x \uparrow y = \overline{x \wedge y}$. Покажем что каждую булеву функцию $n$ переменных можно представить выражением особого вида: Р-выражением.

<u>Определение 1.</u> Р-дерево /подрезанное дерево/, это бинарное дерево, каждая не концевая вершина которого связана хотя бы одной из двух исходящихиз нее грань, с вершиной которой соответствует переменная, либо непосредственно либо через одну вершину. Р-дерево закончено если каждой концевой вершине соответствует переменная данной функции.

<u>Определение 2.</u> Выражение булевой функции соответствующее законченому Р-дереву называется Р-выражение.

<u>Теорема</u> Каждую булеву функцию $n$ переменных можно представить Р-выражением в бинарных операторах Шеффера[1].

Доказательство обосновано на следующих лемах.

<u>Лема 1.</u> Если $f$ и $g$ заданные булевы функции $n$ переменных, то можно отыскать булевы функции $\varphi$ и $\psi$ такие что $(g\uparrow\psi)\uparrow\varphi=f$, при чем $g\uparrow\psi \overset{\text{опр}}{=} \mathop{K}\limits_{j=1}^{2^n}(g^j \uparrow \psi^j)$, $g^j, \psi^j \in L_2$.

Доказательство. Обозначим через $s$ координаты которые могут иметь любое значение из $L_2$ /свободные координаты/ и пусть $\dot{z}\uparrow\bar{s}=1$. Значения $\psi^j$ и $\varphi^j$ определенны в таблице.

| $f^j$ | $g^j$ | $\psi^j$ | $\varphi^j$ |
|---|---|---|---|
| 0 | 0 | s | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | s | 0 |
| 1 | 1 | s | z |

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1) Анналогичный результат действителен для оператора Пирса.

Лема 2. По булевой функцие $f$ $n$ переменных и любой ее переменной $x_i$, можно отыскать булевы функции $F$, $\psi$ и $\varphi$ которые не зависят существенным образом от $x_i$, такие что

$$(F\uparrow(x_i\uparrow\psi))\uparrow(x_i\uparrow\varphi)=f$$

Доказательство. Утверждение лемы выполняется для следующих векторов, записанных посредством переменной $x_i$.

$$F=\mathop{K}_{j=0}^{2^{n-i}-1}\Big(\mathop{K}_{r=1}^{2^{i-1}} f_0^{j2^i+r}\mathop{K}_{r=1}^{2^{i-1}} F_1^{j2^i+2^{i-1}+r}\Big),\ \text{где}\ F_1^{j2^i+2^{i-1}+r}=f_0^{j2^i+r}$$

$$\psi=\mathop{K}_{j=0}^{2^{n-i}-1}\Big(\mathop{K}_{r=1}^{2^{i-1}} \psi_0^{j2^i+r}\mathop{K}_{r=1}^{2^{i-1}} \psi_1^{j2^i+2^{i-1}+r}\Big),\ \text{где}\ \psi_0^{j2^i+r}=\psi_1^{j2^i+2^{i-1}+r}$$

$$\varphi=\mathop{K}_{j=0}^{2^{n-i}-1}\Big(\mathop{K}_{r=1}^{2^{i-1}} \varphi_0^{j2^i+r}\mathop{K}_{r=1}^{2^{i-1}} \varphi_1^{j2^i+2^{i-1}+r}\Big),\ \text{где}\ \varphi_0^{j2^i+r}=\varphi_1^{j2^i+2^{i-1}+r}$$

при чем координаты векторов $\varphi$ и $\psi$ определяются из условия

$$\left(F_1^{j2^i+2^{i-1}+r}\uparrow\overline{\psi}_1^{j2^i+2^{i-1}+r}\right)\uparrow\overline{\varphi}_1^{j2^i+2^{i-1}+r}=f_1^{j2^i+2^{i-1}+r}$$

с использованием таблицы лемы 1.

Теперь, доказательство теоремы становится очевидным. Алгоритм отыскания Р-выражения можно обосновать на повторном применению лемы 2, пока $F$, $\varphi$ и $\psi$ не окажутся функциями одной переменной. Однако, Р-выражение меньшей длины получается если лему 2 использовать только тогда когда к данной функцие не применимы части а/ или б/ следующей лемы 3. Точнее, часть б/ используется только если не применима а/.

Лема 3. Пусть $f$ и $x_i$ данные векторы.

а/ Если не существуют координаты векторов $f$ и $x_i$, такие что $f^j=x_i^j=0$, то функция

$$\varphi=\mathop{K}_{j=0}^{2^{n-i}-1}\Big(\mathop{K}_{r=1}^{2^{i-1}} \varphi_0^{j2^i+r}\mathop{K}_{r=1}^{2^{i-1}} \varphi_1^{j2^i+2^{i-1}+r}\Big),\ \text{где}\ \varphi_0^{j2^i+r}=\varphi_1^{j2^i+2^{i-1}+r}=\overline{f}_1^{j2^i+2^{i-1}+r}$$

не зависит от $x_i$ и удовлетворяет условию $x_i\uparrow\varphi=f$.

б/ Если для $f$ выполнено: если $f_0^{j2^i+r}=1$ тогда $f_1^{j2^i+2^{i-1}+r}=1$,

то функции $F$ и $\Psi$ ,определенны следующим образом

$$F=\mathop{K}_{j=0}^{2^{n-i}-1}\Big(\mathop{K}_{r=1}^{2^{i-1}}F_0^{j2^i+r}\mathop{K}_{r=1}^{2^{i-1}}F_1^{j2^i+2^{i-1}+r}\Big),\ \text{где } F_0^{j2^i+r}=F_1^{j2^i+2^{i-1}+r}=\bar{f}_0^{j2^i+r}$$

$$\Psi=\mathop{K}_{j=0}^{2^{n-i}-1}\Big(\mathop{K}_{r=1}^{2^{i-1}}\psi_0^{j2^i+r}\mathop{K}_{r=1}^{2^{i-1}}\psi_1^{j2^i+2^{i-1}+r}\Big),$$

$$\text{где } \psi_0^{j2^i+r}=\psi_1^{j2^i+2^{i-1}+r}=\begin{cases} f_1^{j2^i+2^{i-1}+r}, & \text{для } f_0^{j2^i+r}=0 \\ S, & \text{для } f_0^{j2^i+r}=1 \end{cases}$$

не зависят от $x_i$ и удовлетворяют $F\uparrow(x_i\uparrow\Psi)=f$.

Доказательство устанавливается непосредственной проверкой.

Может казатся что такой порядок применения лем 2 и 3 приведет к минимальному Р-выражению, но это не так. Поэтому становится вопрос отыскания минимальных Р-выражений.

Пример. Для вектора $f=11111001$ булевой функции трех переменных, указаный порядок применения лем приводит к следующему Р-выражению: $f=((x_1\uparrow(x_1\uparrow x_2))\uparrow(x_2\uparrow(x_1\uparrow x_1)))\uparrow x_3$. Однако, это не минимальное Р-выражение, так как следующее выражение также является Р-выражением данной функции

$$f=((x_1\uparrow x_3)\uparrow(x_2\uparrow x_3))\uparrow(x_1\uparrow x_2).$$

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# HOMOMORPHISMS OF NETS

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Homomorphisms between nets of the same dimension and degree are considered in this paper.

0. Let $\mathcal{P}$ be a nonempty set, n a positive integer ($n \neq 1$) and $\mathbb{B}_1,\dots,\mathbb{B}_k$ ($k \geq n+1$) a collection of disjoint subsets of the boolean of $\mathcal{P}$ . The elements of $\mathcal{P}$ are called points, the sets $\mathbb{B}_1,\dots,\mathbb{B}_k$ classes and the elements of the classes are blocks.

The structure $N = (\mathcal{P}; \mathbb{B}_1,\dots,\mathbb{B}_k)$ is called an n-dimensional net with a degree k (or simply an $[n,k]$ - net) if the following statements are satisfied.

(I) If P is a point and $i \in \{1,\dots,k\}$, then there exits exactly one block $b_i \in \mathbb{B}_i$ such that $P \in b_i$.

(II) Each sequence of blocks $b_{i_1},\dots,b_{i_n}$ from different classes contains exactly one common point.

It is shown in [1] that $\mathbb{B}_1,\dots,\mathbb{B}_k$ have the same cardinal which is called the ordre of the given net.

Let $N = (\mathcal{P}; \mathbb{B}_1,\dots,\mathbb{B}_k)$ and $N' = (\mathcal{P}'; \mathbb{B}_1',\dots,\mathbb{B}_k')$ be two n-dimensional nets with the same degree k. A mapping $f: \mathcal{P} \to \mathcal{P}'$ is said to be a homomorphism from N into N' if it satisfies the following condition:

(H) For each block $b_\nu \in \mathbb{B}_\nu$ there exists a $b_\nu' \in \mathbb{B}_\nu'$ such that $f(b_\nu) = \{f(P) \mid P \in b_\nu\} \subseteq b_\nu'$.

An isomorphism is a bijective homomorphism such that $f^{-1}$ is also a homomorphism.

If $\mathcal{P} \subseteq \mathcal{P}'$ and the embedding mapping from $\mathcal{P}$ into $\mathcal{P}'$ is a homomorphism, then N is said to be a subnet of N'.

Further on by a net we will always mean an $[n,k]$ - net.

1. If $f$ is a homomorphism from a net $N$ into a net $N'$ such that $|f(b)| = 1$ for a block $b$ of $N$, then $f$ is a constant homomorphism, i.e. $|f(\mathcal{P})| = 1$.

Proof. Let $f$ be constant on a block $b \in \mathbb{B}_i$, and let $f(b) = \{P'\}$. If $j \neq i$ and $b_j \in \mathbb{B}_j$, then $b \wedge b_j \neq \emptyset$, and this implies that $P' \in f(b_j)$. Therefore there exists a $b_j' \in \mathbb{B}_j'$ such that $f(b_j) \subseteq b_j'$ and $P' \in b_j'$, for each $b_j \in \mathbb{B}_j$. If $P \in \mathcal{P}$, then $P$ belongs to a block of $\mathbb{B}_j$ and thus $f(p) \in b_j'$.

Assume now that $j_1 < j_2 < \dots < j_n$ and $j_\nu \neq i$. Then for each $\nu$ there is a block $b_{j_\nu}' \in \mathbb{B}_{j_\nu}'$ such that $P' \in b_{j_\nu}'$ and $f(P) \in b_{j_\nu}'$ for each $P \in \mathcal{P}$. This implies that $P'$ and $f(P)$ are in $b_{j_1}' \wedge \dots \wedge b_{j_n}'$, and therefore $f(P) = P'$ for each $P \in \mathcal{P}$.

2. Let $f$ be a homomorphism from a net $N$ into a net $N'$, and let $b_s^1, b_s^2 \in \mathbb{B}_s$ be such that $f(b_s^1) \subseteq b_s'$, $f(b_s^2) \subseteq b_s'$ for some $b_s' \in \mathbb{B}_s'$. Then $f(b_s^1) = f(b_s^2)$.

Proof. Assume that $f(b_s^1) \neq f(b_s^2)$, and $Q' \in f(b_s^1) \setminus f(b_s^2)$. There exists a point $P \in b_s^1$ such that $Q' = f(P)$. Let $i_1 < i_2 < \dots < i_{n-1}$ and $i_\nu \neq s$ for each $\nu \in \{1,\dots,n-1\}$. By (I) there exist $b_{i_1}, \dots, \dots, b_{i_{n-1}}$ such that $P \in b_i$, $b_{i_\nu} \in \mathbb{B}_{i_\nu}$ for each $\nu \in \{1,\dots,n-1\}$. And, by (II) there exists a unique $R \in \mathcal{P}$ such that $R \in b_{i_1} \wedge \dots \wedge b_{i_{n-1}} \wedge b_s^2$. Then: $f(R) \in f(b_{i_1}) \wedge \dots \wedge f(b_{i_{n-1}}) \wedge f(b_s^2) \subseteq b_{i_1}' \wedge \dots \wedge b_{i_{n-1}}' \wedge b_s'$, and also $Q' = f(P) \in b_{i_1}' \wedge \dots \wedge b_s'$, where $b_{i_\nu}'$ are such that $f(b_{i_\nu}) \subseteq b_{i_\nu}'$. Thus we have $Q' = f(R) \in f(b_s^2)$, and this is impossible for we have assumed that $Q' \notin f(b_s^2)$.

3. If $f$ is a surjective homomorphism from a net $N = (\mathcal{P}; \mathbb{B}_1, \dots, \mathbb{B}_k)$ into a net $N' = (\mathcal{P}'; \mathbb{B}_1', \dots, \mathbb{B}_k')$, then:

$$b_\nu \in \mathbb{B}_\nu \Rightarrow f(b_\nu) \in \mathbb{B}_\nu'$$

and for each $\nu \in \{1,\dots,k\}$ and $b_\nu' \in \mathbb{B}_\nu'$ there exists a $b_\nu \in \mathbb{B}_\nu$ such that $f(b_\nu) = b_\nu'$.

Proof. If $b_s \in \mathbb{B}_s$ then there exist a $b_s' \in \mathbb{B}_s'$ such that $f(b_s) \subseteq b_s'$.

Assume that $Q' \in b_s' \setminus f(b_s)$. Then there exists a $P \in \mathcal{P}$ such that $f(P) = Q'$, for $f: \mathcal{P} \to \mathcal{P}'$ is a surjective. There exists a unique block $b_s^1 \in B_s$ such that $P \in b_s^1$. Then $Q = f(P) \in f(b_s^1)$ and therefore $f(b_s^1) \subsetneq b_s'$. Thus $f(b_s) \subsetneq b_s'$ and $f(b_s^1) \subsetneq b_s'$ and by 2 this implies that $f(b_s) = f(b^1)$, but this is impossible for $Q' \in f(b_s^1)$ and $Q' \notin f(b_s)$. Therefore $b_s' \setminus f(b_s) = \emptyset$ i.e. $b_s' = f(b_s)$.

Assume now that $b_s' \in B_s'$. We have to show that there is a $b_s \in B_s$ such that $f(b_s) = b_s'$. Let $P' \in b_s'$, then $P' = f(P)$ for some $P \in \mathcal{P}$ ; there is a unique $b_s \in B_s$ such that $P \in b_s$, and thus $P' \in f(b_s) \cap b_s'$, which implies $f(b_s) = b_s'$.

4. If $f: N \to N'$ is a bijective homomorphism, then it is an isomorphism.

Proof. We have to show that $f^{-1}: N' \to N$ is also a homomorphism.

If $b_\nu' \in B_\nu'$, then there is $b_\nu \in B_\nu$ such that $f(b_\nu) = b_\nu'$, and this implies that $f^{-1}$ is a homomorphism.

5. Let f be a homomorphism from a net $N = (\mathcal{P}; B_1, \ldots, B_k)$ into a net $N' = (\mathcal{P}'; B_1', \ldots, B_k')$ such that for any i,j: $1 \le i < j \le k$ and $b_i \in B_i$, $b_j \in B_j$ we have $f(b_i) \ne f(b_j)$. Then $N'' = (f(\mathcal{P}); f(B_1), \ldots, f(B_k))$ is a subnet of $N'$.

Proof. First, by the assumed property of f, we have that $f(B_i) \cap f(B_j) = \emptyset$ if $i \ne j$. Also by 1, we have that $|f(b_i)| > 1$ for each $i \in \{1, \ldots, k\}$ and $b_i \in B_i$. Namely if we had $|f(b_i)| = 1$ for some $i \in \{1, \ldots, k\}$ and $b_i \in B_i$, then we would get that f is constant and then $f(b_i) = f(b_j)$ for any i,j and $b_i \in B_i$, $b_j \in B_j$.

Let $P' \in f(\mathcal{P})$ and $1 \le i \le k$. Then there is a $P \in \mathcal{P}$ such that $P' = f(P)$ and therefore by (I) there is a unique $b_i \in B_i$ such that $P \in b_i$; this implies that $P' \in f(b_i)$.

Assume now that $P' = f(P^1)$, $P^1 \in b_i^1$; then $P' \in f(b_i) \cap f(b_i^1)$ whence follows that there is a $b_i' \in B_i'$ such that $f(b_i) \subsetneq b_i'$, $f(b_s^1) \subsetneq b_i'$ and by 2 we set $f(b_i) = f(b_i^1)$. So we have proved that $N''$ satisfies (I).

Finally assume that $1\le i_1<\ldots<i_n\le k$ and $b_{i_\nu} \in B_{i_\nu}$. Then there is a unique point $P \in \mathcal{P}$ such that $P \in b_{i_\nu}$ for each $\nu$, and thus $P' = f(P) \in f(b_{i_\nu})$. This implies that the condition (II) is satisfied.

Therefore N" is an n-dimensional k-net. Clearly the embedding mapping from N" into $N'$ is a homomorphism.

If $f:N \to N'$ is a surjective homomorphism, then $f(b_i) \neq f(b_j)$ for $i\neq j$ is satisfied and moreover then we have $N' = N''$.

6. Let f be a surjective homomorphism from a net $N = (\mathcal{P};B_1,\ldots,B_k)$ into a net $N' = (\mathcal{P}';B_1',\ldots,B_k')$ and let $\alpha = \ker f$. If $P \in \mathcal{P}$, then by $\overline{P}$ is denoted the $\alpha$-equivalence class containing P; if b is a block in N, then the set $\{\overline{Q}\,|\,f(Q) \in f(b)\}$ is denoted by $\overline{b}$ and $\overline{B}_i = \{\overline{b}_i\,|\,b_i \in B_i\}$. Then $N/\alpha = (\mathcal{P}/\alpha;\ \overline{B}_1,\ldots,\overline{B}_k)$ is a net isomorphic with $N'$.

Proof. If $\overline{P} = \overline{Q}$ and $\overline{P} \in \overline{b}$, then we have $f(Q) = f(P) \in f(b)$ and thus $\overline{Q} \in \overline{b}$, i.e. $\overline{b}$ is well defined. Assume that $\overline{b}_i = \overline{b}_j$, where $b_i \in B_i$ and $b_j \in B_j$. Then $f(P) \in f(b_i)$ iff $f(P) \in f(b_j)$ and therefore $f(b_i) = f(b_j)$ which is possible only if $i=j$. This shows that if $i\neq j$, then $\overline{B}_i \cap \overline{B}_j = \emptyset$.

Let $\overline{P} \in \mathcal{P}/\alpha$ and $i \in \{1,\ldots,k\}$. Then there is a unique $b \in B_i$ such that $P \in b$ and thus $f(P) \in f(b)$, i.e. $\overline{P} \in \overline{b} \in \overline{B}_i$.

Assume that $\overline{P} \in \overline{b} \cap \overline{c}$, where $\overline{b},\overline{c} \in \overline{B}_i$; then $b,c \in B_i$ and $f(P) \in f(b) \cap f(c)$ and this implies that $f(b) = f(c)$, i.e. $\overline{b} = \overline{c}$.

