QUALITATIVE BEHAVIOR AND EXACT TRAVELLING NONLINEAR WAVE SOLUTIONS OF THE KDV EQUATION

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I would like to dedicate this thesis to my loving father, mother, brothers, sisters, wife and my kids. Today, I dedicate them this important professional achievement because without their presence, support, and comprehension I would not have achieved my goal.

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First of all I would like to express my praises and sincere thanks to almighty Allah, start with no end and end will never start, for his divine assistance and guidance, which gave me the ability to succeed. I Thank you Allah, for life, health, and the energy that you have given me to reach my professional goals.

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Abstract

Most phenomena in the scientific field and other domains can be described and classified as nonlinear diffusion equation which normally results from natural phenomena that appear in our daily lives such as water waves at the beach caused by wind or tides, the movement of a ship, or by raindrops; the same applies to other physical and mathematical phenomena. In this study, we tried to find a solution to this kind of equations.

IN CHAPTER ONE, we gave brief history of the beginning of the study of waves and we talked about some famous scientists who were interested in this field.

IN CHAPTER TWO, we highlighted the diversity and classification of equations in terms of: Linear, Non-linear, Dispersive and Non-dispersive.

IN CHAPTER THREE, we introduced the Painlevé method and we applied it into the KdV and modified KdV equations, and in addition to that, we were able to find analytic solutions for these equations.

IN CHAPTERS FOUR AND FIVE, we showed several methods of scheme difference, we focused our study on the non-linear term of the KdV equation.

IN CHAPTER SIX, we gave some examples of the scheme difference methods and we applied them by Matlab programs. Moreover, our work is supported by pictures and figures.

CHAPTER SEVEN shows the future works, we enhanced the work by Appendix.

Apstrakt

Većina fenomena u mnogim naučnim i drugim oblastima se mogu opisati i klasifikovati kao linearne i nelinearne diferencijalne jednačine, koje su obično rezultat prirodnih fenomena koji se pojavljuju u našim svakodnevnim životima, kao na primer, talasi vode na plaži, prouzrokovani vetrom, plimom, pokretima broda ili kapljicama kiše. Isto se može primeniti na druge fizičke i matematičke fenomene. U ovom radu pokušaćemo da pronađemo rešenje za ovu vrstu jednačina.

U prvom poglavlju data je kratka istorija istraživanja talasa, gde su pomenuti neki od poznatih naučnika koji su se bavili ovim pitanjem.
U drugom poglavlju ukazano je na raznolikost i klasifikaciju jednačina: linearne, nelinearne, disperzivne i nedisperzivne.
U trećem poglavlju uveden je Painlevé metod koji je primenjen na KdV i Modifikovanoj KdV jednačini. Pored toga, pronađena su analitička rešenja za ove jednačine.
U četvrtom i petom poglavlju prikazano je nekoliko metoda šeme razlika, fokusirajući se na proučavanje nelinearnih KdV jednačina.
U šestom poglavlju dati su pojedini primeri metoda šema razlika koje su primenjene uz pomoć Matlab programa.
Sedmo poglavlje se bavi budućim radovima, dok Dodatak sadrži dodatne informacije.

Contents

List of Figures

List of Tables

Chapter 1

Historical Perspective

1.1 Introduction

The concept of nonlinearity was fundamentally changed at end of 20^{th} century. The reason that caused this kind of transformation was the detection of solitons and the discovery of strange attractors Hereman [16]. We can use chaos theory and strange attractors to perceive and interpret mercurial natural occurrences better. However, the soliton theory is very useful in the interpretation of those natural occurrences that are unvarying and surprisingly easy to anticipate, even when the conditions are likely to cause their alteration. A solitary wave which maintains its form and speed after nonlinear interaction with other solitary waves or arbitrary commotions is called a soliton Ali [2]. The analysis of soliton impelled the complete integrability theory and the creation of solutions to numerous nonlinear differential equations. Partial differential equations are characterized by exceptional features. These include illimitable number of generalized symmetries, incalculable number of conservation laws, and if the variables are changed, Painlevé property, Darboux and Bäcklund transformations Hereman [16]. PDEs can also illustrate natural occurrences interesting to physicists such as population dynamics and molecular dynamics, system of reaction-diffusion, chemical reactions nonlinear system and material science as well. When it comes to material, PDEs are widely applied, especially in elastic materials and solid mechanics. If the complete integrability of PDEs is analyzed, the character of their solutions can be comprehended. Various techniques could be used for purpose of solving integrable nonlinear PDEs Ali [2].

1.2 Subject Definition

When people think about water waves, waves at the seashore caused by tides or wind or by a movement of a boat, or those created by throwing a stone in a lake or by raindrops, are the ones that usually come across their minds. All of them fall into different categories Mostafa [23]. This thesis addresses the equation of shallow water waves.

Fig. 1.1 Periodic shallow water waves, Lima coast

The depth of water here is less than the waves of the commotion of the free surface. In addition to that, this thesis will mention gravity waves in which resilience operates as the restoring force. The thesis will devote only a little of its space to capillary effects and waves which take the surface tension as their primary restoring force. Shallow water waves used to attract attention of British and French mathematicians back in 18^{th} and early 19^{th} century. But one of the most important figures in this field is Stokes (1847). He is considered to be one of the innovators of hydrodynamics. It was he who derived the equation for calculation of the motion of inelastic fluid without viscosity, which is exposed to constant vertical force, where the fluid is placed between an impervious bottom and free surface above. Making these basal equations starting point to further assumptions can lead to the derivation of numerous shallow wave models. They have wide application in atmospheric

and oceanography science.
Shallow water wave equations are divided into two main types. Shallow water wave models with wave diffusion are usually completely integrable cnoidal wave solutions.
The second types are classical shallow water wave models without diffusion. This thesis will include a few experiments and observations.

1.3 Historical Aspect

The pioneer in solitary wave in shallow water observation, John Scott Russell was an engineer and naval architect from Scotland. He dealt with observation of motion of boats and experimented with them, since he worked for the Union Canal Company. His main goal was to create a more efficient canal boat [16].
As Russell himself wrote once, he was conducting an experiment where a boat was pulled by two horses along a narrow channel. But something unusual happened when the boat stopped all of a sudden; the mass of water moving along with the boat did not do so. It was drawn around the prow of the boat rocking around it. Than it suddenly went on in great speed, forming a large solitary elevation, a visible, round assessment of water. This heap of water went on in the same direction along the channel without reducing its velocity or form. Russell was observing the wave as he was riding a horse and he noticed that it was still moving with speed of thirteen and a half kilometers per hour, it kept its original form of about nine and a quarter meters long, 0.4 meters high, 0.3 meters wide. Its height reduced, bit by bit, and after about one and half to three kilometers it disappeared in the channel. It was in (August 1834), that Russell had the opportunity to find this amazing phenomenon, which he called the "Wave of Translation" Hereman [16].

Being in so impressed by this phenomenon, Russell decided to simulate it by making a water tank he would use to observe it and its characteristics. Everything was described in detail in "John Scott Russell" (1808-1882), biography written by Emmerson (1977), as well as in many articles written by other scientists, such as Bullough (1988), Darrigol (2003) and Craik (2004), who admired Russell and his contribution.

Two researchers who particularly admired Russell were Diederik Korteweg, a professor from Holland, and his student Gustav de Vries (1895). These two managed to derive PDE which illustrates the solitary waves that Russell had noticed. The equation ,which was named after Korteweg and de Vries, was actually already seen in seminal work written by

Boussinesq (1872,1877) and Rayleigh. Until (1965) solitary waves were treated as insignificant in the area of nonlinear wave research. It was not until then, when Kruskal and Zabusky became conscious that KdV equation appears as the continuum ultimate of one dimensional enharmonic grid. This equation was used to study “thermalization”. Ferma, Ulam and Pasta (1955) wanted to know how energy was diffused between the various fluctuations in the grid. These two scientists imitated a solitary wave collision in a nonlinear crystal grid. They noticed that the waves maintain original form and speed. After colliding they only go through phase change, the faster ones make progress, while the slower ones slacken. Taking these particles in consideration they thought of "solitons" a word that illustrates these elastic colliding waves. This was described in detail in a narrative writtren by Zabusky (2005) Ali [2].

Since the 1970s, many scientists devoted a lot of their attention to the research about

Fig. 1.2 Experience, solitary waves, University of Heriot-Watt, Scotland UK

the KdV equation and other solitary wave equations. They are fascinated by their physical characteristics. What they find particularly intriguing are completely integrable models that require the Inverse Scattering Transform (IST). [1] IST method is thoroughly described

[1]The Inverse Scattering Transform (IST) is a method of solution which can be applied to a number of

by Ablowitz et al (1974), Segur and Ablowitz (1981), and Clarkson and Ablowitz (1991). Completely integrable models belong to Hamiltonian system of infinite dimension. In addition to that, a group of researchers who got together at Herriot-Watt University managed to duplicate a solitary wave. However their wave was smaller than the one which Russell had spotted earlier, see Figure (1.2).

1.4 Application of the Korteweg-de Vries equation

The phenomena of nonlinear waves are the subjected to intense study at present times, in various fields of applied mathematics, as well as engineering and physics, for example in radio physics, acoustics, optics, hydrodynamics, plasma physics…etc The phenomena of nonlinear waves are the subjected to intense study at present times, in various fields of applied mathematics, as well as engineering and physics, for example in radio physics, acoustics, optics, hydrodynamics, plasma physics…etc .

Korteweg-de Vries equation or KdV equation is the principal nonlinear wave equation. It was derived by two scientists, Korteweg and de Vries, for the purpose of describing how one dimensional waves behave in shallow water, when their amplitude is small but finite. Lately, the KdV equation has been employed for describing different sorts of phenomena, such as bubble liquid mixture waves, warm plasma waves, acoustic wave behavior in enharmonic crystals, ion-acoustic waves and magnetohydrodynamics.

This section deals with the rise of the Korteweg-de Vries equation as a true model which governs the development of waves regarding media where weak nonlinear effects are studied. Four examples will be quoted: the first appears in plasma physics in which the Korteweg-de Vries equation directs the long compressive wave evolution in a plasma of hot electrons and cold ions. The second example is the shallow water wave issue, while the third one arises in meteorology where the nonlinear Rossby wave propagation through rotating homogenous fluid is analyzed. The last case slightly differs from the previous two, since the second space dimension is found in the initial equations, while the final KdV equation coefficients are integrals over y. One more example was derived from the electric circuit principle which uses a nonlinear capacitance. Here, we obtain the generalized KdV equation of the pth order, with capacitance depending non-linearity. We will use this example to show how, under particular circumstances, a modified Korteweg-de Vries equation can appear. The simplicity of the KdV equation structure is well known, since it is the equation of single scalar value with two independent and one dependent variable. Nevertheless, the

nonlinear partial differential equations which have soliton solutions.

initial equations of most physical systems motion are complex, and they usually involve quite a few dependent variables. This is the reason we need to employ a procedure which would reduce equation sets of this kind to less complicated forms, that is, perturbation procedure. In order to apply this technique, all variables are scaled to dimensionless structure, while the dependent variables are expanded with regard to a parameter of perturbation ε. The next section will illustrate this method through the fact that the KdV equation governs the ion-acoustic waves [2].

1.5 Absolutely Integrable Shallow Water Wave Equations

Absolutely integrable partial differential equations concerning water waves have appeared in many different levels of approximation in the theory of shallow water waves. The essential part in the derivation of these equations belongs to the four length scales, see Figure (1.3). where λ represents the length between two consecutive peaks of the water wave.

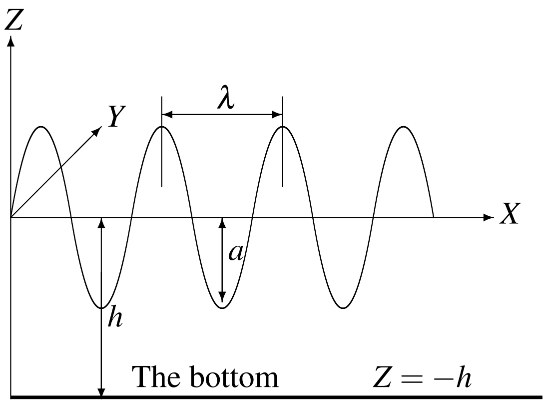


Fig. 1.3 Wave on the surface water

The amplitude a is there to measure how high the wave is, which is actually the alternating distance between untouched water to the peak of the water wave. h represents how deep the water is from the very bottom to the motionless water surface. The $Y-$axis represents how long the water wave is. It goes along the highest point of the wave and its at right angles to the $(X,Z)-$plane. If we presume that waves are created in invariable depth of the water (h is a constant), and if we overlook disintegration, the equations under study in this chapter include some generic characteristics and limitations which allows them to be mathematically manageable (Segur 2007). They are used to illustrate: (i) shallow water waves ($i.e$ $h \ll \lambda$), (ii) small amplitude waves ($i.e$ $a \ll h$), one-dimensional water waves (traveling along $X-$axis), or, (iii) wavering two-dimensional waves (with only insubstantial component $Y-$axis), in addition to that, the size of effects must be similar.

1.6 The Korteweg-de Vries Equation

The first purpose of KdV equation was to illustrate long wavelength shallow water waves of small amplitude. During the process of KdV equation derivation, the two scientists Korteweg and Vries made a presumption: in the Y-direction, the motion is constant in the peak of the wave Gardner [13]. Consequently, the wave surface uplift that goes over h level, diffusing along X-direction, becomes a function of the X-horizontal and of time T, i.e $Z = \eta(X,T)$. Horizontal position represents the position that goes along the canal.
When we take physical factors into consideration the KdV equation

$$\frac{\partial \eta}{\partial T} + \sqrt{gh}\frac{\partial \eta}{\partial X} + \frac{3}{2}\frac{\sqrt{gh}}{h}\eta\frac{\partial \eta}{\partial X} + \frac{1}{2}h^2\sqrt{gh}\left(\frac{1}{3} - \frac{\tau}{pgh^2}\right)\frac{\partial^3 \eta}{\partial X^3} = 0, \tag{1.1}$$

where h represents the constant depth of water, g is used to describe gravitational acceleration about ($9.81 m/sec^2$ sea level), p stands for the density, and τ illustrates the surface tension. τ/pgh^2 or the Bond number, is actually the dimensionless parameter which provides us with the volume of surface tension relative strength and the gravitational force.

If we mention only the first two terms in equation (1.1) than $c = \sqrt{gh}$ will illustrate the associated linear long wave. This is actually, the top speed possible when it comes to the propagation of water waves of small amplitude incited by gravity. The speed of this diffusion of solitary waves of infinitesimal amplitude, which is shown in equation (1.1) is higher to some extent. Russell derived the formula according to which the speed equals $\sqrt{g(h+\kappa)}$ Gardner [13]. The height of the crest of the solitary waves above the surface of calm water is represented by κ. According to Bullough (1988), Russell's relative speed and the actual speed are of a very similar value, the only distinction between these two can be found by a term of $O(\kappa^2/h^2)$. It is possible for the KdV to be modified in dimensionless variables as:

$$u_t + \alpha u u_x + u_{xxx} = 0, \tag{1.2}$$

In this case subscripts indicate partial derivative. It is possible to reduce the parameter α to any real number where $\alpha = \pm 1$ or $\alpha = \pm 6$ are the usual values.

In order to describe the progress of time of the diffusion of waves in one direction, the term u_t is used. That is the equation (1.2) is referred to as an evolution equation. In order to illustrate the wave steepening, αuu_x is used, and the term u_{xxx} accounts for the diffusion of the wave. It is possible for the linear first order term $\sqrt{gh}\frac{\partial\eta}{\partial X}$ to be eliminated from (1.1) using an elementary transformation. Reciprocally, we can add it to (1.2). It is possible to poise the nonlinear elevation of the water wave by diffusion. If that happens, these harmonizing effects will lead to a firm solitary wave. This kind of wave water will have particle-like characteristics. The amplitude of a solitary wave is definite. It moves at invariable speed and permanent shape across a considerably long distance. This differentiates from a group of capillary waves of small amplitude that propagate in concentric manner. These diffuse as they continue their movement.

$$u(x,t) = \frac{w-4\kappa^3}{\alpha\kappa} + \frac{12\kappa^2}{\alpha}\text{sech}^2(\kappa x - wt + \sigma) \tag{1.3}$$

$$= \frac{w-8\kappa^3}{\alpha\kappa} + \frac{12\kappa^2}{\alpha}\tanh^2(\kappa x - wt + \sigma), \tag{1.4}$$

depict the solution of a solitary wave in a closed-form. κ illustrates the wave number. The angular frequency is represented by w and the arbitrary constants by σ. The result of $\lim_{x\to\pm\infty} u(x,y) = 0$ for any time leads to $w = 4\kappa^3$. Consequently equations (1.3) and (1.4) truncate to:

$$\begin{aligned} u(x,t) &= \frac{12\kappa^2}{\alpha}\text{sech}^2(\kappa x - 4\kappa^3 t + \sigma) \\ &= \frac{12\kappa^2}{\alpha}\left[1 - \tanh^2(\kappa x - 4\kappa^3 t + \sigma)\right], \end{aligned} \tag{1.5}$$

The equation (1.2) can be solved by form:

$$u(x,t) = \frac{w - 4\kappa^3(2m-1)}{\alpha\kappa} + \frac{12\kappa^2 m}{\alpha} cn^2(\kappa x - wt + \sigma, m), \tag{1.6}$$

this periodic solution can be referred to as the cnoidal wave solution, because the Jacobi ellipsoidal cosine function, cn, is used where the modulus is $m(0 < m < 1)$.

The wave number κ represents the diameter of cnoidal oscillations. Each module wave has three phases. They are shown in Figure (1.4), where $t = 0$ and $\kappa = 2$, $\alpha = 6$, $w = 16$, $\sigma = 0$ and $m = 3/4$. If we use $\lim_{m\to 1} cn(\zeta, m) = \text{sech}(\zeta)$ property, we can corroborate that equation (1.6) truncates to equation (1.3) where m has a tendency. This means that single oscillations amplify to infinity leaving behind a solitary wave of one single pulse. KdV equation is widely common in all books that refer to soliton theory.

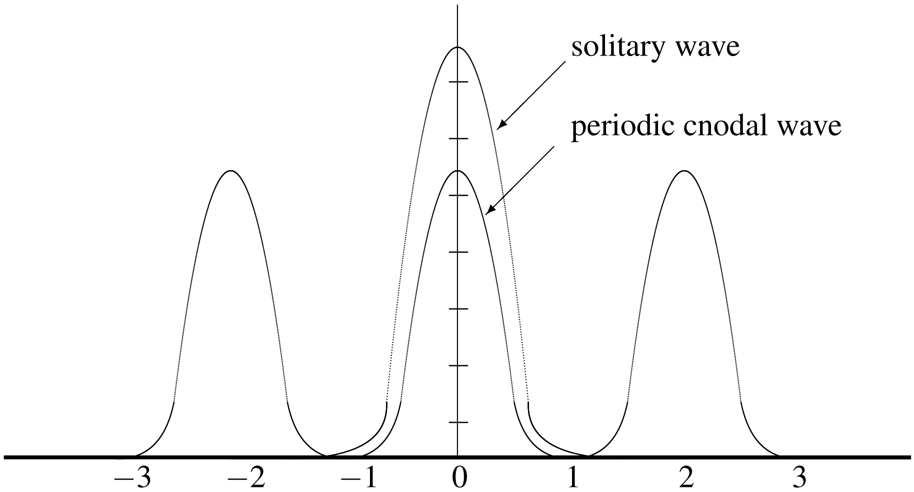


Fig. 1.4 Periodic cnoidal wave and solitary wave

1.7 Korteweg-de Vries Soliton

The discovery that KdV equation was limiting equation that defined the continuous system was made by Martin D. Kruskal and Norman J. Zabusky. They detected this while they conducting the analysis of this mass-and-spring model. They were observing the results of the tendency of the spring length toward zero. In their later work these two scientists used computer simulation to analyze KdV equation. While exploring the interaction of multiple solitary waves, they noticed that the waves were colliding and separating without any change in their velocity, shape or size. The only thing that changed was the position they would have, provided that the collision had not occurred at all. Because of the fact that these waves behaved in the manner of light particles, they were named solitons Brauer [7]. The following expression gives us an ordinary analytical form of the KdV soliton.

$$u(x,t) = 3\omega \text{sech}^2\left[\frac{1}{2}\sqrt{\omega}(\xi - \xi_0)\right]. \tag{1.7}$$

This soliton possesses ω amplitude, width and a starting position at ξ_0, where $\xi_0 = x_0$. It is centered ξ, where $\xi = x + \omega t$. Therefore, this soliton is a travelling wave with a constant amplitude, width and the speed of ω.

It is not complicated to get this solution from KdV equation. We will presume that $u(x,t)$ is the solution of travelling wave in equation (1.2),(where $\alpha = 1$) of the form:

$$u(x,t) = f(x+\omega t) = f(\xi), \tag{1.8}$$

where ω represents the traveling wave constant speed.By substituting equation (1.8) into equation (1.2), we get the ODE:

$$\omega \frac{df}{d\xi} - f\frac{df}{d\xi} - \frac{d^3 f}{d\xi^3} = 0. \tag{1.9}$$

If we perform the substitution of equation (1.8) into equation (1.2), (where $\alpha = 1$) consecutively, it is possible to write this as a perfect derivative,

$$\frac{d}{d\xi}\left[\omega f - \frac{1}{2}f^2 - \frac{d^2 f}{d\xi^2}\right] = 0. \tag{1.10}$$

The integration of equation (1.10) results in

$$\omega f - \frac{1}{2}f^2 - \frac{d^2 f}{d\xi^2} = A. \tag{1.11}$$

In this case A represent an arbitrary constant. If we multiply equation (1.11) by $\frac{df}{d\xi}$, we obtain

$$\omega f\frac{df}{d\xi} - \frac{1}{2}f^2\frac{df}{d\xi} - \frac{d^2 f}{d\xi^2}\frac{df}{d\xi} = A\frac{df}{d\xi}.$$

It is possible to rewrite this as an exact derivative,

$$\frac{d}{d\xi}\left[-\frac{1}{2}\left(\frac{df}{d\xi}\right)^2 - \frac{1}{6}f^3 + \frac{1}{2}\omega f^2 - Af\right] = 0.$$

If we perform the integration, we get the differential equation of the first order,

$$\frac{1}{2}\left(\frac{df}{d\xi}\right)^2 = -\frac{1}{6}f^3 + \frac{1}{2}\omega f^2 - Af + B. \tag{1.12}$$

It is necessary to search for this solution where $u_{xxx} \to 0, u_x \to 0$ and $u \to 0$ as $|x| \to \infty$. On the contrary, we obtain solutions of the periodic traveling wave, called *cnoidal waves*. Therefore, we chose $A = 0$ and $B = 0$ from equations (1.11) and (1.12). It is possible to rewrite equation (1.12) as equation

$$\left(\frac{df}{d\xi}\right)^2 = f^2\left(\omega - \frac{1}{3}f\right).$$

We use the separation of variables technique to obtain,

$$\int \frac{df}{f\sqrt{\omega - \frac{1}{3}f}} = \int d\xi. \tag{1.13}$$

By integration of (1.13) left side by the use of transformation,

$$f = 3\omega \mathrm{sech}^2\theta. \tag{1.14}$$

This yields

$$\omega - \frac{1}{3}f = \omega(1 - \mathrm{sech}^2\theta) = \omega \tanh^2\theta. \tag{1.15}$$

and

$$df = -3\omega \mathrm{sech}^2\theta \tanh\theta d\theta. \tag{1.16}$$

The substitution of equation (1.14) - (1.16) into (1.13) results in

$$\xi - \xi_0 = -2\int \frac{3\omega \mathrm{sech}^2\theta \tanh\theta}{3\omega \mathrm{sech}^2\theta(\sqrt{\omega}\tanh\theta)} d\theta.$$

We simplify it to

$$\xi - \xi_0 = -\frac{2}{\sqrt{\omega}}\int d\theta,$$

or

$$\theta = -\frac{1}{2}\sqrt{\omega}(\xi - \xi_0).$$

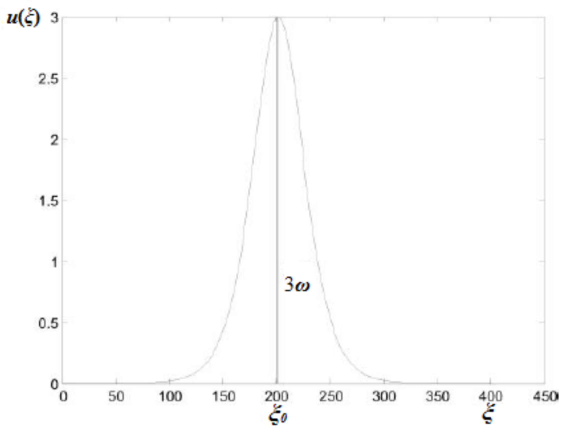
In the end, we substitute θ into equation (1.14), which produces

$$u(\xi) = 3\omega \mathrm{sech}^2\left[\frac{1}{2}\sqrt{\omega}(\xi - \xi_0)\right]. \tag{1.17}$$

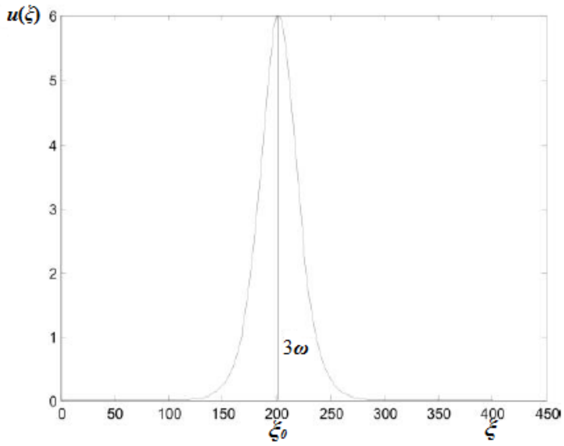
Since $u(x,t) = f(x + \omega t)$, we get:

$$u(x,t) = 3\omega \mathrm{sech}^2\left[\frac{1}{2}\sqrt{\omega}(x + \omega t - \xi_0)\right],$$

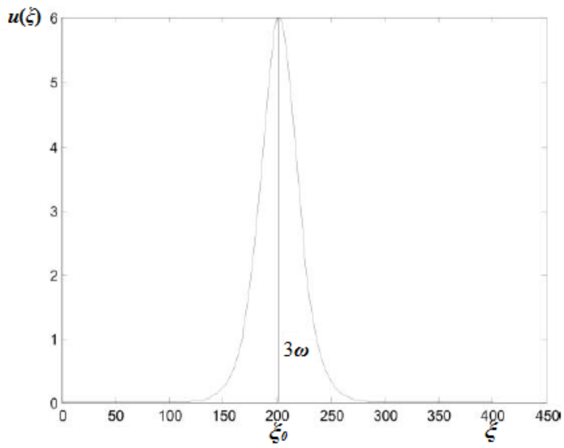
where ω is wave velocity and the starting position is ξ_0. The exact solution (1.17) of KdV equation (1.2) where $\alpha = 1$. Representation the non-dispersive traveling wave. By using random numbers as input ω is the value of speed, the initial wave is ξ_0 and wave length computation in MATLAB. Figure (1.5) describe the behavior of the soliton. We have seen from Figure (1.5) that the height of packet increases with speed. To appoint whether it is in matter of fact the solution demands further evaluation of the KdV equation that includes a parameter time t. It has executed by numerical calculation Aminuddin and Sehah [3].



