Mathematical Institute
Serbian Academy of Science and Arts

Veljko A. Vujičić

PREPRINCIPLES OF MECHANICS

Redactor of second edition

PHd Dragomir Zeković

professor at the Faculty of Mechanical Engineering
University of Belgrade

2015.

Belgrade

PREFACE

The monograph is the result of many years of my University lecturing as well as participating in discussions at scientific conferences on problems of the science about motion of bodies. Moreover, it is a reciprocating result since lecturing on analytical mechanics, theory of oscillation, theory of motion stability, of tensor calculus and differential geometry, or even engineering mechanics, has arisen in me some justified doubts that impelled me to test the knowledge first acquired in my graduate studies and later taught to others and used in preparing my scientific papers, namely, the knowledge in accordance with the current professional world literature.

I have accepted the fact - and used it as a starting point - that analytical mechanics, or, more generally, mechanics, is an exact natural science; it is as exact as mathematics or, even more precisely, it is even more exact than it if its assertions claim not only mathematical proofs, but also verification by nature and by human practice, as well as proofs of technology. Exact to perfection, mechanics is a mathematical theory about harmonious motion of the celestial bodies and, at the same time, about often rough human engineering practice. Its founders have written that geometry is part of mechanics (Isaac Newton) or that mechanics is part of (mathematical) analysis (Lagrange). It has been developed and perfected to exactitude. At the same time, it almost goes without saying that everything has already been solved in this branch of natural science. The assertions (principles, laws, theorems) of the theory of mechanics are accepted and learnt almost as the laws of nature. Mechanics is as old as material and written relicts that testify about the history of findings about motion and rest of bodies; at the same time, it is as contemporary as the novelty itself since everything new that is being created, made or unmade cannot be separated from it.

On the basis of the above-mentioned views, several questions logically arise: What else can still be added to this science? What is the use of additional writings published in hundreds of periodic scientific journals? What about contributions to the body motion theory, if this theory is logically, perceptively and experimentally harmonious and finite to perfection? What makes possible discussions about accordance of all the assertions referring to the nature of things? These and many other questions, objections, incongruous statements and philosophical qualifications and classifications[1] that have accompanied the development of theoretical mechanics are the ones that this book is trying to give answers to. It considerably changes the knowledge about mechanics and in mechanics, namely its starting philosophical assumption, its mathematical-logical conception and the basic and derived concepts that seemed to be clear. Besides, the preprinciples are introduced; the laws of dynamics are given different meanings and definitions; the principles of mechanics, each in its own turn, are shown as sufficient for invariant development of the whole theory of mechanics; the concepts of definitions, laws, principles, theorems and lemmas in mechanics have been differentiated. As a consequence, the axioms or laws of motion of the classical mathematical theory of mechanics have been omitted. Even the generality of the law about the mutual bodies' attraction has been subdued to questioning. The concepts of particular assertions in mechanics, namely those that comprise the names of their authors, are replaced by new terms associated with the respective meanings or concepts so that they could be more easily understood by the reader; the other reason for their replacement being the fact that the historical evidence relative to the development of mechanics gives various data concerning the contributions of the distinguished authors of theoretical mechanics. All the presented innovations or modifications have not been made for their own sake. The level of skill in the history of the development of mechanics has depended upon the possession and development of the mathematical knowledge as well as operational aspects of various theories. The factor of the validity estimate has been and still is the logical verification of the mental modeling of mechanical objects as well as the confirmation of the deduced relations in nature - by

[1]See, for instance, P.V. Harlamov, [8]

observing and measuring changes of the natural processes. Starting from the universally accepted statement that analytical mechanics is a harmonious symphony of natural sciences, I kept on noticing, year after year, some incongruities in the theory both in its initial assertions and in the mathematical analysis of motion. Discussions at scientific seminars and conferences have deepened the differences in knowledge and understanding of the mathematical assertions of mechanics leading to opposite views or complete misunderstanding of both the essence of mechanics and of the meaning of the mathematical symbolism describing the motion of bodies. Moreover, the basic and derived concepts, postulates, axioms, laws, principles, constraints, transformations ... are by no means singularly present in the standard mechanics. In view of all this, a new logical structure of mechanics is proposed here; it can be briefly presented by the following scheme:

This structure has attempted to separate the rational core of the classical mechanics while, at the same time, eliminating redundant conjunctions, mathematical simplifications and, most of all, apparent innovations of mass modernization. The preprinciple of existence has defined the subject matter in mechanics as well as the dominant mathematical dimension directed to it, without any justified doubt about the existence of other mental worlds in mechanics. This does not imply that the knowledge about the motion of bodies is completed; rather, it is an attempt to grade levels of knowledge from intuitive ones to more complex or even the most subtle mathematical proofs and conclusions. By stressing the differences with respect to the standard professional and scientific literature in the field of mechanics, no particular book by one or a group of authors was kept in mind, unless it is precisely quoted in the very text of this book; any possible coincidence or difference left unquoted is unintentional or unbiased. Not once was the writing of this book, especially of some of its parts, accompanied with doubts about the legitimacy and accuracy of the presented assertions, regardless of the deduced and repeated proofs or many reviews by prominent experts when some of its results had been published in scientific journals before appearing now in this monograph. This is something that will be well understood by all the eminent authors of original works in the domain of natural sciences. What was needed, in addition to ever insufficient knowledge, was courage ("gift for all sorts of mischief") in order to avoid a highly grandiose proposition about inertia coordinate systems or to modify the "law" of mechanical energy change, or to stick to the assertion that the standard calculus devastates the tensor character of the mechanical systems' differential equations of motion, or to discard the principle of solidification (freezing) of variable constraints or to change many other things that represent the subject and programs of academic studies throughout the world. In view of all this, it is rather difficult to exclude any possibility of transgression in this book. Each argument proving this, based upon the preprinciples introduced here, as well as every omission, pointed out to me, will be regarded as an authorized contribution that I will publicly acknowledge with gratitude.

The manuscript of this book has been read in whole or partially by Božidar Vujanović, Corresponding Member of the Serbian Academy of Sciences and Arts, Ranislav Bulatović, Corresponding Member of the Montenegro Academy of Sciences and Arts, Dr Slaviša Prešić and Dr Zoran Marković, Professors of the Mathematical Faculty of Belgrade University. I have accepted most of their remarks, helpful for further improvement of the text of the monograph. I am most grateful for their friendly assistance and deeply indebted for their precious time and for their contribution to the publication of the book. The manuscript was first partially and then completely arranged and aptly prepared by Dragan Urošević to whom I am sincerely grateful for assistance and cooperation.

Belgrade, September 26, 1997 Veljko A. Vujičić

PREFACE TO THE ENGLISH EDITION

Before the monograph was to be published in its Serbian edition by the Institute for Textbook Publication, Belgrade, the manuscript had been translated into English by Dragana R. Mašović, Associate Professor, Faculty of Philosophy, Niš, in july, 1998. Besides, regarding the Serbian and the english editions, the author would like to stress that he had made only a few changes in the mathematical text, namely in some of the denotation for the sake of adapting them to the English-speaking public.

The translation was read by Prof. Dr. Vladan Đorđević, Member of the Serbian Academy of Sciences, to whom the author and the publishers owe a great debt of gratitude. His suggestions, referring to the strictly scientific terminology, were almost wholly accepted by the author.

The author's thanks are also due to the technical editor Dr. Dragan Blagojević, who prepared and completely arranged the text for publication.

The author would like to thank the Mathematical Institute of Serbian Academy of Sciences and Arts as well as the Institute for the Textbook Publication, Belgrade, and the Ministry for Science an Technology of the Republic of Serbia for its financial support to the publication of monograph in English.

March 29, 1999. Veljko Vujičić

PREFACE TO THE SECOND EDITION

The first edition of the book PREPRINCIPLES OF MECHANICS has been unsuccessfully requested in bookstores and a number of libraries. Dr Dragomir Zeković, Professor at the Faculty of Mechanical Engineering in Belgrade proposed that a second edition be published, or I should say e-version of the first edition to make the book available to all interested readers.

By meticulous and professional reading of the book as well as rare giftedness Dr Dragomir Zeković has noticed a multitude of misprints and other errors ranging from commas and full stops to very complex mathematical relations, and proposed corrections. He has specified issues of mathematics and mechanics with precision, completely and at a high level, in accordance with authors' attitudes related to the subject matter of the monograph.

The contents and length of the text have remained the same, as of the first edition. E-version of the monograph was prepared by Dragan Urošević.

This second improved e-edition is officially approved by the publishers of the first edition - Zavod za izdavanje udžbenika i drugih izdanja, Belgrade and Mathematical Institute of Serbian Academy of Sciences and Arts, without whose assistance this monograph would not exist.

Belgrade, 2015. Veljko Vujičić

0. PREPRINCIPLES

The compound phrase **preprinciple** *or foreprinciple is here applied as an explicit statement whose truthfulness is not subject to re-questioning, but which theoretical mechanics as a natural science (philosophy) about motion of bodies starts from.*

The preprinciples are the basic starting point in the theory of mechanics which is here understood as one of the sciences about nature, instead of an abstract mathematical theory with no determined interpretation. Before proceeding to discussing mechanics, it should be stated that the preprinciples, as defined above, provide for its distinction from, for instance, geometry which is today no longer considered as a science about real space, but as an abstract formal theory that allows for different, equally valuable interpretations. The preprinciples express the gnoseological assumption that mechanics has its determined interpretation as a science about the motion of real bodies.

The requirement for clarity assumes that the preprinciples can be and are expressed both orally and in a written form, with no previously introduced concepts and definitions; in this way, it is easy and simple to understand the formulated determinations, consistent with the empirically acquired knowledge or hints, all of which being of interest for the theory of mechanics. While describing the motion of bodies the preprinciples represent such assertions that are themselves obvious; hence they neither provoke questions nor do they require answers since it is assumed that the answer to accept would be the one given to himself or to others by the very person who posed the question. Therefore, mechanics starts from the accepted assertion which is not called into doubt at any level of knowledge.

Wider implications of the preprinciples can be grasped by studying mechanics as a whole. The preprinciples are considered accurate in mechanics until opposed either by a new discovery or experimentally or even by a newly-discovered phenomenon in nature. If and when the scientific assertion, brought into accord with natural phenomena, appears to be contradictory to the preprinciples, it can be modified, tog ether with the corresponding assumptions of thus envisioned mechanics. The preprinciples stressed here are the following: those of *existence*, of *casual determinacy* and of *invariance.*

The **knowledge** about motion of bodies dates from ancient times. It has been preserved by genetic inheritance, forms of human practice and a multitude of various records ranging from a millennia-old till the present day ones. The historians of science point to five millennia old records dealing with the motion of bodies. The existing referential literature about the motion of bodies is so large that it considerably exceeds the limits of one congruous rational theory. Even the attempts at formal generalization have reached the sophistication level at which it is impossible to see the knowledge that man needs about the motion of bodies. Numerous definitions that cannot be refuted from the standpoint of the author's right to define his own concepts have first given rise to disparities among the theories of essential concepts which have, in their turn, caused a final split among the existing theories.