Finally, let $\overline{b}_{i_\nu} \in \overline{B}_{i_\nu}$ and $1\le i_k<\ldots<i_n\le k$. Then there exists a unique point P such that $P \in b_{i_1} \cap \ldots \cap b_{i_n}$ and thus $f(P) \in f(b_{i_1}) \cap \ldots \cap f(b_{i_n})$, i.e. $\overline{P} \in \overline{b}_{i_1} \cap \ldots \cap \overline{b}_{i_n}$. Conversely, if $\overline{Q} \in \overline{b}_{i_1} \cap \ldots \cap \overline{b}_{i_n}$, then $f(Q) \in f(b_{i_1}) \cap \ldots \cap f(b_{i_n})$ and this implies that $f(P) = f(Q)$, i.e. $\overline{P} = \overline{Q}$. Thus we have proved that $N/\alpha$ is a net. Clearly the canonical mapping $\overline{f}:\overline{P} \to f(P)$ is an isoporphism from $N/\alpha$ onto $N'$.

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# A NEAR-RINGS OF D-AFFINE TYRE

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In this paper we consider one special type of the near-rings with defect. We study some substructure of the near-rings of this type. Also, we give certain characterizations of the defect of such kind the near-rings.

A (left) zero-symmetric near-ring R is a set with two binary operations + and . , such that (R,+) is a group ( not necessarily abelian), (R,.) is a semigroup, the left distributive law holds, i.e.

$$x(y+z)=xy+xz \text{ for all } x,y,z\in R$$

and $ox=o$ for all $x\in R$, where o is the neutral element of (R,+). A right ideal of R is a normal subgroup of (R,+) such that

$$(x+a)y-xy\in A \text{ for all } a\in A,\ x,y\in R.$$

A subgroup B of (R,+)is a right R-subgroup if $BR\subseteq B$. A subgroup M of (R,+) is an invariant subgroup if $RM\subseteq M$ and $MR\subseteq M$ (see [3]).

Let (S,.) be a multiplicative subsemigroup of the semigroup (R,.) whose elements generate (R,+). The normal subgroup D of the group (R,+) generated by the set

$$D_S=\{d\in R:(\exists x,y\in R)(\exists s\in S)\ (x+y)s=xs+ys+d\}$$

is called the defect of distributivity of the near-ring R (see [1])

Let R be a near-ring with the defect D and let S be a multiplicative subsemigroup of (R,.) whose elements are distributive, i.e.

$$(x+y)s=xs+ys \text{ for all } x,y\in R \text{ and } s\in S.$$

<u>Definition</u>. The near-ring R with the defect D is said a near-ring of D-affine type, if R=L+D, where (L,+) is a subgroup of (R,+) generated by S. If (R,+) is an abelian group, then we say that R is an abelian near-ring of D-affine type.

Every $r \in R$ have the form $r=\sum_i (\pm s_i)+d$ ($s_i \in S$, $d \in D$). Clearly, L is a distributively generated near-ring. If R is an abelian near-ring of D-affine type, then all $r \in R$ have the form $r=s+d$ ($s \in S$, $d \in D$). In this case L is a subring of the near-ring R. If $L \cap D=\{o\}$, then by first isomorphism theorem we have $R/D \cong L$.

Examples. 1) Every distributively generated near-ring is a near-ring of 0-affine type.

2) Let R be a distributively generated near-ring and let A be an additive group. On the set RXA we define the operations + and . as follows:

$$(r_1,a_1)+(r_2,a_2)=(r_1+r_2,a_1+a_2)$$

$$(r_1,a_1)(r_2,a_2)=\begin{cases}(r_1r_2,a_2), & \text{if } a_1 \neq o\\ (r_1r_2,o), & \text{if } a_1 = o\end{cases}$$

for all $r_1,r_2 \in R$, $a_1,a_2 \in A$. From straightforward calculation it follows that (RXA,+,.) is a near-ring of D-affine type, where $D=\{(o,a):\ a \in A\}$.

Theorem 1. Let R be a near-ring of D-affine type. The normal subgroup A of the group (R,+) is a right ideal of R if and only if $AS \subseteq A$ and for all $x \in R$, $d \in D$, $a \in A$ hold $(x+a)d-xd \in A$.

Proof. Let A be a normal subgroup of (R,+) and $(x+a)d-xd \in A$ for all $x \in R$, $a \in A$, $d \in D$. It suffices to show that statement is true for all $r \in R$ of the form $r=s+d$ ($s \in S$, $d \in D$). Thus, for all $x,y \in R$, $a \in A$, where $y=s+d$ ($s \in S$, $d \in D$) we have

$$\begin{aligned}(x+a)y-xy&=(x+a)(s+d)-x(s+d)\\ &=(x+a)s+(x+a)d-(xs+xd)\\ &=xs+as+(x+a)d-xd-xs.\end{aligned}$$

Since $as \in A$ and $(x+a)d-xd \in A$ it follows $(x+a)y-xy \in A$ for all $x,y \in R$, $a \in A$, i.e. A is a right ideal of R.

Conversselly is immediate.

Definition. Let R be a near-ring and let D be a subset of R. The normal subgroup A of $(R,+)$ is a right D-ideal if for all $x \in R$, $a \in A$, $d \in D$ hold $(x+a)d-xd \in A$.

Every right ideal of R is a right D-ideal. The converse isn't true. For example, if R is a distributively generated near-ring and A is a normal subgroup of $(R,+)$ which isn't R-subgroup. Then A is a right O-ideal, but isn't a right ideal of R.

Definition. Let R be a near-ring and let B be a subset of R. The subgroup B of $(R,+)$ is a right D-subgroup if for all $b \in B$, $d \in D$ hold $bd \in B$.

Clearly, every right D-ideal is a right D-subgroup. From the definition of the right D-ideal and by using Theorem 1, we have

Theorem 1'. Let R be a near-ring of D-affine type. The subset A of R is a right ideal of R, if and only if $AS \subseteq A$ and A is a right D-ideal.

Theorem 2. Let R be a near-ring of D-affine type.

a) All right ideals A of R are of the form $A=I'+D'$, where $I'S \subseteq I'$, $D'S \subseteq D'$ and A is a right D-ideal.

b) All R-subgroups B of R are of the form $B=I''+D''$, where $I''S \subseteq I''$, $D''S \subseteq D''$ and B is D-subgroup.

Proof. a) Since $AS \subseteq A$, the result follows as an immediate consequence of the Theorem 1'.

b) For all $b=i''+d'' \in B$ and $r \in R$, where $r=s+d$, $(s \in S,\ d \in D)$ we have

$$br=(i''+d'')(s+d)$$

$$br=(i'' + d'')s+(i''+ d'')d$$

$$br=i''s+d'' s + (i'' + d'')d$$

$$br=b_1+b_2+b_3 \in B \text{ , } (i'' s=b_1 \in B,\ d'' s=b_2 \in B, (i''+d'')=b_3 \in B)$$

By induction on k, where $r=\sum_k(\pm s_k)+d$ ($s_k \in S$, $d \in D$), we complete the proof.

Corollary 3. let R be an abelian near-ring of D-affine type such that every normal D-subgroup is a right D-ideal of R. Then every right R-subgroup is a right ideal of R.

Denote by A(R) the annihilator of R, i.e. $A(R)=\{a \in R: ra=o$ for all $r \in R\}$. The following theorems give certain characterizations of the defect D.

Theorem 4. Let R be a near-ring of D-affine type. Then $D \subseteq A(R)$ if and only if R is a distributively generated near-ring.

Proof. If $D \subseteq A(R)$, then for all $x,y,r \in R$, where $r=\sum_i(\pm s_i)+d$ ($s_i \in S$, $d \in D$), we have $(x+y)d=o=xd+yd$. Hence, d is a distributive element and every $r \in R$ we can write in the form $r=\sum_i(\pm s_i')$, where $s_i'$ are a distributive elements.

Converselly, if R is a distributively generated near-ring, then $D=\{o\}$, that is $D \subseteq A(R)$.

Theorem 5. Let R be a near-ring of D-affine type. If D is a subring of R, then $D^2 = \{o\}$.

Proof. By definition of the defect D, for all $d_k \in D$ we have

$$d_k=\sum_{k_j}(r_{k_j} \pm d_{k_j}' - r_{k_j})$$

($r_{k_j} \in R$, $d_{k_j}'=-(x_{k_j}r_{k_j}+y_{k_j}r_{k_j})+(x_{k_j}+y_{k_j})r_{k_j}$ , $x_{k_j}$ , $y_{k_j}$ , $r_{k_j} \in R$).

It sufficies to show that statement is true for all $r_{k_j}$ of the form

$r_{k_j}=s_j+d$, $(s_j\in S,\ d\in D)$. Thus,

$$d'_{k_j}=-y_{k_j}r_{k_j}-x_{k_j}r_{k_j}+(x_{k_j}+y_{k_j})(s_j+d)$$

$$d'_{k_j}=-y_{k_j}r_{k_j}-x_{k_j}r_{k_j}+(x_{k_j}+y_{k_j})s_j+(x_{k_j}+y_{k_j})d$$

$$d'_{k_j}=-y_{k_j}r_{k_j}-x_{k_j}r_{k_j}+x_{k_j}s_j+y_{k_j}s_j+(x_{k_j}+y_{k_j})d$$

$$d'_{k_j}=-y_{k_j}r_{k_j}-x_{k_j}r_{k_j}+x_{k_j}(r_{k_j}-d)+y_{k_j}(r_{k_j}-d)+(x_{k_j}+y_{k_j})d$$

$$d'_{k_j}=-y_{k_j}r_{k_j}-x_{k_j}r_{k_j}+x_{k_j}r_{k_j}-x_{k_j}d+y_{k_j}r_{k_j}-y_{k_j}d+(x_{k_j}+y_{k_j})d$$

$$d'_{k_j}=-y_{k_j}r_{k_j}-x_{k_j}d+y_{k_j}r_{k_j}-y_{k_j}d+(x_{k_j}+y_{k_j})d$$

Hence, $$d_id'_{k_j}=d_i(-y_{k_j}r_{k_j}-x_{k_j}d+y_{k_j}r_{k_j}-y_{k_j}d+(x_{k_j}+y_{k_j})d)$$

$$d_id'_{k_j}=-d_iy_{k_j}r_{k_j}-d_ix_{k_j}d+d_iy_{k_j}r_{k_j}-d_iy_{k_j}d+(d_ix_{k_j}+d_iy_{k_j})d.$$

Thus, $d_id'_{k_j}=o$, because the defect D is an ideal, i.e. a right R-subgroup, and by assumtation D is a subring of R. Since $D^2=\sum_{i,k} d_id_k, (d_i,d_k\in D)$, we have

$$d_id_k=\sum_{k_j}(d_ir_{k_j}\pm d_id'_{k_j}-d_ir_{k_j})=o$$

Thus, $D^2=\{o\}$ and this finiches the proof.

For example, a near-ring of $\Delta$-endomorphism of the group $(Z_6,+)$, where $\Delta=\{o,3\}$ (see table 2 of [2]) is an abelian near-ring of $\mathcal{D}$-affine type with $S=\{f_o,f_1,f_2,f_3,f_4,f_5\}$ and the defect $\mathcal{D}=\{f_o,f_9,f_{12},f_{14}\}$. Since $\mathcal{D}$ as a subnear-ring is distributive, we have $\mathcal{D}^2=\{o\}$. However, a near-ring (19) (see [3], p. 341) is an abelian near-ring of D-affine type with $S=\{o,3\}$ and $D=\{o,2,4\}$. But D as a subnear-ring isn't distributive and hence $D^2\neq\{o\}$.

For all $x,y\in R$ and $m\in M$ ($M\subseteq R$) we call the element $-(xm+ym)+(x+y)m$ the distributor of $x$ and $y$ with respect to $m$ and denote it by $[x,y,m]$. Denote by $D_M(R)$ the normal subgroup of $(R,+)$ generated by

$$\{[x,y,m] : x,y\in R, m\in M\}.$$

<u>Theorem 6</u>. Let $R=L+M$ be a near-ring with the defect $D$, where $(L,+)$ is a subgroup of $(R,+)$ generated by multiplicative subsemigroup $(S,.)$ of distributive elements of $R$. If $M$ is a subset of additive center of $(R,+)$ and $RM\subseteq M$, then $D\subseteq D_M(R)$.

<u>Proof.</u> Let $d\in D$ and $d=-(xr+yr)+(x+y)r$, $(x,y,r\in R)$. It sufficies to take $r\in R$ of the form $r=s+m$, $(s\in S, m\in M)$. Thus,

$$d=-(x(s+m)+y(s+m))+(x+y)(s+m)$$

$$d=-(xs+xm+ys+ym)+(x+y)s+(x+y)m$$

$$d=-ym-ys-xm-xs+xs+ys+(x+y)m$$

$$d=-ym-xm+(x+y)m\in D_M(R).$$

<u>Theorem 7</u>. Let $R=L+M$ be a near-ring with defect $D$, where $(L,+)$ is a subgroup of $(R,+)$ generated by multiplicative subsemigroup $(S,.)$ of distributive elements of $R$ and let $M$ be an invariant subgroup of $R$. If $M$ is a subset of additive center of $(R,+)$ and $D_M(R)$ is a right $M$-subgroup, then $D_M(R)$ is an ideal of $R$.

<u>Proof</u>. Let $r=\sum_i(\pm s_i)+m$, $(s_i\in S, m\in M)$ and

$$a=-(xb+yb)+(x+y)b\in D_M(R),\ (x,y\in R,\ b\in M)$$

Then

$$ar=(-(xb+yb)+(x+y)b)(\sum_i(\pm s_i)+m)$$

$$ar=\sum_i(-(x(\pm bs_i)+y(\pm bs_i))+(x+y)(\pm bs_i))+(-(xb+yb)+(x+y)b)m\in D_M(R).$$

Thus, $D_M(R)$ is a $R$-subgroup. The result follows by using Theorem 6 and Lemma 3.2 of [1] .

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# SOME NONAXIOMATIZABLE CLASSES OF SEMIGROUPS

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To prove nonaxiomatizability results for some classes of models, we regularly use three well known theorems of model theory: compactness theorem, Löwenheim-Skolem theorem and ultraproduct theorem.

COMPACTNESS THEOREM. (Gödel-Malcev) A theory T has a model iff every finite subset of T has a model.

LÖWENHEIM-SKOLEM THEOREM. If theory T has infinite models, then it has models of any power $\geq \| L \|$ .

$\| L \|$ is a power of the language L and is defined by $\| L \| = \omega + \mathrm{card} L$ .

ULTRAPRODUCT THEOREM (Łoś). Let $(A_i)_{i \in I}$ be a family of models for L and let F be an ultrafilter over I. Then for any formula $\varphi(x_1,\dots,x_n)$ of L and any $a_1,\dots,a_n \in \prod A_i$

$$\prod{}_F A_i \models \varphi[a_1^F,\dots,a_n^F] \quad \text{iff} \quad \{i \in I \mid A_i \models \varphi[a_1(i),\dots,a_n(i)]\} \in F$$

(We assume that in $\prod A_i$ and $\prod_F A_i$ i runs through index set I).

Proofs of these theorems and details of ultraproduct construction can be found in any standard textbook on model theory such as [1] .

In the sequel we use these theorems to give typical nonaxiomatizability proofs in the case of semigroups. We believe that careful reader can easily produce his own nonaxiomatizable classes of semigroups.

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DEFINITION: A semigroup S, with zero 0, is nil if:

(N) $\quad \forall x \exists n \, (x^n = 0)$

(N) is a sentence of so called $\omega$-logic. We assume that m,n,p,q,r are variables for natural numbers and that $x,y,x_1,\dots,x_n$ are variables for semigroup elements

We should state explicitely that for purely practical reasons we do not include 0 among natural numbers. Otherwise we should write in all formulas that all variables are different from 0.

It is natural to ask if there exists a set of first order axioms for a class of all nil semigroups. Using compactness theorem we shall prove that there is no such a set. In order to do that, we shall construct a semigroup satisfying all first order sentences true in all nil semigroups but which is not nil.

THEOREM 1. The class of all nil semigroups is nonaxiomatizable.

Proof: Let $a_n$ be a generator of a free semigroup $F_n$ and $I_n = \{w \in F_n \mid |w| \geqslant n\}$. By $|w|$ we denote the length of the word w from $F_n$. It is easy to prove that $I_n$ is an ideal of $F_n$ and that $F_n/I_n$ is nil.

Let us denote $F_n/I_n$ by $S_n$ and $I_n$ by $0_n$. The semigroup $S = \sum S_n$ is also nil with a zero $0 = (0_1, 0_2, \ldots)$. By $\sum S_n$ we denote a subsemigroup of $\prod S_n$ with elements a such that only finitely many $a(m)$ $(m \in N)$ are different from $0_m$.

Let $L = \{\cdot, 0, a\}$ and ThS be the set of all sentences in a language $\{\cdot, 0\}$ true in S and:

$(N_n)$ $\quad a^n \neq 0 \quad (n \in N)$

Let $T = \mathrm{ThS} \cup \{(N_n) \mid n \in N\}$ and $T_0$ some finite subset of T. $T_0$ is contained in some theory $T_r = \mathrm{ThS} \cup \{(N_n) \mid n < r\}$. $(S, a_r) \models T_r$ so $T_r$ and $T_0$ are consistent theories, and by compactness theorem so is T. Let $(S', a)$ be a model for T. $S' \equiv S$ (S and S′ satisfy the same set of first order sentences) since S′ is a model of ThS but S′ is not nil because it is a model for all $(N_n)$ $(n \in N)$.

If the class of all nil semigroups was axiomatizable, from $S \equiv S'$ it would follow that S′ is nil, a contradiction.

Example 1. A semigroup S is power-joined (see [2] ) if:

(PJ) $\quad \forall xy \exists mn \; (x^m = y^n)$

Adapting the proof of Th1 we can prove that the class of all power-joined semigroups is nonaxiomatizable.

Example 2. S is a 3PJ-semigroup (see [3] ) if it is a semigroup in which:

(BPJ) $\quad \forall xy \forall mn \exists pq \, ((xy)^p = (x^m y^n)^q)$

For BPJ-semigroups a theorem similar to Th1 holds.

In [4] a more general result is proved, of which the above results are simple consequences.

DEFINITION: Let S be a semigroup. A semigroup defined on power set of S, with a product defined by:

$$A \cdot B = \{a \cdot b \mid a \in A,\ b \in B\}$$

when $A \neq \phi$, $B \neq \phi$ and $A \cdot B = \phi$ otherwise, we denote by $2^S$. A semigroup isomorphic to $2^S$ is called a global of S. If S is a group, then any semigroup isomorphic to $2^S$ is called a global of a group.

An interesting result about globals of finite groups can be found in [5].

Using Löwenheim-Skolem theorem, we prove:

THEOREM 2. The class of all globals of groups is nonaxiomatizable.
Proof: Let L be an expansion of the language $\{\cdot\}$. If the class of all globals of groups is axiomatizable by a theory T in L, then (by Löwenheim-Skolem theorem) there are models for T of all cardinals $> \|L\|$. But if $\kappa$ is a limit cardinal, there is no global of a group, with exactly $\kappa$ elements, which is a contradiction.