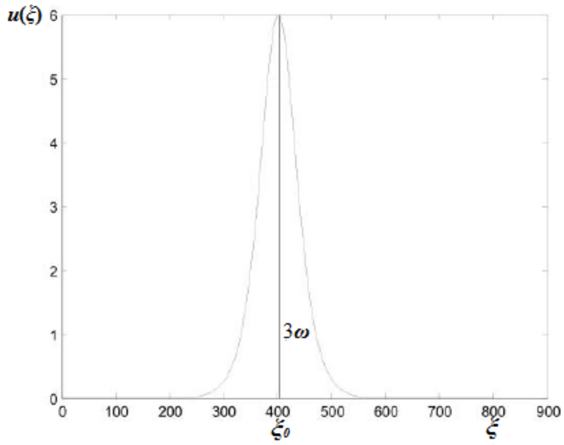
(a) Soliton wave with parameter $\omega = 1$, $\xi_0 = 200$ and $\xi = 400$.



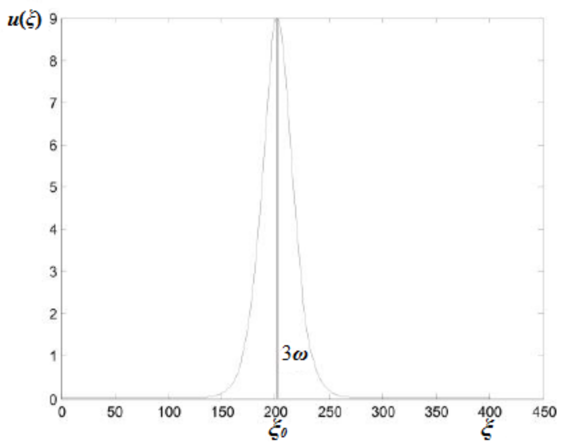
(b) Soliton wave with parameter $\omega = 1$, $\xi_0 = 400$ and $\xi = 800$.



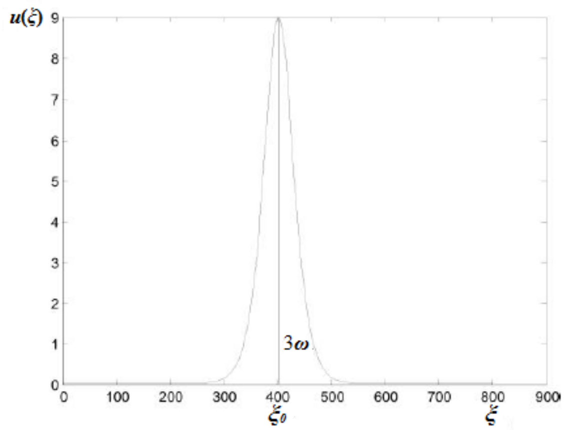
(c) Soliton wave with parameter $\omega = 2$, $\xi_0 = 200$ and $\xi = 400$.



(d) Soliton wave with parameter $\omega = 2$, $\xi_0 = 400$ and $\xi = 800$.



(e) Soliton wave with parameter $\omega = 3$, $\xi_0 = 200$ and $\xi = 400$.



(f) Soliton wave with parameter $\omega = 3$, $\xi_0 = 400$ and $\xi = 800$.

Fig. 1.5 Soliton wave with parameters, speed ω, initial wave packet ξ_0 and wave length.

Chapter 2

Travelling Waves

2.1 Introduction

The perturbation that travels and dispatches energy from one place to another is one of the most important concepts in applied mathematics and physics and it's called a wave. This phenomenon has been involved in a variety of discoveries in physics especially in classical mechanics Pindza [28].

Two significant effects diffusion and non-linearity study in this chapter. The combination of these two effect can lead to their canceling, allowing the potential travelling of waves. After that this chapter will concentrate on the evolution equations family, especially on KdV family. The solutions of the travelling wave of the KdV equation generalized third order will be the subject of our interest. In other words, with the help of the travelling wave transformation, we will breakdown the study the nonlinear partial differential equations into nonlinear ordinary differential equations study. The essential components in getting the precise travelling waves solutions of the KdV family will be the method called Fan sub-equation.

2.2 Dispersive and Non-dispersive Waves

Dispersive wave is a wave which diffuses as it travels. In most cases this is the conduct of a water wave which is localized. The study of this phenomenon of natural disasters, for example tsunami Pindza [28]. In the next subsection we will address linear and nonlinear waves and will also notice that the combination of the two effect results in travelling waves candidates.

2.2.1 Linear Non-dispersive Waves

Analysis the famous one dimensional wave equation.

$$u_{tt} + c^2 u_{xx} = 0, \tag{2.1}$$

where $u(t,x)$ represents some wave associated property and c^2 represent a constant wave speed (each wave phase velocity) here $u(t,x) = f(x-ct) + g(x+ct)$, is the solution of (2.1) where f and g are arbitrary functions.
The most principal solution of (2.1) is:

$$u(t,x) = e^{i(kx-\omega t)}, \tag{2.2}$$

under the assumption that u is periodic with wave number k and ω is the angular frequency. From (2.2) is selected due to the fact that each wave solution of it is superposed plane is physically attainable because at $x \to \pm\infty$, it stays bounded on both boundaries the exponential solution,

$$u(t,x) = e^{\pm(Kx-\Omega t)},$$

satisfies (2.1) as well, but we can notice divergence at one boundary. As a consequence of that, this solution is rejected in the linear PDEs theory. One method that can be used for the verification of whether the wave is non-dispersive or not is to pinpoint the relationship between K, the wave number that satisfies the PDE and angular frequency ω. This relationship is called the dispersion relation.

Definition 1 *The ratio of the wave number frequency $c_p(k) = \frac{\omega}{k}$, is the phase velocity.*

Definition 2 *The rate of the frequency change regarding the wave number $c_g(k) = \frac{\partial \omega}{\partial k}$, is the group velocity.*

Definition 3 *provided that, the wave is dispersive,*

$$\frac{\partial^2 \omega}{\partial k^2} \neq 0.$$

In order to get that dispersion relation of equation (2.1) it is necessary to substitute (2.2) into (2.1),

$$\omega = \pm ck, \tag{2.3}$$

where ω is the frequency, K is the represent the wave number and C is the wave crests velocity. It is very simple to confirm that $\frac{\partial^2 \omega}{\partial k^2} = 0$, from (2.3). this indicates that all superposed waves have equal speed which implies that the wave solution is non-dispersive. We refer to this as the behavior of travelling wave.

2.2.2 Linear Dispersive Waves

We will analysis the next linear equation of the third order,

$$u_t + \kappa u_{xxx} = 0, \tag{2.4}$$

will prove the group velocity and,

$$c_g(k) = 3\kappa k^2.$$

Our goal will be to discover the nature of the wave directed by (2.2). Provided that, we

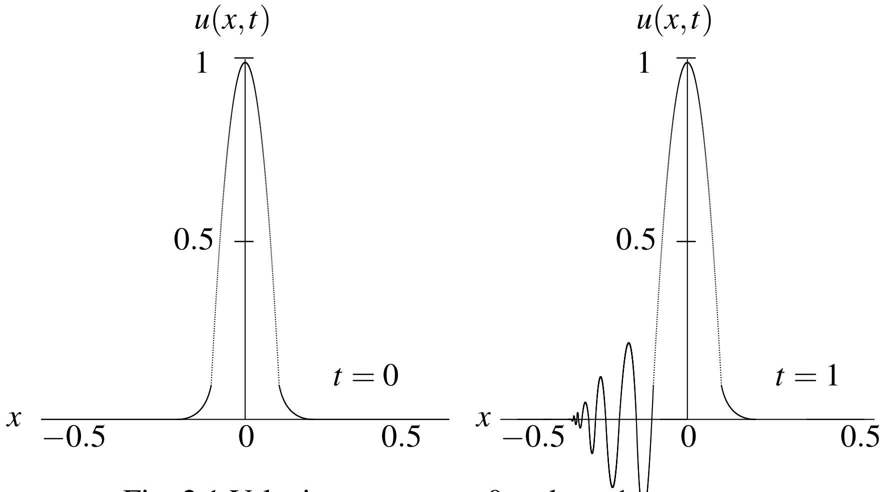


Fig. 2.1 Velocity waves, $t = 0$ and $t = 1$

substitute by (2.2) into (2.4), we will get the dispersion relation.

$$\omega(k) = \kappa k^3, \text{ and } c_p(k) = \kappa k^2,$$

provided the phase velocity.It is obvious that there is difference between the phase velocity and group velocity. This type of wave is called dispersive, meaning that the wave alters its shape as travels. See Figure (2.1). [1]

[1] Figure (2.1): Graphical illustration of equation (2.4) with initial conditions $u(x,0) = \sec(x)$, and $\kappa = 1$ at specific time , $t = 0$ and $t = 1$.

2.2.3 Non-linear Non-dispersive Waves

Suppose the in viscid Burgers equation,

$$u_t + \alpha u u_x = 0. \tag{2.5}$$

Where α is an arbitrary constant is representing by Lagrange's technique is used to get the solution of (2.5). The characteristic equation of (2.5) are,

$$\frac{du}{ds} = 0, \tag{2.6}$$

$$\frac{dx}{ds} = \alpha u, \tag{2.7}$$

$$\frac{dt}{ds} = 1, \tag{2.8}$$

In cases that the independent function $\psi(t,x,u)$ and $\phi(t,x,u)$ are found as,

$$\begin{aligned} \psi_t + \alpha u \psi_x &= 0, \\ \phi_t + \alpha u \phi_x &= 0, \end{aligned} \tag{2.9}$$

then the relation,

$$F(\phi, \psi) = 0, \quad \text{where } F \text{ represent an arbitrary function,}$$

or correspondingly,

$$\phi = f(\psi) \qquad \text{where } f \text{ represent an arbitrary function,}$$

Would give us the solution of (2.5). discovering the solution of (2.6), results in,

$$u = \phi = \text{ constant,} \tag{2.10}$$

If we dismiss ds from (2.7) and (2.8) we will get,

$$\frac{dx}{dt} = \alpha u. \tag{2.11}$$

By integrating (2.11) leads to,

$$\psi = x - \alpha\phi t = \text{constant}. \tag{2.12}$$

It is very simple to verify that (2.10) and (2.12) give the solution into (2.9), that leads to the conclusion.

Is the general solution of (2.5) is,

$$u(x,t) = f(x - \alpha ut). \tag{2.13}$$

where f is an arbitrary function.
It is our intention to point out how the increase of t causes a progressive the deformation of the wave profile (2.13) Bhatnagar [6]. in order to do this, we will perform the analysis of the alteration of the slope of $u(x,t)$ as t escalates. After this the analysis similar to Bhatnagar's follows,

$$u_x(x,t) = (1 - \alpha u_x(x,t)t)f_\xi, \text{ with } \xi = x - \alpha ut. \tag{2.14}$$

Yields the first derivative of (2.13) concerning x. After the application of algebraic simplifications (2.14) produces,

$$u_x(x,t) = \frac{f_\xi}{1 + t\alpha f_\xi},$$

that represents the u-profile slope at (x,t) point, regarding the initial profile slop at ξ. In this case $\xi = x$ at $t = 0$, provided that $f_\xi < 0$, then $u_x(x,t)$ is finite at $t = \frac{-1}{\alpha f_\xi}$. From this we can conclude that on the condition that at some points ξ the initial profile possesses a negative slope, for $t > T = (\frac{-1}{\alpha f_\xi})_{min}$, in the vicinity of a point $x_0 = \xi_0 + \alpha T f(\xi_0)$, the solution stops being single valued. Where ξ_0 represents a point of minimum value for $\frac{-1}{\alpha f_\xi}$.
We intend to detect the alteration in the u slope, at $\xi = \xi_0$ when $t > T$. Then $X_0(t)$ will be ξ position regardless of t time. Also the value of t can be calculated by,

$$t = T + \varepsilon = \left(\frac{-1}{\alpha f_\xi}\right)_{min} + \varepsilon,$$

where $|\varepsilon| \ll 1$.
This leads to,

$$u_x(X_0(t), T + \varepsilon) = \left(\frac{f_\xi}{1 - t\alpha f_\xi}\right)_{\xi = \xi_0, t = T + \varepsilon}.$$

When algebraic simplifications, we get:

$$u_x(X_0(t), T+\varepsilon) = \frac{1}{\alpha\varepsilon}.$$

as a result of this, we have,

$$u_x(X_0(T-0), T-0) = -\infty \text{ and } u_x(X_0(T+0), T+0) = +\infty.$$

Finally, we reach the conclusion that the wave profile $u(x,t)$ is a subject of a progressive deformation t increases.

2.2.4 Non-linear Dispersive Waves

It was shown in the previous sections that dispersion and non-linearity, they do not permit traveling waves. It is our goal to analysis the conditions under which the combination of non-linearity and dispersion can lead to traveling waves. Provided that, we adjoin dispersion and nonlinear term to our determining wave equation, we will get,

$$u_t + \alpha u u_x + \beta u_{xxx} = 0,$$

even though, it is widely familiar that KdV equation possesses traveling wave solutions, we will analysis this issue from different perspective. We will examine a localized traveling wave in motion at constant speed.

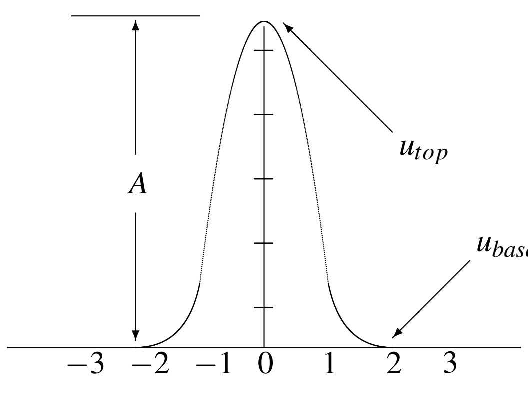


Fig. 2.2 Local travelling wave

We will assume the symmetry of the localized traveling wave solution around the maximum amplitude point. It is possible to reach the proximate value of u by u_{top} function in the vicinity of the maximum amplitude. In this case the u_{xxx}, dispersion term, will equal zero. So the u_{top} function will satisfies (2.5). when near the base of the wave where it the possible to disregard the nonlinear term, because u small, we can approximate the solution by u_{bases}. In the neighborhood of the localized traveling wave base, u_{base} satisfies (2.4) on each respective side.

$$c_p(k) = \frac{\omega}{k} = \beta k^2 \text{ and } c_g(k) = \frac{\partial \omega}{\partial k} = 3\beta k^2.$$

illustrate the group velocity and the phase velocity.

As a consequence of this, there is a difference of velocity between the bottom and the top of the wave above. However a contradiction appear in the argument that the wave above is traveling wave. The reason of the nonlinear of wave equation and the fact that the wave superposition principle ceases to be valid. Consequently, it is necessary to chose:

$$u(x,t) = e^{-\Psi} \text{ as } \Psi \to +\infty \text{ and } u(x,t) = e^{\Psi} \text{ as } \Psi \to -\infty,$$

where $\Psi = Kx - \Omega t$, the solution of the wave at its base could be,

$$u_{base}(x,t) = e^{\pm(Kx-\Omega t)}, \tag{2.15}$$

By substituting (2.15) into (2.4), we get single nonlinear dispersion relation from equation (2.3),

$$\Omega = \beta K^3.$$

$$c_{top} = \alpha A \quad \text{and} \quad c_{base} = \beta K^2,$$

are respective top and base phase velocity, A is used to illustrate the wave maximum amplitude. Leveling the velocities of the top and the base of the wave leads to the relation,

$$\beta k^2 = \alpha A.$$

It is possible to get some traveling wave solutions because the equal speed of the top and the base of the wave. This shown in Figure (2.5).

The effect of the non-linearity and the dispersion of the KdV equation (1.2).

Linearity Non-dispersion

$$u_t + cu_x = 0,$$

where c is constant.
The initial profile is transferred at constant speed without any change of shape. Collisions can not have place since every the initial profile travels in the same velocity.

Linearity Dispersion

$$u_t + \beta u_{xxx} = 0,$$

the solution can expresses in the form:

$$u = a\exp\left(i(kx - wt)\right) = a\exp\left(i(x - ct)k\right).$$

where w is the frequency, k is a wave number and $c = \frac{w}{k}$ is the speed of the traveling wave. this instructs to the dispersion relation $w = \beta k^3$, i.e. the set velocity be based on the wave number. The impact of the dispersion on the wave is to make a wave packed dispersal out as it travels. This dispersion basics out the possibility of solitary waves.

Non-linearity Non-dispersion

$$u_t + \alpha uu_x = 0.$$

The term αu plays the role of the wave velocity.Since this velocity be based on the solution itself, we may anticipate that portions of the wave profile at u is massive will move more rapidly than portions of the wave near the edge of the profile where $u \to 0$. thus the portion with passive u will exceed the portion with smaller u.

Non-linearity Dispersion

$$u_t + \alpha uu_x + \beta u_{xxx} = 0. \tag{2.16}$$

If there is any balance between non-linearity and dispersion, then we get a solution that travels without alteration of shape.

- The generalized KdV equation has the form:

$$u_t + \alpha u^n u_x + \beta u_{xxx} = 0, \tag{2.17}$$

where $n = 1.2, \ldots,$ (n is positive integer). The most important condition after $n = 1$ is $n = 2$ the resulting becomes the form:

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \tag{2.18}$$

it is called modified Korteweg-de Vries (mKdV) equation. Furthermore, the sign of the non-linear term it possible to change obtain the non-trivial alternative:

$$u_t - \alpha u^2 u_x + \beta u_{xxx} = 0. \tag{2.19}$$

Note that alteration the sign of the nonlinear term in the KdV equation itself produces nothing new, since the resulting equation is decreased to (2.16) by alteration the sign of u Miles [21].

2.3 Travelling Soliton Solutions

2.3.1 Introduction

The phenomena of nonlinear waves are the subjected to intense study at present times, in various fields of applied mathematics, as well as engineering and physics, for example in radio physics, acoustics, optics, hydrodynamics, plasma physics $\cdots$ etc Ali [2].

Korteweg-de Vries equation or KdV equation is the principal nonlinear wave equation. It was derived by two scientists, Korteweg and de-Vries Philos [27], for the purpose of describing how one dimensional waves behave in shallow water, when their amplitude is small but finite. Lately, the KdV equation has been employed for describing different sorts of phenomena, such as bubble liquid mixture waves, warm plasma waves, acoustic wave behavior in anharmonic crystals, ion-acoustic waves and magnetohydrodynamics.

2.3.2 Application of the Korteweg-de Vries equation

This section deals with the rise of the Korteweg-de Vries equation as a true model which governs the development of waves regarding media where weak nonlinear effects are studied. Four examples will be quoted: the first appears in plasma physics in which the Korteweg-de Vries equation directs the long compressive wave evolution in a plasma of hot electrons and cold ions. The second example is the shallow water wave issue, while the third one arises in meteorology where the nonlinear Rossby wave propagation through rotating homogenous fluid is analyzed. The last case slightly differs from the previous two, since the second space dimension is found in the initial equations, while the final KdV equation coefficients are integrals over y. One more example was derived from the electric circuit principle which uses a nonlinear capacitance. Here, we obtain the generalized KdV equation of the pth order, with capacitance depending non-linearity. We will use this example to show how, under particular circumstances, a modified Korteweg-de Vries equation can appear. The simplicity of the KdV equation structure is well known, since it is the equation of single scalar value with two independent and one dependent variable. Nevertheless, the initial equations of most physical systems motion are complex, and they usually involve quite a few dependent variables. This is the reason we need to employ a procedure which would reduce equation sets of this kind to less complicated forms, that is, perturbation procedure. In order to apply this technique, all variables are scaled to dimensionless structure, while the dependent variables are expanded with regard to a parameter of perturbation α. The next section will illustrate this method through the fact that the KdV equation governs the ion-acoustic waves Pindza [28].

2.3.3 Single Soliton Solutions

The ability of the Korteweg-de Vries equation to produce stable travelling wave solutions represents one of its most interesting characteristics. These can be solitary waves we refer to as solitons, but also the cnoidal wave that is actually a sinusoidal wave generalization. We obtain these if we put:

$$u(x,t) = U(X) \quad , \quad X = x - ct, \tag{2.20}$$

where, c illustrates the constant velocity of wave, since it propagates along the x-axis positive direction. By substituting (2.20) into the general KdV equation (2.17), we obtain ordinary differential equation:

$$-cU^{'} + \alpha U^{n}U^{'} + \beta U^{'''} = 0. \tag{2.21}$$

A differentiation concerning X is denoted by a prime. After integrating (2.21) immediately, we obtain:

$$\beta U^{''} = cU - \frac{\alpha}{n+1}U^{n+1} + a_1 \tag{2.22}$$

where a_1 the constant of integration. After multiplying (2.22) by $U^{'}$ and integrate, we get:

$$\frac{\beta}{2}\left(U^{'}\right)^2 = \frac{c}{2}U^2 - \frac{\alpha}{(n+1)(n+2)}U^{n+2} + a_1 U + a_2, \tag{2.23}$$

where a_2 is the constant of integration. For the solution of (2.23) to be real, it is necessary for the right side to be non-negative, which means that:

$$\int \frac{\left(\frac{\beta}{2}\right)^{\frac{1}{2}} dU}{\left(\frac{c}{2}U^2 - \frac{\alpha}{(n+1)(n+2)}U^{n+2} + a_1 U + a_2\right)^{\frac{1}{2}}} = \pm(x - ct). \tag{2.24}$$

There are two kinds of solution of equation (2.24), the first are cnoidal waves, expressed through Jacobi elliptic functions (for the precise form and more details see Pindza [28]), while the second are solitary waves A. Scott and McLaughlin [1]. The next step is the derivation of the equation's (2.24), its a solution regarding the solitary waves. For the purpose of this let $U^{'}, U^{''}$ and $U \to 0$ as $|x| \to \infty$. As a result of this, when the constants of integration a_1 and a_2 equal zero:

$$U^{'} = U\left(\frac{1}{\beta}(c - \frac{2\alpha}{(n+1)(n+2)}U^n)\right)^{\frac{1}{2}}, \tag{2.25}$$

suppose,

$$y = (1 - \mu U^n)^{\frac{1}{2}} \quad \text{where} \quad \mu = \frac{2\alpha}{c(n+1)(n+2)},$$

then,

$$U = \left(\frac{1-y^2}{\mu}\right)^{\frac{1}{n}} \quad \text{and} \quad dU = -\frac{2y}{n\mu}\left(\frac{1-y^2}{\mu}\right)^{\frac{1-n}{n}}. \tag{2.26}$$

If substitute (2.26) into (2.25), we get:

$$-\frac{2}{n}\left(\frac{\beta}{c}\right)^{\frac{1}{2}} \int \frac{dy}{1-y^2} = \int dX. \tag{2.27}$$

By integrating (2.27), we get:

$$\ln\left(\frac{1-y}{1+y}\right) = n\left(\frac{c}{\beta}\right)^{\frac{1}{2}} X + c_1 \tag{2.28}$$

To find the initial condition is applied, let $c_1 = -n\left(\frac{c}{\beta}\right)^{\frac{1}{2}} x_0$, which implies that we can write (2.28) as:

$$\frac{1-y}{1+y} = \exp\left[n\left(\frac{c}{\beta}\right)^{\frac{1}{2}}(X - x_0)\right].$$

Suppose that $Z = n\left(\frac{c}{\beta}\right)^{\frac{1}{2}}(X - x_0)$, we obtain:

$$y = \frac{1-\exp(Z)}{1+\exp(Z)}.$$

By the equation (2.26) where $U^n = \frac{1-y^2}{\mu}$, we get:

$$\begin{aligned} U^n &= \frac{1}{\mu}\left[\frac{(1+\exp(Z))^2 - (1-\exp(Z))^2}{(1+\exp(Z))^2}\right] \\ &= \frac{1}{\mu}\frac{4\exp(Z)}{(1+\exp(Z))^2}, \end{aligned}$$

on the other hand,

$$\begin{aligned} U^n &= \frac{1}{\mu}\frac{4}{(\exp(Z/2)+\exp(-Z/2))^2}, \\ &= \frac{1}{\mu}\text{sech}^2\left(\frac{Z}{2}\right). \end{aligned}$$

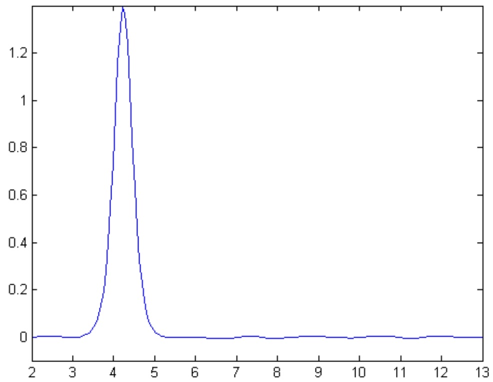
This can be reduced to

$$u^n(x,t) = \frac{c(n+1)(n+2)}{2\alpha}\text{sech}^2\left[\frac{n}{2}\sqrt{\frac{c}{\beta}}(x - ct - x_0)\right]. \tag{2.29}$$

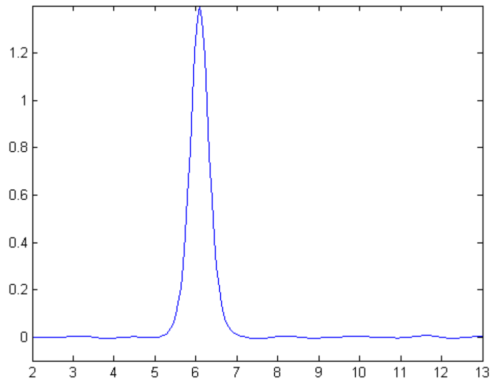
There is a familiar solution when $n = 1$:

$$u(x,t) = \frac{3c}{\alpha}\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{c}{\beta}}(x - ct - x_0)\right].$$

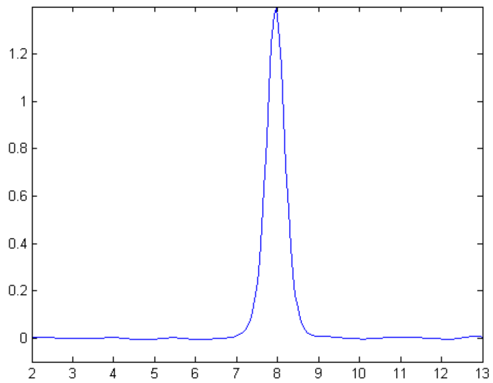
A soliton that has amplitude $3c/\alpha$ is described by (2.29).
Its velocity is proportional to its amplitude. Therefore, a smaller soliton moves more slowly



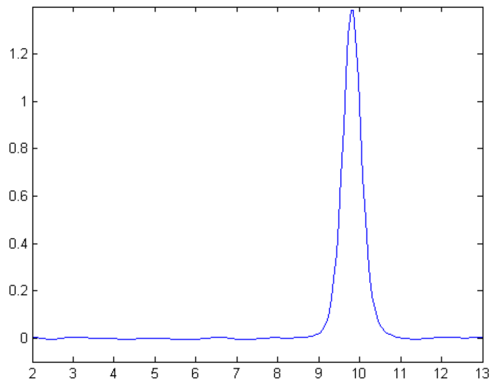
(a) One Soliton, $t = 1$



(b) One Soliton, $t = 2$



(c) One Soliton, $t = 3$



(d) One Soliton, $t = 4$

Fig. 2.3 One Soliton at the time: t=1,2,3 and 4

than the bigger one. Here $\sqrt{\beta/c}$ is proportional to the width of the soliton, while the x_0 constant acts as a phase shift.