A rough mathematical description giving intellectually simplified models of natural objects is often used for explaining the body's state of motion in a way unfaithful to reality. Besides, hundreds of theorems about the motion of body that are annually published in numerous scientific and professional journals contain incongruous "truths". This is sufficiently provoking for a debate concerning the idea of "the proved truthfulness".

What is presented here is an attempt to give a new systematization of the rational core of mechanics, able to eliminate incongruity and vagueness of the existing theories. This has required, among other things, that some habitual and accepted knowledge about principles, laws, theorems and axioms should be averted, given up or at least modified. It seems logical to expect that such an approach should cause detachment or aversion, especially among older connoisseurs of mechanics, namely, those who have accepted its laws and assertions as indisputable laws of nature. In accordance with the preprinciples, as well as for the sake of greater clarity, the basic issues of this study are explained by the mathematical apparatus with which it is much easier to prove the completion of the preprinciples, especially that of invariance [62].

The knowledge about the motion of bodies is expressed by the introduced concepts and mathematical relations. The findings are elaborated, meaning that the general knowledge is not given once and for all; hence they do not have to be the same and equally true. The assertions about the motion of bodies, introduced and deduced in this mechanics, considerably differ from many others in numerous works of mechanics, especially in the part describing the motion of the body system with variable constraints.

Preprinciple of Existence (Ontological Assumptions)

On the basis of the inherited, existing and acquired **knowledge** mechanics starts from the fact that there are:

bodies, distance and **time**.

The **existence of a body** is manifested in the theoretical mechanics as a body mass for which the denotation m and its dimension M, ($\dim m = \mathrm{M}$) are accepted. Consequently, every existing body has its mass. This is the property by which the body existing in mechanics differs from the geometrical concept of the body characterized by volume V (Lat., Volumen). The difference is fundamental since the body mass is not even quantitatively identical with its volume whose dimension is derived by means of the dimension of length L, $\dim V = \mathrm{L}^3$. Every body whose motion is studied in mechanics has its mass regardless of how small it is or of the size of its volume. The body of no matter how small volume V has a finite mass m. Likewise, each part of the body has its mass. A part of the body of volume ΔV has mass Δm. If many bodies or parts of the bodies are dealt with, their masses are successively denoted with the indices m_ν, Δm_ν $(\nu = 1, 2, \dots,)$ that are to be read in the following way: "mass of the ν-th body" or "mass of the ν-th part of the body". No matter what natural numbers are added to the index ν, $\nu \in \mathbb{N}$, masses m_ν are always determined with positive real numbers $\mathbb{R}$, concrete by units of mass M dimension.

The **existence of distance** is identified everywhere: among particles, celestial bodies or between various points on the pathway that the body moves along, as well as between the place of the body and the place of observation. It is denoted by the letter l (Lat. longus)and is measured in units of dimension of length L. Though it is directly perceived and observed, inherited, acquired and understood, the distance between the body's place or position cannot be simply determined. In order to confirm this assertion it is sufficient to mention the following distances: between two airplanes in the air, two ships on the sea, two vehicles on the rough terrain or two pedestrians in the city, etc. The distances are also the subject of other sciences, especially of metrology ($\mu\varepsilon\tau\rho\omega\nu$ - measure, measuring standard, $\lambda o\gamma\iota\alpha$ - Sciences), astrometry ($\alpha\sigma\tau\rho\alpha$ - star), geometry ($\gamma\eta$ - Earth) and topology ($\tau\omega\pi\omega\sigma$ - place) since they depend upon the shape of the medium which the body's positions belong to. Any common trait can, therefore, be deduced only for very small distances between the adjoining points; even so, only under the conditions that the backgrounds against which the distances are being observed are not degenerative. The positions of two bodies, no matter how small particles they can happen to be, cannot coincide; instead, their distance must be different from zero despite the seemingly obvious fact that there is no distance between two bodies touching each other.

Regardless of how small a particle is, it is not a point; the starting point in determining the distance should be a singular point of the particle or of the body in general, namely, the one that can be adjoined by mass of the particle or of the body in general in such a way that the whole body mass is concentrated at this point which thus becomes a fictitious *mass center*. It is for this reason that this point is called the *mass point* or *material point.* In this way the question of the bodies' distance is reduced to the concept of the distance between points.

The concept of the mass or material point is different from the geometrical concept of the point not only by the fact that the mass point is characterized by mass; it differs from the particle by the fact that distance between the two particles always exist and is not equal to zero, since the particles, in addition to their mass centers, also have boundary points of their volume. In this way the mass or material point is also represented by the position $(m, \boldsymbol{r})$. The geometrical points can coincide, so that their distance can be equal to zero.

The mass point position with respect to any chosen observation point can be described by position vector $\boldsymbol{r}$, $\boldsymbol{r} \in \mathbf{R}_3$ where the symbol $\mathbf{R}_3$ implies a set of real trivectors or in numbers $r := (r^1, r^2, r^3) \in \mathbb{R}^3$ that are connected with three linearly independent vectors called the base or coordinate vectors denoted by the letters: $\boldsymbol{e} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, $\boldsymbol{э} := (\boldsymbol{э}_1, \boldsymbol{э}_2, \boldsymbol{э}_3)$ or $\boldsymbol{g} := (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$. The notation $\boldsymbol{e}$ will be used for orthonormal vectors of unit intensity $\boldsymbol{e}_i$, $(i = 1, 2, 3)$, $|\boldsymbol{e}_i| = 1$, while $\boldsymbol{э}_i$ will be used for other unit vectors of rectilinear coordinate systems.

Beside the assumption that they are unit and orthogonal, there is another assumption that $\boldsymbol{e}_i$ change neither direction nor sense; instead, they are assumed to be constant:

$$\boldsymbol{e}_i = \mathbf{const}. \tag{0.1}$$

This assumption concerning the constancy of the base vectors direction has no place in the philosophy of the body motion since all the bodies on which the vector base is chosen are moving. Hence mechanics introduces this assumption conditionally as will be later discussed regarding the introduction of the velocity definition and explanation of the inertia force.

Relative to base $\boldsymbol{e}$, position vector $\boldsymbol{r} \in \mathbf{R}_3$ can be written in its simple form in the following way:

$$\boldsymbol{r} = r^1\boldsymbol{e}_1 + r^2\boldsymbol{e}_2 + r^3\boldsymbol{e}_3 =: r^i\boldsymbol{e}_i, \tag{0.2}$$

where the iterated indices, both subscript and superscript, denote addition till the numbers taken by the indices; $(r^1, r^2, r^3) \in \mathbb{R}^3$ are coordinates of vector $\boldsymbol{r}$, while $r^1\boldsymbol{e}_1 = \boldsymbol{r}_1, \ldots, r^3\boldsymbol{e}_3 = \boldsymbol{r}_3$ are *covectors* or *components* of the given vector. Scalar multiplication of vector $\boldsymbol{r}$ by vectors $\boldsymbol{e}_j$ $(j = 1, 2, 3)$, that is, $\boldsymbol{r} \cdot \boldsymbol{e}_j = \delta_{ij} r^i = r_j$, gives the jth projections r_j of vector $\boldsymbol{r}$ upon the directions of the jth vectors $\boldsymbol{e}_j$. Only with respect to base $\boldsymbol{e}$, vector r^j coordinates are identical to its projections r_j or to coordinates r_j of covector $\boldsymbol{r}_j$ since it is:

$$\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}. \tag{0.3}$$

Observed from any point O which the position vectors start from, the directed distance between any two immediately close points M_1 and M_2 is determined by the difference between vectors $\boldsymbol{r}_2 - \boldsymbol{r}_1 = \Delta\boldsymbol{r}$, where $\boldsymbol{r}_2 = \overrightarrow{OM_2}$, $\boldsymbol{r}_1 = \overrightarrow{OM_1}$ and

$$\Delta\boldsymbol{r} = \overrightarrow{M_1M_2} = \left(r_2^i - r_1^i\right)\boldsymbol{e}_i = \Delta r^i\boldsymbol{e}_i. \tag{0.4}$$

Quantity $|\Delta\boldsymbol{r}| = \Delta s$ can be called the *metric distance* or *distance* (Lat., spatium) or space metrics:

$$\dim s = \mathrm{L}\,. \tag{0.5}$$

Time is denoted by the letter t (Lat., tempus), while its dimension T,

$$\dim t = \mathrm{T}\,.$$

It is continuous and irrevocable. In the mathematical description it can be represented by a numerical straight line or an ordered multitude of concrete numbers, while the multitude of their units is represented by real numbers $\mathbb{R}$, $t \in \mathbb{R}$.

Once the existence of time is accepted, the existence of motion, change, duration, the past, the present and the future is also accepted.

Preprinciple of Casual Determinacy

Distances, their changes and other factors of the body motion are explicitly determined throughout the whole of time, in the future as in the past, and with as much accuracy as the determinants of motion are known at any particular moment of time.

This preprinciple of mechanics prefigures that mechanics as a theory of the body motion is an accurate science in the mathematical sense, while as an applied science, it is so accurate as the data which are of importance for motion are accurately measured at one particular moment of time. In other words, mechanics is an accurately conceived theory, almost to perfection, while in engineering practice it is as much applicable as it is known, depending on the needs and technical capabilities of those applying it.

The concept of the body motion comprises: walking, driving, sailing, swimming, flying, jumping, breaking,... and all other gerunds that refer to displacement and changes of distance or changes of the position vector in time.

Preprinciple of Invariance

Neither motion nor properties of the body motion depend upon the form of statement: the determined truth about motion, once it is written in some linguistic form, is equally contained in the written output of some other form or some other alphabet.

The preprinciple of existence states that there are *mass*, time and *distance*, determined by concrete real numbers m and t as well as real vector $\Delta\boldsymbol{r}$. This preprinciple of invariance or independence of formalities allows for mass, as well as time, to be denoted by some other letters, let's say $\bar{m}$ and $\bar{t}$, which do not change the nature of numbers m and t, and for which there must be $\bar{m} = m$ and $\bar{t} = t$ in the whole correspondence. The same stands for distance $\Delta\boldsymbol{r}$. No matter where the origin of coordinates from which the position vector begins is chosen, let's say $\boldsymbol{\rho}$, there is an equality

$$\Delta\boldsymbol{r} = \Delta\boldsymbol{\rho},$$

so that distance $\Delta\boldsymbol{r}$ does not depend on the form of writing. This is even more expressed in the coordinate form, in which the choice of forms is considerably larger, such as

$$\Delta\boldsymbol{r} = \sum_{i=1}^{3} \left(\Delta r^i\right) \boldsymbol{e}_i = \Delta r^i \boldsymbol{e}_i = \Delta y^i \boldsymbol{e}_i = \Delta z^i \boldsymbol{\mathfrak{z}}_i = \Delta\rho^j \boldsymbol{g}_j = \cdots \; .$$

As such, all the three realities $m \in \mathbb{R}$, $t \in \mathbb{R}$ and $\Delta\boldsymbol{r} \in \boldsymbol{R}_3$ are invariants, m and t being scalar ones, while $\Delta\boldsymbol{r}$ is a *vector invariant*.