Consequently, the class of all globals of groups is nonaxiomatizable.

The following, more general theorem, has the same proof as Th2.

DEFINITION: Let K be a class of semigroups. We say that S is K-global if $S \cong 2^G$ and $G \in K$.

THEOREM 3. Let K be a class of semigroups with an infinite model. Then a class of all K-globals is nonaxiomatizable.

Example 3. Th3 holds for the following classes K (of course the list is not exhaustive):

the class of all regular semigroups
the class of all inverse semigroups
the class of all cyclic groups
the class of all bands
the class of all free semigroups
the class of all nil semigroups

Example 4. Th3 holds for an axiomatizable class $K$ of semigroups with arbitrary large finite models, since it is well known that a theory with arbitrary large finite models, has an infinite model.

Example 5. Th3 holds for every nontrivial variety K of semigroups. A variety is trivial if it satisfies the axiom $\forall xy\ (x = y)$.

Since K is variety there are free K-semigroups. Among them there is a free K-semigroup with infinitely many free generators. Clearly this semigroup is infinite if K is not trivial.

Example 6. Th3 also holds in all logics where Löwenheim-Skolem theorem holds. Examples of such logics are $\omega$-logic, weak second order logic and $L_{\omega_1\omega}$ .

DEFINITION: S is palindromic semigroup (see [6] ) if it satisfies:

(P) $\exists n > 1\ \forall x_1 \dots x_n\ (x_1 \dots x_n = x_n \dots x_1)$

THEOREM 4. A class of all palindromic semigroups is nonaxiomatizable.

Proof: Let $a_n, b_n$ be generators of a free semigroup $F_n$ and $I_n = \{ w \in F_n \mid |w| \geq n \}$. As in Th1 $I_n$ is an ideal and $F_n / I_n$ is palindromic.

Let F be the Fréchet filter over N (a set of all subsets of N with finite complements) and G some ultrafilter containing F. Let also $S = \prod_G S_n$. We prove that S is not palindromic.

Let $x_1 = \dots = x_{n-1} = (a_1, a_2, \dots)^G$ and $x_n = (b_1, b_2, \dots)^G$.

Then:

$$x_1 \dots x_n = (0, \dots, 0, a_{n+1}^{n-1} b_{n+1}, a_{n+2}^{n-1} b_{n+2}, \dots)^G$$

$$x_n \dots x_1 = (0, \dots, 0, b_{n+1} a_{n+1}^{n-1}, b_{n+2} a_{n+2}^{n-1}, \dots)^G$$

and $\{ m \in N \mid x_1(m) \dots x_n(m) \neq x_n(m) \dots x_1(m) \} = \{ n+1, n+2, \dots \} \in F \subset G$

so $x_1 \dots x_n \neq x_n \dots x_1$ .

Since this is valid for all $m \in N$, S is not palindromic and by ultraproduct theorem, the class of all palindromic semigroups is nonaxiomatizable.

Example 7. The class of all finite bands is nonaxiomatizable.

Let $S_n$ be the left zero semigroup with n elements and F a nonprincipal ultrafilter. $\prod_F S_n$ is a left zero semigroup not isomorphic to any of $S_n$ $(n \in N)$, since F is nonprincipal. There are no other finite left zero semigroups except $S_n$ $(n \in N)$, so $\prod_F S_n$ is infinite.

Nonaxiomatizability follows.

Example 8 (D. Blagojević). The class of all semigroups with finitely many idempotents is nonaxiomatizable.

The proof of example 7 is also applicable to example 8.

Example 9 (D. Blagojević). The class of all regular semigroups in which every element has only finitely many inverses

Since $xyx = x$ in all left zero semigroups, the proof of example 7 is again applicable.

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# COMPATIBLE SUBASSOCIATIVES

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The notion of compatible n-subsemigroup of an n-semigroup, introduced in [1], and almost all the results on compatibility obtained there, can be generalized for J-subassociatives of a J-associative in a straightforward way. In this paper we shall consider these questions for J-associatives in some details.

## §1. Preliminaries

Let $\underline{A} = (A;F)$ be an algebra with the carrier A and a nonempty set of finitary operations, $F = F_2 \cup F_3 \cup \ldots \cup F_n \cup \ldots$, where $F_n$ consists of the n-ary operations of F. If $f \in F_{n+1}$ and $f:(x_0,x_1,\ldots,x_n) \mapsto y$, then it is written $y = fx_0x_1 \ldots x_n$.

The semigroup $\underline{A}^\wedge$ with a presentation

$$\langle A;\{a = a_0a_1\ldots a_n | a = fa_0a_1\ldots a_n \text{ in } \underline{A}\}\rangle$$

is called the *universal semigroup* for $\underline{A}$. Denoting by $a^\wedge$ the element of $A^\wedge$ determined by $a \in A$ and putting $^\wedge: a \to a^\wedge$, we obtain a mapping from A into $A^\wedge$. The algebra $\underline{A}$ is called a *semigroup algebra* if the mapping $^\wedge$ is injective.

If $\phi:\underline{A} \to \underline{A}'$ is a homomorphism, then there exists a unique homomorphism $\phi^\wedge:\underline{A}^\wedge \to \underline{A}'^\wedge$ such that $\phi^\wedge(a^\wedge) = \phi(a)$ for any $a \in A$. Clearly, if $\phi$ is an epimorphism (isomorphism), then $\phi^\wedge$ is also an epimorphism (isomorphism), but it may happen $\phi$ to be a monomorphism and $\phi^\wedge$ not to be such one (Ex. 1), §3). A monomorphism $\phi:\underline{A} \to \underline{A}'$ is said to be *compatible* if $\phi^\wedge:\underline{A}^\wedge \to \underline{A}'^\wedge$ is also a monomorphism. And, a subalgebra $\underline{B}$ of $\underline{A}$ is said to be *compatible* in $\underline{A}$ if the embedding monomorphism $\varepsilon:\underline{B} \to \underline{A}$ is compatible.

The subject of this paper are compatible subassociatives of an associative. Namely, an F-algebra $\underline{A} = (A;F)$ is called an F-associative if it satisfies all the identities that hold in the class of semigroup F-algebras, i.e. if the general associative law holds in $\underline{A}$. An F-associative is called an F-group if $(A,f)$ is an n-group for each $f \in F_n$. It is well known that any F-group is a semigroup F-algebra ([2]).

In studying associatives, it is convenient to consider the submonoid $J = J_F$ of the additive monoid of nonnegative integers generated by the set $\{n-1 \mid F_n \neq \emptyset\}$. If $d_F$ is the greatest common divisor of the elements of $J_F$, then the following result holds: Every F-associative is a semigroup associative if and only if $d_F \in J_F$, and then an F-associative is in fact a $(d_F+1)$-semigroup. We note also that the associative law implies that for each $n \in J_F$ we have an "associative product"

$$[\ ]: (x_o,x_1,\ldots,x_n) \to [x_o x_1 \ldots x_n]$$

in an F-associative $\underline{A}$, where $[x_o] = x_o$. This is the reason why an F-associative is called a J-associative and the operational symbols are not used. The notions: J-subassociative, J-subgroup, ideal of a J-associative have usual meaning.

A J-associative is said to be cyclic if it is generated by one of its elements. The structure of cyclic J-associatives is described in [5].

## §2 Properties of compatible subassociatives

Denote by $\mathcal{L}(A)$ the set of all J-subassociatives of a J-associative A and by $\mathcal{C}(A)$ the set of all compatible J-subassociatives of A. The following statements hold:

**2.1.** $B \in \mathcal{C}(A) \Leftrightarrow B\hat{}$ is a subsemigroup of $A\hat{}$. □

**2.2.** $B \in \mathcal{L}(A) \Rightarrow \mathcal{L}(B) \cap \mathcal{C}(A) \subseteq \mathcal{C}(B)$. □

**2.3.** $B \in \mathcal{C}(A) \Rightarrow \mathcal{C}(B) \subseteq \mathcal{C}(A)$. □

**2.4.** $\mathcal{C}(A)$ is inductive, i.e. if $\{B_i \mid i \in I\}$ is a chain in $\mathcal{C}(A)$, then $B = \bigcup_i B_i \in \mathcal{C}(A)$. □

2.5. If $\varphi \in \mathrm{Aut}A$, $B \in \mathcal{L}(A)$ and $C = \varphi(B)$, then

$$B \in \mathcal{C}(A) \iff C \in \mathcal{C}(A). \square$$

2.6. $B \in \mathcal{L}(A)$, $A \setminus B$ is an ideal in $A \Longrightarrow B \in \mathcal{C}(A)$.

Note that the sufficient condition in 2.6 is not necessary (Ex. 4), §3). $\square$

2.7. If G is a J-subgroup of a semigroup J-associative A, then $G \in \mathcal{C}(A)$. $\square$

If $A = \langle a \rangle = \{a^{n+1} | n \in J\}$ is an infinite cyclic J-associative, then $A^\wedge$ is the free semigroup generated by a (3.1 in [5]). The theorem 4.1 od [1] is true for J-associatives too:

2.8. A J-subassociative B of an infinite cyclic J-associative A is compatible in A if and only if B is cyclic. $\square$

Using the fact that every J-subassociative C of a finite J-group G is a J-subgroup of G, as well as 2.7 and 2.8 it can be proved the following proposition:

2.9. Let $A = \{a^{n+1} | n \in J\}$ be a finite cyclic J-associative, let P be its periodic part and C be a J-subassociative of A.

i) If $C \subseteq P$, then $C \in \mathcal{C}(A)$.

ii) Let $C \nsubseteq P$ and let k be the least integer such that $b = a^{k+1} \in C$. If there exists $q \in J$ such that $c = a^{q+1} \in C$, $k+1 \nmid q+1$ and $q < s$, then $C \notin \mathcal{C}(A)$.

(Here, $s = \min\{n \in J | (\exists m \in J)\ m \neq n,\ a^{n+1} = a^{m+1}\}$, and the periodic part of A is $P = \{x | x \in A,\ x = a^{n+1}$ for infinitely many $n \in J\}$.)

§3. Examples

Below we give four examples which can be also found in [1], p.p. 26, 28. Ternary associatives, i.e. J-associatives with $J = \{2k | k \geq 0\}$ in all of them are considered.

1) Let $A = \{a,b,c\}$, $B = \{a,b\}$ and a ternary operation be defined on A by:

$$[ccc] = b \text{ and } [xyz] = a \text{ if } \{a,b\} \cap \{x,y,z\} \neq \emptyset.$$

Then A is a J-associative and B is a J-subassociative of A. The free coverings $A^\wedge$ and $B^\wedge$ are given by the following multiplication tables:

$A^\wedge$:

| | a | b | c | α | β | γ |
|---|---|---|---|---|---|---|
| a | α | α | α | a | a | a |
| b | α | α | β | a | a | a |
| c | α | β | γ | a | a | b |
| α | a | a | a | α | α | α |
| β | a | a | a | α | α | α |
| γ | a | a | b | α | α | β |

$B^\wedge$:

| | a | b | u | v |
|---|---|---|---|---|
| a | u | u | a | a |
| b | u | v | a | a |
| u | a | a | u | u |
| v | a | a | u | u |

$|A^\wedge| = 6$, $|B^\wedge| = 4$.

The extension $\varepsilon^\wedge$ of the embedding monomorphism $\varepsilon: B \to A$ is not a monomorphism, for $\varepsilon^\wedge(u) = \varepsilon^\wedge(v) = \alpha$ but $u \neq v$. Thus $B \notin \mathcal{C}(A)$.

2) Let $A = \{a,b,c,d,e\}$ and a ternary operation $[\ ]$ be defined on A by:

$$\{x,y,z\} \cap \{c,d,e\} \neq \emptyset,\ (x,y,z) \neq (e,e,e) \implies [xyz] = c,$$

$$x,y,z \in \{a,b\} \implies [xyz] = a$$

and $[eee] = d$. Then A is a J-associative, $B = \{a,b\}$ and $C = \{c,d\}$ are two isomorphic J-subassociatives and

$$A^\wedge = \{a,b,c,d,e,aa,bb,cc,ee,be,eb,de\},\ |A^\wedge| = 12,$$

(aa=ab=ba, cc=ac=ca=ad=da=ae=ea=bc=cb=cd=dc=dd=ec=ce, de=ed);

$$B^\wedge = \{a,b,aa=ab=ba,bb\},\qquad |B^\wedge| = 4;$$

$$C^\wedge = \{c,d,cc=cd=bc,dd\},\qquad |C^\wedge| = 4.$$

Therefore $B \in \mathcal{C}(A)$ and $C \notin \mathcal{C}(A)$, for cc=dd in $A^\wedge$ but cc≠dd in $C^\wedge$.

Thus isomorphism, in general, do not preserve the compatibility.

3) The set $A = \{1',1'',3,5,7,\ldots\}$ with the ternary operation $[xyz] = \psi(x) + \psi(y) + \psi(z)$, where the mapping $\psi: A \to \mathbb{N}$ is defined by $\psi(1') = 1 = \psi(1'')$, $\psi(a) = a$ for all $a \neq 1', 1''$, is a ternary semigroup, i.e. J-associative and $B = \{1',3,5,\ldots\}$, $C = \{1'',3,5,\ldots\}$ are J-subassociatives. The free coverings $A^\wedge$, $B^\wedge$, $C^\wedge$ of A,B,C, respectively, are given by:

$A^\wedge = \{1', 1'', (1',1'), (1',1''), (1'',1'), (1'',1'')\} \cup \{3,4,5,6,\ldots\}$,

$B^\wedge = \{1', (1',1'), 3,4,5,6,\ldots\}$,

$C^\wedge = \{1'', (1'',1''), 3,4,5,6,\ldots\}$,

where

$$1^i * 1^j = (1^i, 1^j), \quad 1^i * (1^j, 1^k) = 3 = (1^j, j^k) * 1^i,$$

$$(1^i, 1^j) * (2+k) = 4+k = (2+k) * (1^i, 1^j),$$

$$1^i * (2+k) = 3+k = (2+k) * 1^i.$$

Thus $B, C \in \mathcal{C}(A)$.

The intersection $D = B \cap C$ is also a subassociative of A, but it is not compatible in A; namely, $\varepsilon^\wedge(3*5) = \varepsilon^\wedge(5*3) = 8$, but $3*5 \neq 5*3$ in D.

4) Consider the additive semigroup of positive integers, $\mathbb{N}(+)$, as a ternary semigroup A, $[xyz] = x+y+z$. The set $B = \{2k+1 \mid k=0,1,2,\ldots\}$ is a ternary subsemigroup of A and $B^\wedge \cong A^\wedge \cong N(+)$. Thus the extension $\varepsilon^\wedge : B^\wedge \to A^\wedge$ of the embedding $\varepsilon : B \to A$ is a monomorphism, i.e. $B \in \mathcal{C}(A)$, but $A \setminus \mathbb{N} = 2\mathbb{N}$ is not an ideal in A.

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# ON A CLASS OF SEMIGROUPS

P.Protić and S.Bogdanović

G.LALLEMENT and M.PETRICH have considered Rees matrix semigroup over monoid in [6]. Using the method of Ǵ.ČUPONA, [2] we give the structural theorem of semigroups in which some ideal is a Rees matrix semigroup over monoid, (Theorem 1.1.).The similar method have used S.MILIĆ and V.PAVLOVIĆ in [7]. In section 2. we consider semigroups in which some quasi-ideal is a group (minimal).These semigroups are considered in [9] (Theorem 5.1 4). In section 3. we consider a class of (m,n)-ideal semigroups.This class is larger than a class of bi-ideal semigroups which are considered by B.TRPENOVSKI in [10], so that it contains a class of semigroups in which all subsemigroups are left ideals which were considered by E.G.SHUTOV, [11] and N.KIMURA,T.TAMURA and R.MERKEL, [4].

We denote by $\langle a\rangle$ a monogenic semigroup generated by element $a$, $|\langle a\rangle|$ is the cardinal of $\langle a\rangle$, $K_a$ is the cyclic subgroup of $\langle a\rangle$. For nondefinied notions we refer to [1,8,9].

## 1. EXTENSION OF REES MATRIX SEMIGROUP

Let $D = D^1$ be a semigroup with a group $U(D) = G$ (a group of units of D) and P is a $\Lambda\times I$-matrix over G. Let $\mathcal{M}(D;I,\Lambda;P)$ be a set of elements $(a;i,\lambda)$, where $a\in D$, $i\in I$, $\lambda\in\Lambda$ and operation is definied by :

$$(a;i,\lambda)(b;j,\mu) = (ap_{\lambda j}b;i,\mu) .$$

Then $\mathcal{M}(D;I,\Lambda;P)$ is a semigroup which we call the Rees matrix semigroup over monoid D, (G.LALLEMENT and M.PETRICH, [6]).

CONSTRUCTION. Let $\mathcal{M}(D;I,\Lambda;P)$ be a Rees matrix semigroup and Q a partial semigroup so that: $(D\times I\times\Lambda)\cap Q = \emptyset$. Let $\xi:p\to\xi_p$ be a mapping from Q into semigroup $\mathcal{T}(I)$ of all mappings from I into I and $\eta:p\to\eta_p$ a mapping from Q into semigroup $\mathcal{T}(\Lambda)$ of all mappings from $\Lambda$

into $\Lambda$. Let for all $p,q \in Q$ be:

(i) $pq \in Q \Longrightarrow \xi_{pq} = \xi_q \xi_p$, $\eta_{pq} = \eta_p \eta_q$

(ii) $pq \notin Q \Longrightarrow \xi_q \xi_p = \text{const.}$, $\eta_p \eta_q = \text{const.}$

Let $h: Q \times I \to D$ be mapping and

(iii) $pq \in Q \Longrightarrow h(pq,i) = h(p, i\xi_q) h(q,i)$

(iv) the term $p_{\lambda i\xi_p} h(p,i) p^{-1}_{\lambda\eta_p i}$ does not depend on $i \in I$ for $p,q \in Q$ ;

this term we denote by $k(p,\lambda)$ .

Let us define a multiplication on $\Sigma = (D \times I \times \Lambda) \cup Q$ with:

(1) $(a;i,\lambda)(b;j,\mu) = (ap_{\lambda j} b;i,\mu)$

(2) $p(a;i,\lambda) = (h(p,i)a; i\xi_p, \lambda)$

(3) $(a;i,\lambda)p = (ak(p,\lambda);i, \lambda\eta_p)$

(4) $pq = r \in Q \Longrightarrow pq = r \in \Sigma$

(5) $pq \notin Q \Longrightarrow pq = (h(p,i\xi_q)h(q,i)p^{-1}_{\lambda\eta_p\eta_q, i}\ ; i\xi_q\xi_p, \lambda\eta_p\eta_q)$ .

<u>NOTATION</u>. $(\Sigma, \cdot) = \mathcal{M}(D;I,\Lambda;P;Q;h,k,\xi,\eta)$ .[1)]

<u>THEOREM 1.1</u>. <u>A semigroup</u> $\Sigma$ <u>contains some ideal which is a Rees matrix semigroup if and only if</u> $\Sigma \cong \mathcal{M}(D;I,\Lambda;P;Q;h,k,\xi,\eta)$ .