Provided that n is odd and that the nonlinear term coefficient in the general KdV equation (2.17) possesses a negative sign, the solution we obtain is negative, or:

$$u^n(x,t) = \frac{c(n+1)(n+2)}{2\alpha}\text{sech}^2\left[\frac{n}{2}\sqrt{\frac{c}{\beta}}(x-ct-x_0)\right].$$

The solution we obtain will not be a solitary wave in case that n is even Fornberg and G.Whitham [12] Miles [21]. The Galerkin's method was used by Chen for the purpose of obtaining the analytic solutions of the KdV equation which is strongly nonlinear:

$$u_t + \alpha u^3 u_x + \beta u_{xxx} = 0.$$

2.3.4 Interaction Between Two Solitons

Let us take two solitary waves positioned on the real line into consideration as an initial condition. The shorter wave is placed to the right of the taller one. With the increase of time, the taller wave has greater velocity and consequently goes toward the shorter one, and finally catches it up. Therefore, they are subjected to a nonlinear interaction. Surprisingly, they keep their shape and velocity after the interaction, while the only thing that is changed is their position, which depends in their initial location. It was Russel who first observed this phenomenon experimentally, while Zabusky and Kruskal Zabusky and Kruskal [36] documented it numerically. Since after nonlinear reactions their form is preserved, and because of the fact that they are similar to particles, these waves were named ”solitons” by Zabusky and Kruskal. The exact two soliton interaction was numerically demonstrated by these two scientists, while the soliton properties were analytically proven by Lax Lax [20]. The KdV equation analytical solution was originally derived by Whitham, Dodd Pindza [28], Wadati C. Gardner [9] and Lamb, where $\alpha = 6.0$ and $\beta = 1.0$, with two solitary waves as the initial condition. The form of this solution is:

$$u(x,t) = 2(\ln(F))_{xx},$$

with,

$$F = 1 + \exp(\eta_1) + \exp(\eta_2) + \mu \exp(\eta_1 + \eta_2),$$

where,

$$\mu = \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2, \tag{2.30}$$

and,

$$\eta_1 = \lambda_i x - \lambda_i^3 t + d_i \text{ where } i = 1,2. \tag{2.31}$$

Prior to interaction the form of the solution is:

$$u(x,t) = \frac{1}{2}\lambda_1^2 \text{sech}^2(\eta_1) + \frac{1}{2}\lambda_2^2 \text{sech}^2(\eta_2 - \Delta),$$

where:

$$\Delta = \ln\left(\frac{1}{\mu}\right).$$

When the interaction is completed, the solution transforms into:

$$u(x,t) = \frac{1}{2}\lambda_1^2 \operatorname{sech}^2(\eta_1 - \Delta) + \frac{1}{2}\lambda_2^2 \operatorname{sech}^2(\eta_2).$$

The λ_1 and λ_2 solitary waves are placed:

• Prior to interaction:

Solitary wave λ_1 is placed on $x = \lambda_1^2 t - d_1/\lambda_1$,

Solitary wave λ_2 is placed on $x = \lambda_2^2 t - (d_2 - \Delta)/\lambda_2$,

• When the interaction is completed:

Solitary wave $\lambda_1$is placed on $x = \lambda_1^2 t - (d_1 - \Delta)/\lambda_1$,

Solitary wave λ_2 is placed on $x = \lambda_2^2 t - d_2/\lambda_2$,

The interaction takes place in the region of:

$$t = -\frac{s_1 - s_2}{\lambda_1^2 - \lambda_2^2} \quad \text{and} \quad x = \frac{\lambda_1^2 s_2 - \lambda_2^2 s_1}{\lambda_1^2 - \lambda_2^2}$$

where $s_1 = d_1/\lambda_1$ and $s_2 = d_2/\lambda_2$.

We define the forward phase shift, as well as the backward one, as:

$$\Delta_1 = \Delta/\lambda_1 \quad \text{and} \quad \Delta_2 = \Delta/\lambda_2,$$

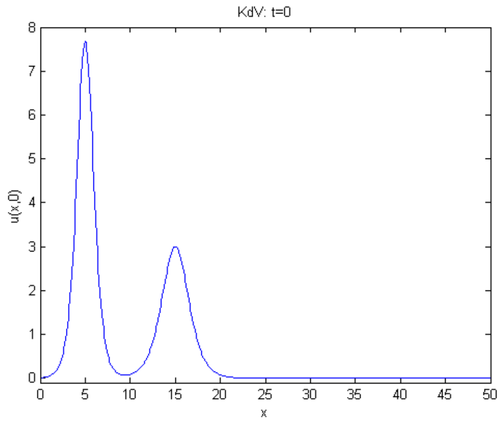
We use (2.17) equation at $t = 0$, for the initial condition. In a similar manner, Taha and Ablowitz found the mKdV equation (2.18) solution, at $\alpha = 6.0$ and $\beta = 1.0$, regarding two solitary waves. The solution has a form:

$$u(x,t) = i(\ln(f^* f))_x. \tag{2.32}$$

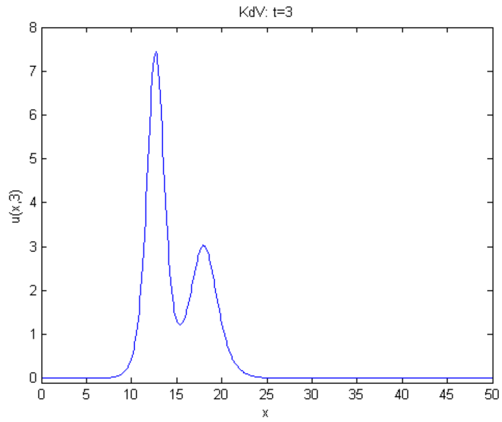
A complex conjugate is denoted by $*$, while

$$f = 1 + i\exp(\eta_1) + i\exp(\eta_2) - \mu\exp(\eta_1 + \eta_2),$$

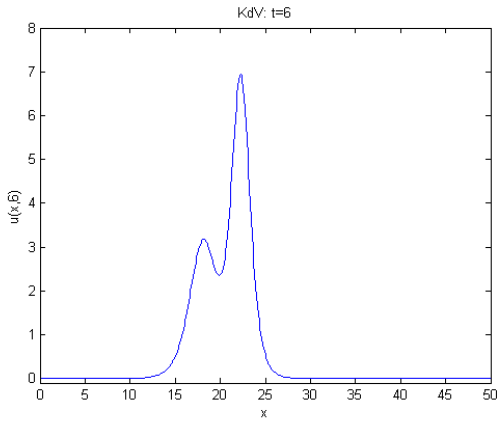
where η_j and μ are defined by equations (2.30) and (2.31), respectively,



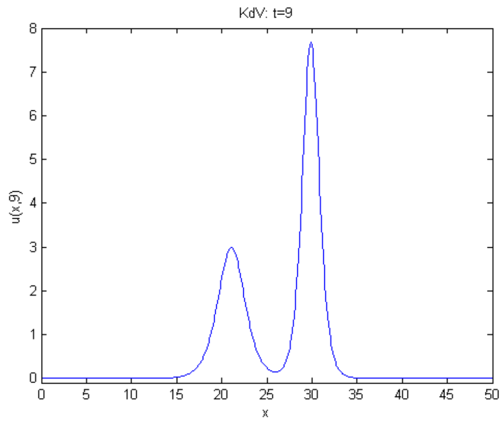
(a) Two Soliton, $t = 0$



(b) Two Soliton, $t = 1$



(c) Two Soliton, $t = 2$



(d) Two Soliton, $t = 3$

Fig. 2.4 Two Soliton at the time: t=0,1,2, and 3

Prior to interaction, the solution has a form:

$$u(x,t) = \lambda_1 \text{sech}(\eta_1) + \lambda_2 \text{sech}(\eta_2 - \Delta).$$

When the interaction is completed, the solution transforms into:

$$u(x,t) = \lambda_1 \text{sech}(\eta_1 - \Delta) + \lambda_2 \text{sech}(\eta_2).$$

We use equation (2.23) equation at $t = 0$, for the initial condition.
The inverse scattering method, was used to show that the N-solitons remain unchanged once the interaction is completed Miles [21].

Generally speaking, the KdV equation's arbitrary initial conditions will transform into a state of oscillatory dispersion propagating to the left and certain sum of solitons propagating to the right. Because of the fact that the solitons velocity depends on their amplitude, eventually we will get a line of solitons propagating to the right, whereas their amplitudes will uniformly increase from the left to the right

We refer to these solutions that do not demonstrate any oscillatory behavior and involve just solitons, as N-soliton solutions or pure soliton solutions.

Chapter 3

Painlevé Analysis

3.1 Painlevé Analysis for Partial Differential Equation

In this chapter we will study Painlevé test of the partial differential equation and we apply that in Korteweg-de Vries equation [1]

$$u_t + \alpha u^n u_x + \beta u_{xxx} = 0, \tag{3.1}$$

where $\beta = 1$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $n = 1, 2$

3.1.1 Introduction

As we have just seen, the ARS [2] conjecture was originally formulated for partial differential equations. We are not going to discuss here whether this conjecture can be turned into a rigorous theorem. This question has been addressed in detail Olver [26]. The conclusion of these studies is that in order to have a chance to find a rigorous proof of the conjecture, a drastic modification of its form would be needed, either on the type of the integrability or the acceptable singular behaviors Mohammad [22].

Although it is nothing more than a conjecture yet, since the formulation of the tests, there has been considerable interest in using the Painlevé property as a means of determining whether a given PDE is integrable. But for the original ARS conjecture to be applicable, one must find all the reductions of a given PDE. Sometimes however, all the reduction one

[1] If $n = 1$ the equation (3.1) is called KdV, and if $n = 2$ the equation (3.1) is called modified KdV

[2] Ablowitz, Ramani and Secure (ARS) algorithm, Painleve property, the recent more comprehensive approach have been developed the singularity structure of nonlinear differential equations.

can find are just too trivial to yield an interesting information. Fortunately Weiss made a major progress in this area Mohammad [22] putting aside the reductions and introducing the Painlevé property for PDEs themselves. In fact, according to Weiss, a PDE will possess the Painlevé property if its solutions are single valued about any non-characteristic singular manifold $\phi(z_1, z_2, \ldots, z_n)$ where z_i are the independent variables Welss [34].

To verify if a PDE has the Painlevé property we expand a solution of a nonlinear PDE about a movable, singular manifold $\phi(z_1, z_2, \ldots, z_n)$. Let $u = u(z_1, z_2, \ldots, z_n)$ be a solution of the PDE and assume that:

$$u = \frac{1}{\phi^p} \sum_{j=0}^{\infty} u_j \phi^j, \tag{3.2}$$

where ϕ and $u_j = u_j(z_1, z_2, \ldots, z_n)$ are analytic functions of $(z_1, z_2, \ldots, z_n)$ in neighborhood of manifold $\phi(z_1, z_2, \ldots, z_n) = 0$. Substitution of (3.2) into the PDE determines the possible value of p and defines the recursion relations for $u_j,\ j = 0, 1, 2, \ldots$. When p is a positive integer, and (3.2) is a valid and general expansion about the manifold $\phi(z_1, z_2, \ldots, z_n) = 0$, then the solution has single value representation about it. [3] If this representation is valid for all allowed movable singularity manifolds, then the PDE has the painlevé property.
For a specific partial differential equation it has necessary to identify all possible values for a p and then find what the form of the resulting ϕ series is [17].

The Weiss algorithm for a PDE with one dependent and two independent variables, consist in looking for the general solution of the PDE in the form:

$$u(t,x) = \frac{1}{\phi(t,x)^p} \sum_{j=0}^{\infty} u_j(t,x) \phi(t,x)^j, \tag{3.3}$$

where p is a positive integer and $\phi(t,x) = 0$ is the equation of a non-characteristic $(\phi_t \phi_x \neq 0)$ singular manifold, and the function $u_j(t,x)$ have to be determined by the substitution of expression (3.3) in the partial differential equation, which becomes:

$$\sum_{j=0}^{\infty} E_j(u_0, u_1, \ldots, u_j, \phi) \phi(t,x)^{j-q} = 0,$$

where q is some positive integer constant. E_j depends on ϕ only.

The algorithm of Weiss, in addition to the three steps of ARS algorithm for ordinary

[3] Manifold conditions are determined by $\phi(z_1, z_2, \ldots, z_n)$ where ϕ is an analytic function of $(z_1 \ldots z_n)$ in a neighborhood of the manifold.

differential equations contains one more step and proceeds in four steps, dealing with the dominant behaviors, the branches , the resonances and the compatibility conditions at resonances, respectively Mohammad [22].

Dominant Behaviors

Substitute in the partial differential equation a solution

$$u(t,x) = u_0(t,x)\phi(t,x)^{-p} \tag{3.4}$$

where p is real number and $\phi(t,x)$ is arbitrary, and find all possible p_i for which two or more terms in each equation balance.

Branches

Solve the equation $E_0 = 0$ for the nonzero value of $u_0(t,x)$. This may lead to several solutions, called branches.

Resonances

Find the value of j for which $u_j(t,x)$ can not be determined from the equation $E_j = 0$. This last equation has usually the form:

$$E_j = (j+1)P(j)\phi_x^k\phi_t^{n-k}u_j + Q(u_0,u_1,\ldots,u_{j-1},\phi) = 0, \;\; j = 0,1,2,\ldots \tag{3.5}$$

where n is the order of partial differential equation, $0 \le k \le n$, and $P(j)$ is a polynomial in j of degree $n-1$. The points where the resonances occur are the zero of $P(j)$ and $j = -1$.

3.1.2 Compatibility Conditions at the Resonances

At a resonance after substitution in (3.5) of the previously computed $u_i(t,x)$, $i \le j-1$. The function Q is either zero, in which case $u_j(x,t)$ can be chosen arbitrarily and the resonance is said to be compatible, or non-zero and the expansion (3.2) does not exist for arbitrary $\phi(t,x)$.
$j = -1$ is always a resonance point, and corresponds to the free singularity manifold function $\phi(t,x)$. The Painlevé property is characterized by the fact that p is a positive integer

and all resonances occur at non-negative integer value of j are compatible.

A point that will emphasize is that ϕ series for nonlinear PDE contains a lot of information about the PDE. For the equations which have the Painlevé property a methods has been developed for finding Bäcklund transformations Welss [33] Welss [35]. An outline and an application of the singular manifold method. As it will be seen in this chapter, for equations that do not satisfy the Painlevé property it is still possible to obtain single valued expansions by specializing the arbitrary functions that appear in the ϕ series expansions. This specialization leads to a system of PDE for the formally arbitrary data. For specific system, and conjectured in general, these equations are integrable. The form of the resulting reducation enables the identification of integrable reductions of the original systems Conte and Musette [11].

3.2 Painlevé Transformation of Non-linear PDEs

Let us truncate the series in (3.2) at the constant term and assume that $u_j = 0,\ j \leq -p+1$, then:

$$u = u_0\phi^{-p} + u_1\phi^{1-p} + \cdots \tag{3.6}$$

Substituting (3.6) in the partial differential equation under the consideration, we obtain the necessary conditions to have a solution of the form (3.6). According to one of these conditions u_p must be a solution of the original PDE. Therefore the relation (3.6) can be taken as Bäcklund transformation that relates solutions u and u_i of the given PDE.
Hence elimination $u_0(t,x), u_1(t,x), \cdots u_p$, between the equations representing the necessary conditions one obtains a PDE for $\phi(t,x)$. Indeed the equations representing the necessary conditions have a property which considerably simplifies the search of their common solutions. The $\phi(x,t)$ equation is invariant under the group of homographic transformations.

$$H : \phi \mapsto \frac{a\phi + b}{c\phi + d}\ ,\ ad \neq bc, \tag{3.7}$$

hence it can be written in terms of two homographic invariant functions

$$S = \{\phi, x\} = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{3.8}$$

called the Schwarzian derivative:

$$C = -\frac{\phi_t}{\phi_x}, \tag{3.9}$$

and,

$$L(\phi) = -\frac{\phi_{xx}}{2\phi_x}, \tag{3.10}$$

which has the dimension of a velocity. The two elementary invariant C and S are linked by the compatibility conditions $(\phi_t)_{xx} = (\phi_{xx})_t$ which finding Conte and Musette [11]

$$S_t + C_{xxx} + 2C_x S + CS_x = 0, \tag{3.11}$$

when expressed in terms of C and S.

The ϕ equation and (3.11) forms a system of nonlinear PDEs in the two invariants C and S. The above analysis reveals that to find $\phi(t,x)$ one must solve this system first. Then the solutions of the ϕ equation is obtained by the help the following two lemmas about the differential equations written in the terms of the Schwarzian derivatives Steep and Euler [30].

Lemma 1 *[22] Let ψ_1 and ψ_2 be two linearly independent solutions of the equation,*

$$\frac{d^2\psi}{dz^2} + P(z)\psi = 0, \tag{3.12}$$

which are defined and holomorphic on some simply connected domain D in complex plane, then $W(z) = \psi_1(z)/\psi_2(z)$ satisfies the equation,

$$\{W : z\} = 2P(z). \tag{3.13}$$

Conversely, *if $W(z)$ is a solution of (3.13) at all point of D, then one can find two linearly holomorphic independent solutions $\psi_1$and ψ_2 of (3.12) such that $W(z) = \psi_1(z)/\psi_2(z)$ in some neighborhood of $z_0 \in D$.*

Lemma 2 *Mostafa [24] The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, namely,*

$$\left\{\frac{aW+b}{cW+d};z\right\} = \{W;z\} \quad , \quad ad \neq bc, \tag{3.14}$$

where a, b, c and d are constants.

3.3 Painlevé Analysis for the KdV Equation

3.3.1 Painlevé Property

In this section we apply Painlevé property in the KdV equation Mostafa [23]:

$$u_t + \alpha u^n u_x + \beta u_{xxx} = 0, \tag{3.15}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta = 1$ and $n = 1$.
The series solution of the partial differential equation is in the form:

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-p}, \tag{3.16}$$

where ϕ is an analytic function that defines a non-characteristic hypersurface S. [4] To determine whether equation (3.15) satisfies Painlevé property we use simplified condition Steep and Euler [30].

$$\phi(t,x) = x + \psi(t) = 0, \tag{3.17}$$

where ψ is an arbitrary function (ϕ is a characteristic of (3.15) if $\partial\phi/\partial x \neq 0$). To find a value of equilibrium point p, by substituting (3.16) into the equation (3.15), where $u_t(t,x) = \partial u(t,x)/\partial t$, $u_x(t,x) = \partial u(t,x)/\partial x$ and $u_{xxx}(t,x) = \partial^3 u(t,x)/\partial x^3$ and by comparing the lowest powers in the produced series, we find $p = 2$. In the neighborhood of the singularity manifold (3.17), the series solution (3.16) will be in the form:

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-2}, \tag{3.18}$$

where $u_0, u_1, \ldots,$ are arbitrary functions. By associating the summation Mostafa [23], we get:

[4] Ward (1984). If S is an analytic non-characteristic complex hypersurface in C^n, then every solution of the PDE which is analytic on $C^n \setminus S$, is meromorphic on C^n

$$\sum_{j=3}^{\infty} u_{j-3,t}\phi^{j-5} + \alpha \sum_{j=0}^{\infty} \left[\sum_{k=0}^{j-1} u_k u_{j-1-k,x} + \sum_{i=0}^{j} u_{j-i} u_i (i-2)\phi_x \right] \phi^{j-5}$$
$$+ \sum_{j=3}^{\infty} u_{j-3,xxx}\phi^{j-5} + \sum_{j=2}^{\infty} 3(j-4) u_{j-2,xx}\phi_x \phi^{j-5}$$
$$+ \sum_{j=1}^{\infty} 3(j-3)(j-4) u_{j-1,x}\phi_x^2 \phi^{j-5} + \sum_{j=2}^{\infty} (j-4) u_{j-2}\phi_{xxx}\phi^{j-5}$$
$$+ \sum_{j=1}^{\infty} 3(j-3)(j-4) u_{j-1}\phi_x \phi_{xx}\phi^{j-5} + \sum_{j=2}^{\infty} 3(j-4) u_{j-2,x}\phi_{xx}\phi^{j-5}$$
$$+ \sum_{j=2}^{\infty} (j-4) u_{j-2}\phi_t \phi^{j-5} + \sum_{j=0}^{\infty} (j-2)(j-3)(j-4) u_j \phi_x^3 \phi^{j-5} = 0, \tag{3.19}$$

To find u_0, then at $j = 0$ in the equation (3.19), we get:

$$u_0 = -\frac{12}{\alpha}\phi_x^2, \tag{3.20}$$

To find u_1, then at $j = 1$ in the equation (3.19), we get:

$$u_1 = \frac{12}{\alpha}\phi_{xx}, \tag{3.21}$$

To find u_2, then at $j = 2$ in the equation (3.19), we get:

$$u_2 = -\frac{1}{\alpha}\frac{\phi_t}{\phi_x} - \frac{4}{\alpha}\frac{\phi_{xxx}}{\phi_x} + \frac{3}{\alpha}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{3.22}$$

Since $p = 2$, by using the technique of truncation, and let $u_j = 0$, for all $j > 2$. Then the series solution $u = \sum_{j=0}^{\infty} u_j \phi^{j-2}$, becomes:

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{3.23}$$

By substituting the equations (3.20) and (3.21), into the equation (3.23), we get:

$$u = u_2 + \frac{12}{\alpha}\frac{d^2}{dx^2}\ln(\phi),$$

This is the relation between u and u_2.

Now, in the equation (3.23), we have to find all coefficients of u_j, where $u_j \equiv 0$ for all $j < 0$.

$$If: \;\; i = 0 \Rightarrow \alpha \sum_{i=0}^{j} u_{j-i} u_i (i-2)\phi_x = 2\alpha\phi_x^3 u_j,$$

and,

$$If: \;\; i = j \Rightarrow \alpha \sum_{i=0}^{j} u_{j-i} u_i (i-2)\phi_x = -\alpha\phi_x^3 (j-2) u_j,$$

Thus, the recursion relation is:

$$(j-4)[j^2 - 5j + (6-\alpha)]\phi_x^3 u_j = -u_{j-3,t} - (j-4)u_{j-2}\phi_t - 3(j-4)u_{j-2,x}\phi_{xx}$$

$$-\alpha \sum_{i=1}^{j-1} u_{j-i} u_i (i-2)\phi_x - u_{j-3,xxx} - 3(j-4)u_{j-2,xx}\phi_x - \alpha \sum_{k=0}^{j-1} u_k u_{j-1-k,x}$$

$$-3(j-3)(j-4)u_{j-1,x}\phi_x^2 - (j-4)u_{j-2}\phi_{xxx} - 3(j-3)(j-4)u_{j-1}\phi_x\phi_{xx}, \qquad (3.24)$$

We note that the coefficients of u_j in the equation (3.24) are $(j-4)$ and $[j^2 - 5j + (6-\alpha)]$, then, in the general of the integer resonance point is $j = 4$. The other resonance points depend on the value of α. For example, if $\alpha = 6$, the resonance points are $j = 0, 4, 5$, and if $\alpha = 12$, the resonance points are $j = -1, 4, 6$ Mostafa [24].

Now at $j = 3$ in the equation (3.24) and using the equations (3.20) and (3.21), we have,

$$u_3 = \frac{1}{\alpha}\frac{\phi_{xt}}{\phi_x^2} + \frac{\phi_{xx}u_2}{\phi_x^2} + \frac{1}{\alpha}\frac{\phi_{xxxx}}{\phi_x^2}, \qquad (3.25)$$

When $j = 4$ in the equation (3.24), since $u_j = 0$ for all $j > 2$, we get $u_4 = 0$.
Then the (KdV1) equation satisfies the Painlevé's property.
When $j = 5$ in the equation (3.24), since $u_j = 0$ for all $j > 2$, we get,

$$u_{2,t} + \alpha u_2 u_{2,x} + u_{2,xxx} = 0, \qquad (3.26)$$

Then u_2 is also a solution of the KdV equation (3.15).

