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M O N D

MODIFICATION OF **N**EWTONIAN **D**YNAMICS

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MOND1 - PREFACE AND PREPRINCIPLES

The title words of this book are contained in an extraordinary Newton's work [1], which among other things contains Newton's RULE IV of causal judgment. In experimental physics the derived assertions on the basis of carried out experiments, disregarding the possible contraries to the assumed, should be considered either as more exact or nearly true, until other exact phenomena are established, by which they are described more exactly, or are excluded. This rule of Newton's theory allows any correction or addition of his natural philosophy by more exact proofs.

1.1. Nature and the science of nature. Nature emerges in a multitude of phenomena, motions and transformations of objects accessible to human vision and perception, known and unknown, newly acquired and inaccessible to cognition. Nature is the emergence, existence and vanishing of things and learning about them; it reflects itself in human consciousness in an indefinite number of patterns. Some people understand this fact and seek to discover the undiscovered, to check the acquired knowledge to the very incompleteness, and others tend to establish belief in academic knowledge, even when they do not understand the lecturing of highly educated scholars. Science is the discovery of new knowledge and correction or modification of established knowledge. Knowledge is comprehensive and always a fonder for science. When science discovers novelties that expand the standardly accepted omniscience, those scientific truths are becoming a part of knowledge, and science is tending to further discover and create new knowledge. Higher education export is not a scientist unters enriches his knowledge, adopting somebody else's knowledge, as long as he becomes confident to check that knowledge regardless of the authorities that have created standard knowledge, or until he engages in searching for or creating new knowledge. So, the author of this contribution views science as a speculative and unique practical creativity, whereas academic degrees represent the level of the existing knowledge attained. Far from science is a person who, irrespective of his scientific title, states that "a new scientific rational result should not be recognized, or published in journals of scientific research, for the reason that nobody does something like that in the world, i.e. what is unknown".

1.2. Astronomy and celestial mechanics. To further facilitate understanding, it is necessary to distinguish between the terms denoted by the subtitle words. It is normal to expect that many astronomers and astrophysicists have their own views, university professors in particular. Without indicating whether those perceptions are right or wrong, the author points to a significant difference in the understanding of Astronomy and Mechanics, even Celestial mechanics. Astronomy is first and foremost the knowledge and science of celestial phenomena and observations. Luminos, emissions and reflection nebulae, stars and accompanying bodies, galaxies,[1] their positions with respect to the position of other stars, motions and relative resting, emergence and disappearance, stability of the cosmic order, and mutual dependence in the motion of celestial objects. Astronomers observed, described and interpreted the nature of motion of the celestial suns long before the mathematical theory about the motion of terrestrial bodies had been established, well before the establishment of rational mechanics. Of critical importance for the application of the theory of mechanics to celestial bodies have been and remained Copernicus' hypotheses and Kepler's astronomical laws: they are referred to as Kepler's laws because they are based on the measurements of observations of the major planetary motions around the Sun; and astronomical because as such they belong to astronomy, a science of the nature of celestial bodies' motion, and as laws in the physical sense of that word, i.e. statement about some attribute of natural existence and motion of the body. Those laws of astronomy that applied to a small number of celestial bodies and practical knowledge about mechanical properties of the body motion on a circular path, especially Galileo's experiments with heavier falling objects, his invention of a telescope with which he saw the Moon much like the surface of the Earth, and Newton's mathematical principles of the science of nature, all this has created a solid foundation for the emergence and development of Celestial mechanics.

1.3. Rational or analytical mechanics. In order to facilitate understanding for all mathematicians, astronomers and astrophysicists, whose field of research is not classical and celestial mechanics, the author of this contribution finds it necessary to point to that academic knowledge required

[1]I. Newton, Phylosophia naturalis principia mathematica, London 3 editions: 1686,1713, 1725. Translated into Russian by A. N. Krilov, Mathematical principles of natural philosophy, Academy of Sciences of SSSR, Moscow-Leningrad,1936.

for easy understanding of this approach in solving the second or inverse problem of mechanics as well as for readily rejecting the unprofessional opponent assertions. The concept of 'mechanics' has long involved a widespread perception that these are machines, and therefore the craft of machine building and repair. That craft, among other things, demonstrated that it is possible to lift or set into motion a considerably bigger object with the lever than without it. A general truth was known that a body once set into motion will not stop moving unless that motion is opposed by something: and many more facts were found out. Master mechanic, who worked with more exactness was a better master than the other craftsman, especially if he was perfect in using his knowledge. With unsurpassed mathematical precision did Isaac Newton manage to generalize three laws of motion, by the virtue of his brilliant genius, using three independent sentences, which like in the fundamentals of geometry he called:
smallskip

Axiomata sive leges motus (Axioms or laws of motion). That wisdom and mental stamina did not describe only artisan operations but predicted future motions and relative resting of the body using mathematical relations. To his opponents' remarks of being unintelligible, he replied in writing: I have written it in such a way so as to make it comprehensible for mathematicians. Thanks to this, natural philosophy has become a strictly precise mathematical science but not a trade. Newton created rational over practical reality or, as he called it, rational mechanics (ratio - reason). On the basis of this greatest scientific work, as its writer indicated by the title, the mathematical principles of natural philosophy were created, later confirmed by all mathematical theories, or used as a basis for testing different mathematical methods.

That greatest work of all natural sciences is composed of 8 definitions, 3 axioms or laws of motion, and a great number of lemmas and theorems. The definitions refer to the mass, the momentum, the force of inertia, a general concept of the force and centripetal force. The axioms or the laws of motion, verbal or in writing, represent the basis of the theory of mechanics, without underlying suspicion or changes. All theorems are proved by the axioms and phenomena. Let us point out once again that there are three axioms or laws of motion and that in his Mathematical Principles Newton does not use the word 'law' for other different statements, as it is common in the post-Newtonian theoretical mechanics. Given that our contribution is related to the generalization of Newton's theory of mutual attraction between two bodies, we could start here from the Newtonian axiomatic theory of body's motion.

The term Modification (here implies: change, modification, or more exactly, determination, or more generally, generalization of the Newtonian and post-Newtonian classical and celestial dynamics in particular, as well as general principles of dynamics. In the title of this ingenious work [1] two words are prominent: MATHEMATICAL PRINCIPLES of natural philosophy - Philosophiae naturalis principia mathematica from where a great book originates containing basic definitions, axioms or laws of motion, lemmas, theorems, tasks, suppositions, phenomena and rules of causal reasoning. The author of this work believes that any deviation from those principled attitudes is a Modification of Newtonian dynamics, which will be termed for short MOND theory - Modified Newtonian Dynamics. It is considerably more general than the term taken from WIKIPEDIA, the free encyclopedia, which links the theory to the name of Mordehai Milgrom (1983) a physicist at Weizmann Institute of Science, Rehovot, Israel.2 The earliest and general modification of Newton's theory was proposed by mathematicians in line with the development of mathematical analysis. The language and relations used in writing the post-Newtonian dynamics are of modernized mathematical analysis. In their analysis the scientists endeavored to keep the nature of dynamic objects from being changed by mathematical transformations. Following the Hamiltonian manner of reducing differential equations of the second kind to twice the number of differential equations of the first kind, an entirely non-Newtonian concept of the 'Dynamic systems" was introduced, where essential properties of the Newtonian dynamics are left out. Prior to gradual generalization of the Newtonian and post-Newtonian mechanics [2]. we find it useful to introduce the reader with a few sentences from the fundamental Newton's work Mathematical Principles of Natural Philosophy - (Philosophia Naturalis Principia Mathematica), (1686, 1713, 1725).

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[2]From Wikipedia, the free encyclopedia. In physics Modified Newtonian dynamics - MOND is a hypothesis that proposed a modification of Newton's law of gravity to explain the galaxy rotation problem. When the uniform velocity of rotation galaxies was first observed, it was unexpected because Newtonian theory of gravity predicts that objects that are farther out will have lower velocities. For example, planets in the Solar System orbit with velocities decrease as their distance from the Sun increases.

MOND was proposed by Mordehai Milgrom in 1983 as a way to model this observed uniform velocity data. Milgrom noted that Newton's law for gravitation force has been verified only where gravitational acceleration is large, and suggested that for extremely low accelerations the theory may not hold MOND theory posits that acceleration is not linearly proportional to force at low values.

MOND stands in contrast to the more widely accepted theory of dark matter. Dark matter theory suggests that each galaxy contains ahead of as yet undeniable type of matter that provides an overall mass description different from the observed description of normal matter. This dark matter modifies gravity so as to cause the uniform rotation velocity data

"The ancients considered mechanics in a twofold respect: as rational (analytical), developed accurately by demonstrations and practical." In this sense rational mechanics is the science of motions which result from any forces, and of the forces which are required for any motions, accurately propounded and demonstrated." "I heartily beg that what I have here done may be read with candor; and that the defects I have been guilty of upon this difficult subject may be not so much reprehended as kindly supplied, and investigated by new endeavors of my readers." (8 March 1686) In a short preface, consisting of 8 lines, to the second edition of the Principia Newton wrote on 28 March 1713 as follows: "In this second edition much scattered material is corrected and some has been added. In Book I, Section II, the discovery of the forces, by which bodies shall be able to revolve in given orbits, is returned easier and more fully." In a SCHOLIUM, inserted between 8 Definitions and The Laws of Motion, Newton explains 'generally familiar concepts of time, space, place and motion ([1], pp 30-37) referred to as Absolute, true or mathematical time and Relative, apparent, and common time, Absolute and relative space. This clarification, without mathematical symbols and assertions, has left little trace in the post-Newtonian mathematical theory of body motion. Moreover, time is considered and referred to as "a natural parameter", while a broader concept of space in mathematics has spread out to various 'spaces', such as Euclidean space, Riemann space, and consequently to Hilbert space, Weyl space, Poincare space, Minkowski space, linear, vector, multi-dimensional, phase, tangential, co-tangential, plane, planar curved, zero space These and other absolute mathematical deviations have overshadowed and are still overshadowing general knowledge about the nature of thing. In his book Newton himself indicates that he was not developing his theory in that direction. His first rule of causal reasoning ([1], p. 502) reads: No more causes of natural things should be admitted than are both true and sufficient to explain their phenomena. Rule IV allows corrections or additions of more exact knowledge and it reads: In experimental physics the derived assertions on the basis of carried out experiments, disregarding possible contraries to the assumed, should be considered either as more exact or nearly true, until other exact phenomena are established, by which they are described more exactly, or are excluded. Historically, a modification of Newton's theory started and went on much earlier but was and still is differently called, such as: Euler-Lagrangian analytical mechanics or Hamiltonian mechanics. Newton grounded his theory in the axioms such as geometry, whereas Euler and Lagrange developed that theory by means of mathematical analysis using their own principles. In his preface Newton says that rational mechanics has two tasks: first, if attributes of motion are known, to determine the force and, second, to determine the force exactly. However, Hamilton in his theory sets just one task - to integrate 2n differential equations, without changing the words and the notion of force, underlying Newton's dynamics. This was the basis for developing a great theory of dynamic systems and noninvariant integration of linear differential equations of the first kind and for studying stability of integrable and nonintegrable systems. In order to enhance the accuracy, let us make a distinction between two notions: Newton's dynamics and Newtonian dynamics.

The notion Newton's will imply what Newton exactly wrote, while Newtonian means what other authors wrote. The simplest example of deviation is Newton's second axiom or law of motion. It is not a small number of authors of classical mechanics who call that axiom the Basic equation of motion and write it in the form:

$$\frac{dm\mathbf{v}}{dt} = \mathbf{F}, \tag{1.1}$$

but according to Newton's axiom or law of motion the accurate one should be written as

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}, \tag{1.2}$$

This makes the difference as will be demonstrated further below. It will be shown that equation (1.1) is incongruous with Newton's second law or axiom of motion. As such, it is not accurate in general, especially as Newtonian general law of body's motion. The approaches mentioned above, as well as a more detailed analysis of Newton's mathematical principles of natural philosophy, and their inconsistent results, required more reliable and better clarified frameworks of rational mechanics, which are presented in the monograph, [3].

1.4. Preprinciples of mechanics. The compound phrase "preprinciple" or "foreprinciple" is here applied as an explicit statement whose truthfulness is not subject to requestioning, but which theoretical mechanics as a natural science (philosophy) about motion of bodies starts from. The preprinciples define the basic starting point of mechanics which is here taken as one of the sciences about nature, instead of an abstract mathematical theory with no determined interpretation. As such, the preprinciples allow for making distinction between mechanics and, for example, geometry which is today no longer considered as a science about real space, but as an abstract formal theory that enables different, equally valuable interpretations. The preprinciples express the gnoseological assumption that mechanics has its determined interpretation as a science about the motion of real bodies.

The requirement for clarity assumes that the preprinciples can be and are expressed both orally and in writing, with no previously introduced concepts and definitions; in this way, it is easy and simple to understand the formulated determinations, consistent with empirically acquired knowledge or hints, all of which being of interest for the theory of mechanics. While describing the motion of bodies the preprinciple represent such assertions that are themselves evident; hence they neither provoke questions nor do they require answers since it is assumed that the answer to accept would be the one given to himself or to others by the very person who posed the question. Therefore, mechanics starts from the accepted assertion which is not called into doubt at any level of knowledge. Broader implications of the preprinciples can be grasped by studying mechanics as a whole. The preprinciples are considered accurate in mechanics until opposed either by a new discovery or experimentally or even by a newly-discovered phenomenon in nature. If and when the scientific assertion, brought into accord with natural phenomena, appears to be contradictory to the preprinciples, it can be modified, together with the corresponding assumptions of thus envisioned mechanics. The preprinciples stressed here are as follows: those of

1. Existence, 2. of Causal determinacy and 3. of Invariance.

The knowledge about motion of bodies dates from ancient times. It has been preserved by genetic inheritance, forms of human practice and a multitude of various records ranging from a millennia-old till the present day ones. The historians of science point to five millennia old records dealing with the motion of bodies. The existing referential literature about the motion of bodies is so large that it considerably exceeds the limits of one congruous rational theory. Even the attempts at formal generalization have reached the sophistication level at which it is impossible to see the knowledge that man needs about the motion of bodies. Numerous definitions that cannot be refuted from the standpoint of the author's right to define his own concepts have first given rise to disparities among the theories of essential concepts which have, in their turn, caused a final split among the existing theories. A rough mathematical description giving intellectually simplified models of natural objects is often used to explain the body's state of motion in a way unfaithful to reality. Besides, hundreds of theorems about the motion of body that are annually published in numerous scientific and professional journals contain incongruous "truths". This is sufficiently provoking for raising the issue of the "proving truthfulness". What is presented here is a new systematization of the rational core of mechanics, able to eliminate incongruity and vagueness of the existing theories. This has required, among other things, that some common and accepted knowledge about principles, laws, theorems, axioms should be averted, given up or modified. It makes sense to expect that such an approach will cause detachment or aversion, especially among older conoisseurs of mechanics, those who have accepted its laws and assertions as indisputable laws of nature. In accordance with the preprinciples and for the sake of greater clarity, the basic issues of this study are explained by the mathematical apparatus with which it is much easier to prove the completion of the preprinciples, especially that of invariance. The knowledge about the motion of bodies is expressed by the introduced concepts and mathematical relations. The findings are evolving, meaning that general knowledge is not given once and for all; hence they do not have to be the same and equally true. The assertions about the motion of bodies, introduced and deduced in this mechanics, considerably differ from many others in numerous works on mechanics, especially in the part describing the motion of the body system with variable constraints.

Ontological assumptions. On the basis of inherited, existing and acquired knowledge, mechanics starts from the fact that there are:

bodies, distance, time

The existence of a body is manifested in the theoretical mechanics as a body mass for which the denotation m and its property or attribute M, (attr m = M) are accepted. Consequently, every existing body has its mass. This is the property by which the body existing in mechanics differs from the geometrical concept of the body characterized by volume L (Lat. Volumen). The difference is fundamental since the body mass is not even quantitatively identical with its volume whose dimension is derived by means of the dimension of length L (Lat. Longus - long), attr $V = L^3$. Each body whose motion is studied in mechanics has its mass regardless of how small it is or of the size of its volume. The body of no matter how small volume V has a finite mass m. Likewise, each part of the body has its mass. A part of the body of volume V has mass m. If many bodies or parts of the bodies are dealt with, their masses are successively denoted with the indices m, that are to be read in the following way: "mass of the body", 'mass of the ν-th part of the body". If it is the ν-th existing particle of mass m, it is to be written "a particle of mass m". No matter what natural numbers are added to the index ν, $(n_i = 1, 2, \dots,)$., masse m are always determined with positive real numbers, designated by units of mass M dimension. The existence of distance is identified everywhere: among particles, celestial bodies or between various points on the pathway that the body is moving along, as well as between the place of the body and the place of observation; it is denoted by the letter l (Lat. Longus) and is measured in units of length L. Though it is directly perceived or observed, inherited, acquired and understood, the distance between the body's place or position cannot be simply determined. In order to confirm this assertion it is sufficient to mention the distances between two airplanes in the air, two vessels on the sea, two vehicles on the road of the rough terrain, two pedestrians in the city. The distances are also the subject of other sciences, especially metrology - measure, measuring standard, science, astrometry - star), geometry, Earth and topology, place, since they depend on the shape of the medium which the body's positions belong to. Any common trait can be, therefore, deduced only for very small distances between the adjacent points; even so, only under the conditions that the backgrounds against which the distances are being observed are not degenerative. The positions of two bodies, no matter how small particles they are, cannot coincide, and therefore their distance must be different from zero although this is contradictory to the obvious fact that there are no distances between the bodies touching each other. Regardless of how small a particle is, it is not a point, but in determining the distance, let us agree, that it should be a singular point of the particle or of the body in general, that is, the one that can be adjoined by the mass of the particle or of the body in general in such a way that the whole body mass is concentrated at this point which thus becomes a fictitious mass center. It is for this reason that this point is called the mass point or material point. In this way the question of the bodies' distance is reduced to the concept of the distance between points. The concept of the mass or material point differs from the geometrical concept of the point not only by the fact that the mass point is characterized by mass; it differs from the particle by the fact that the distance between the two particles always exists and does not equal zero, because the particles, in addition to their mass centers, also have boundary points of their volume. In this way, the mass or material point is represented by the mass and position (m, r). The geometrical points can coincide, and therefore their distance can be equal to zero. The mass point position relative to any chosen observation point can be described by the position vector $\mathbf{r}, r \epsilon\ R^3$, where the symbol R^3 implies a set of real tri-vectors or in numbers $r := (r^1, r^2, r^3)$ that are connected with three linearly independent vectors to be called the base or coordinate vectors and denoted by the letters $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ or $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$. The notation $\mathbf{e}$ will be used for orthogonal vectors of unit intensity $\mathbf{e}_i$, (i = 1, 2, 3), $e_i = 1$, will be used for other unit vectors of rectilinear coordinate systems. In addition to the assumption that they are unit and orthogonal, there is another assumption that $\mathbf{e}_i$ change neither direction nor sense, and consequently they are constant. $\mathbf{e}_i = const$. Note that this assumption about

the constancy of the base vectors direction has no place in the philosophy of the body motion because all bodies on which the vector base is chosen are moving. Mechanics introduces this assumption conditionally to be discussed below in the introduction of the velocity definition and explanation of the inertia force. Relative to base **e**, position vector $\mathbf{r}\epsilon R^3$ can be written in its simplest form in the following way

$$C\mathbf{r} = r_1\frac{m\mathbf{v}}{dt} = \mathbf{F}, \quad \mathbf{e}_1 + r_2\frac{m\mathbf{v}}{dt} = \mathbf{F}, \quad \mathbf{e}_2 + r_3\frac{m\mathbf{v}}{dt} = \mathbf{F}$$

where the iterated indices, both subscript and superscript, denote addition till the numbers taken by indices; $(r_1, r_2, r_3)\epsilon^3$ are coordinates of vector **r**, and $r_1\mathbf{e}_1 = \mathbf{r}_1, \ldots, r_3\mathbf{e}_3 = \mathbf{r}_3$ are covectors or components of the given vector. Scalar multiplication of vector r by vectors $\mathbf{e}_j (j = 1, 2, 3)$, i.e. $r^j\mathbf{e}_j$, gives the j-th projections r_j of vector r upon the directions of the j-th vectors e_j. Only with respect to base e, vector rj coordinates are identical to its projections r_j or to covariant coordinates r_j, because it is

$$\mathbf{e}_1 = \text{const.} \tag{1.3}$$

Note that this assumption about the constancy of the **base vectors direction** has no place in the philosophy of the body motion because all bodies on which the all bodies on which th vector base is chosen are moving. Mechanics introduces this assumption conditionally to be discussed below in the introduction of the velocity definition and explanation of the inertia force.

Relative to base **e** position vector $\mathbf{r} \in \mathbf{R}_n$ can be written in its simplest from the following way

$$\mathbf{r} = r^1\mathbf{e}_1 + r^2\mathbf{e}_2 + r^3\mathbf{e}_3 + = \sum r^i\mathbf{e}_i = r^i\mathbf{e}_i, \tag{1.4}$$

where the iterated induces, both subscript and superscript denote addition till the numbers taken by indices, both the iterated induces, both subscript and superscript, denote ddition till the numbers taken by indices: (r^1, r^2, r^3) are coordinates of vector **r**, and $r^1\mathbf{e}_1 = \mathbf{r}_1, \ldots, r^3\mathbf{e}_3 = \mathbf{r}_1$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

Observed from any point O, which the position vectors start from, the directed distance between any two immediately close points M_1 and M_2 is determined by the difference between vectors $r^2 - r^1 = \rho$, where $\mathbf{r}_2 = \vec{OM}_2\mathbf{r}_1 = \vec{OM}_1$ and

$$\Delta\mathbf{r} = M_1\vec{M}_2 = (r_2^i - r_1^1)\mathbf{e} = \Delta\mathbf{r} = N_1\vec{M}_2 = (r_2^i - r_1^1) = \Delta r^i{}_i.$$

Quantity $\Delta s = \Delta\mathbf{r}$ can be called the metric distance or distance ds (Lat. spatium - space, interspace, distance) or distance Δs,

$$attrs = L. \tag{1.6}$$

Time is denoted by the letter t (Lat. tempus), while its attribute T. It is continuous and irrevocable. In the mathematical description it can be represented by a numerical straight linear an ordered multitude of concrete numbers, while the multitude of their units is represented by real numbers$R$ end t.

Once the existence of time is accepted, the existence of motion, change, duration, the past, the present, the future is also accepted.

$$\gamma_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}. \tag{1.7}$$

Time is denoted by the letter t (Lat. tempus), while its attribute T (atr. attr.$t = T$).

It is continuous and irrevocable. In the mathematical description it can be represented by a numerical straight line or an ordered multitude of concrete numbers, while the multitude of their units is represented by real numbers, Once the existence of time is accepted, the existence of motion, change, duration, the past, the present, the future is also accepted.

Preprinciple of casual determinacy. Distances, their changes and other factors of the body motion are explicitly determined throughout the whole of time, in the future as in the past, and with as much accuracy as the determinants of motion are known at any particular time moment. This preprinciple of mechanics prefigures that mechanics as a theory of body motion is an accurate science in the mathematical sense, while as an applied science, it is so accurate as the data which are of importance for motion are accurately measured at one particular time moment. In other words, mechanics is an accurately conceived theory, almost to perfection, and in engineering practice it is as much applicable as it is known, depending on the needs and technical capabilities of those applying it. The concept of the body motion comprises: walking, driving, sailing, swimming, flying, jumping, breaking and all other gerunds that express displacement and changes of distance or changes of the position vector in time.

Preprinciple of invariance. Neither motion nor properties of the body motion depend upon the form of statement: the determined truth about motion, once it is written in some linguistic form, is equally contained in the written output of some other form or some other alphabet. The preprinciple of existence states that there are mass, time and distance, determined by concrete real numbers m and t and real vector $\mathbf{r}$.

This preprinciple of existence or independence of formalities allows for mass, as well as time, to be denoted by some other letters, which do not change the nature of numbers m and t, and for which there must be in the whole correspondence. The same holds for distance r. No matter where the origin of coordinates from which the position vector begins is chosen, let's say $\mathbf{v}$, there exists an equality $\mathbf{r}$, and therefore distance r does not depend on the form of writing. This is even more expressed in the coordinate form, where the choice of forms is considerably larger, such as,

$$\mathbf{r} = y^i \mathbf{e}_i = x^i \mathbf{g}_i. \tag{1.8}$$

As such, all the three realities R, t, and $\delta\rho$. and m and t invariants, m and t are invariants, and t, being scalar ones, and $\mathbf{r}$ is a vector invariant. All other factors of the body motion are also invariantly expressed in various coordinate systems.

MOND2 - BASIC DEFINITIONS OF MECHANICS

Mathematical logical theory of natural sciences requires the determination (definition) of essential concepts by. of which the whole theory, such as that of mechanics, develops. Newton set out his mathematical principles of motion from eight basic definitions to determine the following concepts:

1. Mass,
2. Impulse or momentum motion,
3. Inertia force,
4. Force in action
5. Centripetal force,
6. Absolute magnitude of centripetal force,
7. Vector of the point;s acceleration,
8. Accelerating magnitude of centripetal force.

Instead of above mentioned Newton's basic definitions and their interpretations, five general definitions are sufficient for developing theoretical mechanics, such as:

1. Velocity,
2. Impulse or momentum motion,
3. Acceleration,
4. Force of inertia'
5. Action of force

Only two of Newton's definitions, the second and third, as evident, coincide with the basic definitions given herein. This approach should not be taken as a negation of Newton's mathematical principles of the theory of body motion but as a modification and improvement of the description of body motion in terms of Newton's IV rule of causal reasoning. Mentioned properties or characteristics have been referred to as dimensions and were written as

$$dim\, m = M, dim\, l = L, dim\, t = T.$$

However, in mathematics the term 'dimension' is most commonly used to denote the number of units in a multitude and in physics it is used as a unit of measurement property, and therefore we will herein refer to the natural properties of the body motion as attributes1 and write them as:

$$attr\, m = M, attr\, l = L, attr\, t = T. \tag{2.1}$$

Obviously, the basic properties or attributes are real numbers and are completely subject to the calculation rules with real numbers. In classical and standard analytical mechanics there are concepts of different properties. Mass, energy and action are thus described by their property numbers, i.e. scalar calculus, and basic concepts: radius vector, velocity, acceleration, impulse, moment of impulse, force and moment of force are all defined by the concept of vector for which there has not been given a unique general definition so far. This can be shown by a number of examples.

In the book ([4] p. 82) it is said: Point and vector are the essential concepts not subject to direct logical definition, three axioms being added:

1^0. There is one point at a minimum.

2^0. For each pair of points A, B, specified in a certain order, only one vector is assigned.

3^0. For every point A and every vector $(\mathbf{x})$ there is one and only one point B, such that

$$\vec{AB} = \mathbf{x}.$$

4^0 (Parallelogram axiom.) If

$$\vec{AB} = \vec{BD}, \tag{2.2}$$

then

$$\vec{AC} = \vec{BD},$$

. In the book [5], a) the concept of vector is introduced by means of the concept of vector space, which is basically contrary to the concept of the agreement of vectors; it is consequently reduced to the concept of parallelogram, or the concept of

a) middle of the pair of points. Vector space (linear space) is a multitude of elements M, called vectors, for which there are two mathematical operations: addition and multiplication of vectors by numbers.
b) scalar product of two vectors is described, and
c) orientation, i.e. the sign of scalar product, $u\dot{u} > 0$, is described.

So, there is no general definition of vectors, but there is one of a "free vector:"

Definition 15.1: For every pair of points (x, y) in the plane of free vector (x, y), there is the transfer of plane Π which translates x into y; such translation is commonly denoted by the symbol (xy).

In the book [6] it is established: In an arbitrary coordinate system the set of coordinates λ^i defines at every point of space a vector whose length, orientation and sense are determined by λ^i. Vector λ at some point P is a diagonal from point P in the parallelogram, whose sides are the lengths of vector $\lambda^1, 0, 0, \lambda^2, 0, 0, \lambda^3, 0, 0$.

In the book ([7], p. 1) it stands on the first page: geometrical, mechanical and physical quantities, whose complete determination, apart from length, requires the knowledge of orientation and sense, are called vectors. A vector is thought of as an oriented line segment with an arrow at one end to denote the sense.

In the book ([6], p.7) it is written: The general concept of a vector - λ^i ordered set of N numbers taken from some number field is called the vector of the N-th order over that field. λA set of all vectors of a certain order over the number field, closed under the operations of addition and multiplication by numbers, is called a linear system (space) of a vector or vector space. A vector is an element of the vector space V, or $v \dots V$ for short.

In the book ([7], p. 23) we can read: A vector in the RN space means two points, so that it is exactly known which point is the first (initial) and which point is the second (terminal) one of a vector. Vectors which determine the position of points with respect to a certain pole are called position vectors.

In the book ([8], p.57) we can read: Vector is a quantity determinable in every coordinate system by three numbers (or functions) A^i which transforms into A_i* under the space coordinate system change, according to the law

$$A_i* = \alpha_i * kA_k. \tag{2.3}$$

In a highly estimed book [9] it is essentially precisely written:

Definition. The vector at point $P = (x_o^1, \dots, x_0^n)$ is called z set of numbers a $(\xi_0^1, \dots, \xi_0^n)$, with respect to the system of coordinates $(x^1, \dots, x^n)$. If two systems of coordinates $(x^1, \dots, x^n)$ and $(z^1, \dots, z^n)$ linked by alpha $x = x(z)$, where $x^i(z_0^1, \dots, z_0^n) = x = x_0^i$, $i = 1, \dots, n$, for a new system of coordinates z that very same vector at the point $z_0^1, \dots, z_0^n$ is specified by another set of numbers $\zeta^1, \dots, \zeta^n$, which are linked by the initial formula (20*)

It should be emphasized that the 'meat' of this definition is in the form of the rule of transformation (20*).

In the 6th edition of a rarely good textbook ([11], pp.18,19) we find the following: *In mathematical physics two types of quantities are encountered: scalar and vector.* Scalar is a quantity which is completely determined by the number and which expresses the relation of that quantity to a corresponding unit of measurement. The term 'vector' derives from Latin word 'where' meaning 'drag', 'tow'. As it is well-known from analytical geometry, cosines of three angles, included by the straight line l of any vector with coordinate axes, related by the relation

$$\cos^2(l, x) + \cos^2(l, y) + \cos^2(l, z) = 1;$$

consequently, vector direction is determined with two numbers, taking into account the numeric meaning of the vector; geometric vector can be represented by the segment AB.

It is interesting to note that in a dictionary [12]13 the term vector is mentioned just once on page 181 as an entry complex plane and is used in a sentence: Sometimes a complex number z is represented by the radius vector of that point.

Vector on the complex plane On the basis of all above mentioned formulations, it can be concluded that vectors can be viewed from two different observation points. According to the first, vectors are numbers subject to certain mathematical operations, whereas the second views vectors as mathematical concepts that have no real numbers and inseparably another two determinants indicating orientation and sense.

If this triplet misses at least one of the three mentioned determinaces, it is not a vector.

2.1. The general concept of vector. *vector is a triplet of numeric value, orientation and sense.*

In the professional and scientific literature it is denoted with a symbol $\mathbf{v}$, or bolded letter v. Numeric value v of the vector $\mathbf{v}$ is a scalar, therefore v can be multiplied by a scalar.

Axioms or rules vectors.

Axiom *I*. The sum of two vectors v_1 and v_2 is the vector $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{d}$ of magnitude d and oriented diagonal closed by addends $\mathbf{v}_1$ and $\mathbf{v}_2$. Axiom *II*. The vector product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in a homogeneous medium is the vector $\mathbf{M}$, orthogonal to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, that is,

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{M}, \quad \mathbf{v}_1 \perp \mathbf{M}, \mathbf{v}_2 \perp \mathbf{M}. \tag{2.4}$$

Axiom *III*. Two vectors are equal if both the numeric value and direction are equal. Inner axiom. The scalar product of two vectors translates vectors into scalars - real numerical functions. Axiom *I* requires that two vectors intersect, and

Axiom *II* enables the displacement of a vector parallel to itself.

Inner axiom allows for leaving vector calculus, base and coordinate vectors being omitted, whereas numeric values of vectors and their coordinates.

Point position vector or radius vector is often used within the general concept of a vector. Our introduced definitions have to be harmonized with the preprinciple. In that regard, let us analyze the velocity vector. Velocities exist in a place where they do not exist apparently. Everything moves. While sitting and being apparently still, we move in two ways: together with the Earth around its center of rotation and together with it around the Sun. The airplane's flight from one place in the east to another place in the west will take less than the other way round by the same clock. Our planet revolves round the Sun at the speed somewhat lower than 30km/sec. It is considerably faster than the speed at which a bullet or a grenade travels. Therefore, velocities exist everywhere around the observer, and in himself/herself too. The preprinciple of existence is satisfied. However, as there is a variety of velocities of the bodies, liquids, clouds and light, mechanics chooses a model that can be used to describe as many motions as possible. The velocity at some point is determinable with mathematical accuracy by definition 1. As such, it is mathematically accurate to the level of perfection, and it is applicable as much as the point position vector is accurately determinable, as well as the time moment at which the velocity is being determined. In this way, the preprinciple of causal determinacy is satisfied. The preprinciple of invariance requires a more thorough analysis, because of the existing incongruent definitions of the concept of a vector in modern mathematics. Vectors are, by definition, linear mathematical objects of wide application. And yet, mathematics and mechanics do not define the concept of a vector in the same way, as has been shown. Let us accept the fact that a point is the basic concept that is not defined. The point is the point, but the point position is not definable in itself, because it is defined by means of other objects. It is an intersection of two lines, or of three planes or surfaces in space; at three distances from two mutually normal walls in a room and height from the floor to... Elementary geometry uses coordinate systems where every point is denoted and determined by three coordinates of its position or by radius vector

2.2. Position vector. Although this is just one of several types of vectors, it is the one most often used to describe the concept of a vector at the point. Consequently, the point position vector is in the focus of our attention, because our approach to this type of vector considerably differs from standard definitions and operations on it. The point is a basic concept, so it is not explained. And yet, let us add, it is what You, estimed reader, understand and know. However, the point position is not such a simple concept. In geometry, it means a single place (point) determined with respect to another object(s) - points, lines straight or curved, planes or surfaces.... with respect to one or more than one observers. For the observation point of a single observer, we are going to call a pole and denote with the letter Q_i is sufficient to know the distance, orientation and sense in order to determine the position of the observed point M; in other words, one vector r is sufficient. The length or distance from one point to another is measured by units of length L (from Latin longus). That vector used to measure the point position is called the point position vector. Standard theory states that the beginning of that vector is at pole Ox and end, denoted by an arrow, is at the respective point M.n doing so, it is necessary to know that the position vector pole is an arbitrary point, so there are arbitrarily many of them and they determine only one position of one real existing point.

Example. Let position vector **r** of point M be an oriented line segment.... where O is the observation point arbitrarily chosen for the beginning of the observed vector. In other words, it is the point position vector or, to put it simply, the point. On the vector axis other points O_i can be chosen for the vector beginning as far as the immediate vicinity of the point, and even point M itself. In doing so, the position of point M will not change. All vectors determine the position of point M, (Fig. 1).

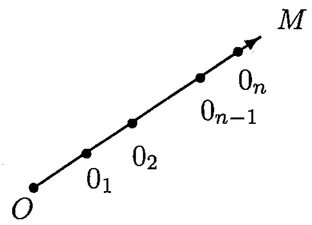

FIGURE 1

A more vivid description is provided by an example of two points (Fig. 2) at mutual distance

In the professional literature Fig.3 is widely used

$$R_A + R_B - F_C = 0, \qquad M_A + M_B - M_C = 0$$

and

$$M_A = lF_C - 2lR_B, \quad M_C = lR_A - lR_B, \quad M_B = 2lR_A - lF_C,$$

indicating that vector addition is performed according to the triangle rule, which is incongruent with the vector addition axiom.

By that axiom I, Fig. 4, the difference (2.5), Fig.5. The start point O should be the intersection of all the three vectors. In order to apply the rule of vector addition, whose axes do not intersect, i.e. to reduce it to the parallelogram diagonal formula, it is necessary to deploy the vector product (refer to V.Vujičić, Statics, Belgrade, 1969) to transfer the vector of diagonal d, i.e. to point O or M_0. So, if point M does not change its position, the change of position vector poles does not affect the change of distance between the two points. However, if the point changes its position

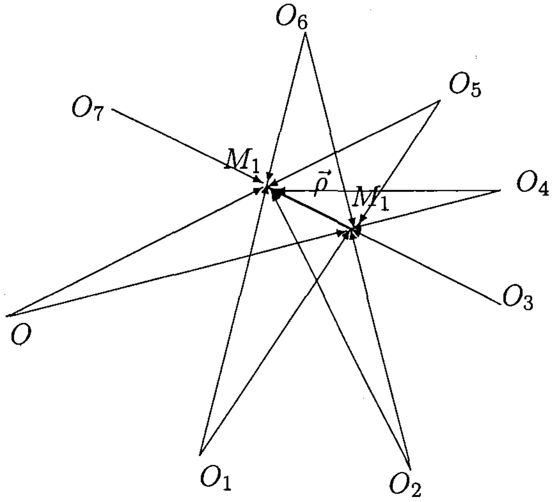

FIGURE 2

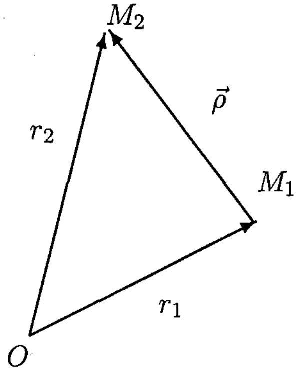

FIGURE 3

anyhow, then it is convenient to choose the vector pole at the initial position of M0 that belongs to the hodograph of a vector function. Such a reflection can be supported by the first theorem of Newton's fundamental work of Mathematical principles of natural philosophy.[3]

Theorem 1. The areas, which revolving bodies describe by radii drawn to an immovable center of force do lie in the same immovable planes, and are proportional to the times in which they are described.

The title itself indicates that the position vector and determines the point position. The start point of that vector, called the pole-orient or

[3]V. Vujičić, Forces of central motion according to Newton and Petronijević, CNOSOS, Belgrade, 2005.

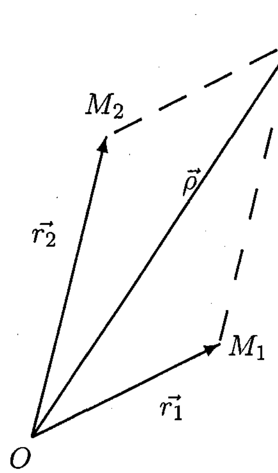

FIGURE 4.

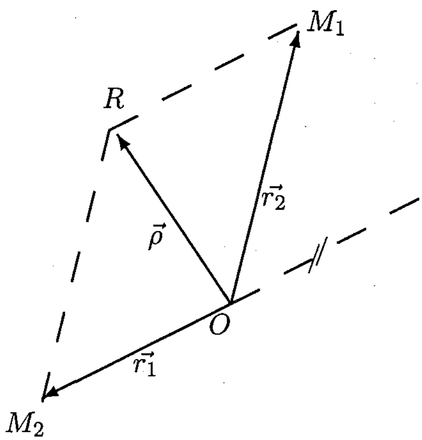

FIGURE 5

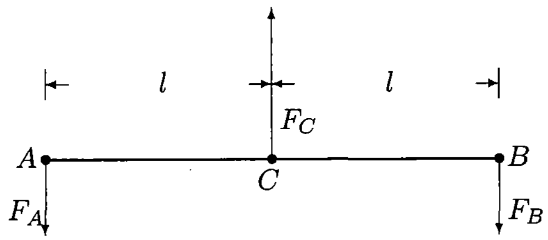

FIGURE 6

observation point, can be arbitrarily assumed to be relatively at rest, and therefore the observation point position is determined according to the chosen point. This is the base of Euclidean geometry of absolute space. Point position is determinable by the intersection of three planes or three surfaces. The intersection for two planes or surfaces determines the straight or curved lines intersecting as such at the pole and representing the rectilinear or curvilinear coordinate systems. Let us denote the coordinates of the point position with respect to the orthogonal rectangular coordinate system by y^1, y^2, y^3, and with respect to the curvilinear coordinate system by x^1, x^2, x^3.

Position vector is invariant with respect to all these and other coordinate systems, so it can be written that:

$$\mathbf{r}(y) = \mathbf{r}(x), \tag{2.6}$$

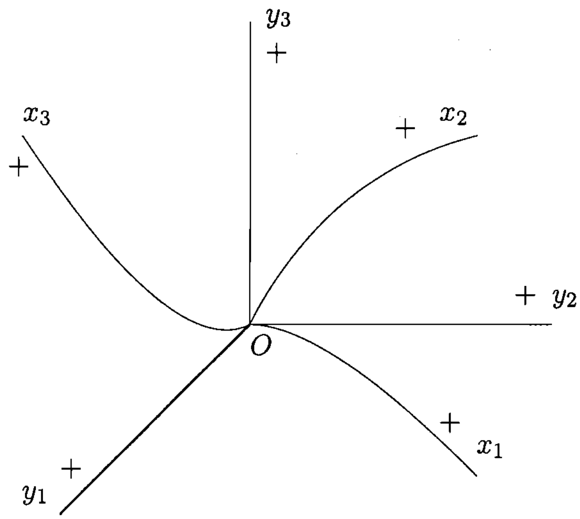

FIGURE 7

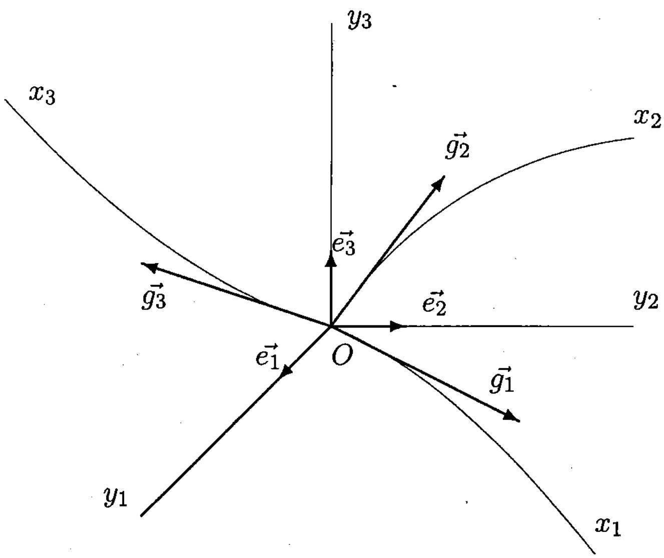

FIGURE 8

or in more detail

$$\mathbf{r} = y^i \mathbf{e}_i = x^i \mathbf{g}_i; \quad (i = 1, 2, 3), \tag{2.7}$$

where the same indices repeated in superscript and subscript denote summation over those indices. No doubt, it follows from here that:

$$\frac{\partial \mathbf{r}}{\partial y^i} = \mathbf{e}_i, \quad \frac{\partial \mathbf{r}}{\partial x^i} = \mathbf{g}_i. \tag{2.8}$$

Based on (2.7) and (2.8) it follows that:

$$\mathbf{r} = y^i \frac{\partial \mathbf{r}}{\partial y^i} == \frac{\partial \mathbf{r}}{\partial x^i}.$$

In the existence of the functions

$$y^i = y^i(x^1, x^2, x^3) \Longleftrightarrow x^i = x^i(y^1, y^2, y^3),$$

it is shown that

$$y^i = \left(\frac{\partial y^i}{\partial x^k}\right)_{\overset{k}{x_o}} x^k, \tag{2.9}$$

because

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial \mathbf{r}}{y^j}\frac{\partial y^j}{\partial x^j} = \frac{\partial y^j}{\partial x^i}\mathbf{e}_j, \tag{2.10}$$

where partial derivatives of the vector-function are considered partial derivatives at points O of a vector.

It should not be overlooked in all above mentioned that partial derivatives of the vector and its coordinates are calculated for an indicated point.

The differential of the position vector $\mathbf{r}$ is indisputable

for condition that $|\frac{\partial y^i}{\partial x^j}|_3^3 \neq 0$ is c.her than zero. It should be noted that solid heavy line denotes the differential of a vector $d\mathbf{r}$, which substantially differs from the differential dr_i of vector coordinates. This non-standard differential of a vector differs from the standard differential of scalar functions $dr^i(t)$. For the derivative of a scalar function $f(t)$ for an independent variable, two mentioned differentials are identified, i.e, $df(x) = Df(x)$. An infinitely small value of the position vector equals the differential ds of displacement s, i.e. $d\mathbf{r}$. Therefore, scalar product of two identical vectors yields

$$d\mathbf{r} \cdot d\mathbf{r} = ds^2.$$

Also, note that the position vector can be written in other forms such as:

$$\mathbf{r} = \mathbf{x}_i + \mathbf{y}_i$$

vectors y^i and x^i are vector coordinate of vector $\mathbf{r}$, whereas scalars y^i and x^i are coordinates of that vector.