<u>PROOF</u>. Let a semigroup $\Sigma$ have an ideal K which is a Rees matrix semigroup.Then $Q = \Sigma \setminus K$ is a partial semigroup so that:

$$\Sigma = K \cup Q = \mathcal{M}(D;I,\Lambda;P) \cup Q\ , \ (K \cong \mathcal{M}(D;I,\Lambda;P)).$$

For $p \in Q$ and $(1;i,\ell) \in K$ is $p(1;i,\ell) = (d;k,s) \in K$ , where $d = h(p,i,\ell)$ , $k = i\xi_{p,\ell}$ , $s = \ell\eta_{p,i}$ , so that

$$\begin{aligned}(h(p,i,\ell);i\xi_{p,\ell}, \ell\eta_{p,i}) &= p(1;i,\ell)p^{-1}_{\ell k};k,\ell) \\ &= (h(p,i,\ell);i\xi_{p,\ell}, \ell\eta_{p,i})(p^{-1}_{\ell k};k,\ell) \\ &= (h(p,i,\ell)p_{\ell\eta_{p,i}k}p^{-1}_{\ell k}; i\xi_{p,\ell}, \ell) .\end{aligned}$$

From this we have $\ell\eta_{p,i} = \ell$ , so that

$$p(1;i,\ell) = (h(p,i,\ell);i\xi_{p,\ell}, \ell) .$$

Further

1) The function k is introduced in order to simplify the proof of Theorem 1.1.

$$(h(p,i,\lambda);i\xi_{p,\lambda},\lambda) = p(1;i,\ell)(p^{-1}_{\ell j};j,\lambda)$$
$$= (h(p,i,\ell);i\xi_{p,\ell},\ell)(p^{-1}_{\ell j};j,\lambda)$$
$$= (h(p,i,\ell);i\xi_{p,\ell},\lambda)\ .$$

From this $h(p,i,\lambda) = h(p,i,\ell)$ , i.e. $h$ does not depend on $\lambda$ and $i\xi_{p,\lambda} = i\xi_{p,\ell}$ , i.e. $\xi$ does not depend on $\lambda$. Hence,

$$p(1;i,\lambda) = (h(p,i);i\xi_p,\lambda)$$

where $h\colon Q\times I \to D$ and $\xi_p : I \to I$ . Similarly

$$(1;i,\lambda)p = (k(p,\lambda);i,\ \lambda\eta_p)$$

where $k\colon Q\times\Lambda \to D$ and $\eta_p\colon \Lambda \to \Lambda$ .

Since

$$(p_{\lambda i\xi_p}h(p,i);i,\lambda) = (1;i,\lambda)(h(p,i);i\xi_p,\lambda)$$
$$= (1;i,\lambda)p(1;i,\lambda)$$
$$= (k(p,\lambda);i,\ \lambda\eta_p)(1;i,\lambda)$$
$$= (k(p,\lambda)p_{\lambda\eta_p i};i,\lambda)$$

it follows that $p_{\lambda i\xi_p}h(p,i) = k(p,\lambda)p_{\lambda\eta_p i}$ , i.e. $k(p,\lambda) = p_{\lambda i\xi_p}h(p,i)p^{-1}_{\lambda\eta_p i}$

so, the term $p_{\lambda i\xi_p}h(p,i)p^{-1}_{\lambda\eta_p i}$ does not depend on $i\in I$ .

For $p\in Q$ , $d\in D$ we have

$$p(d;i,\lambda) = p(1;i,\ell)(p^{-1}_{\ell i}d;i,\lambda)$$
$$= (h(p,i);i\xi_p,\ell)(p^{-1}_{\ell i}d;i,\lambda)$$
$$= (h(p,i)p_{\ell i}p^{-1}_{\ell i}d;i\xi_p,\lambda)$$
$$= (h(p,i)d;i\xi_p,\lambda)$$
$$(d;i,\lambda)p = (dk(p,\lambda);i,\ \lambda\eta_p)\ .$$

For $p,q\in Q$ ; $pq\in Q$ we have

$$(h(pq,i);i\xi_{pq},\lambda) = (pq)(1;i,\lambda) = p(q(1;i,\lambda)) = p(h(q,i);i\xi_q,\lambda)$$
$$= (h(p,i\xi_q)h(q,i);i\xi_q\xi_p,\lambda)$$

hence

$$h(pq,i) = h(p,i\xi_q)h(q,i) \quad \text{and} \quad i\xi_{pq} = i\xi_q\xi_p\ .$$

Similarly

$$k(pq,\lambda) = k(p,\lambda)k(q,\ \lambda\eta_p) \quad \text{and} \quad \lambda\eta_{pq} = \lambda\eta_p\eta_q\ .$$

For $p,q \in Q$ ; $pq \notin Q$ we have

$$pq = (d;i,\lambda) = (d;i,\lambda)(p^{-1}_{\lambda k};k,\lambda) = p(q(p^{-1}_{\lambda k};k,\lambda) = p(h(q,k)p^{-1}_{\lambda k};k\xi_q,\lambda)$$

$$=(h(p,k\xi_q)h(q,k)p^{-1}_{\lambda k};k\xi_q\xi_p,\lambda)$$

from this

$$d = h(p,k\xi_q)h(q,k)p^{-1}_{\lambda k} \quad , \quad i = k\xi_q\xi_p .$$

Thus $\xi_q\xi_p$ = const.

Similarly

$$pq = (d;i,\lambda) = (p^{-1}_{\ell i};i,\ell)(d;i,\lambda) = ((p^{-1}_{\ell i};i,\ell)p)q = (p^{-1}_{\ell i}k(p,\ell);i,\ell\eta_p)q$$

$$= (p^{-1}_{\ell i}k(p,\ell)k(q,\ell\eta_p);i,\ell\eta_p\eta_q)$$

from this

$$d = p^{-1}_{\ell i}k(p,\ell)k(q,\ell\eta_p) \quad , \quad \lambda = \ell\eta_p\eta_q .$$

Thus $\eta_p\eta_q$ = const.

Since

$$d = p^{-1}_{\ell i}k(p,\ell)k(q,\ell\eta_p) = p^{-1}_{\ell i}k(pq,\lambda) = h(p,k\xi_q)h(q,k)p^{-1}_{\ell\eta_p\eta_q,k}$$

we have that d does not depend on k and $\ell$, therefore pq is given by (5) . By this we established that $\Sigma \cong \mathcal{M}(D;I,\ \ ;P;Q;h,k,\xi,\eta)$ .

The converse of the theorem is obvious.

The proof of the next theorem we omited.

<u>THEOREM 1.2.</u> <u>Two semigroups</u> $\mathcal{M}(D;I,\Lambda;P;Q;h,k,\xi,\eta)$ <u>and</u> $\mathcal{M}^*(D^*;I^*,\Lambda^*;P^*;Q^*;h^*,k^*,\xi^*,\eta^*)$ <u>are isomorphic if and only if there exist an isomorphism</u> $\omega:D \to D^*$ , <u>a mapping</u> $i \to u_i$ <u>from</u> I <u>into</u> $U(D^*) = G^*$ , <u>a mapping</u> $\lambda \to v_\lambda$ , <u>from</u> $\Lambda$ <u>into</u> $U(D^*) = G^*$, <u>a bijektive mapping</u> $\varphi: I \to I^*$ , <u>a bijective mapping</u> $\psi: \Lambda \to \Lambda^*$ , <u>a partial isomorphism</u> $\Omega:Q \to Q^*$ <u>and the following conditions are satisfied</u>:

(1) $$p_{\lambda i}\omega = v_\lambda p^*_{\lambda\psi,i\varphi} u_i$$

(2) $$\xi_p\varphi = \varphi\xi^*_{p\Omega}$$

(3) $$\eta_p\psi = \psi\eta^*_{p\Omega}$$

(4) $$h(p,i)\omega = u^{-1}_{i\xi_p}h^*(p\Omega,i\varphi)u_i$$

(5) $$k(p,\lambda)\omega = v_{\lambda}\, k^{*}(p\Omega, \lambda\psi)v^{-1}{}_{\lambda}\, \eta_{p} .$$

## 2. SEMIGROUPS IN WHICH SOME QUASI-IDEAL IS A GROUP

The nonempty subset A of a semigroup S is a *quasi-ideal* of S if $AS \cap SA \subseteq A$ , [9].

*THEOREM 2.1. Some quasi-ideal of a semigroup* S *is a group if and only if* $S \cong \mathcal{M}(G;I,\Lambda;P;Q;h,k,\xi,\eta)$ , *where* G *is a group*.

*PROOF*. If some quasi-ideal $G_i$ of a semigroup S is a group, then $G_i$ is the minimal quasi-ideal, (Theorem 5.3. [9]). Let K be the union of all minimal quasi-ideals of S . Then K is completely simple kernel of S , (Theorem 5.14. [9]). Since $K \cong \mathcal{M}(G;I,\Lambda;P)$ , [1] , it follows from this that $S \cong \mathcal{M}(G;I,\Lambda;P;Q;h,k,\xi,\eta)$.

The converse is trivial.

*COROLLARY 2.1. A semigroup* S *has an ideal which is a rectangular band if and only if* $S \cong \mathcal{M}(G;I,\Lambda;P;Q;h,k,\xi,\eta)$ *where* $|G| = 1$ .

*COROLLARY 2.2. A semigroup* S *is completely simple if and only if it is the union of its minimal quasi-ideals*.

## 3. (m,n)-IDEAL SEMIGROUPS

A subsemigroup A of a semigroup S is an *(m,n)-ideal* of S if $A^m S A^n \subseteq A$ , where $m,n \in N \cup \{0\}$, $(A^{\circ}S = SA^{\circ} = S)$, [5]. S is an *(m,n)-ideal semigroup* if all of its subsemigroups are (m,n)-ideals. (1,1)-ideal semigroup is called *bi-ideal semigroup*. There exists (2,1)-ideal semigroup which is not bi-ideal. For example, semigroup given by the table:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 3 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 3 | 4 | 3 | 4 | 6 | 7 | 8 | 9 | 4 |
| 6 | 3 | 4 | 3 | 4 | 7 | 8 | 9 | 4 | 4 |
| 7 | 3 | 4 | 3 | 4 | 8 | 9 | 4 | 4 | 4 |
| 8 | 3 | 4 | 3 | 4 | 9 | 4 | 4 | 4 | 4 |
| 9 | 3 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |

*LEMMA 3.1. If* S *is an (m,n)-ideal semigroup then homomorphic image of* S *is the (m,n)-ideal semigroup and any of subsemigroup from* S *is the (m,n)-ideal semigroup*.

*LEMMA 3.2. If* S *is an (m,n)-ideal semigroup, then* $(\forall a \in S)(a^m S a^n \subseteq \langle a \rangle)$.

*PROOF.* Let S be an (m,n)-ideal semigroup and a an element of S , then $a^m S a^n \subseteq \langle a\rangle^m S \langle a\rangle^n \subseteq \langle a\rangle$.

The subset R of a partial semigroup Q is a <u>partial subsemigroup</u> of Q if $x,y \in R$; $xy \in Q$ implies $xy \in R$ . The partial subsemigroup R of the partial semigroup Q is an (m,n)-<u>ideal</u> of Q if $R^mQR^n \subseteq Q$ implies $R^mQR^n \subseteq R$ . If all partial subsemigroups of a partial semigroup Q are (m,n)-ideals of Q , then we call Q a <u>partial (m,n)-ideal semigroup</u>.

A class of (m,n)-ideal semigroup is a subclass of a class of semigroups which are described by Corollary 2.1., i.e. the following theorem holds:

<u>THEOREM 3.1.</u> <u>If S is an (m,n)-ideal semigroup, then</u> $S = E \cup Q$ , <u>where E is a rectangular band and ideal of S ; Q is a partial (m,n)-ideal semigroup</u>.

<u>PROOF.</u> Let S be an (m,n)-ideal semigroup and $a \in S$ .Let $\langle a \rangle$ be an infinite semigroup and $B = \{ a^{2k} : k \in N \}$. It is clear that B is a subsemigroup of $\langle a \rangle$ .By Lemma 3.1. we have that B is an (m,n)-ideal of $\langle a \rangle$ . So $a^{2m}a a^{2n} \in B^m \langle a \rangle B^n \subseteq B$ ,which is impossible.Hence, $\langle a \rangle$ is finit for every $a \in S$ and $E \neq \emptyset$ . Let $e \in E$ and $x \in S$ , then by Lemma 3.2. we have $eSe \subseteq \{e\}$,i.e.

(1) $$exe = e .$$

It follows by (1) and by Proposition 3.2. [3] that E is a rectangular band and obviously it is an ideal of S . Let Q = S E be a partial semigroup and A be a partial subsemigroup of Q , $A^mQA^n \subseteq Q$ . Then $B = \langle A \rangle$ is an (m,n)-ideal of S and $A^mQA^n \subseteq B^mSB^n \subset B$ and $A^mQA^n \subseteq B \setminus E = A$ . Hence, Q is a partial (m,n)-ideal semigroup.

<u>REMARK.</u> If S is an (m,n)-ideal semigroup it is periodic.

<u>THEOREM 3.2.</u> <u>Let Q be a periodic partial (m,n)-ideal semigroup</u>, E <u>a rectangular band</u>, $Q \cap E = \emptyset$ <u>and</u> $f: Q \to E$ <u>a homomorphism (partial).Let us put</u> $f(e) = e$ <u>for every</u> $e \in E$ <u>and</u> $f: S = Q \cup E \longrightarrow E$ <u>so that</u> $f|_Q$ <u>is a homomorphism.We define an operation on S by</u>

$$xy = \begin{cases} xy \text{ as in } Q \text{ , if } x,y \in Q \text{ and } xy \text{ is definied in } Q \\ f(x)f(y) \text{ otherwise} . \end{cases}$$

<u>Then S is an (m,n)-ideal semigroup</u>.

<u>PROOF.</u> Let the conditions of the theorem be satisfied, B a subsemigroup of S , $B^* = B \setminus E$ , $b = x_1x_2\dots x_m, (x_1,x_2,\dots,x_m \in B)$, $c = y_1y_2\dots y_n$ $(y_1,y_2,\dots,y_n \in B)$. Suppose that $b \in Q$ , $s \in Q$ , $c \in Q$ . Then $bsc \in Q$ and thus $bsc \in B^{*m}QB^{*n} \subseteq B^* \subseteq B$ . It is clear that $bsc \notin E$ . If $b \in B \cap E$ , $s \in Q$ , $c \in Q$ , then $bsc = f(bs)f(c) = f(b)f(s)f(c) = f(b)f(c) = bc \in B$ . If $b \in Q$ , $s \in E$ , $c \in Q$ , then $bsc = f(bs)f(c) = f(b)f(s)f(c) = f(b)f(c) = f(bc)$ . Now,if $bc \notin E$ , then $bsc = f(bc) = bc \in B$ . If $bc \in E$ , then there exists $k \in N$ so that $(bc)^k = e \in B \cap E$ and $f(bc) = (f(bc))^k = f(bc)^k = f(e) = e \in B$ . Hence, $bsc = f(bc) = e \in B$ . The other cases can be considered in a similar way.

We will mention some more characteristics of the (m,n)-ideal semigroups.

LEMMA 3.3. *If S is an (m,n)-ideal semigroup, then for every $a \in S$ is* $|\langle a\rangle| \leq 2m+2n+1$ .

PROOF. $\langle a\rangle$ is a finit semigroup for every $a \in S$ . Let e be the idempotent of $\langle a\rangle$ and p be the least natural number so that $a^p = e$ . If $x \in K_a$ , then $x = exe = e$ , (Theorem 3.1.).Hence, $K_a = \{e\}$. Suppose that $p > 2m+2n+1$ and $B = \{a^2, a^4, a^6, \ldots, a^p = e\} \subseteq \langle a\rangle$. The set B is a subsemigroup of $\langle a\rangle$ , and $a^{2m}aa^{2n} \in B^m\langle a\rangle B^n \subseteq B$ which is a contradiction.

LEMMA 3.4. *Let e be an idempotent of an (m,n)-ideal semigroup S . Then e is a zero in* $S(e) = \{x \in S : (\exists p \in N)\ x^p = e\}$.

PROOF. Directly follows.

THEOREM 3.3. *If S is an (m,n)-ideal semigroup, then* $S = \bigcup_{e \in E} S(e)$, *where S(e) are disjoint maximal unipotent subsemigroups of S* .

PROOF. Let $x,y \in S(e)$ . Then there exist $p,q \in N$ such that $x^p = y^q = e$ . By Lemma 3.2. it follows $(xy)^m e(xy)^n \subseteq \langle xy\rangle$. From this we have $e \in \langle xy\rangle$ , i.e. there exist $r \in N$ such that $(xy)^r = e$ . It is easily to verify that S(e) are maximal disjoint semigroups.

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# SOME CHARACTERIZATIONS OF BANDS OF POWER JOINED SEMIGROUPS

Stojan Bogdanović

A semigroup S is called power joined if for each pair of elements $a,b\in S$ there exist $m,n\in N$ with $a^m=b^n$. We say that a semigroup S is a band of power joined semigroups if there exists a congruence $\rho$ such that $S/\rho$ is a band and each class mod$\rho$ is a power joined semigroup. Bands of power joined semigroups are studied by T.NORDAHL, [5] in medial case and by author, [1] in general case. In the present paper we give some new characterizations of bands of power joined semigroups.

For nondefinied notions we refer to [2,3,6,7,8].

Let S be a semigroup. We define a relation $\xrightarrow{p}$ on S as follows:

$$a \xrightarrow{p} b \quad \text{iff} \quad (\exists m,n\in N)(a^m=b^n).$$

THEOREM 1. *Let S be a semigroup. Then the following conditions are equivalent:*

(A) S *is a band of power joined semigroups*.

(B) $(\forall a,b\in S)(ab \xrightarrow{p} a^2b \xrightarrow{p} ab^2)$.

(C) $(\forall a,b\in S)(\forall m,n\in N)(ab \xrightarrow{p} a^mb^n)$.

PROOF. (A)$\Rightarrow$(B). Let S be a band Y of power joined semigroups $S_\alpha$, $\alpha\in Y$. For $a\in S_\alpha$, $b\in S_\beta$ we have $ab,a^2b,ab^2\in S_{\alpha\beta}$ and thus (B).

(B)$\Rightarrow$(C). Let S satisfy condition (B). Then we have

$$ab \xrightarrow{p} a^2b=(ab)b \xrightarrow{p} (ab)b^2=ab^3 \xrightarrow{p} \cdots \xrightarrow{p} ab^n \xrightarrow{p} a^2b^n =$$
$$= a(ab^n) \xrightarrow{p} a^2(ab^n)=a^3b^n \xrightarrow{p} \cdots \xrightarrow{p} a^mb^n .$$

Hence, the condition (C) holds.

(C)$\Rightarrow$(A). This follows by Theorem 1. [1].