3.3.2 Analytic Solution

In this section, we follow the idea to derive analytic solution. They are invariant under this transformation,

$$H:\phi \longrightarrow \frac{a\phi+b}{c\phi+d} \quad \text{where} \quad ad-bc\neq 0,$$

They are the Schwartzian derivative

$$S(\phi)=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{3.27}$$

and dimension of velocity,

$$C(\phi)=-\frac{\phi_t}{\phi_x}, \tag{3.28}$$

Furthermore, we define,

$$L(\phi)=-\frac{\phi_{xx}}{2\phi_x}, \tag{3.29}$$

The compatibility of S and C given by,

$$S_t+C_{xxx}+2C_xS+CS_x=0, \tag{3.30}$$

Now, by using the equations (3.22) and (3.25), we obtain:

$$\alpha\phi_x u_3=\frac{\phi_{xt}}{\phi_x}+\frac{\alpha\phi_{xx}}{\phi_x}\left(-\frac{1}{\alpha}\frac{\phi_t}{\phi_x}-\frac{4}{\alpha}\frac{\phi_{xxx}}{\phi_x}+\frac{3}{\alpha}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right)+\frac{\phi_{xxxx}}{\phi_x},$$

Since, $u_j=0$ for all $j>2$, we get:

$$\frac{\phi_t\phi_{xx}}{\phi_x^2}-\frac{\phi_{xt}}{\phi_x}=\frac{\phi_{xxxx}}{\phi_x}-\frac{4\phi_{xx}\phi_{xxx}}{\phi_x^2}+3\left(\frac{\phi_{xx}}{\phi_x}\right)^3, \tag{3.31}$$

Then, by comparing both sides of the equation (3.31) with equations (3.27) and (3.28), we observe:

$$C_x=S_x. \tag{3.32}$$

Now, by using the equations (3.27), (3.28) and (3.29), then the equation (3.22), becomes:

$$u_2=\frac{1}{\alpha}C-\frac{4}{\alpha}S-\frac{12}{\alpha}L^2 \tag{3.33}$$

We derive the equation (3.33), to find $u_{2,t}, \alpha u_2u_{2,x}$ and $u_{2,xxx}$ and substitute them into the equation (3.26), then:

$$\frac{1}{\alpha}C_t - \frac{4}{\alpha}S_t - \frac{24}{\alpha}LL_t - \frac{1}{\alpha}CC_x - \frac{4}{\alpha}CS_x - \frac{24}{\alpha}CLL_x - \frac{4}{\alpha}SC_x + \frac{16}{\alpha}SS_x$$
$$+\frac{96}{\alpha}SLL_x - \frac{12}{\alpha}L^2C_x + \frac{48}{\alpha}L^2S_x + \frac{288}{\alpha}L^3L_x + \frac{1}{\alpha}C_{xxx} - \frac{4}{\alpha}S_{xxx}$$
$$-\frac{24}{\alpha}LL_{xxx} - \frac{72}{\alpha}L_xL_{xx} = 0, \tag{3.34}$$

To eliminate L, by using the relations, $L_t = -CL_x - LC_x + \frac{1}{2}C_{xx}$ and $L_x = -L^2 - \frac{1}{2}S$, and by the equation (3.32) then, $L^2C_x - L^2S_x = 0$ and $LS_{xx} - LC_{xx} = 0$. Then the equation (3.34), becomes:

$$C_t - 4S_t - 3CC_x - 6SC_x - 3C_{xxx} = 0, \tag{3.35}$$

and by substituting S_t in the equation (3.30) into the equation (3.35), we get:

$$C_t - 4(-C_{xxx} - 2C_xS - CC_x) - 3CC_x - 6SC_x - 3C_{xxx} = 0,$$

leads to:

$$C_t + C_{xxx} + 2C_xS + CC_x = 0. \tag{3.36}$$

By comparing the equations (3.30) with (3.36), and using the equation (3.32), we get:

$$C_t = S_t, \tag{3.37}$$

Then by the equations (3.32) and (3.37), we get, $C = S + K$ where K is constant.

For $K = 0$, we get:

$$C = S, \tag{3.38}$$

By substituting $C = S$ into the equation (3.35), we get:

$$C_t + 3CC_x + C_{xxx} = 0,$$

Or otherwise,

$$S_t + 3SS_x + S_{xxx} = 0, \tag{3.39}$$

This is Korteweg-de Vries like equation KdV Mostafa [24].

3.3.3 Exact Solution

Solution for constant S.

The constant functions $S = \pm 2\lambda^2$ where λ is constant, are solutions of the Korteweg-de Vries like equation (3.39) Mostafa [24].

$$\left\{\frac{aW+b}{cW+d};z\right\} = \{W;z\} \quad \text{where} \quad ad-bc \neq 0,$$

where a,b,c and d are constants.

Case A:

For $S = -2\lambda^2$, we have,

$S = \{\phi, x\} = -2\lambda^2$. Hence $P(x) = -\lambda^2$ in (3.13), and two linearly independent solutions are:

$$\Psi_1 = E(t)e^{\lambda x} + F(t)e^{-\lambda x} \quad , \quad \Psi_2 = G(t)e^{\lambda x} + H(t)e^{-\lambda x}$$

Therefore by **Lemma 1** and **Lemma 2**, we obtain:

$$\phi(t,x) = \frac{E(t)e^{\lambda x} + F(t)e^{-\lambda x}}{G(t)e^{\lambda x} + H(t)e^{-\lambda x}} \quad \text{where} \quad EH - FG \neq 0, \tag{3.40}$$

By using the equations (3.28) and (3.38), then:

$$C = S = -\frac{\phi_t}{\phi_x} = -2\lambda^2, \tag{3.41}$$

Now, to find the equation of coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, we derive $\phi(t,x)$ in the equation (3.40), once respect to t and once respect to x and substituting them into the equation(3.41), we obtain:

$$\phi_t = \frac{[G(t)E'(t) - E(t)G'(t)]e^{2\lambda x} + [H(t)F'(t) - F(t)H'(t)]e^{-2\lambda x}}{\left[G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right]^2}$$

$$+\frac{G(t)F'(t) - F(t)G'(t) + H(t)E'(t) - E(t)H'(t)}{\left[G(t)e^{\lambda x} + H(t)e^{-\lambda x}\right]^2},$$

and,

$$\phi_x = \frac{2\lambda[H(t)E(t) - G(t)F(t)]}{[G(t)e^{\lambda x} + H(t)e^{-\lambda x}]^2},$$

Then, the equation (3.41) becomes:

$$C = \frac{[G(t)E^{'}(t) - E(t)G^{'}(t)]e^{2\lambda x} + [H(t)F^{'}(t) - F(t)H^{'}(t)]e^{-2\lambda x}}{-2\lambda[H(t)E(t) - G(t)F(t)]}$$

$$+ \frac{G(t)F^{'}(t) - F(t)G^{'}(t) + H(t)E^{'}(t) - E(t)H^{'}(t)}{-2\lambda[H(t)E(t) - G(t)F(t)]} = -2\lambda^2.$$

Then,

$$(G(t)E^{'}(t) - E(t)G^{'}(t))e^{2\lambda x} + (H(t)F^{'}(t) - F(t)H^{'}(t))e^{-2\lambda x} + G(t)F^{'}(t)$$

$$-F(t)G^{'}(t) + H(t)E^{'}(t) - E(t)H^{'}(t) = 4\lambda^3(H(t)E(t) - G(t)F(t)).$$

This leads to a system of nonlinear ordinary differential equation in coefficients $E(t), F(t), G(t)$ and $H(t)$, then:

(a) $GE^{'} - EG^{'}$

(b) $HF^{'} - FH^{'}$

(c) $GF^{'} - GF^{'} + HE^{'} - EH^{'} = 4\lambda^3(HE - GF)$

Particular solutions of (a) and (b) are:

$$E(t) = AG(t) \quad \text{and} \quad F(t) = BH(t)$$

where A and B are real arbitrary constants. By substituting these into (c), we get:

$$B(G(t)H^{'}(t) - H(t)G^{'}(t)) + A(H(t)G^{'}(t) - G(t)H^{'}(t)) = 4\lambda^3 H(t)G(t)(A - B),$$

then:

$$\frac{H^{'}(t)}{H(t)} - \frac{G^{'}(t)}{G(t)} = -4\lambda^3,$$

By integrating the above, we get:

$$\frac{H(t)}{G(t)} = \exp(-4\lambda^3 t),$$

Then the equation (3.40), becomes:

$$\phi(t,x) = \frac{AG(t)\exp(\lambda x) + BG(t)\exp(-4\lambda^3 t - \lambda x)}{G(t)\exp(\lambda x) + G(t)\exp(-4\lambda^3 t - \lambda x)},$$

Which leads to:

$$\phi(t,x) = \frac{Ae^{\lambda\xi_1} + Be^{-\lambda\xi_1}}{e^{\lambda\xi_1} + e^{-\lambda\xi_1}}, \quad \text{where} \quad \xi_1 = x + 2\lambda^2 t$$

$$= \frac{(A+B)\cosh\lambda\xi_1 + (A-B)\sinh\lambda\xi_1}{2\cosh\lambda\xi_1}$$

Then:

$$\phi(t,x) = K_1 + K_2\tanh\lambda\xi_1, \tag{3.42}$$

where K_1 and K_2 are arbitrary constants, and $K_1 = (A+B)/2$ and $K_2 = (A-B)/2$. For $K_1 = 0$, and by substituting the equation (3.42) into the equation (3.22), we obtain:

$$u_2 = -\frac{1}{\alpha}\frac{2K_2\lambda^3\text{sech}^2\lambda\xi_1}{K_2\lambda\,\text{sech}^2\lambda\xi_1} - \frac{4}{\alpha}\frac{-2K_2\lambda^3\text{sech}^4\lambda\xi_1 + 4K_2\lambda^3\text{sech}^2\lambda\xi_1\tanh^2\lambda\xi_1}{K_2\lambda\,\text{sech}^2\lambda\xi_1}$$

$$+\frac{3}{\alpha}\frac{4K_2^2\lambda^4\text{sech}^4\lambda\xi_1\tanh^2\lambda\xi_1}{K_2^2\lambda^2\text{sech}^4\lambda\xi_1}$$

Then:

$$u_2 = \frac{12\lambda^2}{\alpha}\left(\text{sech}^2\lambda\xi_1 - \frac{1}{2}\right), \text{ where } \xi_1 = x + 2\lambda^2 t,$$

Hence $u_2(t,x)$ is the first exact solution for KdV equation (3.15).

Now, by the equations (3.20), (3.21), (3.23) and (3.42), we obtain:

$$u = \frac{-12}{\alpha}\frac{\phi_x^2}{\phi^2} + \frac{12}{\alpha}\frac{\phi_{xx}}{\phi} + u_2,$$

$$= \frac{-12}{\alpha}\frac{K_2^2\lambda^2\text{sech}^4\lambda\xi_1}{K_2^2\tanh^2\lambda\xi_1} - \frac{24}{\alpha}\frac{K_2\lambda^2\text{sech}^2\lambda\xi_1\tanh\lambda\xi_1}{K_2\tanh\lambda\xi_1} + u_2,$$

$$= \frac{-12\lambda^2}{\alpha}\text{sech}^2\lambda\xi_1\left(\text{csech}^2\lambda\xi_1 + 2\right) + u_2,$$

Then:

$$u = -\frac{12\lambda^2}{\alpha}\left(\text{csech}^2\lambda\xi_1 + \frac{1}{2}\right), \quad \text{where} \quad \xi_1 = x + 2\lambda^2 t,$$

Hence $u(t,x)$ is the second exact solution for KdV equation (3.15) Mostafa [24].

Case B:

For $S = 2\lambda^2$, we have:

$S = \{\phi, x\} = 2\lambda^2$. Hence $P(x) = -\lambda^2$ in (3.13), and two linearly independent solutions are:

$$\Psi_3 = E(t)e^{\lambda ix} + F(t)e^{-\lambda ix} \quad , \quad \Psi_4 = G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}$$

Therefore, from **Lemma 1** and **Lemma 2**, obtain:

$$\phi(t,x) = \frac{E(t)e^{\lambda ix} + F(t)e^{-\lambda ix}}{G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}} \quad \text{where} \quad EH - FG \neq 0, \tag{3.43}$$

By using the equations (3.28) and (3.38), then:

$$C = S = -\frac{\phi_t}{\phi_x} = 2\lambda^2. \tag{3.44}$$

Now to find the equation of coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, we derive $\phi(t,x)$ in the equation (3.43), once respect to t and once respect to x and by substituting them into the equation(3.44), we obtain:

$$C = \frac{[G(t)E^{'}(t) - E(t)G^{'}(t)]e^{2\lambda ix} + [H(t)F^{'}(t) - F(t)H^{'}(t)]e^{-2\lambda ix}}{-2i\lambda[H(t)E(t) - G(t)F(t)]}$$

$$+ \frac{G(t)F^{'}(t) - F(t)G^{'}(t) + H(t)E^{'}(t) - E(t)H^{'}(t)}{-2i\lambda[H(t)E(t) - G(t)F(t)]} = 2\lambda^2.$$

This leads to a system of nonlinear ordinary differential equations with coefficients $E(t), F(t), G(t)$ and $H(t)$, then

(a) $GE^{'} - EG^{'}$

(b) $HF^{'} - FH^{'}$

(c) $GF^{'} - GF^{'} + HE^{'} - EH^{'} = -4i\lambda^3(HE - GF)$

Particular solutions of (a) and (b) are:

$$E(t) = MG(t) \quad \text{and} \quad F(t) = NH(t)$$

where M and N are real arbitrary constants.

By substituting these into (c), we get:

$$\frac{H(t)}{G(t)} = \exp(4i\lambda^3 t),$$

Then the equation (3.43), becomes:

$$\phi(t,x) = \frac{MG(t)\exp(\lambda ix) + NG(t)\exp(4\lambda^3 it - \lambda ix)}{G(t)\exp(\lambda ix) + G(t)\exp(4\lambda^3 it - \lambda ix)},$$

Which leads to:

$$\phi(t,x) = \frac{Me^{\lambda i\xi_2} + Ne^{-\lambda i\xi_2}}{e^{\lambda i\xi_2} + e^{-\lambda i\xi_2}}, \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t$$

$$= \frac{(M+N)\cos\lambda\xi_2 + (M-N)\sin\lambda\xi_2}{2\cos\lambda\xi_2}$$

Then:

$$\phi(t,x) = K_3 + K_4 \tan\lambda\xi_2, \tag{3.45}$$

where $K_3 = (M+N)/2$ and $K_4 = (M-N)/2$ are arbitrary constants.

For $K_3 = 0$, by substituting the equation (3.45) into the equation (3.22), we get:

$$\widehat{u}_2 = \frac{2}{\alpha}\frac{K_4\lambda^3\sec^2\lambda\xi_2}{K_4\lambda\sec^2\lambda\xi_2} - \frac{4}{\alpha}\frac{2K_4\lambda^3\sec^4\lambda\xi_2 + 4K_4\lambda^3\sec^2\lambda\xi_2\tan^2\lambda\xi_2}{K_4\lambda\sec^2\lambda\xi_2}$$

$$+\frac{3}{\alpha}\frac{4K_4^2\lambda^4\sec^4\lambda\xi_2\tan^2\lambda\xi_2}{K_4^2\lambda^2\sec^4\lambda\xi_2},$$

Then:

$$\widehat{u}_2 = -\frac{12\lambda^2}{\alpha}\left(\sec^2\lambda\xi_2 - \frac{1}{2}\right), \text{ where } \xi_2 = x - 2\lambda^2 t,$$

Hence $\widehat{u}_2(t,x)$ is the third exact solution for KdV equation (3.15).

Now, by the equations $(3.20), (3.21), (3.23)$ and (3.45), we get:

$$\widehat{u} = \frac{-12}{\alpha}\frac{\phi_x^2}{\phi^2} + \frac{12}{\alpha}\frac{\phi_{xx}}{\phi} + \widehat{u}_2,$$

Then:

$$\widehat{u} = -\frac{12\lambda^2}{\alpha}\left(\csc^2\lambda\xi_2 - \frac{1}{2}\right), \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t,$$

Hence $\widehat{u}(t,x)$ is the fourth exact solution for KdV equation (3.15).

3.4 Painlevé Analysis for the modified KdV Equation

3.4.1 Painlevé Property

In the present section we apply Painlevé equation in the modified KdV equation [23].

$$u_t + \alpha u^n u_x + \beta u_{xxx} = 0, \tag{3.46}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta = 1$ and $n = 2$.

The series solution of the partial differential equation is in the form:

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-p}, \tag{3.47}$$

where ϕ is an analytic function that defines a non-characteristic hypersurface S. [5] To determine whether equation (3.46) satisfies Painlevé property we use simplified condition Steep and Euler [30].

$$\phi(t,x) = x + \psi(t) = 0, \tag{3.48}$$

where ψ is an arbitrary function (ϕ is a characteristic of (3.46) if $\partial\phi/\partial x \neq 0$). To find a value of equilibrium point p, by substituting (3.47) into the equation (3.46), where $u_t(t,x) = \partial u(t,x)/\partial t$, $u_x(t,x) = \partial u(t,x)/\partial x$ and $u_{xxx}(t,x) = \partial^3 u(t,x)/\partial x^3$ and by comparing the lowest powers in the produced series, we find $p = 1$. In the neighborhood of the singularity manifold (3.48), the series solution (3.47) will be in the form:

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-1}, \tag{3.49}$$

where $u_0, u_1, \ldots,$ are arbitrary functions. By associating the summation Mostafa [23], we get:

$$\sum_{j=3}^{\infty} u_{j-3,t}\phi^{j-4} + \sum_{j=2}^{\infty} (j-3)u_{j-2}\phi_t\phi^{j-4}$$
$$+\alpha \sum_{j=0}^{\infty} \left[\sum_{k=0}^{j} \sum_{i=0}^{k} u_{j_k} u_{k-i} u_i (j-k-1)\phi_x + \sum_{k=0}^{j-1} \sum_{i=0}^{k} u_{j-k-1,x} u_{k-i} u_i \right] \phi^{j-4}$$

[5] This is simply an extension of the partial differntial equation (3.17) which we have studied in section 3.3.1

$$+\sum_{j=3}^{\infty} u_{j-3,xxx}\phi^{j-4}+\sum_{j=2}^{\infty} 3(j-3)u_{j-2,xx}\phi_x\phi^{j-4}+\sum_{j=2}^{\infty} 3(j-3)u_{j-2,x}\phi_{xx}\phi^{j-4}$$
$$+\sum_{j=1}^{\infty} 3(j-2)(j-3)u_{j-1,x}\phi_x^2\phi^{j-4}+\sum_{j=1}^{\infty} 3(j-2)(j-3)u_{j-1}\phi_x\phi_{xx}\phi^{j-4}$$
$$+\sum_{j=2}^{\infty}(j-3)u_{j-2}\phi_{xxx}\phi^{j-4}+\sum_{j=0}^{\infty}(j-1)(j-2)(j-3)u_j\phi_x^3\phi^{j-4}=0, \tag{3.50}$$

To acquire u_0, then at $j=0$ in the equation (3.50), we obtain:

$$u_0 = i\sqrt{\frac{6}{\alpha}}\,\phi_x, \tag{3.51}$$

To acquire u_1, then at $j=1$ in the equation (3.50), we obtain:

$$u_1 = -\frac{i}{2}\sqrt{\frac{6}{\alpha}}\frac{\phi_{xx}}{\phi_x}, \tag{3.52}$$

Since $p=1$, by using the technique of amputating, and let $u_j \equiv 0$, for all $j>1$. Then the series solution $u=\sum_{j=0}^{\infty} u_j\phi^{j-1}$, becomes:

$$u = \frac{u_0}{\phi}+u_1. \tag{3.53}$$

To acquire u_2, then at $j=2$ in the equation (3.50), we obtain:

$$u_2 = \frac{i}{\sqrt{6\alpha}\phi_x}\left[\frac{\phi_t+\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right], \tag{3.54}$$

Now, in the equation (3.50), we must find all coefficients of u_j, where $u_j \equiv 0$ for all $j<0$.

If $k=0$:

$$\Rightarrow \alpha\phi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k} u_{k-i}u_i\right](j-k-1)u_{j-k} \;=\alpha\phi_x u_0^2(j-1)u_j.$$

If $i=k$:

$$\Rightarrow \alpha\phi_x\sum_{k=0}^{j}\left[\sum_{i=0}^{k} u_{k-i}u_i\right](j-k-1)u_{j-k} \;=-\alpha\phi_x u_0^2 u_j.$$

If $k = j$:

$$\Rightarrow \alpha\phi_x \sum_{k=0}^{j} \left[\sum_{i=0}^{k} u_{k-i}u_i\right] (j-k-1)u_{j-k} \quad = -\alpha\phi_x u_0^2 u_j.$$

The relation becomes:

$$(j-3)\left[j-(\frac{3}{2}\pm\sqrt{\frac{1}{4}-\alpha})\right]\phi_x^3 u_j = -u_{j-3,t} + \alpha\phi_x u_0 \sum_{i=1}^{j-1} u_{j-i}u_i - (j-3)u_{j-2}\phi_t,$$

$$-\alpha\sum_{k=1}^{j-1}\left[\sum_{i=0}^{k} u_{k-i}u_i\right] u_{j-k}(j-k-1)\phi_x \; - \alpha\sum_{k=0}^{j-1}\left[\sum_{i=0}^{k} u_{k-i}u_i\right] u_{j-k-1,x} - u_{j-3,xxx},$$

$$-3(j-3)u_{j-2,x}\phi_{xx} - (j-3)u_{j-2}\phi_{xxx} - 3(j-2)(j-3)\left[u_{j-1,x}\phi_x^2 + u_{j-1}\phi_x\phi_{xx}\right],$$

$$-3(j-3)u_{j-2,xx}\phi_x \;. \tag{3.55}$$

We observe that the all coefficients of u_j in the equation (3.55) are $(j-3)$ and $\left[j-(\frac{3}{2}\pm\sqrt{\frac{1}{4}-\alpha})\right]$, then, in the generic of the integer resonance point is $j = 3$. The other resonance points are contingent on the value of α. For instance, if $\alpha = -6$, the resonance points are $j = -1, 3, 4$.

Now, at $j = 3$, and by using the equations (3.51), (3.52) and (3.55), we get,

$$-u_{0,t} - u_{0,xxx} + 2\alpha\phi_x^2 u_1 u_2 - 2\alpha\phi_x u_0 u_1 u_2 + \alpha u_0^2 u_{2,x} - 2\alpha\phi_x u_0 u_1 u_{1,x}$$

$$-\alpha u_{0,x}u_1^2 - 2\alpha u_0 u_2 u_{0,x} = 0,$$

but, $u_j \equiv 0$ for all $j > 1$, we get.

$$u_{0,t} + u_{0,xxx} + 2u_0 u_1 u_{1,x} + \alpha u_{0,x}u_1^2 = 0,$$

Inconsistent at the resonance point $j = 3$, this means that the modified Korteweg-de Vries equation (3.46), does not satisfy the Painlevé's property.

Now, at $j = 4$ in the equation (3.55), we have,

$$-u_{1,t} - u_{1,xxx} - \phi_t\phi_2 - 3\phi_x u_{2,xx} - 3\phi_{xx}u_{2,x} - \phi_{xxx}u_2 - 6\phi_x^2 u_{3,x} - 6\phi_x\phi_{xx}u_3$$

$$+\alpha\phi_x^3 \sum_{k=1}^{3} u_{4-i}u_i - \alpha\phi_x \sum_{k=1}^{3}\left[\sum_{i=0}^{k} u_{k-i}u_i\right](3-k)u_{4-k} + \alpha\sum_{k=0}^{3}\left[\sum_{i=0}^{k} u_{k-i}u_i\right] u_{3-k,x} = 0, \tag{3.56}$$

By implementing the equation (3.51) into the equation (3.56), and $u_j \equiv 0$ for all $j > 1$, we get,

$$u_{1,t} + \alpha u_1^2 u_{1,x} + u_{1,xxx} = 0, \tag{3.57}$$

Then u_1 is also a solution of the modified KdV equation (3.46).

3.4.2 Analytic Solution

In this section, we pursue the project to derive analytic solution. They are unvarying under this transmutation,

$$T : \phi \longrightarrow \frac{a\phi + b}{c\phi + d} \quad \text{where} \quad ad \neq bc,$$

The Schwartzian derivative.

$$S(\phi) = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{3.58}$$

The dimension of velocity,

$$C(\phi) = -\frac{\phi_t}{\phi_x}, \tag{3.59}$$

The compatibility of C and S described by:

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \tag{3.60}$$

By comparing the equations (3.58) and (3.59) with the equation (3.54), and, $u_j \equiv 0$ for all $j > 1$, we observe:

$$C = S, \tag{3.61}$$

By substituting $S = C$ into the equation (3.51), we get:

$$S_t + 3SS_x + S_{xxx} = 0, \tag{3.62}$$

This is Korteweg-de Vries(KdV) like equation.

3.4.3 Exact Solution

Solution for constant S.

The functions of constant $S = \pm 2\lambda^2$ where λ is a constant, are solutions of the Korteweg-de Vries like equation (3.62).

The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, the form:

Case A:

For $S=-2\lambda^2$, we get,
$S=\{\phi,x\}=-2\lambda^2$. Hence $f(x)=-\lambda^2$ in (3.13), and two linearly independent solutions are:

$$\Phi_1=E(t)e^{\lambda x}+F(t)e^{-\lambda x} \quad , \quad \Phi_2=G(t)e^{\lambda x}+H(t)e^{-\lambda x}$$

Therefore by **Lemma 1** and **Lemma 2**, obtain:

$$\phi(t,x)=\frac{E(t)e^{\lambda x}+F(t)e^{-\lambda x}}{G(t)e^{\lambda x}+H(t)e^{-\lambda x}} \quad \text{where} \quad EH\neq FG, \tag{3.63}$$

By using the equations (3.59) and (3.61), then:

$$C=S=-\frac{\phi_t}{\phi_x}=-2\lambda^2, \tag{3.64}$$

Now, to find the differential equation of coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, we derive $\phi(t,x)$ in the equation (3.63), to get $\phi_t(t,x)$ and $\phi_x(t,x)$, and substituting them into the equation (3.64), we obtain:

$$\phi_t=\frac{[G(t)E^{'}(t)-E(t)G^{'}(t)]e^{2\lambda x}+[H(t)F^{'}(t)-F(t)H^{'}(t)]e^{-2\lambda x}}{\left[G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right]^2}$$

$$+\frac{G(t)F^{'}(t)-F(t)G^{'}(t)+H(t)E^{'}(t)-E(t)H^{'}(t)}{\left[G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right]^2},$$

and,

$$\phi_x=\frac{2\lambda[H(t)E(t)-G(t)F(t)]}{[G(t)e^{\lambda x}+H(t)e^{-\lambda x}]^2},$$

Then, the equation (3.64) becomes:

$$C=\frac{[G(t)E^{'}(t)-E(t)G^{'}(t)]e^{2\lambda x}+[H(t)F^{'}(t)-F(t)H^{'}(t)]e^{-2\lambda x}}{-2\lambda[H(t)E(t)-G(t)F(t)]}$$

$$+\frac{G(t)F^{'}(t)-F(t)G^{'}(t)+H(t)E^{'}(t)-E(t)H^{'}(t)}{-2\lambda[H(t)E(t)-G(t)F(t)]}=-2\lambda^2.$$

Then,

$$(G(t)E^{'}(t)-E(t)G^{'}(t))e^{2\lambda x}+(H(t)F^{'}(t)-F(t)H^{'}(t))e^{-2\lambda x}+G(t)F^{'}(t)$$

$$-F(t)G^{'}(t)+H(t)E^{'}(t)-E(t)H^{'}(t)=4\lambda^{3}(H(t)E(t)-G(t)F(t)).$$

This takes us to a system of nonlinear ordinary differential equation in all coefficients $E(t), F(t), G(t)$ and $H(t)$, then:

(a) $GE^{'}-EG^{'}$

(b) $HF^{'}-FH^{'}$

(c) $GF^{'}-GF^{'}+HE^{'}-EH^{'}=4\lambda^{3}(HE-GF)$

particular solutions of (a) and (b) respectively are:

$$E(t)=AG(t) \quad \text{and} \quad F(t)=BH(t)$$

where A and B are real arbitrary constants. By using (a), (b) and (c), we get:

$$B(G(t)H^{'}(t)-H(t)G^{'}(t))+A(H(t)G^{'}(t)-G(t)H^{'}(t))=4\lambda^{3}H(t)G(t)(A-B),$$

then:

$$\frac{H^{'}(t)}{H(t)}-\frac{G^{'}(t)}{G(t)}=-4\lambda^{3},$$

By integrating, we get:

$$\frac{H(t)}{G(t)}=\exp(-4\lambda^{3}t),$$

then the equation (3.63), becomes:

$$\phi(t,x)=\frac{AG(t)\exp(\lambda x)+BG(t)\exp(-4\lambda^{3}t-\lambda x)}{G(t)\exp(\lambda x)+G(t)\exp(-4\lambda^{3}t-\lambda x)},$$

which leads to:

$$\phi(t,x)=\frac{Ae^{\lambda\xi_1}+Be^{-\lambda\xi_1}}{e^{\lambda\xi_1}+e^{-\lambda\xi_1}}, \quad \text{where} \quad \xi_1=x+2\lambda^{2}t$$

$$=\frac{(A+B)\cosh\lambda\xi_1+(A-B)\sinh\lambda\xi_1}{2\cosh\lambda\xi_1}.$$

Then:

$$\phi(t,x) = K_1 + K_2 \tanh \lambda \xi_1, \tag{3.65}$$

where K_1 and K_2 are arbitrary constants, and $K_1 = \frac{A+B}{2}$ and $K_2 = \frac{A-B}{2}$.