All other factors of the body motion are also invariantly expressed in various coordinate systems.

I. BASIC DEFINITIONS

By means of the previously accepted concepts as well as the introduced notations it is both possible and necessary to determine (define) some of the essential concepts of mechanics.

Definition 1. Velocity. *The boundary value of the ratio between distance and time interval* Δt*, for which the material point moves from one position* $\boldsymbol{r}(t)$ *to another position* $\boldsymbol{r}(t+\Delta t)$ *immediately close to it, that is, the natural derivative of the position vector with respect to time*

$$\frac{d\boldsymbol{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} \stackrel{def}{=} \boldsymbol{v} \tag{1.1}$$

is called the velocity of the point.

Velocity is, therefore, *a vector and its nature is invariant.* Depending on our need for more specific determination, there are other formulations such as: velocity vector, momentous velocity of the material point, velocity vector at particular position, or, even more completely, the velocity vector of the material point's motion at one moment or position; nor is the expression the velocity of the material point position change considered contradictory, if the position implies the position vector.

Much more important than the formulation itself is the fact that the velocity definition establishes a relation between distance and time. The velocity dimension is derived from the velocity definition, being

$$\mathrm{Dim}\, v = \mathrm{L\,T}^{-1} \tag{1.2}$$

The position vector now becomes time-dependent; hence, it follows that:

$$\boldsymbol{r}(t) = r^i(t)\boldsymbol{g}_i(t) \rightarrow \boldsymbol{v} = \frac{dr^i}{dt}\boldsymbol{g}_i + r^i \frac{d\boldsymbol{g}_i}{dt}, \tag{1.3}$$

and this relation opens up the problem of accuracy in mechanics as well as of the necessity to make relative the body motion theory which has definition 1.

According to the preprinciple of casual definiteness, relation (1.1) should be used for an explicit determination of velocity if the vector of functions $\boldsymbol{r}(t)$ is known, and vice versa, of the position vector if vector $\boldsymbol{v}(t)$ is known [36]. It follows from relation (1.1) that:

$$\boldsymbol{r}(t) - \boldsymbol{r}(t_0) = \int_{t_0}^{t} \boldsymbol{v}(t)dt \tag{1.4}$$

or

$$r^i(t)\boldsymbol{g}_i(t) - r^i(t_0)\boldsymbol{g}_i(t_0) = \int_{t_0}^{t} v^i(t)\boldsymbol{g}_i(t)dt,$$

that is,

$$\left[r^i(t) - r^k(t_0)g_k^i(t_0,t)\right]\boldsymbol{g}_i(t) = \int_{t_0}^{t} \boldsymbol{v}(t)dt, \tag{1.5}$$

where $g_k^i(t_0,t) : \boldsymbol{g}(t_0) \to \boldsymbol{g}(t)$.

Therefore, definition (1.1) can also be written in the following form:

$$\frac{d}{dt}\left[\boldsymbol{r}(t) - \boldsymbol{c}\right] = \boldsymbol{v}, \quad \boldsymbol{c} = \mathbf{const},$$

which shows that the velocity of the point's motion does not depend upon the choice of the position vector pole in the same base.

An underlying difficulty in determining the point's velocity emerges in the previous choice of the base vectors system which also implies the pole and direction of these vectors. They can be assumed as constant vectors, but, objectively, all the bases which are the base for base vector system $\boldsymbol{g}_i$, move; consequently, vectors $\boldsymbol{g}_i$ change in time. For human existence and for observing the way the bodies move, the base is the Earth which, just like the other planets, moves; so, its relative speed with respect to the Sun, as well as its angular velocity, are measured or calculated till sufficient accuracy is achieved. Regardless of the directions chosen for the base vectors' axes $\boldsymbol{e}_i$, $\boldsymbol{\ni}_i$, $\boldsymbol{g}_i$, including the directions of the Earth as the "immobile" star, they cannot be invariable due to the Earth's motion. In order to reduce the relation to a scalar form, vectors $\dfrac{d\boldsymbol{g}_i}{dt}$ should be expressed by means of base vectors $\boldsymbol{g}_i(t)$. Let it be:

$$\dot{\boldsymbol{g}}_j = \frac{d\boldsymbol{g}_j(t)}{dt} = \omega_j^i \boldsymbol{g}_i(t), \tag{1.6}$$

where ω_j^i are, for the time being, indefinite coefficients of vector $\dot{\boldsymbol{g}}_i$ resolution. On the basis of relation (1.2) it follows that the coefficient $\dot{\boldsymbol{g}}_i$ dimension is time to the power of minus one, that is,

$$\dim \omega_j^i = \mathrm{T}^{-1} \tag{1.7}$$

Quantities ω of the dimension T^{-1} are called *angular velocities*, circular frequencies or frequencies.

Substituting relation (1.6) into (1.3) it is obtained:

$$\boldsymbol{v} = \left(\frac{dr^i}{dt} + \omega^i_j r^j \right) \boldsymbol{g}_i = v^i \boldsymbol{g}_i, \tag{1.8}$$

it follows from the above relation that, due to the independence of the vector $\boldsymbol{g}_i$, the velocity vector coordinates are:

$$v^i = \frac{dr^i}{dt} + \omega^i_j r^j \quad (i, j = 1, 2, 3). \tag{1.9}$$

According to the preprinciples, the solutions of this differential equation's system for known velocities $v^i(t)$ must be equal to solutions (1.5); the integrating operations must be elaborated so that the conditions for casual determince and invariance should be satisfied. This is provided for by the covariant or tensor integral, under the condition that double-dotted tensor $g^i_k(t_0, t)$ and base vectors $\boldsymbol{g}_i(t)$ are known.

For the constant base vectors such as $\boldsymbol{g} = \boldsymbol{e}$ relation (1.6) reduces to a system of homogeneous equations:

$$\omega^1_j \boldsymbol{e}_1 + \omega^2_j \boldsymbol{e}_2 + \omega^3_j \boldsymbol{e}_3 = 0,$$

from which it follows that $\omega^i_j = 0$, so that the velocity vector coordinates (1.9) are in this case:

$$v^i = \frac{dr^i}{dt}. \tag{1.10}$$

This clearly shows that the vector coordinates of the material point's velocity differ with respect to various base vectors. Due to the invariance preprinciple as well as definition (1.1) it can be written:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \left(\frac{dr^i}{dt} + \omega^i_j r^j \right) \boldsymbol{g}_i = \frac{Dr^i}{dt} \boldsymbol{g}_i = \frac{dr^i}{dt} \boldsymbol{e}_i = v^i \boldsymbol{g}_i, \tag{1.11}$$

which satisfies the form equality and corresponds to the expression "the natural derivative" used in the definition. Regarding the fact that nine above-mentioned coefficients ω^i_j are unknown, if we start from the general assumption that each base moves and that the coefficients cannot be determined – contrary to the preprinciple of casual determinces – it is natural that the coordinate vectors that can be related to some base vectors, not likely to change in time, should be chosen as coordinate vectors.

Therefore, in order to determine the material point's position as well as the points' distance by relations (0.2) and (0.4), in addition to condition (0.3), what should be introduced here is the condition that the base vectors do not change with time:

$$\frac{d\boldsymbol{e}_i}{dt} = 0. \tag{1.12}$$

The choice of once oriented base constant vectors provides for setting up other oriented coordinate systems, including curvilinear ones, that can be brought into mutual mapping and in relation to which velocity is invariant.

Coordinate Systems. The concept of coordinate system here implies an ordered set of real numbers and a set of mutually independent vectors that are called coordinate vectors. The coordinate vectors differ from the base ones only in the sense that the base ones are previously determined with respect to objects, while the coordinate ones are determined with respect to the base ones. If the coordinate ones are original, then they are base coordinate vectors. On the basis of the base vectors, originally chosen as in relations (0.2), (0.4) and (1.3), it is possible to introduce other coordinate systems $x = \left(x^1, x^2, x^3\right)$, $(x^i \in \mathbb{R})$ in which the material point's position is explicitly mapped while the velocity has a general invariant form.

Any other rectilinear coordinate system can be chosen as well, let's say $(z, \mathbf{э})$, whose directions change in time with respect to base system $(y, \boldsymbol{e})$. The two systems' ratio is determined by the relations:

$$y^i = \gamma^i_\alpha z^\alpha, \;\; \boldsymbol{e}_i = \frac{\partial z^\alpha}{\partial y^i} \mathbf{э}_\alpha = \bar{\gamma}^\alpha_i \mathbf{э}_\alpha, \;\; \gamma^i_\alpha \bar{\gamma}^\beta_i = \delta^\beta_\alpha.$$

The velocity vector can be represented by the equation:

$$\begin{aligned}\boldsymbol{v} &= \frac{d}{dt}\left(y^i \boldsymbol{e}_i\right) = \dot{y}^i \boldsymbol{e}_i = \left(\dot{\gamma}^i_\alpha z^\alpha + \gamma^i_\alpha \dot{z}^\alpha\right) \bar{\gamma}^\beta_i \mathbf{э}_\beta = \\ &= \left(\dot{\gamma}^i_\alpha \bar{\gamma}^\beta_i z^\alpha + \delta^\beta_\alpha \dot{z}^\alpha\right) \mathbf{э}_\beta = \left(\dot{z}^\beta + \omega^{*\beta}_\alpha z^\alpha\right) \mathbf{э}_\beta = v^\beta \mathbf{э}_\beta\end{aligned}$$

where

$$\omega^{*\beta}_\alpha = \dot{\gamma}^i_\alpha \bar{\gamma}^\beta_i = -\omega^\beta_{*\alpha} = -\gamma^i_\alpha \dot{\bar{\gamma}}^\beta_i$$

are anti-symmetrical coefficients and $*$ denotes the empty place of an index, since it is

$$\frac{d}{dt}\left(\gamma^i_\alpha \bar{\gamma}^\beta_i\right) = \dot{\gamma}^i_\alpha \bar{\gamma}^\beta_i + \gamma^i_\alpha \dot{\bar{\gamma}}^\beta_i = \dot{\delta}^\beta_\alpha = 0.$$

It follows that the velocity vector coordinates with respect to the coordinate rotary system $(z, \mathbf{э})$

$$v^\beta = \dot{z}^\beta + \omega^{*\beta}_\alpha z^\alpha. \tag{1.13}$$

By comparison with expression (0.2), it can be seen that $\boldsymbol{r}$ is a function of y^i coordinates, and through them, it is also a function of x coordinates, so that $\boldsymbol{r} = \boldsymbol{r}\left(y\left(x\right)\right) = y^i\left(x\right) \boldsymbol{e}_i$. According to definition (1.1), the velocity vector is:

$$\boldsymbol{v} = \dot{y}^i \boldsymbol{e}_i = \frac{\partial \boldsymbol{r}}{\partial y^i} \frac{\partial y^i}{\partial x^k} \dot{x}^k = \dot{x}^k \frac{\partial y^i}{\partial x^k} \boldsymbol{e}_i = \dot{x}^k \boldsymbol{g}_k = v^k \boldsymbol{g}_k. \tag{1.14}$$