The symbolism $u\ \mathcal{T}\ v$, where u and v are words over some alphabet means that for all substitutions of variables by elements of S, the resulting elements are $\tau$-equivalent. We denote by $\mathcal{B}$ the class of all bands in which the identity $u = v$ holds. If $S \in \mathcal{B}$, then S is a $\mathcal{B}$-band. The following condition

(D) $$(\forall x_1, x_2, \ldots, x_k \in S)(\, u \,\tau\, v\,)$$

where $\{x_1, x_2, \ldots, x_k\}$ is the set of all variables in u and v is necessary in the following:

PROPOSITION 1. *S is a $\mathcal{B}$-band of semigroups in a class $\mathcal{S}$ if and only if S is a band of semigroups in $\mathcal{S}$ and if $\tau$ is the band congruence induced by the decomposition of S then the condition (D) holds.*

PROOF. Trivial.

THEOREM 2. *Let S be a semigroup. Then the following conditions are equivalent:*

(E) *S is a $\mathcal{B}$-band of power joined semigroups.*

(F) *(B) and (D), where $\tau = \xrightarrow{p}$.*

(G) *(C) and (D), where $\tau = \xrightarrow{p}$.*

PROOF. Follows immediately by Theorem 1. and Proposition 1.

THEOREM 3. *Let S be a semigroup. Then the following conditions are equivalent:*

(E1) *S is a semilattice of power joined semigroups.*

(F1) $(\forall a,b \in S)(\, ab \xrightarrow{p} a^2b \xrightarrow{p} ab^2 \xrightarrow{p} ba\,)$.

(G1) $(\forall a,b \in S)(\forall m,n \in N)(\, ba \xrightarrow{p} a^m b^n\,)$.

PROOF. (E1)$\Rightarrow$(F1). By Theorem 2. we have $ab \xrightarrow{p} a^2b \xrightarrow{p} ab^2$ and $ab \xrightarrow{p} ba$ and thus (F1). By hypothesis and by Theorem 1. we have that (F1) implies (G1). (G1)$\Rightarrow$(E1). Follows by Theorem 1. (see also Theorem 2. in [1]).

A subsemigroup B of a semigroup S is a *bi-ideal* of S if $BSB \subseteq B$, [8]. A semigroup S is a *band of bi-ideals* $B_i$, $i \in Y$ if $S = \bigcup_{i \in Y} B_i$, $B_i \cap B_j = \emptyset$, $i \neq j$ and $B_i B_j \subseteq B_k$.

PROPOSITION 2. *S is a rectangular band of semigroups in a class $\mathcal{S}$ if and only if S is a band of bi-ideals from $\mathcal{S}$.*

PROOF. Let S be a rectangular band Y of semigroups $S_\beta$, $\beta \in Y$

and $S_\beta \in \mathcal{S}$ . Then for each $\alpha \in Y$ we have

$$S_\alpha S S_\alpha = S_\alpha (\bigcup_{\beta\in Y} S_\beta) S_\alpha = \bigcup_{\beta\in Y} S_\alpha S_\beta S_\alpha \subseteq \bigcup_{\beta\in Y} S_{\alpha\beta\alpha} \subseteq S_\alpha .$$

Hence, S is a band of bi-ideals from $\mathcal{S}$ .

Conversely,let S be a band of bi-ideals $B_i \in \mathcal{S}$ $(i \in I)$. Let $\tau$ be the congruence relation on S induced by the decomposition of S . For $a \in B_i$ , $b \in B_j$ we have $aba \in B_i B_j B_i \subseteq B_{iji}$ and $aba \in B_i B_j B_i \subseteq B_i$ . So $B_i = B_{iji}$ and therefore $\tau$ is a rectangular band congruence.

EXAMPLE 1. A semigroup S is a bi-ideal semigroup if all its subsemigroups are bi-ideals.If S is a bi-ideal semigroup,then $S = \bigcup_{e\in E} S^{(e)}$ , where $S^{(e)}$ , $e \in E(S)$ are unipotent bi-ideal semigroups,

$$S^{(e)}S^{(f)} \subseteq S^{(ef)}$$

and $S^{(e)} \cap S^{(f)} = \emptyset, e \neq f$, [9] . Hence,any bi-ideal semigroup is a rectangular band of unipotent bi-ideal semigroups.

EXAMPLE 2. Let S be a semigroup.Then S is a completely simple if and only if S is a rectangular band of groups, [2]. Hence completely simple semigroup is band of its minimal bi-ideals,(the converse is also true).

THEOREM 4. Let S be a semigroup.Then the following conditions are equivalent:

(1) S is a rectangular band of power joined semigroups.

(2) $(\forall a,b,c \in S)( abc \xrightarrow{P} ac )$ .

(3) S is a band of power joined bi-ideals.

PROOF. (1)$\Leftrightarrow$(2). This is the Theorem 3. in [1]. (1)$\Leftrightarrow$(3). This follows by Proposition 2.

A semigroup S is a band of left (right) ideals $L_i$ , $i \in Y$ if $S = \bigcup_{i\in Y} L_i$ , $L_i \cap L_j = \emptyset$ , $i \neq j$ .

PROPOSITION 3. S is a left (right) zero band of semigroups in $\mathcal{S}$ if and only if S is a band of right (left) ideals from $\mathcal{S}$.

PROOF. Similarly to the proof of Proposition 2.

THEOREM 5. Let S be a semigroup.Then the following conditions are equivalent:

(1) S is a left zero band of power joined semigroups.

(2) $(\forall a,b \in S)( ab \xrightarrow{P} a )$ .

(3) S is a band of power joined right ideals.

PROOF. (1)$\Leftrightarrow$(2). This is the Corollary from [1]. (1)$\Leftrightarrow$(3). Follows by Proposition 3.

It is clear that dualy theorem to the Theorem 5. holds.

A band S is a left (right) regular if in S the identity $ax = axa$ ($xa = axa$) holds, [6].

THEOREM 6. Let S be a semigroup. Then the following conditions are equivalent:

(1) S is a left regular band of power joined semigroups.

(2) $(\forall a,b \in S)(\ ab \overset{p}{\overline{\quad}} a^2b \overset{p}{\overline{\quad}} ab^2 \overset{p}{\overline{\quad}} aba\ )$ .

(3) $(\forall a,b \in S)(\forall m,n \in N)(\ ab \overset{p}{\overline{\quad}} a^mb^n \overset{p}{\overline{\quad}} aba\ )$ .

(4) $(\forall a,b \in S)(\forall m,n \in N)(\ ab \overset{p}{\overline{\quad}} a^mb^na^m\ )$ .

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3). Follows by Theorem 2. (3)$\Rightarrow$(4). It follows from $ab \overset{p}{\overline{\quad}} aba$ that $a^mb^n \overset{p}{\overline{\quad}} a^mb^na^m$ and since $ab \overset{p}{\overline{\quad}} a^mb^n$ we have (4) . (4)$\Rightarrow$(1). Assume that $a \overset{p}{\overline{\quad}} b$ . Then by (4) we have $ab \overset{p}{\overline{\quad}} a$ . Hence, each class mod $\overset{p}{\overline{\quad}}$ is a power joined semigroup. Suppose $a \overset{p}{\overline{\quad}} b$ and $c \in S$ . Then $ac \overset{p}{\overline{\quad}} a^mc^ka^m$ and $bc \overset{p}{\overline{\quad}} b^nc^kb^n$ . It follows from this that

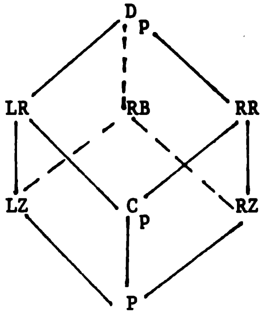
$$ac \overset{p}{\overline{\quad}} a^mc^ka^m \overset{p}{\overline{\quad}} b^nc^kb^n \overset{p}{\overline{\quad}} bc\ .$$

Similarly we obtain $ca \overset{p}{\overline{\quad}} cb$ . Consequently $\overset{p}{\overline{\quad}}$ is a congruence and since $a \overset{p}{\overline{\quad}} a^2$ , for every $a \in S$ we have that S is a band of power joined semigroups. From (4) we have that $S/\overset{p}{\overline{\quad}}$ is a left regular band.

SOME NOTES. Let $C$ , $C_r$ , $C_\ell$ , $C_t$ , $C_p$ denote the class of semigroups which are semilattices of archimedean, right archimedean, left archimedean, t-archimedean and power joined semigroups respectively. Also let $D$ , $D_r$ , $D_\ell$ , $D_t$ , $D_p$ denote the class of semigroups which are bands of arcimedean, right archimedean, left archimedean, t-archimedean and power joined semigroups respectively: Using the implication scheme of M.S.PUTCHA, [7] and Theorems 1. and 3. we have the following implication scheme:

$$\begin{array}{ccccccccccccc}
D_p & \rightarrow & D_t & \rightarrow & D_\ell & \rightarrow & D & \leftarrow & D_r & \leftarrow & D_t & \leftarrow & D_p \\
\uparrow & & \uparrow & & \uparrow & & \updownarrow & & \uparrow & & \uparrow & & \uparrow \\
C_p & \rightarrow & C_t & \rightarrow & C_\ell & \rightarrow & C & \leftarrow & C_r & \leftarrow & C_t & \leftarrow & C_p
\end{array}$$

On the other hand let P , LZ , RZ , LR , RR , RB denote the class of semigroups which are power joined, left zero bands of power joined semigroups, right zero bands of power joined semigroups, left regular bands of of power joined semigroups, right regular bands of power joined semigroups and rectangular bands of power joined semigroups respectively. We then have the following strict implication scheme:

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# A NEW TYPE OF BINARY GRIDS AND RELATED COUNTING PROBLEM

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A binary (m, n)- D -grid is defined as an array of $m$ rows and $n$ columns formed from $mn$ square cells each of which is divided into two congruent rectangulars. Thus, Fig. 1 is a (3, 4)- $D$ -grid.

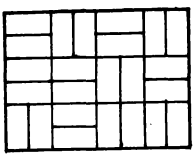

Fig. 1

In fact, each $D$ - grid gives a tilling of some rectangular with dominos in a special way; the rectangular is divided into $mn$ unit squares and each unit square is covered with two dominos.

Two $(m, n)$ -$D$ -grids are said to be equivalent iff they can be transformed one into the other by rigid motion in the space.

In this paper we determine the number $N_d(m, n)$ of non- equivalent $(m, n)$- $D$ - grids, for arbitrary natural numbers $m$ and $n$ .

By $\mathcal{X}_{(m,n)}$ we denote the set of all $(m,n)$- $D$- grids. Once $m$ and $n$ are specified, we write simply $\mathcal{X}$ instead of $\mathcal{X}_{(m,n)}$. If $\mathcal{A}$ is a set, then the cardinality of $\mathcal{A}$ is $|\mathcal{A}|$ . By $\lceil x \rceil$ we denote the smallest integer $\geq x$ . It is clear that

$$|\mathcal{X}_{(m,n)}| = 2^{mn} \tag{1}$$

Any rigid motion in the space by which two $(m,n)$ - $D$-grids $(m \neq n)$ can be transformed one into the other, reduces to one of the transformations from the group of simmetries of the rectangular (Fig. 2):

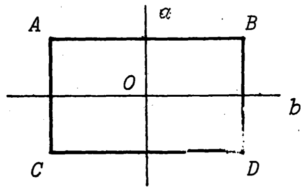


Fig. 2

$i$ - identical transformation,

$a$ - symmetry with respect to the vertical axis,

$b$ - symmetry with respect to the horizontal axis,

$c$ - symmetry with respect to the center $0$ of rectangular (central symmetry).

Let $t(x)$ denote the grid into which the grid $x$ is transformed by applying the transformation $\mathcal{X}$. We shall consider the following subsets of $\mathcal{X}$ :

$$\mathcal{A} = \{ X / a(X) = X\} \ ,$$
$$\mathcal{B} = \{ X / b(X) = X\} \ ,$$
$$\mathcal{C} = \{ X / c(X) = X\} \ .$$

If $m = n$, we have square $D$ - grids. Now, in addition to $i$, $a$, $b$ and $c$, the group of rigid motions which transform a square into itself contains four additional transformations (Fig. 3):

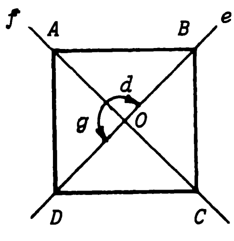


Fig. 3

$d$ - rotation about the center $0$ of the square through angle $\varphi = 90^{\circ}$,

$e$ - symmetry with respect to the diagonal $BD$ ,

$f$ - symmetry with respect to the diagonal $AC$ ,

$g$ - rotation about the center $O$ through angle $\varphi=-90^{\circ}$.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be subsets of $\mathcal{X} = \mathcal{X}_{(n,n)}$ , defined as for rectangular grids. We consider also the following subsets of $\mathcal{X}$:

$$\mathcal{D} = \{X \,/\, d(X) = X\} \,,$$
$$\mathcal{E} = \{X \,/\, e(X) = X\} \,,$$
$$\mathcal{F} = \{X \,/\, f(X) = X\} \,,$$
$$\mathcal{G} = \{X \,/\, g(X) = X\} \,.$$

In [2] we have generalised some results of Hoffman given in [1] , for an other type of binary grids. In the same way as in [2] , it can be proved that for $D$ - grids the following statement is true:

Lemma 1.

(i) If $m \neq n$ , then

$$N_d(m,n) = \frac{1}{4}\,(\,|\mathcal{X}| + |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}|\,)\,; \qquad (2)$$

$$\text{(ii)}\ N_d(n,n) = \frac{1}{4}\,|\mathcal{D}| + \frac{1}{8}(\,|\mathcal{X}| + |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + |\mathcal{E}| + |\mathcal{F}|\,). \qquad (3)$$

Now, we are going to determine the numbers $|\mathcal{A}|$, $|\mathcal{B}|$, $|\mathcal{C}|$, $|\mathcal{D}|$, $|\mathcal{E}|$, and $|\mathcal{F}|$.

Lemma 2. For arbitrary natural $m$, $n$, and for corresponding subsets $\mathcal{A}$ , $\mathcal{B}$ , $\mathcal{C}$ of $\mathcal{X} = \mathcal{X}_{(m,n)}$ :

$$\text{(i)}\quad |\mathcal{A}| = 2^{m\left\lceil \frac{n}{2} \right\rceil}\,, \qquad (4)$$

$$\text{(ii)}\quad |\mathcal{B}| = 2^{\left\lceil \frac{m}{2} \right\rceil \cdot n}\,, \qquad (5)$$

$$\text{(iii)}\quad |\mathcal{C}| = 2^{\left\lceil \frac{mn}{2} \right\rceil}\,. \qquad (6)$$

Proof. (i) For $n = 2k$ ($k \in N$), an arbitrary $D$-grid $X$ from $\mathcal{X}_{(m,n)}$ can be represented in the form

| | |
|---|---|
| $P$ | $Q$ |

where $P$ and $Q$ are $(m,k)$ - $D$-grids. Now, $X \in \mathcal{A}$ iff $Q = a(P)$. It means that $X \in \mathcal{A}$ is determined by its $(m,k)$ -$D$ - subgrid $P$ containing $mk = m\,\frac{n}{2} = m\left\lceil\frac{n}{2}\right\rceil$ cells, hence follows the statement.

For $n = 2k + 1$ $(k \geq 0)$, an arbitrary $D$ - grid $X$ from $\mathcal{X}_{(m,n)}$, can be represented in the form

| | | |
|---|---|---|
| $P$ | $I$ | $Q$ |

where $P$ and $Q$ are $(m,k)$ - $D$-grids and $I$ is a $(m,1)$- $D$ - grid. Now, $X \in \mathcal{A}$ iff $Q = a(P)$. Since $a(I) = I$ for arbitrary $(m,1)$- $D$- grid $I$, it follows that $X \in A$ is determined by its $D$ - subgrids $P$ and $I$ containing $m(k+1) = m\,\frac{n+1}{2} = m\left\lceil\frac{n}{2}\right\rceil$ cells, and (4) is proved.

(ii) (5) can be proved in the same way.

(iii) $X \in \mathcal{C}$ iff any two cells situated symmetricaly with respect to the center $O$ of $X$ are of the same sort (either ⊟ or ◫ ). If either $m$ or $n$, is even, $X \in \mathcal{C}$ is determined by $\frac{mn}{2} = \left\lceil\frac{mn}{2}\right\rceil$ pairs of cells. If both $m$ and $n$ are odd, the central cell is symmetrical to itself and it can be of arbitrary sort. In that case, $X \in \mathcal{C}$ is determined by $\frac{mn-1}{2} + 1 = \left\lceil\frac{mn}{2}\right\rceil$ pairs of cells (central cell included), and the statement follows.

<u>Remark.</u> If $m = n$, (4), (5) and (6) become:

$$|\mathcal{A}| = |\mathcal{B}| = 2^{n\cdot\left\lceil\frac{n}{2}\right\rceil}, \tag{7}$$

$$|\mathcal{C}| = 2^{\left\lceil\frac{n^2}{2}\right\rceil}. \tag{8}$$

<u>Lemma 3. For arbitrary natural</u> $n$, <u>and for corresponding subsets</u> $\mathcal{D}, \mathcal{E}, \mathcal{F}$ <u>of</u> $\mathcal{X} = \mathcal{X}_{(n,n)}$ :

$$\text{(i)} \quad |\mathcal{E}| = |\mathcal{F}| = 0, \tag{9}$$

$$\text{(ii)} \quad |\mathcal{D}| = (1+(-1)^n)\cdot 2^{\frac{n^2}{4}-1}. \tag{10}$$

<u>Proof.</u> (i) $X \in \mathcal{E}$ iff each cell situated on the diagonal $BD$ (Fig 3) is transformed into itself by applying

the transformation $e$ , but it is impossible because $e($⊟$)=$◫ and $e($◫$)=$⊟ . Hence, $\mathcal{E}=\emptyset$. Similarly, $\mathcal{F}=\emptyset$.

(ii) If $n$ is even $(n=2k,\ k\geqslant 1)$ , then $X\in\mathcal{X}_{(n,n)}$ can be represented in the form

| | |
|---|---|
| $P$ | $Q$ |
| $S$ | $R$ |

where $P, Q, R$ and $S$ are $(k,k)$ - $D$-grids. Now, $X\in\mathcal{D}$ iff $Q=d(P)$, $R=d(Q)=d^2(P)$ and $S=d(R)=d^3(P)$ . It means that $X\in\mathcal{D}$ is determined by its $D$ - subgrid $P$ , containg $\frac{n^2}{4}$ cells.

If $n$ is odd $(n=2k+1,\ k\geq 0)$ , then $X\in\mathcal{D}$ iff the central cell (with the center $O$ - Fig. 3) is transformed into itself by applying the transformation $d$ ; but it is im - possible because $d($◫$)=$⊟ and $d($⊟$)=$◫ ). In that case, $\mathcal{D}=\emptyset$, hence follows the statement.