For $K_1 = 0$, and by substituting the equation (3.65) into the equation (3.52), we obtain:

$$u_1 = -i\sqrt{\frac{6}{\alpha}} \frac{-K_2\lambda^2 \text{sech}^2\lambda\xi_1 \tanh\lambda\xi_1}{K_2\lambda \text{sech}^2\lambda\xi_1}$$

Then:

$$u_1 = i\sqrt{\frac{6}{\alpha}}\, \lambda \, \tanh\lambda\xi_1 \quad \text{where} \quad \xi_1 = x + 2\lambda^2 t.$$

Hence $u_1(t,x)$ is the first exact solution for modified KdV equation (3.46).

Now, by the equations (3.51) , (3.52) , (3.53) and (3.65), we obtain:

$$u = \frac{i\sqrt{\frac{6}{\alpha}}\, K_2\lambda \, \text{sech}^2\lambda\xi_1}{K_2\lambda \, \tanh\lambda\xi_1} + u_1,$$

Then:

$$u = i\lambda\sqrt{\frac{6}{\alpha}} \, \coth\lambda\xi_1, \quad \text{where} \quad \xi_1 = x + 2\lambda^2 t,$$

Hence $u(t,x)$ is the second exact solution for modified KdV equation (3.46).

Case B:

For $S = 2\lambda^2$, we get:
$S = \{\phi, x\} = 2\lambda^2$. Hence $f(x) = -\lambda^2$ in (3.13), and two linearly independent solutions are:

$$\Phi_3 = E(t)e^{\lambda ix} + F(t)e^{-\lambda ix} \quad , \quad \Phi_4 = G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}$$

Therefore, **Lemma 1** and **Lemma 2** obtain:

$$\phi(t,x) = \frac{E(t)e^{\lambda ix} + F(t)e^{-\lambda ix}}{G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}} \quad \text{where} \quad EH \neq FG, \tag{3.66}$$

By using the equations (3.59) and (3.61), then:

$$C = S = -\frac{\phi_t}{\phi_x} = 2\lambda^2. \tag{3.67}$$

Now, to find the differential equation of coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, we derive $\phi(t,x)$ in the equation (3.66), to get $\phi_t(t,x)$ and $\phi_x(t,x)$, and by substituting them into the equation (3.67), we obtain:

$$C = \frac{[G(t)E^{'}(t) - E(t)G^{'}(t)]e^{2\lambda ix} + [H(t)F^{'}(t) - F(t)H^{'}(t)]e^{-2\lambda ix}}{-2i\lambda[H(t)E(t) - G(t)F(t)]}$$

$$+ \frac{G(t)F^{'}(t) - F(t)G^{'}(t) + H(t)E^{'}(t) - E(t)H^{'}(t)}{-2i\lambda[H(t)E(t) - G(t)F(t)]} = 2\lambda^2.$$

This takes us to a system of nonlinear ordinary differential equations in all coefficients $E(t), F(t), G(t)$ and $H(t)$, then:

(a) $GE^{'} - EG^{'}$

(b) $HF^{'} - FH^{'}$

(c) $GF^{'} - GF^{'} + HE^{'} - EH^{'} = -4i\lambda^3(HE - GF)$

particular solutions of (a) and (b) respectively are:

$$E(t) = MG(t) \quad \text{and} \quad F(t) = NH(t)$$

where M and N are real arbitrary constants.

By substituting these into (c), we get:

$$\frac{H(t)}{G(t)} = \exp(4i\lambda^3 t),$$

Then the equation (3.66), becomes:

$$\phi(t,x) = \frac{MG(t)\exp(\lambda ix) + NG(t)\exp(4\lambda^3 it - \lambda ix)}{G(t)\exp(\lambda ix) + G(t)\exp(4\lambda^3 it - \lambda ix)},$$

which leads to:

$$\phi(t,x) = \frac{Me^{\lambda i\xi_2} + Ne^{-\lambda i\xi_2}}{e^{\lambda i\xi_2} + e^{-\lambda i\xi_2}}, \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t$$

$$= \frac{(M+N)\cos\lambda\xi_2 + (M-N)\sin\lambda\xi_2}{2\cos\lambda\xi_2}.$$

Then:

$$\phi(t,x) = K_3 + K_4\tan\lambda\xi_2, \tag{3.68}$$

where K_3 and K_4 are arbitrary constants, and $K_3 = \frac{M+N}{2}$ and $K_4 = \frac{M-N}{2}$.
For $K_3 = 0$, by substituting the equation (3.68) into the equation (3.52), we get:

$$\widehat{u}_1 = -i\sqrt{\frac{6}{\alpha}}\,\frac{K_4\lambda^2 \sec^2\lambda\xi_2\,\tan\lambda\xi_2}{K_4\lambda\sec^2\lambda\xi_2}$$

Then:

$$\widehat{u}_1 = -i\lambda\sqrt{\frac{6}{\alpha}}\,\tan\lambda\xi_2 \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t$$

Hence $\widehat{u}_1(t,x)$ is the third exact solution for modified KdV equation (3.46).

Now, by the equations (3.51), (3.52), (3.53) and (3.68), we get:

$$\widehat{u} = \frac{i\sqrt{\frac{6}{\alpha}}\,K_4\lambda\sec^2\lambda\xi_2}{K_4\lambda\tan\lambda\xi_2} + \widehat{u}_1,$$

then:

$$\widehat{u} = i\lambda\sqrt{\frac{6}{\alpha}}\,\cot\lambda\xi_2, \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t$$

Hence $\widehat{u}(t,x)$ is the fourth exact solution for modified KdV equation (3.46).

Chapter 4

Numerical Solution of the KdV Equation

4.1 Introduction

The Korteweg-de Vries equation which is the nonlinear PDE of the third order is of the form:

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{4.1}$$

where α and β are positive parameters. $u(x,t)$ has described the elongation of the water wave at x distance and t time. the nonlinear term, $\alpha u u_x$ is like to the familiar wave equation cu_x term. this means that as long as u does not alteration too much, the wave diffuses with a speed symmetrical to αu. The nonlinear term $\alpha u u_x$ introduces the potentially of shock wave into the general solution. The third order part βu_{xxx} produces dispersive extending that can completely the narrowing occasioned by the nonlinear term Ali [2].
The Korteweg-de Vries equation can be solved numerically by Zabusky and Kruskal method, therefore, we using finite difference scheme $x = i\Delta x$ and $t = j\Delta t$, the grid of which in Figure (4.1).

The discrete variables the derivatives in the equation (4.1) are given by:

$$\begin{aligned}\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} &+ \frac{\alpha}{6\Delta x}(u_{i+1,j} + u_{i,j} + u_{i-1,j})(u_{i+1,j} - u_{i-1,j}) \\ &+ \frac{\beta}{2(\Delta x)^3}(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}) = 0,\end{aligned}$$

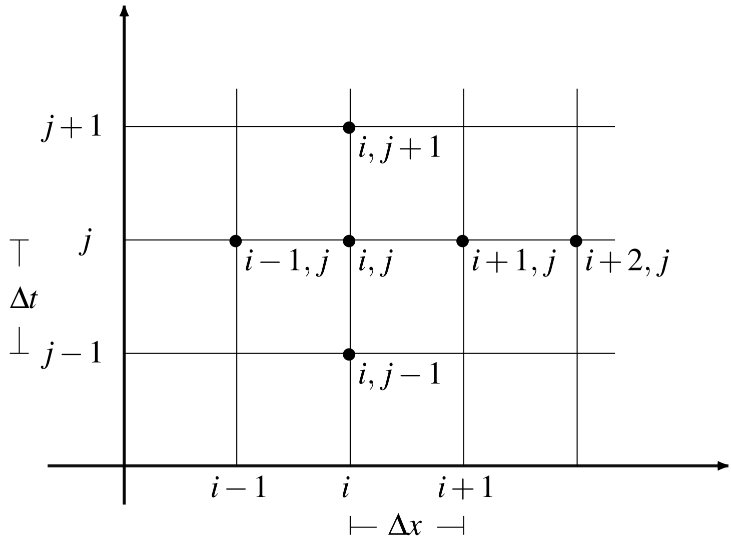


Fig. 4.1 Implicit scheme of the finite difference methods

then:

$$\begin{aligned} u_{i,j+1} &= u_{i,j-1} - \frac{\alpha\Delta t}{3\Delta x}(u_{i+1,j}+u_{i,j}+u_{i-1,j})(u_{i+1,j}-u_{i-1,j}) \\ &- \frac{\beta\Delta t}{(\Delta x)^3}(u_{i+2,j}-2u_{i+1,j}+2u_{i-1,j}-u_{i-2,j}). \end{aligned} \quad (4.2)$$

For numerical solution of KdV (4.2), the initial time step, where ($j=0$), we apply the forward difference scheme, we have avoid $u_{i,-1}$ in the time derivative. The discretized equation, becomes:

$$\begin{aligned} u_{i,1} &= u_{i,0} - \frac{\alpha\Delta t}{6\Delta x}(u_{i+1,0}+u_{i,0}+u_{i-1,0})(u_{i+1,0}-u_{i-1,0}) \\ &- \frac{\beta\Delta t}{(\Delta x)^3}(u_{i+2,0}-2u_{i+1,0}+2u_{i-1,0}-u_{i-2,0}). \end{aligned}$$

An implicit scheme (Goda scheme) for approximating the equation (4.2) was extended to the KdV equation for every value α and β.

$$\begin{aligned} \frac{u_{i,j+1}-u_{i,j}}{\Delta t} &+ \frac{\alpha}{6\Delta x}[u_{i+1,j+1}(u_{i,j}+u_{i+1,j}) - u_{i-1,j+1}(u_{i,j}+u_{i,j-1})] \\ &+ \frac{\beta}{2(\Delta x)^3}(u_{i+2,j+1}-2u_{i+1,j+1}+2u_{i-1,j+1}-u_{i-2,j+1}) = 0, \end{aligned}$$

Now, we can determine the system of linear equation to solve at every time step, by using iteratively ascending order from $(i=1)$ to $(i=m-1)$:

For $(i = 1)$**:**

$$[6(\Delta x)^3]u_{1,j+1} + [-(6\beta\Delta t - \delta\alpha(\Delta x)^2\Delta t)]u_{2,j+1} + [3\beta\Delta t]u_{3,j+1}$$
$$= [6(\Delta x)^3]u_{1,j} + [3\beta\Delta t]u_0 - [6\beta\Delta t - \gamma\alpha(\Delta x)^2\Delta t]u_0,$$
where $\delta = (u_{i,j} + u_{i+1,j})$ and $\gamma = (u_{i,j} + u_{i-1,j})$.

For $(i = 2)$**:**

$$[6\beta\Delta t - \gamma\alpha(\Delta x)^2\Delta t]u_{1,j+1} + [6(\Delta x)^3]u_{2,j+1} + [-(6\beta\Delta t - \delta\alpha(\Delta x)^2\Delta t)]u_{3,j+1}$$
$$+[3\beta\Delta t]u_{4,j+1} = [6(\Delta x)^3]u_{2,j} + [3\beta\Delta t]u_0$$

For $(3 < i < m-2)$**:**

$$-[3\beta\Delta t]u_{i-2,j+1} + [6\beta\Delta t - \gamma\alpha(\Delta x)^2\Delta t]u_{i-1,j+1} + [6(\Delta x)^3]u_{i,j+1}$$
$$-[(6\beta\Delta t + \delta\alpha(\Delta x)^2\Delta t)]u_{i+1,j+1} + [3\beta\Delta t]u_{i+2,j+1} = [6(\Delta x)^3]u_{i,j}$$

For $(i = m-2)$**:**

$$-[3\beta\Delta t]u_{m-4,j+1} + [6\beta\Delta t - \gamma\alpha(\Delta x)^2\Delta t]u_{m-3,j+1} + [6(\Delta x)^3]u_{m-2,j+1}$$
$$-[(6\beta\Delta t + \delta\alpha(\Delta x)^2\Delta t)]u_{m-1,j+1} = [6(\Delta x)^3]u_{m-2,j} - [3\beta\Delta t]u_m$$

For $(i = m-1)$**:**

$$-[3\beta\Delta t]u_{m-3,j+1} + [6\beta\Delta t - \gamma\alpha(\Delta x)^2\Delta t]u_{m-2,j+1} + [6(\Delta x)^3]u_{m-1,j+1}$$
$$= [6(\Delta x)^3]u_{m-1,j} + [(6\beta\Delta t + \delta\alpha(\Delta x)^2\Delta t)]u_m - [3\beta\Delta t]u_m,$$
where $\delta = (u_{i,j} + u_{i+1,j})$ and $\gamma = (u_{i,j} + u_{i-1,j})$.

Now,we can write the system of linear equation in the matrix form:

$$\begin{pmatrix} 6(\Delta x)^3 & B & 3\beta\Delta t & 0 & 0 & 0 & 0 & \cdots & 0 \\ E & 6(\Delta x)^3 & B & 3\beta\Delta t & 0 & 0 & 0 & \cdots & 0 \\ -3\beta\Delta t & E & 6(\Delta x)^3 & B & 3\beta\Delta t & 0 & 0 & \cdots & 0 \\ 0 & -3\beta\Delta t & E & 6(\Delta x)^3 & B & 3\beta\Delta t & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -3\beta\Delta t & E & 6(\Delta x)^3 & B \\ 0 & 0 & 0 & 0 & \cdots & 0 & -3\beta\Delta t & E & 6(\Delta x)^3 \end{pmatrix}$$

$$\times \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \\ \vdots \\ u_{m-2,j+1} \\ u_{m-1,j+1} \end{pmatrix} = \begin{pmatrix} 6(\Delta x)^3 u_{1,j} + (3\beta\Delta t - E)u_0 \\ 6(\Delta x)^3 u_{2,j} - 3\beta\Delta_t u_0 \\ 6(\Delta x)^3 u_{3,j} \\ 6(\Delta x)^3 u_{4,j} \\ \vdots \\ 6(\Delta x)^3 u_{m-2,j} - 3\beta\Delta t u_m \\ 6(\Delta x)^3 u_{m-1,j} - (3\beta\Delta t + B)u_m \end{pmatrix},$$

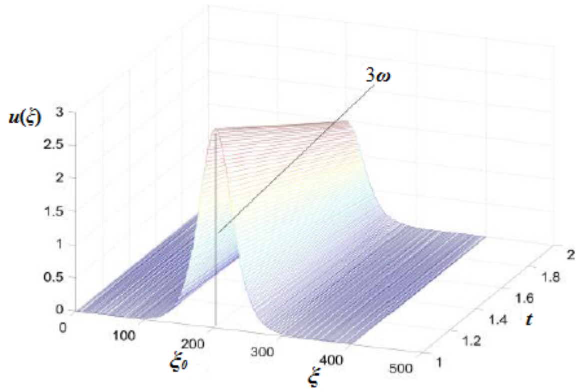
where $E = 6\beta\Delta t - \gamma\alpha(\Delta x)^2\Delta t$ and $B = -(6\beta\Delta t - \delta\alpha(\Delta x)^2\Delta t$,
where $\delta = (u_{i,j} + u_{i+1,j})$ and $\gamma = (u_{i,j} + u_{i-1,j})$.

The numerical solution of equation (4.1) is detected by calibrating the value of u in the last matrix Kolebaje and Oyewande [19]. In this calculation, the number of column of elements increases with t time. The wave packets $u(x,t) = x + \omega t$,(see Section 7, Chapter One). In this section we have calculated previous parameters with a parameter t time, we using the exact solution (1.17). The constant parameters used in this calculation are $\Delta x = 3500$, $\Delta t = 0.01$ and some random numbers instead of variables such as, initial wave ξ_0, speed ω, wave length u and time t in MATLAB some graphical illustration (see Figure (4.2)).

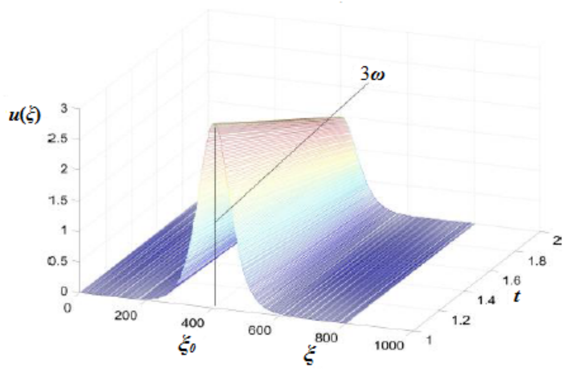
We have seen the surface in Figures (4.2), describe that there are some changes of soliton waves $u(\xi,t)$ that occurs due to time t. The figures show that the amplitude of soliton wave wrenches after an extension of time t for the longest Aminuddin and Sehah [3].

4.2 Explicit Finite Difference Methods

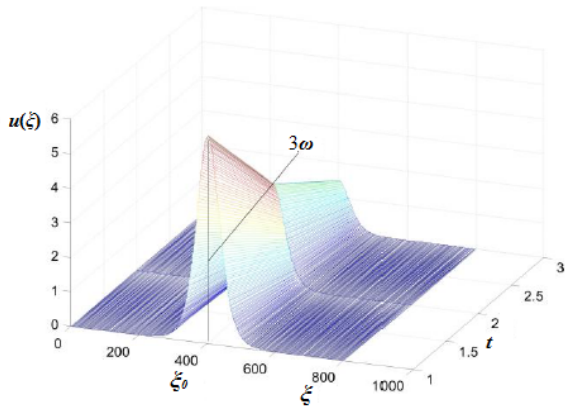
Both methods, implicit and explicit method as well, use a mesh on the spatial domain of N points of equal distance. Unless otherwise is told, the distance between the points is



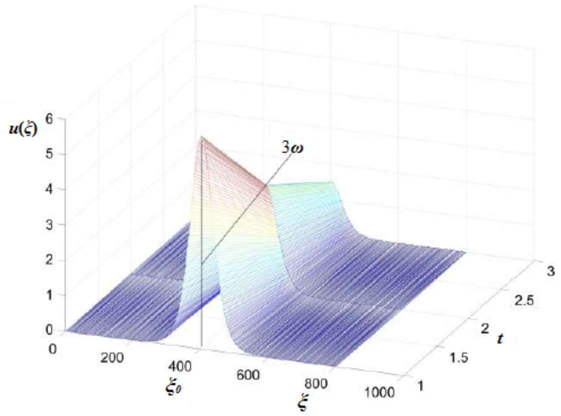
(a) Soliton Wave with parameters $\omega = 1$, $\xi_0 = 200$ $t = 2$ and $\xi = 400$.



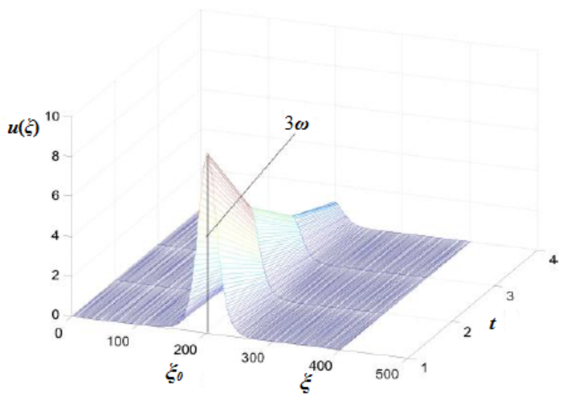
(b) Soliton Wave with parameters $\omega = 1$, $\xi_0 = 400$ $t = 2$ and $\xi = 800$.



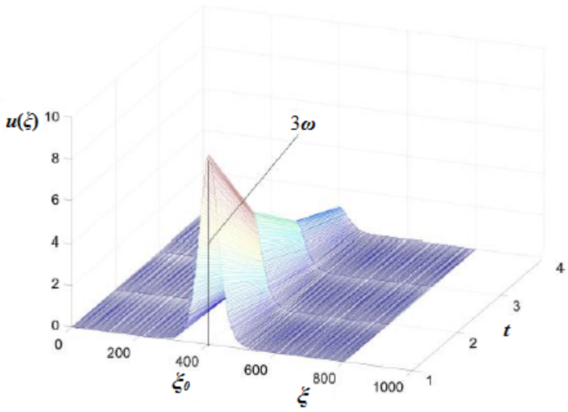
(c) Soliton Wave with parameters $\omega = 2$, $\xi_0 = 200$ $t = 3$ and $\xi = 400$.



(d) Soliton Wave with parameters $\omega = 2$, $\xi_0 = 400$ $t = 3$ and $\xi = 800$.



(e) Soliton Wave with parameters $\omega = 2$, $\xi_0 = 200$ $t = 4$ and $\xi = 400$.



(f) Soliton Wave with parameters $\omega = 2$, $\xi_0 = 400$ $t = 4$ and $\xi = 800$.

Fig. 4.2 3D. Soliton Wave.

$h = \Delta x$. The uniformed discretization of the temporal domain is performed in $k = \Delta t$. The u function subscripts and its superscripts refer to spatial steps and temporal steps respectively. For instance $u_{j+2}^{n-1} \approx u(x+2\Delta x, t-\Delta t) = u(x+2h, t-k)$.

4.2.1 Zabusky and Kruskal

The scientists who managed to publish the numerical results of the soliton interaction regarding KdV equation, were Zabusky and Kruskal,

$$\begin{aligned} &u_t + \alpha u u_x + \beta u_{xxx} = 0, \\ &\quad x \in (-\infty, \infty)\ ,\ t > 0, \end{aligned} \tag{4.3}$$

where initial condition:

$$u(x,0) = \cos(\pi x).$$

The KdV equation numerical solution often causes the presence of β in literature. Zabusky

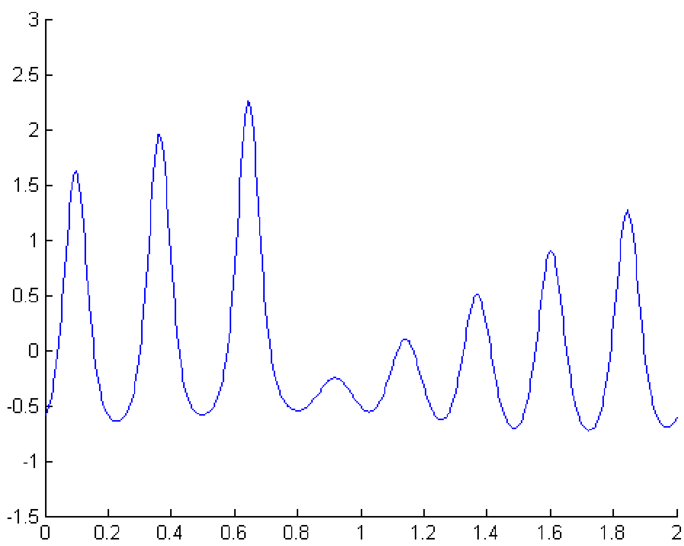


Fig. 4.3 Zabusky & Kruskal experiment, $u_t + \alpha u u_x + \beta u_{xxx} = 0$, $u(x,0) = \cos(\pi x)$, where $\alpha = 1$ and $\beta = (0.022)^{\frac{1}{2}}$

and Kruskal devoted a lot of their attention to the recurrence of initial condition, as well as the soliton interaction. They concluded that it is necessary to be emphasized that, through the nonlinear interaction, all solitons are built up to their initial state and they arrive in almost equal phase. As this process make progress, another recurrence happens. However, the second recurrence is not as good as first one. It was Goda who analyzed this phenomenon again in (1977). Since this thesis will not deal with the method derivation, it is assumed that at least the third derivative of the central finite difference approximation is known.

$$
\begin{aligned}
f^{'}(x_0) &\approx \frac{1}{2h}(f_1 - f_{-1}),\\
f^{'''}(x_0) &\approx \frac{1}{2h^3}(f_2 - 2f_1 + 2f_{-1} - f_{-2}),
\end{aligned}
$$

regarding function f at a x_0, point where uniform mesh space is $h := \Delta x$. The values $f(x_0 + yh)$ are referred to by the subscripts y. Both variables give a second order finite difference approximation. When the approximation of u is concerned, Zabusky and Krushal decided to use a three-point average

$$
f(x_0) \approx \frac{1}{3}(f_1 + f_0 + f_{-1}).
$$

After applying approximation as shown, we obtain the finite difference method,

$$
\begin{aligned}
\frac{1}{2\Delta t}(u_j^{n+1} - u_j^{n-1}) &= -u.\frac{1}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)\\
&\quad -\beta^2\frac{1}{2\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n),\\
\frac{1}{2\Delta t}(u_j^{n+1} - u_j^{n-1}) &= -\frac{1}{3}(u_{j+1}^n + u_j^n + u_{j-1}^n).\frac{1}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)\\
&\quad -\beta^2\frac{1}{2\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n),\\
u_{j+1}^n &= u_j^{n-1} - \frac{\Delta t}{3\Delta x}(u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-1}^n)\\
&\quad -\beta^2\frac{\Delta t}{\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n). \qquad (4.4)
\end{aligned}
$$

So there is an explicit finite difference method for solving Korteweg-de Vries equation. [1] There is a truncation error of $O((\Delta t)^2) + O((\Delta x)^2)$ in this method. So in order to reach stability it is important that

$$
\frac{\Delta t}{\Delta x}| - 2u_{max} + \frac{1}{(\Delta x)^2}| \leq \frac{2}{3\sqrt{3}}
$$

Unfortunately, this method requires a small time step step, as we can conclude from the

[1] To apply the uncentered scheme $u_j^1 + u_j^0 - \frac{\Delta t}{6\Delta x}(u_{j+1}^0 + u_j^0 + u_{j-1}^0)(u_{j+1}^0 - u_{j-1}^0) - \beta^2\frac{\Delta t}{2(\Delta x)^3}(u_{j+2}^0 - 2u_{j+1}^0 + 2u_{j-1}^0 - u_{j-2}^0)$, for the initial time step.

stability condition. So, it is one of the slowest methods available up to now. However, it has one huge advantage. It is an accurate method which compares favorably to other techniques.