Invariant relations (1.5) and (1.7) indicate that coordinates y^i and $4x^i$ are function-correlated $y^i = y^i(x^1, x^2, x^3)$ and vice versa $x^i = x^i(y^1, y^2, y^3)$ in the region of those functions' continuity. Thus, there are differentials

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j \leftrightarrow dx^j = \frac{\partial x^j}{\partial y^i} dy^i, \tag{2.11}$$

for condition $|\partial y^i partial x^j| \neq 0$. Note that indices i over coordinate y^i of vector $\mathbf{r}$ in transformation (1.10) emerge as free indices, which makes them different from summation indices repeated in transformation

$$y^i = \left(\frac{\partial y^i}{\partial x^j}\right)_{x_M} x^j., \tag{2.12}$$

because

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial \mathbf{r}}{y^j}\frac{\partial y^j}{\partial x^j} = \frac{\partial y^j}{\partial x^j}\mathbf{e}_j, \tag{2.13}$$

where partial derivatives of the vector-function are considered partial derivatives at at point O of a vector.

It should not be overlooked in all above mentioned that partial derivatives of the position vector $\mathbf{r}$ is indisputable for condition that

2.3. Velocity definition. Velocity at some point and time t, whose position is determined by position vector $\mathbf{r}$, is change of that vector with respect to time, that is,

$$\mathbf{d}y^i = \frac{\partial y^i}{\partial x^j} dx^j \longleftrightarrow dx^j = \frac{\partial x^j}{\partial y^i} dy^i, \tag{2.14}$$

$$\lim_{\Delta t \to 0} \frac{\mathbf{r}}{+\Delta t) - \mathbf{r}(t)} \Delta t = \frac{d\mathbf{r}}{dt}, \longleftrightarrow \mathbf{v}_0 = LT^{-1}.$$

and

$$attr\,\frac{d\mathbf{r}}{dt}, \Longrightarrow v_0 = LT^{-1}. \tag{2.15}$$

and for $|\frac{\partial y^i}{\partial x^j}|_3^3 \neq 0$.

The concept of velocity is widely used in mechanics, in formulations such as: velocity vector, momentous velocity of the material point, angular velocity, relative velocity vector, All velocities at a certain position are vectors. This fact makes mechanics a linear science of motion or relative rest of the body if it is subject to vector calculus at a certain time moment. Velocity is a vector invariant. As such, it can be written in the forms as follows:

2.4. Motion impulse definition. The product of mass m of the material point and its velocity vector v is called the motion impulse of material point P. In accordance with the preprinciples, velocity definition and definition above given, the motion impulse can be written in the following forms:

$$\mathbf{p} = m\mathbf{v} = m\dot{y}^i\mathbf{e}_i = m\dot{x}^i\mathbf{g}_i. \longrightarrow atr\, MLT^{-1}. \tag{2.16}$$

In the exposition below, special emphasis will be placed on coordinates p_i of impulse p, as well as on base vectors that are point position functions. The impulse vector coordinates are equal to the respective products $p_i = \mathbf{p}\cdot\mathbf{g}_i = mg_{ij}\dot{x}^j = a_{ij}\dot{x}^j$. They are measured or calculated up to the required accuracy of masses m and velocity coordinates. However, for the motion impulse it is clear enough that the velocity vector coordinates used are projections of that vector upon the axis coordinates, i.e. where, as evident, the material points differ with respect to the base scalar products of impulse vector p and coordinate vectors. Due to the preprinciple of invariance and determinacy of vector $\mathbf{g}_j$, it follows:

$$\mathbf{p}\cdot\mathbf{g}_i = m\dot{x}^i\mathbf{g}_i\cdot\mathbf{g}_j = mx^i g_{ij} = a_{ij}\dot{x}^j = p_i. \tag{2.17}$$

Such projections, denoted by the subscript index, represent motion impulses, which are also called, due to the subscript index, covariant coordinates of the impulse. It should be noted that tensor $a_{ij}(m, x)$ differs from geometric metric tensor gij. Tensor aij satisfies the equality of the geometric metric tensor form, in property L; tensor a_{ij} contains masses of material points, and therefore as such it corresponds to the term mass, material, or inertia tensor.

2.5. Acceleration definition. The natural derivative of the velocity vector with respect to time is called the vector of the point's acceleration and is written as:

$$\mathbf{w} = \frac{d\mathbf{v}}{dt}, \quad attrw = LT^{-2}. \tag{2.18}$$

In mathematical analysis and kinematics there are some disagreements with respect to different coordinate systems and their coordinate vectors that have to be overcome herein. When differentiating the velocity vector, one encounters the scalar coordinates of the velocity vector but also coordinate vectors occurring as the coordinate functions of the point position. If the coordinate vectors are the base ones that are invariable, the second derivatives of the point position vector coordinates are equal to ordinary derivatives with respect to time of the scalar coordinates of the velocity vector. However, if the coordinate vectors are the functions of point coordinates and by means of them the functions of time too, then they are subject to differentiation as complex functions. If the x coordinate is an angle, then in accordance with a definition and its corresponding relations acceleration vector$w$ (Latin, acceleration) is $w = LT^{-2}$. In differential geometry and analytical mechanics there is a widely spread approach that "acceleration vector is not a tensor in the tensor sense", meaning that it does not maintain the natural property of acceleration. This misconception needs to be clarified. It is not deniable that acceleration can be written in the following forms:

$$\mathbf{w} = w^i(y)\mathbf{e}_i = \ddot{y}^i\mathbf{e}_i = w^i(x)\mathbf{g}_i = w^i(x)\frac{\partial \mathbf{r}}{\partial x^i} = w^i(x)\frac{\partial \mathbf{r}}{\partial y^j}\frac{\partial y^j}{\partial x^i} = w = \frac{\partial y^j}{\partial x^i}\mathbf{e}^j.$$

This shows conclusively enough that acceleration and its coordinates transform in accordance with the tensor calculus. Yet, the relation deriving from above

$$\ddot{y}^i = \frac{\partial y^i}{\partial x^j}(\frac{dv^j}{dt} + \Gamma^j_{ik}(x)v^j\frac{dx^k}{dt})$$

suggests, at first sight, the remark that left-hand and right-hand sides are not symmetrical in the tensor sense - on the right-hand side there no correlation coefficient $\Gamma^i_{jk}(y)$. That remark is formally justifiable, but not essentially, because correlation coefficients $\Gamma^i_{jk}(y)$ equal zero, considering that

$$\frac{d}{dt}\frac{\partial \mathbf{r}}{\partial y^j} = \frac{d}{dt}\mathbf{e}_j = 0.$$

So, this objection is unacceptable and the remark that the velocity vector is not a vector in the tensor sense is disregarded. An acceleration vector invariant is proved. Accordingly

$$\ddot{y}^i = w^k\frac{\partial y^i}{\partial x^k} => w^i(y)\mathbf{e}_i = w^i(x)\mathbf{g}_i.$$

acceleration is invariant under tensor or linear transformations, i.e. differentials, that is,

$$w_i(x) = \left(\frac{\partial y^j}{\partial x^i}\right)_{x^i_o} w_j(y); \quad w_j(y) = \ddot{y}_j.$$

Let us demonstrate above stated with a simple **example** of point acceleration with respect to the cylindrical system of coordinates $y^1 = \rho\cos\theta, y^2 = \rho\sin\theta, y^3 = \zeta$. which are correlated with coordinates y^i as the following functions:

$y^1 = \rho\cos\theta, y^2 = \rho\sin\theta,\ y^3 = \zeta.$

$$\ddot{y}_1 = (\ddot{\rho} - \rho\dot{\theta}^2)cos\theta + rho\ddot{\theta} - 2\dot{\rho}\dot{\theta})\sin\theta = \ddot{\rho} - \rho\dot{\theta}^2,$$

$$\ddot{y}_2 = (\ddot{\rho} - \rho\dot{\theta}^2)\sin\theta + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\cos\theta = \rho\ddot{\theta} + 2\dot{\rho}\dot{\theta},$$

$$\ddot{y}_3 = \ddot{\zeta}.$$

The square of the acceleration is

$$w^2 = \delta_{ij}\ddot{y}^i\ddot{y}^j - (\ddot{\rho} - \rho^2)^2 + \rho^{2\cdot} + \frac{2}{\ddot{\rho}}^2 + \ddot{\zeta}^2.$$

The same results are obtained by means of Christopher's symbols, the corresponding Christopher's symbols being calculated or known and equaling zero for coordinates y^1, y^2, y^3, equal zero, the acceleration is invariant with respect to tensor transformations as much as the velocity.

2.6. Inertia force definition. The product of the material point's mass m and acceleration $\mathbf{w}$, which is equal but directed opposite to acceleration $\mathbf{w}$, is called the material point's inertia force. If the inertia force is denoted by the letter $\mathbf{I}_F$ or simply I, the definition can be written in a shorter form

$$\mathbf{p} = m\mathbf{v} = m\dot{y}^i\mathbf{e}_i = m\dot{x}^i\mathbf{g}_i. \longrightarrow attr\, MLT^{-1}.$$

The product of the object's mass and acceleration vector is the inertia force, i.e. it follows that (2.15). This significant definition establishes the property of every force. In accordance with relations (2.14) and (2.16), it can be written F by means of the product MLT, that is,

$$\mathbf{I} = -m\mathbf{w} = -m\frac{d\mathbf{v}}{dt}. \tag{2.19}$$

As such, the force changes velocity in time, where from it is evident that *vector coordinates are inertia forces proportional to acceleration*, whose proportionality factor is the mass of the object or body. Mass is a representative of every body as $I^i = -m\frac{Dv^i}{dt}$ of the totality and is related to a single point - the center of mass or the center of inertia

$$-m\Big(\frac{dv^i}{dt} + \Gamma^i_{jk}v^j\frac{dx^k}{dt}\Big) = m\frac{Dv^i}{dt}. \tag{2.20}$$

That point is called the material or mass point. It differs from the geometrical or topological point in that that besides position L it represents the mass of the body M (2.17) where

$$a_{ij}(m, x) = a_{ij}\frac{Dv}{dt}.$$

2.7. The action of the force definition. The action of the force is a natural integral invariant with respect to all coordinate systems y, x, q.

$$\mathcal{A}(\mathbf{F}(y)) = \mathcal{A}(\mathbf{F}(x)) = \mathcal{A}(\mathbf{F}(q)),$$

such as

$$\mathcal{A}(\mathbf{F}) = \int_{t_0}^{t}\Big(\int_{y_0}^{y} F_i(y)dy^i\Big)dt = \int_{t_0}^{t}\Big(\int_{x_0}^{x} F_i(x)dx^i\Big)dt = \int_{t_0}^{t}\Big(\int_{q_0}^{q} Q_k dq^k\Big)dt,$$

where Q_k are generalized forces corresponding to generalized independent coordinates q.

For inertia forces $\mathbf{I} = -m\frac{d\mathbf{v}}{dt}$ the action of the inertia force is

$$\mathcal{A}(\mathbf{I}) = -\int_{t_0}^{t}\Big(\int m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}\Big)dt = -\int_0^t m\frac{v^2}{2}dt.$$

We point out that here we give our definition of the action of the force, whose property is $attr\, A(\mathbf{F}) = ML^2T^{-1}$. The term action indicates that something is being done or has been done in a time interval, such as: the action of medication during 7 days, a three-month's action of aviation, the Sun's action from 11am to 4pm,. In the professional literature we encounter the following: "the force is the action", the action of the force, the action in Lagrangian mechanics, the amount of action, the action in Hamiltonian mechanics, "14, but there are textbooks of mechanics, i.e. of physics for secondary schools or universities, where therm "action" cannot be found in the register of physical quantities either. The Institute for textbooks and teaching aids from Belgrade has published a color overview of "International system of measuring units" adopted in 1960 by the General assembly for weights and measures. In that overview there is neither the word "action" nor its measure. A more comprehensive second edition of "The engineering-mechanical engineering manual" was published in 1992 in three volumes, but action does not exist, nor does 'the magnitude of action". This brief remark about the concept of the action leads to the question: Is the concept of action inessential for physics or is it nonuniform, and as such it is not accepted in scientific and professional community; or, is it renamed by other terms, for example, momentum, whose physical dimensions equal the product of properties, such as mass M, length L and time T^{-1}, i.e. $\mathrm{attr}\, ML2T^{-1}$. This physical property, from the historic and essentially mechanics' viewpoint, and physics' too, represents the attribute of the action, which makes us wonder why the concept of action is not included in mentioned physical properties. Such vagueness is eliminated by our fifth definition, which characterizes the physical property of action A, as incongruity between Newton's and Leibniz's concept of the action. The incongruity of a significant concept of the action requires more detailed explanations [2]. From the historic viewpoint as well as that of essential mechanics, and therefore of physics, represents a dimension of the action. Consequently, our first modification does not change the properties of the action according to Lagrange and Hamilton, which by means of integral variational principles modify the definitions of the fundamentals of classical analytical theory about the motion of the body, so it remains surprising that the concept of the action is not found among mentioned physics' quantities. The first and general modification of Newton's theory was proposed just by mathematicians, along with the development of mathematical analysis. The author of this monograph shows that the action of the force after Euler is comprehensive in mechanics, like analysis in mathematics. Newton wrote his Principia very logically and axiomatically significant for the theory, history and phenomenology of physics, using symbols of Euclidean geometry of segment lines and their relationships. Post-Newtonian dynamics is written using the language and relations of modernized mathematical analysis. In doing so, efforts were made not to change 14 On the nature of the action (refer, for example, to [3]) the nature of dynamic properties by mathematical transformations.

Leibniz's formal action. In the anthology of the variational principles of mechanics a lot has been written about the principles of mechanics, but there is not much deviation ([15], p. 782). In the epilogue and notes, L.S.Polak the editor of that significant work writes: "The first formulation of the concept of action, entitled Actio Formalist, was proposed by Leibniz (Leibniz Gottfried Wilhelm) during his stay in Italy in 1669. Formal action is measured by the product of mass, velocity and length. As such, the dimension of **Leibniz's action** is equal to the expression.

Newton's action force. In 1686 Newton neither defines nor interprets the action, but uses it to define the concept of force: Def. IV. An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line. Newton clarifies his definition by the sentence: This force consists in the action only; and remains no longer in the body when the action is over.

Maupertuis's least action. After Newton, on 20 February 1740, Pierre Louis Maupertuis presented his paper entitled "Agreement of several natural laws that had hitherto seemed to be incompatible" [15] in the Paris Academy of Sciences and published in "Histoire de l' Academie de Science de Paris" in 1744. Let us single out the sentences referring to the concept of Action. "When a body is transported from one point to another, it involves an action. This action depends on the speed of the body and on the distance it travels. However, the action is neither the speed nor the distance taken separately." The least action is the true expense of Nature. Two years later (1746), in his work "The laws of movement and of rest deduced from metaphysical principle" [16] Maupertuis clearly defines the least action as the product of the mass of the body involved, the distance it had travelled and the velocity at which it was travelling. Euler's action of forces.

Euler actions of forces. In 1748 Euler introduced the concept of momentary actions of forces, and then formulated The sum of all momentary actions that a body is subjected to for finite time, i.e. it equals

$$\int dt\Big(\int V dv + \int V' dv' + \int V'' dv'' + \dots\Big)$$

and has the property of the product of $massm, velocityv$ and distance s that is traversed

$$attr\,(mvs) = ML^2T^{-1},$$

which is in accordance with the action of the Leibniz and Maupertuis least action.

Lagrange's action. In his book Analytical mechanics Lagrange implies the concept of action in the same way as Maupertuis, i.e. the sum of products of the masses, velocities and distances, in the form 15Mecanique Analitique par M.U.Lagrange, ([2], pp. 159-166)

$$M\int uds + M'\int u'ds' + M''\int u''ds'',$$

where M, M', M'' are masses of bodies, u, u', u'' are velocities and ds, ds', ds'' are distances traversed. Since $ds = vdt$ the previous expression can be written in the form:

$$\int (Mu^2 + M'u'^2 + M''u''^2)dt = \int 2E_k dt$$

where $2Ek$ is "the living force", i.e. double kinetic energy.

"Planck's quantum of action". In Sommerfeld's paper entitled "Planck's quantum of action and its universal meaning in molecular physics" (1911), it is written: We arrive at a more accurate proposition for the energy magnitude : time if we follow the term quantum of action, very successfully chosen by Planck. "Universal constant chosen as a means of theoretical and experimental investigations of radiation does not emerge as the quantum of energy (erg dimensions) but as the quantum of action:

$$h = 6,55 \times 10^{-27} erg\,sec,$$

which has the dimension (energy x time) and figures in expression

$$\int p_i dq = n_i h,$$

where n_i is a whole number and h is Planck's constant.

MOND3 - ACTION AND REACTION FORCES

The Newtonian theory is primarily founded on Newton's Axioms or Laws of motion, as he himself called them. Here and there, the term axiom is omitted as a statement or requirement taken to be logically true, which requires no proof, and therefore it is more common to refer to the laws of motion in nature. However, Newton himself does not use the term law for other statements but uses the terms such as lemmas, theorems, propositions, phenomena, rules, tasks. Newton's laws are the foundation of theoretical and applied mechanics. Yet, it proved that Newtonian mechanics did not respond positively, either physically or mathematically, to all motions and phenomena in nature. After Newton, to enrich the theory of motion in nature and human practice, several principles were established. Relying on the concept of the action of force, which is incongruent with Newtonian force, we are laying down here a general principle of the action of forces to achieve agreement and generalization of all knowledge acquired to date about the motion of the body. From the viewpoint of logic, the concept of law is distinct from the concepts of equations, lemmas or theorems irrespective of the writings found in the textbooks of physics or mechanics. Newton made a clear distinction between those concepts. Only three of his axioms he referred to as laws, calling upon the proofs provided by his predecessors Copernicus, Huygens, Kepler and Galileo. Newton used that knowledge, laws, phenomena and causal reasoning to describe the attributes of motion by means of propositions and theorems. Laws are derived by applying previously known mathematical relations, as it is done with theorems. In academic and professional literature, for instance, the formulation of Newton's second law we encounter reads: The alteration of motion is proportional to the force and takes place in the direction of the force. And there follows wrong explanation of the sentence: "The motion in the second law implies the momentum - the product of the mass m of the body and its speed $\mathbf{v}$, i.e. $m\mathbf{v}$." Furthermore, it is added: "In the vector form Newton's second law reads:

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F}^*, \tag{3.0}$$

where $\mathbf{F}$ is a force vector and represents the resultant of all forces acting upon the body. When it is assumed that mass does not change during motion, $m = const.$, and that $m > 0$, as Newton tacitly assumed, Newton's second law is reduced to the form:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}, \tag{3.1}$$

which represents the fundamental vector equation of dynamics. Considering a critical importance of this law, it should be noted the following:

Equation (3.0) does not represent Newton's axiom or law. The text of the law, as obvious, does not mention the concept of momentum $m\mathbf{v}$, which figures in that equation. 1. Anyway, an axiom or a law cannot be arbitrarily changed, because based on the law many statements (theorems) of dynamics can follow, as they follow from the law (3.1).

2. The assumption on whether the mass is like this or like that cannot change the law. On the contrary, it will follow from the law, what mass can be like.

3. Moreover, equation (3.0) is not correct physically, because it does not describe adequately the corresponding motion of the rockets or bodies with reaction forces. This error might have occurred as a result of not reading Newton's works carefully enough. Newton defined the 'momentum' but not the concept of 'motion' to formulate the second axiom or law of motion. However, in clarifying his Definition $VIII$ Newton writes: "Accelerative force (read: acceleration; author's note) stands in the relation to the motive (read: force) as velocity does to momentum. Indeed, momentum is proportional to velocity (mass), and the motive force is proportional to acceleration (and mass)." He did not define the concept of motion, but in a lengthy SCHOLIUM he says that time, space, place and motion are well known to all. Yet, item IV of the SCHOLIUM indicates that "motion is the translation of a body from one place into another". The change of momentum.

Equation derived based on the law (3.1), but not vice versa. Formally, if is added to the left-hand side of that equation (3.1), that is, $\frac{dm}{dt}\mathbf{v}$, will be

$$m\frac{d\mathbf{v}}{dt} + \frac{dm}{dt}\mathbf{v} = \mathbf{F} + \frac{dm}{dt}\mathbf{v},$$

it is obtained

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}^*. \tag{3.2}$$

where it is then $\mathbf{F}^* = \mathbf{F} + \frac{m}{dt}\mathbf{v}$. This conclusion indicates that forces are not formal numbers but diverse vector causal agents of motion, whose common property is MLT^2. We do not change those laws here, we generalize them by a single principle of mechanics.

3.1. Principle of action and reaction forces. In a volume of mentioned and unmentioned serious scientific and professional papers the concept of action is most commonly associated with the principle of least action. However, action is one of the adopted properties of a moving body, which is determined by basic properties of mechanics. In mechanics the principle is a statement used to establish a general provision, rule, relation about being in motion or at relative rest of any object, both small and the smallest one. In order to bring into congruity Newton's Definition IV of the force, as an action, and his third axiom or the law of ACTION AND REACTION with Euler's principle [16] of the action of forces, the author of this work has clearly formulated the concept of the action of forces F by the formula

$$\mathcal{A}(\mathbf{F}) = \int_{t_0}^{t_1} \Big(\int_{r_0}^{r_1} \mathbf{F} d\mathbf{r} \Big) dt \tag{3.3}$$

at the distance $r_1 - r_0$ for the $t_1 - t_0$.

In this case, distinction should be made between the concept of action and the concept of acting, which constitutes mentioned indefinite integral

$$\mathcal{A}(\mathbf{F}) = \int_{t_0}^{t} \Big(\int_{r_0}^{r} \mathbf{F}) d\mathbf{r} \Big) dt. \tag{3.4}$$

This definition corresponds to all forces attacking at a single material point, including the defined inertia force, which is the only one innate to body forces, by which a material point resists the action of all other forces. For the sake of that distinctive characteristic of inertia force, shorter and more striking importance of that force, we introduce the concept of Reaction.

Definition: The action of negative inertia force I of the material point of mass m represents the reaction of the material point, of a general form:

$$\mathcal{A}(\mathbf{I}) = \int_{t_0}^{t} \Big(\int_{r_0}^{r} \mathbf{I} d\mathbf{r} \Big) dt$$

Based on above statement, a link between Leibniz's action of forces and Newton's concept of force is established and a general Principle of action and reaction is formulated: The action of a force is equal to the action of a material point.

According to above mentioned, mathematical expression of this principle is as follows:

$$\mathcal{A}(\mathbf{F}) = \mathcal{A}(\mathbf{I}),$$

respectively

$$\mathcal{A}(\mathbf{I}) = \int_{t_0}^{t} \Big(\int_{s_0}^{s} \mathbf{I} \cdot d\mathbf{r} \Big) dt = \int_{t_0}^{t} \Big(\int_{0}^{r} \mathbf{F} \cdot d\mathbf{r} \Big) dt. \tag{3.5}$$

All three Newton's laws of motion and other valid principles of analytical mechanics follow from this principle. The proof that the whole theory of rational mechanics can be derived based on this principle of forces, and preprinciples, will suffice to show how Newton's basic laws of mechanics result from it.

3.2. Axioms or laws of motion. This is Newton's title for his basic laws of mechanics taught at all schools of general education and obligatory in vocational education of the motion and relative rest of the body. However, some opponents find the term axiom to belong to the 17th century language, so it is pointless to use it now, whereas others wonder what a law is. For many university educated and very distinguished scientists these two terms have the same meaning. However, the author of this work makes logical distinction between them. The author uses the word axiom (Greek $\alpha\iota\zeta$) as a reasonable starting point, the truth that does not require argumentation, it is an unprovable truth, an unspectable truth and as such it is adopted. The theory developed upon adopted axioms is truthful as much as the axioms are, because theoretical approaches are proved by means of them. No matter which and how much the deviation from the axiom, it is no longer axiomatic rational theory of mechanics. Now, let us present Newton's axioms both in words and by mathematical equations.

I. Newton's first axiom or law of motion reads: Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it. In Latin: "Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum suum mutare."

From the equation for the principle (3.5), it follows that: If $\mathbf{v} = corv ==$ 0, then

$$\mathbf{F} = \sum \mathbf{F}_i = \mathbf{0}, \tag{3.6}$$

and vice versa, which makes Newton's first law relativized by the preprinciples of existence and causal determinacy, because absolute rest of the body does not exist.

II. The second basic law or axiom reads: *The alteration of motion is ever proportional to the motive force impressed and is made in the direction in which that force is impressed.* A more reliable formulation in Latin is: "Mutationem motus proportionalem esse vi motrici impressae, et fireri secundum lineam rectam qua vis illa imprimatur." It follows from the expression for the principle of action and reaction forces (3.5), as a sufficient condition, that Newton's second law (3.1) is:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}. \tag{3.7}$$

Note again that not rare is the case that writers of classical mechanics translate and understand the words "mutationem motus" as "the change of momentum", which changes Newton's second law. This probably comes from the fact that Newton defined "momentum" as "quantitas motus" (the quantity of motion), but did not define the concept of "motion". In the

Serbian language the term "motion" implies the change of the body's position during some time, which is in accordance with the definition of velocity. Therefore, "mutationem motus" means the change of the speed of motion. In clarifying his Definition VIII, that is, before writing the axiom or law of motion, Newton put down: "Accelerative force (read: acceleration; author's note) stands in the relation to the motive (read: force) as velocity does to momentum. Indeed, momentum is proportional to velocity and mass, and the motive force is proportional to acceleration and mass in general, the weight of the body will be constantly proportional to the mass of the body and the acceleration."

III. The third law reads: To every action there is always opposed an equal or opposite reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. A more reliable formulation in Latin is: "Actioni contrariam semper et aequalem esses reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi."

By comparing Newton's third axiom and our principle of *action and reaction forces, it should be first pointed out that in his*

Definition IV Newton wrote: "*the force is an action*", and in that respect he explains the concept of action. He introduced neither action, nor reaction by his definition. Our Definition 5 includes both concepts. From that logical deducing, the same statements about the principle of action and reaction do not have the same meaning in Newton's third axiom or law either. Mathematically, Newton's third axiom or law is written simply for two bodies by the equation:

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{3.8}$$

This is additional equation to the Newton's second law equation (3.7).

However, the principle of action and reaction for the motion of two material points states that:

$$\int_{t_0}^{t}\int_{r_{01}}^{r} \mathbf{F}_1 \cdot d\mathbf{r}_1 dt = \int_{t_0}^{t}\int_{r_{01}}^{r} \mathbf{I}_1 \cdot d\mathbf{r}_1 dt, \tag{3.9}$$

$$\int_{t_0}^{t}\int_{r_{02}}^{r} \mathbf{F}_2 \cdot d\mathbf{r}_2 dt = \int_{t_0}^{t}\int_{r_{02}}^{r} \mathbf{I}_2 \cdot d\mathbf{r}_2 dt. \tag{3.10}$$

These are large and essential differences.

First and foremost, a crucial difference relates to the properties of forces and actions. The property of a force is MLT^{-2} (1.7), and the property of action is $\mathrm{ML}^2\mathrm{T}^{-1}$. In other words, those are different attributes of motion.

The second objection to Newton's third law refers to the independence of Newton's first and third axiom. The first axiom states that the body remains in its state of rest, or in uniform rectilinear motion. According to equation (3.8) this could mean that the sum of forces of two bodies' mutual actions is zero, and furthermore that two bodies are in mutual uniform motion in a straight line, which is contrary to the condition of manifestation of the nature of things. Two bodies can move independently of one another, can approach, remain at the same place, distance, or move away.

The simplest experiment, most convincing and readily available, is: if you drop an object from your hand, it will move rapidly toward the earth. But, if you tie that object with some string, whose upper part you are holding in your hand, the object will be at rest if your hand is at rest, or will move if you move your hand.

In accordance with mentioned three Newton's axioms or laws of motion, the outcomes of the Principle of action and reaction forces for the existing motions of the system of material points can have more general and precise applications than those of Newton's axioms. Also, the Principle of action and reaction forces encompasses other principles of mechanics, such as: the principle of equilibrium, the principle of work, the principle of action, the principle of compulsion, which occur as the result of our principle. All those principles have been developed on manifolds or systems of material points, and therefore it is necessary to point first to modification of the system with variable constraints. Mentioned simple example of an experiment does not represent only two bodies but has an additional material object - a particular distance constraint. So, this is not about two independent bodies but about the system of two material points linked by some real constraint that can be represented by equation

$$\mathbf{r}_2 - \mathbf{r}_1 = \rho(t). \tag{3.11}$$

It is well known in mechanics that such ideal constraints are hiding force $\mathbf{R}$, most commonly called the reaction of constraint, that is,

$$\mathbf{R} = -\lambda\, grad\, f, \tag{3.12}$$

where

$$f = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \rho^2} = 0 \tag{3.13}$$

and $\rho = |\mathbf{r}_2 - \mathbf{r}_1|$. With condition (3.11), we have two equations of motion

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \mathbf{F}_1, \tag{3.14}$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{F}_2, \tag{3.15}$$

where those vector equations can be written in the form of scalar differential equations of motion

$$m_1\ddot{x}_1 = \lambda(x_2 - x_1) = X_1,$$
$$m_1\ddot{y}_1 = \lambda(y_2 - y_1) = Y_1,$$
$$m_1\ddot{x}_1 = \lambda(z_2 - z_1) = Z_1;$$

$$m_2\ddot{x}_2 = -\lambda(x_2 - x_1) = X_2,$$
$$m_2\ddot{y}_2 = -\lambda(y_2 - y_1) = Y_2,$$
$$m_2\ddot{z}_2 = -\lambda(z_2 - z_1) = Z_2.$$

In more detail, "The forces of constraints" [3] Obviously, there are 6 differential and one finite constraint equation (3.13) by means of which 6 coordinates of the force vectors and one multiplier of constraints λ can be determined. By comparing the right-hand sides of equations, due to explicit meaning of parameter λ, it is obtained that is

$$X_2 = -X_1, \quad Y_2 = -Y_1 =, \quad Z_2 = -Z_1,$$

or

$$\mathbf{F}_1 = -\mathbf{F}_2,$$

which means that the forces of mutual action are equal in magnitude and direction and opposite in sense. This is in accordance with Newton's third axiom. According to above mentioned, it has been proved that all three Newton's axioms or laws derive from our principle of action and reaction forces, with additional constraint (3.11) or explanation for distinguishing between the concept of force and the concept of the action of force. The second derivative with respect to time of distance ρ (3.11) is reduced to:

$$\frac{d^2\boldsymbol{\rho}}{dt^2} = \frac{d^2\mathbf{r}_2}{dt^2} - \frac{d^2\mathbf{r}_1}{dt^2}.$$

Considering (3.14) and (3.15), it is obtained:

$$\frac{F_2}{m_2} - \frac{\mathbf{F}_1}{m_1} = \frac{d^2\boldsymbol{\rho}}{dt^2},$$

or, in accordance with (3.8)

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\boldsymbol{\rho}}{dt^2}.$$

The principle of the action of force satisfies all three preprinciples. The preprinciple of existence and causal determinacy are accurate as much as our first four vector definitions, while the preprinciple of invariance is reduced here to scalar invariant, and as such to tensor invariant. Indeed, subintegral scalar product $\mathbf{F} \cdot d\mathbf{r}$ is the elementary work of the force on the displacement $d\mathbf{r}$, that is,

$$\mathbf{F} \cdot d\mathbf{r} = F^i \mathbf{e}_i \cdot dy^j \mathbf{e}_j = g_{ij} X^i dx^j = X_j dx^j,$$

were X^i and Y^i vector coordinates, and $X_j dx^j$. In the same way, (3.11) there is transformation of the covariant coordinates of inertia force of the material point of mass m,

$$X_i = \frac{\partial y^j}{\partial x_i} Y_j.$$

Thus, invariance of the principle of the action of forces is reduced to

$$I_i(x) = m \frac{\partial y^j}{\partial x_i} I_j(y).$$

Thus, invariance of the principle of the action of forces is reduced to

$$\int_{t_0}^{t} \Big(\int_{x_0} I_i dx^i \Big) dt = \int_{t_0}^{t} \Big(\int_{x_0}^{x} dx^i \Big) dt$$

$$\int_{t_0}^{t} \Big(\int_{y_0}^{y} m w_i(y) dy^i \Big) dt = \int_{t_0}^{t} \Big(\int Y_i dx^i \Big) dt,$$

that is

$$\int_{t_0}^{t} \Big(\int I_i(x) dx^i \Big) dt = \int_{t_0}^{t} \Big(\int I_i(y) dy^i \Big) dt,$$

as well as

$$\int_{t_0}^{t} \Big(\int X_i(x) dx^i \Big) dt = \int_{t_0}^{t} \Big(\int Y_i(y) dy^i \Big) dt.$$

3.3. Manifold and a system of material points. Manifold, concerning the preprinciple of existence, denotes a large number of elements, more than one, whereas a system can, but need not, mean the element if it is conditioned by some connections. The definitions indicate explicitly enough that the concepts of 'manifold' and 'system' are not identical. It is justifiable to be doubtful whether there exists a single point, in itself, without neighborhood, or neighborhood boundaries. Certainly not, because the boundary is some kind of connection. In that regard, a single material point together with some connection constitutes a system. "Manifold" as a set of real numbers is undeniable in mechanics, but not a set of all rational numbers. A system of material points.

The second part of Newton's third axiom formulation refers to two bodies, i.e. to two material points which are mutually attracted. This indicates that, besides two bodies, there exist some relations connecting them, as the equation of actions

$$\mathcal{A}(\mathbf{F}) = \int_{t_0}^{t_1}\int_{r_0}^{r_1} \mathbf{F}_1 \cdot d\mathbf{r}dt = -\int_{t_0}^{t_1}\int_{r_0}^{r_1} \mathbf{F}_2 \cdot d\mathbf{r}dt = -\mathcal{A}(\mathbf{F}_2), \tag{3.16}$$

and the equation of directed distances

$$\boldsymbol{\rho} = \mathbf{r}_2 - \mathbf{r}_1.$$

The concept of a system indicates that there exists the motion of the material point or material points along with other factors that determine and restrict motion, respectively. Such objects or programs are referred to in mechanics as constraints, which are described by various mathematical equations and inequalities. Depending on the type of equations and functions that figure in them, in the literature of classical mechanics the constraints are represented by different terms, such as: finite, geometric, differential, kinematic, holonomic, bilateral, restrained, nonholonomic, smooth and real, linear and nonlinear, scleronomic and rheonomic, in a vector or coordinate form. For brevity and easier general presentation herein, the concept of constraint will imply, in addition to differential equations and integral equations (3.16), all mathematical relations in the form of equations or inequalities used to describe manifested or programmed motion of a system of material points. For example: (3.17) where functions

$$f_\mu(y_1, y_2, \dots, y_{3N}) = 0, \tag{3.17}$$

are continuous regular, dimensionally homogeneous in the region S, and differentiable with respect to time t,

$$\frac{df_\mu}{dt} := \dot{f}_\mu = \sum_i^3 N\frac{\partial f_\mu}{\partial y^i}\dot{y}^i = b_{i\mu}\dot{y}^i = 0, \tag{3.18}$$

in the neighborhood of each point y_o^i. A linear system of kinematic equations is obtained, by means of which the k coordinate of velocities $\dot{y}^k$ can be determined, depending on the rest of $3N-k$ coordinates of velocities $\dot{y}^\alpha$; $\alpha = 1, 2, \dots, n = 3N - k$.

In order to make the previous proof even more clarified, let us observe a simpler system of linear, mutually independent homogeneous algebraic equations

$$f_\mu = a_{\mu 1}y_1 + a_{\mu 2}y_2 + \cdots + a_{\mu 3N}y_{3N} = 0, \quad \mu = 1, \dots, k \leqslant 3N,$$

which can be always written in the form

$$\sum_{i=1}^{3N-k} a_{\mu i}y_i = -\sum_{k=1}^{n} a_{\mu k}y_k : \quad n = 3N - k. \tag{3.19}$$

It is evident from here that for conditions $|a_{\mu k}| \neq 0$, it is possible to determine the k coordinates of y^i by means of the $n = 3N - k$ independent coordinates. Indices k denote the number of constraint equations to determine 3N unknown position functions $y(t)$ and coordinates of forces Y_j, which is insufficient for solving the primary system, without additional conditions. Mechanics solves this by providing (most commonly, experimentally) constraints, programs and moving conditions as if well-known, while forces generating constraints are determined by the method of Lagrange's multipliers of constraints.

Prior to solving the mechanical system motion with more general constraints, let us point out some other properties of linear constraints (3.19). According to the preprinciple of invariance, mathematical transformations do not change mechanical constraints. Simply, it means that if we introduce curvilinear coordinates x instead of Cartesian coordinates, we obtain the system

$$f_\mu(x^1, x^2, \ldots, x^{3N}) = 0, \tag{3.20}$$

without changing their property. Change in the second derivative with respect to time is significantly expressed; instead of linear relations, we obtain nonlinear equations which indicate that the forces generating constraints are proportional to the second derivative $\ddot{x}^i$.

Misunderstanding, not to say incongruity, is present in a view on dependent and independent coordinates as well as on generalized dependent and independent coordinates, for which the most common notation is q^α and for corresponding generalized forces it is Q_α. To avoid misunderstanding in this contribution, we stress the following: the letters y^i are used to denote $3N$ Cartesian rectilinear independent and dependent coordinates of the position of N material points; xi denote corresponding curvilinear coordinates; a mix of all mentioned $3N$ coordinates can be called general coordinates, which has to be stressed, and writing the systems of stated constraints being mandatory. The author of this work always implies that generalized coordinates are independent coordinates obtained from equations (3.19) and he denotes them with the letters q^α, of which there are $n = 3N - k$. The significance is twofold. First, equations $y^i = y^i(q^1, \ldots, q^n)$ substitute constraints (3.20), so they are sometimes called *parametric constraints*. Second, by substituting $y^i = y^i(q^1, \ldots, q^n)$ into constraint equations (3.19), nothing else is obtained but identities

$$f_\mu(y^i) = f_\mu\big(y^i(q^1, \ldots, q^n)\big) = f_\mu(q^1, \ldots, q^n) = 0.$$

Compared to independent generalized coordinates, many formulas and equations of motion are expressed in a shorter and simpler form, such as:

Generalized velocities

$$\dot{q}^\alpha = \left(\frac{dq^\alpha}{dt}\right)_M,$$

generalized impulses

$$p_\alpha = a_{\alpha\beta}\dot{q}^\beta,$$

generalized accelerations,

$$\left(\frac{D\dot{q}^\alpha}{dt}\right)_M = \left(\frac{d\dot{q}^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\frac{\dot{q}^\gamma}{dt}\right)_M,$$

Kinetic energy

$$E_k = \frac{1}{2} = a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = a^{\alpha\beta}p_\alpha p_\beta,$$

Differential covariant equation of the system motion,

$$a_{\alpha\beta}\frac{D\dot{q}^\alpha}{dt} = Q_\alpha,$$

where $a_{\alpha\beta} = a_{\beta\alpha}(m_1, \ldots, m_N; q^1, \ldots, q^n)$ is *inertia tensor* or *inertia or mass tensor*, but not metric, as referred to by many authors.

Such harmony of motion description is present in the whole analytical mechanics of the system with constraints of the form (3.17). However, if the constraint functions are explicitly dependent of coordinates x or y, and of time t, that is,

$$f_\mu(y_1, \ldots, y_{3N}, t) = 0, \quad \mu = 1, \ldots, k,$$

everything changes in a standard theory, which is impermissible according to the preprinciples of invariance.

3.4. Systems with variable constraints. [96] In case that finite constraints

$$f_\mu(x^1, \ldots, x^{3N}, t) = 0; \quad \mu = 1, \ldots k, \tag{3.21}$$

depend, apart from coordinate functions $y(t) \in E^{3N}$, and of time t, the conditions of velocity and acceleration are considerably changed, number of addends in equations is increased, as evident by the following:

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^\alpha}\dot{y}^\alpha + \frac{\partial f_\mu}{\partial t} = grad_\nu f_\mu \cdot \mathrm{v}_\nu + \frac{\partial f_\mu}{\partial t} = 0. \tag{3.22}$$

This means that there is one more addend $\frac{\partial f_\mu}{\partial t}\frac{\partial f_\mu}{\partial t}$ each of change than it is the case with geometric constraints. Variable constraints must satisfy the dimensional equation in the course of time, i.e. they must be dimensionally homogeneous. In order to achieve homogeneity between the coordinates y and time t, it is necessary that these quantities be connected by some parameter of dimensions L and T. So, in mechanical constraints time occurs as an independent variable in the structure of functions which contain dimensional parameters, and therefore variables or moving constraints, in accordance with definition (3.21), are written in the form

$$f_\mu(y,\tau) = 0 \qquad (\mu = 1,\ldots,k),$$

where $\tau = \tau(t)$ is a real function of time with determined real coefficients that have physical properties. For brevity, instead of the function $\tau(t)$ with determined coefficients, let us introduce additional coordinate y^0, so that it fulfills the condition $f_0 = y^0 - \tau(t) = 0$.

Using the coordinate y^0, constraint equations (3.21) can be written in the form

$$f_\mu(\tilde{y}) = 0, \quad \tilde{y} = (y^0, \underbrace{y^1,\ldots,y^{3N}}_{y}),$$

and the first and second derivatives with respect to time are:

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y}\dot{y} + \frac{\partial f_\mu}{\partial y^0}\dot{y}^0 = 0 \tag{3.23}$$

$$\ddot{f}_\mu = \frac{\partial^2 f_\mu}{\partial \tilde{y}\partial \tilde{y}}\dot{\tilde{y}}\dot{\tilde{y}} + \frac{\partial f_\mu}{\partial \tilde{y}}\ddot{\tilde{y}} =$$
$$\frac{\partial^2 f_\mu}{\partial y \partial y}\dot{y}\dot{y} + 2\frac{\partial^2 f}{\partial y^0 \partial y}\dot{y}\dot{y}^0 + \frac{\partial^2 f}{\partial y^0 \partial y^0}\dot{y}^0\dot{y}^0 + \frac{\partial f_\mu}{\partial y}\ddot{y} + \frac{\partial f_\mu}{\partial y^0}\ddot{y}^0 = 0.$$

The last relation can be written for short

$$\frac{\partial f_\mu}{\partial y}\ddot{y} + \frac{\partial f_\mu}{\partial y^0}\ddot{y}^0 = \Phi\left(\tilde{y},\dot{\tilde{y}}\right), \tag{3.24}$$

where the composition of the function Φ is evident. By incorporating $\ddot{y}$ from differential equations of motion

$$m\ddot{y} = Y + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y}, \tag{3.25}$$

into equation (3.21) it is obtained

$$\frac{\partial f_\mu}{\partial y}\sum_{\sigma=1}^{k} \lambda_\sigma \frac{\partial f_\sigma}{\partial y} = m(\Phi - \frac{\partial f_\mu}{\partial y^0}\ddot{y}^0) - Y\frac{\partial f_\mu}{\partial y}.$$

Solutions for unknown multipliers of constraints show that reaction forces of rheonomic constraints do not depend only of coordinates y and velocities $\dot{y}$, but also of acceleration $\ddot{y}^0$ and inertia force $-m\ddot{y}^0$, which occurs due to change of constraints with respect to time. This indicates that it is not only formal writing of a single additional coordinate but identifying a single existing force that has been lost due to ignoring a rheonomic coordinate. Constraints in equations (3.21) can be written in the parametric form

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^0, q^1, \dots, q^n), \quad n = 3N - k, \tag{3.26}$$

where $q = (q1, \dots, q^n)$ are independent generalized coordinates and q^0 is a rheonomic coordinate that satisfies the equation

$$q^0 - \tau(t) = 0. \tag{3.27}$$

By reducing finite constraints to the parametric form (3.26), the number of differential equations is reduced by the number of constraints, and constraint forces $\mathbf{R}_N$, are eliminated, which considerably facilitates task solving. The velocities of the ν-th material points, in accordance with definition (1), are

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^1}\dot{q}^1 + \dots b + \frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot{q}^n = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha, \tag{3.28}$$

where

$$\mathbf{g}_{\nu\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}$$

are coordinate vectors; index ν designates the number of the material point, while index α the number of independent coordinates $q^\alpha, \alpha = 0, 1, \dots, n$. Summing for index ν deploys the summing σ_ν, whereas summing for coordinates α indices denotes the repetition of the same letter in the same expression as a subscript and superscript index.