<u>Theorem.</u> (i) If $m\neq n$ , then

$$N_d(m,n)=2^{mn-2}+2^{m\lceil\frac{n}{2}\rceil-2}+2^{\lceil\frac{m}{2}\rceil\cdot n-2}+2^{\lceil\frac{mn}{2}\rceil-2};$$

$$\text{(ii) } N_d(n,n)=2^{n^2-3}+2^{n\cdot\lceil\frac{n}{2}\rceil-2}+2^{\lceil\frac{n^2}{2}\rceil-3}+(1+(-1)^n)\cdot 2^{\frac{n^2}{4}-3}$$

<u>Proof</u>. (i) Follows from (1), (2), (4), (5) and (6).

(ii) Follows from (1), (3), (7), (8), (9) and (10).

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# PERFECT PERMUTATIONS AND RELATED COUNTING PROBLEM

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Let $Q$ and $Q'$ denote the sets of all permutations of the elements of the sets $N_q = \{0, 1, 2, \dots, q-1\}$ and $N'_q = N_q \setminus \{0\}$, respectively. Two permutations from $Q$, $p = a_0 a_1 \dots a_{q-1}$ and $p' = a'_0 a'_1 \dots a'_{q-1}$, are said to be equivalente (we write $p \sim p'$) iff there exists $t \in N_q$ such that for arbitrary $i \in N_q$, $a'_i - a_i \equiv t \pmod q$. The equivalence relation $\sim$ defines a partition of the set $Q$ into the classes of cardinality $q$. For each class we take as its representative the permutation in which $a_0 = 0$. Let $Q_0$ denote the set of all such representatives.

We associate with each permutation $p = a_0 a_1 \dots a_{q-1} \in Q$ a word $w(p) = b_1 b_2 \dots b_{q-1}$ over the alphabet $N'_q$, in such a way that for $i = 1, 2, \dots, q-1$, $b_i \equiv a_i - a_{i-1} \pmod q$. It is clear that two permutations $p$ and $p'$ from $Q$ are equivalent iff $w(p) = w(p')$.

A permutation $p \in Q$ is called perfect iff $w(p) \in Q'$.

Let $N(q)$ denote the number of perfect permutations in the set $Q_0$.

<u>Lemma.</u> *If $p = a_0 a_1 \dots a_{q-1} \in Q$ is a perfect permutation, then*

$$a_{q-1} - a_0 \equiv \binom{q}{2} \pmod q.$$

<u>Proof</u>. $a_{q-1} - a_0 \equiv \sum_{i=1}^{q-1} (a_i - a_{i-1}) \pmod q \equiv$

$$\equiv \sum_{i=1}^{q-1} i \pmod q \equiv \binom{q}{2} \pmod q .$$

<u>Corollary 1.</u> *If $q$ is odd, then* $N(q) = 0$.

<u>Corollary 2.</u> *If $q$ is even, then*

$$a_{q-1} \equiv a_0 + \frac{q}{2} \pmod q .$$

In order to determine the numbers $N(q)$ for some even integers, we developed a computer algorithm. The results can be formulated in the form of the following statement:

<u>Theorem</u>. (a) $N(2) = 1$ ,
(b) $N(4) = 1$ ,
(c) $N(6) = 4$ ,
(d) $N(8) = 24$ ,
(e) $N(10) = 288$ ,
(f) $N(12) = 3856$ .

Now, we formulate two problems (the second being the generalisation of the first) and a conjecture.

<u>Problem 1</u>. Determine $N(14)$.

<u>Problem 2.</u> Find a formula for $N(2m)$ , $m \in N$.

<u>Conjecture</u>. For arbitrary natural $m$, $m \geq 3$ , $N(2m)$ is an even number.

Perfect permutations are of some importance in the theory of horizontally complete latin squares and in the coding theory (see [1] and [2]). Namely, a latin square of order $q$ such that its entry belonging to $i$- th row and to $j$ -th column is $a_{j-1} + i - 1 \pmod q$ , where $a_0 a_1 \ldots a_{q-1}$ is a perfect permutation from $Q$ , is horizontally complete.

Using computer, we found four latin squares of order 10,

T A B L E

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 8 | 2 | 4 | 9 | 7 | 3 | 6 | 5 |
| 1 | 2 | 9 | 3 | 5 | 0 | 8 | 4 | 7 | 6 |
| 2 | 3 | 0 | 4 | 6 | 1 | 9 | 5 | 8 | 7 |
| 3 | 4 | 1 | 5 | 7 | 2 | 0 | 6 | 9 | 8 |
| 4 | 5 | 2 | 6 | 8 | 3 | 1 | 7 | 0 | 9 |
| 5 | 6 | 3 | 7 | 9 | 4 | 2 | 8 | 1 | 0 |
| 6 | 7 | 4 | 8 | 0 | 5 | 3 | 9 | 2 | 1 |
| 7 | 8 | 5 | 9 | 1 | 6 | 4 | 0 | 3 | 2 |
| 8 | 9 | 6 | 0 | 2 | 7 | 5 | 1 | 4 | 3 |
| 9 | 0 | 7 | 1 | 3 | 8 | 6 | 2 | 5 | 4 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 9 | 2 | 8 | 6 | 1 | 3 | 7 | 4 | 5 |
| 1 | 0 | 3 | 9 | 7 | 2 | 4 | 8 | 5 | 6 |
| 2 | 1 | 4 | 0 | 8 | 3 | 5 | 9 | 6 | 7 |
| 3 | 2 | 5 | 1 | 9 | 4 | 6 | 0 | 7 | 8 |
| 4 | 3 | 6 | 2 | 0 | 5 | 7 | 1 | 8 | 9 |
| 5 | 4 | 7 | 3 | 1 | 6 | 8 | 2 | 9 | 0 |
| 6 | 5 | 8 | 4 | 2 | 7 | 9 | 3 | 0 | 1 |
| 7 | 6 | 9 | 5 | 3 | 8 | 0 | 4 | 1 | 2 |
| 8 | 7 | 0 | 6 | 4 | 9 | 1 | 5 | 2 | 3 |
| 9 | 8 | 1 | 7 | 5 | 0 | 2 | 6 | 3 | 4 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 7 | 6 | 4 | 8 | 3 | 9 | 1 | 2 | 5 |
| 1 | 8 | 7 | 5 | 9 | 4 | 0 | 2 | 3 | 6 |
| 2 | 9 | 8 | 6 | 0 | 5 | 1 | 3 | 4 | 7 |
| 3 | 0 | 9 | 7 | 1 | 6 | 2 | 4 | 5 | 8 |
| 4 | 1 | 0 | 8 | 2 | 7 | 3 | 5 | 6 | 9 |
| 5 | 2 | 1 | 9 | 3 | 8 | 4 | 6 | 7 | 0 |
| 6 | 3 | 2 | 0 | 4 | 9 | 5 | 7 | 8 | 1 |
| 7 | 4 | 3 | 1 | 5 | 0 | 6 | 8 | 9 | 2 |
| 8 | 5 | 4 | 2 | 6 | 1 | 7 | 9 | 0 | 3 |
| 9 | 6 | 5 | 3 | 7 | 2 | 8 | 0 | 1 | 4 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 4 | 6 | 2 | 7 | 1 | 9 | 8 | 5 |
| 1 | 4 | 5 | 7 | 3 | 8 | 2 | 0 | 9 | 6 |
| 2 | 5 | 6 | 8 | 4 | 9 | 3 | 1 | 0 | 7 |
| 3 | 6 | 7 | 9 | 5 | 0 | 4 | 2 | 1 | 8 |
| 4 | 7 | 8 | 0 | 6 | 1 | 5 | 3 | 2 | 9 |
| 5 | 8 | 9 | 1 | 7 | 2 | 6 | 4 | 3 | 0 |
| 6 | 9 | 0 | 2 | 8 | 3 | 7 | 5 | 4 | 1 |
| 7 | 0 | 1 | 3 | 9 | 4 | 8 | 6 | 5 | 2 |
| 8 | 1 | 2 | 4 | 0 | 5 | 9 | 7 | 6 | 3 |
| 9 | 2 | 3 | 5 | 1 | 6 | 0 | 8 | 7 | 4 |

given in TABLE, which have the following properties:

(i) each of them is horizontally complete;

(ii) all 320 ordered triples of three adjacent (in a row) elements are different.

Problem 3. Does there exist more then four latin squares of order 10 satisfying these properties?

The answer is affirmative if there exists more then four perfect permutations over the set $\{0,1,...,9\}$ such that all their subwords of length 3 are different.

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# ON AFFINE STEINER TERNARY ALGEBRAS

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Abstract. In this paper, we investigate affine Steiner ternary algebras (ASTA's) and give their representation by the unipotent abelian groups, for which a derived loop is used. For a finite ASTA , the elements and quadruples of the associated Steiner quadruple system (SQS) are respectively the points and planes of an affine space over GF(2). It implies an elementary proof of Cameron's result which states that affine geometry is characterized among SQS's by a symmetric difference property.

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# SOME COMBINATORIAL SEARCH PROBLEMS

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The problem of ascertaining the minimum number of weighings which suffice to determine the defective coin in a set of $n$ coins of the same appearance, given an equal arm balance and the information that there is precisely one defective coin present, is well known. A large number of solutions exist, some based upon sequential procedures and some not. References to many papers on this question can be found in [2] and [7].

It is interesting that the corresponding problem for more than one defective coin has attracted little attention. The problem is of singnificance because it represents one of the simplest examples of a sequential testing problem replete with the difficulties of combinatorial nature. For some discussion of these matters in greater detail, see [1], [3], [5] and [6]. In [3] a systematic way is indicated in which the theory of dynamic programming can be used to provide a computational solution to the determination of optimal and suboptimal testing policies. The problem for two coins was also investigated by Cairns in [4]. Apart from these papers, little appear to be written on the subject.

We denote with $\mu_2(n)$ the maximum number of steps in an optimal program for two defective coins problem.

We proved the following statement:

<u>Theorem</u>. $\lceil \log_3 \binom{n}{2} \rceil \leq \mu_2(n) \leq \lceil \log_3 \binom{n}{2} \rceil + 1.$

An infinite set of $n$'s is determined for which the lower bound is reached and the corresponding procedures are constructed inductively. Some results of Cairns are improved and his conjecture that $\mu_2(n)$ has one of the values $2k-1$, $2k$, $2k+1$, depending on $n$, for $3^{k-1} < n \leq 3^k$, is shown to be slightly incorrect. It follows that for $3^{k-1} < n \leq 3^k$, $\mu_2(n)$ has one of the values $2k-2$, $2k-1$, $2k$, depending on $n$.

Proof of Theorem and further details are given in [8].

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# SUBALGEBRAS OF ABELIAN TORSION GROUPS

G.Čupona,G.Vojvodić and S.Crvenković

An $\Omega$-algebra $\underline{A}=(A;\Omega)$ is said to be an $\Omega$-subalgebra of a semigroup S if $A\subset S$ and there is a mapping $\omega\longmapsto\bar{\omega}$ of $\Omega$ into S such that $\omega(a_1,\dots,a_n)=\bar{\omega}a_1\dots a_n$ for each n-ary operator $\omega\in\Omega$ and any $a_1,\dots,a_n\in A$.If $\underline{C}$ is a class of semigroups,then by $\underline{C}(\Omega)$ is denoted the class of $\Omega$-algebras which are $\Omega$-subalgebras of semigroups belonging to $\underline{C}$.Here we give corresponding descriptions of the classes ABTG($\Omega$) and $A_m(\Omega)$,where ABTG is the class of abelian torsion groups and $A_m$ the class of abelian groups in which each element has an order which is a divisor of m($m\geq 2$ is a given integer).

1.First,we will give a description of ABTG($\Omega$).

Theorem 1.*Let* $\Omega\neq\Omega(1)$ ($\Omega(n)$ *is the set of* n-*ary operators belonging to* $\Omega$).*An* $\Omega$-*algebra* $\underline{A}=(A;\Omega)$ *belongs to* ABTG($\Omega$) *iff it satisfies the following conditions:*

(*) *For every* $m,n\geq 1,\omega'\in\Omega(m),\omega''\in\Omega(n),i\in N_m=\{1,2,\dots m\}$ *and permutation* $\nu\mapsto i_\nu$ *of* $N_m$ *the following identity equations are satisfied:*

$$\omega'(x_1,\dots,x_m)=\omega'(x_{i_1},\dots,x_{i_m}),$$

$$\omega'\omega''(x_1,\dots,x_{m+n-1})=\omega''\omega'(x_1,\dots,x_{m+n-1})=$$
$$=\omega'(x_1,\dots,x_{i-1},\omega''(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{m+n-1});$$

(**) *There is a mapping* $m:z\mapsto m(z)$ *of* $A\cup\Omega$ *into the set of positive integers such that:*

$$\omega_1^{i_1}\dots\omega_p^{i_p}(a_1^{\alpha_1},\dots,a_q^{\alpha_q})=\omega_1^{j_1}\dots\omega_p^{j_p}(a_1^{\beta_1},\dots,a_q^{\beta_q}),$$

*for any* $\omega_\nu\in\Omega(n_\nu),a_\lambda\in A$ *and nonnegative integers* $i_\nu,j_\nu,\alpha_\lambda,\beta_\lambda$ *such that:*

$$i_\nu\equiv j_\nu(\mathrm{mod}\,m(\omega_\nu)),\quad \alpha_\lambda\equiv\beta_\lambda(\mathrm{mod}\,m(a_\lambda))$$

*and:*

$$1+i_1n_1+\dots+i_pn_p=\alpha_1+\dots+\alpha_q$$
$$1+j_1n_1+\dots+j_pn_p=\beta_1+\dots+\beta_q.$$

Proof. In the first place, it is clear that if $\underline{A}$ is a subalgebra of an abelian semigroup S then (*) is satisfied. (In [4] it is shown that the converse is also satisfied). If $S\in$ ABTG and if for each $a\in A(\omega\in\Omega)$, $m(a)(m(\omega))$ is the order of $a(\bar\omega)$ in S, we obtain that the condition (**) is satisfied.

Assume now that $\underline{A}=(A;\Omega)$ is an $\Omega$-algebra which satisfies the conditions (*) and (**). If $z\in A\cup\Omega$ then $C_z$ denotes the cyclic group with a generator z and order m(z). Further on, let $\mathcal{H}$ be the free product $\mathcal{H}=\bigsqcup_{z\in A\cup\Omega} C_z$ in the class of abelian groups. (We use a multiplicative notation.)

If $u=au'\in\mathcal{H}$ and $a=\omega(a_1,\dots,a_n)$ in $\underline{A}$, then we write $u\vdash\omega a_1\dots a_nu'$, and also $\omega a_1\dots a_nu'\dashv u$. Let $u\vdash\!\dashv v\Leftrightarrow u\vdash v$ or $u\dashv v$. Further on, denote by $\approx$ the reflexsive and transitive extension of $\vdash\!\dashv$, i.e.:

$$u\approx v\Leftrightarrow(\exists u_0,u_1,\dots,u_p\in\mathcal{H})\,u=u_0,v=u_p,p>0 \text{ and } u_{i-1}\vdash\!\dashv u_i$$

for each $i\in\{1,\dots,p\}$.

Then, clearly, $\approx$ is a congruence on $\mathcal{H}$, and $\omega(a_1,\ldots,a_n)=$ $=a$ in $\underline{A} \Rightarrow \omega a_1\ldots a_n \approx a$.

We will show that:

($\Delta$) $\quad a,b \in A \Rightarrow (a \approx b \Rightarrow a=b)$,

and this will complete the proof of Theorem 1..

First we introduce the notion of $\Omega$-word. Namely, an element $w \in \mathcal{H}$ is said to be an $\Omega$-word iff

$$w=\omega_1^{i_1}\omega_2^{i_2}\ldots\omega_p^{i_p}a_1^{\alpha_1}\ldots a_q^{\alpha_q},$$

and $1+i_1n_1+\ldots+i_pn_p=\alpha_1+\ldots+\alpha_q$, where $\omega_\nu \in \Omega(n_\nu+1), i_\nu, \alpha_\lambda > 0$.

Then,

$$\omega_1^{i_1}\ldots\omega_p^{i_p}(a_1^{\alpha_1},\ldots,a_q^{\alpha_q})=a \in A,$$

and we say that $a=[w]$ is the "value" of $w$.

We note that by (**) the value of an $\Omega$-word $w$ is uniquelly determined.

Clearly, ($\Delta$) is a consequence of the following proposition

($\Delta\Delta$) Let $u,v \in \mathcal{H}$ be such that $u \vdash\!\dashv v$. If $u$ is an $\Omega$-word then $v$ is also an $\Omega$-word and $[u]=[v]$.

<u>Proof.</u> Let $u=\omega_1^{i_1}\ldots\omega_p^{i_p}a_1^{\alpha_1}\ldots a_q^{\alpha_q}, \omega_\nu \in \Omega(n_\nu+1)$ and $1+i_1n_1+\ldots+$ $+i_pn_p=\alpha_1+\ldots+\alpha_q$.