It follows from these invariant relations that coordinate vectors $\boldsymbol{g}_k$ for the system of x coordinates are derived by base vectors $\boldsymbol{e}_i$ by means of the covariant relations

$$\boldsymbol{g}_i = \frac{\partial y^k}{\partial x^i}\boldsymbol{e}_k = \frac{\partial \boldsymbol{r}}{\partial x^i} = \boldsymbol{g}_i\,(x)\,, \tag{1.15}$$

as well as the metric tensor

$$g_{ij} := \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \frac{\partial \boldsymbol{r}}{\partial x^i}\cdot\frac{\partial \boldsymbol{r}}{\partial x^j} = \delta_{kl}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}. \tag{1.16}$$

Accordingly, velocity vector $\boldsymbol{v} = \dfrac{d}{dt}\left(r^i\boldsymbol{g}_i\right)$ can be reduced to the general form:

$$\boldsymbol{v} = \frac{dr^i}{dt}\boldsymbol{g}_i + r^i\frac{\partial \boldsymbol{y}_i}{\partial x^k}\frac{dx^k}{dt} = \left(\frac{dr^i}{dt} + r^j\Gamma^i_{jk}\frac{dx^k}{dt}\right)\boldsymbol{g}_i = \nabla_k r^i\dot{x}^k\boldsymbol{g}_i = v^i\boldsymbol{g}_i$$

where $\Gamma^i_{jk}(x)$ are the coefficients connecting the coordinate vectors $\boldsymbol{g}_i$ and their partial derivatives with respect to x coordinates, namely:

$$\frac{\partial \boldsymbol{g}_j}{\partial x^k} = \Gamma^i_{jk}(x)\boldsymbol{g}_i(x). \tag{1.17}$$

It follows that the velocity vector coordinates in any system of coordinates $(x, \boldsymbol{g})$ can be written in the form:

$$v^i = \frac{Dr^i}{dt} = \frac{dr^i}{dt} + r^j\Gamma^i_{jk}\dot{x}^k, \tag{1.18}$$

where $\dfrac{Dr^i}{dt}$ denote natural derivatives of the position vector coordinates with respect to time, while

$$\nabla_k r^i = \frac{\partial r^i}{\partial x^k} + r^j\Gamma^i_{kj}$$

is a covariant derivative of these vector's coordinates with respect to the point's position coordinates [36], [49].

The projections of velocity vector $\dot{y}_i$ upon the axes of base vectors $\boldsymbol{e}_i$, as scalar products of vector $\dot{v}_i$ and base vectors $\boldsymbol{e}_i$, are equal to the velocity vector coordinates $\dot{y}^i$:

$$\dot{y}_i = \delta_{ij}\dot{y}^j,$$

while v_i projections upon the axes of the coordinate vectors $\boldsymbol{g}_i$ are linear homogeneous forms of the velocity vector coordinates:

$$v_i = g_{ij}v^j = g_{ij}\frac{Dr^j}{dt} = g_{ij}\dot{x}^j$$

where $g_{ij}(x)$ is metric tensor (1.16).

The velocity square, as a scalar invariant, can now be written in the following form:

$$v^2 = \delta_{ij}\dot{y}^i\dot{y}^j = g_{ij}\dot{x}^i\dot{x}^j = g_{ij}\frac{Dr^i}{dt}\frac{Dr^j}{dt}. \tag{1.19}$$

Regarding the fact that element ds of path $s(t)$:

$$ds^2 = g_{ij}dx^i dx^j = g_{ij}Dr^i Dr^j$$

it follows that the magnitude of the velocity vector $\boldsymbol{v}$ is simply determined as a derivative of the path with respect to time, that is,

$$v = \frac{ds(t)}{dt}. \tag{1.20}$$

Therefore, it can be proved that *covariant derivative* $\nabla_i r_j$ *of the projections* r_j *of the point position vector upon the* j*th coordinate direction with respect to* x^i *coordinate is equal to the respective coordinates of metric tensor* g_{ij}. with respect to respective indices.

Namely, if $\boldsymbol{r} = y^k\boldsymbol{e}_k$ vector is scalarly multiplied by $\boldsymbol{g}_j$ vector, the projection of the position vector upon the j-th coordinate axis $\boldsymbol{r}\cdot\boldsymbol{g}_j = r_j = y^k\left(\boldsymbol{e}_k\cdot\boldsymbol{g}_j\right)$ is obtained or:

$$r_j = y^k\frac{\partial y^l}{\partial x^j}\left(\boldsymbol{e}_k\cdot\boldsymbol{e}_l\right) = \delta_{kl}y^k\frac{\partial y^l}{\partial x^j}.$$

Regarding relation (1.16), it follows:

$$\frac{\partial r_j}{\partial x^i} = \delta_{kl}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} + \delta_{kl}y^k\frac{\partial^2 y^l}{\partial x^i\partial x^j} = g_{ij} + \delta_{kl}y^k\frac{\partial y^l}{\partial x^m}\Gamma^m_{ij} = g_{ij} + r_m\Gamma^m_{ij}.$$

This also confirms the assertion that it is

$$\nabla_i r_j = \frac{\partial r_j}{\partial x^i} - r_m\Gamma^m_{ij} = g_{ij}.$$

By partial differentiating metric tensor (1.16) with respect to all the coordinates and summing up, it is obtained that $\Gamma_{ij,k}(x^1,x^2,x^3)$ are Christopher's symbols for the given metrics:

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{1.21}$$

For the coordinate system z in which $g_{ij} = \ni_{ij} = \text{const}$ all the symbols Γ^m_{ij} are equal to zero, so that

$$\nabla_i r_j = \frac{\partial r_j}{\partial z^i} = \ni_{ij},$$

which more clearly points to the relation between the position vector and the metric tensor.

The previous relations can be related to the *base vectors' covariant derivatives* with respect to the coordinates

$$\nabla_k \boldsymbol{g}_j = \frac{\partial \boldsymbol{g}_j(x)}{\partial x^k} - \Gamma^i_{jk}(x)\boldsymbol{g}_i(x) = 0 \tag{1.22}$$

which are very important for describing the base vectors and their changes in time. Just as relations (1.15) establish the ratio between base vectors $\boldsymbol{e}$ and the subsequently introduced coordinate $\boldsymbol{g}$, so the covariant derivative $\nabla_k \boldsymbol{g}_j$ stands in a direct relation with conditions (1.12). The derivatives of relations (1.15) with respect to time, due to condition (1.13) are:

$$\frac{d\boldsymbol{g}_i}{dt} = \frac{\partial^2 y^k}{\partial x^j \partial x^i}\dot{x}^j \boldsymbol{e}_k.$$

It is always possible to introduce such functions $\Gamma(x)$ so that it is

$$\frac{\partial^2 y^k}{\partial x^j \partial x^i} = \Gamma^\lambda_{ij}\frac{\partial y^k}{\partial x^\lambda}$$

thus, it is obtained

$$\frac{d\boldsymbol{g}_i}{dt} - \boldsymbol{g}_\lambda \Gamma^\lambda_{ji}\frac{dx^j}{dt} = \frac{D\boldsymbol{g}_i}{dt} = \nabla_j \boldsymbol{g}_i \dot{x}^j = 0. \tag{1.23}$$

These are the conditions which, just like conditions (1.12), show that coordinate vectors $\boldsymbol{g}_i$ are covariantly constant:

$$y^i = \gamma^i_\alpha z^\alpha, \ \boldsymbol{e}_i = \frac{\partial z^\alpha}{\partial y^i}\textbf{ɘ}_\alpha = \bar{\gamma}^\alpha_i \textbf{ɘ}_\alpha, \ \gamma^i_\alpha \bar{\gamma}^\beta_i = \delta^\beta_\alpha$$

The velocity vector is

$$\boldsymbol{v} = \dot{y}^i \boldsymbol{e}_i = \left(\dot{z}^\beta + \omega^{*\beta}_\alpha z^\alpha\right)\textbf{ɘ}_\beta = v^\beta \textbf{ɘ}_\beta$$

where $\omega^{*\beta}_\alpha = \dot{\gamma}^i_\alpha \bar{\gamma}^\beta_i = -\omega^\beta_{*\alpha} = -\gamma^i_\alpha \dot{\bar{\gamma}}^\beta_i$ are anti-symmetrical coefficients. It follows that the velocity vector coordinates with respect to the coordinate inverse system $(z, \textbf{ɘ})$ are:

$$v^\beta = \dot{z}^\beta + \omega^{*\beta}_\alpha z^\alpha = \frac{Dz^\beta}{dt}.$$

This clearly shows that the velocity vector coordinates are varied regarding various coordinate vectors. Due to the preprinciple of invariance as well as the casual definiteness of the statement about "natural derivative" from the definition of velocity, it is natural that the chosen coordinate vectors should be the ones that can be related to some base vectors (0.1), invariable in time.

Once base vectors $\boldsymbol{e}_i$ are chosen, other oriented coordinate vectors $\boldsymbol{g}_i$ can be chosen, including curvilinear ones, for which the natural derivatives (1.23) will be valid.

Definition 2. Motion Impulse. *The product of mass m of the material point and its velocity vector $\boldsymbol{v}$ is called the motion impulse of material point $\boldsymbol{p}$.*

In accordance with the preprinciples, the velocity definition and the above-given definition, the motion impulse can be written in the following way:

$$\begin{aligned} \boldsymbol{p} = m\boldsymbol{v} = m\dot{y}^i\boldsymbol{e}_i = m\frac{Dz^i}{dt}\boldsymbol{\ni}_i = mv^i\boldsymbol{g}_i = \\ = m\frac{Dr^i}{dt}\boldsymbol{g}_i = m\frac{\partial \boldsymbol{r}}{\partial x^i}\dot{x}^i = m\dot{x}^i\boldsymbol{g}_i. \end{aligned} \tag{1.24}$$

Further on, special emphasis will be put on p_i projections of this vector upon coordinate directions $\boldsymbol{g}_i$:

$$p_i = \boldsymbol{p}\cdot\boldsymbol{g}_i = mg_{ij}\dot{x}^j = a_{ij}\dot{x}^j, \tag{1.25}$$

where

$$a_{ij} = mg_{ij} = m\frac{\partial \boldsymbol{r}}{\partial x^i}\cdot\frac{\partial \boldsymbol{r}}{\partial x^j} = a_{ji}\,(m,x) \tag{1.26}$$

is *inertia tensor.*

In the coordinate system $(z,\boldsymbol{\ni})$, in accordance with (1.13) it will be:

$$p_i = \ni_{ij}\left(\dot{z}^j + \omega_k^{*j}z^k\right) = \ni_{ij}\frac{Dz^j}{dt},$$

where $\omega_k^{*j} = 0$ for $k = j$, while $\ni_{ij} = m\boldsymbol{\ni}_i\cdot\boldsymbol{\ni}_j$. Therefore, in such a system, there is the material point's motion impulse, regardless of the fact that the points do not move with respect to this coordinate system:

$$p_i = \ni_{ij}\omega_k^{*j}z^k = \omega_{ik}^{*}z^k = -\omega_{*ik}z^k.$$

It should be noted that inertia tensor $a_{ij}(m,x)$ differs from the metric tensor $g_{ij}(x)$. The basic physical dimensions of the impulse vector are:

$$\dim\boldsymbol{p} = \mathrm{M\,L\,T^{-1}}$$

but its coordinates and projections can also have other dimensions.