4.3 Implicit Finite Difference Method

4.3.1 Hopscotch method of Greig and Morris

After observing $u_t + uu_x + u_{xxx}$ KdV equation, we note that

$$uu_x = \frac{1}{2}(u^2)_x \tag{4.5}$$

we perform the approximation of $\frac{1}{2}u_x^2$ with the use of central difference approximation. After that $w(x,t) = \frac{1}{2}u^2(x,t)$, this leads to

$$\begin{aligned}\left(\frac{u^2}{2}\right)_x &= w_x,\\ &\approx \frac{1}{2\Delta x}(w_{j+1}^n - w_{j-1}^n),\\ &= \frac{1}{2\Delta x}\left[\frac{(u_{j+1}^n)^2}{2} - \frac{(u_{j-1}^n)^2}{2}\right],\end{aligned}$$

which possess $O(\Delta x^2)$ truncation error

After applying a forward difference method on time, where the linear term is processed by a cetral difference scheme, the explicit method is obtained

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(w_{j+1}^n + w_{j-1}^n\right) - \frac{\Delta t}{2(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right),$$

where $w = \frac{u^2}{2}$.

Finally, the algebraic system is obtained Greig and Morris [15],

$$AU^{m+1} = K,$$

$$\text{where } A = \begin{pmatrix} 1 & \frac{P\beta}{2h^2} & 0 & \cdots & 0 \\ -\frac{P\beta}{2h^2} & 1 & \frac{P\beta}{2h^2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{P\beta}{2h^2} & 1 & \frac{P\beta}{2h^2} \\ 0 & \cdots & 0 & -\frac{P\beta}{2h^2} & 1 \end{pmatrix}, U^{m+1} = \begin{pmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n-2} \end{pmatrix}^{m+1}$$

and $K = [k_1, k_3, \ldots, k_{N-2}]$

where $p = \frac{\Delta t}{\Delta x} = \frac{k}{h}$ There is $O((\Delta t)^2) + O((\Delta x)^2)$ truncation error, which can also be found in the finite difference method by Zabusky-Kruskal. In order to reach stability there is a condition

$$\frac{\Delta t}{(\Delta x)^3} \leq |\frac{1}{(\Delta x)^2 u_{max} - 2}|$$

However, the condition of Zabusky-Kruskal's method is more restrictive than this one.

4.3.2 Goda's Scheme

In Goda's scheme a forward difference is applied on time, while the u_{xxx} term uses a central difference scheme. The nonlinear term uu_x however, uses the combination of methods

$$\begin{aligned} uu_x &\approx \frac{1}{2\Delta x}\left(u_{j+1} - u_{j-1}\right) u_j, \\ &= \frac{1}{2\Delta x}(u_{j+1}u_j - u_{j-1}u_j), \\ &\approx \frac{1}{2\Delta x}(u_{j+1}u_{(j1)} - u_{j-1}u_{(j2)}). \end{aligned}$$

We decided to omit the time step references temporarily in order to keep simplicity. The approximation of the function u two occurrences is done differently. The approximation of first one is done by $u_{(j1)} \approx \frac{1}{2}(u_j + u_{j+1})$ forward explicit average, while the second is approximated by $u_{(j2)} \approx \frac{1}{2}(u_j + u_{j-1})$ backward explicit average. Provided that the $u_{(1)}$ and $u_{(2)}$ approximations are left out, we can say that Goda's method is mainly implicit. If we perform Goda's approximation for $u_t + uu_x + u_{xxx} = 0$, KdV equation is:

$$\begin{aligned} &\frac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) + \frac{1}{\Delta x}\left(u_{j+1}^{n+1}(u_j^n + u_{j+1}^n) - u_{j-1}^{n+1}(u_j^n + u_{j-1}^n)\right) \\ &\quad + \frac{1}{2(\Delta x)^3}\left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}\right) = 0. \end{aligned} \tag{4.6}$$

This method possess $O(\Delta t) + O((\Delta x)^2)$ truncation error and an unconditional stability, which leads to the conclusion that any choice of Δt yields stability.

4.4 Fourier Method (Pseudospectral Method)

This method compares favorably with finite difference methods in several cases. It is a method of global approximation. It is not necessary to approximate spatial derivatives because of the Fourier transform properties F for function f derivatives $F\left(\frac{d^n f}{dx^n}\right) = (ik)^n F(f)$, so the algorithm requires less number of grid points. Generally, the number of computations and the computing memory can be considerably reduced in a given problem Anhaouy [4].

4.4.1 Fornberg and Whitham Method

We reconsider standard KdV equation given in (4.3) in the form:

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \qquad x \in [-p, p]. \tag{4.7}$$

In this method the Fourier transform is used. This causes the spatial domain $[-p, p]$ normalization of the $[0, 2\pi]$ since the variable changes $x \to x\pi/p + \pi$, then the equation (4.7), becomes:

$$u_t + \frac{\alpha\pi}{p} u u_x + \frac{\beta\pi^3}{p^3} u_{xxx} = 0, \qquad x \in [0, 2\pi]. \tag{4.8}$$

We know the inverse Fourier transform operator is in the form:

$$\frac{d^n u}{dx^n} = F^{-1}(ik)^n F(u), \;\; n = 1, 2, \cdots$$

By using this with $n = 1$ and $n = 3$, then the equation (4.8), becomes:

$$u_t = -u\frac{i\alpha\pi}{p} F^{-1}[kF(u)] + \frac{i\beta\pi^3}{p^3} F^{-1}[k^3 F(u)], \tag{4.9}$$

We consider for any integer $N > 0$,

$$x_j = j\Delta x = \frac{2\pi j}{N}, \quad j = 0, 1, \cdots N-1.$$

We transform the solution $u(x,t)$ of the equation (4.8) into the discrete Fourier space, we get,

$$\hat{u}(k,t) = F(u) = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j,t)e^{-ikx}, \quad -\frac{N}{2} \leq k \leq \frac{N}{2} - 1.$$

Then, the inverse Fourier transform for the above equation, becomes:

$$u(x_j,t) = F^{-1}(\hat{u}) = \sum_{k=-N/2}^{N/2-1} \hat{u}(k,t)e^{ikx}, \quad 0 \leq j \leq N-1.$$

When we perform the approximation of the solution of equation (4.9) Anhaouy [4], we get,

$$\frac{du(x_j,t)}{dt} = \frac{-i\alpha\pi}{p}u(x_j,t)F^{-1}[kF(u)] + \frac{i\beta\pi^3}{p^3}F^{-1}[k^3F(u)], \quad 0 \leq j \leq N-1.$$

or,

$$u_j^{n+1} - u_j^{n-1} = -2i\Delta t\left[\frac{-i\alpha\pi}{p}u(x_j,t)F^{-1}[kF(u)] + \frac{i\beta\pi^3}{p^3}F^{-1}[k^3F(u)]\right], \tag{4.10}$$

where $0 \leq j \leq N-1$.
Then, the equation (4.9) can be written in the vector form:

$$U_j = F(U),$$

where $U = [u(x_0,t), u(x_1,t), \cdots, u(x_N,t)]^T$ and F defines the right hand side of equation (4.10).

4.4.2 Taha and Ablowitz

Several different methods were compared by Taha and Ablowitz, regarding the KdV equation numerical computation Taha and Ablowitz [31]:
- Greig-Morris Hopscotch method.
- Zabusky-Kruskal scheme.
- Goda's scheme.
- Proposed local scheme.
- Kruskal's scheme.
- Tappert's split step Fourier method.
- Whitham and Fornberg's pseudospectral method.

It was Kruskal who suggested the scheme based on his idea where the equation $u_t + u_{xxx} = 0$

in the dispersion term u_{xxx} is approximated by:

$$\frac{1}{2(\Delta x)^3}(u_{j+2}^{n+1} - 3u_{j+1}^{n+1} + 3u_j^{n+1} - u_{j-1}^{n+1})$$
$$+\frac{1}{2(\Delta x)^3}(u_{j+1}^{n} - 3u_j^{n} + 3u_{j-1}^{n} - u_{j-2}^{n}).$$

The base for the proposed local scheme is an inverse scattering transform Johansen [18]. The obtained results were compared to the results obtained by previously mentioned methods just for the single soliton solution. This comparison revealed that the proposed local scheme was the most precise Nouri and Sloan [25]. After the period of seven years Sloan and Nouri analyzed the proposed local scheme, regarding two soliton solution. However, Chan and Kerkhoven Chan and Kerkhoven [10] developed a dominating Fourier pseudospectral method. The results revealed that the pseudospectral scheme is less efficient than the local scheme on the single soliton problem. However, it is more efficient on the more complicated two soliton problem. Nevertheless, there are not large differences in calculation. The obtained results agree with the statement by Taha and Ablowitz that, for equations which can be solved by IST (inverse scattering transform), there are approximations provided by the finite difference schemes, that were established on IST.

4.4.3 Chan and Kerkhoven's Semi-implicit Scheme

In one of the methods developed by Chan and Kerkhoven a leapfrog method was used for the nonlinear term $-3(u^2)_x$, while for the linear term u_{xxx} they used Crank Nicholson method. Up to now the semi-implicit scheme of Chan and Kerkhoven has been the most efficient method for solving the KdV equation.

We approximate the linear term as:

$$\begin{aligned} u_{xxx} &\approx \frac{1}{2}(u_j^{n+1} + u_j^n)_{xxx}, \\ F(u_{xxx}) &\approx \frac{1}{2}F[(u_j^{n+1} + u_j^n)_{xxx}], \\ &\approx -ik^3\frac{1}{2}F(u_j^{n+1} + u_j^n), \end{aligned}$$

while the nonlinear term is approximated as:

$$-3F((u^2)_x) \approx -3ik\frac{\pi}{p}F[(u_j^n)^2].$$

The result of this is:

$$\frac{1}{2\Delta t}F(u_j^{n+1}-u_j^{n-1})-3ikF((u_j^n)^2)-ik^3F(u_j^{n+1}+u_j^n)=0,$$
$$F(u_j^{n+1}-u_j^{n-1})-6ik\Delta tF((u_j^n)^2)-2ik^3\Delta tF(u_j^{n+1}+u_j^n)=0,$$
$$F(u_j^{n+1})-F(u_j^{n-1})-6ik\Delta tF((u_j^n)^2)-2ik^3\Delta t\left[F(u_j^{n+1})+F(u_j^n)\right])=0,$$

$$(1-2ik^3\Delta t)F(u_j^{n+1})=F(u_j^{n-1})+6ik\Delta tF((u_j^n)^2)+2ik^3\Delta tF(u_j^n),$$

$$F(u_j^{n+1})=\kappa(k)[(F(u_j^{n-1})+6ik\Delta tF((u_j^n)^2)+2ik^3\Delta tF(u_j^n)],$$
$$u_j^{n+1}=F^{-1}\left[\kappa(k)(F(u_j^{n-1})+6ik\Delta tF((u_j^n)^2)+2ik^3\Delta tF(u_j^n))\right],$$

where $\kappa(k)=\frac{1}{1-2ik^3\Delta t}$. Is required per time step in this method. There is also one more requirement for stability,

$$(\Delta t)^2<\frac{3\sqrt{3}}{2}\frac{1}{|\alpha|^3},$$

where α stands for coefficient of the nonlinear term.

4.5 Finite Element Methods

4.5.1 Petrov and Galerkin Method

The method of Petrov and Galerkin was used by Christie and Sanz-Serna in order to perform the evaluation of KdV_+ equation Sanz-Serna and Christie [29]. The Petrov-Galerkin method resembles the Galerkin method. However, there is one exception, where the basis functions are allowed for the trial and the test functions to differentiate. In the method of Christie and Sanz-Serna, the common hat functions were used for the trial function u while Hermite cubic polynomials were used for the test function v. They claim that with the use of Petrov-Galerkin method, they are enabled to employ C^0 interpolant, which takes less computational effort than with the case of the standard Galerkin approach based on Hermite cubics or B-splines.

The KdV_+ equation is multiplied by test function v and then the dispersion is integrated by parts twice.

$$
\begin{aligned}
&u_t + uu_x + \beta u_{xxx} = 0,\\
&vu_t + vuu_x + \beta vu_{xxx} = 0,\\
&\int_{-\infty}^{\infty} vu_t dx + \int_{-\infty}^{\infty} vuu_x dx + \beta \left(vu_{xx}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v_x u_{xx} dx \right) = 0,\\
&(u_t, v) + (uu_x, v) + \beta \left(vu_{xx}|_{-\infty}^{\infty} - u_x v_x|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u_x v_{xx} dx \right) = 0,\\
&(u_t, v) + (uu_x, v) + \beta \left(vu_{xx}|_{-\infty}^{\infty} - u_x v_x|_{-\infty}^{\infty} + (u_x v_{xx}) \right) = 0,
\end{aligned}
$$

the L_2 inner product is donated by $(\cdot\ , \cdot)$ where $(f, g) = \int_{-\infty}^{\infty} f(x)g(x)dx$.

It was implicitly demanded above that $v(x) \in C^1$. Ander the condition that v is taken as a hostage, there is a demand that $v(x) \to 0$ as $|x| \to \infty$, while it was retrieved that

$$
(u_t, v) + (uu_x, v) + \beta(u_x, v_{xx}) = 0. \tag{4.11}
$$

If we employ the finite element methods the equally spaced mesh $x_0 < x_1 < \ldots < x_n$ is introduced, with h spacing, while finite elements are used in spatial domain for test and trial functions as well.

$$
\begin{aligned}
U(x,t) &= \sum_{i=0}^{n} U_i(t)\phi_i(x),\\
v(x,t) &= \sum_{j=0}^{n} v_j(t)\psi_j(x).
\end{aligned}
$$

The compact support is required for the trial functions ϕ_i. The approximate solution is denoted by U. After that the equation (4.11) transforms into:

$$
\begin{aligned}
&(U_t, v_j\psi_j) + (UU_x, v_j\psi_j) + \beta(U_x, (v_j\psi_j)_{xx}) = 0,\\
&(U_t, v_j\psi_j) + (UU_x, v_j\psi_j) + \beta(U_x, v_j(\psi_j)_{xx}) = 0,\\
&v_j(U_t, \psi_j) + v_j(UU_x, \psi_j) + \beta v_j(U_x, (\psi_j)_{xx}) = 0,\\
&(U_t, \psi_j) + (UU_x, \psi_j) + \beta(U_x, (\psi_j)_{xx}) = 0,
\end{aligned}
$$

where $j = 0, \cdots, n$. The next step is the choice of the trial function $\phi(x)$, which will be the common piecewise linear function called the hat function at every node x_i. So, $\phi(x_i) = \delta_{ij}$, the more common Kronecker delta function. So, when $i = j$ then $U_i(x_i, t) = U_i(t)\phi_i(x_i) = U_i(t)\delta_{ij} = U_i(t)$. The next step is the choice of the test function $\psi(x)$. Because of the fact that the Petrov-Galerkin method is now used instead of the Galerkin method, it is not necessary to use one and the same function as both trial and test functions. We have

$$\psi_i(x) = \psi\left[\frac{x-x_0}{h} - i\right].$$

The five points approximate is required for u_{xxx} in order to achieve accuracy. Therefore, there has to be support for ψ on $[-2,2]$, which implies the presence of cubic polynomials in every interval $[i,i+1]$ when $i=-2,-1,0,1$. After the choice of Hermite cubic interpolation, we get:

$$\rho(x) = \begin{cases} x(|x|-1)^2 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma(x) = \begin{cases} (|x|-1)^2(2|x|+1) & \text{if } |x| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where,

$$\begin{aligned} &\rho'(0) = \sigma(0) = 1, \\ &\sigma(1) = \sigma(-1) = 0, \\ &\rho(-1) = \rho(0) = \rho(1) = 0, \\ &\sigma'(-1) = \sigma'(0) = \sigma'(1) = 0, \\ &\rho'(-1) = \rho'(1) = 0. \end{aligned}$$

In the end

$$\begin{aligned} \psi(x) &= \alpha_{-1}\sigma(x+1) + \alpha_0\sigma(x) + \alpha_1\sigma(x-1) + \gamma_{-1}\rho(x+1) \\ &+ \gamma_0\rho(x) + \gamma_1\rho(x-1), \end{aligned}$$

where $\gamma_i = \psi'(i)$ and $\alpha_i = \psi(i)$.
When these are available to us, this system

$$(U_t, \psi_j) + (UU_x, \psi_j) + \beta(U_x, (\psi_j)_{xx}) = 0,$$

transforms into

$$
\begin{aligned}
&+\frac{\beta}{h^3}(-\gamma_1 U_{i+2}+(2\gamma_1-\gamma_0)U_{i+1}+(2\gamma_0-\gamma_1-\gamma_{-1})U_i \\
&+(2\gamma_{-1}-\gamma_0)U_{i-1}-\gamma_{-1}U_{i-2}) \\
&+\frac{1}{60h}[(9\alpha_1+2\gamma_1)U_{i+2}^2+(12\alpha_1+\gamma_1)U_{i+1}U_{i+2} \\
&+(9\alpha_0+2\gamma_0-6\gamma_1)U_{i+1}^2+(12\alpha_0-12\alpha_1+\gamma_0+\gamma_1)U_iU_{i+1} \\
&+(9\alpha_{-1}-9\alpha_1-6\gamma_0+2\gamma_{-1}+2\gamma_1)U_i^2 \\
&-(12\alpha_0-12\alpha_{-1}-\gamma_0-\gamma_{-1})U_{i-1}U_i-(9\alpha_0-2\gamma_0+6\gamma_{-1})U_{i-1}^2 \\
&-(12\alpha_{-1}-\gamma_{-1})U_{i-2}U_{i-1}-(9\alpha_{-1}-2\gamma_{-1})U_{i-2}^2] \\
&+\frac{1}{60}[(9\alpha_1+2\gamma_1)\acute{U}_{i+2}(9\alpha_0+42\alpha_1+2\gamma_0)\acute{U}_{i+1} \\
&+(42\alpha_0+9\alpha_1+9\alpha_{-1}-2\gamma_1+2\gamma_{-1})\acute{U}_i \\
&+(9\alpha_0+42\alpha_{-1}-2\gamma_0)\acute{U}_{-1}+(9\alpha_{-1}-2\gamma_{-1})\acute{U}_{i-2}]=0,
\end{aligned}
\tag{4.12}
$$

when $i=0:n$. A partial derivative with respect of t is denoted by $\acute{U}$. We establish U that lies at the mesh as zero, or in other words $U_{-2}=U_{-1}=U_{n+1}=U_{n+2}=0$.
According to Taylor it is necessary to set α and γ relationship in a following way,

$$
\begin{aligned}
&\alpha_{-1}+\alpha_0+\alpha_1=1, \\
&\gamma_{-1}-\gamma_1=1, \\
&\gamma_{-1}+\gamma_0+\gamma_1=0.
\end{aligned}
\tag{4.13}
$$

Since it is required for the test functions to be symmetric, more limitations are present because of the conservation properties Sanz-Serna and Christie [29],

$$\alpha_{-1}=\alpha_1 \ , \ \gamma_{-1}=-\gamma_1 \text{ and } \gamma_0=0. \tag{4.14}$$

We can notice from equations (4.13) and (4.14) that $\gamma_{-1}=\frac{1}{2}$ and $\gamma_1=-\frac{1}{2}$. Because γ_{-1},γ_0 and γ_1 possess these characteristics, only one free parameter is available α_1. Then test functions $\psi(x)$ depending on the single parameter $\alpha=\alpha_1$ Sanz-Serna and Christie [29], are considered.
Therefore, Petrov-Galerkin method is interpreted via every term

$$
\begin{aligned}
U_t &= \frac{1}{60}(9\alpha-1)\acute{U}_{i+2}+(9+24\alpha)\acute{U}_{i+1}+(44-66\alpha)\acute{U}_i \\
&+ (9+24\alpha)\acute{U}_{i+1}+(90\alpha-1)\acute{U}_{i-2},
\end{aligned}
$$

$$
\begin{aligned}
UU_x &= \frac{1}{120h}(18\alpha-2)U_{i+2}^2+(24\alpha-1)U_{i+2}U_{i+1}+(24-36\alpha)U_{i+1}^2 \\
&+ (23-72\alpha)U_{i+1}U_i-(23-72\alpha)U_{i-1}U_i-(24-36\alpha)U_{i-1}^2 \\
&- (24\alpha-1)U_{i-1}U_{i-2}-(18\alpha-2)U_{i-2}^2,
\end{aligned}
$$

and,

$$U_{xxx}=\frac{1}{2h^3}(U_{i+2}-2U_{i+1}+2U_{i-1}-U_{i-2}).$$

4.5.2 The Modified Petrov and Galerkin Method

The modified Petrov and Galerkin method used for the KdV equation is very simple. Actually, the approach for approximation used in it and in the method of Morris and Greig, is the some. We write the nonlinear term uu_x as $(\frac{u^2}{2})_x$ and we perform approximation by:

$$\frac{1}{48h}(12\alpha-1)U_{i+2}^2+(14-24\alpha)U_{i+1}^2-(14-24\alpha)U_{i-1}^2-(12\alpha-1)U_{i-2}^2,$$

which gives this method the accuracy of the fourth order. [2] The obtained results were amazing. In comparison to other methods, a mPG method showed to be superior because of the accuracy of the fourth order in space, the mPG method error shrank faster than in standard PG method.

4.6 Numerical Methods Summary

As we already said in previous chapters, it is possible to write the nonlinear term as $\pm uu_x$, or $\pm 6uu_x$. In the table given below we refer to the equation as $\pm KdV$, or $KdV_{\pm}$, respectively. Therefore, the Korteweg-de Vrise is $u_t+\alpha uu_x+u_{xxx}$.

[2] The initial condition $f(x)=3C\text{sech}^2(kx+d)$ was used by Christic and Sanz-Serna for parameters $\alpha=\frac{1}{6},\beta=0.000484,\sqrt{C/4\beta}$ where $C=0.3$.

Non-linear part	Method's Name	Domain	Reference
$+uu_x$	Zabusky & Kruskal	$\mathbb{R}$	[18]
$+uu_x$	Goda's scheme	$\mathbb{R}$	[14]
$+uu_x$	Hopscotch of Greig & Morris	$\mathbb{R}$	[15]
$-6uu_x$	Semi-implicit scheme	$[0,2\pi]$	[10]
$+6uu_x$	Proposed local scheme	$\mathbb{R}$	[31]
$+uu_x$	Petrov & Galerkin method	$\mathbb{R}$	[29]
$+uu_x$	Modified Petrov & Galerkin method	$\mathbb{R}$	[29]
$+6uu_x$	Fornberg & Whithams pseudospectral	$[0,2\pi]$	[18]
$+6uu_x$	Local discontinuous Galerkin method	$\mathbb{R}$	[32]
$-6uu_x$	Time and space collocation	$\mathbb{R}$	[8]

Table 4.1 Numerical methods summary

Chapter 5

Preferred Experimental Methods

5.1 Introduction

The following methods have been used by the author. If the method is starred $(*)$, this implies that the method is modified by the author.

• The Finite Difference Method of Zabusky-Kruskal $(*)$.
• Fornberg & Whitham Pseudospectral Method.
• Pseudospectral Method.

It is important to note that the graphs illustrating analytical solution are denoted by a solid line while the graphs illustrating the experimental solution are denoted by a dashed line.

5.2 The Finite Difference Method of Zabusky-Kruskal

Analogous to previously considered Zabusky & Kruskal method is applied to +KdV or $u_t + 6uu_x + u_{xxx} = 0$. Uniform mesh is assumed in the spatial domain, where the step is of Δx size. The approximation of the KdV equation is the same as before, with the fact that the coefficient $+6$ is added before the nonlinear term. We decided to omit the interim steps,

$$u_t + 6uu_x + u_{xxx} = 0,$$

that leads to

$$\begin{aligned} u_j^{n+1} &= u_j^{n-1} - 2\frac{\Delta t}{\Delta x}(u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-1}^n), \\ &- \frac{\Delta t}{\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n), \end{aligned}$$

demands that,

$$\frac{\Delta t}{\Delta x}| - 2u_{max} + \frac{1}{(\Delta x)^2}| \leq \frac{2}{3\sqrt{3}},$$

for the purpose of stability.

In order to make it simple, the first step $u(x,\Delta t)$ together with initial condition were written to the program (the appendices include the copy of it).