The vector (3.28), as obvious, has $n+1$ independent coordinate vectors. Consequently, the impulse vector of the $\nu - th$ material point of mass m of the observed system is

$$\mathbf{p}_\nu = \mathbf{p}_\nu = m_\nu \mathbf{v}_\nu = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha.$$

Scalar multiplication of above relation by coordinate vectors $\frac{\partial \mathbf{r}_\nu}{\partial q^\beta}$ yields coordinate impulses

$$p_{\nu\beta} = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta}\dot{q}^\alpha, \quad \alpha, \beta = 0, 1, \dots, n.$$

Considering that $p_{\nu\beta}$ are scalar quantities, it is possible to add them

$$p_\beta := \sum_{\nu=1}^{N} p_{\nu\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta}\dot{q}^\alpha = a_{\alpha\beta}\dot{q}^\alpha, \tag{3.29}$$

from where it is evident that $a_{\alpha\beta}$ is the inertia tensor of the whole system

$$a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} = a_{\alpha\beta}(m_1, \dots, m_N; q^0, q^1, \dots, q^n). \tag{3.30}$$

Using relations (3.29) the concept of generalized impulses of the system of material points is introduced. *Generalized impulses occur as linear homogeneous forms of generalized velocities*, which is in accordance with the basic definition of impulse. Considering that the determinant of the inertia tensor $a_{\alpha\beta}$ is, in a general case, different from zero, it is possible to determine generalized velocities $\dot{q}^\alpha$ as linear homogeneous combinations of generalized impulses, such as

$$\dot{q}^\alpha = a^{\alpha\beta} p_\beta, \tag{3.31}$$

where $a^{\alpha\beta}$ is the countervariant inertia tensor. If constraints are not obviously dependent of known functions of time τ, rheonomic coordinate q^0 does not occur, and therefore in all expressions (3.28) - (3.29) the coordinates $q^0, \dot{q}^0$ i p_0. The form of the impulse (3.39) does not change, except that indices $\alpha = 0, 1, \ldots, n$ do not take the values from 0 to n, but from 1 to n. In order to make it distinguishable in the text below, let Greek indices $\alpha, \beta, \gamma, \delta$ take values from 0 to n, while Latin ones i, j, k, l take values from 1 to n $(i, j, k, l = 1, 2, \ldots, n)$. With such indices, it can be written

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i,$$

or covariantly

$$\begin{aligned} p_i &= a_{0i}\dot{q}^0 + a_{ij}\dot{q}^j = a_\alpha \dot{q}\alpha, \\ p_0 &= a_{00}\dot{q}^0 + a_{0j}\dot{q}^j = a_{0\alpha}\dot{q}^\alpha, \\ \dot{q}^i &= a^{0i}p_0 + a^{ij}p_j = a^{i\alpha}p_\alpha, \\ \dot{q}^0 &= a^{00}p_o + a^{0j}p_j = a^{0\alpha}p_\alpha. \end{aligned}$$

$$2E_k = a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = a^{\alpha\beta}p_\alpha p_\beta, \quad \alpha, \beta = 1, \ldots, n+1. \tag{3.32}$$

Accordingly, the quadratic form of kinetic energy E_k also obtains the invariant form (3.32) which is considerably different from standard non-invariant form

$$2E_k = a_{ij}\dot{q}^i\dot{q}^j + 2b_i\dot{q}^i + c, \quad i, j = 1\ldots, n.$$

Veretennikov and Sinicyn in their book "Method of variable action" point out that incongruity is eliminated by the approach proposed by Vujicic, [35]. For the case of finite geometric constraints that do not contain explicit time, the rheonomic coordinate is equal to zero, and therefore the expression (3.32) is reduced to known homogeneous quadratic form

$$2E_k = a_{ij}\dot{q}^i\dot{q}^j = a^{ij}p_i p_j, \quad i, j = 1, \ldots, n. \tag{3.33}$$

Generalized coordinates $q^1, \ldots, q^n$ and generalized impulses $p_1, \ldots, p_n$ are also called "Hamiltonian coordinates". This is not only the formal side of the problem.

The papers and monograph [19] [21] contain a more extensive overview and proved changes in classical analytical mechanics of time-dependent systems. Shorter and credible, it is evident from the review in the Prologue. In standard classical mechanics it is considered that the total mechanical energy integral cannot be obtained as for the systems whose constraints are time-independent. The exception is Painleve's energy integral

$$E_{k,2} - E_{k,0} + \mathrm{P} = const.$$

However, the previous scheme provides, as work [21], a considerably more extensive picture. It is shown that the Painleve integral does not occur as energy integral but as one from a multitude of cocyclic integrals. One of the university professors has given a "counter-example"

$$f(x, y, t) = y - tx = 0$$

with the following commentary: "Behold, if this can be solved according to a modified theory by V.V., I admit I do not know mechanics." This example, not a counter one, but a nice, simple and instructive example called "counter-example" by the opponent shows that he understood neither essence nor formal procedure of Vujičić modification of the theory of mechanics of a system with variable constraints. It can be readily proved:

First, the equation of the "counter-example" is not dimensionally homogeneous, and as such cannot be the constraint of mechanical systems.

Second, only in case that time t is multiplied by unit angular velocity ω or unit frequency, which have the property T^{-1}, the rheonomic coordinate obtains a simple form $y^0 = \omega t$ a equation (3.34) the form $f(x, y, y^0) = 0$ and $y^0 = t(t) = 0$.

What's more important is that this modification of the theory of rheonomic systems produced significant results. Writing a book "Preprinciples of mechanics" later, the author used the example of the "problem of two bodies", as a system of two material points of masses m_1 and m_2 and their existing distance $\rho(t)$, representing an explicit example of the rheonomic constraint, to determine Newton's gravitational force, which could not be denied. However, instead of expected familiar expression for Newton's universal gravitation

$$F = k\frac{m_1 m_2}{\rho^2}, \tag{3.34}$$

the author has obtained a completely different formula

$$F = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}\frac{m_1 m_2}{\rho}, \tag{3.35}$$

which was surprising. He thought that he had made some algebraic error in his calculations, but could not find it. There was no error, nor could he doubt the accuracy of "the most magnificent law of nature that a mortal man could grasp", [M. Milanković]. In a two months' preoccupation with literature browsing and checking his own calculations, he faced a disappearance of his manuscript. This made him communicate his result at a scientific seminar of the Department of Mechanics, Mathematical Institute, Serbian Academy of Sciences and Arts, for live check, hoping that some of the people in the audience would notice and point out a likely mistake, because he himself was wondering how it was possible that his result had not been detected 300 years back. There were no remarks, however diverse prologues were not missing out of professional meetings

A more general and mathematically stricter proof the author submitted at the meeting of Serbian Scientific Society on 22 May 1997 published in "Scientific Review, Series: Science and Engineering, 24, pp. 61-67 (1997), entitled "A Possible Reconsideration of Newton's Gravitation Law".

Basically, the problem involved the following task: there are two material points, of masses m1 and m2, connected by mutual distance $\rho(t)$, which varies in the course of time, that is,

$$f = (x_2 - x_1)^2 + \cdots (z_2 - z_1)^2 - \rho^2 = 0;$$

It is necessary to determine the magnitude of force by which the forces are acting upon one another. Considering that the Lagrangian method of constraints is included in every course in mechanics at the university, the solution of the task was sought just in this way, because the force sought should be the reaction of the constraint:

$$\mathbf{R} = -\lambda\, grad\, f,$$

so that differential equations of motion of two material points are:

$$m_1\ddot{x}_1 = \lambda\frac{\partial f}{\partial x_1} = \frac{\lambda}{\rho}(x_1 - x_2),$$

$$m_1\ddot{y}_1 = \lambda\frac{\partial f}{\partial y_1} = \frac{\lambda}{\rho}(y_1 - y_2),$$

$$m_1\ddot{z}_1 = \lambda\frac{\partial f}{\partial z_1} = \frac{\lambda}{\rho}(z_1 - z_2),$$

$$m_2\ddot{x}_2 = \lambda\frac{\partial f}{\partial x_2} = \frac{\lambda}{\rho}(x_1 - x_2),$$

$$m_2\ddot{y}_2 = \lambda\frac{\partial f}{\partial y_2} = \frac{\lambda}{\rho}(y_1 - y_2),$$

$$m_2\ddot{z}_2 = \lambda\frac{\partial f}{\partial z_2} = \frac{\lambda}{\rho}(z_1 - z_2).$$

The first derivative of the constraint $\dot{f} = 0$ with respect to time is:

$$(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + \cdots + (z_1 - z_2)(\dot{z}_1 - \dot{z}_2) = \rho\dot{\rho},$$

and the second derivative:

$$\ddot{f} = v^2 + (x_1 - x_2)(\ddot{x}_1 - \ddot{x}_2) + \cdots + (x_1 - x_2)(\ddot{x}_1 - \ddot{x}_2) - \dot{\rho}^2 - \rho\ddot{\rho} = 0,$$

where

$$v_{or}^2 = (\dot{x}_1 - \dot{x}_2)^2 + \cdots + (\dot{z}_1 - \dot{z}_2)^2.$$

Substituting the second derivatives $\ddot{x}_1, \dots \ddot{z}_2$. from mentioned Lagrangian equations of the first kind into the previous relation, it is obtained:

Substituting thus obtained λ multiplier backwards into differential equations of motion, the following system of differential equations of motion of the two material points is obtained:

$$m_1\ddot{x}_1 = \chi\frac{m_1m_2}{\rho^2}(x_1 - x_2),$$

$$m_1\ddot{y}_1 = \chi\frac{m_1m_2}{\rho^2}(y_1 - y_2),$$

$$m_1\ddot{z}_1 = \chi\frac{m_1m_2}{\rho^2}(z_1 - z_2),,$$

$$m_2\ddot{x}_2 = \chi\frac{m_1m_2}{\rho 2}(x_1 - x_2),$$

$$m_2\ddot{y}_2 = \chi\frac{m_1m_2}{\rho^2}(y_1 - y_2),$$

$$m_2\ddot{z}_2 = \chi\frac{m_1m_2}{\rho^2}(z_1 - z_2).$$

The first derivative of the constant $\dot{f} = 0$ with respect to time is:

$$(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + \cdots + (z_1 - z_2)(\dot{z}_1 - \dot{z}_2) = r\dot{\rho},$$

The right-hand sides of these equations represent coordinates of the vector of forces $\mathbf{F}_1$ and $\mathbf{F}_2$, so the magnitudes of forces are:

$$F_1 = \chi\frac{m_1m_2}{\rho(t)}, \quad F_2 = -\chi\frac{m_1m_2}{\rho(t)}, \tag{3.36}$$

where

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}. \tag{3.37}$$

If another constraint is added $f_2 = z_1 - z_2 - C_2 = 0$, the form of the formula does not change, but it is logical and evident that orbital velocity will be

$$v_{or}^2 = (\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2. \tag{3.38}$$

Note: Such modification of classical analytical mechanics of rheonomic systems urged some authoritarian experts in Lagrangian mechanics to introduce more than one additional coordinate in rheonomic systems, which is not in agreement with the previous procedure, i.e. with the theory of independent coordinates. Prior to demonstrating why such procedure is precarious let us give a very simple example, which in itself shows that this procedure is ungrounded in alleged generalization of the Lagrangian formalism.

Let there be 2 ordinary independent finite equations

$$y_1 + 2y_2 - 3y_3 = ax,$$

$$2y_1 - y_2 - 3y_3 = b + c\sin\omega x,$$

where $y_i = y_i(x)$,and $i = 1,2,3$, are functions of independent variable x and a, b, c, ω are real numbers.

The number of independent functions $y(x)$ is to be determined. First. Commonly, the sum of the observed equations, as obvious, is:

$$f(x) = ax + b + c\sin\omega x$$

It follows from here that:

$$y_2(x) = 3y_1(x) - f(x),$$

where $f(x) = ax + b + c\sin\omega x$ is known function of independent variable x and a, b, c, ω are real numbers.

Also, when considering a system of N material points linked by k rheonomic constraints

$$f_\mu\big(y_1(t), \dots, y_{3N}(t), \tau_\mu(t)\big) = 0, \quad \mu = 1, \dots k < 3N, \tag{3.39}$$

it is reduced to a multitude $3n-k+1$ of independent coordinates $q^0, q^1, \dots, q^n$ of which $q^0(t)$ is known function of time that is contained in the constraint equations. Let us show now that our principle of action and reaction forces also includes other integral principles of action in mechanics.

3.5. Euler's principle of the action of forces. Our definition of the action of forces

$$\mathcal{A} = \int\Big(\int \mathbf{F}\cdot d\mathbf{r}\Big)dt$$

conforms to Euler's sum of all momentary actions of forces [24]

$$\int dt\Big(\int V dv + V'dv' + V''dv'' \dots\Big), \tag{3.40}$$

where V, V', V'' are forces, while dv, dv', dv'' are the elements of the path and dt is the element of time. Inertia force is primary among these forces; let these be Eulerian symbols for forces

$$\mathbf{V} = \mathbf{I} = -m\frac{d\mathbf{v}}{dt},$$

and let the others be

$$\mathbf{V}' = \mathbf{F}_1, \ldots, \mathbf{V}'' = \mathbf{F}_2, \ldots$$

our principle of action and reaction forces, and vice versa. Euler formulated the principle of least action over the concept of the sum of all momentary actions, saying: *A body takes the path at which the sum of all momentary actions (3.40) has a minimum.*

Here is a new general principle for free motion of the body subjected to the action of any forces, whose accuracy becomes true only if we reflect upon the concept of action that I have established" (Euler, note by V.Vujičić) ([15], p. 76).

Let us point out again what Euler writes and let us not forget the last words of the quotation: "Let us reflect upon the concept of action that I have established." (Euler, note by V. Vujičić) This is Euler's principle of least action, which follows from our principle of action and reaction.

3.6. Lagrange's general principle. Lagrange began his work[4] with the sentence: "Mr. Euler founded his principle in accordance with which for the trajectories described by the bodies affected by central forces, the integral of velocities multiplied by the elements of the arc of a curve should have a minimum." "I am endeavoring here to generalize that principle and its application for the solution of all tasks of Dynamics."

General principle.*Let there be as many bodies as needed M, M', M'', which mutually interact in any manner, and which are moreover animated by central forces proportional to any functions of these distances; let s, s', s'' denote the spaces traveled by these bodies in time t and let $u, u', u'', \ldots$ be their speeds at the end of this time; the formula*

$$M\int u ds + M\int u' ds' + \int u'' ds'', \ldots \tag{3.41}$$

will always be a maximum or a minimum.

[4]Lagrange, Application de la method exposee dans la memore precedent a la solution de differentes problemes de dynamique. Tom 2: Miscellanea Taurinesia pour 1760-1761.

In clarifying his general principle Lagrange mentions the action of force in the following formula

$$\frac{u^2}{2} = h - \int (Pdp + Qdq + Rdr + \dots).$$

and

$$\mathcal{A} = \int E_k dt = \frac{1}{2}\int a_{\alpha\beta}\dot{q}^\alpha q^\beta dt = \frac{1}{2}\int a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta dt = \frac{1}{2}\int p_\alpha dq^\alpha,$$

In this respect, mentioned Lagrange's principle refers to the systems with potential forces. Compared to Euler's principle of action, significant differences are noticeable. In Euler we have forces, but in Lagrange speeds figure instead of forces. Euler's action has a minimum, whereas Lagrange's principle has a minimum and a maximum. Besides, Euler's action of force is reduced to action

$$\mathcal{A} = \int E_k dt = \frac{1}{2}\int a_{\alpha\beta}\dot{q}^\alpha q^\beta dt = \frac{1}{2}\int a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta dt = \frac{1}{2}\int p_\alpha dq^\alpha,$$

whereas Lagrange's action is twofold larger

$$A = 2\mathcal{A} = \int 2E_k dt = \int p_\alpha dq^\alpha. \tag{3.42}$$

As such, Lagrangian action and Lagrange's action principle occurs as a result of the principle of action and reaction forces. In proving generality of his principle ([15], pp. 123-124), Lagrange calls upon and conditions himself to the "principle of living forces". It is only with this condition that

$$\sum m_i v_i^2 = 2\int \sum Q_i dq^i \dots, \tag{3.43}$$

the principle of action and reaction forces will be satisfied

$$\delta \int L(q, \dot{q}) dt = 0. \tag{3.44}$$

where

$$L = E_k - E_p = L(q, \dot{q}). \tag{3.45}$$

In that case, Lagrange's principle is reduced to where (3.45) is the so-called Lagrangian function.

3.7. Hamilton's general method. In analytical mechanics, particularly in theoretical physics, Hamilton's general method is widely used in dynamics, where different notions are deployed: Hamilton's action or Hamiltonian action, Hamilton's canonical differential equations, Hamilton's function, which in itself indicates the importance of this method. Our intention is to prove that "Hamilton's principle" follows from our principle of action and reaction forces, as has been proved that Lagrange's principle is valid only for the condition that the law of conservation of energy exists. But, prior to producing the proof, let us present some of Hamilton's assessments: Of the finest scientists, Lagrange has perhaps done more than any other analyst, to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make his great work a kind of scientific poem. Yet the science of the action of forces in time and space suffered another modification. However, when that law of a minimum, or better to call it, of stationary action is applied to particular actual motions of the systems, its purpose is to obtain by the rules of the calculus of variations the differential equations of motion of the second kind, which can always be obtained in another way. It seems that this was why Lagrange, Laplace and Poisson underestimated, not without reason, the usefulness of that principle with the state-of-art of dynamics in those days. It might happen that the second principle introduced by Hamilton, by means of this work entitled the Law of Varying Action, where we transfer from actual motion to another, dynamically virtual, motion by varying the end positions of the systems and in general the quantity H, which serves to express by means of the function not only of differential equations of motion but also of their middle and definite integrals, encounter different evaluation. Here, we start from Lagrange's principle, reported by Hamilton as the first one, which makes things easier for us, because we have already evaluated Lagrange's principle as the result of our principle of action and reaction forces, with the condition that there exists the law of conservation of energy. It is possible only for that condition to reduce the Lagrangian action to the form (3.40), where $L = Ek - Ep$. With the law of change in energy, Hamilton introduces his function H, which represents the sum of kinetic and potential energy,

$$E_k(p, q) + E_p(q) = H(p, q), \longrightarrow E_k = H - E_p.$$

In that respect, Lagrange's action is reduced to

$$attr\, \mathcal{A} = \int_{to}^{t} (2E_k - H)dt = \int_{to}^{t} p_\alpha dq^\alpha - Hdt,$$

which is here referred to as Hamilton's action, due to the presence of Hamilton's function H and Hamilton's variables p, q that imply generalized impulses and generalized coordinates, for which Hamilton assumed that function H need not be a constant. Indeed, if we add or subtract action (3.44) E_k, it is obtained

$$L = E_k - E_p + E_k - E_k = 2E_k - (E_k + E_p) = 2E_k - H.$$

So, function L is only what Hamilton assumed that function H need not be a constant. In that respect, Hamilton's action is more general than Lagrange's principle of action and reaction forces.

MOND4 - GRAVITATION OR ATTRACTION BETWEEN BODIES

In most primary and secondary schools across the world, small or large, or universities, the teaching of physics, mathematics and engineering includes *Newton's Law of Universal Gravitation*, as a law of nature that applies to all objects available to humans and those unavailable in the overall universe. The Law is globally accepted today too, but a modification of Newton's Dynamics involves exactly this Law. It is modification of Newton's Law of Universal Gravitation that inspired the title Modification of Newton's Dynamics, or MOND theory for short. Considering that many specialists adopt Newton's Law as a law of natural attraction between all bodies that are really existing, a number of comprehensible proofs would be needed to alter or replace the Law. Hence, let us first give some relevant Newton's statements on the basis of which it 'was' or 'was not' possible to prove the Law of Universal Gravitation:

$$F = -k\frac{m_1 m_2}{\rho^2}, \tag{4.1}$$

where k is so-called 'universal gravitational constant', whose numeric value is:

$$k = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}.$$

The first chapter of the university textbook FUNDAMENTALS OF CELESTIAL MECHANICS by M. Milanković ([28], p.30) was titled Newton's law of gravitation and its first paragraph Kepler's laws, which refer to the major planets of the solar system, and as such are inseparable from the astronomy of the solar system. Consequently, a question is imposed: What is meant by the title CELESTIAL MECHANICS. The question is not insignificant. The second work to quote is by a distinguished and recognized specialist[5], which represents a complement to the above mentioned attitude: "The foundation of classical mechanics is constituted of Newton's axioms or laws of motion. On that basis, with additional Newton's law of universal gravitation (4.1), Celestial mechanics is built up."

These two approaches are not contradictory. A view that planetary motion is reduced to the motion of material points implies axioms and theorems of a theory. In the theory, the statement (4.1) is one of several Newton's theorems. However, as the formula (4.1) prevails in the textbooks and scientific literature we will often herein refer to that "Newtonian law", without overlooking the remark that this is just one of several Newton's theorems.

In the Preface of his epochal work **"Philsophiænatural principia mathematica"** Newton writes: *It is the task of mathematicians to find*

[5] V.V.Beletski, Ocherki o dvizhenii kosmicheskih tel. Nauka, Moskva, 1972

such a force, which would retain with accuracy a given body moving along a specified orbit at a given speed, and vice versa, to find that curvilinear path on which a body is positioned by a specified force, which leaves a specified location at a specified speed.[6]

In section II, Book I titled OF THE INVENTION OF CENTRIPETAL FORCES, Newton wrote his first statement:

Theorem I.*The areas, which revolving bodies describe by radii drawn to an immovable center of force do lie in the same immovable planes, and are proportional to the times in which they are described.*

Theorem IV. *The centripetal forces of bodies, which by equable motions describe different circles, tend to the centers of the same circles; and are one to the other as the squares of the arcs described in equal times applied to the radii of the circles.*

Corollary 1. *Since those arcs are as the velocities of the bodies, the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely.* Mathematically, in symbols it is

$$F = m\frac{v^2}{R}, \tag{4.2}$$

where m is the factor of proportionality.

Corollary 2.*And since the periodic times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inversely, the centripetal forces are in a ratio compounded of the radii directly, and the duplicate ratio of the periodic times inversely.*

Corollary 6 is conditional, which means that it has an additional condition which states: If the periodic times are in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces will be in the duplicate ratio of the radii inversely; and the contrary.

That condition of the Corollary of IV can be written in mathematical symbols for short:

$$\frac{R^3}{T^2} = const.$$

[6]When the author began his lecture at a Serbian-Bulgarian astronomers' meeting by Milanković' explicit sentence: "Planetary motion is reduced to the motion of material points, which have masses of individual planets. This is a starting point of our today's Celestial mechanics," one of the participants asked: "Why do you quote Milanković, whereas an anonymous reviewer thought that his name should not be associated with celestial mechanics. However, the lecturer maintained his claim that the present textbook is better and easier to use than all others, which he used to prepare his exam in celestial mechanics.

It is is evident by above stated that different constants can be introduced for different proportionality levels, but the essence of the centripetal force magnitude remains the same: it is proportional to the radius of the orbit and inversely proportional to the square of periodic time of motion along the circular path.

In Corollary 6 of Theorem *IV* Newton states that the centripetal force will be inversely proportional to the square of the radius of the circle if the third Kepler's law exists in nature, i.e. if:

$$K = \frac{R_1^3}{T_1^2} = \frac{R_2^3}{T_2^2} = \cdots = \frac{R_n^3}{T_n^3}.$$

Considering that this theorem refers to different circles with different radii, the formula of the theorem should be written more accurately in the form:

$$F = m\frac{v_i^2}{R_i} = m\frac{4\pi^2 R_i^2}{R_i T_i^2}, \tag{4.3}$$

or if bodies of different masses m_i are on different circles:

$$F = m_i\frac{v_i^2}{R_i} = m_i\frac{4\pi^2 R_i^2}{R_i T_i^2}. \tag{4.4}$$

These are physical, i.e. mechanical properties of the material point's motion along a circular line. The property of the force, as obvious, remains the same in different formulas. In accordance with the preprinciples of invariance, that property will not be changed if the previous formula, or formulas, are multiplied by *a* dimensionless unit quotient $R_i^n/R_i^n = 1$, that is

$$F = m_i\frac{v_i^2}{R_i^{n+1}}R_i^n T^2 = m_i = m_i\frac{4\pi^2}{R_i^2}K = m_i\frac{\mu}{R_i^2},$$

where

$$K = \frac{R_i^3}{T_i^2} = const. \tag{4.5}$$

is Kepler's constant, and

$$\mu = \frac{4\pi^2 R_i^3}{T_i^2} = const. \tag{4.6}$$

is Gauss's constant.

It is evident that different constants can be introduced for different proportionality levels, but the essence of the centripetal force magnitude remains proportional to the radius of the circle, and inversely proportional to the square of the time period of motion along a circular line. Proposition. The corollaries of the theorem, as well as the entire body of Newton's Principia indicate explicitly that Newton's concept of proportional did not imply only one particular value of a real number but a constant proportionality factor of relationship between two functions; the constant proportionality factor, most commonly referred to as a constant, can be also the function of invariable values of objects' properties.

Typical examples are Kepler's constant and Gauss's constant (4.5) and (4.6). Newton clarified the additional condition for Corollary 1 of Theorem *IV* saying that: The case of the 6*th* Corollary obtains in the celestial bodies (as Sir Christopher Wren, Dr. Hook, and Dr. Halley have severally observed), and therefore in what follows, I intend to treat more at large than those things which relate to centripetal force decreasing in a duplicate ratio of the distances from the centers.

In Book III titled Of the SYSTEM OF THE WORLD in Hypothesis 1 Newton writes:

The center of the system of the world is immovable, and

Theorem XI: *The common center of gravity of the earth, the sun, and all the planets is immovable.*

Theorem XII: *The sun is agitated by continual motion, but never recedes far from the common center of gravity of all the planets.*

This assumption and the theorem are significant for Newton's theory of gravitation, which necessarily includes Newton's law of gravitation (4.1), make us call it *the Newtonian theory of gravitation.*

Clarification of the concept of constant proved to be necessary, because some prominent-for their title scientists state that the constant, among which is the 'universal gravitational constant', has only a single value of the natural number, even though this problem was extensively treated in the work [3].[7]

How much reliable that theory is with the "Newtonian law of gravitation" is very well shown by the book Physics and Astronomy of the Moon [5]. On the first page titled THE MOTION OF THE MOON IN SPACE the author Andrea Dupree, among other things, writes: "Lunar theory has developed completely differently from other planetary theories." "The Moon under simultaneous attraction of the Earth and Sun revolves around the

[7]This had gone as far as the incomprehension of challenging. To prove that a constant is only a single number, a distinguished university professor presented at a congress the example $y = cx$ using it to prove his statement by equation, but overlooking that for a single value, e.g. $c = 2$, his equation $y = 2x$ represents only one straight line, and for $c = const.$ his equation $y = cx$ represents a family of straight lines in plane xy, which pass through the coordinate origin. Nor did he remember that of the angle which the straight line closes with axis x.

Earth, far from the Keplerian." The previous conclusion about the complexity of the Moon orbiting the Earth, as well as many opponent conclusions, indicate that it is necessary to more explicitly separate the motion of the bodies as material points in respect of Newton's mathematical theory from post-Newtonian theoretical mechanics founded on the principles. When comparing formulas (4.1) and (4.2) such difference is obvious, which is not easy to explain in a simple way. Let us focus on the commentary of Corollary 1, which is written by the relation

$$F = k\frac{v^2}{r}, \tag{4.7}$$

where k is some proportionality factor, for the time being, v is the magnitude of velocity, by which the material point for the time interval T describes a circular line of circumference $2r$, that is $2r\pi$,

$$v = \frac{2r\pi}{T}.$$

Accordingly,

$$F = k\frac{4\pi^2 r^2}{rT^2} = k\frac{4\pi^2}{T^2}r.$$

Considering that by definition attr $F = MLT - 2$, it follows that. from where we find that the proportionality factor has the property of mass m, i.e. atr $k = M$. Therefore, the formula (4.7) can be written in the form:

$$F = m\frac{v^2}{r}.$$

From the stated formulas it is evident that different constants can be introduced for different proportionality levels, but essentially the magnitude of the centripetal force is directly proportional to the square of the velocity of the body and inversely proportional to the radius of the circle r.

Conclusion. This example, as well as the overall body of Newton's Principia, show that Newton did not imply the same real number by the concept of proportional, but the constant factor of relationship between two functions of invariable properties, most often called a constant.

Conditional agreement. It is obvious that formulas (4.1) and (4.2) differ considerably not only in the proportionality factor but also in qualifying the law of gravitation. Let us commence from Newton's second law or axiom, sometimes referred to as the 'basic equation of motion', in the vector form:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}, \tag{4.8}$$

where $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and $\mathbf{r}$ is the material point position vector. In order to determine the magnitude F_r of the force F which acts in the direction of the position vector, scalar multiplication of this equation by unit vector is sufficient that is

$$\mathbf{r}_0 = \frac{\mathbf{r}}{r};$$

i.e.

$$\left(m\frac{d\mathbf{v}}{dt}\right)\cdot \mathbf{r}_o = \mathbf{F}\cdot \mathbf{r}_o = F_r. \tag{4.9}$$

It follows that:

$$\frac{d}{dt}\mathbf{v}\cdot\frac{\mathbf{r}}{r} = \frac{d}{dt}\big(\mathbf{v}\cdot\frac{\mathbf{r}}{r}\big) - \mathbf{v}\cdot\frac{d}{dt}\frac{\mathbf{r}}{r}\big) = \frac{F_r}{m},$$

it follows that

$$\mathbf{v}\cdot\frac{\mathbf{r}}{r} = \frac{d\mathbf{r}}{dt}\big(\mathbf{r}\cdot\frac{\mathbf{r}}{r} - \mathbf{r}\frac{d}{dt}\big(\frac{\mathbf{r}}{r}\big).$$

and furthermore

$$\mathbf{v}\cdot\frac{\mathbf{r}}{r} = \frac{d\mathbf{r}}{dt}\big(\mathbf{r}\cdot\frac{\mathbf{r}}{r} - \mathbf{r}\frac{d}{dt}\big)\big(\frac{\mathbf{r}}{r}\big).$$

Substituting in the initial scalar product of the vector equation of motion, it is obtained that

$$F_r = m\frac{\dot{r}^2 + r\ddot{r} - v^2}{r}. \tag{4.10}$$

So, this is the magnitude of the force acting along the direction of the position vector of the observed material point, directed towards the center.

To better understand our approach to applying classical mechanics to the Solar System, we will observe the example of a system of two material points, of which one is immovable or moves at constant velocity v; this statement is allowable according to Newton's hypothesis 1 and Theorems *XI* and *XII*. On the basis of Newton's first axiom, which no doubt says that:*Every body continues in its state of rest or of uniform motion in a straight line, until acted upon by a force to change that state,* it can be concluded that a given body can be acted upon by several forces if their sum equals zero, i.e. if the body is not acted upon by no matter which or what type of force, it remains in the equilibrium state.

On the basis of the second axiom d, which doubtlessly states that: *The alteration of motion is ever proportional to the motive force and moves in the direction of the right line in which that force acts,* there follows that the resultant force equals zero if in a given example:

$$\mathbf{v} = \mathbf{C}, \longrightarrow \mathbf{r} = \mathbf{r}_0 + \mathbf{C}(t - t_0).$$

This indicates that motion is taking place along a straight line.

The first sentence of the third axiom, which states that to *every action there is always opposed an equal and opposite reaction*, generates interpretation that the inertia force also equals zero in reaction. Such a result is acceptable in human practice and can be locally verified, where Euclidean geometry applies - where there exists a straight line. However, from the viewpoint of the preprinciple of existence in celestial spaces, i.e. celestial mechanics, the existence of the immovable straight line can neither logically nor experimentally be proved.

The first axiom gains in generality if the phrase in a straight line is omitted: it is more general because this means one condition less. This conclusion is clarified by a similar, but not identical,

Example. *Material point in uniform motion.* The notion uniform motion here implies the motion, whose magnitude of velocity is constant, that is $\frac{ds}{dt}v = \frac{ds}{dt} = c = const.$, $\boldsymbol{\tau} = \frac{\mathbf{v}}{v}$.

In such motion Newton's first axiom indicates that the force is not acting upon the material point in the direction of a tangent, but it could be some other force.

From the second axiom and first statement of the third axiom of uniform motion, it follows that:

$$m\frac{d\mathbf{v}}{dt} = mv\frac{d\boldsymbol{\tau}}{dt} = m\frac{v^2}{\rho_k}\mathbf{n} = \boldsymbol{F} = F_n\mathbf{n}.$$

where n is unit vector of the main normal and ρ_k is radius of the trajectory's curve.

So: The material point is in uniform motion along a curved line acted upon by some force directed towards the center of the curve (centripetal force)

$$F_n = m\frac{v^2}{r_k}. \tag{4.11}$$

This **example** is in full agreement with Newton's Theorem IV (Book 1) and its corollaries.

Newton devotes particular attention to Corollary 6 emphasizing that it is significant for celestial bodies, as independently noted by Wren, Hooke and Galileo. Here, we also encounter ([1], p.81) Newton's statement: "This is the centrifugal force, with which the body impels the circle, and to which the contrary force, where with the circle continually repels the body towards the center, is equal."4 On the basis of present-day basic knowledge, the formula (3.3) can be written for a circular line in the forms as follows:

$$F_n = mK_rv^2 = m\frac{4\pi^2R^2}{RT^2} = m\frac{4\pi^2R^3}{R^2T^2} = m\frac{4\pi^2}{T^2}\frac{K}{R^2} = \mu\frac{m}{R^2},$$

where there are common terms: K_r - curve of the path, $K = \frac{R^3}{T^2}$ - Kepler's constant, and $\mu = \frac{4\pi^2}{T^2}K$ - Gauss's constant. Hence, for the same physical quantity different constants can be introduced, and for the same constants their different numeric values. If, for instance, the formula

$$F = m\frac{4\pi^2 R^3}{R^2 T^2} = \mu\frac{m}{r^2},$$

where

$$\mu = \frac{4\pi^2 R^3}{T^2} = const.,$$

$$F = m\frac{4\pi^2 R^3}{R^2 T^2} = \mu\frac{m}{r^2},$$

is multiplied by unity

$$\mu = \frac{4\pi^2 R^3}{T^2} = const.,$$

is multiplied by unity $\frac{M}{M}$, the previous formula will not essentially change, but the form and the proportionality factor will look different:

$$F = Mm\frac{4\pi^2 R^3}{MR^2 T^2} = f\frac{Mm}{R^2},$$

where the proportionality factor is now

$$f = \frac{4\pi^2 R^3}{MT^2}, \qquad art\, f = L^3 M^{-1} T^{-2}.$$

The aim of emphasizing this sentence is to deny some assertions that Newton did not use the notion centrifugal force, nor is the centrifugal force a force. This proportionality factor, which Professor M. Milanković, in his book "Fundamentals of Celestial Mechanics", 2nd ed., Nauchna knjiga, Belgrade, 1955, ([28], p. 44)," denotes with the letter f and writes that it has the same value for all planets and represents a constant that applies to the entire solar system and expresses a general property of matter accumulated in that part of the universe."

However, the manner in which we have arrived at that proportionality factor does not produce a unique conclusion. Actually, we have simultaneously multiplied and divided equation (4.11) by the number M, without determining the value of that number, nor its property, which means that other propositions could have been taken. This is of particular importance, because we haven't yet considered simultaneous motion of a two-body system. That subject will be discussed afterwards. Now, while considering the determination of the force acting upon a single body's motion, if such a body exists in nature, let us discuss as follows:

Task: Determine the magnitude of force acting upon the material point, opposite in direction along the position vector to its pole.

The vector differential equation of motion of the material point:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F},$$

has the velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\mathbf{r}_0 dr}{dt} = \dot{r}\mathbf{r}_0 + r\dot{\theta}\boldsymbol{\tau}_0, \quad \boldsymbol{\tau}_0 \perp \mathbf{r}_0, \quad \mid \boldsymbol{\tau} \mid = 1,$$

where$\mathbf{r}_0$ unit vector of the position vector of some point and τ_0 is unit tangent vector to the circle. Consequently, acceleration vector too is decomposed into radial and transverse acceleration, that is,

$$\mathbf{w} = \frac{d\mathbf{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_2.$$

Scalar multiplication of this differential equation by unit vector $\mathbf{r}_0 = \mathbf{e}$ yields the required magnitude of force in the form:

$$F_r = m(\ddot{r} - r\dot{\theta}^2),$$

alternatively, considering that $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$, in the form

$$F_r = m\frac{\dot{r}^2 + r\ddot{r} - v_{or}^2}{r}.$$

Obviously, mass m is here the proportionality factor, which would remain the same assuming that $r = const.$, but in that case it could be reduced to other proportionality factors via algebraic calculations, as well as via previously mentioned relations (4.4).

Note. Further comprehension of the application of mechanics to the motion of celestial bodies points out the fact that previous examples and assumptions refer to the motion of a single, of any, and therefore of every individual material point of mass m. However, it is not easy to notice, nor assume, that only one body is moving independently of others. That is why the agreed basic object of the theory of celestial mechanics is a two-material-point system. The emphasis is placed on 'a two-point system', not on two individual points. The term system indicates that material points are connected in some way, affect each other's motion, have their program of motion, and that it is not sufficient to write two differential equations of motion:

$$m_i\frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i, \quad i = 1, 2. \tag{4.12}$$

without mathematical conditions coupling differential equations of motion. These two differential equations contain four unknown vectors $\mathbf{v}_1, \mathbf{v}_2; \mathbf{F}_1, \mathbf{F}_2$. To obtain any solution, two existing independent conditions should be added. The conditions can be imposed by the program, but here let's find them as generally existing in the nature of things. Newton's third axiom contains the condition

$$\mathbf{F}_1 = -\mathbf{F}_2, \tag{4.13}$$

and the second condition is manifest, observational and logical, existing as a distance between those material points, which can be reduced to vector equation

$$\mathbf{r}2 - \mathbf{r}_1 = \rho. \tag{4.14}$$

Now, the motion of the system of two bodies, as material points is complete. As such, it enables determination of the forces if velocity and distances are known, i.e. velocity on the path s, or determination of velocities and positions of material points if forces are known or specified.

4.1. The inverse two-body task. In standard theory of celestial mechanics, the concept of the two-body problem implies determination of paths and velocities of motion, at given Newton's gravity forces, which are inversely proportional to the squares of distances, or according to Hamilton, at potential energy inversely proportional to the distance between material points. In both cases the task is solved by means of integral calculus, which in specific cases does not produce finite and invariantly accurate solutions. This problem was a challenging task for many mathematical and mechanical giants. Why has it remained the problem but not the task? The notion problem implies a scientific task of doubtful solution to a challenging issue. If this is true, isn't doubt cast over centuries-long visible solution of planetary motion? Unlike the predecessors, who started from Newton's Law of Universal Gravitation (4.1), here we are solving the basic task of checking the validity of Newton's Law, i.e. we are solving the inverse task of mutual interaction between two bodies and determining the force with which the two material points are mutually attracted. That task was easy to solve by means of Lagrange's multipliers of constraints [6], but as it proved the majority of specialists were not familiar with the method, so let's choose a shorter one, and it is the vector calculus.

According to Newton's second axiom or law of motion, there are two vector differential equations:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_1, \quad m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_2. \tag{4.15}$$

Without loss of generality, let's differentiate equation

$$\dot{\rho} = \dot{\mathbf{v}}_2 - \dot{\mathbf{v}}_1. \tag{4.16}$$

Substituting acceleration and from differential equations of motion (4.13) into the previous relation, it is obtained:

$$\frac{\mathbf{F}_2}{m_2} - \frac{\mathbf{F}_1}{m_1} = \ddot{\rho},$$

respectively, considering the above relation,

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{m_1 m_2}{m_1 + m_2}\ddot{\rho}, \tag{4.17}$$

or

$$-\mathbf{F}_2 = \mathbf{F}_1 = -\frac{m_1 m_2}{m_1 + m_2}\ddot{\rho}. \tag{4.18}$$

This is a simple, but significant relation. It cannot be denied for any two material points in mechanics that further determination and interpretation of the forces of mutual interaction between two celestial bodies cannot be focused on this relation. It says at first sight that the forces by which one body acts upon the other are proportional to the change of relative velocity of one body relative to the other, or proportional to accelerated change of mutual distance. Simply put, it is not a problem of any kind, but a simple task that is reduced to identifying vector distance between inertia centers of these bo dies, the proportionality factor being a reduced mass

$$\frac{m_1 m_2}{m1 + m_2}$$

Note that mentioned result refers to any two material points, and as such it is more general than the formula of Newton's Law of Gravitation. So, the forces of mutual interaction between two bodies are proportional to accelerations $\ddot{\rho}$. In order to compare them to "Universal gravitational force", scalar multiplication of vector equation of motion by unit vector $\rho_0 = frac\rho\rho$ is necessary and sufficient. As shown, a formula for the magnitude of force F of mutual interaction between any two material points at distance ρ is obtained in the form:

$$F_\rho = \frac{m_1 m_2}{m_1 + m_2}\frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho}, \tag{4.19}$$

where $v_{or} = v_2 - v_1$. Note that this formula is considerably more general than formula (4.1) and that it symbolizes Newton's theorems on mutual attraction between two bodies, without taking into account Kepler's Third Law.

Coordinate method. The same result is obtained when respective motion of the material point along a circular path of radius $\rho = \rho(t)$ is observed relative to the rectangular system of coordinates x and y in the plane (xy), or in the space (xyz), i.e. when there is the constraint

$$f(x,y) = (x_2 - x_1)^2 + (y_2 - y_1)^2 - \rho^2 = 0.$$

Differential equations of motion of the observed points are The first and second derivative of function f with respect to time are

$$\dot{f} = 2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + 2(y_2 - y_1)(\dot{y}_2 - \dot{y}_1) = 2\rho\dot{\rho}, \tag{4.20}$$

(4.21)

$$\ddot{f} = (\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2 + (x_2 - x_1)(\ddot{x}_2 - \ddot{x}_1) + (y_2 - y_1)(\ddot{y}_2 - \ddot{y}_1) = \dot{\rho}^2 + \rho\ddot{\rho}.$$

If we substitute derivatives $\ddot{x}, \ddot{y}$ from differential equations of motion into previous equations (4.21), we obtain

$$(x_2 - x_1)\frac{X_2}{m_2} - (y_2 - y_1)\frac{Y_2}{m_1} = \dot{\rho}^2 + \rho\ddot{\rho}. \tag{4.22}$$

Taking into account equation (4.12), according to which $X_1 = X_2$; $Y_1 = Y_2$, previous equation is reduced to

$$F = \sqrt{X_2^2 + Y_2^2} = \frac{m_1 m_2}{m_1 + m_2}\frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho}, \tag{4.23}$$

where $v_{or}^2 = (\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2$ is orbital velocity of the motion of the point. For the case when $\rho = R = const.$ it is obtained that which is in accordance with Newton's Theorem IV, that is,

For the conditions of Kepler's First and Second Law, or for the Third law only, previous formula is reduced to:

$$f = \frac{4\pi^2 a^3}{(m_1 + m_2)T^2}. \tag{4.24}$$

So, from previous statements it explicitly follows that the gravitational force does not depend on distances only, but on planetary parameters: masses, mean distances and rotation periods. The difference is not only formal. In applying our formula to solving the problem of two to three bodies, the Sun, Earth and Moon, it eliminates the paradox generated by Newton's formula for gravity force.

4.2. Paradoxes of the theory of lunar motion. In a large-circulation not[8] there appears a question: **Why doesn't the Moon fall into the Sun?** "The question may seem naive", the author writes, "but when the readers learn that the Sun attracts the Moon by a larger force than the Earth, they expose suspicion and surprise." Using simple calculations he

[8]Ya. I. Pereleman, "Zanimateljnaya astronomiya", str. 64.

shows that the attraction force of the Sun is greater than the attraction force of the Earth, $\frac{330000}{160000}$, by two times. A higher mathematical level book gives a more specific information: Sun's gravity is stronger by 2.5 times than that of the Earth. Note that such paradox is caused by the theory of a widely known formula (4.1). Specifically, according to that formula, force. with which the Sun attracts the Moon of mass m is

$$F_\odot = -k\frac{M_\odot m}{\rho_\odot^2}, \tag{4.25}$$

and the magnitude of force $F_\oplus$ with which the Earth attracts the Moon is

$$F_\oplus = -k\frac{M_\oplus m}{\rho_\oplus^2}. \tag{4.26}$$

The ratio of these two quantities is

$$\frac{F_\odot}{F_\oplus} = \frac{M_\odot}{M_\oplus}\frac{\rho_\oplus^2}{\rho_\odot^2}.$$

For the known numeric values it follows that $M_\odot = 19891 \times 10^{26}$ kg, $M_\oplus = 597 \times 10^{22}$ kg; $\rho_\odot = 1496 \times 10^8 m$, $\rho_\oplus = 384.4 \times 10^6 m$ follows that $F_\odot \approx 2.1820 F_\oplus$ which indicates that the magnitude of the attraction force of the Sun $F_\odot$ is greater by over 2 times than the attraction force of the Earth to the Moon. Hence, the theory of Newton's gravity force in the observed case for two bodies lead to unacceptable dynamical paradox. No wonder that there are comments by reputable specialists for lunar astronomy. "Lunar theory - one of the most difficult problems of celestial mechanics - has been developing completely differently from other planetary theories." ([43], p. 9). Such statements too readily bring into question both Kepler's and Newton's basics of celestial mechanics.

Our approach to the problem commences from the axiom of classical mechanics, by means of which we have obtained that radial acceleration is

$$w_\rho = \frac{\dot\rho^2 + \rho\ddot\rho - v_{or}^2}{\rho},$$

without referring to Kepler's laws.