If x coordinate is an angle, then it is:

$$\dim p_i = \mathrm{M\,L^2\,T^{-1}}\,.$$

Inertia tensor a_{ij} sets up a relation between impulse and velocity at any position. Its essential content is mass which exists for every body or material point as well as in all coordinate systems.

Example 1. In an orthogonal rectilinear coordinate system$(y,\boldsymbol{e})$, the inertia tensor coordinates are equal to the point's mass since it is

$$a_{ij} = m\delta_{ij} = \begin{cases} m & i = j \\ 0 & i \neq j. \end{cases} \tag{1.27}$$

However, the following relations are valid in other coordinate systems [36]:
Cylindrical: $x^1 = \rho$, $x^2 = \varphi$, $x^3 = z$; $y^1 = \rho\cos\varphi$, $y^2 = \rho\sin\varphi$, $y^3 = z$,

$$a_{ij} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Spherical: $x^1 = \rho$, $x^2 = \varphi$, $x^3 = \theta$; $y^1 = \rho\sin\varphi\cos\theta$, $y^2 = \rho\sin\varphi\sin\theta$, $y^3 = \rho\cos\varphi$

$$a_{ij} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2\sin^2\varphi \end{pmatrix}.$$

Rotatory-ellipsoitic: $x^1 = \xi$, $x^2 = \eta$, $x^3 = \theta$; $y^1 = b\,\mathrm{ch}\,\xi\sin\eta\cos\theta$, $y^2 = b\,\mathrm{ch}\,\xi\sin\eta\sin\theta$, $y^3 = b\,\mathrm{ch}\,\xi\cos\eta$,

$$a_{ij} = mb^2\begin{pmatrix} \mathrm{sh}^2\,\xi & 0 & 0 \\ 0 & \mathrm{ch}^2\,\xi & 0 \\ 0 & 0 & \mathrm{ch}^2\,\xi\sin^2\eta \end{pmatrix}.$$

Rotatory-parabolidic: $x^1 = \xi$, $x^2 = \eta$, $x^3 = \theta$; $y^1 = \xi\eta\cos\theta$, $y^2 = \xi\eta\sin\theta$, $y^3 = \frac{1}{2}\left(\xi^2 - \eta^2\right)$.

$$a_{ij} = m\begin{pmatrix} \xi^2+\eta^2 & 0 & 0 \\ 0 & \xi^2+\eta^2 & 0 \\ 0 & 0 & \xi^2\eta^2 \end{pmatrix}$$

Bipolar: $x^2 = \eta, x^3 = \theta; (0 \le \xi \le \pi, -\infty < \eta < \infty, 0 \le \theta \le 2\pi); y^1 = b\frac{\sin\xi\cos\theta}{\mathrm{ch}\,\eta - \cos\xi}, y^2 = b\frac{\sin\xi\sin\theta}{\mathrm{ch}\,\eta-\cos\xi}, y^3 = b\frac{\mathrm{sh}\,\eta}{\mathrm{ch}\,\eta-\cos\xi}$,

$$a_{ij} = mb^2\begin{pmatrix} \frac{1}{(\mathrm{ch}\,\eta-\cos\xi)^2} & 0 & 0 \\ 0 & \frac{1}{(\mathrm{ch}\,\eta-\cos\xi)^2} & 0 \\ 0 & 0 & \frac{\sin^2\xi}{(\mathrm{ch}\,\eta-\cos\xi)^2} \end{pmatrix}$$

Cylindrical-orthogonal: $x^1, x^2, x^3 = z$; $y^1 = f^1\left(x^1, x^2\right)$, $y^2 = f^2\left(x^1, x^2\right)$, $y^3 = x^3$,

$$a_{ij} = m\begin{pmatrix} \left(\frac{\partial f^1}{\partial x^1}\right)^2 + \left(\frac{\partial f^2}{\partial x^1}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\partial f^1}{\partial x^2}\right)^2 + \left(\frac{\partial f^2}{\partial x^2}\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The inertia tensor forms a positive definite matrix whose determinant is other than zero. During the transformation from one coordinate system into other ones, constraints should be looked for in mapping and degeneration of the figure, instead of in the nature of the inertia tensor. Regarding the fact that its determinant is different from zero, it is possible, by means of relation (1.25), to determine the velocity vector $\dot{x}^i$ coordinates as homogeneous linear functions of the impulse

$$\dot{x}^i = a^{ij} p_j \tag{1.28}$$

where $a^{ij}(x)$ are countervariant coordinates of the inertia tensor. Relations (1.28) and (1.27) are existing, determinable and invariant with respect to all possible mappings from one coordinate system into the other one.

It should also be noted that a_{ij} inertia tensor changes during the motion if mass $m(t)$ of the material point changes in time. This relevant fact points to a considerable qualitative difference between the inertial a_{ij} and the metric tensor g_{ij}. If this fact is neglected, general conclusions about the motion of the celestial as well as the elementary bodies may be wrong. This will be more clearly seen in further presentation of this theory.

Definition 3. Acceleration. *The natural derivative of the velocity vector with respect to time is called the vector of the point's acceleration.*

This definition is replaced by a shorter written form:

$$\boldsymbol{a} \stackrel{\text{def}}{=} \frac{d\boldsymbol{v}}{dt}, \quad \dim \boldsymbol{a} = \mathrm{L\,T}^{-2}. \tag{1.29}$$

Acording to the definition and respective relations of the velocity vector (1.1)–(1.29), the *acceleration vector* $\boldsymbol{a}$ (Lat. acceleratio) can be written in many ways:

$$\begin{aligned} \boldsymbol{a} &= \frac{Dv^i}{dt}\boldsymbol{g}_i = \frac{D^2 r^i}{dt^2}\boldsymbol{g}_i \ddot{y}^i \boldsymbol{e}_i \\ &= \left(\frac{dv^i}{dt} + \Gamma^i_{jk} v^j \frac{dx^k}{dt}\right)\boldsymbol{g}_i = a^i \boldsymbol{g}_i, \end{aligned} \tag{1.30}$$

and its coordinates:

$$a^i = \frac{dv^i}{dt} + \Gamma^i_{jk} v^j \frac{dx^k}{dt} = \frac{Dv^i}{dt}. \tag{1.31}$$

At the same time, it is necessary to know with respect to what coordinate vectors $\boldsymbol{g}_i$ or metric tensor g_{ij}, coefficients Γ^i_{jk} are calculated. If relations (1.15) between base $\boldsymbol{e}_i$ and coordinate vectors $\boldsymbol{g}_i$ from previous invariant relations (1.31) are taken into consideration, it is easy to notice that the acceleration vector coordinates can be mapped from one coordinate system y into the other one x if matrix determinant $\left(\frac{\partial y}{\partial x}\right)$ is different from zero, since the relations are derived:

$$\ddot{y}^i = a^k \frac{\partial y^i}{\partial x^k} \quad \text{i} \quad a^i = \ddot{y}^k \frac{\partial x^i}{\partial y^k}. \tag{1.32}$$

Regarding practice, the acceleration analysis with respect to the natural system of the coordinate vectors $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)$, which are unit and orthogonal is of special interest. Let the vector

$$\boldsymbol{\eta}_1 = \boldsymbol{\tau} = \lim_{\Delta s \to 0} \frac{\Delta \boldsymbol{r}}{\Delta s} = \frac{d\boldsymbol{r}}{ds}$$

determine the direction and sense of the tangent on the pathway s at some moment of time t and let it coincide with the velocity direction at this moment of time; $\boldsymbol{\eta}_2 \equiv \boldsymbol{n}$ is directed with respect to the principal normal toward the center of the (first) pathway curve, while $\boldsymbol{\eta}_3 \equiv \boldsymbol{b}$ is directed with respect to the (second normal) binormal.

Regarding base vectors $\boldsymbol{e}_i$e, the coordinate vectors can be determined by means of linear relations:

$$\boldsymbol{\eta}_i = \alpha_i^k \boldsymbol{e}_k \quad \longrightarrow \quad \boldsymbol{e}_k = \bar{\alpha}_k^i \boldsymbol{\eta}_i, \quad \left|\alpha_i^k\right| \neq 0$$

where α_i^k are cosines of respective angles formed by vectors $\boldsymbol{\eta}_i$ and $\boldsymbol{e}_k$.

The velocity vector with respect to the natural trihedron can be represented by the expression:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = v\boldsymbol{\tau} \tag{1.33}$$

where v, as can be seen from (1.20), is velocity vector magnitude.

Since $\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1 \longrightarrow \dfrac{d\boldsymbol{\tau}}{ds} \cdot \boldsymbol{\tau} = 0$, where from it follows that vector $\dfrac{d\boldsymbol{\tau}}{ds}$ is perpendicular to $\boldsymbol{\tau}$, it can be written that $\dfrac{d\boldsymbol{\tau}}{ds} = \dfrac{d\boldsymbol{\tau}}{d\theta}\dfrac{d\theta}{ds} = \varkappa \boldsymbol{n}$, where $\varkappa$ is the curvature of a curve. According to definition (1.29), the acceleration vector can be resolved along tangent $\boldsymbol{\tau}$ and principal normal $\boldsymbol{n}$, namely:

$$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{\tau} + v^2 \frac{d\boldsymbol{\tau}}{ds} = \frac{dv}{dt}\boldsymbol{\tau} + \frac{v^2}{\rho_k}\boldsymbol{n} = a_\tau \boldsymbol{\tau} + a_n \boldsymbol{n}, \tag{1.34}$$

where ρ_k is pathway curve radius, while $\boldsymbol{n}$ is a unit vector of the principal normal, so that

$$a_\tau = \frac{dv}{dt} \tag{1.35}$$

the acceleration vector coordinate directed with respect to the tangent (tangent acceleration) and

$$a_n = \frac{v^2}{\rho_k} \tag{1.36}$$

the acceleration coordinate $\boldsymbol{a}$ is directed with respect to the principal normal (normal acceleration). Relation (1.34) clearly shows that only one acceleration component, namely, $\boldsymbol{a}_n = a\boldsymbol{n}$, which belongs to the tangent vector field or $\boldsymbol{a}_\tau = \dot{v}\boldsymbol{\tau}$ belonging to the osculatory one, perpendicular to the tangent plane does not determine the acceleration vector.