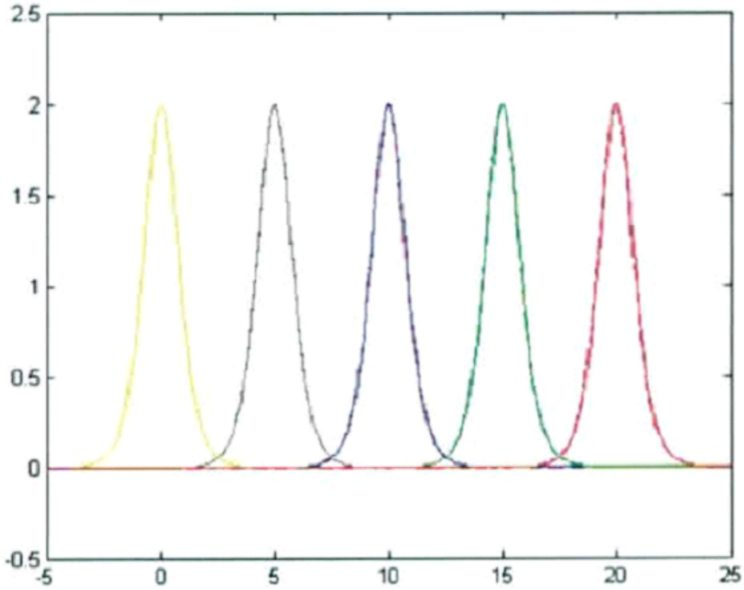


Fig. 5.1 Zabusky Kruskal, Finite Difference Method, where: $\Delta t = 3.8641e-004, N = 2^9$, plotted at $2\Delta t$ and $\frac{1}{4}T, \frac{1}{2}T$ and T, where $T = 5.0$. The error of these parameters, $E = 0.0881$

5.2.1 Modified Zabusky-Kruskal

Let us take the central difference approximation to $+6uu_x$ nonlinear term into consideration. Rather than the central different approximation up to the first derivative or the three points average, the approximation is performed differently. Notice that

$$6uu_x = 3(u^2)_x. \tag{5.1}$$

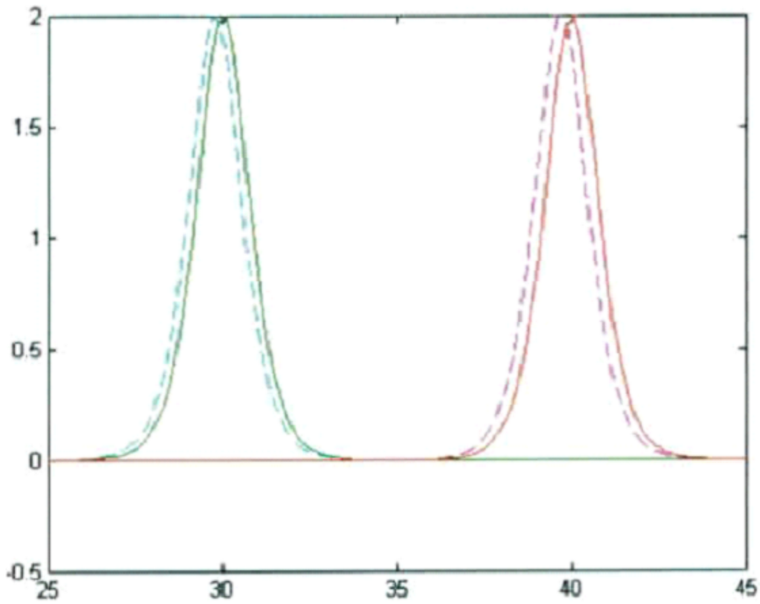


Fig. 5.2 Zabusky Kruskal, Finite Difference Method, where: $\Delta t = 3.8641e-004$, $N = 2^9$, plotted at $\frac{3}{4}T$ and T, where $T = 10$. The error of these parameters, $E = 0.4661$

The approximation of $3(u^2)_x$ is done with the use of the central difference approximation. It is defined that $w(x,t) = u^2(x,t)$. After that, we get

$$\begin{aligned} 3(u^2)_x &= 3w_x, \\ &\approx \frac{3}{2\Delta x}(w^n_{j+1} - w^n_{j-1}) \\ &= \frac{3}{2\Delta x}((u^n_{j+1})^2 - (u^n_{j-1})^2) \end{aligned}$$

that possess $O(\Delta x^2)$ truncation error, or precisely

$$\Delta x^2 |3u_x u_{xx} + uu_{xxx}|.$$

The modified Zabusky-Kruskal method becomes:

$$\begin{aligned} u^{n+1}_j &= u^{n-1}_j - 3\frac{\Delta t}{\Delta x}(u^n_{j+1})^2 - (u^n_{j-1})^2) \\ &- \frac{\Delta t}{\Delta x^3}(u^n_{j+2} - 2u^n_{j+1} + 2u^n_{j-1} - u^n_{j-2}). \end{aligned}$$

The obtained result was quite motivating. The same inputs were used as in the Zabusky and Kruskal method. When time $T = 5.0$, $\Delta t = 3.8641e-004$ and $N = 2^{-9}$. At this time T, there was an error 0.0503. However, for the equal inputs, the primary the Zabusky-Kruskal method gave an error of 0.0881. This means that there is an increase of 43% in accuracy when the modified Zabusky-Kruskal method is compared to the original finite difference method of Zabusky-Kruskal.

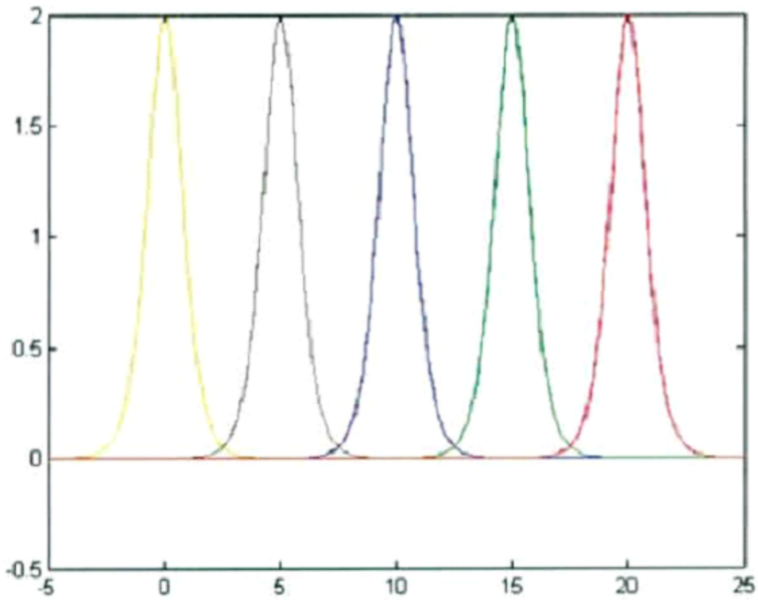


Fig. 5.3 Modified Zabusky Kruskal, Finite Difference Method, where: $\Delta t = 3.8641e-004, N = 2^9$, plotted at $2\Delta t$, where $\frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T$ and T where $T = 5.0$. The error of these parameters, $E = 0.0503$

5.3 Fornberg & Whitham Pseudaspectral Method

At first it is necessary to develop the code that employs a pseudospectral method in order to show Fornberg and Whitham pseudospectral method. If we normalize +KdV to $[0, 2\pi]$ (4.8), we get:

$$u_t + 6\frac{\pi}{p}uu_x + \frac{\pi^3}{p^3}u_{xxx} = 0,$$
$$u(0,t) = u(2\pi,t).$$

As mentioned (4.4.2), the method is,

$$u_j^{n+1} = u_j^{n-1} - 12iu\Delta t\frac{\pi}{p}F^{-1}\{kF(u)\} + 2i\Delta t\frac{\pi^3}{p^3}F^{-1}\{k^3F(u)\}, \tag{5.2}$$

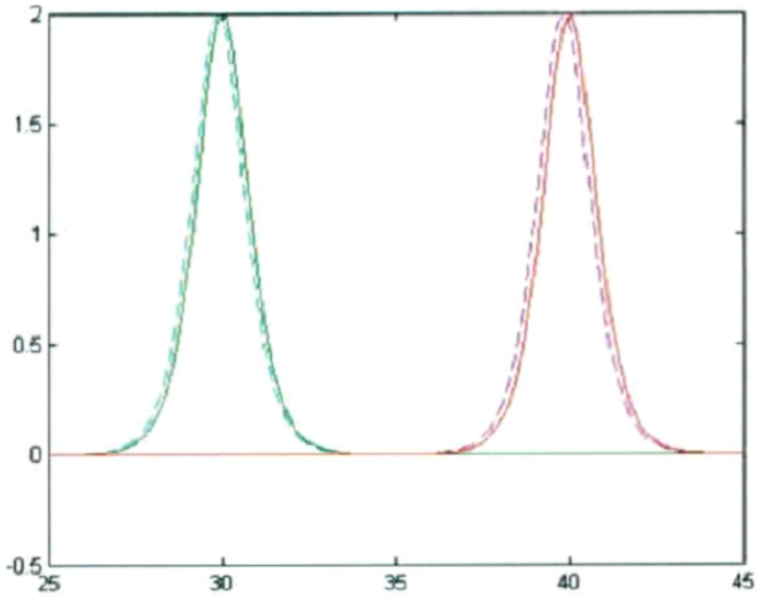


Fig. 5.4 Modified Zabusky Kruskal, Finite Difference Method, where: $\Delta t = 3.8641e - 004, N = 2^9$, plotted at $\frac{3}{4}T$ and T where $T = 10.0$. The error of these parameters, $E = 0.2717$

where the identification of the first u_j^{n-1} is performed analytically.

Here, there are inputs $\Delta t = \frac{(\Delta x)^3}{x^3} \approx 0.0323(\Delta x)^3$, the initial condition $u(x_j, 0) = 2\text{sech}^2(x_j)$, $T = 1.0$, where $u(x_j, \Delta t) = \text{sech}^2(x_j - 4\Delta t)$ is analytically determined with $N = 2^7$, the series of uniformly spaced points.
[1] The comparison error experimental $U(x,t)$ error and the exact solution obtained at $T = 1.0$, that is $u(x_j, 1.0) = 2\text{sech}^2(x_j - 4)$. There is an error equal 6.1×10^{-3} produced by the program. This error was computed for all j by $||U(x_j, T) - u(x_j, T)||_\infty$.
Both u and U solutions were plotted one across another. An unbroken line illustrate the true solution while a dashed line illustrates the experimental function U. Provided that the program is at the first estimation $U(x_j, 2\Delta t)$ and at $\frac{1}{4}T, \frac{1}{2}T, 3\frac{3}{4}T$ and T times, the solution is plotted, where T illustrates the time at which the calculation of the solution is performed.
Since the error is so small, it is obvious that is program can do the estimation of exact solution u when a single soliton solution is concerned.
However, is this possible when two soliton solution is concerned? Or two soliton interacting by moving at various speeds?.

[1]Two initial data time steps are required by the algorithm. There is the solution input at $t = \Delta t$, or $u(x_j, \Delta t) = U(x_j, \Delta t) = 2\text{sech}^2(x_j - \Delta t)$, it possible to employ a forward Euler to calculate $U(x_j, \Delta t)$.

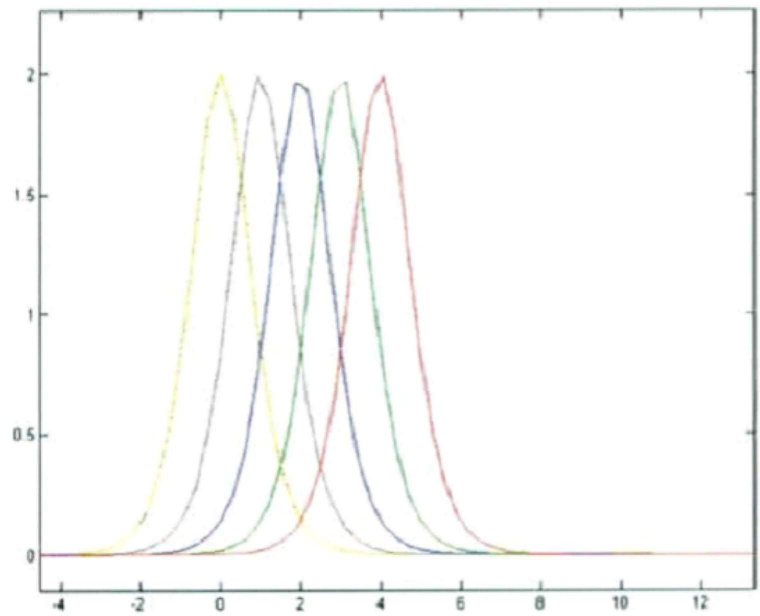


Fig. 5.5 Pseudospectral Method, where $N = 2^7$, plotted at $\frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T$ and T where $T = 1.0$

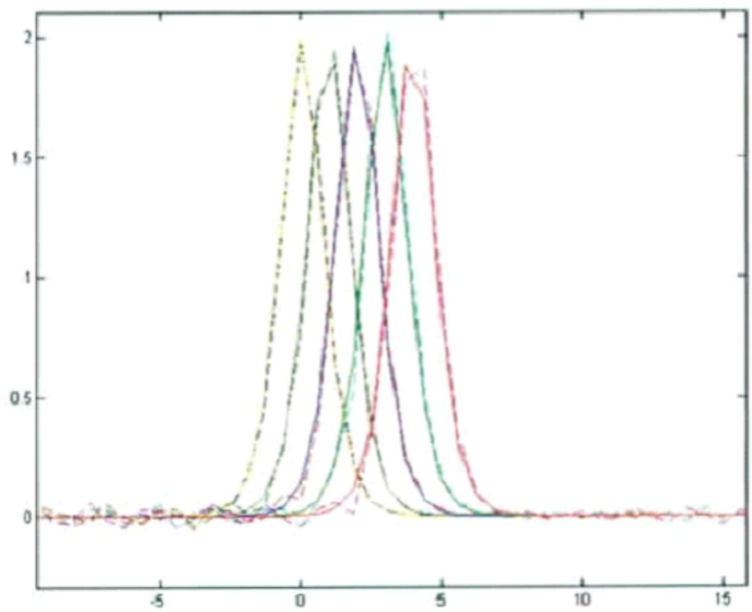


Fig. 5.6 Pseudospectral Method, reduced the number of points to $N = 2^6$, the method is less accurate with lower points

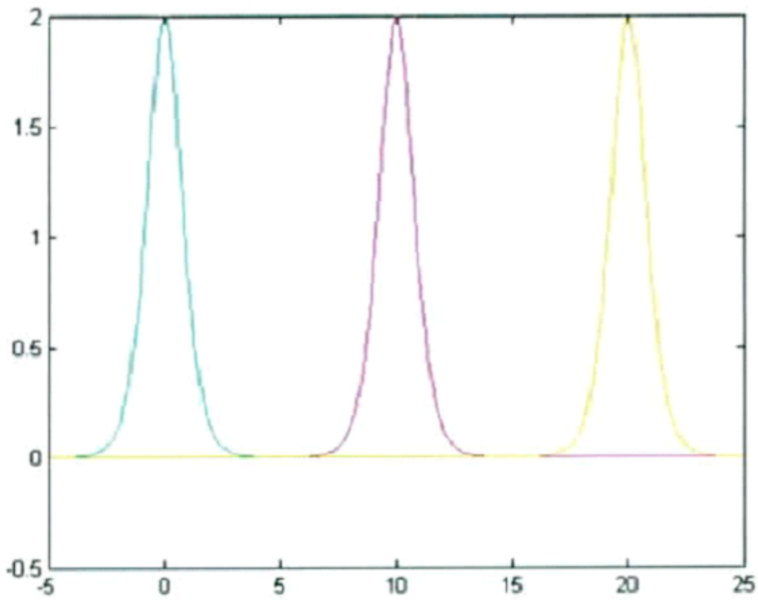


Fig. 5.7 Pseudospectral Method, where $N = 2^9$, plotted at $t = 2\Delta t$ where $\Delta t = 5.1903e-005$, and $\frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T$ and T where $T = 1.0$. The error: $E = 3.1994e-004$

The program was employed again. However, this time initial condition was replaced by

$$u(x,t) = -4\frac{4\cosh(2x)+\cosh(4x)+3}{3(\cosh(x)+\cosh(3x))^2}.$$

[2] The partition of the solitons is illustrated by Figure (5.8). There is a graph of u and U when $T = \frac{1}{2}T, T$, where $N = 2^8$. This methods yields an error of 2.51×10^{-2}.

Because of the fact that the pseudospectral method we have is experimentally functioning u_{xxx}, approximation is changed in Fornberg and Whitham steps.

$$-2i\Delta t\frac{\pi^3}{p^3}F^{-1}\{k^3F(u)\} \rightarrow -2iF^{-1}\left[\sin\left(\frac{\pi^3k^3}{p^3}\Delta t\right)F(u)\right].$$

After this change is implemented, the error 6.2×10^{-3} is retrieved for $(N = 2^7)$, regarding a single soliton. When two solitons are concerned, there is 2.80×10^{-2} error for $(N = 2^8)$. This implementation is faster to some extent than (5.2).

[2]The first time step was also given to the program
$U(x,\Delta t) = -12\frac{3+4\cosh(2x-8\Delta t)+\cosh(4x-64\Delta t)}{(3\cosh(x-28\Delta t)+\cosh(3x-36\Delta t))^2}$.

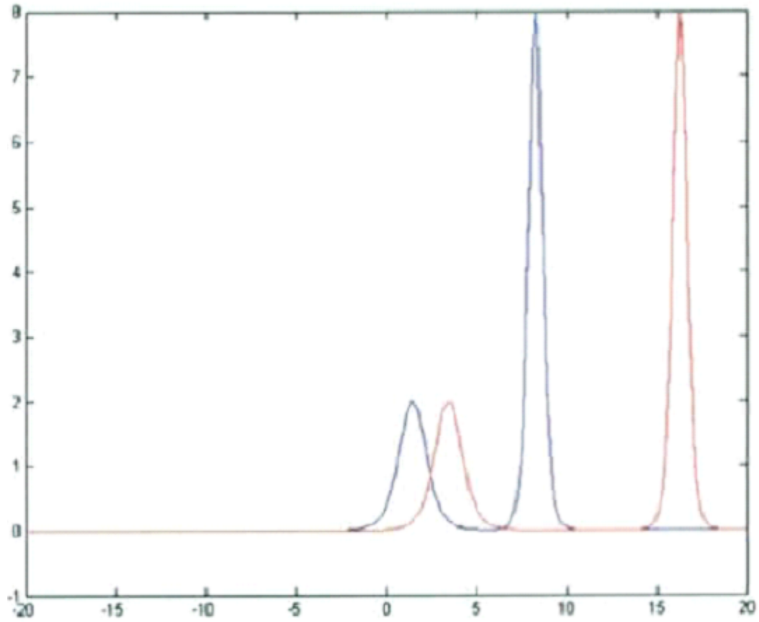


Fig. 5.8 Take notice, u and U are similar

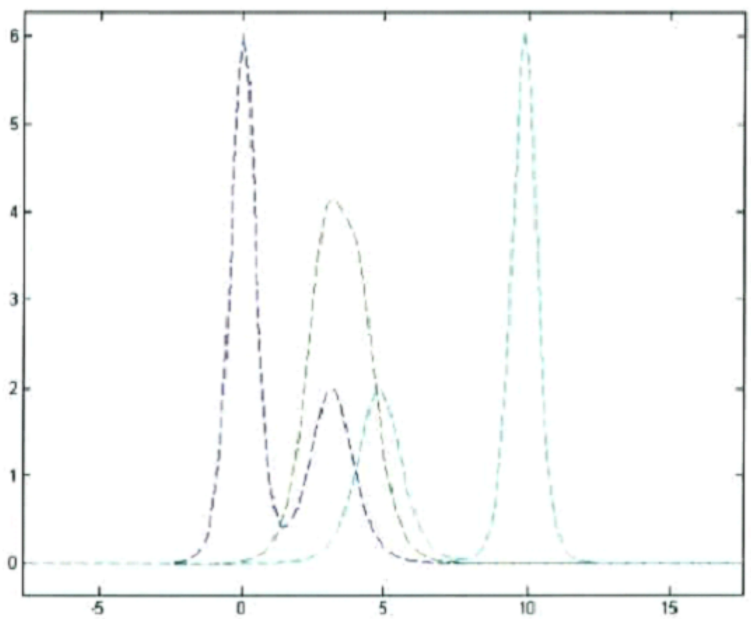


Fig. 5.9 The large soliton overtakes the smaller soliton

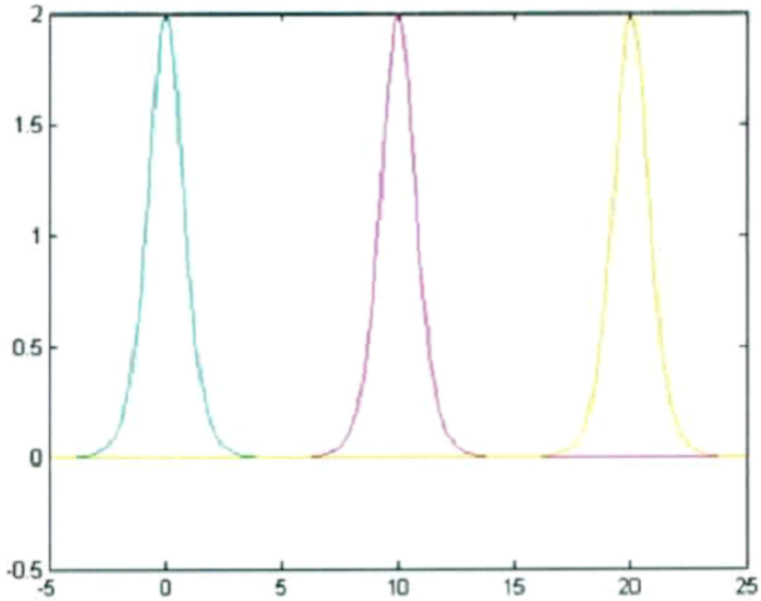


Fig. 5.10 Fornberg-Whitham and Pseudospectral Method, where $T = 5.0$, $\Delta t = 2.3459e-004$ and $N = 2^9$. Plotted at $t = 2\Delta t$ and $\frac{1}{2}T, T$. The error when $T = 0.0015$

Chapter 6

Finite Difference Scheme

6.1 Introduction

The Korteweg-de Vries equation which is a non-linear third order of partial differential equation given by:

$$u_t + \alpha u u_x + \beta u_{xxx} = 0. \tag{6.1}$$

The original plan was to create and implement the most transparent finite difference scheme that could be easily conceived. The approximations of the lowest order were used with u_x and u_{xxx}, and then the explicit Euler time-step procedure was implemented Ascher [5]. Where $\alpha = 6$ and $\beta = 1$ in the equation (6.1). Then the first phase of the original scheme is:

$$\frac{u_j^{n+1} - u_j^n}{h} + 6u_j^n \left(\frac{u_{j+1}^n - u_j^n}{\Delta x} \right) + \left(\frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{(\Delta x)^3} \right) = 0.$$

For the purpose of explicit time stepping, it is necessary to find the solution for u_j^{n+1}. We get,

$$u_j^{n+1} = u_j^n + \frac{6h}{\Delta x}(u_{j+1}^n - u_j^n)u_j^n + \frac{h}{2(\Delta x)^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n). \tag{6.2}$$

It is obvious that the third term coefficient $\frac{h}{2(\Delta x)^3}$ is quite large, provided that the time step and the size of the spatial grid are small and of an equal order. So the first direct result we get about the KdV finite difference scheme is

$$\text{Keep } \frac{h}{2(\Delta x)^3} \text{ small.} \tag{6.3}$$

The fact that there is an instability that is hindering the opportunity to obtain the precise solution is visible, if we consider the matrix-vector form. If M is the matrix, that gives us chance to rewrite equation (6.2)

$$u^{n+1} = Mu^n,$$

we have to point out that,

$$u^n = M^{n-1}u^{n-1}, \tag{6.4}$$

gives a solution at all time steps K. At any value of Δx and h, the spectral radius surpasses the value of (6.1) which leads to instabilities that grow limitlessly. After this, we will try to apply a technique applied by Driscoll and Fornberg in their work that discussed dispersive nonlinear wave equation. [1] They applied an implicit time stepping to linear part of the equation, while applying the explicit time stepping to the nonlinear term. The scheme's stability was improved to some extent, but as expected, not enough Ascher [5].

However, after certain modifications of Δx and h, and after an apprehensive low order implementation, the use of two point difference scheme for the explicit calculation of uu_x nonlinear term, as well as the seven point difference scheme for the implicit calculation of u_{xxx} linear term, a rather quick scheme appeared. This scheme was so brilliant that it provided an insight into the essential facts the future generations will use to approximate the nonlinear dispersive wave equation as KdV

$$\text{Nonlinear} \leftrightarrow \text{Explicit} \quad \text{and} \quad \text{Linear} \leftrightarrow \text{Implicit}. \tag{6.5}$$

It is necessary to mention that the stability must be improved whenever possible. This transformation resulted in a linear solve at every time step. However, since solving a KdV with the use of a finite difference scheme is so difficult, it is necessary to use this opportunity. In addition to this, it is necessary to perform the stencils optimization just like the discrete dispersion relation.

It is necessary to pay close attention to avoid over compensating with the finite difference stencil of a high order on the linear term, because this can lead to a specifically high cost of computation of the implicit inversion. However, we must try to get as close as possible to the point where the discrete dispersion relation is the closest to its continuous correspondent. Because of the fact that, due to certain stencils, there is no precise formula for the calculation of the discrete dispersion relation, we get certain errors and trials as well. How-

[1] A fast Spectral Algorithm for Nonlinear Wave Equations with linear Dispersion, April (1999).

ever, its calculation is already familiar:

First the scheme has to be linearized. After that, we perform the substitution of u for discrete solutions of traveling waves. At the end find the solution of w, regarding k. The true dispersion relation is obtained together with the error, depending on the chosen stencils. We tend to reduce the error term order.

When this case is concerned, the seven-point stencil functions appropriately, because the condition of the dispersion relation is fulfilled and a simple matrix is provided for finding the solution at every time step.

Finally, some numerical results are considered. The appendix provides us with the full code, but this scheme's general context will be discussed. At first, the following equation [2] must be solved:

$$u_t + uu_x + u_{xxx} = 0,$$

the initial condition

$$u(0) = 3A^2 \text{sech}^2\left(\frac{A(x+2)}{2}\right) + 3B^2 \text{sech}^2\left(\frac{Bx}{2}\right), \tag{6.6}$$

regarding $[-\pi, \pi]$ domain, discretized to 256 uniformly spaced points, where the integration is performed from $0 \le t \le 0.016$ where the size of the time steps is $h = 7.4798 \times 10^{-7}$. The short time interval was imposed on us to consider since there is the explicit relationship between the time-step size and the size of the spectral grid. The boundary conditions in this case are *dirichlet*. Here, the code is completed in an appropriate period of time, where it is required about 26.108 seconds to perform an integration from $t = 0$ to $t = 0.016$, while the total of 21391 time-steps is required. Lets take the solution waterfall plot into consideration: Even though the formula for finding the KdV exact solution has not been explicitly calculated from the initial condition (6.6), it is noted that the solution is visually the same as the one obtained from the spectral method Ascher [5].

This provides us with rough impression of precision. Although, we cannot claim the solution is highly precise we can say that it is stable and consistent over these two methods.