Without loss of generality, let's introduce onto that plane a polar coordinate system $\rho, \theta, \rho_0, \theta_0$ relative to which there exists radial velocity $\dot\rho$ and transverse velocity $\rho\dot\theta$. It is well known that with respect to that system of coordinates, radial acceleration has the form because

$$w_\rho = \ddot\rho - \rho\dot\theta^2 \quad \text{and} \quad v_{or}^2 = \dot\rho^2 + \rho^2\dot\theta^2.$$

The inverse proof also holds. It is well known and easily provable that radial acceleration corresponds to covariant derivative of radial velocity

$$w_\rho = \frac{D\dot{\rho}}{dt} = \ddot{\rho} - \rho\dot{\theta}^2 = \ddot{\rho} - \frac{\rho^2\dot{\theta}^2}{\rho}.$$

In the literature it has been shown how much radial accelerations of a satellite are at different altitudes H above the Earth's surface, according to standard formula

$$\gamma = g\frac{R^2}{\rho^2}, \tag{4.27}$$

as well as according to the formula

$$\ddot{\rho} = \gamma^* = g\frac{v^2}{\rho^?}, \tag{4.28}$$

as shown by the following scheme:

altitudes	velocities	accelerations	accelerations
H km	v km/s	γ	γ^*
0	7,91	981,0	982,3
100	7,84	948,9	950,0
1000	7,35	732,1	733,0
10000	4,93	148,4	148,4
100000	1,94	3,5	3,5
384400	1,02	0,002693	0.002706

Note that the last column of the table refers to mean velocity of the Moon's motion around the Earth and its mean distance from the center of the Earth. It indicates that the magnitude of force F of mutual interaction between two moving bodies of masses m_1 and m_2 is in accordance with Newton's basic laws of dynamics. For constant distance between the centers of their masses, it can be written that

$$F = -\frac{m_1 m_2}{m_1 + m_2}\frac{v_{or}^2}{\rho}. \tag{4.29}$$

which complies with Newton's Theorem IV.

4.3. Elimination of the lunar paradox. In accordance with above stated and formula, the force with which Earth of mass $M_\oplus = 5,97 \times 10^{24}$kg attracts the Moon of mass $m = 0,0739 \times 10^{24}$ kg, at mean distance $\rho = 384400km$ and mean velocity $v = 1,02km/s$ would be equal to

(4.30)

$$F_\oplus = -\frac{M_\oplus m}{M_\oplus + m}\frac{v_{or}^2}{\rho_\oplus} = 0,987839876 \cdot \frac{(v_\oplus + 1,02 - v_\oplus)^2}{384400} m = 0,0026736\, m,$$

because

$$\frac{M_\oplus}{M_\oplus + m} = 0,987839878.$$

For the Sun of mass $M_\odot = 1,9891 \times 10^{30}$ kg and the Moon of mass$m$, the force of attraction will be

$$F_\odot = -\frac{M_\odot m}{M_\odot + m}\frac{v_{or}^2}{\rho_\odot}. \quad (4.31)$$

For known numerical values[9]

$$\frac{M_\odot}{M_\odot + m} = \frac{19891 \times 10^{26}}{19891 \times 10^{26} + 0,000735 \times 10^{26}} = 0,999999, \quad (4.32)$$

and formula (4.31) shows that:

$$F_\odot = 0,999999\frac{v_{or}^2}{149,6 \times 10^6}m. \quad (4.33)$$

Further calculations, as evident from (4.31), depend of numeric value of the Sun's velocity, and in astronomy that number can be determined by a single number. In the books we encounter the following: "All stars (that belong to our galaxy - the Milky Way), including our Sun, are moving relative to each other at mean velocity of 30km/s, i.e. at the velocity at which our planet moves along its orbit." In the books of higher mathematical level [118], [119], [120] the velocity of solar motion in km/s is

$$V_0 = 20$$

determined more accurately And even more accurately in the work [118], relying upon the book by P. G. Kulikovsky ([120], p. 78), the velocity is given in km/s for the Sun:

$$V_\odot = 19,6$$

and for given motion Let's calculate the Sun's gravity force in km/h for 'standard velocity'

$$V_\odot = 19,5.$$

At critical position A, for distance ρ. and reduced mass, we obtain the magnitude of force with which the Sun attracts the Moon at Sun's velocity of

Compared to the magnitude of gravity force of $2,6736436 \times 10^{-3}m$, with which the Earth attracts the Mon, that is,

$$F_\oplus = 0.00267306,$$

[9]Ya. I. Pereleman, "Zanimateljnaya astronomiya", p. 64.

we obtain that the Earth's gravity force is stronger than the corresponding Sun's gravity force by more than 4 times, assuming that paths are circular lines.

Elliptical motion. It is well known that high accuracy has been established for the fact that elliptical paths of the Moon and Earth differ a little from circular trajectories. Approximation in calculations is even greater, when it is well known that Kepler's laws refer to mean distances. This is clearly indicated by the eccentricity of the Moon's orbit $e = 0,0549$ and eccentricity of the Earth's orbit $e = 0,0681$. To keep the Moon on elliptical path, it is necessary and sufficiently for radial acceleration to be equal to zero, that is,

$$w_\rho = \ddot{\rho} - \rho\dot{\theta}^2 = 0,$$

that is

$$\ddot{\rho} = \rho\dot{\theta}^2. \tag{4.34}$$

The component of transverse acceleration w_θ, with respect of Newton's third axiom, is equal to zero, that is,

$$w_\theta = \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) = 0.$$

Consequently, as it is well known

$$\rho^2\dot{\theta} = C = \frac{2\pi ab}{T},$$

where T is sidereal time of Moon's revolution.

Furthermore, based on previous equations, it is obtained that

$$\gamma = \ddot{\rho} = \rho\frac{4\pi^2a^2b^2}{\rho^4T^2} = \frac{4\pi^2a^2b^2}{a^3T2} = \frac{4\pi^2}{T^2}a(1-e^2) = 0,0027136,$$

amounting approximately to 0,002706555 in mentioned work, which is obtained for circular motion. For lunar motion at mean distance from the Sun (along a mean trajectory at the distance of the Earth from the Sun) we obtain even more approximate results, considering that the eccentricity of the Earth is smaller than the eccentricity of the Moon.

So, the considered dynamical paradox of the theory of lunar motion has been fairly accurately eliminated, irrespective of Newton's universal formula of gravity force and a two-body theory. It is reasonably theoretical but not realistic result, because the Sun and Moon do not represent an isolated system of two material points. It is certainly a more realistic.

4.4. The inverse three-body problem. Newton's task of three-body system - Sun-Earth-Moon, but this is not inseparable from other planets,determining the force with which the Sun and Earth simultaneously affect lunar motion belongs to a familiar three-body problem. Let's try to solve this problem too by means of our formula for the gravity force that acts between any two material points. Similar to the view of mathematical two-body problem, here the notion 'three-body' problem implies considerations of mutual interaction between three material points.

In a general case, let's observe three points: M_1, M_2, M_3,

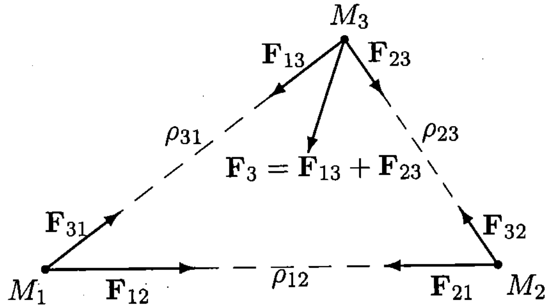

FIGURE 9

Gravity sphere In that way, both forces acting simultaneously upon the Moon and their sum were determined. For the force in the form (4.38), let's conclude: first, it differs considerably from a corresponding expression for Newton's force and, second, formula (4.38) is not based on Kepler's laws, as indicated by quoted contemporary astrophysicists.

For the Moon's extreme positions in the points $A(t = 0), C(t = \frac{T}{2})$ it is easy to calculate the magnitudes of forces, which show meaningful and interesting results, as evident from (4.40). The performed calculations deployed well known quantities $m3$ is mass of the Moon;, used in solving a two-body problem.

$$F_\odot = \kappa\frac{m_1 m_3}{\rho^2}. \quad F^*_\odot = \chi\frac{m_1 m_3}{\rho}$$

	$F_\odot$	$F^*_\odot$
A	0,0059605 m_3	0.000577 m_3
C	0,0058995 m_3	0.000854 m_3
	$F_\oplus$	$F^*\oplus$
A, B, C	0,002695 m_3	0.002673 m_3

The forces of the Sun and Earth acting upon the Moon According to the general view of the notion gravitation, the author of this work implies that gravitational sphere or gravity sphere is space limited by the sphere, where the gravity force of one body, whose center of inertia is in the center of the sphere, is stronger than gravity forces of other bodies outside that sphere. Specifically, the Earth's gravity sphere is the space around the Earth (as a material point) where the Earth's force of attraction is stronger than the gravity forces of other bodies, including the Sun's gravity force.

4.5. Modification of Earth's gravity sphere. Earth's gravity sphere is a space around the Earth (as a material point) where in the Earth's force of attraction is stronger than the gravitational forces of other bodies, including the gravitational force of Sun. The formula that determines the radius ρ of the so called sphere of influences (gravy sphere) of the Earth's gravity in this case is ([1], p. 196),

$$\rho = r\sqrt[5]{(m_1/M)^2}, \tag{4.41}$$

where r is the distance between the Earth and Sun, $m_1 = M_\oplus$ is the mass of the Earth, and $M_\odot \approx 333000\, m_1$ is the mass of the Sun. The size of this radius of the Earth's sphere amounts approximately to

$$\rho = 917\,000\, km \tag{4.42}$$

or ([2], p.108) 923 000km.

Verification of the formula (4.41) with the use of the Newton's formula of "universal gravitational force"

$$F = \kappa\frac{m_1 m_2}{\rho^2}. \tag{4.43}$$

Led to a paradoxical result. According to formula (4.41), at the boundary of the Earth's gravity sphere, it should be $F_\oplus = F_\odot$. However, the calculation shows the opposite. And indeed, let us show this with some more details.

Let it be assumed that: $m_1 = M_\oplus$ is the mass of the Earth, $m_2 = M_\odot$ is the mass of the Sun, and m is the mass of any body at the boundary $\rho_\oplus = x = 917\,000$ km. For the above mentioned assertions of the book the mass of the Sun is $M_\odot \approx 333\,000 M_\oplus$, whereas a tabulated distance of the Earth from the Sun is $\rho_\odot = a = 149\,600\,000$ km.

First. The Sun and the Earth act at the same time on a body having the mass m in a critical boundary point at the distance $\rho_\oplus = x$ with the forces according to Newton's formula:

$$F_\oplus = \kappa\frac{M_\oplus m}{x^2}, \qquad F_\odot = \kappa\frac{M_\odot m}{(r-x)^2}.$$

Therefore, in a critical point $\rho_\oplus = x$, it should be

$$F_\odot = \kappa\frac{M_\odot\, m}{(149\,600\,000 - x)^2} = 1,506335 \times 10^{-11}\kappa M_\oplus m. \tag{4.44}$$

and

$$F_{\oplus} = \kappa \frac{M_{\oplus} m}{x^2} = 0,11892 \cdot 10^{-11} \kappa M_{\oplus} m.$$

This shows that, according to Newton's formula, the gravitational force of the Sun at the distance of 917 000 km from the center of the Earth is more than 12 times greater than the value of the Earth's gravitational force, i.e.

$$F_{\odot} = 12,666\,611\, F_{\oplus}.$$

However, this is not in compliance either with the definition of gravity sphere, or with the phenomena in the nature. The Moon moves around the Earth at an average distance of 384 400 km, under the dominant attraction of the Earth, not the Sun.

According to the Newton's formula, gravitational forces (3) at the boundary $\rho = x = 917\,000$ km of the gravisphere of the Earth are:

$$F_{\odot} = \kappa \frac{M_{\odot} m}{(149\,600\,000 - x)^2} = 1,5063 \cdot 10^{-11}. \tag{4.45}$$

and

$$F_{\oplus} = \kappa \frac{M_{\oplus} m}{x^2} = 0,11892 \cdot 10^{-11}$$

The ratio of these forces is

$$F_{\oplus} : F_{\odot} = \frac{1}{917000^2} : \frac{333000}{148683000^2} = 0.11892 \cdot 10^{-11} : 1.5063 \cdot 10^{-11} = 0.0789495.$$

This would imply that the Earth's force of attraction at the boundary of its gravity sphere is significantly less, $F_{\oplus} = 0,789 F_{\odot}$, than the Sun's gravitational force, which represents dynamical paradox.

The second. Let's determine the boundary of the Earth's gravity sphere with the use of a strict procedure, by means of the universal gravity formula (4.43).

According to the Newton's gravity theory (4.43),

$$F_{\oplus} = \kappa \frac{M_{\oplus} m}{x^2}, \quad F_{\odot} = \kappa \frac{M_{\odot} m}{(\rho - x)^2},$$

would follow, so that it should be:

$$\frac{M_{\oplus} m}{x^2} = \frac{M_{\odot} m}{(r - x)^2},$$

or for $M_{\odot} = 333\,000 M_{\oplus}$

$$\frac{(\rho - x)^2}{x^2} = \frac{M_{\odot}}{M_{\oplus}} = 333\,000.$$

Further calculation gives:

$$(\rho - x)^2 = (577,6152\, x)^2,$$

i.e.

$$\rho - x = 577,6152\,x^2,$$

or

$$\rho = 578,0652\,x,$$

and from there, for $\rho = 149\,600\,000\,\text{km}$, it follows that

$$x = 258\,795,993\,km.$$

This is contradictory to the fundamental laws of dynamics, as well as the actual state of the motion of the Moon around the Earth at an average distance of 384 400 km, and particularly the formula (1), which demonstrates the radius of the sphere of the Earth's gravity. Doubt about the validity of the Newton's formula is increased by a facts from the very above mentioned book ([1], p. 193-201). According to the Newton's formula (1) it follows that the acceleration of gravity depends not only on the distance, but it is asserted that at the first cosmic velocity of 7,91 km/s, a body will escape from the Earth's attraction and will rotate around the planet Earth under an assumption that the resistance of the medium is ignored. At the second cosmic velocity $v_{or} = 11,19$ km/s, a missile will leave the area of the Earth's gravisphere.

The third. In his historical and still unequalled work "mathematical principles of natural philosophy" Is. Newton tells with his Theorem IV, Consequence 6 (Volume 1), as well as with Theorem VII (Volume 3), that he was acquainted with: normal acceleration of Huygens, Kepler's laws, as well as Galileo's measurement of the acceleration $g \approx 9,81$ m/s of body falling under gravity. Those facts confirmed his mathematical principles. By comparing the formula of the Earth's gravitational force (3) with gravity $G = -mg$, it followed

$$g = \kappa \frac{M_\oplus}{R^2},$$

where R is a radius of the Earth at the equator, whereas κ is "universal gravity constant". This is sufficient to calculate, even today, the value of the acceleration of gravity at any distance x from the center of the Earth,

$$\gamma = \kappa \frac{M_\oplus}{x^2},$$

from where it follows

$$\gamma^* = \frac{R^2}{x^2} g.$$

Based on this formula, in the university textbook we find a table ([1], p. 194). At the first sight of the formula (1) and the table, disharmony of the formula and the table 1 is evident; the formula clearly demonstrates that the acceleration depends only on the distance, whereas in the table, the dependency on the speed and distance is clear

Altitude	Velocity	Acceleration
H km	v km/s	γ
0	7,91	981,0
100	7,84	948,9
1000	7,35	732,1
10000	4,93	148,4
100000	1,94	3,5

Table 1.

These provable facts point to the verification of the Newton's formula of the value for the force of mutual attraction of two bodies (4.43) exactly from the perspective of his axioms of mechanics. This is shown in several ways in papers [1], which also can be easily verified here.

4.6. Modification of universal gravity formula. In communicated and published book and papers [2],[3],[6], it is demonstrated that our formula of mutual action of two bodies has a form,

$$F_\rho = \frac{\dot\rho^2 + \rho\ddot\rho - v_{or}^2}{m_1 + m_2}\frac{m_1 m_2}{\rho}, \tag{4.46}$$

or, in Simić's form

$$F_\rho = \frac{m_1 m_2}{m_1 + m_2}\frac{\frac{d}{dt}(\rho\dot\rho) - v_{or}^2}{\rho}. \tag{4.47}$$

For the escaping boundary of the attraction of a body having a mass of m and the body having a mass of M, it will be

$$\frac{Mm(\dot\rho^2 + \rho\ddot\rho - v_{or}^2)}{(M+m)\rho} = 0, \tag{4.48}$$

or, in Simić's form,

$$\frac{d}{dt}(\rho\dot\rho) - v_{or}^2 = 0. \tag{4.49}$$

For average speeds of planets or satellites, average speeds $v_{or} = const.$ are usually considered, the equation (9) shows a relation between the distance

$$\rho = \sqrt{v_{or}^2 t^2 + bt + h} \tag{4.50}$$

and speeds in the state when the force of mutual attraction equals zero. For the purpose of clearer and more straightforward comprehension of this assertion, let us mention that formula (7), in relation to the natural coordinate system, can be reduced to a simpler form. It is sufficient to observe that it is

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2$$

so as to reduce the formula (4.47) to a form

$$F_\rho = \frac{m_1 m_2}{m_1 + m_2}(\ddot{\rho} - \rho\dot{\theta}^2). \tag{4.51}$$

In the state of motion where $F_\rho = 0$, the known formula for normal acceleration follows

$$\ddot{\rho} = \rho\dot{\theta}^2 = \frac{v^2}{\rho}, \tag{4.52}$$

as well as formula for the force of mutual attraction

$$F^{**} = \frac{m_1 m_2}{m_1 + m_2}\frac{v^2}{\rho}, \tag{4.53}$$

where $\rho = R = const.$ It has been shown (See above mentioned Table 1) what the radial accelerations of the satellites are at different altitudes H above the Earth according to the standard formula

$$\gamma = g\frac{R^2}{\rho^2}, \tag{4.54}$$

as well as the formula

$$\gamma^* = \frac{v^2 c}{c} \tag{4.55}$$

which follows from the formula

$$F_\oplus = F_\odot; \tag{4.56}$$

Altitude	Velocity	Acceleration	Acceleration
H km	v km/s	γ	γ^*
0	7,91	981,0	982,3
100	7,84	948,9	950,0
1000	7,35	732,1	733,0
10000	4,93	148,4	148,4
100000	1,94	3,5	3,5
384400	1,02	0,002693	0.002706

Table 2.

Let's note that the last type of table refers to the average speed of the Moon's motion around the Earth and its average distance from the center of the Earth.

By the application of formula (12) to the motion of the Moon in relation to the Sun and in relation to the Earth, it has been proven that the gravitational force of the Earth, which acts on the Moon, is greater than the corresponding force of the Sun. In this way, dynamical paradox in the theory of the Moon's motion has been removed. It is logical that it is possible to determine the boundary of the Earth's gravity sphere in the same way.

Using this procedure, we obtain a significant modification of the Earth's gravitation sphere (1) and (2). Starting from the aforementioned definitional of the gravity sphere of two bodies, let us find the boundary x of the gravisphere of the Earth in relation to the gravitational force of the Sun for that same body. By the very nature of things and by mathematical logics, initial relation of that task is that the gravitational force of the Earth is greater than, and at the boundary of the sphere $\rho = x$ is equal to, the Sun's gravitational force, i.e., where :

$$F_\oplus = \frac{M_\oplus m}{M_\oplus + m}\frac{v^2_{or\oplus}}{x}, \quad v_{or\oplus} < 1km,$$

$$F_\odot = \frac{M_\odot m}{M_\odot + m}\frac{v^2_{or\odot}}{a-x}, \quad v_{or\odot} = 29,8 - (19,5 + 0,3) = 10,$$

Ratio of the gravitational forces $F_\oplus$ and $F_\odot$ at the boundary of the Earth's gravisphere is:

$$\frac{F_\oplus}{F_\odot} \equiv \frac{v^2_{or\oplus}}{x} : \frac{v^2_{or\odot}}{a-x} = 1.$$

From here, it follows that

$$x = \frac{a}{1 + \left(\frac{v_{or\odot}}{v_{or\oplus}}\right)^2}. \tag{4.57}$$

Value of the fraction which is derived, depends, as we can see, on the ratio of the orbital speeds of bodies in relation to the Sun and the Earth at the boundary x of the Earth's gravisphere. Let us analyze that for our needs.

First

$$v_{or\odot} \neq v_{or\oplus},$$

because it is

$$v_{or\odot} = v_\oplus \pm v_{or\oplus} - v_\odot; \quad v_\oplus \neq v_\odot.$$

The second: for $v_{\oplus} = 1$ is

$$x = \frac{a}{1 + v_{or\odot}^2}.$$

The third: For $v_{or} > 1$ the value of the fraction is decreased, and already for $v_{or} > 1$ the fraction (17) is decreased, and for $v_{or} < 1$ it is increased. In view of the fact, let as choose $v_{or} = 1$. As it can be seen, the boundary of the Earth's gravisphere depends on the ratio of the speeds of two bodies in relation to the Earth $v_{or\oplus}$ and in relation to the Sun $v_{or\odot}$. Usually the speed $v_{or\oplus}$ is not known, so that we are left only with a hypothetical analysis on the basis of the average standard data. The speed of the Sun $v_{or\odot}$is even less known. Speeds of the Sun in relation to various groups of stars [8]. The standard speed of the Sun is usually taken to be $v_{\odot} = 20\,000$ km/s. Since the mean speed of the Earth's motion around the Sun is $v_{\oplus} \approx 30\,000$ m/s. In this state of motion, it is

$$v_{or\odot} \approx v_{\oplus} - v_{\odot} = 10\,km/s.$$

For this logical choice and numerical values of the standard quantities:

$$\frac{M_{\oplus}\,m}{M_{\oplus} + m} = 0,987,$$

$$\frac{M_{\odot}\,m}{M_{\odot} + m} = 0,999,$$

$$a = 149\,600\,000 km, \quad M_{\odot} = 333\,000\,M_{\oplus},$$

it is obtained that the radius of the gravi sphere of the Earth is $x = 1\,481\,188$ km, or

$$x \approx 1\,481\,000 km.$$

Therefore, for the standard data which are taken, the radius of the geosphere of the Earth is $x = 1481188$ km, or

$$x \approx 1481000 km$$

Therefore, for the standard data which are taken, the radius geosphere the geosphere of the Earth is significantly greater than the radius (2).

Corollary. In the first part of this paper it is proven that the formula of the gravitational sphere of the Earth (1) has not been derived on the basis of the Newton's formula (3). By direct calculation with the use of the formula (3) it is shown that the formula leads to the results, which are not in accordance with the nature of the motion between the Sun and the Earth. Convincing example is the motion of the Moon, for which the formula (3) leads to paradoxical dynamic result of the Newton's gravity theory.

With the use of the formula (6) for the mutual attraction of two bodies, the above mentioned paradox in the theory of the Moon's motion is removed and one solution to the problem of three bodies (Sun-Earth-Moon) is obtained. That was a reason to consider the boundary (2) of the gravisphere of the Earth in this paper. Approximately correct result for the radius of the Earth's gravisphere on the basis of the formula (16) amounts to 1 400 000 km, which is considerably different from the value (2). In this analysis, difficulty in choosing the Sun's speed is emphasized. Based on our formulas (4) and (5) radius of the Earth's gravity sphere is obtained (17). By this formula, it is easy to determine the speeds that condition the result of the Tisserand's boundary of geosphere

$$917\,000 = \frac{a}{1 + \left(\frac{10}{v_{or\oplus}}\right)^2},$$

or

$$917\,\frac{100}{v^2_{or\oplus}} = 149\,600 - 917.$$

It follows that

$$v_{or\oplus} = 0,785\, km/s.$$

At various values of speeds, which have been taken in the consideration by the author of this paper, it is shown that the equation (4.18) gives interesting indicators of the permissible ratio of the orbital speeds of bodies in relation to the Sun and the planet Earth, as material points.

$$F_\oplus = \kappa\frac{M_\oplus m}{x^2}. \tag{4.57}$$

$$F_\odot = \kappa\frac{M_\odot m}{x(r-x)^2}. \tag{4.58}$$

Therefore, in a critical point $\rho_\oplus = x = 917\,000$, it should be

$$F_\odot = \kappa\frac{M_\odot\, m}{(149\,600\,000 - x)^2} = 1,5063\cdot 10^{-11}\kappa M_\odot m.$$

and

$$F_\oplus = \kappa\frac{M_\oplus\, m}{x^2} = 0,11892\cdot 10^{-11}\kappa M_\oplus m.$$

'This shows that, according to Newton's formula, the gravitational force of the Sun at the distance of 917 000 km from the center of the Earth is more than 12 times greater than the value of the Earth's gravitational force, i.e.

$$F_\odot = 12,666611\, F_\oplus \Longleftrightarrow F_\oplus = 0,0789478\, F_\odot.$$

However, this is not in compliance either with the definition of gravity sphere, or with the phenomena in the nature. The Moon moves around the Earth at an average distance of 384 400 km, under the dominant attraction of the Earth, not the Sun.

Let's determine the boundary of the Earth's gravity sphere with the use of a strict procedure, by means of the universal gravity formula (4.42). According to the Newton's formula, gravitational forces (4.43) at the boundary $\rho = x = 917\,000$ km of the gravisphere on the Earth are:

$$F_{\odot} = \kappa \frac{M_{\odot} m}{(149\,600\,000 - x^2} = 1,5063 \cdot 10^{-11}.$$

and

$$F_{\oplus} = \kappa \frac{M_{\oplus} m}{x^2} = 0,11892 \cdot 10^{-11}.$$

The racio of these forces is

$$F_{\oplus} = \kappa \frac{1}{91700^2} : \frac{333000}{148683000^2} : \frac{333000)}{148683000^2} = 0.11892 \cdot 10^{-11} = 0.789495.$$

This would imply that the earth's force of attraction at the boundary of this gravity sphere is significantly less, $F\oplus = 0.789\ F\odot$, than the Sun's gravitational force, which represents dynamical paradocs.

The second.

$$\frac{M_{\oplus} m}{x^2} = \frac{M_{\odot} m}{(r - x)^2},$$

or for $M_{\odot} = 333\,000 M_{\oplus}$ follow $(\rho - x)/x)^2 = M_{\odot}/M_{\oplus} = 333\,000$.

Further calculation gives: $(\rho - x)^2 = (577,6152\,x)^2$, i.e. $\rho - x = 577,6152\,x$, or $\rho = 578,0652\,x$, and from there, for $\rho = 149\,600\,000$ km, it follows that

$$x = 258\,795,993\text{km}.$$

This is contradictory to the fundamental laws of dynamics, as well as the actual state of the motion of the Moon around the Earth at an average distance of 384 400 km, and particularly the formula (1), which demonstrates the radius of the sphere of the Earth's gravity. Doubt about the validity of the Newton's formula is increased by a fact from the above mentioned book. According to the Newton's formula (1) it follows that the acceleration of gravity depends not only on the distance, but it is asserted that at the first cosmic velocity of 7,91 km/s, a body will escape from the Earth's attraction and will rotate around the planet Earth under an assumption that the resistance of the medium is ignored. At the second cosmic velocity $v_{or} = 11,19$ km/s, a missile will leave the area of the Earth's gravity sphere.

4.7. Modification of the theory of gravity. In the papers [5, 7, 8, 9] author demonstrated that our formula of mutual action of two bodies has the form

$$F_{\rho} = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2} \frac{m_1 m_2}{\rho} = M^* \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho} = F^* + F^{**}. \tag{4.59}$$

where we introduced notations:

$$M^* = \frac{m_1 m_2}{m_1 + m_2}, \quad F^* = M^* \frac{\dot{\rho}^2 + \rho\ddot{\rho}}{\rho}, \quad F^{**} = M^* \frac{v_{or}^2}{\rho}.$$

For the escaping boundary of the attraction of a body having a mass of m and the body having a mass of M, it will be

$$M^* \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho} = 0,$$

or in Simić's form

$$\frac{d}{dt}(\rho\dot{\rho}) - v_{or}^2 = 0.$$

For the purpose of clearer and more straightforward comprehension of this assertion, let us mention that formula (4.46), in relation to the natural coordinate system, can be reduced to a simpler form. It is sufficient to observe that it is $v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2$ so as to reduce the formula (4.46) to a form

$$F_\rho = M^*(\ddot{\rho} - \rho\dot{\theta}^2).$$

In the state of motion where $F_\rho = 0$, the known formula for normal acceleration follows

$$\ddot{\rho} = \rho\dot{\theta}^2 = \frac{v^2}{\rho},$$

as well as formula for the force of mutual attraction

$$F^{**} = M^* \frac{v^2}{\rho}, \tag{4.60}$$

where $\rho = R = const.$

It has been shown what the radial accelerations of the satellites are at different altitudes H above the Earth according to the standard formula $\gamma = gR^2/\rho^2$, as well as the formula $\gamma^* = v^2/\rho$, which follows from the formula (4.46).

Altitude	Velocity	Acceleration	Acceleration
H km	v km/s	γ	γ^*
0	7,91	981,0	982,3
100	7,84	948,9	950,0
1000	7,35	732,1	733,0
10000	4,93	148,4	148,4
100000	1,94	3,5	3,5
384400	1,02	0,002693	0.002706

Let's note that the last type of table refers to the average speed of the Moon's motion around the Earth and its average distance from the center of the Earth.

By the application of formula (4.47) to the motion of the Moon in relation to the Sun and in relation to the Earth, it has been proven that the gravitational force of the Earth, which acts on the Moon, is greater than the corresponding force of the Sun. In this way, dynamical paradox in the theory of the Moon's motion has been removed, [4.47]. It is logical that it is possible to determine the boundary of the Earth's gravity sphere in the same way.

Using this procedure, we obtain a significant modification of the Earth's gravitation sphere. Starting from the aforementioned definitional of the gravity sphere of two bodies, let us find the boundary x of the gravity sphere of the Earth in relation to the gravitational force of the Sun for that same body. By the very nature of things and by mathematical logics, initial relation of that task is that the gravitational force of the Earth is greater than, and at the boundary of the sphere $\rho = x$ is equal to, the Sun's gravitational force, i.e.,

$$F_{\oplus} = F_{\odot}, \tag{4.60}$$

where :

$$F_{\oplus} = \frac{M_{\oplus} m}{M_{\oplus} + m} \frac{v_{or\oplus}^2}{x}, \quad v_{or\oplus} < 1\,\text{km/s},$$

$$F_{\odot} = \frac{M_{\odot} m}{M_{\odot} + m} \frac{v_{or\odot}^2}{a - x}, \quad v_{or\odot} = 29,8 - (19,5 + 0,3) = 10\,\text{km/s}.$$

Ratio of the gravitational forces $F_{\oplus}$ and $F_{\odot}$ at the boundary of the Earth's gravity sphere is:

$$\frac{F_{\oplus}}{F_{\odot}} \equiv \frac{v_{or\oplus}^2}{x} : \frac{v_{or\odot}^2}{a - x} = 1.$$

From here, it follows that

$$x = \frac{a}{1 + (v_{or\odot}/v_{or\oplus})^2}. \tag{4.61}$$

Value of the fraction which is derived, depends, as we can see, on the ratio of the orbital speeds of bodies in relation to the Sun and the Earth at the boundary x of the Earth's gravity sphere. Let us analyze that for our needs.

First: $v_{or\odot} \neq v_{or\oplus}$, because it is $v_{or\odot} = v_{\oplus} \pm v_{or\oplus} - v_{\odot}$; $v_{or\oplus} \neq v_{\odot}$.

The second: For $v_{\oplus} = 1$ is $x = a/(1 + v_{or\odot}^2)$.

The third: for $v_{or} > 1$ the value of the fraction is decreased, and already for $v_{or} > 1$ the fraction (8) is decreased, and for $v_{or} < 1$ it is increased. In view of the fact, let as choose $v_{or} = 1$. As it can be seen, the boundary of the Earth's gravy sphere depends on the ratio of the speeds of two bodies in relation to the Earth $v_{or\oplus}$ and in relation to the Sun $v_{or\odot}$. Usually the velocity $v_{or\oplus}$ is not known, so that we are left only with a hypothetical analysis on the basis of the average standard data. The velocity of the Sun $v_{or\odot}$is even less known. Speeds of the Sun in relation to various groups of stars [4]. The standard velocity of the Sun is usually taken to be $v_{\odot} = 20\,000$ km/s. Since the mean velocity of the Earth's motion around the Sun is $v_{\oplus} \approx 30\,000\,\text{km/s}$. In this state of motion, it is

$$v_{or\odot} \approx v_{\oplus} - v_{\odot} = 10\,\text{km/s}. \tag{4.62}$$

For this logical choice and numerical values of the standard quantities (see for example [3]:

$$\frac{M_{\oplus} m}{M_{\oplus} + m} = 0,987; \quad \frac{M_{\odot} m}{M_{\odot} + m} = 0,999;$$
$$a = 149\,600\,000km, \quad M_{\odot} = 333\,000\,M_{\oplus},$$

it is obtained that the radius of the gravitation sphere of the Earth is $x = 1\,481\,188$ km, or

$$x \approx 1\,481\,000\,\text{km}. \tag{4.63}$$

Therefore, for the standard data which are taken, the radius of the gravitation sphere of the Earth is significantly greater than the radius $x = 917\,000$km, and expressly than $x = 258\,795$km.

Conclusion. In the first part of this paper it is proven that the formula of the gravitational sphere of the Earth (4.41) has not been derived on the basis of the Newton's formula (2). By direct calculation with the use of the formula (4.42) it is shown that the formula leads to the results, which are not in accordance with the nature of the motion between the Sun and the Earth. Convincing example is the motion of the Moon, for which the formula (2) leads to paradoxical dynamic result of the Newton's gravity theory.

With the use of the formula (4.46) for the mutual attraction of two bodies, the above mentioned paradox in the theory of the Moon's motion is removed and one solution to the problem of three bodies (Sun-Earth-Moon) is obtained. That was a reason to consider the boundary of the gravity sphere of the Earth in this paper. Approximately correct result for the radius of the Earth's gravity sphere on the basis of the formula (9) amounts to **1 400 000** km, which is considerably different from the value 917 000 km and 258795km.

MOND5 - KEPLER-NEWTON'S LAW OF GRAVITATION

Previous statements concerning our modification of the fundamentals of celestial mechanics differ substantially from classical theory of gravitation, with embedded Kepler's laws. In book [35][10], the author Andrea Dupree, among other things, writes: The Moon being under simultaneous attraction by the Earth and Sun moves around the Earth in the orbit, far away from the Keplerian. At the seminar on the history of mathematics, mechanics and astronomy, there have recently been organized a number of discussions about Kepler's laws, particularly Kepler's third law, ended by a lecture in December 2011. Kepler's laws were accepted as the laws of nature, so it was to be shown as follows: If all three Kepler's laws can be derived by the help of Newton's gravity law, then Newton's law is the law of nature, and as such it cannot be modified. The first sentence indicates awareness of the open problem. That these laws are coupled is undeniable, so let's try to find the undeniable solution in this exposition. To this end, we will present only those statements that contribute to easier understanding of proofs, using Kepler's laws and "generalized Kepler's laws". In the book ASTRONOMY AND COSMOLOGY, a short chronological guide from ancient times to the present day, Kiev, 1967 [36] by S.A. Seleshnykov it is written down:

1609. *A great work by J. Kepler was published in Prague, The New Astronomy, Based upon Causes, or Celestial Physics, Treated by Means of Commentaries on the Motions of the Star Mars, from the Observations of Tycho Brahe, gent.*

1. The planets move in elliptical orbits with the sun at one focus.

2. Radius vectors sweep out equal areas in equal times.

1618. The Five Books of the HARMONY OF THE WORLD, 15 May 1618. The third law of planetary motion is:

3. *periodic times are proportional to the cubes of the semi-major axes of their orbits.*

In mathematical symbols it can be written as:

$$\frac{T_1}{T_2} = (\frac{a_1}{a_2})^{3/2}. \tag{5.1}$$

1686 Newton writes: Any planet, in accordance with Copernicus's hypothesis 2, orbits the Sun in an ellipse, with the Sun at one focus. Phenomenon IV. Stellar orbital periods of major planets, and also of the Sun around the Earth, and vice versa, are proportional to the semi-cube of their mean (note,

[10] Physics and Astronomy of the Moon, 2nd ed, Ed. By Zdenchek Kopal, Dept. of Astronomy, Univ. of Manchester; Academic Press, NY and London, 1971. Translated into Russian. Revised by Leykin, MIR, Moscow,1973, Ch.1, p.9.

distances from the Sun. This Kepler's finding is recognized by everybody, as Newton writes ([1], p. 508).

In his university textbook of celestial mechanics [28] M. Milankovic devoted the first paragraph of Chapter 1 to Kepler's laws:

I.All planets circle the Sun in elliptical orbits; the Sun is at the common focus of the ellipses.

II.Radius vector drawn from the Sun to the planet sweeps out equal areas in equal periods of time.

III. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

In this rarely easy-to-understand and rather short textbook (98 pages), Milankovic demonstrates Kepler's second law using the mathematical formula

$$r^2\frac{d\nu}{dt} = C,$$

where ν is the angle called true anomaly.

He formulates the third law in a simple and explicit manner, using the equation

$$\frac{a^3}{T^2} = \kappa, \tag{5.2}$$

where K is the same number for all planets.

Later, in 1983, the author of the book ([37], p. 48) writes: "Kepler's third law is expressed in the form

$$T_1^2 : T_2^2 = a_1^3 : a_2^3,$$

which is readily derived from

$$T^2 = \frac{4\pi^2 a^3}{\lambda}, \tag{5.3}$$

and further on *"The squares of the orbital periods* (of planets around the Sun, satellites around the planets)*are proportional to the cubes of the semi-major axes of their orbits* (mean distances of a movable object from the central object).

"Note this extension of Kepler's third law. As evident, this formulation differs somehow from previous formulations of Kepler's third law, which refers to the planets of the solar system, but not to the satellites orbiting the planets, especially to any of the two bodies. In addition, Kepler's law is not derived, but is set up on the basis of natural phenomena. The formula is said to be mathematical expression of Kepler's law. This is not as simple as it might seem if it is not known what λ represents in the expression. In truth, on page 45 of his book the author writes that ? is the so-called characteristic gravitational constant related to a specific body as a gravitational parameter of the body of mass m_1

$$\lambda = \kappa^2 m_1 \tag{5.4}$$

In that case, equation (5.3) holds for any planet of mass m_2,. From the previous text of the book we learn that k^2 is "a universal gravitational constant" (the same number), and m_1 is mass of a designated body; if it is about the revolving of the planets around the Sun, it will be considered that $m_2 = M$, i.e. the mass of the Sun. In that case, equations ([37], p. 48) are as follows:

$$T_1^2 = \frac{4\pi^2}{\lambda} a_1^3, \quad T_2^2 = \frac{4\pi^2}{\lambda} a_2^3.$$

Division of two equations yields Kepler's law set up for revolving of the planets around the Sun. Other additions in formulations represent generalization of Kepler's laws.

Proof or generalization of Kepler's laws. In books ([19], pp. 374-375) and ([37], p. 54) we read: "If two planets, whose masses are m1 and m2, are observed traveling around the Sun in elliptical orbits, with semi-major axes a1 and a2, it will be"

$$T_1^2 = \frac{4\pi^2}{\lambda} a_1^3, \quad T_2^2 = \frac{4\pi^2}{\lambda} a_2^3. \tag{5.5}$$

However, now $\lambda_1 = k^2(M + m_1)$ and $\lambda_2 = k^2(M + m_2)$. Dividing T_1^2 by T_2^2, it is obtained

$$\frac{T_1^2}{T_2^2} = \frac{M + m_2}{M + m_1} \frac{a_1^3}{a_2^3}, \tag{39**}$$

and this is the improved Kepler's law that holds when the central body is not immovable, i.e. when the mass of one body is not substantially smaller than the mass of the other body.

This approach is sustainable only if λ_1 and λ_2 1 are the same numbers, which is not the case in the planetary system, and it will be proved further below in considerations of the force of attraction between two bodies.

Corrections of Kepler's laws can be encountered in other authors too. Let us quote a very interesting and highly professional book "A general theory of revolving of the Earth", whose authors Z.S.Erzanov and A.A.Kolybaev write: Kepler's:

First (generalized) law. *Unperturbing orbit of the point M2 relative to the point M_1 represents the second-order curve, where at one of the focuses the point M_1 is located and its focal axis directed along the Laplace vector l.*

Kepler's **second** (generalized) law. *Sector velocity of the point M_2 motion relative to the point M_1 remains constant over the whole time of motion, and an area of a sector described by the radius vector* $\mathbf{r}$ *of the point M_2 changes proportional to time.*

Kepler's **third** (generalized) law. *In an unperturbing elliptical orbiting of two material points the products of the squares of orbital periods and the sums of masses of the central and material point that is moving, are proportional to the cubes of the major axes of the orbits.*

Mathematically presented, it is more explicit:

$$T_1^2(M_0 + M_1)/T_2(M_0 + M_2) = a_1^3 : a_2^3.$$

Newton's theorems of mutual attraction between two bodies are largely founded on Kepler's laws. It is even possible to come across the term the Keplerian-Newtonian theory. Here, we are trying to prove that there is a mismatch in the connection between Kepler's laws and Newton's mathematical theory, leading to unsustainable conclusions. In the scientific literature there are generalizations or corrections of Kepler's laws, and having in mind this fact, it is sensible to check how much and in what way this affects Newton's theorems of the body motion.

5.1. Newton's law of gravitation. Professor Milankovic ends the second paragraph of his book with the formula

$$P = f\frac{m_1 m_2}{r^2}, \tag{5.6}$$

describing it like this: Every particle of matter in the universe attracts every other particle with a force that is directed toward these particles, and its intensity is proportional to the product of the masses m_1 and m_2 of the particles, and inversely proportional to the square of the distance r between them.

In hat case, f is the proportionality factor, a universal constant, denoted by the formula

$$f = 4\pi^2\frac{a^3}{T^2}\frac{1}{m_1},$$

respectively

$$f = 4\pi^2\frac{a^3}{T^2}\frac{1}{m_1}, \tag{5.7}$$

where M is the mass of the Sun. At the same time, the formula of universal "Newton's law of gravitation" has widespread use in the form

$$F = k^2\frac{m_1 m_2}{r^2}, \tag{5.8}$$

were k, si for example ([37], p. 39) 3"Nauka", Moscow, 1984, pp.92-95. (5.8) where k (refer to, for example ([37], p.30),

$$F = f\frac{Mm}{\rho^2}, \tag{5.9}$$

where is we find the magnitude of the gravitational force in the form (5.9) where

$$f = 4\pi^2\frac{a^3}{T^2}\frac{1}{M+m} \tag{5.10}$$

The faculty professor mentioned above states that [29] is

$$F = k^2\frac{Mm}{\rho^2}, \qquad k^2 = 6,67\times 10^{-11}.$$

There is a big difference. The proportionality factor (5.10) changes from one planet to the other, such that

$$f_i = \frac{4\pi^2}{M+m_i}\frac{a_i^3}{T_i^2}.$$

Dividing f over f_i, it is obtained

$$\frac{f}{f_i} = \frac{a^3}{a_i^3} = \frac{M+m_i}{M+m}\frac{T_i^2}{T^2}.$$

Only provided that the proportionality factors f and f_i were equal, the relation of "improved Kepler's law" could be obtained. However, it is obvious that f and f_i therefore previous relation, referred to as "generalized Kepler's law",, nor is it correct. Kepler's law does not include masses, so it is sufficient to state that the corrected Kepler's law is not Kepler's law. In addition, generalization or modification of Kepler's laws cannot be founded on mathematical transformations; the laws are formulated based on observations and identification of measured data on planets' motion of the solar system.

5.2. Gravity forces. The significance of Kepler's laws has been emphasized by Newton's describing the motion of the body. This is interpreted by MilankoviĆ better than by anyone else. First, he observes the motion of a single body, in which Kepler's laws for calculating the gravitational constant come to the fore with geometrical accuracy. Afterward, Milankovic solves two-body motion, where there occurs change in the constant proportionality factor, when formulating gravity forces.

Let us commence from Corollary 1 of Newton's Theorem *IV* ([1], p. 78) that Newton himself based his proof on of a general theorem (Book *III*, Theorem *VIII*, p. 519).

Theorem *IV*. *If there are two homogeneous spheres mutually gravitating one to the other, equidistant from their centers, the gravitation of each sphere by the other is inversely proportional to the square of distance between their centers.*

The centripetal forces of bodies, which by equable motions describe different circles, tend to the centers of the same circles; and are one to the other as the squares of the arcs described in equal times applied to the radii of the circles.

Corollary 1. *Since those arcs are as the velocities of the bodies the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely.*

The corollary written down mathematically represents the centripetal force F in the formula form:

$$F_i = m_i \frac{v_i^2}{R_i}, \tag{5.11}$$

(where R_i are radii of the radius, as denoted by Newton. If T_i is used to denote time intervals over which the material point describes a full circle, the above formula can be written in the form

$$F_i = m_i \frac{4R_i^2\pi^2}{T_i^2 R_i}, \tag{5.12}$$

alternatively, without changing neither magnitude nor property, the force F_i can be written

$$F_i = m_i \frac{4\pi^2 R_i^2 R_i^n}{T_i^2 R_i^n} = m_i \frac{4\pi^2 R_i^{n+2}}{T^2 R_i^{n+1}}. \tag{5.13}$$

Corollary 6. *If the orbital periods are in a sesquiplicate ratio of the radii, the centripetal forces are in the duplicate ratio of the radii inversely; and the contrary.*

The Corollary can be written in the form

$$F_i = m_i \frac{4R_i^3\pi^2}{T_i^2} \frac{1}{R_i^2} = m_i 4\pi^2 \frac{R^3}{T^2} \frac{1}{R_i^2} = m_i 4\pi^2 \frac{K}{R_i^2},$$

where, as obvious, $K = \frac{R_i^3}{T_i^2}$ is Kepler's third law if R_i were the semi-major axis of an ellipse. Note also the consequential fact that all formulas (5.11), (5.12) and (5.13) can be reduced to direct proportionality of the radius of a circular line, or at different nonlinear proportionality factors of the radius. This teaches us again that Newton did not imply that the proportionality factor is the same number for the whole universe. If it were that $R = a$, which is not the case, but $R \approx a$, we could talk of the motion on the ellipse and approximate accuracy of the law of gravitation. However, it cannot be talked like that, so we continue to seek the solution. It is well known in differential geometry and analytical mechanics that normal acceleration is equal to

$$w_n = \frac{v^2}{R_k}, \tag{5.14}$$

where R_k is the radius of curvature of the path. This is in total compliance with mentioned Newton's Theorem *IV*, because the center of curvature of the circular line is exactly its center, which is not the case with an ellipse. On the semi-major axis of the ellipse the curvature radius is $R_k(A) = \frac{b^2}{a}$, and at the point B on the semi-minor axis it is $R_k(B) = \frac{a^2}{b}$. Consequently, normal accelerations at points A and B are not equal for $a \approx b$. This subject matter is more extensively discussed by Newton's

Proposition *X*. problem *V*

A body is moving on the ellipse; to find the law of the centripetal force directed to the center of the ellipse.