Definition 4. Inertia Force. *The product of the material point's mass m and the vector, which is equal but directed opposite to acceleration vector $\boldsymbol{a}$, is called the material point's inertia force.*

If the inertia force is denoted by the letter $\boldsymbol{I}_F$ or simply $\boldsymbol{I}$, the definition can be written in a shorter form:

$$\boldsymbol{I}_F \stackrel{\text{def}}{=} -m\boldsymbol{a} = -m\frac{d\boldsymbol{v}}{dt}. \tag{1.37}$$

Hence it follows that

$$\dim I_F = \mathrm{M\,L\,T}^{-2}\,.$$

According to relations (1.30) and (1.31), it can be written:

$$\boldsymbol{I}_F = I^i\boldsymbol{g}_i = -ma^i\boldsymbol{g}_i = m\frac{Dv^i}{dt}\boldsymbol{g}_i = -m\ddot{y}^i\boldsymbol{e}_i, \tag{1.38}$$

where it can be seen that the vector coordinates of inertia force

$$I^i = -m\frac{Dv^i}{dt} = -m\left(\frac{dv^i}{dt} + \Gamma^i_{jk}v^j\frac{dx^k}{dt}\right). \tag{1.39}$$

By scalar multiplication of vector (1.38) and coordinate vector $\boldsymbol{g}_j$ the projections of the inertia force vectors upon the j-th coordinate axes are obtained, namely,

$$\begin{aligned} I_j = I_F \cdot \boldsymbol{g}_j = -a_{ij}I^i = -a_{ij}\frac{Dv^i}{dt} = \\ = -a_{ij}\left(\frac{dv^i}{dt} + \Gamma^i_{lk}v^l\frac{dx^k}{dt}\right), \end{aligned} \tag{1.40}$$

where a_{ij}, as in relation (1.26), is inertia tensor. It is clear from relation (1.40) that the inertia force can have many addends; also, I_j projections upon y coordinate axes, depending on a_{ij}, can have dimensions different from $\mathrm{M\,L\,T}^{-2}$. That is why I_j coordinates can be called *generalized inertia forces.*

Regarding the natural system of coordinates, in view of relations (1.34), it follows that:

$$\boldsymbol{I}_F = I^\tau\boldsymbol{\tau} + I^n\boldsymbol{n} = -m\frac{dv}{dt}\boldsymbol{\tau} - m\frac{v^2}{\rho_k}\boldsymbol{n}.$$

It is obvious from this equation that tangent coordinate I^τ of the inertia force

$$I^\tau = -m\frac{dv}{dt}, \tag{1.41}$$

while the coordinate on the principal normal of the curve

$$I^n = -m\frac{v^2}{\rho_k}.$$

Therefore, two last expressions show in a more obvious way than relations (1.39) that the inertia force can exist even in the case when the velocity magnitude is constant $v = \text{const}$. Only in the case that the velocity vector $\boldsymbol{v} = \mathbf{const}$, that is, that the velocity changes neither magnitude nor direction, does it follow from relation (1.37) that the inertia force is equal to zero. It can be seen, from the relation for the velocity square (1.19), that $v = \text{const}$ if all the velocity coordinates $\dot{y}^i$ with respect to the base system $\boldsymbol{e}$ are constant values. Since base vectors $\boldsymbol{e}_i$ are constant with respect to time, it also follows that the velocity vector is constant ($\boldsymbol{v} = \mathbf{const}$). Consequently, as in the definition of inertia force, it follows that the bodies moving at constant velocity $\boldsymbol{v}$ do not produce inertia force. The coordinate systems that can be related to such bodies are called inertia coordinate systems.

The initial point of the force vector is called the dynamic point (Greek, $\delta\upsilon\nu\alpha\mu\iota\sigma$ – force). The material and the dynamic points geometrically coincide, but the concept of the material point implies mass, while the same material point, when it is called a dynamic point, is related to inertia force, or, more generally, some force acting at that point. In some parts of mechanics only relations between forces are discussed with no concept of mass. In this case, it is more natural and rational to use the concept of dynamic point.

II. LAWS OF DYNAMICS

The word *dynamics* is derived from the Greek word ($\delta\iota\nu\alpha\mu\iota\kappa\eta$) meaning "a science about forces", while the term *laws of dynamics* implies *formulations and definitions used for determining particular forces with accuracy of mathematical functions up to the concrete constant.* In this study, the knowledge necessary for the formulations that make up the laws of dynamics is acquired on the basis of experiments and measurements in nature and human practice so that no other logical proofs of their truthfulness are needed. They can be expressed in an oral or written way, in words or by mathematical formulae that satisfy the preprinciples of mechanics.

By the definition of inertia force, the dimensions of force are determined and thus, the laws of dynamics as well; in accordance with this definition all other forces are formulated as vector invariants having a dimension $\mathrm{M\,L\,T^{-2}}$.

The phrase "up to the concrete constant" implies concrete real numbers determined by various measurements of experimental or natural phenomena. They are called *dynamic parameters* in order to emphasize that they are comprised by the forces' functions.

If the difference between the expressions *determination-definition* and *determination-law* is not sufficiently clear, it should be stressed that the *definition* is a product of human mind as well as of the desire for singular accuracy, while the laws of dynamics use words or formulae of the previously defined concepts in order to state particular repetitive properties of the body motion with accuracy up to the dimension constant of the dynamic parameters.

All the forces, including the defined inertia force (1.37), appear as interactions of some bodies being related to other bodies. One body, that is, one non-free material point, can be only conditionally "separated" from others in such a way that "separation" in mechanics is abstracted by forces. The laws of natural sciences, as well as the laws of dynamics describe not only particular repetitive and measurable properties of one material point's constraint with a multitude of others. The mental deliverance from constraints is achieved in dynamics by abstraction by forces, formulated by particular laws. Nature is much more complex than mechanical models; still, these models can be used to determine numerous movements of the body with great mathematical accuracy. Mechanics can use one concept of the *material* or *dynamic point* to describe a position change of all the bodies, from the celestial ones to the bodily molecules. And such a multitude is so great that it is hardly comprehensible. The mass of the outer space is considered to be so great that 10^{23} stars of the same magnitude as the Sun can be formed, while the composition of the Earth comprises about 4×10^{51} protons and neutrons (see [9]). In this theory of motion man can, in some cases (like that of the parachute), be regarded as a material point, though it is accepted that it consists of, on average, 10^{16} of cells that are, in their turn, regarded as having a structure of $10^{12} - 10^{14}$ atoms each. The number of entities stand in some proportion with the possibilities of mutual association. For example, in a molecule of DNA which consists of $10^8 - 10^{10}$ atoms, the atom distribution and their mutual relations exceed any countable multitude; this, in its own right, makes particular specific sciences introduce simpler models upon which they can carry out their research. Mechanics finds it sufficient to deal with the concepts of the material and dynamic points.

Law of constraints

It is from classical mechanics and its related sciences about nature that knowledge can be acquired as to the ways in which the bodies affect each other through real objects that are called the constraints. The present findings do not point to any particular body, out of a multitude of bodies, that can be isolated and exist by itself, namely, without being affected by other bodies. Still, this assertion cannot be made about the whole multitude of bodies whose boundaries have not been discovered yet; neither has the multitude in its wholeness. In the observed rational or practical locality only limited sets of constraints are known. Many constraints or particular sets of constraints can be abstracted by means of particular mathematical relations used for connecting essential attributes of motion as positions of the material points $x = (x^1, \ldots, x^n)$, velocities $\dot{x} = (\dot{x}^1, \ldots, \dot{x}^n)$ or impulses $p = (p_1, \ldots, p_n)$, as well as time t, by means of geometrical or kinematic parameters $\varkappa$.

Example 2. A body M_1 of mass m_1 is lying or is moving along the horizontal smooth plate. This body is connected with another body M_2 of mass m_2 by some attachment (fiber, rope, thread) passing through a smooth opening O on the plate (Fig. 1).

Fig. 1

Therefore, two bodies with known masses are given; their motion is bound by means of two constraints: one of them being a smooth plate, while the other is the fiber connecting them. For the mathematical description of these constraints it is appropriate to introduce either Descartes rectilinear coordinate system $Oxyz$ or a cylindrical system of coordinates ρ, φ, z with the coordinate origin O, so that it is:

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z. \tag{E2.1}$$

In both the coordinate systems the "plate" can be represented by the relation:

$$f_1 := z_1 - C = 0. \tag{E2.2}$$

However, the second constraint in the coordinate system $Oxyz$ is represented by the equation:

$$f_2 := \sqrt{x_1^2 + y_1^2} + |z_2| - l = 0, \tag{E2.3}$$

while, relative to system $O\rho\varphi z$, it can be represented by the equivalent equation:

$$f_2 = \rho_1 + \mid z_2 \mid - l = 0. \tag{E2.4}$$

It is understandable that at some transverse velocity the body M_1 will move along a circular line:

$$x_1^2 + y_1^2 = l^2, \quad z_1 = z_2 = 0. \tag{E2.5}$$

This will happen, among other possibilities, when the boundary point of body M_2 coincides with point C. Such an equation also represents the case in which the constraint is not taken to be fiber $\overline{M_1 C}$, but a circular wire of radius l along which body M_1 glides. The mechanical difference is relevant. The wire will resist the motion if it is not ideally smooth; this does not happen with the fiber. In the case of smoothness both the constraints can be abstracted by the constraint force which is called the constraint reaction and is most often denoted by the letter $\boldsymbol{R}$.

This example can clearly differentiate the concept of the "constraints" in mathematics and mechanics. It is customary in mathematics to consider every relation establishing some sort of ratio between the observed mathematical parameters as "constraint"; consequently, it includes (E2.1), (E2.2), (E2.3) or (E2.4) and (E2.5). In mechanics, as can be seen in this example, the constraints are (E2.2), (E2.3), (E2.4) or (E2.5). Therefore, each mathematical relation, as in example (E2.1), will not be called the constraint. The difference is not just formal. Constraints (E2.2) and (E2.3) or (E2.4) produce forces, so that constraint (E2.2) can be abstracted by some vector $\boldsymbol{R}_1$, while relations (E.3) or (E.4) are abstracted by some other vector $\boldsymbol{R}_2$. However, "mathematical constraints" and those similar to them ("substitution", "transformation of coordinates", "mapping") do not produce any forces or other physical phenomena.

It depends upon the relation between these forces, that is, upon the relation of bodies and their association, as well as upon the inherited motion (position and velocity) whether body M_1, for instance, will move in the plate plane along the pathway having either the shape of a straight line to which point C of the circumference $\rho_1 = \text{const}$, or that of a falling or rising helix or some other curved line.

In the case that the plate also moves so that the constraint should be of the form

$$f_1 := z_1 - \tau(vt) = 0,$$

where v is a velocity parameter or in the case that fiber l changes in time, constraints (E.2) and (E.3) or (E.4) would be written in the form

$$f_1(z_1, \tau) \geq 0 \tag{E2.6}$$

or

$$f_2(x_1, y_1, z_1, z_2, \tau) \geq 0. \tag{E2.7}$$

This simple example becomes more complex if it is not assumed that the plate plane is ideally smooth - as indeed happens in practice - and that the surrounding medium is not empty, but existent. Then the structure of the force vectors becomes more complex.