Assume first that $u \vdash v$. Then $u=\omega_1^{i_1}\ldots\omega_p^{i_p}a_1^{\beta_1}a_2^{\alpha_2}\ldots a_q^{\alpha_q}$, $\beta_1 \equiv \alpha_1 (\mathrm{mod}\, m(a_1)), \beta_1 \geqslant 1 \quad a_1=\omega_1(a_1^{\gamma_1},\ldots,a_q^{\gamma_q}), \gamma_\lambda \geqslant 0$,

$$v=\omega_1^{i_1+1}\omega_2^{i_2}\ldots\omega_p^{i_p}a_1^{\beta_1+\gamma_1-1}a_2^{\alpha_2+\gamma_2}\ldots a_q^{\alpha_q+\gamma_q}.$$

Let $\omega \in \Omega(n+1), n \geqslant 1$. Then $v=\omega^{sm(\omega)}a_1^{tm(a_1)}v$, for each $s$, $t \geqslant 0$ and it can be easily seen that there exist $s,t \geqslant 0$ such that $1+sm(\omega)n+(i_1+1)n_1+i_2n_2+\ldots+i_pn_p=tm(a_1)+\beta_1+\gamma_1-1+\alpha_2+$ $+\gamma_2+\ldots+\alpha_q+\gamma_q$, and this will imply that $v$ is also an $\Omega$-

-word. Moreover, we shall have:

$$[v]=\omega^{sm(\omega)}\omega_1^{i_1+1}\omega_2^{i_2}\ldots\omega_p^{i_p}(a_1^{tm(a_1)+\beta_1+\gamma_1-1},a_2^{\alpha_2+\gamma_2}\ldots a_q^{\alpha_q+\gamma_q})$$
$$=\omega^{sm(\omega)}\omega_1^{i_1}\ldots\omega_p^{i_p}(a_1^{tm(a_1)+\beta_1},a_2^{\alpha_2},\ldots,a_q^{\alpha_q})=$$
$$=\omega_1^{i_1}\ldots\omega_p^{i_p}(a_1^{\alpha_1},a_2^{\alpha_2},\ldots,a_q^{\alpha_q})=[u]\ .$$

Consider now the case $u\dashv v$. Namely, we can assume that $u=\omega_1^{j_1}\omega_2^{i_2}\ldots\omega_p^{i_p}a_1^{\beta_1}\ldots a_q^{\beta_q}$, $j_1\geqslant 1, j_1\equiv i_1(\mathrm{mod}\, m(\omega_1)), \beta_\lambda\equiv\alpha_\lambda(\mathrm{mod}\ m(a_\lambda))$, and $v=\omega_1^{j_1-1}\omega_2^{i_2}\ldots\omega_p^{i_p}a_1^{\beta_1-\gamma_1+1}a_2^{\beta_2-\gamma_2}\ldots a_q^{\beta_q-\gamma_q}$, where

$a_1=\omega_1(a_1^{\gamma_1},\ldots,a_q^{\gamma_q})$. We can also assume that $n_k\geqslant 1$ for some $k\in\{1,\ldots,p\}$. Now it can be easily seen that there exist $s_1,s_2,\ldots,s_p,t_1,\ldots,t_q\geqslant 0$ such that

$$1+(j_1-1+s_1m(\omega_1))n_1+(i_2+s_2m(\omega_2))n_2+\ldots+(i_p+s_pm(\omega_p))n_p=$$
$$=(\beta_1-\gamma_1+1+t_1m(a_1))+(\beta_2-\gamma_2+t_2m(a_2))+\ldots+(\beta_q-\gamma_q+t_qm(a_q))\ .$$

Then:

$$v=\omega_1^{j_1-1+s_1m(\omega_1)}\ldots\omega_p^{i_p+s_pm(\omega_p)}a_1^{\beta_1-\gamma_1+1+t_1m(a_1)}a_2^{\beta_2-\gamma_2+t_2m(a_2)}\ldots a_q^{\beta_q-\gamma_q+t_qm(a_q)}$$

and

$$[v]=\omega_1^{j_1-1+s_1m(\omega_1)}\ldots\omega_p^{i_p+s_pm(\omega_p)}(\omega_1(a_1^{\gamma_1},\ldots,a_q^{\gamma_q}),a_1^{\beta_1-\gamma_1+t_1m(a_1)},\ldots,a_q^{\beta_q-\gamma_q+t_qm(a_q)})=$$
$$=\omega_1^{j_1+s_1m(\omega_1)}\ldots\omega_p^{i_p+s_pm(\omega_p)}(a_1^{\beta_1+t_1m(a_1)},\ldots,a_q^{\beta_q+t_qm(a_q)})=$$
$$=\omega_1^{i_1}\ldots\omega_p^{i_p}(a_1^{\alpha_1},\ldots,a_q^{\alpha_q})=[u]\ .$$

This completes the proof of $(\Delta\Delta)$, and thus of Theorem1. as well.

**Corollary.** *If* $\Omega\setminus\Omega(1)\neq\emptyset$, *then the class* $A_m(\Omega)$ *is a variety.*

**Proof.** In this case we have that $m:z\mapsto m(z)=m$ is a constant, and thus in (**) we have a system of identities.

2. Consider now the case when $\Omega=\Omega(1)$ consists of only unary operators. The following example shows that the conditions (*), (**) are not suficient.

Example. Let $\Omega=\Omega(1)$ and let $\omega_0$ be a fixed element of $\Omega$. Let $A=\{1,2,3,4,5\}$ and the algebra $\underline{A}=(A;\Omega)$ be defined by:

$$\omega_{0_{\underline{A}}}=(123)(45),\ \omega_{\underline{A}}=1_A \quad \text{if} \quad \omega\neq\omega_0.$$

The algebra $\underline{A}$ satisfies the conditions (*), (**). Namely, the condition (*) reduces to the commutativity of the semigroup generated by the transformations which are interpretations of the operators from $\Omega$. And, if we put $m(\omega_0)=6, m(\omega)=m(a)=1$ for each $\omega\in\Omega$, $\omega\neq\omega_0$ and each $a\in A$, we obtain that (**) is also satisfied. But $\underline{A}$ does not belong to ABTG$(\Omega)$, for if $\underline{A}$ were an $\Omega$-subalgebra of a group $G\in$ ABTG then we would have $\bar{\omega}_0^2 4=4$, but $\bar{\omega}_0^2 1=3$, which is impossible. (Namely, $\bar{\omega}_0^2 4=4$ implies that $\bar{\omega}_0^2$ is the identity of the group G.)

Theorem 2. *Let* $\Omega=\Omega(1)$. *An* $\Omega$*-algebra* $\underline{A}=(A;\Omega)$ *belongs to* ABTG$(\Omega)$ *iff it satisfies the following conditions:*

(*´) $\omega'\omega''(x)=\omega''\omega'(x)$, *for any* $\omega',\omega''\in\Omega, x\in A$;

(**´) *There is a mapping* $m:\omega\mapsto m(\omega)$ *of* $\Omega$ *into the set of positive integers such that* $\omega^{m(\omega)}(x)=x$, *for any* $\omega\in\Omega$, $x\in A$;

(***´) $\underline{A}$ *satisfies any quasiidentity of the following form:*

$$\omega_1\ldots\omega_p(x)=\omega_1'\ldots\omega_q'(x)\Rightarrow\omega_1\ldots\omega_p(y)=\omega_1'\ldots\omega_q'(y),$$

Proof. Clearly, the conditions $(*')$, $(**')$ and $(***')$ are necessary. The suficiency is a corollary of the following

Lemma. Let $\Gamma$ be a commutative group of permutations on a set A such that:

$$\omega'(x)=\omega''(x)\Rightarrow\omega'(y)=\omega''(y).$$

Define a relation $\approx$ on A by:

$a\approx b \Leftrightarrow (\exists\phi\in\Gamma)b=\phi(a)$. Then, (i) $\approx$ is an equivalence in A.

(ii) If B is a subset of A such that $(\forall a\in A)(\exists! b\in B)a\approx b$ then the mapping $\xi: (\omega,b)\longmapsto\omega(b)$ is a bijection from $\Omega\times B$ into A, such that $\xi(\omega'\omega'',b)=\omega'(\xi(\omega'',b))$.

(iii) If $K$ is an abelian group generated by B and if $G=\Gamma\times K$, then by putting $\xi(\omega,b)=(\omega,b)$, we obtain that the algebra $(A;\Gamma)$ is a $\Gamma$-subalgebra of G.

The proof of the Lemma is obvious.

3. Here we will make some remarks and state some problems.

First, we note that if we try to generalize Theorem 1. for abelian periodical semigroups, then we get the result that this generalization is not true. And, we do not know if the corresponding analogy of Theorem 1. holds for the class of commutative semigroups with the propertiy $(\forall x)(\exists m>0)x^{m+1}=x$.

The similar situation arises if we try to generalize Theorem 2..

We also note that we do not know any convenient description of the class $\underline{C}(\Omega)$ if $\underline{C}$ is one of the following classes of semigroups:

(a) idempotent semigroups;

(b) periodic groups;

(c) groups;

(d) inverse semigroups;

(e) regular semigroups;

(f) completely simple semigroups.

In other words we think that the well known Kurosh's problem of characterisations of $\underline{C}(\Omega)$ is until now solved only for a few classes of semigroups, namely only if $\underline{C}$ is one of the following classes of semigroups:

1) the class of semigroups [1];

2) the class of commutative semigroups [4];

3) the class of cancelative semigroups [5];

4) the class of nilpotent semigroups [6];

5) the class of semilattices [2];

6) $\underline{C}_{1,m}$, i.e. the class of commutative semigroups that satisfies the identity $x^{m+1}=x$ [3];

7) ABTG;

8) $A_m$.

We would like also to mention the problem of finding the set of varieties $\underline{C}$ of semigroups such that $\underline{C}(\Omega)$ is also a variety for all $\Omega$ or for some $\Omega$.

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# PROPER SUBSEMIGROUPS OF A SEMIGROUP

S.Milić and S.Crvenković

Let L be the elementary language of the semigroup theory and $\phi(x,y)$ a formula in the language L.

Denote with $S_\phi$ a class of semigroups satisfying the condition

(*) $(\forall x)(\exists y)\phi(x,y)$.

A semigroup S belongs to a class $QS_\phi$ of semigroups if each proper subsemigroup of S belongs to $S_\phi$.

In [4] E.S.Ljapin introduced the concept of a basis class for some class of semigroups.

In the present article we consider some classes $QS_\phi$ and their basis class.

First, we have the following theorem.

<u>Theorem 1</u>. [5] Let S be a semigroup. Then, the following two propositions are equivalent.

(i) $QS_\phi$ has a basis class relative to the class of all semigroups.

(ii) $QS_\phi \subset S_\phi$.

If $\phi(x,y)$ is one of the following formulas

(1) $x=xyx$;

(2) $x=xyx \wedge xy=yx$;

(3) $x=yx^2$;

(4) $x=x^2y$,

we have:

<u>Proposition 1</u>. [5] Let S be a semigroup. The following conditions are equivalent.

$\bar{1}$ $S \in QS_\phi$.

$\bar{2}$ $S$ is monogenic of an index of 2 or $(\forall x \in S)x^n=x$ for some integer $n \geq 2$.

It follows immediately that $QS_\phi$ does not have a basis clas.

If $m$ and $r$ are the index and period, respectively, of the monogenic semigroup $\langle a\rangle$ we denote by $K_a=\{a^m,a^{m+1},\dots,a^{m+r-1}\}$ the subgroup of $\langle a\rangle$.

Denote with $\Pi$ the class of semigroups $S_\phi$ where $\phi(x,y)\equiv xy=y$. The class $\Pi$ has a basis class (see [4]). According to [5], $Q\Pi$ does not have a basis class relative to the class of all semigroups.

Let $\phi(x,y)$ be the following formula

(1´) $x^m=y^m \wedge yx=x^{m+1}y \wedge x^n=x$,

where $m,n$ are positive integers, $n>1$. $S_\phi$ is the class $S^*_{m,n}$ (see [2]). If $S \in S^*_{m,n}$ we denote by $[\{x,y\}]$ the semigroup generated with $x,y \in S$ such that the formula (1´) holds. $[\{x,y\}]$ is a finite group. We have the following:

<u>Theorem 2.</u> 2 Let $S$ be a semigroup. Then

$S \in S^*_{m,n} \Leftrightarrow (\forall x \in S)(\exists y \in S)([\{x,y\}] \in S^*_{m,n})$.

According to [3], the set $M$, of all groups $[\{x,y\}] \in S^*_{m,n}$ which can not be represented as a union of proper subgroups of the same type, is a basis class for $S^*_{m,n}$.

<u>Theorem 3.</u> Let $S$ be a semigroup. $S \in QS^*_{m,n}$ iff one and only one of the conditions hold.

(1) $(\forall a \in S)a=a^{(m,n-1)+1}$, where $(m,n-1)$ is the GCD of $m,n$.

(2) $S$ is a cyclic group $|S|=p^\alpha$, $p^{\alpha-1} \mid (m,n-1)$ and $p^\alpha \nmid (m,n-1)$, where $p$ is a prime number and $\alpha$ nonnegative integer.

(3) $S$ is a monogenic semigroup with an index of 2 and $r \mid (m,n-1)$, where $r$ is a period of the semigroup $S$.

<u>Proof.</u> Let $S \in QS^*_{m,n}$ and let $x \in S$. If $S$ is not monogenic then $\langle x\rangle \in S^*_{m,n}$. It follows that $x^m=e_{\langle x\rangle}=x^{n-1}$ i.e. $x^{(m,n-1)+1}=x$. If $S$ is monogenic i.e. $S=\langle x\rangle$, then it holds that

(a) $S$ is a cyclic group

or

(b) $S$ is a monogenic semigroup.

Consider (a).Let $|S|=p_{i_1}^{\alpha_1}p_{i_2}^{\alpha_2}\dots p_{i_k}^{\alpha_k}$.Then $<a^{p_{i_j}}>$ is a subgroup of $S(p_{i_j}$ are primes,$j=1,\dots k)$.$<a^{p_{i_j}}>\in S^*_{m,n}$ and

$$a^{p_{i_1}(m,n-1)}=\dots=a^{p_{i_k}(m,n-1)}=e_S$$

so that $a^{(m,n-1)}=e_S$ i.e. $|S|\,\big|\,(m,n-1)$ (so that (1) holds).
If $|S|=p^{\alpha}$ ($\alpha\in N$,p-prime) then $<a^p>\in S^*_{m,n}$.We have

$$(a^p)^{(m,n-1)}=(a^p)^{p^{\alpha-1}}=e_S.$$

For $\alpha=1$ and $p\nmid(m,n-1)$ (2) follows immediately.For $\alpha>1$ we have

$$(a^p)^{((m,n-1),p^{\alpha-1})}=e_S$$

i.e. $p^{\alpha-1}\mid(m,n-1)$.
If $S=<a>$ is a monogenic semigroup i.e. (b) holds, then S is finite (otherwise $S\notin S^*_{m,n}$) and $S=\{a,a^2,\dots,a^{\bar{m}+r-1}\}$. $K_a'=K_a\cup\{a^{\bar{m}-1}\}$ is a subsemigroup of S and is not regular ($K_a'^2\neq K_a'$) so that $K_a'\notin S^*_{m,n}$,$\bar{m}\geq 3$.It follows that $\bar{m}=2$ i.e. $a^2=a^{2+r}$.$K_a$ is a cyclic group of order r generated with the element $a^{r+1}$.Further on, $<a>\in S^*_{m,n}$ and

$$(a^{r+1})^{(m,n-1)}=(a^{r+1})^{r}=e_S.$$

It follows that $((m,n-1),r)=r$ i.e. $r\,|\,(m,n-1)$.

If (1) holds,it follows that $a^{(m,n-1)}$ is a unity for a.Then we have $a^m=e_a=a^{n-1}$ so that $S\in S^*_{m,n}$.

Let condition (2) hold.Then $<a^{p^{\beta}}>\in S^*_{m,n}$ as

$$(a^{p^{\beta}})^{(m,n-1)}=(a^p)^m=e\quad(1\leq\beta\leq\alpha\ )$$

so every subgroup of S belongs to $S^*_{m,n}$ i.e. $S\in QS^*_{m,n}$.

(3).Subsemigroups of S are $K_a$ and subgroups of $K_a$.As $r\,|\,(m,n-1)$ it follows that

$$(a^{r+1})^{(m,n-1)}=e$$

so we see that every element itself can be taken to be y in formula (1´).It follows that $S\in QS^*_{m,n}$.

From the definition of the class $S^*_{m,n}$ it fallows that $x^{n-1}=e_x$.Let p be a prime number such that $p>n-1$.Then $C_p\in QS^*_{m,n}$ and $C_p\notin S^*_{m,n}$.Therefore,$QS^*_{m,n}$ does not have a basis class relative to the class of all semigroups.

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# ON A CLASS OF SEMIGROUPS AND ITS CHARACTERISATION

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In this paper semigroups containing a Rees matrix subsemigroups as an ideal are considered. For such semigroups a structural description is given by the Theorem 1. This class of semigroups coincides with the class of semigroups having at least one minimal left ideal and at least one minimal right ideal [Clifford A. H., Theorem 3.1. in [1]].

Let $\mathcal{M}[I,G,J,P]$ be a Rees matrix semigroups, where I and J are non empty sets, G is a group, P is a $J \times I$ matrix with entries $p_{ji}$ in G; T a partial semigroup and let

$$S = T \cup \mathcal{M}[I,G,J,P]$$

where $(I \times G \times J) \cap T = \emptyset$ . Let

$$\xi : T \to \mathcal{T}(I) \text{ and } \eta : T \to \mathcal{T}(J)$$

be mappings, where $\mathcal{T}(I)$ and $\mathcal{T}(J)$ are semigroups of all mappings I in I and J in J, respectively, i.e.

$$\xi : p \to \xi_p \quad \text{and} \quad \eta : p \to \eta_p \ ,$$

such that for all $p,q \in T$ is fulfilled:

(a) If $pq \in T$ then $\xi_{pq} = \xi_q \xi_p$ and $\eta_{pq} = \eta_p \eta_q$ ;

(b) If $pq \notin T$ then $\xi_q \xi_p$ is a constant mapping and $\eta_p \eta_q$ is constant mapping;

Let

$$\phi : T \times I \to G$$

be a mapping which satisfies :

(c) If $pq \in T$ then $\phi(pq,i) = \phi(p,i\xi_q)\phi(q,i)$ ;

(d) $p_{j,i\xi_p}\phi(p,i)p^{-1}_{j\eta_p,i} = \Psi(p,j)$ , i.e. does not depend upon $i \in I$.

Let us define the multiplication in S by:

(i) $(i,x,j)\cdot(k,y,\ell) = (i,xp_{jk}y,\ell)$ ,

(ii) $p\cdot(i,x,j) = (i\xi_p,\phi(p,i)x,j)$ ,

(iii) $(i,x,j)\cdot p = (i,x\Psi(p,j),j\eta_p)$ ,

where $p \in T$, $(i,x,j)$, $(k,y,\ell) \in I \times G \times J$ ;

(iv) If $pq = r$ in T then $pq = r$ in S;

if $pq \notin T$ then

$$pq = (i\xi_q\xi_p,\phi(p,i\xi_q)\phi(q,i)p^{-1}_{j\eta_p\eta_q,i},j\eta_p\eta_q).$$

Let us denote the groupoid (S,.) by $\mathcal{N}[I,G,J,P;T,\phi,\xi,\eta]$. It is directly verified that $\mathcal{N}[I,G,J,P;T,\phi,\xi,\eta]$ is a semigroup with a Rees matrix subsemigroup $\mathcal{M}[I,G,J,P]$ which is an ideal in $\mathcal{N}$.

Now we can state

Theorem 1. In a semigroup S there is an ideal which is a completely simple subsemigroup (without zero) if and only if a semigroup S is isomorphic to a semigroup $\mathcal{N}[I,G,J,P;T,\phi,\xi,\eta]$ .

Proof. See [2].

From Theorem 3.1. and the Theorem 3.2. from [1] and Theorem 1. of this paper, we have.

Theorem 2. A semigroup S contains at least one minimal left ideal and at least one minimal right ideal if and only if S is isomorphic to a semigroup $\mathcal{N}[I,G,J,P;T,\phi,\xi,\eta]$ .

Problem. Give a structural description of a semigroup which contains a Rees matrix subsemigroup as a left ideal.

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# ON ONE REPRESENTATION OF GENERALIZED EQUIVALENCES

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In this article we describe the representation of (n+1)-ary equivalence relations on S [1], by means of the system of k-ary ($2 \le k \le n$) relations (specially by binary relations).

* * *

1. [1] (n+1)-ary relation $\rho$ on S is an *equivalence relation*[1] on S, $|S| \geq n$, $n \in N$, iff it satisfies:

(1) $(\forall a_1,\dots,a_n \in S)\ ((a_1,\dots,a_n,a_1) \in \rho)$[2] ;

(2) $(\forall a_1,\dots,a_{n+1} \in S)((a_1^{n+1}) \in \rho$[3] $\Longrightarrow (a_{\pi(1)},\dots,a_{\pi(n+1)}) \in \rho)$,
for each $\pi \in \{1,\dots,n+1\}!$ ; and

(3) $(\forall a_0, a_1,\dots,a_{n+1} \in S)((a_0,a_1^n) \in \rho \wedge (a_1^n, a_{n+1}) \in \rho \wedge$
$(a_i \neq a_j$, for $i \neq j$, $i,j \in \{1,\dots,n\}) \Longrightarrow (a_0, a_1^{n-1}, a_{n+1}) \in \rho)$.