By geometrical procedure ([1], pp. 88,89) with the help of a drawing of the ellipse in two ways, Newton proved that the sought centripetal force is directly proportional to the distance of the point on the ellipse to its center. So, it is the same as in mentioned uniform motion of the material point along a circular line. This is to be confirmed, but here the motion cannot be uniform along the arcs as in the circle. Indeed, in order to provide a more explicit proof, the equations of central ellipse are described by equations

$$x = a\cos\theta(t), \quad y = b\sin\theta(t). \tag{5.15}$$

The first and second derivatives with respect t time are

$$\dot{x} = a\sin\theta\dot{\theta}, \quad \dot{y} = b\cos\theta\dot{\theta}.$$

$$\ddot{x} = -a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta},$$

$$\ddot{y} = -b\sin\theta\dot{\theta}^2 - b\cos\theta\ddot{\theta}.$$

Furthermore, it follows

$$\ddot{x}^2 + \ddot{y}^2 = a^2(\cos^2\dot{\theta}^4 + \sin^2\theta\ddot{\theta}) + b^2(\sin^2\theta\dot{\theta}^4 + \cos^2\theta\ddot{\theta}) = r^2\dot{\theta}^4 + r^2\ddot{\theta} = w^2,$$

racio, respectively

$$w = r\sqrt{\dot{\theta}^4 + \ddot{\theta}^2}.$$

It is obvious from here that the square of accelerated motion on the ellipse has two addends ([29], p. 29), the same as in a circle: the square of normal acceleration $R^2\dot{\theta}^4$ and the square of tangential acceleration $R\ddot{\theta}$. This would mean that motion on the ellipse, in a general case, is non-uniform. At the condition that $\theta = \omega t$, $\omega = \frac{2\pi}{T}$ tangential acceleration is $R\ddot{\theta} = 0$, and therefore $w_n = R\omega^2$. In that case, the centripetal force sought, acting on the material point that is moving on the ellipse, is

$$F_r = m\omega^2 R = m\omega^2 \frac{R^2}{R} = m\frac{v^2}{R}$$

directly proportional to the square of velocity v^2 and inversely proportional to the radius R provided that angular velocity w is a constant, and circular motion is uniform.

5.3. Motion on the eccentric ellipse. By using a likewise procedure (refer to, for example ([7], p. 194), let us determine the force acting on the planet, of mass m, which moves on the ellipse, in whose one focus the Sun of mass M is located. Let the focus with the Sun be at distance c from the center of the ellipse, and let the planet's distance from the Sun be ρ. With respect to that focus, the coordinates of the planet's center on the ellipse are:

$$x = (a-c)\cos\varphi(t) = a_k\varphi(t), \quad y = b\sin\varphi(t).$$

where, for providing a more explicit proof, the notation $a_k = a - c.$ is introduced. Now, using the procedure from the previous example of the central ellipse (refer to, for example ([37], p. 112)

$$\rho^2 = a_k^2\cos^2\varphi + b^2 sin^2\varphi, \qquad w = \rho\sqrt{\dot{\varphi}^4 + \ddot{\varphi}^2},$$

for uniform circular motion $\varphi = \omega t$, $\omega = \frac{2\pi}{T}$, it follows that $\varphi = \omega t$, $\omega = \frac{2\pi}{T}$, and furthermore $\varphi = \omega t$, $\omega = \frac{2\pi}{T}$, following

$$w_\rho = \rho\omega^2, \qquad w_\rho = \frac{\omega^2\rho^3}{\rho^2} = \frac{\omega^2 a_k^3}{\rho^2} = \frac{\omega^2(a-c)^3}{\rho^2},$$

where $a_k = a - c$.

Let us express the line segment $c = ae$ by means of the eccentricity of the major planets.

Planets eccentricity e: Mercury 0.20561 Venus 0.00682 Earth 0.01675 Mars 0.09331 Jupiter 0.04833 Saturn 0.05589 Uranus 0.04634 Neptune 0.00900

Afterward, let us calculate deviations from Kepler's third law for each planet, using the relation

$$\frac{4\pi^2 a_k^3}{T^2} = \frac{4\pi^2(a-c)^3}{T^2} = \frac{4\pi^2\big(a(1-e)\big)^3}{T^2} = \frac{4\pi^2 a^3(1-e)^3}{T^2}.$$

Planets Kepler's constant K Mercury 0.500130K Venus 0.000672K Earth 0.950586K Mars 0.794875K Jupiter 0.861904K Saturn 0.833503K Uranus 0.866732K Neptune 0.097322K

Mean deviation of the constant K of Kepler's law is **0.613220**. Hence, the application of Kepler's third law is only approximately accurate, like Newton's law of gravitation derived from Kepler's law. The relations of improved Kepler's third law are obviously related to two-body systems. In that case, according to some specialists, we have Newton's laws in the form

$$F_1 = f_1 \frac{Mm_1}{\rho^2} \quad and \quad F_2 = f_2 \frac{Mm_2}{\rho^2}, \tag{5.16}$$

where forces f_1 and f_2 are determined by formulas (5.10) and according to the others ([29], p. 187)

$$F_1 = k^2 \frac{Mm_1}{\rho^2} \quad and \quad F_2 = k^2 \frac{Mm_2}{\rho^2}, \tag{5.17}$$

where

$$k^2 = 6,67 \times 10^{-8} cm^3 g^{-1} sec^{-2}.$$

Dividing constants f_1 and f_2, it is again obtained

$$\frac{f_1}{f_2} = \frac{a_1^3}{(M+m_1)T_1^2} : \frac{a_2^3}{(M+m_2)T_2^2}.$$

It is only for the case that $f1 = f2$, which cannot be, it would follow:

$$\frac{T_1^2}{T_1^2} = \frac{(M+m_2)a_1^3}{(M+m_1)a_2^3}.$$

But this is not the case either in theory or in practice, so this allegedly improved Kepler's third law is not Kepler's law, nor is it correct. Previous analysis contains mainly Kepler's third law and Newton's law of gravitation determined by formulas (5.16) and (5.17).

However, according to Newton, the basic tasks of mechanics are:

1. to find the force if motion is known, and

2. to accurately determine motion if the force is known. Solving the task based on Newton's axioms, the author of the booklet [40]7 determined the force of mutual interaction between two bodies in two to three ways by using Newton's axioms. Instead of the forces

$$F_\rho = f \frac{Mm_2}{\rho^2} \quad or \quad F_\rho = k^2 \frac{Mm_2}{\rho^2}$$

a more general formula was found for mutual interaction between two bodies in the form

$$\mathcal{F} = \frac{m_1 m_2}{m_1 + m_2} \frac{\dot\rho^2 + \rho\ddot\rho - v_{or}^2}{\rho} = \chi \frac{m_1 m_2}{\rho}. \tag{5.18}$$

where ρ is the distance between two material points.

It is noticeable for the condition that the distance does not change, i.e.$\rho = const.$ from where Corollary 1 of Newton's Theorem IV follows, and our analysis of the application of Kepler's third law started from Corollary

6 of that Theorem. Using our previous formula (5.18) for mutual attraction between two bodies, we have solved the problem of the lunar motion paradox.

5.4. Differential equations of planetary motion. Previously emphasized discrepancy in proportionality factors, i.e. whether f and k^2 are the same number valid for the whole universe, or they differ from one planet to the other, lead to substantial difference in differential equations of planetary motion. According to book ([37], p. 49) the Sun

$$m\frac{d^2\mathbf{r}}{dt^2} = -f\frac{m(M+m}{r^3}\mathbf{r}, \tag{5.19}$$

whereas, according to book ([19], p.375)

$$m\frac{d^2\mathbf{r}}{dt^2} = -k\frac{m(M+m)}{r^3}\mathbf{r}, \tag{5.20}$$

] Since the left-hand sides of equations are equal, the right-hand sides should be equal too, but they are not, because $f \neq k^2$. Given that

$$f = \frac{4\pi^2 a^3}{(M+m)T^2},$$

differential equation of planetary motion (5.19) is reduced to whereas, equation (3.20) remains unchanged

$$m\frac{d^2\mathbf{r}}{dt^2} = -\kappa f\frac{m(M+m)}{r^3}.$$

This means that the magnitude of the force attracting the planets towards the Sun, according to some specialists, is equal

$$F = m\frac{4\pi^2 a^3}{T^2 r^2} = m\frac{\mu}{r^2}, \quad \mu = \frac{4\pi^2 a^3}{T^2},$$

$$m\frac{d^2\mathbf{r}}{dt^2} = -f\frac{m(M+m)}{r^3}\mathbf{r} = -m\frac{4\pi^2 a^3}{T^2}\frac{\mathbf{r}}{r^3};$$

while that same force, according to those adhering to the standards, is

$$F = -k^2\frac{m(M+m)}{r^2}.$$

Based on above presented, it can be concluded that "universal gravitational constant" is not universal, and it is only by equalizing with the proportionality factor f that differential equation of planetary motion is invariant, and therefore valid in terms of the preprinciple of invariance.

Lastly, the question asked at the seminar was: What are the differences between provisions given in books? Some more important attitudes could be singled out:

Group 1 Group 2

Newton's law of gravitation

$$F = f\frac{m_1 m_2}{r^2} \qquad F = k^2\frac{m_1 m_2}{r^2}$$

$$f = \frac{4\pi^2 a^3}{(m_1+m_2)}T^2 \qquad k = 6,67\times 10^{11} m^3 kg^{-1} T^{-2}.$$

$$\mathbf{F} = \frac{m_1 m_2}{m_1+m_2}\ddot{\rho} \qquad \mathbf{F} = k^2\frac{m_1 m_2}{r^3}\mathbf{r}$$

Differential equations of planetary motion

$$F = f\frac{m_1 m_2}{r^2} \qquad F = k^2\frac{m_1 m_2}{r^2}$$

$$f = \frac{4\pi^2 a^3}{(m_1+m_2)}T^2 \qquad k = 6,67\times 10^{11} m^3 kg^{-1} T^{-2}.$$

$$\mathbf{F} = \frac{m_1 m_2}{m_1+m_2}\ddot{\rho} \qquad \mathbf{F} = k^2\frac{m_1 m_2}{r^3}\mathbf{r}$$

Kepler's third law

$$\frac{a^3}{T^2} = K = konst. \qquad \frac{T_1^2}{T_2^2} = \frac{M+m_2}{M+m_1}\frac{a_1^3}{a_2^3}.$$

Conclusion: 1.Kepler's third law is approximately accurate for planetary motion with mean deviation of **0.0613220.**

2. By our formula, the gravitational force between two bodies, which is not derived by means of Kepler's laws, but based on Newton's axiom, is

$$F_c = \mathcal{M}\frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho} = \mathcal{M}\Big(\ddot{\rho} - \frac{v_{or}^2}{\rho}\Big),$$

where $\mathcal{M} = \frac{Mm}{M+m}$ is a reduced mass. Here, it is quite apparent that the force of mutual interaction between two bodies equals zero if

$$\frac{d^2\rho}{dt^2} = \frac{v^2}{\rho},$$

that is, if *the centripetal force is equal to the centrifugal force.*

Our formulas that describe spontaneous and programmed motion of two bodies do not depend on Kepler's laws, but they can be derived from these laws, for specific conditions, and therefore "the Newtonian law of gravitation" is only provisionally correct if it is based on Kepler's laws.

5.5. Four-dimensional spaces of classical and celestial mechanics. The concept of relative space will be explained by the help of the notion of a system, which implies a single or several points connected by one or more than one geometrical or rheonomic constraints. The distance from one to the other point is measured by length, whose property or essence is expressed by the symbol L (Lat. longus). Many authors dealing with differential geometry consider the motion of points as well. However, the concept of motion is a part of Kinematics (movement, motion). Consequently, Kinematics is a part of rational mechanics just as Geometry is a part of mathematics. *Geometric point* is a basic notion of geometry, therefore it is unnecessary to explain it. However, the position or place of a point is defined in various ways, most commonly by means of three measures of the same geometric attribute L. Since vector is defined as the triplet of numeric value, orientation and sense, the position of a point can be defined relative to any observation point by means of one point position vector,**r**. Kinematic point differs from geometric point in that it is set into motion or is moving, and motion cannot be separated from the concept of existing time. Time is not geometrically "*a* natural parameter", but it is an independent variable, denoted by the letter t, possessing the property T. The basic notion used in kinematics is velocity defined as the distance s moved per unit of time t. Since the distance has the property L and time has the property T, the basic kinematics' notion v has a physical dimension or property LT^{-1}. The difference between geometry and kinematics shows the difference between their properties, that is,

$$L \neq LT^{-1}.$$

In order to provide the proof for the subtitle of this section, let us recollect that there is a simple constraint between geometry, i.e. line segment s, whose property or attr is $s = L$, and kinematics, established by means of velocity v, attr $v = LT^{-1}$, line segment and time t, that is,

$$s = vt \longrightarrow L = \frac{L}{T}T.$$

If is it even assumed that $|v| = 1$, the distance travelled and time cannot be equalized, because

$$attrs \neq attr\, t.$$

At constant speed $v = v_0$ it follows that $ds = v_0 dt$, because attr $v_0 = LT-1$, but $ds = dt$ not at all. These indicators inspire us to seek a four-dimensional geometric-kinematic position of some point.

Standard and modified kinematics metrics. In analytical mechanics there is an established view that the mechanical system composed of N points M_ν $(\nu = 1, 2, \ldots N)$, whose positions are defined by orthonormal coordinates y and linked by $k < 3N$ of mutually independent geometric constraints

$$(5.21) \qquad f_\mu(y_1, \ldots, y_{3N}) = 0, \qquad |\frac{\partial f_\mu}{\partial y_\nu}| \neq 0, \quad \mu = 1, \ldots, k,$$

have 3N - k DOF, i.e. point positions can be defined by means of $n = 3N - k$ independent generalized coordinates $q^1, \ldots, q^n$. Note that in the literature there is not a unique notion of generalized coordinates. "Generalized coordinates denoted by the letters q^i can represent in general all coordinates of point positions in different coordinate systems. However, the notion of *independent generalized coordinates* q^α implies those rectilinear or curvilinear coordinates of independent solutions for equations (5.21). Constraints are objects and as such they are invariant relative to linear transformations of rectilinear coordinates into curvilinear coordinates $x^1, , x^n$, and therefore relative to generalized coordinates $q^1, \ldots q^n$,

$$(5.22) \qquad \mu(y^1, \ldots, y^{3N}) = f_\mu(x^1, \ldots c, x^{3N}) = 0,$$

$$(5.23) \qquad f_\mu(q^1, \ldots, q^n) \equiv 0.$$

Mentioned condition that follows, based on the theorem of implicit functions, has explicit meaning in kinematics, and it reads that constraints (5.21) should satisfy the conditions of velocities

$$\frac{f_\mu}{dt} = \frac{\partial f_\mu}{\partial y^i}\frac{dy^i}{dt} = \frac{\partial f_\mu}{\partial y^i} v^i_\mu = 0.$$

Metrics of such systems of linked points is described in geometry by invariant expressions

$$(5.24) \qquad ds^2 = \delta_{ij} dy^i dy^j = g_{ij}(x) dx^i dx^j = g_{\alpha\beta}(q) dq^\alpha dq^\beta.$$

Kinematic constraints. If the coordinates of the points or their constraints change in time, it is common in standard mechanics and analytical geometry to represent the constraints (5.21) by functions

$$(2.25) \qquad f_\mu(y_1 \ldots, y_{3N}, t)$$

At the same time, it is "found" that the number of DOF, i.e. the number of independent coordinates, is increased by 1, i.e.$n = 3N - k + 1$, where t is denoted as the $(n+1)th$ coordinate. However, the expression (5.25) allows dimensional non-homogeneity, leading to incorrect conclusions. For example, equations (5.25) include the form of the functions: $f = y1 + y2 + t = 0$, which is impermissible, because $(y) \neq atr(t)$.

In work [20] and monograph [21] it is shown that kinematic equations of constraints should be written in the form

$$f_\mu(y_1(t),\ldots,y_{3N}(t),\tau(t))=0. \tag{5.26}$$

and not in the form (5.25). The function $\tau(t)$ is a known objective indicator of constraints' changing in time. If several constraints change in different ways, such as

$$f_\mu(y^1(t),\ldots,y^{3N}(t),\tau_\mu(t))=0, \tag{5.27}$$

it is always possible to choose one known function q^{n+1}, from the set $\tau(t)$, so that t can be defined as a function of q^{n+1}; even from one equation of constraint, where, it is for example,

$$\tau(t)=a+b\sin\omega t,$$

and

$$q^{n+1}=\tau(t)=a+b\sin\omega t\longrightarrow t=\frac{1}{\omega}\arcsin\frac{q^{n+1}-a}{b},$$

$$q^{n+1}=b\sin\omega\, t\longrightarrow t=\frac{1}{\omega}\arcsin\frac{q^{n+1}}{b},$$

$$q^{n+1}=\sin\omega\, t\longrightarrow t=\frac{1}{\omega}\arcsin q^{n+1},$$

$$q^{n+1}=\omega\, t\longrightarrow t=\frac{q^{n+1}}{\omega}.$$

Substituting t into (1.7), it is obtained

$$f_\mu(y^1,\ldots,y^{3n}). \tag{5.28}$$

Let us point out that equations (1.7) differ from equations (2.25) in that they characteristically homogenize equations of constraints, as well as that the analysis of solutions can provide the effects of motion, depending on the $(n+1)th$ coordinate. On the grounds of above presented, we can draw

Conclusion 1. There is a system of N kinematic points, in our minds and every where around us. Position vectors of those points relative to an arbitrary observation pole are

$$\mathbf{r}_\nu=y^1_\nu\mathbf{e}_1+y^2_\nu\mathbf{e}_2+y^3_\nu\mathbf{e}_3.\quad \nu=1,2,\ldots N, \tag{5.29}$$

and there are $k<3N$ constraints (5.28). The metrics of such system is *four-dimensional* of the form

$$d\sigma^2=g_{ij}dq^idq^j,\quad i,j=1,2,3,4. \tag{5.30}$$

Hence, if, e.g., $N=2, k=3$, it is a system with 4 DOF. Let us prove it, first, on a two-body problem. That is to say, if there are two points and three constraints (5.26), it is a 4-DOF system.

Four-dimensional metrics for two bodies. The significance of this statement is underlying Newton's third axiom. The positions of two existing points M_1 and M_2 which are connected by three kinematic scalar constraints can be defined by means of three independent coordinate functions $y(t)$ and one known function of time $\tau(t)$. Let us demonstrate this in a convincing and shorter manner. The positions of observed points are defined by position vectors, where $\mathbf{e}_i$ are base vectors $|\mathbf{e}_i| = 1$,

$$\mathbf{r}_2 - \mathbf{r}_1 = \boldsymbol{\rho}, \tag{5.31}$$

therefore

$$\mathbf{m}\rho = a\mathbf{e}_1 + b\mathbf{e}_2 + \tau(t)\mathbf{e}_3.$$

From here there follow three scalar constraints

$$y_2^1 - y_1^1 = a, \quad y_2^2 - y_1^2 = b, \quad y_2^3 - y_1^3 = \tau(t).$$

Accordingly, it is obvious that coordinates of the vector $\mathbf{r}_1$

$$y_1^1 = y_2^1 - a, \quad y_1^2 = y_2^2 - b, \quad y_1^3 = y_2^3 - \tau(t).$$

depend on coordinates of the vector $\mathbf{r}_2$ and one function $\tau(t)$.

If independent generalized coordinates are denoted

$$y_2^1 = q^1(t), \quad y_2^2 = q^2(2), \quad y_2^3 = q^3(t), \quad \tau(t) = q^4,$$

we will have metrics

$$d\sigma^2 = \frac{\partial \mathbf{r}_2}{\partial q^\alpha} dq^\alpha \cdot \frac{\partial \mathbf{r}_2}{\partial q^\beta} dq^\beta + \frac{\partial \boldsymbol{\rho}}{\partial q^\alpha} dq^\alpha \cdot \frac{\partial \boldsymbol{\rho}}{\partial q^\beta} dq^\beta =$$

$$(\frac{\partial \mathbf{r}_2}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_2}{\partial q^\beta} + \frac{\partial \boldsymbol{\rho}}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{\rho}}{\partial q^\beta}) dq^\alpha dq^\beta = dq_1^2 + dq_2^2 + dq_3^2 + dq_4^2.$$

If coordinate $q4 = v_0 t$, it will be

$$d\sigma^2 = dq_1^2 + dq_2^2 + dq_3^2 + v_0^2 dt^2, \tag{5.32}$$

where v_0 is velocity of dimension LT^{-1}, and therefore coordinate $q^4 = v_0 t$ has dimension of length L. Metrics indicates that there are two points, whose positions are defined by 4 functions of time, $q^i = q^i(t), (i = 1, 2, 3, 4)$.

The Sun planetary system. In classical and celestial mechanics the concept of "two bodies" is related, first of all, to the Sun-planet concept. Let us denote the center of inertia of the Sun by the point $M_1 = M_\odot$ and the centers of inertia of the planets by points M_ν. With respect to the arbitrary observation point, the position vector of the Sun is denoted by $\mathbf{r}_o dot$ and position vectors of N planets, like in above text, by $\mathbf{r}_\nu; (\nu = 2, \ldots, N)$. The distances of the planets from the Sun are

$$\mathbf{r}_p - \mathbf{r}_\odot = \rho_p. \tag{5.33}$$

Given that all position vectors have a common observation pole, they can be reduced to relation (5.33) where

$$\mathbf{r}_p = \sum_{\nu=2}^{N} \mathbf{r}_\nu, \quad \boldsymbol{\rho}_p = \sum_{\nu=2}^{N} \rho_\nu(t).$$

In this way, the central planetary system of the Sun is reduced to relation

$$\mathbf{r}_p - \boldsymbol{\rho}_\odot = \boldsymbol{\rho}_p$$

where $\boldsymbol{\rho}_\odot = a\mathbf{e}_1 + b\mathbf{e}_2 + v_0 t\mathbf{e}_3$.

Since all mentioned vectors have the same vector base $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, as well as it is obtained, as in previous example, the metrics of the form

$$d\sigma^2 = dq_1^2 + dq_2^2 + dq_3^2 + dq_4^2. \tag{5.34}$$

or

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 1, 2, 3, 4. \tag{5.35}$$

If we use some other non-orthogonal or curvilinear coordinate systems x^i instead of orthonormal coordinate systems y^i, proved relations are changed only in the form of the metric tensor coordinates. Given that vectors are invariant with respect to linear transformations, base vectors are transformed by the law It is well known that metric tensor develops by scalar product of coordinate vectors, that is. Consequently, the sought space metrics has the form of the formula(5.34), that is, where tensor equals matrix. The covariant coordinate of tensor can be different from unity, depending on the choice of the form of the function $q^0(t)$ and $\mathrm{R}q^{n+1}$, respectively.

Conclusion 2. Metrics of the Sun planetary system, with $3N + 1 - 3$ constraints of the form (5.27) or (5.28), where the Sun moves at constant velocity v^0, is four-dimensional **relative spaces**. Our four-dimensional geometric form (5.34) was presented first at the seminars on philosophy and history at the Mathematical Institute of Serbian Academy of Sciences and Arts on 11 December 2011. At the seminar of Dept. of Mathematics of Mathematical Institute of Serbian Academy of Sciences and Arts, 16 November 2012, a lecture was given titled "Four-dimensional spaces with geometric and kinematic constraints; arguments vs negative review". The author challenged reviewer's opinion and attendees pointed out that it was not a proper review. A seminar manager and editor-in-chief of the journal published by Mathematical Institute and an academic, prior to ending the discussion, concluded that the paper was correct but derived by means of well-known mathematics. In other words, it did not contain novel mathematical contributions, so the paper could not be published in the Journal, which otherwise publishes works concerning pure mathematics only. The author did not oppose the determined editor-in-chief, although this paper was based exactly on the author's significant mathematical contribution, but the editor maintained that the Journal and other publications of Mathematical Institute publish only papers on pure mathematics. Judging by editor's attitude, even Albert Einstein would not be able to publish his famous theory of relativity in the respective Journal [43]. Indeed, the author8 of mentioned paper derived the metric form in two ways. (5.34) Much earlier Einstein's invariant was published in the form (5.35) These two quadratic forms are seemingly equal but substantially different. Tò make our proof more comprehensive and acceptable, we will quote I. Newton [1], D. Hilbert [48] and A. Einstein [49]. I. Newton [1] "PROPOSITION: Absolute space is in its existence without relation to anything, it remains always equal and immovable." "Relative space is a measure or any other limited part which is defined by our senses according to its position with respect to other objects and which is accepted as immovable space in everyday life."[11]

D.Hilbert [47]: "Let $w_s(s = 1,2,3,4)$ be any spacetime coordinates" (Hilbert wrote). The quantities W_s characterized by state in w_s are:

1) the first ten, introduced by Einstein, gravitational potentials, and $g_{\mu\nu} = 1,2,3,4)$, which have symmetrical tensor character relative to any transformations of (Mirot's) parameters w_s.

2) four electrodynamic potentials q_s, which are vector transformed.

In work Albert Einstein [49]. **Hamilton's Principle and the General Theory of relativity**, the first paragraph A variational principle and field equations of gravitation and of matter.

In the third paragraph: The properties of gravitational field equations deriving from the invariant theory, Einsten wrote: "Let us allow now that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad mu, \nu = 1,2,3,4. \tag{5.36}$$

represents eigen invariant. Thus, the character of transformation $g_\mu \nu$ is established. About the character of q_ϱ which describes matter, we do not make any assumptions." The provided quotation is sufficient proof that our metrics (5.34) of four-dimensional space, attr. L, differs from Einstein's four-dimensional invariant, characterized by the attribute space LL, matter M and time T.

However, it should not be overlooked that analogous to Einstein's invariant (5.35) in classical analytical mechanics of Lagrange and Hamilton

[11] V. Vujičic, Four Dimensional Spaces with Kinematic Constraint, Proceedings, 4th Int. Congress of Serbian Society of Mechanics, pp. 153-158, 2013. "In their appearance and size absolute and relative spaces are equal, but numerically they do not remain equal. Thus, for instance, if the Earth is observed non-stationary, the space of our air, towards the Earth, always remains the same, representing part of the absolute space and, second, looking at where the air has passed, to put it consequently accurately, it means that space is continuously changing."

there is a differential invariant

$$d\sigma^2 = a_{\alpha\beta} dq^\alpha dq^\beta, \quad \alpha, \beta = 1, \dots n. \tag{5.37}$$

where q^α are generalized independent coordinates of dynamical systems, while tensor

$$a_{\beta\alpha} = \sum_\nu m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q\alpha} = a^{\beta\gamma} = a_{\beta\alpha}(m_1, \dots, m_N; q^1, \dots q^n, t)$$

contains the properties of space L, mass of matter M and time T. Using that tensor, we can write down Kinetic energy E_k in invariant forms

$$2E_k = a_{\beta\alpha} \frac{dq^\alpha}{dt} \frac{dq^\beta}{dt} = a^{\beta\gamma} p_\beta p_\gamma,$$

where $p_\gamma = a_{\gamma\alpha} \dot{q}^\alpha$ are generalized impulses. For the system composed of N material points connected by means of k stationary constraints, the position of a system is defined by $3N-k$ generalized coordinates, whereas the state of the system's motion is defined by n coordinates q and n generalized impulses. If one, more than one, or all constraints change during motion, the number of independent coordinates, as well as the number of impulses p_γ is increased to $n+1$. Consequently, as in metrics form (5.24) and (5.37), the system with $3N+1-k$ varying constraints is reduced to four-dimensional relative space, as well as the invariant (5.35).

On identical basis of rheonomic constraints, four-dimensionality of deformation tensor11 has been proved, the deformation tensor being of the form [44]

$$\begin{pmatrix} \epsilon_{00} & \epsilon_{01} & \epsilon_{02} & \epsilon_{03} \\ \epsilon_{10} & \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{20} & \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{30} & \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}.$$

MOND6 - MODIFICATION OF PRINCIPLES MECHANIC

6.1. The principle of work. The concept of the work of the force is fundamental in mechanics. Section 3 of this book, dealing with the action and reaction forces, contains the differential for the force of the work

$$dA = \mathbf{F} \cdot d\mathbf{r} = X dx + Y dy + Z dz, \tag{6.1}$$

where $dA, d\mathbf{r}, dx, dy, dz$ are mathematically truth differentials as infinitely small quantities. Such work called the principle of work is most commonly referred to as elementary work on possible displacements. This statement points out the disparity of possible and differential displacements in expression (6.1) and is written in the form

$$\Delta A = \mathbf{F}\Delta\mathbf{r}. \tag{6.2}$$

Accordance with the preprinciple of invariance, it follows that differentially small work

$$\Delta A = \mathbf{F} \cdot \Delta\mathbf{r}.$$

$dA, d\mathbf{r}, dx, dy, dz$ is also a scalar invariant. This work is also often called *elementary work of the forces on real displacement.* The phrase "on real displacement" emphasizes the difference from the other hypothetical and arbitrarily small work of the forces on any possible small displacement $\Delta\mathbf{r}$,

By the concept of possible displacement, one implies any small deviation from the real position of the material point, which that point could have realized. The concept is even more general than the differential dr of the position vector. To put it simply, it is any hypothetically achievable distance at possible displacement. In practice, it could be taken as tested factual or contemplative small displacement. Quantity is not accurately determinable, it is arbitrarily small, from negligibly small to some finite, which can be assumed to be possible quantity. Analytically, the concept may be considered a difference between position vector of possible point $\mathbf{r}$ displacement and vector of undisplaced or specified position $\mathbf{r}$, $\Delta\mathbf{r} := \mathbf{r}(x + \Delta x) - \mathbf{r}(x)$. Following the example of the formula of finite increments, the vector function $\mathbf{r}$ can be expressed in the analytical form

$$\Delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial y^i}(y^{*i} - y^i) = \frac{\partial\mathbf{r}}{\partial y^i}\Delta y^i \tag{6.3}$$

as well as

$$\Delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial x^j}\frac{\partial x^j}{\partial y^i}\Delta y^i + \cdots = \frac{\partial\mathbf{r}}{\partial x^j}\Delta x^j + \cdots = \frac{\partial\mathbf{r}}{\partial q^\alpha}\Delta q^\alpha + \cdots$$

$$\mathbf{F} \cdot \Delta\mathbf{r} = Y_i\Delta x^i = X_i\Delta x = Q_\alpha\Delta q^\alpha, \tag{6.4}$$

where Δy, Δx, Δq are coordinates of the vector of possible displacement in various coordinate systems. It is these coordinates of vector r that are most commonly referred to as possible displacements. Analogously to elementary work on real displacement (6.1), formula (6.2) will be called work on possible displacements.

Formula (6.1) is a scalar invariant, like (6.2), but due to possible and real displacement, the preprinciple of existence is satisfied. The invariant form

$$\mathbf{F} \cdot \Delta \mathbf{r} = Y_i \Delta y^i = X_i \Delta x^i = Q_\alpha \Delta q^\alpha \tag{6.5}$$

satisfies the preprinciple of invariance, while relations (6.3) and (6.4) define the level of accurate determinacy, and therefore satisfy the preprinciple of determinacy as well.

Being scalar quantities, they enable summation

$$\sum_{k=1}^{N} \mathbf{F}_\nu \cdot \Delta \mathbf{r}_\nu = \sum_{k=1}^{3N} Y_k \Delta y^k = \sum_{\beta=1}^{n} Q_\beta \Delta q^\beta, \tag{6.6}$$

which makes up the total of the work of all forces $\mathbf{F}_\nu, (\nu = 1, \ldots, N)$ on possible displacements. Formulation of the principle of work. The essence of the principle of work has been known (according to Galileo) since Aristotle as "the golden rule of mechanics", and afterward as "the principle of possible displacements", "the principle of possible variations", "the fundamental basis of the equations of mechanics', "the principle of virtual work", "the D'Alembert-Lagrange principle", . One of the most severe mathematical analysts of classical mechanics A. M. Lyapunov writes: "The principle of possible displacements was familiar to Galileo, and then Wallis and Johann Bernoulli used it too. However, the first general proof of the principle was laid down by Lagrange, who established the basis of analytical mechanics. Later, it was proved by Poisson, Cauchy and others, although the best proof is considered to be that of Lagrange's." In the present approach to the theory of the motion of a body, the principle is not proved but, as pointed out in the preprinciples [3] about the concept of the principle of mechanics (p.74), the principle is a truthful proof, verbal or written, and therefore being either the former or the latter, as much accurate as it is possible to tell the most, based on the level of knowledge. Formulation of the principle encompasses its generality. Instead of providing the proof, its interpretation and demonstration is applied to various systems. In short, the principle of work can be expressed by the following sentence. The total work done by the forces on possible displacements is null and void, and in the presence of unilateral constraints, nonpositive. Mathematical expression is even shorter: (6.7) For a mathematically educated reader, the following sentence may be more explicit: The total work done by all forces on all independent possible displacements equals zero, and for the system with unilateral constraints nonpositive. Relation

$$\sum_{\nu=1} \mathbf{F}_\nu \cdot \Delta \mathbf{r}_\nu \leqslant 0. \tag{6.7}$$

is very general, but not directly operational. Its application requires strict mathematical analysis implying, first of all, understanding of constituent elements. Limited arbitrariness of possible displacements is described. Vectors $\mathbf{F}$ contain the properties or attributes as components of the inertia force $\mathbf{I}_\nu$ of the ν-th material point and main vectors of all other forces $\mathbf{F}_{\nu k}$ acting in the ν-th point, i.e. $\mathbf{F}_\nu = \sigma_k \mathbf{F}_{\nu k}$. Accordingly, without loss of generality of the relation (6.7), the principle can be written in the form

$$\sum_{\nu=1}^{N} (\mathbf{I}_\nu + \mathbf{F}_\nu) \cdot \Delta \mathbf{r}_\nu \leqslant 0 \tag{6.8}$$

In the thus written principle it is implied that in vectors $\mathbf{F}_\nu$, as pointed out, all forces, except the inertia force, are contained, as well as the reactions of constraints, in accordance with the law of constraints [3]. This means that the reactions of μ constraints are represented by the forces

$$\mathbf{R}_\nu = \sigma_{\nu\mu}.$$

If the reactions of constraints are not calculated a priori, as above mentioned, the relations describing the constraints should be added to the relation (6.8), that is,

$$\sum_{\nu=1}^{N} (\mathbf{I}_\nu + \mathbf{F}_\nu) \cdot \Delta \mathbf{r}_\nu = 0 \tag{6.9}$$

$$f_\mu(\mathbf{r}, \mathbf{v}, \tau) \geqslant 0. \tag{6.10}$$

As for the signs of equality and inequality, the difference is noticeable between the relations (6.9) and (6.8); the sign of inequality from (6.8) is encompassed by the relations (6.10). For the case of bilateral constraints represented by the forces, the relation of the principle (6.8) is written in the form (6.9), and for the case when constraints are not calculated in relation (6.8), relations (6.9) and (6.10) should be written in the form

$$\sum (\mathbf{I}_\nu + \mathbf{F}_\nu) \cdot \Delta \mathbf{r}_\nu = 0. \tag{6.11}$$

$$f_\mu(\mathbf{r}, \mathbf{v}, \tau) = 0. \tag{6.12}$$

Starting from the fact that constraints are more commonly written in the coordinate form, let us observe the application of the principle for some mechanical systems relative to the Cartesian coordinate system $y = (y^1, y^2, y^3)$.

Static systems. By the concept of "static system", one here implies N points of application $M_\nu = (1, \ldots, N)$ of forces $\mathbf{F}_\nu = Y_\nu^i \mathbf{e}_i$ $(1, 2, 3)$ which are linked by k finite constraints (2.5). These constraints are written more specifically

$$f_\mu(y_1^1, y_1^2, y_1^3, \ldots, y_N^1, y_N^2, y_N^3) = 0, \tag{6.13}$$

or by formalizing indices $y_\nu^1 = y^{3\nu-2}$, $y_\nu^2 = y^{3\nu-1}$, $y_\nu^3 = y^{3\nu}$,

$$f_\mu(y^1, \ldots, y^{3N}) = 0. \tag{6.14}$$

Such system $\mathbf{I}_\nu = 0$, and therefore the relations (6.11) and (6.12) can be written in the following coordinate form

$$Y_\alpha \Delta y^\alpha := Y_1 \Delta y^1 + \cdots + Y_{3N} \Delta y^{3N} = 0. \tag{6.15}$$

$$f_\mu = f_\mu(y^1, \ldots, y^{3N}) = 0. \tag{6.16}$$

First, we conclude that the non-ideal factor of the constraint is represented by the force contained in the forces Y_α, while relations (6.16) describe idealization of the constraints. Developing in a series for possible displacements of those constraints in the neighborhood of equilibrium positions of points $M_\nu(y = b)$, it is obtained, in addition to the linear form (6.15), another k linear forms for Δy, such as

$$f_\mu(y) - f_\mu(b) = a_{\mu,\alpha} \Delta y^\alpha = a_{\mu 1} \Delta y^1 + \cdots + a_{\mu 3N} y^{3N} = 0, \tag{6.17}$$

where

$$a_{\mu\alpha} = \frac{\partial f_\mu}{\partial y^\alpha}. \tag{6.18}$$

So, relations (6.15) and (6.16) are reduced to $k + 1$ linear equations

$$Y_\alpha \Delta y^\alpha = 0, \tag{6.19}$$

$$a_{\mu\alpha} \Delta y^\alpha = 0, \qquad (\mu = 1, \ldots, k < 3N), \tag{6.20}$$

where there figure $3N$ mutually dependent displacements Δy^{3N}. Given that relation (6.19) contains independent possible displacements, this task can be further solved in two ways, with the aim of eliminating dependent possible displacements, as follows:

a. by direct solution of equations,

b. by introducing undetermined multipliers of constraints.

Solving for dependent possible displacements. If possible displacements are divided into dependent $\Delta y^1, \ldots, \Delta y^k$ and independent ones: then addends in equations (6.15) and (6.16) are divided into those with dependent and independent possible displacements

$$Y_\nu \Delta y^\nu + Y_\beta \Delta y^\beta = 0, \ \nu = 1, \ldots, k, \tag{6.21}$$

$$a_{\mu\nu} \Delta y^\nu + a_{\mu\beta} y^\beta = 0, \qquad \beta = k+1, \ldots, 3N. \tag{6.22}$$

Substituting

$$\Delta y^\nu = -a^{\mu\nu} a_{\mu\beta} \Delta y^\beta = b^\nu_\beta, \Delta y^\beta; \quad |a_{\mu\nu}| \neq 0,$$

a single relation with independent displacements is obtained, such as

$$(Y_\beta - Y_\nu b^\nu_\beta) \Delta y^\beta = 0; \tag{6.23}$$

Due to independence of displacements Δy^β, it follows that the system of observed forces will be in equilibrium in the presence of constraints (6.16) if it satisfies the following system of $3N - k$ algebraic equations

$$Y_\beta - Y_1 b^1_\beta - \ldots b - Y_k b^k_\beta = 0. \tag{6.24}$$

As obvious from the system of equations, it is possible to define $3N - k$ coordinates of the force vector.

Undetermined multipliers of constraints. If each of the equations (6.17) is multiplied by a corresponding multiplier λ_σ and then summed for index μ, the systems of $k+1$ equations (6.19) and (6.20) are reduced to two equations

$$Y_\alpha \Delta y^\alpha = 0, \quad \sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} \Delta y^\alpha = 0. \tag{6.25}$$

The sum of the two relations

$$\left(Y_\alpha + \sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} \right) \Delta y^\alpha = 0, \tag{6.26}$$

also enables, as in the previous method, to eliminate dependent possible displacements $\Delta y^1, \ldots, \Delta y^k$. Given that λ_σ are for the time being undetermined multipliers, it is permissible to elicit the conditions that delete k multipliers $\lambda + \mu$ from equations (6.26), so that it is

$$Y_\sigma + \sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\sigma} = 0 \quad \sigma = 1, \ldots, k. \tag{6.27}$$

There remain k equations (6.26) with $3N-k$ independent displacements, such as

$$\left(Y_\beta+\sum_{\mu=1}^{\mu=k}\lambda_\mu\frac{\partial f_\mu}{\partial y^\beta}\right)\Delta y^\beta=0. \tag{6.28}$$

From here, as from (6.23), another $3N-k$ equations are obtained of the form (6.27). In this way, the system of $3N$ equations of force are obtained for the solution of a static task

$$Y_\alpha+\sum_{\mu=1}^{\mu=k}\lambda_\mu\frac{\partial f_\mu}{\partial y^\alpha}=0 \quad (\alpha=1,\dots,3N).$$

with k equations of constraints

$$f_\mu(y^1,\dots y^{3N})=0$$

with k equations of constraints.

Rheonomic systems. As in the previous static system, the principle of work is also applied for the mechanical system with variable constraints. Without loss of generality, for brevity, let us assume that the constraints are specified by equations of constraints

$$f_\mu(y^0;y^1,\dots,y^{3N})=0, \quad y^0=\tau(t), \tag{6.29}$$

where $\tau(t)$ is a known function of time. Developing the function in a series, analogously to (6.17), it is shown that there are $3n+1$ possible displacements $\Delta y^0,\Delta y^1,\dots,\Delta y^{3N}$. Indeed,

$$\Delta f_\mu=\frac{\partial f_\mu}{\partial y^0}\Delta y^0+\frac{\partial f_\mu}{\partial y^i}\Delta y^i=0, \quad (i=1,\dots,3N).$$

The principle of work states about all possible displacements and work of corresponding forces on the displacements. So, here, apart from works on possible displacements $Y_i\Delta y^i$, the work on possible displacement Δy^0, i.e. $Y_0\Delta y^0$ should be added. Thus, for such a system with variable constraints, instead of relations (6.15) and (6.16), here we have the system of equations

$$Y_\alpha\Delta y^\alpha=Y_0\Delta y^0+Y_i\Delta y^i=0, \tag{6.30}$$

$$f_\mu(y^0,y)=f_\mu(y^0,y^1,\dots,y^{3N})=0, \tag{6.31}$$

$(\alpha=0,1,\dots,3N;i=1,\dots,3N)$. From here, using the same procedure as from (6.25) to (6.29), another additional equation is obtained

$$Y_0+\sum\lambda_\mu\frac{\partial f_\mu}{\partial y^0}=0. \tag{6.32}$$

The force

$$Y_0 = -\sum \lambda_\mu \frac{\partial f_\mu}{\partial y^0} \tag{6.33}$$

is also evident in more general relations (6.30).

The system with unilateral and bilateral constraints. The principle of work, expressed by relation (6.1), states that the sign of inequality is related to the unilateral constraints. For the case of unilateral constraints only, the principle states that the work on possible displacements is less than zero, that is,

$$\sum_{\nu=1}^{\nu=N} \mathbf{F}_\nu \cdot \Delta \mathbf{r}_\nu \leqslant 0 \tag{6.34}$$

and for bilateral constraints, as shown

$$\sum_{\nu=1}^{\nu=N} \mathbf{F}_\nu \cdot \Delta \mathbf{r}_\nu = 0. \tag{6.35}$$

Consider simultaneous presence of bilateral constraints

$$f_\mu(\mathbf{r}_1, \ldots, \mathbf{r}_N) = 0 \quad \mu = 1, \ldots, k \tag{6.36}$$

and of unilateral

$$\Phi_\sigma(\mathbf{r}_1, \ldots, \mathbf{r}_N) \geqslant 0 \quad \sigma = 1, \ldots, l, \tag{6.37}$$

provided that $k + l < 3N$. Let us

$$Y_\alpha \Delta y^\alpha = \Delta c, \;\; \Delta c \leqslant 0 \tag{6.38}$$

$$\sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} \Delta y^\alpha = 0. \tag{6.39}$$

$$\sum_{\sigma=1}^{l} \chi_\sigma \frac{\partial f_\mu}{\partial y^\alpha} \Delta y^\alpha = \sum \chi_\sigma \Delta c_\sigma, \tag{6.40}$$

where $\Delta c_\sigma \geqslant 0$ or $\Delta c_\sigma \leqslant 0$.

$$\sum_{\alpha=1}^{3N} \left(Y_\alpha + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} + \sum_{\sigma=1}^{\sigma=l} \chi_\sigma \frac{\partial f_\sigma}{\partial y^\alpha} \right) \Delta y^\alpha = \delta c + \sum \chi_\sigma \delta_\sigma \tag{6.41}$$

The sum of these equations (6.41) leads to deriving necessary and sufficient number of equations for solving the task. As in the case of bilateral or restrained constraints of dependent $k+1$ possible displacements $Deltay^{\alpha}$, let us exclude by request that multipliers λ_μ and χ_σ be such that

$$(6.42)\quad Y_i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i} + \sum_{\sigma=1}^{l} \chi_\sigma \frac{\partial f_\sigma}{\partial y^i} = 0, \quad i = 1, \dots, k; k+1, \dots, k+l.$$

The rest of $3N-(k+1)$ coefficients with independent possible displacements Δy^j, $(j = k+l+1, \dots, 3N)$ will also equal zero, that is,

$$(6.43)\quad Y_j + \sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial y^j} + \sum_{\sigma=1}^{\sigma=l} \chi_\sigma \frac{\partial f_\sigma}{\partial y^j},$$

in order that in accordance with the principle be $\Delta c + \sum \chi_\sigma \Delta c_\sigma = 0$. However, since in accordance with (6.60) and (6.61), $\Delta c < 0$, it follows that

$$(6.44)\quad \sum_{\sigma_1}^{l} \chi_\sigma \Delta c_\sigma > 0.$$

Taking into account the independence of undetermined multipliers of constraints, there follow additional conditions for equations (6.42) and (6.43) that χ_σ and Δc_σ have the same sign.