In the most general case, the constraints linking N bodies M_ν, $(\nu = 1, 2, \ldots, N)$ can be written by means of k relations

$$f_\mu\left(y_1, \ldots, y_N, \dot{y}_1, \ldots, \dot{y}_N, \tau(t)\right) \geq 0, \quad y_i \in E^{3N}, \tag{2.1}$$

where τ is some known function of time, or in the form

$$f_\mu\left(x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N, \tau(t)\right) \geq 0, \quad (\mu = 1, \ldots, k) \tag{2.1a}$$

since, as already stressed, the constraints are objects that are invariant regarding any mathematical transformations. Considering the fact that in the literature about the constraints' mapping (2.1) from one coordinate system y into the other x, or vice versa, there is much vagueness or incomprehension. The previous sentence should be repeated in the following form:

$$f_\mu\left(y,\dot{y},\tau(t)\right)_{y=y(x)} = f_\mu\left(x,\dot{x},\tau(t)\right), \quad \left\|\frac{\partial y}{\partial x}\right\| \neq 0 \tag{2.2}$$

In words, it states that the constraint equations written with respect to one coordinate system $(y, \boldsymbol{e})$ can be also written with respect to the other coordinate system $(x, \boldsymbol{g})$ in the region in which there is explicit mapping between them. The constraints can be scalar or vector invariant.

Relations (2.1) *in which are real differentiable functions in the observed region, namely, those pointing to boundaries of motion in a particular way are considered as constraint relations or, shorter, constraints.*

Therefore, constraints are dynamic objects that, together with material or dynamic bodies, make up the system of material, or, consequently, dynamic points. According to relations' structure (2.1), functions f_μ of the constraints are also most often classified as:

- unilateral or unconstraining

$$f_\mu \geq 0. \tag{2.3}$$

- bilateral or constraining

$$f_\mu = 0. \tag{2.4}$$

- geometric and finite

$$f_\mu(y) = 0. \tag{2.5}$$

- kinematic or differential

$$f_\mu\left(y,\dot{y},\tau(t)\right) \geq 0. \tag{2.6}$$

- invariable and fixed

$$f_\mu\left(y\right) \geq 0. \tag{2.7}$$

- variable or moveable[1]

$$f\left(y,\dot{y},\tau(t)\right) = 0. \tag{2.8}$$

[1]There are other terms used in literarure; these are, most often: unilateral (2.3), bilateral (2.4), holonomic (2.5), holonomic differential and non-holonomic (2.6), scleronomic, stationary or autonomous (2.7), rheonomic, nonstationary or non-autonomous (2.8).

It can be noticed that all the finite constraints can be written in differentiated form by differentiation. But, it is not always possible to reduce the originally given differential constraints to the finite ones. For this reason, the writing of differential constraints (2.6) contains differential integrable - finite or holonomic, differential non-integrable – non-holonomic constraints. Any particular classification implies that the signs of equality and inequality in (2.3) and (2.4) are taken into consideration simultaneously with the function class (2.5) – (2.8).

However, much more essential is the classification of all the mechanical constraints into *real constraints* and *ideally smooth*, or, simply, *smooth constraints*.

As it happens, all the constraints are real and cannot be ideally smooth. If one constraint is classified as "ideally smooth", it means that in mechanics it is desirable to stress that its dynamic factors (friction, resistance, hardness, elasticity) that are not described by differentiable functions f are either neglected or described by other functions. The general property of all constraints is marked by the determinant that will be called the *law of constraints*.

The constraints restrict displacement of the material points as forces; they are abstracted by the constraints' reactions; k *constraints* $f_\mu = 0$ $(\mu = 1, \dots, k)$, *that constrain the motion of some* ν-th *material point, are abstracted by vector sum*

$$\boldsymbol{R}_\nu = \sum_{\mu=1}^{k} \boldsymbol{R}_{\nu\mu} \tag{2.9}$$

of constraints' reactions $\boldsymbol{R}_{\nu\mu}$.

Vector (2.9) is called the *resultant of the constraints' reactions* of the ν-th point.

The vector-function of the constraints' reactions can be completely or partially determined for some classes of constraints. The most important task is to determine the nature of given constraints.

For example, constraint (E2.2) is a unilateral geometrical finite invariable and fixed. However, relation (E2.2) does not give information which is essential for motion, namely, whether constraint (E2.2) is real or ideally smooth. One force will act upon body M_1 if the plate surface is rough; another force will affect it if the plate surface is polished and dry; another if it has the same polish, but it is covered with a thin layer of fluid; another if the air above the plate is rarefied or if it is a gas in its liquid state.

The real constraints, in addition to relation (2.1), have a multitude of properties which the constraints' reactions will depend on. For the sake of an easy, but, at the same time, more comprehensive solution of the given problem, constraint reaction $\boldsymbol{R}_\nu$ is always possible to be resolved into two components, namely, one of them $\boldsymbol{R}_\nu^\tau$ belonging to the tangent plane at the contacting vth point of the body, while the other $\boldsymbol{R}_\nu^N$ is perpendicular to that tangent plane.:

$$\boldsymbol{R}_{\nu\mu} = \boldsymbol{R}_{\nu\mu}^\tau + \boldsymbol{R}_{\nu\mu}^N. \tag{2.10}$$

Forces $\boldsymbol{R}_{\nu}^{\tau}$ appear as a result of the constraint's friction or the medium resistance which always exists. Its magnitude is experimentally determined; it is generalized by the friction law and the law of medium resistance. If given force R^{τ} is negligibly small, $R^{\tau} \approx 0$, and thus, with no effect upon motion, or if $\boldsymbol{R}^{\tau} \neq 0$ is taken into account, independently of the constraint, then the geometrical constraints are considered as ideally smooth and substituted by reaction R whose direction, with respect to the pathway, is determined by constraint gradient

$$\boldsymbol{R}_{\nu\mu}^{N} = \lambda_{\mu} \operatorname{grad}_{\nu} f_{\mu} \tag{2.10a}$$

where λ_{μ} is a certain constraints' multiplier.

Laws of Friction

1. *At the contacting point between the body and the constraint friction force $\boldsymbol{R}_{\nu}^{\tau}$ sets up and affects geometrical constraints at the contacting point; if the bodies touch each other with their surfaces, the friction force' point of action is considered to be the geometrical center of the contacting surfaces.*

2. *The boundary value of dry friction force $R_{\max}^{\tau}$ at rest is proportional to the magnitude of force N, perpendicular to the constraint, that is,*

$$R_{\max}^{\tau} = \mu R^{N}, \quad R^{N} = -N, \tag{2.11}$$

where μ is a tabular coefficient of the sliding friction and rest, depending on the body structure (material point), way of treatment (smoothness) and other states (humidity, temperature, etc.) of the rubbing surfaces, but not on the size of these surfaces.

3. *The friction force of the real constraints appears in the general case as a function of velocities and positions*

$$R_{M}^{\tau} = R^{\tau}(y, \dot{y} = 0) + R_{K}^{\tau}(y, \dot{y}) \gtrless 0. \tag{2.12}$$

Laws of Medium Resistance and Thrust

The bodies have a boundary contact with the surrounding real medium which can also be considered as a constraint. The fluid medium affects the body by resistance force which, in a way similar to the friction force, appears as a function of the contact velocity or of connecting fluid particles and bodies, as well as a force of pressure or thrust.

1. *Each element up to the surface $d\sigma$ of the body is affected by force $\boldsymbol{p}_{n} d\sigma$ where $\boldsymbol{p}_{n}$ is the surface force density directed with respect to the normal of surface elements $d\sigma$.*

2. *The principal forces' vector*

$$\boldsymbol{F} = \int_{\sigma} \boldsymbol{p}_{n} d\sigma \tag{2.13}$$

can be expressed as a vector sum of resistance force $\boldsymbol{F}^{\tau}$*, directed opposite to velocity* $\boldsymbol{v}$*, and thrust force* $\boldsymbol{F}^{N}$*, perpendicular to* $\boldsymbol{v}$*.* [22, pp. 133, 454, 455].

Law of Elasticity of Materials

The body whose constraints between particles have the property of regaining their original shapes after any kind of deformation is called elastic, while the restitution force is regarded as the force of elasticity.

At small strains $\boldsymbol{F}$ *of an elastic body, elasticity force* $\boldsymbol{F}$ *is proportional to strain* $\boldsymbol{\varepsilon}$*, that is,*

$$\boldsymbol{F} = -k\boldsymbol{\varepsilon}, \tag{2.14}$$

where k *is the restitution factor.*

Law of Reaction Thrust

Mass flow $\dot{m} = \dfrac{dm}{dt}$*, that departs from the body of mass* $m(t)$ *in time* t *and at velocity* $\boldsymbol{u}$ *with respect to the body affects this body by a reaction force*

$$\boldsymbol{\Phi} = \dot{m}\boldsymbol{u} \tag{2.15}$$

Law of Gravity

Thousands of years devoted to observing and studying positions and motion of the celestial bodies, as well as of satellites in their interactions, offer the following findings:

- many bodies in the outer space apparently preserve for good their positions or repetitive apparent motion with respect to each other,
- their mutual distances are either constant or change periodically,
- there are particular centers around which the bodies move along helical pathways leading towards the gravitational center.

Briefly, *The bodies are mutually connected by the forces that induce particular motions with respect to each other, namely, the motions that depend on their masses, distances and kinematic characteristics of motion.*

This general assertion that can be considered as the general law of gravity does not provide sufficient information either about the constraint or about the force of mutual connections. It only says that there are mutual forces and motions of the bodies. The theoretical assumptions of classical mechanics about the celestial bodies' motion, of natural and artificial objects within the solar system have been confirmed so far or modified with sufficient reliability in practice. The solar system here implies all the bodes existing either permanently or for a limited amount of time in the space in which, under any kinetic circumstances, the dominant influence upon their motion is that of the Sun, either indirectly or through local gravitational fields of the planets. Of interest in this study are Kepler's laws and Newton's force of gravity.

Kepler's Laws

I. *The planets describe elliptical pathways around the Sun; it is at the common focus of all these ellipses that the Sun is located* [14, p. 29].
II. *The radius-vector, drawn from the Sun to the planet, covers equal surfaces in equal time intervals.*
III. *The time squares of some planets' revolution around the Sun are proportional to the third degrees of the great semiaxes of their pathways.*

Note. It can be noticed that all the three laws do not speak about force directly; neither do they determine the mutual attraction forces. For this reason, none of them forms the laws of dynamics on its own. However, on the basis of all the three laws Newton was able to determine the magnitude of the mutual attraction force (2.18).