2. In [1] it is proved that (n+1)-ary equivalence relation $\rho$ on S, induces on S *partition of type n* [2], and vice versa.

3. Starting with (1) and using (2), we get the following:

($\bar{1}$) $(\forall a_1,\dots,a_{j-1},a_{j+1},\dots,a_{n+1} \in S)((a_1^{i-1}, a_i, a_{i+1}^{j-1}, a_i, a_{j+1}^{n+1}) \in \rho)$, for each $i, j \in \{1,\dots,n+1\}$, $i \neq j$, i.e. if $\rho$ satisfies (1), then $\rho$ is reflexive [3].

4. It is obvious that each (n+1)-ary relation $\rho$ on S can be represented by the system of (n+1-i)-ary relations $\rho_{a_1^i}$ , $i \in \{1,\dots,n\}$, $a_1,\dots,a_i \in S$, such that

(4) $(x_1^{n+1-i}) \in \rho_{a_1^i}$ iff $(a_1^i, x_1^{n+1-i}) \in \rho$,

---

1) *For n > 1: generalized equivalence.*

2) *(1,n+1) - reflexive* [3].

3) $a_1^{n+1}$ *stands for* $a_1,\dots,a_{n+1}$; *see also* [3].

and that each such system of $\rho_{a_1^i}$, $a_1,\dots,a_i \in S$, defines, by (4), an (n+1)-ary relation $\rho$ on S.

a) If $|S| = m \in N$, then the above mentioned system consists of at most $m^i$ relations $\rho_{a_1^i}$. Note that some of them may be described by the same set.

b) It follows also that each collection of at most $m^i$ (n+1-i)-ary relations $\rho_{a_1^i}$, $i \in \{1,\dots,n\}$, $a_1,\dots,a_i \in S$, $|S| = m \in N$, determines one (n+1)-ary relation $\rho$ on S, defined by (4).

* * *

*LEMMA 1.*

*If $\rho$ is (n+1)-ary relation on S, defined by*

*(a) $(a_1^{n+1}) \in \rho$ iff there are $i, j \in \{1,\dots,n+1\}$, $i \neq j$, such that $a_i = a_j$,*

*then $\rho$ is (n+1)-ary equivalence relation on S.*

Proof:

$\rho$ is reflexive and symmetric (satisfies ($\bar{1}$) and (2)) by its definition (a).

Let $(a_0^n) \in \rho$ and $(a_1^{n+1}) \in \rho$, $a_i \neq a_j$, $i,j \in \{1,\dots,n\}$, $i \neq j$. Then

$a_0 = a_k$, for some $k \in \{1,\dots,n\}$, and

$a_{n+1} = a_p$, for some $p \in \{1,\dots,n\}$, by (a) and by the assumption.

Then $a_0 = a_k$, for some $k \in \{1,\dots,n-1\}$, or $a_{n+1} = a_p$, for some $p \in \{1,\dots,n-1\}$, and if neither is satisfied, then $a_0 = a_{n+1} = a_n$. In every case it follows that $(a_0^{n-1}, a_{n+1}) \in \rho$, proving that $\rho$ is transitive.

*THEOREM 2.*

*Let $\rho$ be (n+1)-ary equivalence relation on S, $n > 1$[1], $|S| > n$, and let $\rho_{a_1^i}$ be (n+1-i)-ary relations on S, defined by (4), $i \in \{1,\dots,n-1\}$[2]. Then the following is satisfied:*

*I The restrictions $\bar{\rho}_{a_1^i}$ of relations $\rho_{a_1^i}$ to the corresponding sets $S\setminus\{a_1,\dots,a_i\}$[3], are (n+1-i)-ary equivalence relations[4];*

*II If $(a_{i+1}^{n+1}) \in \rho_{a_1^i}$, then*

*$(a_{\pi(i+1)},\dots,a_{\pi(n+1)}) \in \rho_{a_{\pi(1)},\dots,a_{\pi(i)}}$, for each $\pi \in \{1,\dots,n+1\}!$ ;*

*III $\rho_{a_1^i} \setminus \bar{\rho}_{a_1^i} = S^{n+1-i} \setminus (S\setminus\{a_1,\dots,a_i\})^{n+1-i}$*

*1) $\rho$ is at least ternary*

*2) $\rho_{a_1^i}$ are at least binary.*

*3) The cases when there are equal elements among $a_1,\dots,a_i$, are also included.*

*4) Because of (2): $\rho_{a_1^i} = \rho_{a_{\pi(1)},\dots,a_{\pi(i)}}$, for each $\pi \in \{1,\dots,i\}!$.*

Proof:

$I_1$. $\bar{\rho}_{a_1^i}$ are reflexive (they satisfy $(\bar{1})$ :

(a) $(a_{i+1}^{j-1}, a_j, a_{j+1}^{k-1}, a_j, a_{k+1}^{n+1}) \in \bar{\rho}_{a_1^i}$ , for $j,k \in \{i+1,\ldots,n+1\}$, $j \neq k$, since (a) is by (4)[5], equivalent to

(b) $(a_1^i, a_{i+1}^{j-1}, a_j, a_{j+1}^{k-1}, a_j, a_{k+1}^{n+1}) \in \rho$, for $j, k \in \{i+1,\ldots, n+1\}$, and (b) is satisfied since $\rho$ is reflexive.

$I_2$. $\bar{\rho}_{a_1^i}$ are symmetric (they satisfy (2)):

Since $\bar{\rho}_{a_1^i} \subseteq \rho_{a_1^i}$ , from $(a_{i+1}^{n+1}) \in \bar{\rho}_{a_1^i}$ it follows that $(a_{i+1}^{n+1}) \in \rho_{a_1^i}$ , and by (4), $(a_1^i, a_{i+1}^{n+1}) \in \rho$. $\rho$ is symmetric and thus

$$(a_1^i, a_{\pi(i+1)},\ldots,a_{\pi(n+1)}) \in \rho$$

for each permutation $\pi \in \{i+1,\ldots,n+1\}!$. From this and the fact that $\{a_1,\ldots,a_i\} \cap \{a_{\pi(i+1)},\ldots,a_{\pi(n+1)}\} = \emptyset$ ,
it follows that

$$(a_{\pi(i+1)},\ldots,a_{\pi(n+1)}) \in \bar{\rho}_{a_1^i} ,$$

for each $\pi \in \{i+1,\ldots,n+1\}!$ .

$I_3$. $\bar{\rho}_{a_1^i}$ are transitive (in the sense of (3)):

Let $(x_0^{n-i}) \in \bar{\rho}_{a_1^i}$ and $(x_1^{n+1-i}) \in \bar{\rho}_{a_1^i}$ , $x_k \neq x_p$ , $k \neq p$, $k,p \in \{1,\ldots,n-i\}$. This is equivalent to $(a_1^i, x_0^{n-i}) \in \rho$ and $(a_1^i, x_1^{n+1-i}) \in \rho$, $x_k \neq x_p$, $k \neq p$, $k, p \in \{1,\ldots,n-1\}$. $\rho$ is symmetric and thus

(c) $(x_0, a_1^i, x_1^{n-i}) \in \rho$ and $(a_1^i, x_1^{n+1-i}) \in \rho$, $x_k \neq x_p$, $k \neq p$, $k,p$ $\{1,\ldots,n-i\}$.

Suppose now that a) $a_1,\ldots,a_i$ are not all mutualy different, and b) $a_1,\ldots,a_i$ are all different.

a) In this case, using Lemma 1., we get

(d) $(x_0, a_1^i, x_1^{n-i-1}, x_{n-i+1}) \in \rho$ .

b) Here we use the fact that $\rho_{a_1^i}$ is the relation on $S\backslash\{a_1,\ldots,a_i\}$, i.e. $\{a_1,\ldots,a_i\} \cap \{x_0,\ldots,x_{n-i+1}\} = \emptyset$, and from (c), by (3), it follows again that

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5) $\bar{\rho}_{a_1^i} \subseteq \rho_{a_1^i}$

$$(x_0, a_1^i, x_1^{n-i-1}, x_{n-i+1}) \in \rho$$

By symmetry (d) implies

$$(a_1^i, x_0^{n-i-1}, x_{n-i+1}) \in \rho .$$

Hence, in both cases we get

$$(x_0^{n-i-1}, x_{n-i+1}) \in \bar{\rho}_{a_1^i} ,$$

proving transitivity (3) for $\bar{\rho}_{a_1^i}$ .

II follows directly from the symmetry of $\rho$.

III$_1$. It is obvious that

$$\rho_{a_1^i} \setminus \bar{\rho}_{a_1^i} \subseteq S^{n+1-i} \setminus (S \setminus \{a_1,\dots,a_i\})^{n+1-i}.$$

III$_2$. The converse is also true, i.e.

$$S^{n+1-i} \setminus (S \setminus \{a_1,\dots,a_i\})^{n+1-i} \subseteq \rho_{a_1^i} \setminus \bar{\rho}_{a_1^i} :$$

If $(x_1^{n+1-i}) \in S^{n+1-i} \setminus (S \setminus \{a_1,\dots,a_i\})^{n+1-i}$, then there is at least one $a_k \in \{a_1,\dots,a_i\}$ such that $x_j = a_k$, $j \in \{1,\dots,n+1-i\}$ , $k \in \{1,\dots,i\}$ .

Now by II, the statement

$$(x_1^{n+1-i}) = (x_1^{j-1}, a_k, x_{j+1}^{n+1-i}) \in \rho_{a_1^i}$$

is equivalent to

$$(a_k, x_2^{j-1}, a_k, x_{j+1}^{n+1-i}) \in \rho_{x_1,\ a_1^{k-1},\ a_{k+1}^i},$$

and hence

$$(x_1^{n+1-i}) \in \rho_{a_1^i} .$$

Since by the definition of $\bar{\rho}_{a_1^i}$ , $(x_1^{j-1}, a_k, x_{j+1}^{n+1-i}) \notin \bar{\rho}_{a_1^i}$ , then it follows that $(x_1^{n+1-i}) \in \rho_{a_1^i} \setminus \bar{\rho}_{a_1^i}$ , proving III$_2$.

III$_1$ and III$_2$ show that III is satisfied, completing the proof of Theorem 2.

When we put i = n-1 in the formulation of Theorem 2., we get the following proposition.

*COROLLARY 3.*

*If $\rho$ is (n+1)-ary equivalence relation on S, $n > 1$, $|S| \geq n$, then the relations $\bar{\rho}_{a_1}{}^{n-1}$ are binary equivalence relations on $S\ \{a_1,\dots,a_{n-1}\}$.*

The following proposition is a direct consequence of Theorem 2., remark a) in 4) concerning the finite sets and the fact that $a_1,\dots,a_i$ represent all variations with repetitions of the class i one the finite set S.

*COROLLARY 4.*

*If $|S| = m \geq n$ $(m,n \in N)$, then each (n+1)-ary equivalence relation on S can be represented by $m^i$ [1)] (n+1-i) - ary relations $\rho_{a_1}{}^{i}$, satisfying I, II and III from Theorem 2.*

*COROLLARY 5.*

*Under the assumptions of Theorem 2., if $a_p \neq a_q$, for some $p,q \in \{1,\dots,i\}$, $p \neq q$, then $\rho_{a_1}{}^{i}$ are universal (n+1-i)-ary relations on S.*

Proof:

$(x_1^{n+1-i}) \in \rho_{a_1}{}^{i}$ for arbitrary $x_1,\dots,x_{n+1-i} \in S$ iff $(a_1^i, x_1^{n+1-i}) \in \rho$ and this is true since $\rho$ is reflexive.

REMARK: From Corollary 5. it follows that $\bar{\rho}_{a_1}{}^{i}$ are universal (n+1-i)-ary relations on $S\ \{a_1,\dots,a_i\}$, since the restriction of an universal relation is universal.

*COROLLARY 6.* [2)]

*Let $\rho$ be (n+1)-ary relation on S, defined by (a), Lemma 1. Then $\bar{\rho}_{a_1}{}^{i}$ (on $S\setminus\{a_1,\dots,a_i\}$) satisfies the same property (a), Lemma 1., if $a_1,\dots,a_i$ are different.* [3)]

Proof:

$(a_{i+1}^{n+1}) \in \bar{\rho}_{a_1}{}^{i}$ is equivalent to

$(a_{i+1}^{n+1}) \in \rho_{a_1}{}^{i}$, $a_1 \dots, a_i \in S$, $a_{i+1},\dots,a_{n+1} \in S\setminus\{a_1,\dots,a_i\}$, that is (by (4)) to

(g) $(a_1^{n+1}) \in \rho$, $a_1,\dots,a_i \in S$, $a_{i+1},\dots,a_{n+1} \in S\setminus\{a_1,\dots,a_i\}$.

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1) *In fact there are $\binom{m+i-1}{i}$ classes of relations, each class consisting of relations described by the same set; see also notice 4) concerning Theorem 2.*

2) *Of Theorem 2. and Lemma 1.*

3) *The case when $a_1,\dots,a_i$ are not all different has been considered in Corollary 5.*

By the definition of $\rho$, some of $a_1,\dots,a_{n+1}$ in (g) must be equal. Since all $a_1,\dots,a_i$ are different and

$$\{a_1,\dots,a_i\} \cap \{a_{i+1},\dots,a_{n+1}\} = \emptyset ,$$

it follows that equal elements are among $a_{i+1},\dots,a_{n+1}$, proving that $\bar{\rho}_{a_1^i}$ satisfies (a), Lemma 1.

The following proposition is a converse of Theorem 2.

*THEOREM 7.*

*Let* $|S| \geq n$, $n \in N$, *and let*

$$\{\rho_{a_1^i} ;\ i \in \{1,\dots, n-1\},\ a_1,\dots,a_i \in S\}$$

*be a collection of $(n+1-i)$-ary relations on $S$, satisfying I, II and III from Theorem 2. Then $\rho$, defined by (4) on $S$, is $(n+1)$-ary equivalence relation.*

Proof:

A. $\rho$ is reflexive:

Suppose that $a_1,\dots,a_{n+1}$ are not all different. We shall consider following two cases:

a) $\{a_1,\dots,a_i\} \cap \{a_{i+1},\dots,a_{n+1}\} = \emptyset$ ; and

b) $\{a_1,\dots,a_i\} \cap \{a_{i+1},\dots,a_{n+1}\} \neq \emptyset$ .

If a) holds, and $a_p = a_q$ for some $p, q \in \{i+1,\dots,n+1\}$ , $p \neq q$, then the reflexivity of $\rho$ follows by the same propery of $\bar{\rho}_{a_1^i}$. But if, assuming a), $a_p = a_q$ for some $p,q \in \{1,\dots,i\}$, and $a_{i+1},\dots,a_{n+1}$ are different, then we have the following:

$(a_1^{n+1}) \in \rho$ is by (4) equivalent to $(a_{i+1}^{n+1}) \in \rho_{a_1^i}$ i.e. to

$(a_{i+1}^{n+1}) \in \rho_{a_1^{p-1},\ a_p,\ a_{p+1}^{q-1},\ a_p,\ a_{q+1}^i}$.

This is, by II, equivalent to

$$(a_p, a_p, a_{i+3}^{n+1}) \in \rho_{a_1^{p-1},\ a_{i+1},\ a_{p+1}^{q-1},\ a_{i+2},\ a_{q+1}^i},$$

which holds if and only if

$$(a_p, a_p, a_{i+3}^{n+1}) \in \bar{\rho}_{a_1^{p-1}, a_{i+1},\ a_{p+1}^{q-1},\ a_{i+2},\ a_{q+1}^i} ,$$

and this proves the reflexivity of $\rho$, since $\rho_{b_1^i}$ are reflexive relations.
Suppose now that b) holds. Then

$$(a_{i+1}^{n+1}) \in \rho_{a_1^i} \text{ if and only if } (a_{i+1}^{n+1}) \in \rho_{a_1^i} \setminus \bar{\rho}_{a_1^i} ,$$

and this is by III equivalent to

$$(a_{i+1}^{n+1}) \in S^{n+1-i} \setminus (S \setminus \{a_1,\ldots,a_i\})^{n+1-i} .$$

Thus, $\rho$ is reflexive by (4) and by the fact that $S^{n+1-i} \setminus (S \setminus \{a_1,\ldots,a_i\})^{n+1-i}$ allways contains at least one element from $\{a_1,\ldots,a_i\}$.

B. $\rho$ is symmetric:

This is the direct consequence of (4) and II.

C. $\rho$ is transitive:

Let $(a_0^n) \in \rho$, $(a_1^{n+1}) \in \rho$, $a_p \neq a_q$, $p \neq q$, $p,q \in \{1,\ldots,n\}$.

Here again we consider two cases: a) $a_0,\ldots,a_{n+1}$ are not all different, and b) $a_0,\ldots,a_{n+1}$ are different elements of S.

If a) is satisfied and $a_0 = a_n$, or $a_{n+1} = a_n$, then by the assumptions we have

(h) $$(a_0, a_1^{n-1}, a_{n+1}) \in \rho.$$

In all other cases included in a), (h) follows from (already proved) reflexivity of $\rho$.

Assume now that b) holds. Then

$$(a_0^n) \in \rho \text{ iff } (a_1^n) \in \bar{\rho}_{a_0^{i-1}} \text{ iff } (a_0, a_{i+1}^n) \in \bar{\rho}_{a_1^i} \text{ and}$$

$$(a_1^{n+1}) \in \rho \text{ iff } (a_{i+1}^{n+1}) \in \bar{\rho}_{a_1^i} .$$

Hence, by the transitivity of $\bar{\rho}_{a_1^i}$, it follows that

$(a_0, a_{i+1}^{n-1}, a_{n+1}) \in \bar{\rho}_{a_1^i}$, and this is equivalent to (h).

A., B. and C. completely prove the theorem.

For i = n-1, Theorem 7 reducesto the following proposition.

*COROLLARY 8.*

*Let $|S| \geq n, n \in N$, and let $\{\rho_{a_1^{n-1}};\ a_1,\ldots,a_{n-1} \in S\}$ be a collection of binary relations on S, such that I, II and III are satisfied. Then the relation $\rho$, defined by (4) on S, is (n+1)-ary equivalence relation.*

Directly from Theorem 7., provided that S is finite, we get the following:

*COROLLARY 9.*

*Let* $|S| = m > n$, $m, n \in N$, *and let*

$$\{\rho^{i}_{a_1} ; i \in \{1,\dots,n-1\}, a_1,\dots,a_i \in S\}$$

*be a collection of* $m^i$ $(n+1-i)$*-ary relations on* $S$*, such that I, II and III from Theorem 2. are satisfied. Then* $\rho$*, defined by (4), is* $(n+1)$*-ary equivalence relation on* $S$*.*

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