Kinetic systems Let us recall that vector functions $\mathbf{F}_\nu$, whose coordinates are Y, contain all active forces $\mathbf{F}$ including the inertia force $-m\frac{d\mathbf{v}}{dt}.\mathbf{F}$. Accordingly, 3N differential equations of motion (6.42) and (6.43) and $k+l$ finite equations of constraints with conditions resulting from (6.8) make up a complete system of relations for solving the motion of the observed system with finite unilateral and bilateral constraints.

Nonholonomic systems. The title implies the system of N material points, whose motion is restricted, among other things, by at least one differential nonintegrable (nonholonomic) constraint. Taking into account mentioned restriction, let them be the constraints

$$(6.45)\quad f_\mu(y^1, \dots, y^{3N}, \dot{y}^1, \dots, \dot{y}^{3N}) = 0.$$

Due to difficulties that occur during developing in a series, the functions in the neighborhood of trajectory $C(y)$ and complexity of possible equations of the constraints, as well as of their kinematic character, and for generality and brevity, we will here apply the method of constraint substitution, in accordance with the law of constraints, by corresponding constraint reaction forces $\mathbf{R}_{\nu\mu}$. That is to say that each constraint acting on the ν-th point is substituted by the resultant vector of the reactions of constraints $\mathbf{R}_{\mu\nu}$, i.e. $\mathbf{R}_\nu = |\sigma| a_{\nu\mu}$. Given previously introduced notation, this can be written more concisely, as well as the other force vectors, using a set of $3N$ coordinates $R_1, \dots, r_3 N$. In such general approach, let us write the principle of work (6.35) in the coordinate form ooo. The system of $3N$ differential equations of motion

$$(6.46) \qquad (I_\alpha + Y_\alpha + R_\alpha)\,\Delta y^\alpha = 0.$$

respectively

$$(6.47) \qquad m_\alpha \ddot{y}_\alpha = Y_\alpha + R_\alpha,$$

contains, among other things, $3N$ unknown reactions of constraints R_α that should satisfy the conditions of acceleration

$$(6.48) \qquad \dot{f}_\mu(y,\dot{y},\ddot{y}) = \frac{\partial f_\mu}{\partial y^\alpha}\dot{y}^\alpha + \frac{\partial f_\mu}{\partial \dot{y}^\alpha}\ddot{y}^\alpha = 0.$$

Substituting $\ddot{y}_\alpha$ from equations (6.46) into previous equations (6.47), k linear equations for R_α are obtained as follows

$$\frac{\partial f_\mu}{\partial y^\alpha}\dot{y}^\alpha + \frac{1}{m_\alpha}(Y_\alpha + R_\alpha)\frac{\partial f_\mu}{\partial \dot{y}^\alpha} = 0.$$

It is possible from here to determine k reactions

$$R_i = R_i(m, y, \dot{y}, Y, R_{k+1}, \ldots, R_{3N}) \quad (i = 1, \ldots, k)$$

depending, among other things, on $3N$ coordinates of forces Y and $3N-k$ reactions R_j ($j = k+1, \ldots, 3N$). Further, substituting Ri into equations (6.46) and (6.47), respectively, in the system of N differential equations of motion there remain $3N-k$ unknown reactions of the constraints. As such, it is possible to determine them from that system, depending on other functions in the equations, or to seek new $3N-k$ conditions that define or determine the remaining $3N-k$ unknown reactions of differential constraints (6.45). Many studies deal with this problem, which is still actual.

First conclusion. The principle of work may be used to derive and develop the relations of dynamic equilibrium. The principle of equilibrium may be used for the same purpose as well. Both principles are equivalent.

Invariant notation of the principle of work. Expressions (6.17), (6.19) and (6.20) indicate that relations (6.30) can be written in an analogous form relative to different coordinate systems. Let it further be $(y, \mathbf{e})$ immovable Cartesian orthogonal coordinate system; $(z, \mathbf{g}_\epsilon)$ rectilinear coordinate system; $(x, \mathbf{g}(x))$ curvilinear coordinate system, and $(q, \mathbf{g}_(q))$ a system of independent generalized coordinates. The same constraints, as shown, are written by invariant form

$$(6.49) \qquad \mathbf{r}_\nu = \mathbf{r}_\nu(q^0, q^1, \ldots, q^m) =: \mathbf{r}_\nu(q)$$

$$(6.50) \qquad q^0 = \tau(t).$$

Possible displacements, according to (6.4), are written, depending on the choice of the coordinate system

(6.51) $$\Delta \mathbf{r}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial y^i}\Delta y^i = \frac{\partial \mathbf{r}_\nu}{\partial z^i}\Delta z^i = \frac{\partial \mathbf{r}_\nu}{\partial x^i}\Delta x^i$$

or

(6.52) $$\Delta \mathbf{r}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\Delta q^\alpha =: \frac{\partial \mathbf{r}_\nu}{\partial q}\Delta q,$$

where $\frac{\partial \mathbf{r}_\nu}{\partial q}$ are coordinate vectors of the ν-th point on the configuration manifold.

The number of possible displacements allows possible changes in constraints

(6.53) $$\Delta f_\mu = \frac{\partial f_\mu}{\partial y}\Delta y + \frac{\partial f_\mu}{\partial \tau}\Delta \tau = \frac{\partial f_\mu}{\partial x}\Delta x + \frac{\partial f_\mu}{\partial \tau}\Delta \tau = 0$$

$$\Delta f_0 = \Delta y^0 - \Delta \tau = 0.$$

If we accept the constraint (6.50), which exists as long as other constraints, by the force R_0, possible changes in constraints (6.53) indicate that there exist $3N + 1$ possible displacements, so that indices in (6.51) and (6.78) take the values $i = 0, 1, \dots, 3N; \alpha = 0, 1, \dots, n$.) Consequently, the basic formulation of the work (6.30) has the following invariants (6.54) as well as (6.55) For the case when constraint functions do not depend explicitly on time, the coordinate $q0$ does not exist, and therefore zero indices $I = 0$, do not exist in relations

(6.54) $$Y\Delta y = Z\Delta z = X\Delta x = Y_i\Delta y^i = Z_i\Delta z^i = X_i\Delta x^i \leqslant 0,$$

$$f_\mu \geqslant 0; \quad \mu = 1, \dots, k < 3N, \; i = 0, 1, \dots, 3N$$

and

(6.55) $$Q\Delta q := Q_\alpha \Delta q^\alpha \leqslant 0 \quad (\alpha = 0, 1, \dots, n).$$

The same invariant forms refer to relation (6.8). Given that the observed relations (6.16) and (6.32) have been previously developed relative to rectilinear coordinates y, we will further below use curvilinear coordinates x and generalized independent coordinates $q \epsilon M$.

The principle of work in curvilinear coordinates. In relations (1.40) it has been shown that the coordinates of the inertial force vector relative to curvilinear coordinate systems are determined by the expressions

(6.56) $$I_i = -a_{ij}\frac{Dv^j}{dt}.$$

Since Xi in relation (6.54) denotes the sum of active forces and inertia forces (6.82), it is

$$\left(X_i - a_{ij}\frac{Dv^i}{dt}\right)\Delta x^i \leqslant 0. \tag{6.57}$$

This follows straightforward from relation (6.32) if it is taken into account that possible displacements are

$$\Delta \mathbf{r}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial x^s}\Delta x^s, \quad (s = 1, 2, 3). \tag{6.58}$$

Substituting into (6.32), it is obtained

$$\sum_{\nu=1}^{nu=N}\left(I_\nu^s\frac{\partial \mathbf{r}_\nu}{\partial x^s} + X_\nu^s\frac{\partial \mathbf{r}_\nu}{\partial x^s}\right)\cdot\frac{\partial \mathbf{r}_\nu}{\partial x^r}\Delta x^r =$$

$$= \sum_{\nu=1}^{nu=N}\left(X_\nu^s\frac{\partial \mathbf{r}_\nu}{\partial x^s} - m_\nu\frac{dv_\nu^s}{dt}\frac{\partial \mathbf{r}_\nu}{\partial x^s}\right)\cdot\frac{\partial \mathbf{r}_\nu}{\partial x^r}\Delta x^r =$$

$$= \sum_{\nu=1}^{N}\left(g_{(\nu)sr}X_\nu^s - a_{(\nu)rs}\frac{dv_\nu^s}{dt}\right)\Delta x^r \leqslant 0.$$

If indices $i, j, \ldots, 3N$; $m_{3k} = m_{3k-1} = m_{3k-2}$, $i = 3\nu = 3\nu - 1 = 3\nu - 2$ are incorporated it follows

$$\left(g_{ij}X^j - a_{ij}\frac{Dv^j}{dt}\right)\Delta x^i \leqslant 0, \tag{6.59}$$

or given that $X_i = g_{ij}X^j$. If displacements are restricted by bilateral or restrained constraints

$$f_\mu(x^1, \ldots, x^{3N}, \tau) = 0 \quad \mu = 1, \ldots, k$$

in relation (6.57) the sign of inequality is left out and by means of the constraint

$$f_0 = x^0 - \tau(t) = 0,$$

to which the force R_0 corresponds, k homogeneous linear equations for possible displacements are obtained

$$\Delta f_\mu = \frac{\partial f_\mu}{\partial x^i}\Delta x^i + \frac{\partial f_\mu}{\partial x^o}\Delta x^0 = 0. \tag{6.60}$$

$$\Delta f_0 = \Delta x^0 + \Delta\tau = 0. \quad (i = 1, \ldots, 3N)$$

Multiplication by corresponding undetermined multipliers λ_μ and λ_0 and summing with

$$X_i = a_{ij}\frac{Dv^j}{dt} \tag{6.61}$$

yields

$$\left(X_i + \sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} - a_{ij}\frac{Dv^j}{dt}\right)\Delta x^i + \left(\lambda_0 + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^0}\right)\Delta x^0 = 0.$$

From here, there follow $3N$ differential equations of motion

$$a_{ij}\frac{Dv^j}{dt} = X_i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i}, \tag{6.62}$$

and the force of change in the constraints and

$$\lambda_0 = -\sum_{\mu=1}^{\mu=N} \lambda_\mu \frac{\partial f_\mu}{\partial x^0} = X_0, \tag{6.63}$$

of which k finite equations of the observed constraints $f_\mu = 0$ should be added.

The principle of work in independent coordinates. Let us write all constraints (6.57) by corresponding reactions of the constraints

$$R_i = \sum_{\mu=1}^{\mu=N} \lambda_\mu \frac{\partial f_\mu}{\partial y^i}, \quad (i = 1, \ldots, 3N), \tag{6.64}$$

and an additional constraint $y^0 - \tau = 0$ by the force R_0. The equation (6.72) will then look more specifically, such as

$$\left(I_i + Y_i + \sum_{\mu=1}^{\mu=k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i}\right)\Delta y^i + R_0 \Delta y^0 = 0, \quad (i = 1, \ldots, n). \tag{6.65}$$

If equations of constraints (6.57) are substituted by the parametric form

$$y^i = y^i(q^0, q^1, \ldots, q^n),\ y^0 = q^0,\ n = 3N - k,$$

and displacements Δy^i by independent possible displacements Δy^α,

$$\Delta y^i = \frac{\partial y^i}{\partial q^\alpha}\Delta q^\alpha \quad (\alpha = 0, 1, \ldots, n),$$

equation (6.65), considering (6.57), is reduced to a new invariant form

$$(I_\alpha + Q_\alpha)\Delta q^\alpha + \sum_{\mu=0}^{\mu=N} \lambda_\mu \frac{\partial f_\mu}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\Delta q^\alpha = 0.$$

$(i = 1, \ldots, n)$. However, since $f_\mu(q^0, q^1, \ldots, q^n) \equiv 0$, it is and

$$\Delta f_\mu(y(q)) = \frac{\partial f_\mu}{\partial y^i}\Delta y^i = \frac{\partial f_\mu}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\Delta q^\alpha = \frac{\partial f_\mu}{\partial q^\alpha} \equiv 0,$$

for $j = 1, \ldots, n = 3N - k$, there follows that the principle of work observed relative to the generalized coordinates has this form

$$(I_j + Q_j)\Delta q^j + (I_0 + Q_0^* + R_0)\Delta q^0 = 0. \tag{6.66}$$

$f_\mu(q^0, q^1, \ldots, q^n) \equiv 0$, to i

$$\Delta f_\mu(y(q)) = \frac{\partial f_\mu}{\partial y^i}\Delta y^i = \frac{\partial f_\mu}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\Delta q^\alpha = \frac{\partial f_\mu}{\partial q^\alpha} \equiv 0.$$

This equation is derived from equations (6.30), it is equivalent to the system of equations (6.30) and (6.31). Due to described character of possible independent generalized displacement Δq, apart from equations (6.30) and (6.31), the principle of work (6.66) indicates correlation between the forces I_j, Q_j, displacements Δq^α and forces I_0, Q_0, R_0.

Second conclusion: As evident from above statements, the principle of work is applicable to a high degree with respect to any coordinate system, retaining the linear invariant scalar form for all coordinate systems, systems of constraints and systems of forces.

6.2. Variational principles. Lagrange's variational principle and its generalization by Hamilton are mentioned in subsection 3. However, the interpretation of the concept of variation is not given, so that variation is often taken as an operator of determining the extremum of an integral (3.43). In order to distinguish between variation and differential, Lagrange was the first1 to introduce the notation δ so that δZ expresses the "differential" of Z, which does not coincide with dZ; if it is possible to have $dZ = mdx$, it is then possible to have $\delta Z = m\delta x$. Here, we develop relation (3.44) in more detail and more extensively, with which we represent the variation of the action functional. In analytical mechanics, theoretical physics, and mathematics as well, by the concept of action one implies more or less accurately defined functional, whose definition does not contain force. That is why, here, as in the principle of work, let us mention various notations for action, for accuracy and explicitness. In book [3] the concept of variation is formulated as: The concept of variation of the function $y(\alpha, x)$ implies the product of the derivative of the function for parameter α and a small perturbation $\delta\alpha$ of that parameter, that is,

$$\delta y = \lim_{\Delta\alpha - 0} \frac{y(\alpha + \Delta\alpha, x) - y(\alpha, x)}{\Delta\alpha}.$$

Example. If the motion of the material point is described by the formula

$$y = gt^2, \quad g = const.,$$

the change of the function for independent variable t can be done at every change of t, by the rule[12]

$$dy = \frac{\partial y}{\partial t} dt = 2gtdt.$$

However, if parameter g is not accurately estimated to the smallest deviation Δg, we write

$$\delta y = \frac{\partial y}{\partial g} \delta g = t^2 \delta g.$$

The change of the constant α for independent variable t equals zero, that is,

$$\frac{\Delta \alpha}{\Delta t} = 0.$$

Likewise, the change of independent variable t for parameter α

$$\frac{\Delta t}{\Delta \alpha} = 0.$$

More generally, differentials of the function $f = f(x_1, \dots, x_n)$, where x_i are independent variables, will be

$$df = \frac{\partial f}{\partial x_i} dx_i,$$

the same as for the function $f = f(x_1, \dots, x_n; \alpha_1, \dots, \alpha_k)$. But variation of that function is $\delta f = \frac{\partial f}{\partial \alpha_i} \delta \alpha_i$. Here, we present in more detail and more extensively the variation of the action functional. In analytical mechanics, theoretical physics, and in mathematics too, the concept of action implies more or less accurately defined functional, whose definition does not contain force. That is why, here, as in the principle of work, let us mention various notations for action, for accuracy and explicitness.

6.3. Variational principle of the action. The action of mechanical system is an integral quantity

$$\mathcal{A} = \int_{t_0}^{t} A(\mathbf{F}) dt. \tag{6.67}$$

where $\mathcal{A}$ is the work of the force $\mathbf{F}$. Physical property of the action is as that of the moment of impulse,

$$\mathcal{A} == ML^2 T^{-1}. \tag{6.68}$$

[12]1 See: [15] Pol. Variational principles of mechanics. V.A.Vujicic, Preprinciples of mechanics, Institute for textbooks and teaching aids and Mathematical institute of Serbian Academy of Sciences and Arts, Belgrade, 1998

This notation also allows for understanding that action is an integral of the product of the work of some force and time interval. Subintegral expression is a scalar invariant, and therefore the action .. can be written in the invariant form

$$\mathcal{A} = \int_{t_0}^{t} A(X)dt, \tag{6.69}$$

alternatively

$$\mathcal{A} = \int_{t_0}^{t} A(Y)dt. \tag{6.70}$$

Just as there are several invariant and equivalent forms of the notation of action, so can the action principle be stated and is stated by various but equivalent sentences. Here, mathematical expression is essential: Variation of the action $\mathcal{A}$ during time $[t_0, t]$ equals zero if the work of the active forces on possible variations for equal time equals zero, that is, if, according to aforementioned, the work of the active forces $\mathbf{F}$ on possible variations is

$$\delta\mathcal{A}(\mathbf{F}) = 0, \quad \longleftrightarrow \quad \delta A = 0, \tag{6.71}$$

and variation of the action

$$\delta\mathcal{A} = \delta \int_{t_0}^{t} E_k dt = \int_{t_0}^{t} \delta E_k dt = -\int_{t_0}^{t} \delta A(\mathcal{I})dt = 0. \tag{6.72}$$

In order to reconcile $\delta\mathcal{D}$ and $\delta A(Y)$, let us multiply (6.70) by the time differential $dt > 0$ and differentiate under the integral sign, that is,

$$\int_{t_0}^{t} \delta A(Y)dt = \delta \int_{t_0}^{t} A(Y)dt. \tag{6.73}$$

Summing (6.71) and (6.72) the action principle is operationalized by relation

$$\delta \int_{t_0}^{t_1} [\mathcal{A}(\mathbf{F}) - \mathcal{A}(\mathbf{I})]dt = 0 \tag{6.74}$$

alternatively

$$\int_{t_0}^{t_1} (\delta E_k + Y_i \delta y^i)dt = 0. \tag{6.75}$$

A more complete and accurate determinacy of the relation (6.74) or (6.75) of the principle can be interpreted by its application to some mechanical systems, from simple to more complex ones.

Kinetic task. Unlike previous static task, the number of forces $\mathbf{F}$ is here extended by the inertia forces and determination of their work, such as kinetic energy

$$E = \frac{1}{2}\sum m_\nu \mathbf{v}_\nu\cdot\mathbf{v}_\nu = \frac{1}{2}g_{ij}(m,y)\dot{y}^i\dot{y}^j = \frac{1}{2}g_{ij}(m,x)\dot{x}^i\dot{x}^j. \quad (i,j = 1,\ldots,3N).$$

Then relation (6.75) becomes

$$(6.76) \qquad \int_{t_0}^{t_1}(\delta E_k + Y_j\delta y^j + \sum_{\mu=1}^{k}\lambda_\mu\delta f_\mu)dt = 0.$$

The essential difference between the action principle and the principle of work is that the former is used to study motion by means of the function of kinetic energy.

Lagrange's variational principle. If all forces, except for the inertia force, acting on the material point of constant mass m, mutually annul, i.e. if the resultant equals zero, the result is that there is only the action

$$\mathcal{A} = \int_{t_0}^{t_1} E_k dt,$$

and, in that case, the variational principle is written as

$$(6.77) \qquad \delta\int_{t_0}^{t} E_k dt = 0.$$

The significance of the formula (6.72) is underpinned by the fact that here and there it is referred to as the *Lagrangian action* and relation (6.72) as the *principle of least action* developed and expanded by the most distinguished and meritorious creators of analytical mechanics: Wolff (1726), Maupertuis (1746), Euler (1748), Lagrange 1760),... Jacobi even wrote that the principle of least action is the mother of the whole analytical mechanics. Relation (6.12) deriving here from a simple example can be obtained from much or increasingly more general observation. If the variation of the work of active forces equals zero, then

$$\int_{t_0}^{t_1}\delta E_k dt = 0, \qquad \int_{t_0}^{t_1}\delta A dt = 0,$$

as well as conversely, that is,

$$(6.78) \qquad \delta A = 0 \Leftrightarrow \delta A = 0.$$

For the system with potential energy, previous relation is reduced to

$$(6.79) \qquad \int_{t_0}^{t_1}(\delta E_k + \delta A(x))dt = \int_{t_0}^{t_1}(\delta E_k - \delta E_p)dt = \delta\int_{t_0}^{t_1} L dt = 0$$

where function $L := E_k - E_p$, is known as *Lagrange's function, Lagrangian,* or *kinetic potential.* It is, actually, Lagrange's variational principle, even though it is also known as Hamilton's principle, and the expression $L := E_k - E_p$, where Lagrange's function $L = E_k - E_p$, known as *the Hamiltonian action*, refers as such only to mechanical systems with potential forces, written in Hamilton's variables p, q. Considering relations (6.78) and (6.79) in a general and modified form, the author has here opted for the phrase action principle, because action is a scalar invariant

$$\mathcal{A} = \int_{t_0}^{t} L dt = \int_{t_0}^{t} (E_k - E_p) dt = \int_{t_0}^{t} (2E_k - H) = \int_{t_0}^{t} \big((p_\alpha \dot{q}^\alpha - H(p,q,t)\big) dt.$$

In that notation Hamilton's principle has the form

$$\delta \int_{t_0}^{t} \big(p_\alpha \dot{q}^\alpha - H(p,q)\big) dt = 0,$$

where $H = E_k - E_p$. Being such, it refers to mechanical systems with potential forces. Given the relations (6.78) and (6.79) in a general and modified form of the principle, the author has also opted here for the phrase The action principle. In applying Hamilton's principle the attention is often directed to the physical meaning of the function L, for which the principle is laid down, and therefore it is for the function L that the term Lagrangian is acceptable for any function depending of used independent coordinates x, its derivatives $\dot{x}$ and time t. Such approach led to some results incongruent with the preprinciples of mechanics, and consequently incongruent with real motion. In order to facilitate comparison of our statements to the standards of classical analytical mechanics, we will demonstrate below a somewhat more detailed application of the action principle (6.74) and (6.72) using configuration manifolds.

6.4. Action on configuration manifolds. Observe N material points of masses m_ν (N). Relative to the arbitrarily chosen pole O and orthonormal coordinate system $(y, \mathbf{e})$, let the position of the $\nu - th$ point be defined by vector$\mathbf{r}_\nu = y^i_\nu \mathbf{e}_i$. Let the point motion be restricted by $k \leqslant 3N$ restrained constraints that can be represented, according $k \leqslant 3N$ to the law of constraints, by vectors $\mathbf{R}^T_\nu$ (resistance, friction,) and by the help of independent equations

$$f_\mu(\mathbf{r}_1, \ldots, \mathbf{r}_N, \tau(t)) = 0 \quad (\mu = 1, \ldots, k). \tag{6.80}$$

alternatively, which is the same,

$$f_\mu(y^1_1, y^2_1, y^3_1; \ldots; y^1_N, y^2_N, y^3_N, \tau(t)) = 0, \textit{respectively} \tag{6.81}$$

$$f_\mu(y^1, \ldots, y^{3N}, y^0) = 0, \quad y^0 = \tau(t). \tag{6.82}$$

Functions f_μ are ideally smooth and regular in the reference frame of the material points. The condition of constraints' independence is in the simplest way reflected by the conditions of velocities on the constraints

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^i}\dot{y}^i + \frac{\partial f_\mu}{\partial y^0}\dot{y}^0 = 0. \tag{6.83}$$

For evidence, these equations can be written in the form

$$\frac{\partial f_\mu}{\partial y^1}\dot{y}^1 + \cdots + \frac{\partial f_\mu}{\partial y^k}\dot{y}^k = +\frac{\partial f_\mu}{\partial y^{k+1}}\dot{y}^{k+1} + \cdots + \frac{\partial f_\mu}{\partial y^{3N}}\dot{y}^{3N} + \frac{\partial f_\mu}{\partial y^0}\dot{y}^0. \tag{6.84}$$

From this, for velocities $\dot{y}$, linear system of equations, it is possible to define k velocities $\dot{y}^1, \ldots, \dot{y}^k$ by the help of the other $3N - k + 1$ velocities $\dot{y}^{k+1}, \ldots, \dot{y}^{3N}, \dot{y}^0$, at condition that the determinant is

$$\frac{\partial f_\mu}{\partial y^m}|_k^k \neq 0 \quad (\mu, m = 1, \ldots, k). \tag{6.85}$$

Many ways, or for short, the manifold of the choice of the sets of coordinates q^α by means of which the position or configuration of the points of a system at an instant of time is determined, indicate that the set of independent coordinates $q := (q^0, q^1, \ldots, q^n) \in M^{n+1}$ is to be called the configuration manifold. Equally, a set of coordinates q and velocities $\dot{q} = (\dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n)^T$ will be termed tangential manifolds TM^{n+1}. Accordingly, the pencil of all velocity vectors in point q will be denoted as $\dot{q} = (\dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n)^T$, which implies $n+1$ coordinate vectors $\mathbf{T}_q M^{n+1}$, at each point on manifolds $n+1$. So, further below we will consider two sets. For brevity, let us introduce the following notations, and in accordance with that too. At this condition and mentioned properties of the functions f it is possible, according to the theorem of implicit functions, to determine from equations (6.82) k dependent coordinates $y^{\cdot} \ldots, y^k$ by the help of the other $3N - k + 1$ coordinates $y^{k+1}, \ldots, y^{3N}, y^0$.. In doing so, the conditions of velocities (6.83) are substituted, in accordance with the definition, by relations

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^1}\dot{q}^1 + \cdots + \frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot{q}^n = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha. \tag{6.86}$$

In such theory, the action principle is

$$\int_{t_0}^{t_1} [Q_\alpha \delta q^\alpha - \delta A(\mathcal{I})]dt = 0 \tag{6.87}$$

where

$$Q_\alpha = Y_i \frac{\partial y^i}{\partial q^\alpha}, \quad (i = 0, 1, \ldots, 3N; \alpha = 0, 1, \ldots, n.)$$

are generalized forces. The work of the inertia forces, for $m_\nu = const.$, may be represented as the kinetic energy, and therefore, considering (6.86) and

(6.83), it is

$$A = -E_k = -\sum_{\nu=1}^{N} \frac{m_\nu}{2} \mathbf{v}_\nu^2 \cdot \mathbf{v}_\nu^2 = -\frac{1}{2} \sum m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \dot{q}^\alpha \dot{q}^\beta = -\frac{1}{2} a_{\alpha\beta}(m_\nu, q) \dot{q}^\alpha \dot{q}^\beta, \quad \dot{q} \in T\mathcal{N}. \tag{6.88}$$

The conditions of velocity (6.73) are substituted, in accordance with the definition (1.1)

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0} \dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^1} \dot{q}^1 + \cdots + \frac{\partial \mathbf{r}_\nu}{\partial q^n} \dot{q}^n =: \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \dot{q}^\alpha. \tag{6.89}$$

In such analysis, the action principle (6.73) is

$$\int_{t_0}^{t_1} [Q_\alpha \delta q^\alpha - \delta A(\mathcal{I})] dt = 0 \tag{6.90}$$

where

$$Q_\alpha = Y_i \frac{\partial y^i}{\partial q^\alpha} \quad i = 0, 1, \ldots, 3N, \quad \alpha = 0, 1, \ldots, n$$

are generalized forces or coordinates of the acting force vectors. The work of the inertia forces, for $m_\nu = const\nu$, is defined as a negative kinetic energy, and therefore, considering (6.86),

$$A(\mathbf{I}) = -E_k = -\frac{1}{2} \sum_{\nu=1}^{N} m_\nu \frac{m_\nu}{2} \mathbf{v}_\nu^2 \cdot \mathbf{v}_\nu^2 = -\frac{1}{2} \sum m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \dot{q}^\alpha \dot{q}^\beta = -\frac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta.$$

6.5. Hamiltonian variational principle. The notation $T^*\mathcal{N}$ here implies $2n+2$ dimensional manifold, consisting of $n+1$ generalized coordinates $q = (q^0, q^1, \ldots, q^n)$ i $n+1$ and $n+1$ generalized impulses $p = (p_0, p_1, \ldots, p_n)$; $p, q \in T^*\mathcal{N}$. The vocabulary which terms $T\mathcal{N}$ the tangential manifold calls the symbol $T^*\mathcal{N}$ the cotangential manifold. In the literature there are, here and there, the phrases such as 'phase space', 'state space', 'Hamilton's variables', etc. If we start from the fact that the state of motion is characterized by the position coordinates of point q^α and coordinates of impulse p_α, then it could be stated here that $T^*\mathcal{N}$ is the state of motion of a system. As $T^*\mathcal{N}$, for $T^*\mathcal{N}$ can be said to be the extended manifold if it is necessary to point out the difference from configuration manifold and its corresponding cotangential manifold. It is more important to understand and accept that are impulses, whose essence is defined by definition 2, than to know the names. In that case, there is mutual linear combination of generalized impulses and generalized velocities

(6.91) $$p_\alpha = a_{\alpha\beta}\dot{q}^\beta \quad \Leftrightarrow \quad \dot{q}^\alpha = a^{\alpha\beta}p_\beta.$$

Further considerations for the action principle on $T^*\mathcal{N}$ consists simply of substituting the velocities $\dot{q}^\alpha$ and considered relations by generalized impulses p_α,

(6.92) $$A = \frac{1}{2}\int_{t_0}^{t} p_\alpha dq^\alpha = \frac{1}{2}\int_{t_0}^{t} p_\alpha \dot{q}^\alpha dt.$$

Kinetic energy

(6.93) $$E_k = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = \frac{1}{2}p_\beta\dot{q}^\beta = \frac{1}{2}a^{\beta\gamma}p_\beta p_\gamma.$$

Hamiltonian action is

$$\mathcal{A} = \int_{t_0}^{t} Ldt = \int_{t_0}^{t}(E_k - E_p)dt = \int_{t_0}^{t}(2E_k - (E_k + E_p))dt = \int_{t_0}^{t}(p_\alpha\dot{q}^\alpha - H)dt$$

where

(6.94) $$H = E_k + E_p = \frac{1}{2}a^{\beta\gamma}p_\beta p_\gamma + E_p(q,t) = H(q,p,t),$$

is Hamilton's function.

If generalized forces Q_α are divided into potential and nonpotential forces P_α such that

$$Q_\alpha = -\frac{\partial E_p}{\partial q^\alpha} + P_\alpha,$$

and substituted into (6.90), it is obtained

$$\int_{t_0}^{t} \delta(p_\alpha\dot{q}^\alpha - H) + P_\alpha\delta q^\alpha dt = 0.$$

Furthermore,

(6.95)

$$\int_{t_0}^{t_1} \delta(p_\alpha\dot{q}^\alpha - H)dt = E\int_{t_0}^{t_1}[\delta p_\alpha\dot{q}^\alpha + p_\alpha\delta\dot{q}^\alpha - (\frac{\partial H}{\partial p_\alpha}\delta p_\alpha + \frac{\partial H}{\partial q^\alpha}\delta q^\alpha)]dt =$$

$$= p_\alpha\delta q^\alpha|_{t_0}^{t_1} + \int_{t_0}^{t_1}[(\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha})]\delta p_\alpha = (\dot{p}_\alpha + \frac{\partial H}{\partial q^\alpha})\delta q^\alpha dt = 0$$

and further

(6.96) $$p_\alpha\delta q^\alpha|_{t_0}^{t_1} + \int_{t_0}^{t_1}[(\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha})\delta p_\alpha + (P_\alpha - \dot{p}_\alpha - \frac{\partial H}{\partial q^\alpha})\delta q^\alpha]dt = 0.$$

If it is taken into account that

(6.97) $$\dot{q}^{\beta} = -\frac{\partial H}{\partial p_{\alpha}} = a^{\alpha\beta} p_{\alpha},$$

it is obtained (6.97).

Due to this, relation (6.96) is reduced to

(6.98) $$p_{\alpha}\delta q^{\alpha}|_{t_0}^{t_1} + \int_{t_0}^{t_1} (P_{\alpha} - \dot{p}_{\alpha} - \frac{\partial H}{\partial q^{\alpha}})\delta q^{\alpha} dt = 0.$$

With the condition that, besides potential, there exist nonpotential forces **P**, from the principle (6.96), the equations will follow

(6.99) $$\dot{p}_{\alpha} = -\frac{\partial H}{\partial q^{\alpha}} + P_{\alpha}, \quad (\alpha = 0, 1, \dots, n),$$

and these are differential equations of the state of motion of the system, constituting the system of $2n + 2$ differential equations

(6.100) $$\dot{p}_i = -\frac{\partial H}{\partial q^i} + P_i, \quad \dot{q}^i = \frac{\partial H}{\partial p_i},$$

(6.101) $$\dot{p}_0 = -\frac{\partial H}{\partial q^0} + P_0, \quad \dot{q}^0 = \frac{\partial H}{\partial p_0},$$

where $P_0 = P_0^* + R_0$. For the case when $P_i = 0$ i $P_0^* = 0$, the function (6.94) can be extended to the total mechanical energy

(6.102) $$E = H + \mathcal{P},$$

so that differential equations of motion can be written in the canonical form

(6.103) $$\dot{p}_{\alpha} = -\frac{\partial E}{\partial q^{\alpha}}, \quad \dot{q}^{\alpha} = -\frac{\partial E}{\partial p_{\alpha}}, \quad \alpha = 0, 1, \dots, n.$$

For the case of invariable constraints of the system, when there is not a rheonomic coordinate q0, equations with index 0 disappear, so that in equations (6.100) indices range from 1 to n.

MOND7 - DETERMINATION OF MOTION

7.1. Vector and tensor integration. The main task of mechanics, as enunciated by Newton, is to determine forces if motion is known, and if forces are known to determine motion - velocities and trajectories. Motion is described by a differential equation of motion

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}. \tag{7.1}$$

Much later, after Newton, Hamilton elicits just one task of mechanics: to integrate the system of $2n$, i.e. $6n$ differential equations of relative motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad i = 1, \dots, n. \tag{7.2}$$

Differences are substantial: the first equation (7.1) is a vector equation, whereas the second ones (7.2) are scalar, where forces are not included, except for the scalar function H which has the property of energy. In Section MOND2 of this book, the difference between the vector and the tensor of the same object is sufficiently emphasized, so there exists an issue of determining motion by the vector and tensor integral. To this end, it is necessary to clarify what vector integral is and what tensor integral is, when determining the same motion of the same object. Let us commence from the simplest example, $\mathbf{F} = 0$.

$$\frac{d\mathbf{v}}{dt} = 0, \rightarrow d\mathbf{v} = \mathbf{0},$$

and

$$\int_{\mathbf{v}_0}^{\mathbf{v}} d\mathbf{v} = \mathbf{v} - \mathbf{v}_0 = \mathbf{0}. \tag{7.3}$$

This is in full agreement with Newton's first axiom or law of motion - a body is at rest, or in uniform motion in the direction of the right line. Considering the first elementary definition of velocity at the point, $=\frac{d\mathbf{r}}{dt}$, from equation (7.3) there follows

$$\int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} = \int_{t_0}^{T} \mathbf{v}_0 dt = \mathbf{r} - \mathbf{r}_0 = \mathbf{v}_{0(t-t_0)}. \tag{7.4}$$

The example of the gravity force $F = G = const$ is similar

$$m\int_0^v d\mathbf{v} = \int_0^t \mathbf{G}dt.$$

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{G}}{m}t, \rightarrow \mathbf{r} = \frac{\mathbf{G}}{2m}t^2$$

Galileo was familiar with this phenomenon owing to his practice with a falling body. The examples indicate that standard vector integrals are applied to vector differentials. In accordance with the principle of invariance, vectors can be decomposed into coordinate vectors in rectilinear y^i and curvilinear x^i coordinates x^i. Thus the position vector $\mathbf{r}$ of the same point $M(y) = M(x)$ can be decomposed into three vectors each

$$\mathbf{r} = y^i\mathbf{e}_i = x^i\mathbf{g}_i. \tag{7.5}$$

Let us repeat integration of (7.3) and (7.4) with base vectors $\mathbf{e}_i$ being constant; their differentials are $d\mathbf{e}_i = 0$, and coordinate vectors $\mathbf{g}_i$ are functions of coordinates x^i . Consequently, their differentials are:

$$d\mathbf{r} = dy^i\mathbf{e}_i = dx^i\mathbf{g}_i + x^i d\mathbf{g}_i(x), \tag{7.6}$$

and relative integrals

$$\mathbf{r} = y^i\mathbf{e}_i = \int \mathbf{g}_i(x)dx^i + \int x^i d\mathbf{g}_i(x),$$

where

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial y^i}, \mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\mathbf{e}_j. \tag{7.7}$$

Considering that

$$\frac{\partial^2 y^i}{\partial x^j \partial x^k} = \Gamma_{ik}\frac{\partial yi}{\partial x^j} \tag{7.8}$$

it follows that it is a differential of the covariant vectors or for short

$$D\mathbf{g}_i = \frac{\partial y^j}{\partial x^i}d\mathbf{e}_j = 0. \tag{7.9}$$

By applying standard vector integral to relation (7.6), it is obtained

$$\mathbf{r} - \mathbf{r}_0 = (y^i - y_0^i)\mathbf{e}_i = \int_{x_0}^{x^i} d(x^i\mathbf{g}_i(x)),$$

which coincides with equations (7.5) for corresponding boundary conditions. However, if only vector coordinates y^i and dx^i are observed, with exclusion of $d\mathbf{g}_i(x)$, the integration task is more complex and does not yield the results complying with the preprinciple of mechanics. This is sufficient to doubt the invariant standard integration on vector coordinates, i.e. tensors.

7.2. On the solutions of differential equations of motion. Integration of differential equations or systems of differential equations of motion and analysis of obtained solutions for known parameters at some instant of time means comprehension of the motion of mechanical objects. There are very few real motions of the body, of the system of bodies in particular, which can be described by finite general analytical solutions of differential equations. Many models of the system, presented in the textbook literature do not reflect accurately the real motion of objects. And yet, mechanics solves successfully, with high accuracy and pretty correct assessment of error size, the problems of all mechanical motions accessible to the human eye, and even more than that. Here, our attention focuses on several conclusions, based on parallel motion of vectors and more general solutions grounded on linear or tensor transformations. Two approaches deriving from differential equation (7.1) - the second axiom of motion, which are rational starting points of Newtonian mechanics, require verification in accordance with the preprinciple of existence.

a. Material points, such as celestial bodies, ballistic missiles, or a falling body, are acted upon by the gravity force, so that the relation, according to the present knowledge about forces, does not satisfy the preprinciple of existence, and therefore it cannot be stated that bodies are in uniform motion in the direction of the straight lines. If we knew, but we don't, at any moment and at any position, the universal gravitation force of all celestial bodies and if we could currently produce opposing forces, the missile would move along a straight line, and this means: if it were, what is not and what is known, which cannot be predicted by our knowledge.

b. Vessels can move on calm waters of the ocean at the velocity of constant magnitude, but not in a straight line, which does not really occur in the ocean.

c. Locally, the technical measurement system, tied to the Earth, can apply the reaction and other forces to the object to make it move at constant velocity, but this does not still lead to the conclusion on the trajectory shape as a straight line; the straight line being the concept of plane geometry is not available for logical-physical experiment, so it is unnecessary to ground mechanics on this fact, especially if the whole theory can be developed without the principle of rectilinear motion.

7.3. Impulse integrals for material point motion. For the material point of constant mass and condition,

$$\mathbf{F} + \mathbf{R} = \mathbf{0}, \tag{7.10}$$

from equation (7.1) it is obtained that the impulse vector of motion is constant,

$$\mathbf{p} = m\mathbf{v}(t) = \mathbf{c} = \mathbf{const.} = m\mathbf{v}(t_0) = \mathbf{p}_0. \tag{7.11}$$

At first sight, this is the simplest first vector integral which solves the task of determination of motion

$$\mathbf{r}(t) = \mathbf{v}(t_0)t + \mathbf{r}(t_0). \tag{7.12}$$

However, relation (7.11) and incongruity with impulse coordinates require more clarification of this essential meaning. Integral (7.11) satisfies and explains the best the preprinciple of determinacy; how much accurately the mass and initial velocity are known at some instant of time t_0, with such accuracy (7.12) is the impulse of motion determined at any other instant of time.

The preprinciple of invariance must be satisfied, so that integral (7.12) - essentially impulse p survives in the present theory. If vector (7.12) is decomposed in a coordinate system $y, \mathbf{e}$ as

$$\mathbf{p} = m\mathbf{v} = m\dot{y}^i\mathbf{e}_i = c^i\mathbf{e}_i = m\dot{y}_0^i\mathbf{e}_i$$

and scalar multiplication is performed by vector $\mathbf{e}_j$, it is obtained

$$p_j(t) = m\dot{y}_j = m\dot{y}_j(t_0) = p_j(t_0). \tag{7.13}$$

Note that these equations are not vector but scalar equations. Allowing parallel shift of base vectors e_i and along with them the transformation of coordinate vectors $\mathbf{g}_k = \frac{\partial y^i}{\partial x^k}\mathbf{e}_i$ for the material point free shift, vector

$$\mathbf{p} = m\dot{x}^k\mathbf{g}_k(x) = m\dot{x}^k(t_0)\mathbf{g}_k(x_0)$$

can undergo scalar multiplication by vector $\mathbf{g}(x)$. Projections of integrals onto coordinate directions

$$p_l(x, \dot{x}) = a_{kl}(x)\dot{x}^k = a_{kl}(x_0, x)\dot{x}^k(t_0) = a_{Kl}a^{KL}p_L = a_l^L p_L, \tag{7.14}$$

where the capital letter in an index denotes relative quantity at the initial instant of time, where as tensor

$$a_{Kl} = m\left(\frac{\partial y}{\partial x^k}\right)_0 \frac{\partial y}{\partial x^l} = mg_{Kl} = m\mathbf{g}_{Kx_0}\cdot\mathbf{g}_l(x) \tag{7.15}$$

Tensor mg_{kl} found in the lecture as "parallel shift tensor".

To satisfy the preprinciple of invariance, integrals should be obtained directly from coordinate forms of the equations of motion. According to the preprinciple of invariance, this relation should also hold for a curvilinear coordinate system. This is confirmed by integration of equations with forces $\mathbf{X}_i + \mathbf{R}_j = 0$. The covariant integral is

$$\int D(a_{ij}v^j) = a_{ij}v^j - A_i = 0, \tag{7.16}$$

where $A_i = g_i^K p_K(t_0)$ is a covariantly constant co-vector; $DA_i dt = 0$. Accordingly, integral (7.14) is the integral of differential equations of motion.

$$p_i(t) = a_{ij}\dot{x}^j = a_{iJ}\dot{x}^J = a_{iJ}a^{JK}p_K = g_i^K p_K(t_0), \tag{7.17}$$

where $g_i^K p_K$ is a covariantly constant vector; $DA_i dt = 0$. Without indicating the possibility of parallel shift of the co-vector, the impulses can be translated from the system of coordinates y to the curvilinear coordinates x. If coordinates x are denoted by indices $x_k; l = 1, 2, 3$, it will follow

$$p_j(t) = p_k \frac{\partial x^k}{\partial y_j} = p_j(t_0) = p_K(t_0) \frac{\partial x^K}{\partial y^j}.$$

Multiplying by matrix $\left(\frac{\partial y^j}{\partial x^l}\right)$ it is obtained

$$p_j(t) \frac{\partial y^j}{\partial x^l} = p_K(t_0) \frac{\partial x^K}{\partial y^j} \cdot \frac{\partial y^j}{\partial x^l} = g_l^K p_K(t) = p_l(t),$$

because

$$g_l^K = \frac{\partial x^K}{\partial y^j} \frac{\partial y^j}{\partial x^l}.$$

Although co-variant integrals satisfy all three preprinciples, such integration is not widespread in mechanics due to 'difficulties' in determining the tensor g_l^K. So, let us seek ordinary first integrals, reduced to constants, but not covariantly constant impulse coordinates.