The two quantities conserved by KdV are taken in consideration. We have two types of

[2] The coefficient six is missing on the nonlinear term. Note that the appropriately scaled KdV can be lent to solutions that solve equations that have the form: $u_t + \alpha uu_x + \beta u_{xxx} = 0$ for every α and $\beta \in \mathbb{R}$.

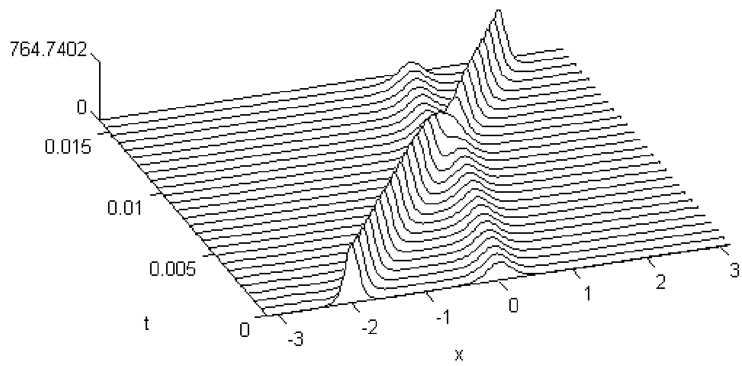


Fig. 6.1 The finite difference method to compute the solution water-fall plot.

conserved quantity:

$$\text{(I)} \int_{-\infty}^{\infty} u dx \quad , \quad \text{(II)} \int_{-\infty}^{\infty} u^2 dx$$

The first type U solution's squared L^1 norm which stays constant throughout time. Quantity number (II) is the solution's squared L^2 norm that also stays constant throughout time. If we compute these quantities and observe how they develop over time or actually how constant they stay, it is possible to assess our scheme's quality. [3] Figure (6.2) illustrates conserved plots for $0 \leq t \leq 0.016$. We can conclude that these quantities have very poor conservation. There is a change in L^1 norm by around 0.64% for this time interval, whereas the performance of L^2 norm was $e - 0043.0794 \times 10^{-4}\%$. The explanation for the possibility of relatively poor quantities as well as the comparison of the obtained results to the ones we got using the spectral method is given in the following section.

6.2 Spectral Method

This method was found to be superior from the very beginning. Its theoretical conception and actualization was supervised by Lloyd Trefethen. This code at first uses the initial con-

[3] It is possible to use the quality parameter only in certain contexts. For example, if we want to calculate KdV solutions for long intervals of time, where it is necessary for the quantities to be conserved.

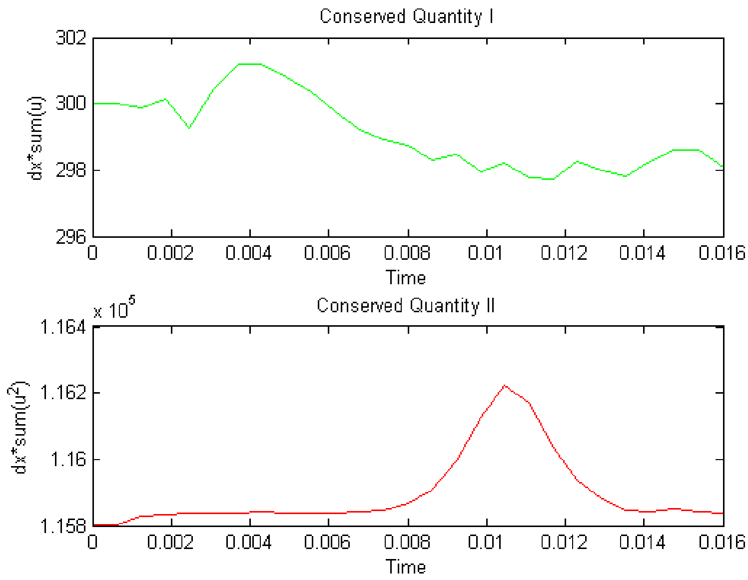


Fig. 6.2 The Finite Difference Method with conservation.

dition's $u(0)$ Fast Fourier Transform, then periodic boundary conditions are defined over $[-\pi.\pi]$, after that the linear term $\hat{u}_{xxx}$ is approximated by $(ik^3)\hat{u}$. At last with the use of Runge-Kutta of the fourth order the next time step is accessed, where $v = e^{-k^3 t}(\hat{u})$ integrating factor is used in Fourier space. The solutions provided by this method are stable and accurate while the problems are solved efficiently. The quantities are conserved quite well. We are going to observe the performace of the scheme regarding total time as well as the accuracy of solution and the quantity conservation Ascher [5].

We can find the full spectral code in the appendix, but we will discuss the scheme's context here. We find the solution for:

$$u_t + uu_x + u_{xxx} = 0,$$

initial condition

$$u(0) = 3A^2 \text{sech}^2\left(\frac{A(x+2)}{2}\right) + 3B^2 \text{sech}^2\left(\frac{Bx}{2}\right), \tag{6.7}$$

over $[-\pi.\pi]$ domain, performing discretization into 256 uniformly spaced points, and integration with $0 \le t \le 0.016$, where time steps are $h = 6.1035 \times 10^{-6}$. The consequence of short interval of time are small time steps where the spacing of the spatial grid is quite tight. That leads to good interaction between the two solitons. The total interval of time is quite limited due to periodic boundary conditions. If the integration is performed for large time, the solitons would meet a boundary and just wrap, which would make them interact with themselves. If we do not pay attention these artificial self interactions can cause a lot of complications. The complication of the code happens very quickly where it is necessary for just 6.259 seconds to pass for the integration from $0 \leq t \leq 0.016$, while the total of 3035 time-steps [4] is required. We will take the solution's waterfall plot into consideration. Once

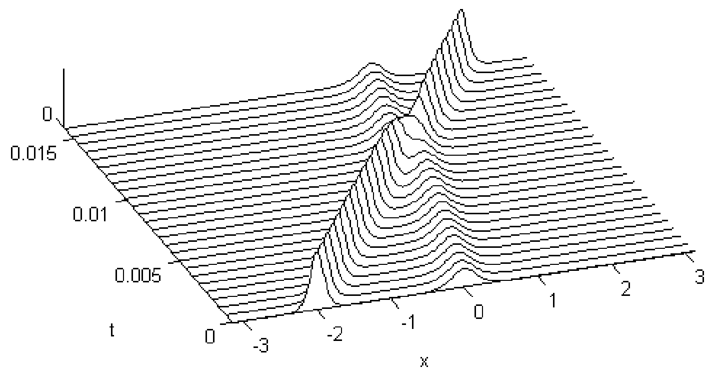


Fig. 6.3 The solution water-fall Plot calculated by the Spectral Method.

again, even though the exact solution formula for the initial value has not been computed, this method, as well as the finite difference method, provide us with the solution consistence which can be used to support its designed accuracy.

In order to consider the conserved quantities, we can start with an in-complex two quantity plot, regarding time over the whole interval.

$$\text{(I)} \int_{-\infty}^{\infty} u dx \quad , \quad \text{(II)} \int_{-\infty}^{\infty} u^2 dx$$

While performing the explicit value computation regarding the change of the quantities

[4]This number is quite small in comparison to the number of the time-steps needed by the finite difference scheme, 21391, for the purpose of stability. That is why the run time is improved.

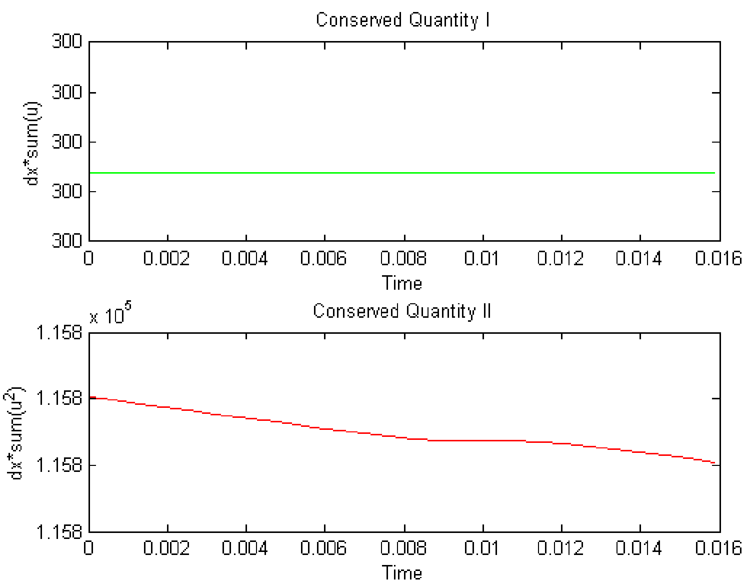


Fig. 6.4 The Spectral Method with conservation.

over the interval of time $0 \leq t \leq 0.016$, we can observe the change of the L^1 norm by just $3.7896 \times 10^{-14}\%$. We are able to hypothesize that the conservation of these two quantities depends two crucial facts. At first, there is no loss of energy due to a very short interval of time, despite the long term aspect. However, there is no possibility to increase the time interval significantly due to the scheme's accuracy and stability regarding this particular initial value problem, having these parameters and these boundary conditions Ascher [5].

The second hypothesis states that the enhanced conservation of the norms is obtained (especially L^1 norm) as a consequence of the periodic boundary conditions regarding spatial domain. Instead of being lost, because of the fact that the solution is coming closer to the domain edge (or Dirichlet Boundary Conditions), any integral quantity portion goes directly to the other boundary through the boundary back, where these change are compensated in the integral value at the boundaries. As we observed in the finite difference scheme, the integral quantity is lost through the domain boundary provided that there is the deficit of periodic boundary conditions.

Chapter 7

Outlook

7.1 Conclusion

• When we applied KdV and modified KdV equation we discovered that the KdV equation satisfies Painleve's property, but mKdV equation does not satisfy Painleve's property. In spite of that, we were able to find analytic solutions for both of them.

• We find that most difference scheme methods have the similar case solution with the exception of the non-linear term, where every method differs and is somehow specific.

7.2 Some Future Work

• The generalized KdV equation as:

$$u_t + \alpha u^n u_x + \beta u_{xxx} = 0,$$

where $n > 2$. We get another type of the KdV equation, commonly referred to as Critical General Korteweg-de Vries equation. It got its name because of the fact that the value $n = 5$ represents a critical point where the solutions can distend into finite time. This means that the (gKdV) solitary waves solutions possess stability provided that $n < 4$. If the value n is $n > 5$.
we will apply Painlevé method into gKdV equation, where $n > 2$.

• In addition, there are some studies about KdV and modified KdV equations which applied by B-spline method and its subsidiaries: Linear Spline, Quadratic Spline, Cubic Spline,... with some results of every one of them. We will perform the comparison of these results with the results of the Painlevé method.

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Appendix A

MATLAB CODES

SOLITON WAVES

A: One Soliton Wave

```
clc
close all
N=512;
L = 30;
a=L/(2*pi);
T=72;
h=2*pi/N;
dt=.005;
x =(h:h:2*pi)';
y=a*x;
u= exp(-2*(y-.25*L).^2)/a;
k=[(0:N/2)';(1-N/2:-1)'];

disp=0.01; %disp=input('disp=?`); 7.0.05 works well

m=disp*a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2=-.5*i*dt*k./(1-i*m);
d3=(.5)*d2;
sol= plot(y,a*u,'Erasemode', 'background');
axis([2 13 -.1 1.4]);

%64 CHAPTER 5. MATLAB CODES

zoom;
%title('KdV: Disintegration of a Gaussian Pulse');
drawnow;
t=1;
while t<=T;

v=real(ifft(d1.*fft(u)+d3.*fft(u.^2)));
w=real(ifft(d1.*fft(u)+d2.*fft(u.^2)));

for n=1:3;
w=v+real(ifft(d3.*fft(w.^2))) ;
end
u=w;
t=t+dt;
set(sol,'ydata',a*u);
end
```

B: Two Soliton Wave

```
%solution of the equation  u_t+ uu_x +u_xxx =0
clc
clear all
N=512;
L = 50;
a=L/(2*pi);
T=10000;
h=2*pi/N;
dt=.005;
x=(h:h:2*pi)';
y=a*x;
c1=0.8;c2=0.5;
u=(12/a)*(c1^2*sech(c1*(y-0.1*50)).^2+c2^2*sech(c2*(y-0.3*50)).^2);
k=[(0:N/2)'; (1-N/2:-1)'];
m=a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2= -.5*i*dt*k./(1-i*m);
d3=.5*d2;
sol= plot(y,a*u,'Erasemode','background');
axis([0 L -.1 8]);
title('KdV: t=15');drawnow
t=0;
xlabel('x');
ylabel('u(x,15)');
%
while t<T;
fftu = fft(u); fftuu = fft(u.^2);
v=real(ifft(d1.*fftu+0.5*d2.*fftuu));
w=real(ifft(d1.*fftu+d2.*fftuu));
for n=1:3;
w=v+real(ifft(d3.*fft(w.^2)));
end
u=w;
t=t+dt;
set(sol,'ydata',a*u);

end
```

C: Experiment of Kruskal Zabusky Method

```
close all
clear;
N=512;
L=2; a=L/(2*pi);
T=1.146;
h=2*pi/N;
dt=.005;
x=(h:h:2*pi)';
y=a*x;
u=cos(pi*y)/a;
k=[(0:N/2)';(1-N/2:-1)'];
disp=.000484; %input('disp=');
m=disp*a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2=-.5*i*dt*k./(1-i*m);
d3=.5*d2;
sol=plot(y,a*u,'Erasemode','background');
axis([ 0 L -1.5 3]);
box off
title('Kruskal-Zabusky Experiment');
text(.8, 2.2, 'u_t+uu_x+\delta^2u_{xxx}=0,   where \delta=0.022')
text(1.0,1.9,'u(x,0)=cos(x)');
drawnow;
t=0;

while t<T;
    fftu=fft(u); fftuu=fft(u.^2);
    v=real(ifft(d1.*fftu+d3.*fftuu));
    w=real(ifft(d1.*fftu+d2.*fftuu));
    for n=1:5;
        w=v+real(ifft(d3.*fft(w.^2)));
    end
    u=w;
    t=t+dt;
    set(sol,'ydata',a*u);
end
```

D: Zabusky-Kruskal Finite Difference Method

```
clear
close all
N=2^9;
T=0.5*0.5;
p=100;
h=2*pi/N;
j=N;
J=[0:1:j-1];
x=2*pi*J/N;
x;
for ii=1:j
    xx(ii)=p*(x(ii)/pi-1);
end
% Initial Condition.
for i=1:j
    u1(i)=2.*(sech(xx(i))).^2;
end
%%dt=h^3/(4+6*(h^2*max(u1)));
dt=3.8641e-004;
for i=1:j
    u2(i)=2.*(sech(xx(i)-4. *dt)).^2;
end
time =dt;
counter=0;
b2=-dt/(h)*2;
v2=dt/(h^3);
while time<(T)
    time=time + dt
    u(1)=u1(1)+b2*(u2(2)+u2(1)+u2(j))*(u2(2)-u2(j))-v2*(u2(3)-
2*u2(2)+2*u2(j)-u2(j-1));

    u(2)=u1(2)+b2*(u2(3)+u2(2)+u2(1))*(u2(3)-u2(1))-v2*(u2(4)-
2*u2(3)+2*u2(1)-u2(j));

    u(j-1)=u1(j-1)+b2*(u2(j)+u2(j-1)+u2(j-2))*(u2(j)-u2(j-2))-v2*(u2(
2*u2(j)+2*u2(j-2)-u2(j-3));

    u(j)=u1(j)+b2*(u2(1)+u2(j)+u2(j-1))*(u2(1)-u2(j-1))-v2*(u2(2)-
2*u2(1)+2*u2(j-1)-u2(j-2));
    for i=3:j-2
        u(i)=u1(i)+b2*(u2(i+1)+u2(i)+u2(i-1))*(u2(i+1)-u2(i-1))
        -v2*(u2(i+2)-2*u2(i+1)+2*u2(i-1)-u2(i-2));
    end
    counter=counter + 1
    u1=u2;
    u2=u;
end

u_ans=2.*(sech(xx-4*time)).^2;

time
error=max(abs(u_ans-u))
plot(xx,u,'*',xx,u_ans,'-')
```

E: Modified Zabusky-Kruskal Finite Difference Method

```
N=2^9;
T=20;
p=100;
h=2*p/N;
j=N;
J=[0:1:j-1];
x=2*pi*J/N;
for ii=1:j
    xx(ii)=p*(x(ii)/pi-1);
end
%set initial condition.
for i=1:j
    u1(i)=2.*(sech(xx(i))).^2;
end
%dt=h^3/(4+6*(h^2*max(u1)));
dt=3.8641e-004;
for i=1:j
    u2(i)=2.*(sech(xx(i)-4.*dt)).^2;
end
time=dt;
counter=0;
b2=-dt/(h)*3;
v2=dt/(h^3);
while time<(T)
    time=time+dt
        u(1)=u1(1)+b2*((u2(2)).^2-(u2(j)).^2)-v2*(u2(3)-2*u2(2)+2*u2(j)-u2(j-
1));

        u(2)=u1(2)+b2*((u2(3)).^2-(u2(1)).^2)-v2*(u2(4)-2*u2(3)+2*u2(1)-
u2(j));

        u(j-1)=u1(j-1)+b2*((u2(j)).^2-(u2(j-2)).^2)-v2*(u2(1)-2*u2(j)+2*u2(j-
2)-u2(j-3));

        u(j)=u1(j)+b2*((u2(1)).^2-(u2(j-1)).^2)-v2*(u2(2)-2*u2(1)+2*u2(j-1)-
u2(j-2));

        for i=3:j-2
          u(i)=u1(i)+b2*((u2(i+1)).^2-(u2(i-1)).^2)-v2*(u2(i+2)-
2*u2(i+1)+2*u2(i-1)-u2(i-2));
        end
        counter=counter+1
        u1=u2;
        u2=u;
end
u_ans=2.*(sech(xx-4*time)).^2;
time
error=max(abs(u_ans - u))
plot(xx,u2,'--',xx,u_ans,'-')
%plot(xx,u_ans,'*')
```

F: Pseudospectral Method

```
N=2^9;
T=10;
p=100;
norm=pi/p;
dx=2*p/N;
dt=(dx)^3/(pi)^3; %\approx 0.0323
j=[0:1:N-1];
x=2*pi*j/N;
for ii=1:N
xx(ii)=p*(x(ii)/pi-1);
end
for ii=1:N
    u0(ii)=2.*sech(xx(ii)).^2;
    u1(ii)=2.*sech(xx(ii)-4.*dt).^2;
end
counter=0;
time=dt;
while time <T
    time
    for ii=1:N
        U(ii)=(-1)^(ii-1)*u1(ii);
    end
    U=fft(U);
    for iii=1:N
        FFT(iii)=(iii-1-N/2)*U(iii);
        FFT3(iii)=(iii-1-N/2)^3*U(iii);
    end
    IFFT=ifft(FFT);
    IFFT3=ifft(FFT3);
    i=sqrt(-1);
    for ii=1:N
        first(ii)=(-1)^(ii-1)*-6*i*norm * u1(ii) * IFFT(ii);
        third(ii)=(-1)^(ii-1)*i * norm^3 * IFFT3(ii);
    end
    %for ii=1:N
      %  u(ii)=u0(ii)+2*(dt*first(ii)+ third(ii));
    %end
    for ii=1:N
        post(ii)=first(ii)+third(ii);
    end
    for ii=1:N
        u(ii)=u0(ii)+2*dt*post(ii)
    end
    for ii=1:N
        u0(ii)=u1(ii);
        u1(ii)=u(ii);
    end
    time=time + dt;
    counter = counter + 1
end
u_ans = 2.*sech(xx-4*(dt*counter)).^2;
error=max(abs(u_ans - u1))
%plot(xx,u, '--',xx,u_ans,'-')
```

G: Fornberg Whitham Pseudospectral Method

```
N=2^9;
T=10;
p=100;
norm=pi/p;
dx=2*p/N;
dt=3*(dx)^3/(2*(pi)^2)-0.00001;
j=[0:1:N-1];
x=2*pi*j/N;
for ii=1:N
xx(ii)=p*(x(ii)/pi-1);
end
for ii=1:N
    u0(ii)=2.*sech(xx(ii)).^2;
    u1(ii)=2.*sech(xx(ii)-4.*dt).^2;
end
counter=0;
time=dt;
while time <T
    time
    for ii=1:N
        U(ii)=(-1)^(ii-1)*u1(ii);
    end
    U=fft(U);
    for iii=1:N
        FFT(iii)=(iii-1-N/2)*U(iii);
        FFT3(iii)=sin(norm^3*(iii-1-N/2)^3*dt)*U(iii);
    end
    IFFT=ifft(FFT);
    IFFT3=ifft(FFT3);
    i=sqrt(-1);
    for ii=1:N
        first(ii)=(-1)^(ii-1)*-6*i*norm * u1(ii) * IFFT(ii);
        third(ii)=(-1)^(ii-1)*i * norm^3 * IFFT3(ii);
    end
    for ii=1:N
        post(ii)=first(ii) + third(ii);
    end
    for ii=1:N
        u(ii)=u0(ii)+2*(dt * first(ii) + third(ii));
    end
    for ii=1:N
        u0(ii)=u1(ii);
        u1(ii)=u(ii);
    end
    time=time + dt;
    counter = counter + 1
end
u_ans = 2.*sech(xx-4*(dt*counter)).^2;
error=max(abs(u_ans - u1))
%plot(xx,u,'--',xx,u_ans,'-')
```

H: Fast Fourier Transform, Finite Difference Scheme

```
clc
clear all
close all
%Solve KdV equation u_t+uu_x+u_xxx=0 on [-pi,pi]
%tintegrating factor v= exp(-ik^)*u-hat.
% twe-soliton initial data:
figure; tic;
N=256; dt=.4/N^2; x=(2*pi/N)*(-N/2:N/2-1)';
dx=x(2)-x(1);
A=16; B=9; %16;
clf, drawnow, set(gcf,'renderer','zbuffer')
u=3*A^2*sech(.5*(A*(x+2))).^2+3*B^2*sech(.5*(B*(x))).^2;
v=fft(u); k=[0:N/2-1 0 -N/2+1:-1]'; ik3=1i*k.^3;
%
% Solve PDE and plot results:
tmax=0.016; nplt=floor((tmax/25)/dt); nmax=round(tmax/dt);
udata=u; tdata=0; h=waitbar(0,'please wait ...');
e1all=dx*sum(u);
e2all=dx*sum(u.^2);
for n=1:nmax
    t=n*dt; g=-.5i*dt*k;
    E=exp(dt*ik3/2); E2=E.^2;
    a=g.*fft(real( ifft( v )).^2);
    b=g.*fft(real( ifft(E.*(v+a/2)) ).^2);   %4th-order
    c=g.*fft(real( ifft(E.*v+b/2) ).^2);     % Runge-Kutta
    d=g.*fft(real( ifft(E2.*v+E.*c) ).^2);
    v=E2.*v+(E2.*a+2*E.*(b+c)+d)/6;
    if mod(n,nplt)==0
        u=real(ifft(v)); waitbar(n/nmax)
        e1=dx*sum(u);
        e2=dx*sum(u.^2);
        e3=dx*sum(-u.^3+0.5*(real(ifft(i*k.*v))).^2);
        e1all=[e1all e1];
        e2all=[e2all e2];
        udata=[udata u]; tdata=[tdata t];
    end
end
toc
waterfall(x,tdata,udata'), colormap(1e-6*[1 1 1]); view(-20,25)
xlabel x, ylabel t, axis([-pi pi 0 tmax 0 max(max(udata))]), grid off
set(gca,'ztick',[0 2000]), close(h), pbaspect([1 1 .13])
figure
subplot(2,1,1);
plot(tdata,e1all','g');
title('Conserved Quantity I');
xlabel('Time'); ylabel('dx*sum(u)');
subplot(2,1,2);
plot(tdata,e2all','r');
title('Conserved Quantity II');
xlabel('Time'); ylabel('dx*sum(u^2)');
```

I: Fast Fourier Transform, Spectral Method

```
%KdV eq. u_t + uu_x + u_xxx = 0 on [-pi,pi]
%integrating factor v = exp(-ik^3t)*u-hat.
%two-soliton initial data:
clear all
close all

tic

N = 256; dt = .4/N^2; x = (2*pi/N)*(-N/2:N/2-1)';
dx = x(2)-x(1);
A = 16; B = 9;

u = 3*A^2*sech(.5*(A*(x+2))).^2 + 3*B^2*sech(.5*(B*(x))).^2;

v = fft(u); k = [0:N/2-1 0 -N/2+1:-1]'; ik3 = i*k.^3;

% Solve PDE and plot results:
tmax = 0.016;
nplt = floor((tmax/25)/dt);
nmax = round(tmax/dt);

udata = u;
tdata = 0;
h = waitbar(0,'please wait...');
e1all = dx*sum(u);
e2all = dx*sum(u.^2);

for n = 1:nmax
t = n*dt; g = -.5i*dt*k;
E = exp(dt*ik3/2); E2 = E.^2;

a = g.*fft(real( ifft( v ) ).^2);
b = g.*fft(real( ifft(E.*(v+a/2)) ).^2);
c = g.*fft(real( ifft(E.*v + b/2) ).^2);
d = g.*fft(real( ifft(E2.*v+E.*c) ).^2);
v = E2.*v + (E2.*a + 2*E.*(b+c) + d)/6;

if mod(n,nplt) == 0
u = real(ifft(v)); waitbar(n/nmax)
e1 = dx*sum(u);
e2 = dx*sum(u.^2);
e1all = [e1all e1];
e2all = [e2all e2];
udata = [udata u];
tdata = [tdata t];
end
end

toc

close(h)
```

```
figure;
waterfall(x,tdata,udata'),
colormap(1e-6*[1 1 1]);
view(-20,25)
xlabel x,
ylabel t,
axis([-pi pi 0 tmax 0 max(max(udata))]),
grid off
set(gca,'ztick',[0 2000]),
pbaspect([1 1 .13])

figure;
subplot(2,1,1);
plot(tdata,e1all','g');
title('Conserved Quantity 1');
xlabel('Time'); ylabel('dx*sum(u)');
subplot(2,1,2);
plot(tdata,e2all','r');
title('Conserved Quantity 2');
xlabel('Time'); ylabel('dx*sum(u^2)');
```

Appendix B

DECLARAION

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Potpisani Attia .A.H Mostafa
broj indeksa 2001/2010

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Ime i prezime autora: *Attia .A.H Mostafa*
Broj indeksa: 2001/2010
Studijski program: ____________________________
Naslov rada: *"Qualitative Behavior and Exact Travelling Nonlinear Wave Solutions of the KdV Equation"*
Mentor: *redovni Prof. dr. Miodrag Mateljević*

Potpisani *Attia .A.H Mostafa*

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