Let us write differential equations of motion (7.1) in a developed form

$$a_{ij} \frac{D\dot{x}^i}{dt} = \frac{D a_{ij} \dot{x}^j}{dt} = \frac{Dp_i}{dt} = \frac{dp_i}{dt} - p_k \Gamma_{ij}^k \frac{dx^j}{dt} = X_j + R_j. \tag{7.18}$$

For conditions

$$X_j + R_j + p_k \Gamma_{ij}^k \dot{x}^j = 0, \tag{7.19}$$

which differ, it should be noted, from conditions (7.10), that first integrals are obtained

$$p_j(t) = const_j = p_j(t_0) \tag{7.20}$$

relative to coordinate system $((x, \mathbf{g})$. So, as in the case of integral (7.4) in the base coordinate system $(y, \mathbf{e})$. These integrals differ considerably from integrals (7.8), and therefore in their essence from integrals (7.13). This is why integrals (7.4) and (7.8) will be referred to as covariant integrals, unlike ordinary integrals (7.11). Ordinary integrals are destructive for the tensor nature of the observed objects.

Example 19. Observe the motion of a material point in parallel relative to rectilinear y^1, y^2, y^3 and cylindrical coordinate system $x^1 := r$, $x^2 := \varphi$, $x^3 := z$. It is well-known[13] that $y^1 = r \cos\varphi$, $y^2 = r \sin\varphi$, $y^3 = z$

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

[13] V. Vujicic, Preprinciples of Mechanics, pp. 164,165.

$$g_{ik} = \begin{pmatrix} \cos(\varphi-\varphi_0) & r_0\sin(\varphi-\varphi_0) & 0 \\ -r\sin(\varphi-\varphi_0) & rr_0\cos(\varphi-\varphi_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Differential equations and integrals for

$$Y + R_Y = 0 \quad \Rightarrow \quad X + R_X = 0$$

and covariant differentiation and covariant integration establish equivalence at the same transformation

$$\frac{d\dot{y}^i}{dt} = \frac{D\dot{y}^i}{dt} = 0 \quad \Leftrightarrow \quad \frac{D\dot{x}^i}{dt} = 0$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\dot{y}^i = \dot{y}^i_0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}^1 = \dot{x}^1_0\cos(x^2 - x^2_0) + \\ \quad + x^1_0\dot{x}^2_0\sin(x^2 - x^2_0) \\ \dot{x}^2 = \dfrac{\dot{x}^1_0\dot{x}^2_0}{x^1}\cos(x^2 - x^2_0) + \\ \quad + x^1_0\sin(x^2 - x^2_0), \\ \dot{x}^3 = \dot{x}^3_0. \end{cases}$$

A shorter, more explicit, general and significant difference between the first impulse integrals $p_i = c_i$ and covariant integrals $p_i = A_i$ shows integration of differential equations (7.4) for the condition that generalized forces are $Q_i = 0$. Let it be for now the motion of a single material point in the curvilinear coordinate system x^1, x^2, x^3, that is,

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}^i} - \frac{\partial E_k}{\partial x^i} = 0, \quad (i = 1, 2, 3). \tag{7.21}$$

These equations can be written in the form

$$\frac{D}{dt}\frac{\partial E_k}{\partial \dot{x}^i} = 0. \tag{7.22}$$

From (7.21) integrals (7.20) are obtained for $\partial E_k/\partial x^i = 0$, and from (7.22) covariant integrals (7.17), because

$$\frac{\partial E_k}{\partial \dot{x}^i} = p_i.$$

Canonical equations (7.2), as evident from

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} + X_i, \quad (i = 1), 2, 3)$$

commonly yield integral impulses of the type (7.20) for the condition that the right-sides of those equations equal zero.

Standard integration and integrals (7.20) are more widespread compared to covariant integrals (7.17). This is mainly due to undeveloped vector and tensor calculus, respectively, in contrast to scalar functions. The convenience of ordinary integration is that constants can be determined, depending on a given initial value of the observed impulse, when the number of impulse integrals is smaller than the number and coordinates of impulses, for instance,

$$x = c_1 = p_1(t_0), \quad p_3(t) = c_3 = p_3(t_0); \qquad p_2 \neq const.$$

This advantage comes to the fore in the system of material points with constraints, especially on manifolds T^*M. The accuracy of both integrations is proved, but for different conditions. Covariant integration is invariant in relation to linear homogeneous transformations of coordinate systems, and therefore reflects the tensor nature of integrals. However, this is not the case with standard integration, nor is it in agreement with the preprinciple of invariance; this indicates that final synthesis results should be checked by comparing them with corresponding results in coordinate systems $(y, \mathbf{e})$.

Example 20. Impulse integrals for motion along a surface. Differential equations of the material point motion along a surface

$$f(y_1, y_2, y_3, y_0) = 0, \quad f_0 = y_0 - \tau(t) = 0 \tag{7.23}$$

are of the form

$$m\ddot{y}_i = Y_i + \lambda \frac{\partial f}{\partial y^i}, \tag{7.24}$$

and

$$\lambda_0 \frac{\partial f_0}{\partial y_0} + \lambda \frac{\partial f}{\partial y_0} = 0. \tag{7.25}$$

From the acceleration conditions, i.e. in a specific case

$$\ddot{f} = \frac{\partial^2 f}{\partial y^k \partial y^l}\dot{y}^k \dot{y}^l + \frac{\partial f}{\partial y_i}\ddot{y}^i + \frac{\partial f}{\partial y_0}\ddot{y}_0 = 0; \quad (k, l = 0, 1, 2, 3;\ i = 1, 2, 3), \tag{7.26}$$

it is obtained that

$$\lambda = -\frac{m(\phi + \frac{\partial f}{\partial y_0}\ddot{y}_0) + \frac{\partial f}{\partial y_i} Y_i}{\frac{\partial f}{\partial y_i}\frac{\partial f}{\partial y_i}} \tag{7.27}$$

where

$$\phi = \frac{\partial^2 f}{\partial y_i \partial y_j}\dot{y}_i \dot{y}_j + 2\frac{\partial^2 f}{\partial y_i \partial y_0}\dot{y}_i \dot{y}_0 + \frac{\partial^2 f}{\partial y_0 \partial y_0}\dot{y}_0 \dot{y}_0. \tag{7.28}$$

It becomes obvious that on the right-hand sides of differential equations of motion (7.24) the inertia force $-m\ddot{y}_0$ figures for the case when the equation of $m\ddot{m}$ and hy_0 surface (7.23) contains the function of time to the power different from unity, and for the case when the power to one constant velocity $\mathbf{v}^0$is present. That is why prior to integration of differential equations it is important to take this fact into account in order to obtain accurate impulse integrals. Pointing out this conclusion does not affect the general proof if it is assumed that the resultant of active forces is not present $Y_i = 0$.

If the multiplier (7.27) also equals zero, impulse integrals (7.23) would exist. Also, if it is assumed that the surface does not change in time, i.e. that equation (7.23) has the form $f(y_1, y_2, y_3) = 0$, it would follow that

$$\lambda = -m \frac{\frac{\partial^2 f}{\partial y^i \partial y^j} \dot{y}_i \dot{y}_j}{\frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^i}} = 0. \tag{7.29}$$

which brings us back to considering the motion along a double-sided immovable surface. However, if that surface changes, it would follow from (7.27) and (7.28) that

$$\lambda = -m \frac{\partial f}{\partial y_0} \ddot{y}_0 + 2 \frac{\partial^2 f}{\partial y^i \partial y^j} \dot{y}^i \dot{y}^j . y_0^2 + \frac{\partial^2 f}{\partial y^i \partial y_j} \dot{y}_i \dot{y}_j \frac{\partial}{f} \partial y_i \frac{\partial f}{\partial y_i}. \tag{7.30}$$

Equalizing with zero would lead to the conclusion that impulses of motion are constant at material point motion along a surface that is in uniform and translational motion in the absence of forces. But, the assumptions are in contradiction with the preprinciple of existence, Galileo's laws and universal gravitation law.

The assumption (7.3) is possible, but in that case multipliers of constraints (7.21) indicate a significant difference between the material point motion along a movable and immovable surface.

Example 21. A heavy point, of mass m, moves along a horizontal smooth plane $f = z - (at)^2 = 0$, $a = const$, which moves horizontally upward. Differential equations of motion (7.24) are

$$m\ddot{x} = 0, \quad m\ddot{y} = 0, \quad m\ddot{z} = -mg + \lambda.$$

If we choose $z_0 = at$, rom relations (8.12), for an auxiliary coordinate, it follows that $a^2 = \sqrt{g/2}$. and therefore there will exist three integrals (8.4) for $a^2 = \sqrt{g/2}$. It need not be proved that for any other different motion of the observed horizontal plane ('lift floor') the first integral $p_z = m\dot{z} = c = m\dot{z}_0$. will not exist. Relative to curvilinear coordinate systems (x, g) the equation of constraint (8.14) is transformed into

$$f(x^1, x^2, x^3, x^0) = 0, \quad x^0 = \tau(t). \tag{7.31}$$

From these equations, for the assumed conditions, covariant impulse integrals (7.8) can be obtained, and for the conditions (7.10) first integrals of the form (7.11) will be obtained. If the observed motion along the surface

(7.22) is determined by means of equations (7.2) where kinetic energy is

$$E_k = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta, \quad \alpha, \beta = 0, 1, 2,$$

and impulses are

$$p_0 = a_{0\beta}\dot{q}^\beta, p_1 = a_{1\beta}\dot{q}^\beta, p_2 = a_{2\beta}\dot{q}^\beta,$$

three covariant integrals will be obtained

$$p_0 = a_{0\beta}\dot{q}^\beta, p_1 = a_{1\beta}\dot{q}^\beta, p_2 = a_{2\beta}\dot{q}^\beta,$$

for the conditions that generalized forces equal zero, $Q_\alpha = 0$, or the first three integrals

$$p_\alpha(t) = c_\alpha = p_\alpha(t_0) \tag{7.32}$$

for the conditions

$$Q_\alpha + \frac{\partial E_k}{\partial q^\alpha} = 0.$$

For the case when constraints (7.31) do not depend explicitly on the time of rheonomic coordinates q^0, and the corresponding impulse, then there are only two impulses (7.32).

Impulse integrals for the rotation motion of the body. For an arbitrary system of material points from the impulse of point change theorem (7.1) covariant impulse integrals are obtained

$$p_\alpha = A_\alpha\left(p(t_0), q(t)\right)$$

where A_α are covariantly constant vectors if generalized forces equal zero.

The first integrals $p_\alpha(t) = c_\alpha = p_\alpha(t_0)$ are sought, which are obtained in the simplest way from differential equations of motion (7.2), from where it is evident that there also exist the first integrals for the conditions $P_\alpha - \frac{\partial H}{\partial q^\alpha} = 0$, $\alpha = 0, 1, \ldots, n$. For $p_0 \neq -H$ from here, as from (7.2), it is proved that $p_0 \neq -H$.

Impulse integrals for the rotational motion of the body Based on relations (7.1) and (7.2), it follows that there exist impulse integrals for the rotational motion of the body, of constant mass, around a stationary point and relative to a stationary orthonormal coordinate system $(\mathbf{y}; \mathbf{e})$,

$$p_i = I_{ij}(t)\omega^j(t) = A_i = I_{ij}(t_0)\omega^j(t_0). \tag{7.33}$$

if moments of forces are $M_i = 0$, $(i, j = 1, 2, 3)$. Similarly, from differential equations

$$M_i = 0, \quad I_{ik} = 0$$

there follows

$$p_1 = I_{11}\Omega^1 = A_1 = c_1,$$
$$p_2 = I_{22}\Omega^2 = A_2 = c_2, \tag{7.34}$$
$$p_3 = I_{33}\Omega^3 = A_2 = c_3,$$

where $c_i = const.$ By squaring these equations and summing, it is obtained

$$(I_{11}\Omega^1)^2 + (I_{22}\Omega^2)^2 + (I_{33}\Omega^3)^2 = c^{2*} \tag{7.35}$$

where $c = constant$

The theorem of change in kinetic energy shows that E_k equals integral

$$E_k = \int S dt + c_1, \tag{7.36}$$

and is constant only if the power S of that system equals zero, and therefore total mechanical energy is constant, that is,

$$E_k + E_p + P(q^0) = c_2 \tag{7.37}$$

if the power of nonpotential forces equals zero. The same integral can be written in the form

$$E_k + E_p = \int R_0(q^0)dq^0 + c_2. \tag{7.38}$$

For the case when constraints are also unchangeable, the right-hand side integral disappears (7.38), and under such condition a well-known "conservation" of energy integral is obtained

$$E_k + E_p = h = const. \tag{7.39}$$

is shown extensively and explicitly in the work by V. Vujicic: Integral test for canonical differential equations.

Each function $f_\mu(q^0, \ldots, q^n; p_0, \ldots, p_n)$, or equation

$$f_\mu(q^0, \ldots, q^n; p_0, \ldots, p_n) = c_\mu, \tag{7.40}$$

is the integral of equations

$$\dot{q}^\alpha = \frac{\partial E}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial E}{\partial q^\alpha} + P_\alpha, \qquad (\alpha = 0, 1, \ldots, n), \tag{7.41}$$

if the derivative with respect to time of the function f_μ equals zero along the phase space trajectory of the system, that is,

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial q^\alpha}\frac{\partial E}{\partial p_\alpha} - \frac{\partial f_\mu}{\partial p_\alpha}\frac{\partial E}{\partial q^\alpha} + P_\alpha \frac{\partial f_\mu}{\partial p_\alpha} = 0 \tag{7.42}$$

alternatively,

$$(f_\mu, E) + P_\alpha \frac{\partial f_\mu}{\partial p_\alpha} = 0, \tag{7.43}$$

where (f_μ, E) are the Poisson brackets for T^*N.

Example 22. Gyroscopic forces are specified by the formula

$$P_\alpha = G_{\alpha\beta}\dot{q}^\beta, \quad G_{\alpha\beta} = -G_{\beta\alpha}.$$

Check if E is the integral of differential equations (7.41). Since $(E; E) \equiv 0$ and

$$G_{\alpha\beta}\dot{q}^\beta \frac{\partial E}{\partial p_\alpha} = G_{\alpha\beta}\dot{q}^\beta \dot{q}^\alpha = 0$$

it follows that there exists the integral

$$E = \frac{1}{2}a^{\alpha\beta}p_\alpha p_\beta + E_p(q^0, q^1, \ldots, q^n) + \int R_0(q^0)dq^0 = c.$$

Analogously, the existence of the energy integral in the presence of nonholonomic constraints of the form $\varphi_\sigma = b_{\sigma\sigma}(q^0, q^1, \ldots, q^n)\dot{q}^\alpha = 0$ is shown.

Example 23. Hamilton's function $H(p_1, \ldots, p_n;\ q^1, \ldots, q^n)$ is not the integral of the starting Hamilton's differential equations in a general case, because $(H, E) \neq 0$. Indeed, it follows that

$$(H, F + P) = (H, H) + (H, P) = (H, P) = \frac{\partial H}{\partial q^\alpha}\frac{\partial}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial P}{\partial q^\alpha} =$$

$$= \frac{\partial H}{\partial q^i}\frac{\partial P}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial P}{\partial q^i} + \frac{\partial H}{\partial q^0}\frac{\partial P}{\partial p_0} - \frac{\partial H}{\partial p_0}\frac{\partial P}{\partial q^0} = \frac{\partial H}{\partial p_0}\frac{\partial P}{\partial q^0} = \dot{q}^0 R_0 \neq 0.$$

It is only for the case when constraints do not depend on time, or when $R_0 = 0$ that Hamilton's function occurs as an integral of a potential mechanical system.

Example 24. By composing differential equations (of rotational motion (7.34) with Ω^i, for $I_{ik} = 0$, $(i \neq k)$, $M_i = 0$, or by gradual multiplication of equations of motion by corresponding angular velocities $\Omega^1, \Omega^2, \Omega^3$, by summing and integration, one obtains the energy integral

$$2E_k = I_{11}(\Omega^1)^2 + I_{22}(\Omega^2)^2 + I_{33}(\Omega^3)^2 = h = const$$

of rotational motion around the center of inertia. Integration and preprinciples. In developing the theory of mechanics, based on some principles of mechanics, it has been demonstrated that the same motions of the same mechanical systems can be described by various differential equations relative to the same or different coordinate systems. For all mentioned systems of differential equations of motion, it has been shown that they are congruent with the preprinciples. The preprinciple of invariance could have been represented for very complex systems of differential equations of motion due to a very developed theory of differential geometry on manifolds and invariance of natural ('covariant' or 'absolute') vector derivative with respect to time. However, in integral calculus and its application in mechanics the attention is insufficiently paid to the issue of invariance of differential expressions integration, of which the most frequent are differential equations of motion. It has been already pointed out that standard integration is destructive for the tensor nature of geometrical and mechanical objects, which is in disagreement with the preprinciples, especially those of determinacy and invariance. Generalization of vectors as an arranged set of functions over a vector base, constituted again by vectors, does not lead to determining the attribute of motion in mechanics neither by differentiating nor by integration, and therefore that generality cannot be the basis for agreement between the derived theory and the prepinciple of determinacy. More general categories of knowledge belong to higher-level mathematics. The examples explicitly indicate the kind of difficulties encountered with the preprinciple of invariance if vector base is not determined and known. There is still the presence of 'truths': 'acceleration is not a vector (in terms of the tensor), 'acceleration vector is not a vector', or 'inertia tensor is not a tensor'. Such theses do not have their place in the theory that starts from the herein introduced preprinciples of existence, determinacy and invariance. In mechanics, it does not exist only one general configuration arrangement - a single general, arranged set of all bodies and their mutual distances, but many diverse sets and subsets, whose problems of motion are not solved in a single way, i.e. uniformly but in a number of equivalent ways. That is why the phrase 'differentiation and integration of tensors on manifolds' is meaningful if the type of manifold is clarified, or valid evidence provided for invariance of differentiation and integration on manifolds. Generality indicates a multitude of diversity, and therefore it is justifiable to seek solutions of general accuracy in terms of the preprinciple of determinacy; the subject matter of solving requires specific and general knowledge. Simple integral, for example

$$f(x) = \int x dx = \frac{1}{2}x^2 + c, \quad c = const.$$

is indefinite or definite to a constant, for if there is no other knowledge about the function $f(x)$ it is impossible to determine what type of curve it is (path, force, energy,...) for uninterrupted multitude of curves for each $c \in R$. It is only when we have at least one data more about $f(x)$ at any point, let's say $f(2) = 2$, will we know what type of line it is. Similar situation is with covariant integrals on metric differential manifolds which are, as evident, present in mechanics. For integral

$$\hat{f} = \hat{\int} g_{ij}(x)v^i(x)dv^j(x) = \frac{1}{2}g_{ij}(x)v^i(x)v^j(x) + A \tag{7.44}$$

or more simply

$$\hat{f} = \int g_{ij}(x) dv^j(x) = g_{ij}(x) v^j + A_i \tag{7.45}$$

it can be said that it is indefinite or definite to a covariantly-constant tensor. (A_i are vector coordinates, A - constant). To the level of knowledge about manifold, which means about metric tensor g_{ij} too and covariantly constant tensor A, at some defined point, the integral sought can be determined. Integral (7.45) is of the energy integral type (7.39), whereas integral (7.45) is of the impulse type (7.33).

Example 25. A system of N material points of constant masses m_ν ($\nu = 1, \dots, N$) and of $3N-2$, $3N-2$ finite constraints $f_\mu(\mathbf{r}_1, \dots, \mathbf{r}_N) = 0$ has two-dimensional manifold ^{2}, whose metric, or more precisely, mass tensor is

$$a_{ij} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} = a_{ji}(q^1, q^2). \tag{7.46}$$

Differential equations of motion for $Q_1 = 0,\ Q_2 = 0$ are

$$\frac{D}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^i}\right) = 0,$$

or considering that

$$\frac{\partial E_k}{\partial \dot{q}^i} = p_i = a_{ij}\dot{q}^j, \quad Dp_i = D(a_{ij}\dot{q}^j) = 0.$$

Covariant integral is

$$\hat{\int} D(a_{ij}(q)\dot{q}^j) = a_{ij}\dot{q}^j - A_= 0. \tag{7.47}$$

where A_i is covariant constant tensor, that is,

$$DA_i = dA_i - A_k \Gamma^k_{ij} dq^j = 0. \tag{7.48}$$

7.4. Superfluous symbols in tensor integration. Here, we start from the definition of a vector, as a tensor, provided by B.A. Dubrovin, S.P. Novikov, A.G. Fomenkoo [1], or taken from the book [2].

Definition 4. The vector at point $P = (x_o^1, \dots, x_0^n)$ is called z set of numbers a $(\xi_0^1, \dots, \xi_0^n)$, with respect to the system of coordinates $(x^1, \dots, x^n)$. if two systems of coordinates $(x^1, \dots, x^n)$ end $(z^1, \dots, z^n)$ linked by alft $x = x(z)$, where $x^i(z_0^1, \dots, z_0^n) = x = x_0^i$, $i = 1, \dots, n$, for a new system of coordinates z that very same vector at the point $z_0^1, \dots, z_0^n$ is specified by another set of numbers$\zeta^1, \dots, \zeta^n$, which are linked by the initial formula

$$\xi^i = \left(\frac{\partial x^i}{\partial \zeta^j}\right)_{\zeta^k = \zeta_0^k} \zeta^j. \tag{1*}$$

Note that a from of transformation law appears as the major provision of a vector, bring a first-rank tensor (1)."

In order to clarify our assertions,let as present some basic knowledge: a point is an essential concept not undergoing logical determination.However, the position of a point is not that simple understanding. I is generally determined with recept to coordinate systems. Let point P be an intersection in a Cartesian coordinate system $P(y^1, y^2, y^3)$ of there straight lines or three planets, or an intersection of three curves of three curved surfaces.

In accordance with the invariance principle [3],

$$P(y^1, y^2, y^3) = P(x^1, x^2, x^3)$$

and

$$\mathbf{r} = y^i \mathbf{e}_i = x^i \mathbf{g}_i(x),$$

at the condition

$$y^i = (\frac{\partial y^i}{\partial x^j})_{P_0} x^j, \ldots, |\frac{\partial y^i}{\partial x^j}|_3^3 \neq 0.$$

Immediately next to the point P_0 the differentials are

$$dy^i = (\frac{\partial y^i}{\partial x^j})_{P_0} dx^j \longleftrightarrow dx^j = (\frac{\partial y^i}{\partial x^j})_{P_0} dy^i,$$

and derivatives for the parameter t

$$\frac{dy^i}{dt} = (\frac{\partial y^i}{\partial x^j})_{P_0} \frac{dx^j}{dt},$$

but since

$$(\frac{\partial y^i}{\partial x^j})_{P_0} \neq 0,$$

the velocity vectors are also created at the point P_0

$$v^i(y) = (\frac{\partial y^i}{\partial x^j})_{P_0} v(x) \longleftrightarrow v^j(x) = (\frac{\partial x^j}{\partial y^i})_{P_0} v^j(y),$$

as well as the acceleration vectors w^i,

$$w^i(y) = (\frac{\partial y^i}{\partial x^j})_{P_0} w^j(x),$$

which are often written in the forme:

$$w^i(y) = \frac{dv^i}{dt} = \frac{d}{dt}[(\frac{\partial y^i}{\partial x^j})\dot{x}^j] = (\frac{\partial^2 y^i}{\partial x^k \partial x^j})\dot{x}^k \dot{x}^j + (\frac{\partial y^i}{\partial x^j})\ddot{x}^j =$$
$$= \Gamma^i_{jk} \dot{x}^k \dot{x}^j (\frac{\partial y^k}{\partial x^i}) + (\frac{\partial y^i}{\partial x^j})\ddot{x}^j = \frac{D^2 y^i}{dt^2},$$

where Γ^i_{jk} symbols by the provision (see, for example [1], str. 258.)

$$\Gamma^i_{jk} = \frac{\partial^2 y^i}{\partial x^k \partial x^j}\left(\frac{\partial y^k}{\partial x^i}\right).$$

However, since $\left(\frac{\partial y^k}{\partial x^i}\right)_{P_0}$ are constants, it follows that

$$\frac{d}{dt}\left[\left(\frac{\partial y^i}{\partial x^j}\right)_{P_0}\right] = 0, \rightarrow \quad \Gamma^i_{jk} = 0.$$

Now, it is clear that the linear partial derivative is invariant with respect to the multi-degree coordinate vectors. This follows from the determination (definition) of a vector as a first-rank tensor.

The system of points. The term the system of points $M_1, \dots, M_N$ impulses a multitude of points linked by geometric constraints continuous in specified region

$$f_\mu(y^1, \dots y^{3N} = 0, \qquad \mu = 1, \dots k \leqslant 3N,$$

and under conditions

$$\frac{\partial f_\mu}{\partial y^i} dy^i = 0,$$

when $|\frac{\partial f_\mu}{\partial y^i}| \neq 0$. Equations $f_\mu = 0$ enable the determination $3N - k = n$ independent coordinates referred to and denoted as independent and generalized coordinates, q^α; $\alpha = 1, \dots, n = 3N - k$. Indeed, further determination of a vector, as first-rank tensor (1), refers to the system of n independent coordinates $q^1, \dots, q^n$ and it is obtained

$$y^l = y^l(q^1, \dots, q^n).$$

Analogous to the relations, it will be

$$y^i = \left(\frac{\partial y^i}{\partial q^\alpha}\right)_{P_0} q^\alpha, \qquad attr\, y = L.$$

Furthermore, with respect to the natural argument t it will be

$$\frac{dy^l}{dt} = \frac{d}{dt}\left(\left(\frac{\partial y^l}{\partial q^\alpha}\right)_{P_0}\frac{dq^\alpha}{dt}\right) = \left(\frac{\partial^2 y^l}{\partial q^\beta \partial q^\alpha}\right)_{P_0}\dot{q}^\alpha \dot{q}^\beta + \left(\frac{\partial y^l}{\partial q^\alpha}\right)_{P_0}\frac{dq^\alpha}{dt}.$$

However, since, like with respect to coordinates x^i, $\left(\frac{\partial y^l}{\partial q^\alpha}\right)_{q_0}$ are constants, it is

$$\frac{d}{dt}\left[\left(\frac{\partial y^i}{\partial q^\beta}\right)_{P_0}\right] = 0,$$

it follows that $G^\gamma_{\alpha\beta} = 0$,

$$v^\alpha(q) = \left(\frac{\partial q^\alpha}{\partial y^k}\right)_{P_0} v^k(y), \qquad attr\, v = LT^{-1};$$

$$w^\alpha(q) = (\frac{\partial q^\alpha}{\partial y^k})_{P_0} w^k(y), \qquad attr\, w = LT{-}2;$$

. **An obvious example.** Differential equation of motion of the mechanical system are written in the form

$$a_{\alpha\beta}(\ddot{q}^\beta + \Gamma^\beta_{\alpha\gamma}\dot{q}^\beta\dot{q}^\gamma) = Q_\alpha.$$

however, since, $\Gamma^\beta_{\alpha\gamma} = 0$, it follows

$$a_{\alpha\beta}\frac{d\dot{q}^\beta}{dt} = Q_\alpha,$$

or

$$\frac{dp_\alpha}{dt} = Q_\alpha,$$

where $p_\alpha = a_{\alpha\beta}\dot{q}^\beta$ impulses, which d denotes an absolute differential.

7.5. Invariant tensor integrals. In accordance with the definition (1) it is found that there exist linear inverse transforms of standard integrals $\int$ and tensor integral $\hat{\int}$ independent of a derivative rank at a particular point of the system. In symbols:

$$\int dy^i = (\frac{\partial y^l}{\partial x^j})_{P_0} \hat{\int} dx^j.$$

7.6. Methods and applications in mechanics. It is well known that tensor integration of differential equation of motion is different from standard integration. This due to the acceptedássertionťhat a non invariant in the linear transform. That is why we are here solving that problem and propose easier and more precise solution of a tensor integral. Based on standard differentiations, tensor calculus should transform coordinate functions from absolute or covariant differentials of those function.

In mechanics a linear transformation is implemented in: coordinates of the position vector,

$$y^i = (\frac{\partial y^i}{\partial x^j})_{P_0} x^j,$$

differentials of coordinate positions at the point,

$$dy^i = (\frac{\partial y^i}{\partial x^j})_{P_0} dx^j,$$

coordinates of the velocity vectors,

$$v_i(y) = (\frac{\partial y_i}{\partial x^j})_{P_0} v^j(x),$$

differentials

$$dv^i(\frac{\partial y^i}{\partial x^j})_{P_0} dx^j,$$

and coordinates of the acceleration vectors

$$\frac{dv^i(y)}{dt} = (\frac{\partial y^i}{\partial x^j})_{P_0} \frac{dv^j(x)}{dt}.$$

where y^i are rectilinear coordinates, x^j are curvilinear coordinates of the position vector at point $x^i_{P_0}$.

However, as generalized coordinates q^α are exactly the solutions of existing constants that are the functions of coordinates y^i or x^i,written using the letters q^α, it means that they are the coordinates of the point P_0.

It can be shown that the same linear transformation that holds for the the differentials of a vector dv^i,holds for tensor integrals,i.e

$$\int dv^i(y) = \frac{\partial y^i}{\partial x^j} \hat{\int} (v^j + A^j),$$

therefore

$$v^i + c^i = \frac{\partial y^i}{\partial x^j}(v^j + A^j),$$

where

$$A^j = \left(\frac{\partial x^j}{\partial y^i}\right)_{y_o} c^i \quad \longleftrightarrow \quad c^k = \left(\frac{\partial y^k}{\partial x^j}\right)_{y_o} A^j.$$

that is

$$v^i(y) + c^i = \frac{\partial y^i}{\partial x^j} v^j(x) + A^j);$$

$$c^k = \frac{\partial y^k}{\partial x^j} A^j - - - - - - A^j(x) = \frac{\partial x^j}{\partial y^k} c^k.$$

7.7. The system of points. A multitude of points M_ν $(\nu = 1, \ldots, N)$, linked by $k < 3N$ finite continuous constraints will be referred to as *the system of points.* Functions f_μ are mutually independent and represent zero rank tensors.

$$f_\mu(y) = f_\mu(x) = 0, \quad \mu = 1,$$

Consequently, their differentials df_μ are equal to absolute differentials $Df_\mu = 0$.

It follows that it is possible to determine y^k or x^k as functions of other $3N - k$ independent coordinates y or x, that are generally referred to generalized independent coordinates q^α $(\alpha = 1, \ldots, n = 3N - k)$,

$$y^k = y^k(q^1, \ldots, q^n), \quad q^\alpha \in M^n$$

where M^n configurational n-dimensional manifold on which

$$dy^i = (\frac{\partial y^i}{\partial q^\alpha})_{q_0} dq^\alpha.$$

This way, the system of $3N$ coordinates y^{3N} and k mutually independent constraints will be $3N - k = n$ independent generalized coordinates q^α of manifold M^n, whose metric tensor is $g_{\alpha\beta}$, $(\alpha, \beta = 1 \ldots, n)$. However, apart from coordinates of the position of points and time t, mechanics contains the essential property of masses m_ν. Mass and velocity are used to determine the concept of momentum the point

$$\mathbf{p}_\nu = m_\nu \mathbf{v}_\nu = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \frac{dq^\alpha}{dt}.$$

In a system of particles, mass m_ν, instead of a spatial metric tensor $g_{\alpha\beta}$ in geometry, there is an inertia or a mass tensor

$$a_{\alpha\beta} = a_{\beta\alpha}(m_1, \ldots, m_N; q^1, \ldots, q^n),$$

that figures in a number of relations in mechanics,.

Scalar multiplication of the impulse $m_\nu \mathbf{v}_\nu$ a coordinate vectors $\mathbf{g}_\nu$ yields a covariant tensor

$$p_\beta = a_{\alpha\beta} \frac{dq^\alpha}{dt} \longleftrightarrow \frac{dq^\beta}{dt} = a^{\alpha\beta} p_\alpha.$$

It is known that the differential of a metric tensor $g_{\alpha\beta}$ equals zero however the differential of an inertia tensor $a_{\alpha\beta,m}$ is equal to zero only if masses are constant quantities.

The inertia force is described by a first-rank tensor

$$I_\alpha = a_{\alpha\beta} \frac{dv^\beta}{dt},$$

where $\frac{dv^\beta}{dt}$ coordinates of the acceleration vector which is not possible to regard identical with standard second derivatives for time t of generalized coordinates $\ddot{q}^\alpha$.

The work of the inertia force is a zero-rank tensor

$$\hat{\int} I_\alpha dq^\alpha = \hat{\int} a_{\alpha\beta} \frac{dv^\beta}{dt} dq^\alpha = \hat{\int} a_{\alpha\beta} v^\alpha dv^\beta =$$

$$\hat{\int} \frac{1}{2} d(a_{\alpha\beta} v^\beta v^\alpha) = \frac{1}{2} a_{\alpha\beta} v^\beta v^\alpha + h,$$

and represents the formula of kinetic energy of the system of particles. The same occurs with other forces.

This proves that integrals also transform linear i.e. by tensor transformations.

For the work of generalized potential forces $Q_\alpha(q^1, \ldots, q^n) = grad_\alpha U$, it follows

$$A = \hat{\int} Q_\alpha dq^\alpha = \hat{\int} grad_\alpha U dq^\alpha = \hat{\int} dU = U + h.$$

Lastly, the Lagrangian second-order differential equations of motion

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^\alpha} - \frac{\partial E_k}{\partial q^\alpha} = Q_\alpha,$$

on the-dimensional manifold are transformed to n covariant equations of the form

$$a_{\alpha\beta}\frac{dv^\beta}{dt} = Q_\alpha,$$

that can be integrated over a tensor, which eliminates difficulties of standard integral calculus nonlinearity and non invariance in analytical mechanics, while establishing inverse transformation of differential d and tensor integral $\hat{\int}$.

MOND8 - THE STABILITY OF MOTION AND REST

8.1. Introductory remarks. "Dynamics is the study of real states of rest and motion of the material systems. Galileo and Newton discovered its principles and demonstrated their plausibility by experiments with heavy falling bodies and interpretation of planetary motion. However, each state of a mechanical system, corresponding to a mathematically strict solution of both rest equations and differential equations of motion, does not correspond to reality." "A general principle for the choice of solution, corresponding to steady states in mechanics, has not been given; there is recognition of the character of science about idealized systems and for each strict application to nature, in principle, every time, the solutions to the stability task have been always sought." "The major problem of the stability of motion in classical theory was solved by Lyapunov". The above mentioned statements given by Nikolai Gurevich Chetaev fully in compliance with the preprinciples of mechanics, as well as the work by V. V. Rumyantsev and A. S. Oziranara [65] represent the best introduction to a brief consideration of the stability of motion of the mechanical systems.

8.2. Differential equations of motion. For the purpose of the generality of mechanical systems, observe $2n+2$ differential equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} + P_\alpha, \tag{8.1}$$

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \alpha = 0, 1, \ldots, n. \tag{8.2}$$

where $H(p_0, p_1, \ldots, p_n; q^0, q^1, \ldots, q^n)$ is the function determined by the formula

$$H = \frac{1}{2} a^{\beta\gamma}(q^0, q^1, \ldots, q^n) p_\beta p_\gamma + E_p(q^0, q^1, \ldots, q^n) \tag{8.3}$$

In the system of equations (8.1) and (8.2), $n+1$ unknown impulses

$$p_\alpha = a_{\alpha\beta}(q^0, q^1, \ldots, q^n)\dot{q}^\beta, \tag{8.4}$$

and n unknown and independent generalized coordinates $q^1(t), \ldots, q^n(t)$ and up to the solution of differential equations (8.1), inconiforce $R_0(q^0)$. Coordinate $q^0(t)$ is specified ahead to the accuracy level of the chosen parameter. The inertia matrix $a_{\alpha\beta}$ is positive definite and its rank is $n+1$. It is easy to prove it by positive-definite kinetic energy E_k. Starting from determinants which show that kinetic energy is

$$Ek = \frac{1}{2} a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta \geqslant 0 \tag{8.5}$$

a homogeneous quadratic form of generalized velocities $\dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n$ or generalized impulses $p_0, p_1, \ldots, p_n$, positive for each $\dot{q}^\alpha$ and equal to zero only for the case of rest, i.e. for $\dot{q}^\alpha = 0$ $(\alpha = 0, 1, \ldots, n)$ or $p_\alpha = 0$. Hence, both matrix $a_{\alpha\beta}$ and its inverse matrix $a^{\alpha\beta}$ are positive definite. Equation

$$\dot{p}_0 = -\frac{\partial H}{\partial q^0} + P_0^* + R_0, \tag{8.6}$$

is the only one of the entire system (8.1) that contains the function R_0, which is possible to avoid by observing only the system of $2n$ differential equations of motion (8.1). Such system of equations is not complete - it does not describe completely the motion of a mechanical system with variable constraints, so it can be referred to as the system of differential equations of motion with respect to one part of the variables. By excluding the auxiliary coordinate q^0, the function (8.5) loses the homogeneity degree 2, which is not in compliance with the preprinciple of invariance. Mechanical systems of material points with constraints independent on time and coordinates q^0, satisfy the same form of differential equations

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + P_i, \tag{8.7}$$

and

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad (i = 1, \ldots, n), \tag{8.8}$$

where the function

$$H = \frac{1}{2} a_{ij}(q^1, \ldots, q^n)\dot{q}^i\dot{q}^j + E_p(q^1, \ldots, q^n) \tag{8.9}$$

contains positive-definite matrix $a_{ij} = a_{ji}$ of the n rank. For the variable-mass systems $m_\nu(t)$, the inertia matrix depends indirectly by masses $m(t)$ on time t as well.

8.3. State of equilibrium and equilibrium position. The concept of the state of equilibrium of a system implies the state of rest of the observed bodies in the observed position $q^\alpha = q_0^\alpha = const$; all generalized velocities equal zero, and considering (8.4) all generalized impulses are $p_a = 0$. Hence, the equilibrium state equations derive from equations (8.7), that is,

$$\left(P_\alpha - \frac{\partial E_p}{\partial q^\alpha}\right)_{p_\alpha = 0} = 0, \tag{8.10}$$

or,

$$Q_\alpha(\dot{q}, q)|_{\dot{q}_0=0} = 0, \tag{8.11}$$

so, solutions of equations (8.10) or (8.11) determine the state of equilibrium of a mechanical system.

Definition 1. The state of equilibrium of a mechanical system implies the set of solutions $q_0^\alpha \in N$ for equations (8.11) and $\dot{q}_\alpha(t) = 0$ or $\dot{p}_\alpha(t) = 0$.

Definition 2. The position of equilibrium of a mechanical system implies the position $q^\alpha = q_0^\alpha$ on a coordinate manifold, whose coordinates satisfy equation (8.11).

Example 26. On a rotational ellipsoid, whose equation in a coordinate system $(y, \mathbf{e})$

$$f(y,t) = c^2(t)(y_1^2 + y_2^2) + a^2(t)y_3^2 - a^2(t)c^2(t) = 0,$$

or relative to generalized coordinates $q^1 = \varphi$, $q^2 = \theta$, $q^0 = a(t)$,

$$y^1 = q^0 \cos\theta \sin\varphi,$$
$$y^2 = q^0 \sin\theta \sin\varphi,$$
$$y^3 = c(q^0) \cos\varphi,$$

there is a point of mass G; axis c of the ellipsoid is vertical, as coordinate y^3.

The equilibrium position of the observed point is determined by $2+1$ equations (8.11), such as

$$y^1 = q^0 \cos\theta \sin\varphi,$$
$$y^2 = q^0 \sin\theta \sin\varphi,$$
$$y^3 = c(q^0) \cos\varphi,$$

It follows that the equilibrium positions at mentioned variable constraint $\varphi = k\pi \quad (k = 0, 1, 2, \ldots, n)$ are at the condition

$$R_0 = \pm G\frac{\partial c}{\partial q^0}, or \quad \frac{\partial c}{\partial q^0} = 0 \to R_0 = 0,$$

and that the axis of ellipsoids along which force G acts does not change. Deviations from solutions $q^\alpha = q_0^\alpha$ i $p_\alpha = 0$, which can be called unperturbed or specified state of equilibrium, describe differential equations of motion (8.7) and (8.8), so they can be considered differential equations of non-equilibrium state, and they can be written in the covariant form

$$\frac{\Delta p_\alpha}{dt} = Q_\alpha, \tag{8.12}$$

$$\dot{q}a^{\alpha\beta}p_\beta, \tag{8.13}$$

implying that perturbations belong to the neighborhood of the point of equilibrium state $q = q_0$, $p = 0$, where right-hand sides of previous equations equal zero

$$Q(0, 0, .., 0) = 0, \tag{8.14}$$

$$a^{\alpha\beta}p_\beta = 0. \tag{8.15}$$

Previous equations (8.8) differ in that from non-equilibrium state equations (8.12)–(8.15). Non-equilibrium position equations $q^\alpha = q_0^\alpha =: b^\alpha = const$, can be considered approximately accurate, sufficient for rough technical practice.

For some other values of $q = b + \Delta$ and $\dot{q} = 0$ forces D_α will not satisfy equations (8.11), except for first-degree accuracy

$$\bar{Q}(q,\dot{q})_{\dot{q}=0} = Q(b+\Delta q, 0) = Qq, 0 + \left(\frac{\partial Q}{\partial q}\right)_{q=b,q=0}\Delta q + \cdots$$

or on account of (8.11)

$$Q_a = \left(\frac{\partial Q_\alpha}{\partial q^\beta}\right)_{q=b}\Delta q^\beta. \tag{8.16}$$

By the analysis of this expression for solutions of $\|\Delta q^\alpha\| \neq 0$, i.e. in terms of the derivative $\left\|\frac{\partial Q_\alpha}{\partial q^\beta}\right\|_b$, it is possible to arrive at certain conclusions on the equilibrium position $q = b$ of the system and its stability.

8.4. Differential equations of perturbed motion. In the professional literature dealing with the motion of the body this term does not always imply the same thing, irrespective of the generality of the title. In the general theory of planetary perturbations these are most generally differential equations of motion1

$$m_\nu \delta\dot{\mathbf{r}} = \mathbf{F}_\nu + \mathbf{G}_\nu \tag{8.17}$$

with perturbation forces added. In describing the motion of a system by equations (8.1) in the absence of forces, equations of perturbed motion are in the form of variation

$$\begin{cases} \frac{d}{dt}\delta p_i = -\frac{\partial^2 H}{\partial q^j \partial q^i}\delta q^j - \frac{\partial^2 H}{\partial p_j \partial q^i}\delta p_j, \\ \frac{d}{dt}\delta q^i = \frac{\partial^2 H}{\partial q^j \partial p_i}\delta q^j + \frac{\partial^2 H}{\partial p_j \partial p_i}\delta p_j. \end{cases} \tag{8.18}$$

In an attempt to derive the equations of perturbed motion, described by covariant equations (8.12), tensor variational equations were derived[14]

$$\frac{D^2\xi^i}{dt^2} + R^i_{jkl}\dot{q}^j\xi^k\dot{q}^l = \nabla_l Q^i \xi^l \tag{8.19}$$

which have not taken a proper place in the stability theory, due to their complex non-linear structure. Differential equations (8.19) are equivalent to differential equations (8.18), where $\xi^i := \delta q^i$, and Q^i, and time t, generalized

[14] Refer to, e.g. M. Milanković, Fundamentals of celestial mechanics, Prosveta, Serbian publishing house, Belgrade, 1947, p.53.

forces dependent on the position q and time $t3$ In the theory of stability of motion the differential equations of perturbed motion are reduced to a general form

$$\frac{d\xi}{dt} = f(t, \xi), \quad \xi \in R^n. \tag{8.20}$$

Equations (8.17) essentially differ from other mentioned equations and the entire theory of planetary perturbations has been elaborated over them. All other mentioned systems of perturbation differential equations are created from basic differential equations of motion by developing into the power series or by varying functions and their derivatives figuring in them. In work [78] it has been proved that variation of the vector projection is not equal to the projection of the variation vector, so instead of equations (8.19), the covariant perturbation differential equations are derived in the form

$$\frac{D\eta_\alpha}{dt} = \psi_\alpha(t, \eta, \xi) \tag{8.21}$$

In order to clarify and evaluate the satisfaction of the preprinciples, let us derive previous equations from the theorem of impulse change (4.1), ie

$$\frac{d}{dt}(m_\nu \mathbf{v}_\nu) = \mathbf{F}_\nu(\mathbf{r}, \mathbf{v}, t). \tag{8.22}$$

Solutions for unperturbed motion are

$$\mathbf{v}_n = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha \quad \text{and} \quad \mathbf{r} = \mathbf{r}(q(t)). \tag{8.23}$$

For any other (perturbed) solution

$$\mathbf{r}^*_\nu = \mathbf{r}_\nu + \xi^\alpha \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}, \tag{8.24}$$

a corresponding impulse is

$$m_\nu \mathbf{v}^*_\nu = m_\nu \frac{d\mathbf{r}^*_\nu}{dt} = m_\nu \left(\mathbf{v}_\nu + \dot{\xi}^\alpha \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} + \xi^\alpha \frac{\partial^2 \mathbf{r}_\nu}{\partial q^\beta \partial q^\alpha}\dot{q}^\beta \right) \tag{8.25}$$

and therefore impulse perturbations, in accordance with (2.25), will be

$$p^*_\gamma - p_\gamma =: \eta_\gamma = \sum_{\nu=1}^{N} m_\nu (\mathbf{v}^*_\nu - \mathbf{v}_\nu) \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\gamma} =$$
$$= \sum_{\nu=1}^{N} m_\nu \left(\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial q^\gamma}\dot{\xi}^\alpha + \xi^\alpha \frac{\partial^2 \mathbf{r}_\nu}{\partial q^\beta \partial q^\alpha}\frac{\partial \mathbf{r}_\nu}{\partial q^\gamma}\dot{q}^\beta \right).$$

However, as there is a link

$$\frac{\partial^2 \mathbf{r}_\nu}{\partial q^\beta \partial q^\alpha} = \Gamma^\delta_{\alpha\beta}\frac{\partial \mathbf{r}_\nu}{\partial q^\delta}, \tag{8.26}$$

considering (8.26), it follows further that

$$\eta_\gamma = a_{\alpha\gamma}\dot{\xi}^\alpha + a_{\gamma\delta}\Gamma^\delta_{\alpha\beta}\xi^\alpha\dot{q}^\beta = a_{\alpha\gamma}(\dot{\xi}^\alpha + \Gamma^\alpha_{\delta\beta}\xi^\delta\dot{q}^\beta) = a_{\alpha\gamma}\frac{D\xi^\alpha}{dt} \tag{8.27}$$

or

$$\frac{D\xi^\alpha}{dt} = a^{\alpha\gamma}\eta_\gamma. \tag{8.28}$$

For solutions of (8.25), differential equations of motion (8.23) are

$$\begin{aligned}\frac{d}{dt}(m_\nu \mathbf{v}^*_\nu) = m_\nu(&\partial_{\delta\alpha}\mathbf{r}_\nu\dot{\xi}^\alpha\dot{q}^\beta + \partial_\alpha r_\nu\ddot{\xi}^\alpha + \\ &+ \partial_{\delta\alpha\beta}\mathbf{r}_\nu\xi^\alpha\dot{q}^\beta\dot{q}^\delta + \partial_{\alpha\beta}\mathbf{r}_\nu\dot{\xi}^\alpha\dot{q}^\beta + \partial_{\alpha\beta}\mathbf{r}_\nu\xi^\alpha\ddot{q}^\beta) = \\ &= F^*_\nu(\mathbf{r}_\nu + \vec{s}\rho_\nu, \mathbf{v}_\nu + \vec{\dot{\rho}}\mathbf{s}_{\nu}, t),\end{aligned}$$

where

$$\partial_\alpha := \frac{\partial\cdot}{\partial q^\alpha}, \quad \partial_{\alpha\beta} = \frac{\partial^2\cdot}{\partial q^\alpha \partial q^\beta}.$$

After scalar multiplication of these equations and equations (8.23) by coordinate vectors $\frac{\partial \mathbf{r}_\nu}{\partial q^\gamma}$, and summing over index ν, it is obtained

$$\begin{aligned}\sum_{\nu=1}^{N} m_\nu(&\partial_\alpha\mathbf{r}_\nu\cdot\partial_\gamma\mathbf{r}_\nu\ddot{\xi}^\alpha + 2\partial_\gamma\mathbf{r}_\nu\cdot\partial_{\alpha\beta}\mathbf{r}_\nu\xi^\alpha\dot{q}^\beta + \\ &+ \partial_\gamma\mathbf{r}_\nu\cdot\partial_{\delta\alpha\beta}\mathbf{r}_\nu\xi^\alpha\dot{q}^\beta\dot{q}^\delta + \partial_\gamma\mathbf{r}_\nu\cdot\partial_{\alpha\beta}\mathbf{r}_\nu\xi^\alpha\ddot{q}^\beta) = \\ &= \sum_{\nu=1}^{N}(\mathbf{F}^*_\nu - \mathbf{F}_\nu)\cdot\frac{\partial\mathbf{r}_\nu}{\partial q^\gamma}.\end{aligned} \tag{8.29}$$

Partial derivatives $\partial_{\delta\alpha\beta}\mathbf{r}_\nu$ figuring in the previous relation can, by means of (8.26), be reduced to

$$\begin{aligned}\partial_{\delta\alpha\beta}\mathbf{r}_\nu &= \partial_\delta(\partial_{\alpha\beta}\mathbf{r}_\nu) = \partial_\delta(\Gamma^l_{\alpha\beta}\partial_l\mathbf{r}_\nu) = \\ &= \partial_l\mathbf{r}_\nu\partial_\delta\Gamma^l_{\alpha\beta} + \Gamma^l_{\alpha\beta}\partial_{\delta l}\mathbf{r}_\nu = \partial_l\mathbf{r}_\nu\partial_\delta\Gamma^l_{\alpha\beta} + \Gamma^l_{\alpha\beta}\Gamma^\mu_{\delta l}\partial_\mu\mathbf{r}_\nu.\end{aligned}$$

where

$$\begin{aligned}\partial_{\delta\alpha\beta}\mathbf{r}_\nu &= \partial_\delta(\partial_{\alpha\beta}\mathbf{r}_\nu) = \partial_\delta(\Gamma^l_{\alpha\beta}\partial_l\mathbf{r}_\nu) = \\ &= \partial_l\mathbf{r}_\nu\partial_\delta\Gamma^l_{\alpha\beta} + \Gamma^l_{\alpha\beta}\partial_{\delta l}\mathbf{r}_\nu = \partial_l\mathbf{r}_\nu\partial_\delta\Gamma^l_{\alpha\beta} + \Gamma^l_{\alpha\beta}\Gamma^\mu_{\delta l}\partial_\mu\mathbf{r}_\nu.\end{aligned} \tag{8.30}$$

where

$$\Psi_\gamma := \sum_{\nu=1}^{N}(\mathbf{F}^*_\nu - \mathbf{F}_\nu)\cdot\frac{\partial\mathbf{r}_\nu}{\partial q^\gamma} = \Psi_\gamma(\xi,\eta,t) \tag{8.31}$$

Equations (8.30) can be further reduced to a shorter form

$$a_{\alpha\gamma}\frac{d}{dt}\left(\dot{\xi}^{\alpha}+\Gamma^{\alpha}_{\sigma\delta}\xi^{\sigma}\dot{q}^{\delta}\right)+a_{\gamma\mu}\Gamma^{\mu}_{\sigma\beta}\left(\dot{\xi}^{\sigma}+\Gamma^{\sigma}_{\alpha\delta}\xi^{\alpha}\dot{q}^{\delta}\right)\dot{q}^{\beta}=\Psi_{\gamma}$$

or, taking into account (8.27)

$$a_{\alpha\gamma}\frac{d}{dt}\left(\frac{D\xi^{\alpha}}{dt}\right)+a_{\gamma\mu}\Gamma^{\mu}_{\sigma\beta}\left(\frac{D\xi^{\sigma}}{dt}\right)\dot{q}^{\beta}=\Psi_{\gamma},$$

that is

$$a_{\alpha\gamma}\frac{d}{dt}\left(\frac{D\xi^{\alpha}}{dt}\right)=\frac{D}{dt}\left(a_{\alpha\gamma}\frac{D\xi^{\alpha}}{dt}\right)=\frac{D\eta_{\gamma}}{dt}=\Psi_{\gamma},$$

which, together with equations (8.28), constitutes $2n+2$ perturbation differential equations (8.21) and (8.22).

Stability of equilibrium state of motion and rest. The title term is not unambiguous, irrespective of previously defined notions of equilibrium state and position.

Determination 1. If at any specified positive real numbers A_{α} and B_{α}, disregarding how small they are not, such positive numbers λ_a and $\bar{\lambda}_a$ and can be chosen for all numeric values of coordinates of the equilibrium state $q_i = q_{i0}$, $p = 0$, subject to restrictions

(8.32) $$\left|q^{i}(t)-q^{i}_{0}\right|<A^{i},\quad |p_i(t)|<B_i$$

and for each time $t > t_0$ satisfy inequalities

(8.33) $$\left|q^{i}(t)-q^{i}_{0}\right|<A^{i},\quad |p_i(t)|<B_i,$$

the equilibrium state $(q^i - q^i_0; p_i = 0)$ of a system is stable with respect to perturbations $q_i \neq q_{i0}$ i $p_i \neq 0$; otherwise it is unstable.

The above determination can be formulated in another way or by relations (8.33). By suitable choice of the origin of the coordinate system at the equilibrium position, the equilibrium state can be represented by zero point on the manifold $T^*\mathcal{N}$, i.e. $q^{\alpha} = 0$, $p_{\alpha} = 0$; (8.14) is then reduced to

(8.34) $$Q_{\alpha}(0,\ldots,0,t)=0.$$

Determination 2. If at any arbitrarily specified number A ¿ 0, disregarding how small it is not, such real number can be chosen, for which all initial positions are restricted by relation

(8.35) $$\delta_{\alpha\beta}q^{\alpha}(t_0)q^{\beta}(t_0)+\delta^{\alpha\beta}p_{\alpha}(t_0)p_{\beta}(t_0)\leqslant\lambda,$$

and for each $t \geqslant t_0$ inequality is satisfied

(8.36) $$\delta_{\alpha\beta}q^{\alpha}q^{\beta}+\delta^{\alpha\beta}p_{\alpha}p_{\beta}<A,$$

unperturbed equilibrium state $p_{\alpha} = 0$, $q^{\alpha} = 0$ is stable; otherwise it is unstable.

The $\delta_{\alpha\beta}$ andi $\delta^{\alpha\beta}$ are the Kronecker symbols. If the equilibrium state stability or unperturbed motion is considered only with respect to a part of variables $2m$ promenljivih $q^1, \ldots, q^m, p_1, \ldots, p_m$, $m < n$, the stability condition (8.36) is reduced to the observed variables

$$\delta_{kl} q^k q^l + \delta^{kl} p_k p_l < A \quad (k, l = 1, \ldots, m) \tag{8.37}$$

Stability criterion. *If for differential equations of motion of a scleronomic system (8.12) and (8.13) can be found positive definite function* $W(t, q^1, \ldots, q^n)$ *such that*

$$\frac{\partial W}{\partial t} + \left(Q_i + \frac{\partial W}{\partial q^i}\right) \dot{q}^i \leqslant 0 \quad (i = 1, \ldots, n) \tag{8.38}$$

the equilibrium state $q = q_0$, $p = 0$ *or* $q = 0$, $\dot{q} = 0$ *is stable.*

Proof. For the assumed existence of function W, the function

$$V = \frac{1}{2} a^{ij}(q^1, \ldots, q^n) p_i p_j + W(q^1, \ldots, q^n, t) \tag{8.39}$$

is positive definite, because kinetic energy is

$$E_k = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j = \frac{1}{2} a^{ij} p_i p_j$$

by definition, positive definite. Time derivative of the function (8.39) is

$$\dot{V} = a^{ij} \frac{Dp_i}{dt} p_j + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial q^i} \frac{dq^i}{dt},$$

because

$$\dot{V} = \frac{dV}{dt} = \frac{DV}{dt} \quad \text{and} \quad \frac{Da^{ij}}{dt} = 0.$$

Taking into account equations (8.12) and (8.13), previous derivative is reduced to

$$\frac{\partial W}{\partial t} + a^{ij} Q_i p_j + \frac{\partial W}{\partial q^i} a^{ij} p_j = \frac{\partial W}{\partial t} + a^{ij}\left(Q_i + \frac{\partial W}{\partial q^i}\right) p_j = \frac{\partial W}{\partial t} + \left(Q_i + \frac{\partial W}{\partial q^i}\right) \dot{q}^i, \tag{8.40}$$

whereby the criterion is proved.

Corollaries

1. If the system is autonomous, the function W should be sought only depending on coordinates, so that the condition (6.38) is reduced to

$$\left(Q_i + \frac{\partial W}{\partial q^i}\right) \dot{q}^i \leqslant 0. \tag{8.41}$$

Such are conservative mechanical systems for which the potential energy $E_p(q^1,\dots,q^n)$ exists. By the choice of just that energy, if it is positive definite, for the function $W, W = E_p$ it is shown

$$a^{ij}\left(-\frac{\partial E_p}{\partial q^i}+\frac{\partial W}{\partial q^i}\right)p_j=\left(-\frac{\partial W}{\partial q^i}+\frac{\partial W}{\partial q^i}\right)\dot{q}^i\equiv 0$$

that the equilibrium state of the system is stable.

2. If generalized forces consist of conservative and any other forces $P_i(q,\dot{q})$, i.e,

$$Q_i=-\frac{\partial E_p}{+}P_i(q,\dot{q}),$$

that is, by choosing again W = Ep, for the stability condition of the system equilibrium state, it is obtained

$$a^{ij}P_ip_j=P_i\dot{q}^i\leqslant 0. \tag{8.42}$$

3. If the system is acted upon by gyroscopic forces

$$P_i=g_{ij}\dot{q}^j=-g_{ji}\dot{q}^j \tag{8.43}$$

the condition of the equilibrium state stability (8.41) is satisfied, because $P_i\dot{q}^i=g_{ij}\dot{q}^j\dot{q}^i\equiv 0$.

4. For dissipative forces $P_i=b_{ij}\dot{q}^j$ condition (8.41) is reduced to the requirement that quadratic function of energy dissipation $R=-b_{ij}\dot{q}^i\dot{q}^j$ should be larger than or equal to zero.

8.5. Generalization of the criterion. Previous theorem also holds for mechanical systems with rheonomic constraints. Condition (8.38) changes only for indices $i,j\doteq 1,\dots,n$, taking the values$\alpha,\beta=0,1,\dots,n$. So, three additional addends are obtained

$$\frac{\partial W}{\partial t}+a^{\alpha\beta}\left(Q_\alpha+\frac{\partial W}{\partial q^\alpha}\right)p_\beta=\frac{\partial W}{\partial t}+a^{ij}\left(Q_i+\frac{\partial W}{\partial q^i}\right)p_j+ \tag{8.44}$$

$$=a^{i0}\left(Q_i+\frac{\partial W}{\partial q^i}\right)p_0+a^{0i}\left(Q_0+\frac{\partial W}{\partial q^0}\right)p_i+a^{00}\left(Q_0+\frac{\partial W}{\partial q^0}\right)p_0\leqslant 0$$

The proof is identical to the previous one, indices in equations (8.12) and (8.13) being retained in the range $0,1,n$. For the case of the system of forces, whose potential energy is $E_p=E_p(q^0,q^1,\dots,q^n)$, a $P_0=P_0^*+R_0$, the function $W=Ep$ can be chosen if E_p is definite-positive function of $q^0,q^1,\dots,q^n$, and therefore the expression (8.43) is reduced to $a^{\alpha\beta}P_\alpha p_\beta=P_\alpha\dot{q}^\alpha=P_i\dot{q}^i+(P_0^*+R_0)\dot{q}^0\leqslant 0$.

Corollaries

1. Expressions (8.38 - 8.43) occur as a consequence of relations (8.44) for the case when constraints are scleronomic, because auxiliary coordinate q^0 disappears.

2. Classical (standard) mode of testing the stability of equilibrium state of a rheonomic system with respect to variables $q^1, \dots, q^n; p_1, \dots, p_n$ can be considered the stability with respect to a part of variables.

6. Necessary additional comment. To prove the criterion (8.38) or (8.44) we commenced from the fact that function (8.39), that is,

$$V = E_k + W(t, q^0, q^1, \dots, q^n) \tag{8.45}$$

is a positive-definite function. Considering the starting assumption that W is a positive-definite function, and E_k is kinetic energy, the issue of the function definiteness V should not be problematic. Yet, a question is raised on the definiteness of kinetic energy. The proof starts from the preprinciple of invariance which states that the attributes of motion do not depend on formal mathematical description and on the expression for kinetic energy of the system

$$2E_k = m_1 v_1^2 + m_2 v_2^2 + \cdots + m_N v_N^2 = \sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu \cdot \mathbf{v}_\nu. \tag{8.46}$$

All masses m_i are positive named real numbers, so it is undeniable that Ek from (8.46) is positive function of $\mathbf{v}_\nu$ and equals zero only if all velocities, i.e. functions $\mathbf{v}_\nu$ equal zero. So, it is true that

$$2E_k = \sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu \cdot \mathbf{v}_\nu \geqslant 0. \tag{8.47}$$

Relative to orthonormal coordinate system $(y, \mathbf{e})$, it follows that

$$2E_k = \sum_{\nu=1}^{N} m_\nu (\dot{y}_{\nu 1}^2 + \dot{y}_{\nu 2}^2 + \dot{y}_{\nu 3}^2) \geqslant 0. \tag{8.48}$$

Nothing will change it we introduce other notation

$$m_{3i} = m_{3i-1} = m_{3i-2}; \quad i = 3\nu - 2,\ 3\nu - 1,\ 3\nu,$$

because (8.48) will be

$$2E_k = \sum_{i=1}^{3N} m_i \dot{y}_i^2 \geqslant 0.$$

In other coordinate systems, say $(z, \mathbf{e})$ or $(x, \mathbf{g})$, between which there are unambiguous point mappings $y^i = y^i(z^1, \dots, z^{3N})$, $y^i = y^i(x^1, \dots, x^{3N})$

or constraints $y^i = y^i(q^0, q^1, \ldots, q^n; n < 3N)$, the quadratic homogeneous form (8.48) will remain unchanged in the forms

$$\sum_{i=1}^{3N} m_i \dot{y}_i^2 = \sum_{i=1}^{3N} m_i \frac{\partial y_i}{\partial z^k}\frac{\partial y_i}{\partial z^l}\dot{z}^k\dot{z}^l = \sum_{i=1}^{3N} m_i \frac{\partial y_i}{\partial x^k}\frac{\partial y_i}{\partial x^l}\dot{x}^k\dot{x}^l =$$
$$= \sum_{i=1}^{3N} m_i \frac{\partial y^i}{\partial q^\alpha}\frac{\partial y^i}{\partial q^\alpha}\dot{q}^\alpha\dot{q}^\beta = \epsilon_{kl}\dot{z}^k\dot{z}^l = g_{kl}(x)\dot{x}^k\dot{x}^l = a_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta \geqslant 0,$$

where ϵ_{kl}, g_{kl}, $a_{\alpha\beta}$ are positive-definite matrices. "Deviation from matrix definiteness" for some values of x or q does not derive from the nature of kinetic energy but from irregularity of transformation matrices (8.20) $\left(\frac{\partial y_i}{\partial x^k}\right)$ or $\left(\frac{\partial y_i}{\partial q^k}\right)$ transition from one coordinate to another. For those values of coordinates x for which the relation $\dot{y}^i = \frac{\partial y^i}{\partial x^\alpha}\dot{x}^\alpha$ is irregular (i.e. nonexistent) the definiteness of matrix a_{ij}, or coordinate forms of kinetic energy cannot be assessed.

Example 27. Kinetic energy of the planar motion of the point, of mass m, can be written relative to the cylindrical coordinate system $\rho, \theta, zeta$, for which there exist the relations

$$y_1 = \rho\cos\theta,\ y_2 = \rho\sin\theta,\ y_3 = z,$$

at the condition $\rho \neq 0$, in the form

$$E_k = \frac{m}{2}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) \geqslant 0 \tag{8.49}$$

or in the plane $z = c = const$,

$$E_k = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right).$$

All three relations for E_k equal zero only if velocities equal zero, because E_k cannot be taken into account, given that $\rho = 0$, is excluded from considerations in the transformation between the observed coordinate systems.

8.6. Invariant criterion of the stability of motion. The phrase 'invariant criterion' stresses a general measure in all coordinate systems for assessing the stability of some unperturbed motion of a mechanical system. As such it encompasses the stability of equilibrium state and position, the stability of stationary motions and the stability of motion of mechanical systems in general, whose perturbation equations are of the coordinate form (8.21) and (8.22).

If for perturbation differential equations (8.21) and (8.22) there exists a positive-definite function W of perturbations $\xi^0, \xi^1, \ldots, \xi^n$ and time t, such that the expression

$$\frac{\partial W}{\partial t} + a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \leqslant 0 \tag{8.50}$$

is smaller than or equal to zero, the state of unperturbed motion of a mechanical system is steady. Proof. Functions Ψ_α, as obvious from (8.31), for unperturbed motion $\xi^\alpha = 0$, $\eta_\alpha = 0$ equal zero $\Psi_\alpha(0,0,t) = 0$. The function

$$V = \frac{1}{2}a^{\alpha\beta}\eta_\alpha\eta_\beta + W(\xi,t) \tag{8.51}$$

is positive definite, because $a^{\alpha\beta}(q^0(t), q^1(t), \dots, q^n(t))$ is positive definite, matrix of the functions in M^{n+1}, a $W(\xi,t)$ for is assumed to be positive-definite function of perturbation M^{n+1}, a $W(\xi,t)$. Being scalar invariant, V is a zero-rank tensor. That is why the ordinary derivative $\frac{dV}{dt}$ is

$$\frac{DV}{dt} = a^{\alpha\beta}\frac{D\eta_\alpha}{dt}\eta_\beta + \frac{\partial W}{\partial \xi^\alpha}\frac{D\xi^\alpha}{dt} + \frac{\partial W}{\partial t} \tag{8.52}$$

which requires to be smaller than or identically equal to zero on perturbations. Substituting natural derivatives from (8.21) and (8.22) into (8.52), it is obtained,

$$\frac{DV}{dt} = a^{\alpha\beta}\frac{D\eta_\alpha}{dt}\eta_\beta + \frac{\partial W}{\partial \xi^\alpha}\frac{D\xi^\alpha}{dt} + \frac{\partial W}{\partial t}$$

is equal to natural derivative (8.52) which requires to be smaller than or identically equal to zero on perturbations.

Substituting natural derivatives from (8.21) and (8.22) into (8.52), it is obtained,

$$\frac{DV}{dt} = \frac{\partial W}{\partial t} + a^{\alpha\beta}\Psi_\alpha\eta_\beta + \frac{\partial W}{\partial \xi^\alpha}a^{\alpha\beta}\eta_\beta,$$

which, with the criterion requirement, is reduced to

$$\frac{\partial W}{\partial t} + a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \leqslant 0. \tag{8.53}$$

Hence, the stability criterion is proved.

If forces $\mathbf{F}^*_\nu$ and $\mathbf{F}$. from (8.31) and if differences $\mathbf{F}^*_\nu - \mathbf{F}_\nu$, do not depend on the position $\mathbf{r}$ and velocity $\mathbf{v}$, nor will the function Ψ_γ depend on t. If the constraints of a mechanical system do not depend on time t. Then and function W disappear $\xi^0, \xi^1, \dots, \xi^n$, i.e. $W = W(\xi^0, \xi^1, \dots, \xi^n)$, so that expression (8.50), i.e. (8.53) is reduced to

$$a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \leqslant 0. \tag{8.54}$$

If the mechanical system's constraints do not depend on time q^0, ξ^0, η_0, ψ_0 vanish, so that expression (8.50) that is to (8.53), is reduced to

$$\frac{\partial W}{\partial t} + a^{ij}\left(\Psi_i + \frac{\partial W}{\partial \xi^i}\right)\eta_j \leqslant 0, \tag{8.55}$$

and (8.54) to

$$a^{ij}\left(\Psi_i + \frac{\partial W}{\partial \xi^i}\right)\eta_j \leqslant 0, \tag{8.56}$$

where functions Ψ_i and W do not depend on ξ^0 and η^0.

All expressions for above mentioned criterion of equilibrium state stability occur as a consequence of the expression (8.53) if ξ and η are considered perturbations of of the equilibrium state. q and p.

8.7. On integrals of perturbation equations. Covariant equations of motion (8.12) or their corresponding perturbation differential equations (8.21) in a developed form and general case have a very complex structure, which makes their integration difficult. However, by applying covariant integration some first covariantly constant integrals are obtained, which can be used to assess the stability of the equilibrium state, as well as the unperturbed motion. We will support this statement by two distinguishing and acceptable examples.

1. Let generalized forces Q_α in equations (8.12) have the function of force $U(q^0, q^1, \ldots, q^n)$. Let us multiply each equation (8.12) by a corresponding differential from equation (8.13) and sum as follows

$$a^{\alpha\beta} p_\beta D p_\alpha = Q_\alpha dq^\alpha = \frac{\partial U}{\partial q^\alpha} dq^\alpha.$$

Since $Da^{\alpha\beta} = 0$, to

$$\frac{1}{2} D\left(a^{\alpha\beta} p_\beta p_\alpha\right) = dU$$

and furthermore

$$\frac{1}{2} a^{\alpha\beta} p_\beta p_\alpha - U = C = const.$$

2. Let right-hand sides of covariant equations (8.21) be linear forms of perturbations of $\xi^1, \ldots, \xi^n$, that is,

$$\Psi_i = -g_{ij}(q^1(t), \ldots, q^n(t))\xi^j,$$

where g_{ij} is a covariantly constant tensor. For thus specified perturbations, equations (8.21) and (8.22) can be written in the covariant form

$$\frac{D\eta_i}{dt} = g_{ij}\xi^j,$$

and in the counter-variant form

$$D\xi^i = a^{ij}\eta_j dt.$$

By complete mutual multiplication and summation over index I, as in the previous example for α, the scalar invariant is obtained

$$a^{ij}\eta_j D\eta_i = -g_{ij}\xi^j D\xi^i.$$

Covariant integration yields

$$\frac{1}{2}a^{ij}\eta_j\eta_i - g_{ij}\xi^j\xi^i = A$$

here A constant, $DA = dA = 0$.

Stability of the whole or piecewise variables. For the stability of motion or equilibrium of mechanical systems, it is more important to note if perturbations in perturbation equations are a consequence of calculation error or they are caused by new changes of forces; the inertia force due to the change of tensor a_{ij}, active forces due to approximate accuracy of dynamic parameters and non-ideally accurate laws of dynamics, produced by formulas of some forces. The laws of dynamics, as viewed herein, are formulated based on the stability processes in terms of mentioned definitions of stability. This means exactly up to the boundary value of the chosen number, irrespective of how small it is not. In differential equations (8.12), especially (8.21), any deviation of the functions or their parameters from real values, no matter how small they are, can but need not affect the stability or non-stability of the observed motion. That is why the stability of mechanical systems with respect to the forces is of crucial importance.

MOND9 - AFTER WORD

Modifications

ISAAC NEWTON (Isaaco Newtono) is the founder of the modern science of rationall mechanics. The introduction of the book emphasizes the acronym MOND, standing for 'modified Newton's dynamics', but the title word MOND actually denotes 'modification of NEWTONIAN dynamics'. The difference is in that the entire classical rational mechanics is Newtonian. Almost all modifiers have used Newton's basic axioms to check their various mathematical relations. Newton wrote axioms or laws, like axioms of geometry, but not the later called 'law of gravitation'. He wrote his axioms using the term THEOREM. Refer to, for example, Book *I*, Theorem *IV*, Corollary 6, or Book *III*, Theorems *VII* and *VIII*. Here, it is moreover shown that theorems of gravitation do not derive genuinely from axioms, but only with addition of Kepler's third law of the major planetary motion, which is only approximately accurate. In short, all approaches to the state of motion and rest are presented herein so as to be readable and understandable for secondary-school, post-secondary-school and university students. Quotations of Newton's rules at the very beginning of the book are to demonstrate in a straightforward manner that we do not change Newton's essential starting points of his natural philosophy.

Newton's Rules of causal judgment in physics.

Rule I.

No more causes of natural things should be admitted than are both true and sufficient to explain their phenomena.

Rule II

. Therefore, the causes assigned to natural effects of the same kind must be, so far as possible, the same.

Rule III.

Those qualities of bodies that cannot be intended and remitted and that belong to all bodies on which experiments can be made should be taken as qualities of all bodies universally.

Rule IV.

In experimental philosophy, propositions gathered from phenomena by induction should be considered either exactly or very nearly true notwithstanding any contrary hypotheses, until yet other phenomena make such propositions either more exact or liable to exceptions.

These rules are not modified here by MOND theory. We modify (Lat. modus - measure, manner), i.e. change, alter, transform, denote more precisely, mathematically adequate measure of the science of nature. We have not started from a priori statement, but from inherited, existing knowledge, acquired

Rational (Lat. rationalis) - phil. reasonable, endowed with reason, reason-based, scientific; math. which can be completely calculated, calculable, expressed without root sign; which can be calculated by whole quantities; (opposite: phil. empirical; math. irrational).

by taking over and attaining the existing logical and mathematical standards, whereby the accuracy of mathematically most precise science of nature has been relativized. Instead of Newton's introductory SCHOLIUM: about time, space, position and motion; absolute and real, three PREPRINCIPLES are herein adopted: 1. Of Existence, 2. Of Invariance, and 3. Of Determinacy. Instead of Newton's 8 basic definitions: I. Amount of matter (mass) II. Quantity of motion (momentum) III. Innate forces (Inertia) IV. Motive (acting) forces V. Centripetal forces VI. Absolute quantities of centripetal force VII. Acceleration quantities of centripetal force VIII. Centripetal forces

We have here formulated only five definitions: 1. Velocities at the point 2. Impulses of motion at the point 3. Accelerations at the point 4. Inertia forces at the point 5. The action of the force.

The first corollary and fig. 1 of Newton's axioms or laws of motion ([1], pp. 41/42), as well as Newton's Theorem 1, are encompassed by our figures 4 and 5, and it reads: Corollary 1. A body acted on by forces acting jointly describes the diagonal of a parallelogram in the same time in which it would describe the sides if the forces were acting separately.

Proof: The intersections of parallelogram sides are denoted by letters A, B, D and C. If only a single force M would act upon a body at point A, in a specified added time the body would move from point A to point B, and if at the same point only another force N would act, and the body would move from point A to point C, at the action of both forces simultaneously, the body would move along the diagonal of the parallelogram $ABCD$ from point A to point D. MOND theory differs from standard rational mechanics, which the following mathematical relations make clear.

Standards	Author's modifications
Laws of motion	Principle of the action of forces
$I.\ \vec{F} = 0, \rightarrow \vec{c};$	
$II.\ m\frac{d\vec{v}}{dt} = \vec{F},$	
$III.\ \mathcal{A} = R.$	$\mathcal{A}(\mathbf{F}) = \mathcal{A}(\vec{I}).$

Time	
Natural parameter	independent variable; attr.T
Space	
Three-dimensional	Four-dimensional; attr L
Distance	
ρl	$\rho(s)$
$ds^2 = \delta_{ij} dy^i dy^j$ $(i, j = 1, 2, 3)$,	$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ $(\mu, \nu = 1, 2, 3, 4)$
Velocity at the point (momentary)	
$\vec{v} = \frac{\partial \vec{r}}{\partial q}\dot{q} + \frac{\partial \vec{r}}{\partial t}$	$\vec{v} = \frac{\partial \vec{r}}{\partial q}\dot{q} + \frac{\partial \vec{r}}{\partial q^0}\dot{q}^0$
$\dot{q} = (\dot{q}^1, \ldots, \dot{q}^n)$	$\dot{q} = (\dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n)$.
Impulses of motion	
$p_i = a_{ij}\dot{q}^j + b_i$,	$p_i = a_{ij}\dot{q}^j + a_{i0}\dot{q}^0$,
$p_0 =: -H$	$p_0 = a_{0j}\dot{q}^j + a_{00}\dot{q}^0$
Accelerations in space	
$w^i = \ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k$	$w^i = \frac{D\dot{q}^i}{dt}, \quad w^0 = \frac{D\dot{q}^0}{dt}$
Generalized forces	
Q	Q, Q_0
Work	
$A(Q) = \int_s Q_i dq^i$	$A = \int_s (Q_i dq^i + Q_0 dq^0)$
Variation principles	
$\delta \int_{t_0}^{t_1} L dt = 0$	$\delta \int_{t_0}^{t_1} (A(\vec{F}) - A(\vec{I}))dt = 0$
Kinetic energy	
$E_k = \frac{1}{2} a_{ij}\dot{q}^i\dot{q}^j + b_i\dot{q}^i + c(q)$,	$E_k = \frac{1}{2} a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta$
Differential equations of motion	
$a_{ij}\ddot{q}^j + \Gamma_{jk,i}\dot{q}^j\dot{q}^k = Q_i$,	$a_{ij}\frac{D\dot{q}^j}{dt} + a_{i0}\frac{D\dot{q}^0}{dt} = Q_i$,
Lagrange's equations	

$$\frac{D}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)=0 \qquad \frac{D}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)=0, \quad \frac{D}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^0}\right)=0$$

Hamilton's equations

$$\frac{D}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)=0 \qquad \frac{D}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)=0, \quad \frac{D}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^0}\right)=0$$

Equations of perturbed motion

$$\frac{d}{dt}\delta p_i = -\frac{\partial^2 H}{\partial q^j \partial q^i}\delta q^j - \frac{\partial^2 H}{\partial p_j \partial q^i}\delta p_j, \qquad \frac{D\eta_\alpha}{dt} = \psi_\alpha(t,\eta,\xi)$$

$$\frac{d}{dt}\delta q^i = -\frac{\partial^2 H}{\partial p^j \partial q^i}\delta q^j - \frac{\partial^2 H}{\partial p_j \partial q^j}\delta q^j + \frac{\partial^2 H}{\partial p_j \partial p_i}\delta p_j, \qquad \frac{D\xi^\beta}{dt} = a^{\alpha\beta}\eta_\alpha$$

Space integration

$$\int(dv^i + \Gamma^i_{jk}v^j dx^k) = ? \qquad \bar{\int} Dv^i = v^i + c^i$$

in point

$$\int g_{il}dv^i = g_{il}v + c_l \qquad \bar{\int} g_{il}dv^i = g_{il}v^i + c_l$$

Invariant generalization stability criterion

Lapunov theory. For differential equations of motion of a scleronomic system (8.12) and (8.13) can be found positive definite function $W(t, q^1, \ldots, q^n)$ such that

$$\frac{\partial W}{\partial t} + \left(Q_i + \frac{\partial W}{\partial q^i}\right)\dot{q}^i \leqslant 0 \quad (i = 1, \ldots, n)$$

the equilibrium state $q = q_0$, $p = 0$ or $q = 0$, $\dot{q} = 0$ is stable.

Newton-Kepler's law of gravitation

$$F = \kappa\frac{m_1 m_2}{\rho^2}, \qquad F = \chi\frac{m_1 m_2}{\rho},$$

$$\kappa = 6,67 \times 10^{-8} cm^3 g^{-1} s^{-2}, \qquad \chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v^2_{or}}{m_1 + m_2}$$

Paradox of Lunar motion

$$F_\odot \approx 2,5F_\oplus \qquad F_\oplus \approx 3,5F_\odot$$

Radius of the Earth's gravy sphere

260000km, or 720000km 1450000km

Planetary space metric

3-metric 4-metric

Considering such and comparisons, the author of this book was asked significant and logical questions at scientific meetings: "Do you find to date assertions of standard mechanics wrong?", "Assuming your assertions are right, how do you explain the fact that it hasn't been noted in practice?" Avoiding the word "wrong" the answer was that the assertions are more general and complete. From Aristotle to Galileo and Newton, respectively, a viewpoint was accepted that a body has uniform motion under the action of a constant force. After Newton wrote his first axiom or law of motion that a body will remain in uniform motion in a straight line in the absence of forces, philosophy was assessing and finally assessed Aristotle's view wrong. Such a rough evaluation was neither proposed by Newton nor by Einstein, who did not call incorrect the postulate of rectilinear motion, but used a more complete and nicer expression saying that "rectilinear motion derives neither logically nor experimentally from experience".

' Example of a beam bridge (the simplest structural form for a bridge) provides a simple answer to another question - even though such object of mechanics is considered, excluding the calculations of axial forces, practice always allows for possibility of displacing one end. So, this fact has been noted in practice. The approach presented herein extends mathematical knowledge to be applicable in the theory of body motion, entailing some other different views of some attributes of motion. The book points out novelties in more detail than it describes familiar and accepted relations. Thus, for example, a detailed distinction between 'material point' and 'particle' is provided, as well as between 'covariant integration' and 'standard integration' of differential equations of rotational motion of a rigid body. It is emphasized that a model of material point can be used to develop the theory applicable to all mechanical objects.

Preprinciples preceding the entire body of the book, the starting point of mechanics is unambiguously determined; its fundamental concepts of existence are mass, distance and time, and thereby its realm of investigations by means of three sets of real numbers and changes of three orientation vectors; the concept of geometric spaces has been abandoned but not the concept of the volume of a body; the possibility of identity of two particles has been ruled out, which geometrically distinguishes the concept of particle from both material and geometric point, thus making redundant 'the law of impenetrability'. The possibility of determination of motion is predicted a priori, however, the accuracy to achieve it by available knowledge of relevant natural parameters at some particular time moment of motion is relativized. Relativized is the knowledge of the state of motion and rest in mechanics, which is described by mathematical relations in various coordinate systems, a prerequisite for invariance that natural properties of motion do not depend on formal manner of describing. Consequently, the preprinciples objectify the subject of theoretical mechanics but relativize its general knowledge; they are the accompanying corrector and verifier of all assertions of the theory of body motion. The basics of mechanics include definitions of five concepts only, which can be employed to upgrade the entire theory of the body motion. In accordance with the preprinciples, initially it was necessary to open up the problem of the choice of base orientation vectors, unchangeable in time. Unlike the definition of velocity by means of boundary values for distance, the definition of acceleration did not include a vector to a vector boundary transfer, so that a standard definition of acceleration as natural source of velocity per time has been accepted. In describing the impulse of motion, emphasis is placed on the importance of inertia tensor and its distinction from geometric metric tensor. This definition, as the others, remains in the entire theory presented later, excluding from current considerations the impulse of motion as a negative energy (Hamiltonian), i.e. the work of the forces. The term 'impulse of motion' is used instead of 'impulse' to make it distinct from 'the impulse of force'. The definition of the inertia force is used to determine the dimension of the force in general, which later comes to the fore when introducing and dimensioning various dynamics parameters, as well as in formulating the laws of dynamics.

The concept of 'law' has been attributed an unambiguous meaning of the determinant of the force; this is more markedly incompatible than Newton's laws, but it is used to make strict distinction between the concept of the law in mechanics and the concepts of the principle and theorem. Dominant place in this chapter is given to the law of constraints, which points out that the relation between material points or particles can be abstracted by the force, i.e. the constraints are the sources of the forces, and mathematical or thought relation termed constraint has to be distinguished from the concept of the mechanically objective existing constraint. A more prominent place is given to Kepler's laws compared to Newton's gravity force. Laws of gravitation represent inherited knowledge about mutual motions of bodies, but the formula for the force by which two bodies attract one another has been altered. Consistent interpretation of the concept of law remains in the After word. For 'the force of mutual attraction' a formula is derived to be used for certain assumptions, to develop a modified formula of Newton's gravity force. By omitting the determinants of other forces, i.e. laws of dynamics, for brevity, a newly introduced concept of the law of gravitation is not challenged. The section The principles of mechanics contains four principles that can each become a basis for developing the entire theory of body motion.

The principle of equilibrium is assigned, not without reason, the largest part of the text despite being based on the smallest number of definitions and consequent determinants. But it is sufficient to embrace all motions of bodies coupled by any constraints in any coordinate systems. And consequently the action of the moment of a couple in a system of material or dynamic points subjected to constraints has been shown. From the principle it has reversely originated the necessity for modification of the formula of gravity force or calling into question the validity of differential equations of motion with multipliers of constraints. By introducing additional definition of the concept of work, the principle of work has been formulated. Unlike the vector invariant of the principle of equilibrium, the principle of work is expressed by scalar invariant, whereby difficulties in summation of the coupled vectors are avoided. Consequently there occurs, apart from potential energy, 'the rheonomic potential' as the negative work of the force of constraint change; also, it is shown that kinetic energy is the negative work of the inertia force. In his own manner the author characterizes the elementary works on real displacement, possible displacement, as well as the work on possible variations. By introducing the additional coordinate - rheonomic coordinate - the principle of rheonomic constraints solidification is abandoned, so the relation for the principle of work is expanded by a single adequate addendum. This has been preceded by a modification of the constraints' variations as well as of the work of a mechanical system with rheonomic constraints. The concept of work was used to define the concept of action, which is the object of a general integral variation principle referred to as the principle of action. For such formulations of the principle, at the unique concept of variation, the well-known classical integral variation principles occur as a consequence. Since the preprinciple of existence takes time as an independent variable, being as such it does not vary, this integral principle is shown to be invariant on expanded configuration manifolds TMn+1 and T*Mn+1 the same as for schleronomic systems on TMn and T*Mn; in other words, it is shown in the form by the same relations for autonomous and non-autonomous systems. A more substantial meaning of this principle is expressed in Chapter IV in the proof for the theorem of optimal control of motion. On the grounds of the first four definitions and the definition of constraint, the differential variational principle of constraint is expressed, which essentially scalarizes the vector invariant of the equilibrium principle. By describing the constraint as a homogeneous quadratic form of the acceleration vector coordinate over the inertia tensor, the possibility of its transformation into any coordinate system has been proved. From the requirement that the constraint has the lowest value on real motion, it is easy to arrive at ordinary scalar differential equations of motion, expressed by the functions of the constraint. Here, it is explicitly explained what the concept of theorem implies in mechanics. Natural derivative with respect to time is employed to prove the theorem of change of impulse and the theorem of change in kinetic energy; they both have, in accordance with the preprinciples, the invariant sense and differ from accepted approaches of analytical mechanics. Determination of motion by the solutions of differential equations of motion is devoted mainly to undeveloped covariant integration, first integrals and covariant integrals; Poisson brackets are extended for rheonomic systems. A short and clear enough attention was focused on the modification of cocyclic energy integrals. The ending of the chapter is a section on The stability of motion and rest that assesses the accuracy and validity of the solutions of differential equations of motion, depending on the observed dynamic and kinematic parameters. In addition to quoting, not without reason, the thoughts of a great and esteemed professor Nikolai Gurevich Chetaev about false modernization, not less topical today than at the time they were written, the author presents general and covariant differential equations of perturbations and the criterion of the stability of equilibrium state of motion and rest of mechanical systems. The book is rightly called a monograph because it constitutes a theoretical entirety based on the author's results published in scientific journals and monographs, referred in bibliography. In short, this theory encompasses all mechanical systems, rigid and deformable bodies being implied. The author's conception of the application of rheonomic coordinate to deformable media is not missing herein. It proved that deformable bodies can be represented as a system of material points with rheonomic constraints and that deformable medium can be modelled as (3+1)-dimensional geometric, kinematic or dynamic with the metrics.

The far-reaching significance of this metrics for learning about the processes in nature the author perceives from the pages of this and other books in mechanics. Metrics inspires the introduction of other adequate examples. More than that, at the end of this book metrics is used to challenge any view that mechanics, as a science of body motion, was finished long ago; on the contrary, new pieces of information about the body motion and interaction between bodies are presented. Grateful recognition. All scientific contributions have been previously presented at scientific seminars of the Mathematical Institute of Serbian Academy of Sciences and Arts, starting from the first one on March 11, 1959: V. Vujičić, Identification of trajectories of variable-mass point with auto-parallels, under the chairmanship of academic M. Tomić, and to the last lecture on March 22, 2014 at the seminar of probability logic titled Numbers and vectors, under the chairmanship of Prof. Dr. M. Rašković having this fact in mind, the author points out that Mathematical Institute of Serbian Academy of Sciences and Arts has provided multiple contribution to science, as evidenced by bibliography.

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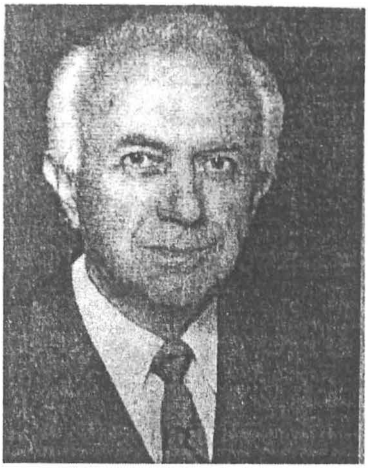
CONTENTS

ABOUT THE AUTHOR



Veljko A. Vujičić was born in 1929, in Nikšić, where he finished elementary and grammar school. In the Yugoslav People's Army, he graduated from the Air Defense Artillery School (Second Class YPA ADAS, Serb: PAVU JNA). Afterwards, he taught the Anti-aircraft Shooting Class in one of the YPA Units for Air Defense NCOs. He graduated from the Faculty of Mathematics and Natural Sciences (FMNS), at the University of Belgrade, in 1957 and got his MSc degree in 1958, after which he interned at the Faculty of Mechanics and Mathematics of the Lomonosov Moscow State University from 1959 to 1960. Having past his doctoral program exams, he defended his PhD thesis in 1961.

He was elected assistant professor at FMNS in Belgrade in 1959. He taught Mechanics of Variable Mass Systems (Rocket Dynamics – Special Course), Statics, the Theory of Oscillation, Analytical Mechanics and Tensor Calculus).

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He was the dean of the Faculty of Mathematics and Natural Sciences at the University of Belgrade, as well as the vice-president and acting director of the Mathematical Institute of the Serbian Academy of Sciences and Arts. He is a member of various national and international scientific societies. He is a member of the International Academy of Nonlinear Sciences based in Moscow, a member of American (South and North) Academy of Mechanics, a member of the European Academy of Arts, Sciences and Humanities. He has been awarded the Order of Labour with Golden Wreath as well as the Order of Labour with Red Banner, and honoured with the Plaque of the City of Belgrade. He has also received the honorary charter of the International Academy of Nonlinear Sciences based in Moscow. The American Biographical Institute places him into top 500 leaders in the 20th century science.