Newton's Gravitational Force

One material point of mass m_μ attracts another material point of mass m_ν ($\mu \neq \nu$) by force $\boldsymbol{F}_{\nu\mu}$ which is proportional to the product of masses m_ν and m_μ, while it is inversely proportional to squared distance $r_{\nu\mu}^2$ of these points, namely,

$$\boldsymbol{F}_{\nu\mu} = -k\frac{m_\nu m_\mu}{r_{\nu\mu}^2}\boldsymbol{e}_{\nu\mu}, \tag{2.16}$$

where $k = \text{const} > 0$, while

$$\boldsymbol{e}_{\nu\mu} = \frac{\boldsymbol{r}_\nu - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\nu\mu}|}.$$

More material points m_μ affect the νth material point of mass m_ν by the resultant attraction force:

$$\boldsymbol{F}_\nu = \sum_{\mu=1} \boldsymbol{F}_{\nu\mu} = -\sum_{\substack{\mu=1\\ \mu\neq\nu}} k\frac{m_\nu m_\mu}{r_{\nu\mu}^2}\boldsymbol{e}_{\nu\mu}. \tag{2.17}$$

The constant k is called as the *universal gravitational constant.* Regarding the importance of the law that Isaac Newton (1643–1727) deduced on the basis of Kepler's laws and published in his ingenious work Philosophia naturalis principa mathematica (Londoni, Anno MDCLXXXVII) and in order to pay homage to my Professor M. Milanković who made available the grandiose Newton's work to a wide reading audience by his book The Celestial Mechanics (Nebeska mehanika, Belgrade, 1935), I would like to quote some of his commentaries on the universal gravitational law. "Every particle of the matter in the outer space attracts every other particle by the force which falls in these particles' straight line while having an intensity which is proportional to the products of masses m_1 and m_2 of these particles; it is, though, inversely proportional to their squared distance r. The magnitude of this force is represented, therefore, by the expression:

$$F = f\frac{m_1 m_2}{r^2}. \tag{2.18}$$

In the above expression, proportionality factor f is one universal constant. The sign "minus" is eliminated from the above expression, since the word "attracts" explicitly determines this force's direction".

"It is Newton's law that finally revealed a thousand years old mystery of the planetary motion; it is from it that new findings came into being. All inequalities of the planetary motion and the Moon became obvious as a natural consequence of this law, as well as a clear expression of the mutual attraction among these celestial bodies. Not only that the nature of these inequalities became explained by it; now they could be computed and traced back into the past or followed into the future. It turned out that, soon after Newton's law had been postulated, that it was also valid for the comets, for all the celestial bodies with no exception, namely, even beyond the solar system. The precession of the equinox that was, so far as we are informed, first stated by Chiparcos, found its full explanation by means of Newton's law as did the Earth's axes nutation that was observed later on. Even the shape of our Earth, especially its flatness due to its rotation, was given its mechanical and geometrical explanation in all details. The same stands for the antique question concerning the rise of the sea tide which turned out to be an immediate consequence of the attraction between the Sun and the Moon. Thus, Newton's law, the most magnificent of all that a mortal man could formulate, turned out to be the general law of nature that all the space is subdued to. It is from this law that another new science came into being, namely, the celestial mechanics.

On the basis of law (2.16) it follows that force $\boldsymbol{F}$, by which the Earth of mass m_zm attracts some body of mass m is determined by the formula:

$$\boldsymbol{F} = -k\frac{mm_z}{(R+h)^2}\boldsymbol{e} = -mg_0\boldsymbol{e} = -m\boldsymbol{g}_0 \tag{2.19}$$

where $R = 6,37 \times 10^8$ cm average Earth's radius, h distance of the observed body from the Earth's surface, and g_0 denotes acceleration

$$g_0 = k\frac{m_z}{(R+h)^2}. \tag{2.20}$$

On Laws of Dynamics

The introductory commentary on the laws of dynamics now becomes much clearer and more concrete. Our starting point that the concept of the "law of dynamics" implies formulations - determinants of forces with accuracy up to some constants - comprises the laws of constraints, friction, medium resistance, elasticity of materials, reactive force, gravitational law as well as the law about the Earth's gravitational force (2.19). Each of these basic laws is used, directly or indirectly, for determining particular forces. Some formulae of these forces can be of greater or smaller generality, but they all comprise either one or a set of constants that allow for or require a more accurate determination for particular objects. The accuracy of these constants, including the forces' formulae, will depend not only on ignorance of the objects' nature, but often on the lack of mathematical knowledge which would otherwise provide calculations with very complex relations. For instance [2], in formula (2.19) the average Earth's radius $R = 6.37 \times 10^8$ cm is taken, while the equatorial one is 6.38×10 cm, [2, p. 17],"h is distance of the body from the Earth's surface", while the concept of distance "from the Earth's surface" is not precise. It can mean one thing when it refers to the Earth's mathematical surface; another thing is when it implies the sea surface at some geographical latitude; another thing is at the bottom or beyond the mountain chains. Finally, even the gravitational constant is subjected to scientific verification for particular gravitational areas. It is logical to expect that the recent development of astronautics and of its applications will contribute to more accurate knowledge about the gravitational force, while the development of other branches of mechanics will depend on making other laws of dynamics more concrete, modified or generalized. This study supports the view that the laws of dynamics are used for determining formulae of particular forces, except for inertia forces introduced by the definition. As could be seen, the laws are more or less general for particular mechanical systems. The considered number of laws is incomplete since, for particular and more concrete body systems, the forces are classified more concretely, including the laws by which they are determined. It is still valid that the laws of dynamics satisfy the preprinciples of mechanics.

On the basis of the fact that the laws of dynamics are determined by observing and measuring in nature and in experimental human practice, it is accepted that the principle of existence is satisfied.

Once the forces' existence is stated, the preprinciple of invariance is an indispensable condition that the mappings of forces' functions, during the transition from one coordinate system into another, should not change the laws of dynamics.

The history of the discovery, along with the measuring and observing practice, of the Newton's theorem on "the world system" itself, as well as the "problem" solving about motion of two bodies (see (3A.70)), suggests that other laws of mutual attraction, different from (2.18), should be set up regarding the preprinciple of casual definiteness.

On Mutual Attraction Force

Two material forces of masses m_1 and m_2 attract each other at mutual distance $\rho(t)$ by the force of magnitude [69], [70]:

$$F = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}\frac{m_1 m_2}{\rho} = \chi\frac{m_1 m_2}{\rho} \tag{2.21}$$

where the meaning of velocity v_{or} is explained by the formula:

$$v_{or}^2 = (\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2 + (\dot{z}_1 - \dot{z}_2)^2.$$

If it is assumed that distance $\rho = r$ does not change in time and if it is assumed that the distance between mass M center of the Sun and the planet center of mass

m, the expression (2.21) is reduced to:

$$F = -\frac{v_{or}^2}{M+m}\frac{Mm}{r}.$$

By introducing another conjunction that it is

$$v_{or}(t) = v_{or}(t_0) = r\Omega$$

where Ω is angular velocity of the planet's revolution around the Sun, the modified ([14] or [15]) Newton's force (2.18) [14] is obtained:

$$F = -\frac{r^2\Omega^2 r}{M+m}\frac{Mm}{r^2} = -\varkappa^*\frac{Mm}{r^2}.$$

The principle of casual definiteness speaks about accuracy up to some constant. The table on page 73 clearly shows the meaning of the concept "up to some constant" used here along with "to the accuracy" of dynamic parameters. The making of accuracy more relative before generalization also refers to other laws of dynamics.

III. PRINCIPLES OF MECHANICS

The concept of the *principle of mechanics* implies here an *expression of general significance, based on the introduced concepts and definitions of mechanics whose truthfulness is not liable to verification.*

The principles of mechanics must be concordant with the preprinciples. smallskip On the basis of the definitions introduced so far the *principle of equilibrium* can be set up now.

By additional defining of "work", "action" and "compulsion", other principles can be introduced, namely, *the principle of work, the principle of action* and *the principle of compulsion.* The principles of mechanics are not themselves sufficient to provide for problem-solving in mechanics without laws of dynamics. The statement of general significance in mechanics, such as the principle of mechanics, represents the basis for developing a whole theory of mechanics, but its application would require knowledge of the laws of dynamics.

3A. PRINCIPLE OF EQUILIBRIUM

A body is in dynamic equilibrium so that the sums of all the forces acting upon particular dynamic points of the body are equal to zero.

This principle can be written in the form of vector equations:

$$\sum_{\mu=1} \mathsf{F}_{\nu\mu} = 0 \quad (\nu = 1, 2, \ldots, N) \tag{3A.1}$$

where index ν denotes the ν-th dynamic point, while index μ denotes the forces exerted upon the ν-th dynamic point.

Material Point. If only one material point is observed, then, instead of a system of many equations (3A.1) there is only a vector one, namely:

$$\sum_{\mu} \mathsf{F}_{\mu} = 0. \tag{3A.2}$$

In accordance to the definition of the inertia force and the laws of dynamics, equations (3A.1) can be written in the form:

$$\boldsymbol{I}_{\nu} + \sum_{i=1} \boldsymbol{F}_{\nu i} = 0, \tag{3A.3}$$

where $\boldsymbol{I}$ is inertia force determined by definition (1.37), while $\boldsymbol{F}_{\nu i}$ are the forces determined by the laws of dynamics (2.2, 2.3, 2.8, 2.9, 2.10, etc.)

Equations (3A.3) allow some forces to be unknown in advance; consequently, they have to be determined by means of definitions (3A.3) depending on the number of forces known on the basis of the laws of dynamics and definition (1.37). This implies that the constraints equations should be added to the equations of the law (3A.1). By substituting inertia force (1.37), force (2.6) and constraint force (2.7) in equation (3A.3), the vector equation of the material point motion is obtained:

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} + \boldsymbol{R}. \tag{3A.4}$$

The following conclusion can be derived from this:

Conclusion 1. *The material point moves at constant velocity* $\boldsymbol{v}$ *if the sum of acting forces* $\boldsymbol{F}$ *and constraint forces is equal to zero, that is,*

$$\sum_{\mu} \boldsymbol{F}_{\mu} + \sum_{\mu} \boldsymbol{R}_{\mu} = 0 \longrightarrow \boldsymbol{v}(t) = \boldsymbol{v}(t_0). \tag{3A.5}$$

With respect to the natural coordinate trihedron (η_1, η_2, η_3) of equation (3A.3) it is easy to write, if forces $\boldsymbol{F}_i$ are resolved along the tangent, the principal normal and the binormal, namely,

$$\sum_{\mu} \boldsymbol{F}_{\mu} = \boldsymbol{F} = F_{\tau}\boldsymbol{\tau} + F_n\boldsymbol{n} + F_b\boldsymbol{b}, \tag{3A.6}$$

$$\sum_{\mu} \boldsymbol{R}_{\mu} = \boldsymbol{R} = R_{\tau}\boldsymbol{\tau} + R_n\boldsymbol{n} + R_b\boldsymbol{b}. \tag{3A.7}$$

By substituting these relations as well as the coordinates of inertia force (1.41) and (1.42) in equation (3A.1), the scalar equations of the material point's motion are obtained in the natural system of coordinates:

$$F_{\tau} + R_{\tau} - m\frac{dv}{dt} = 0 \tag{3A.8}$$

$$F_n + R_n + I_n = 0 \tag{3A.9}$$

$$F_b + R_b = 0. \tag{3A.10}$$

From motion equations (3A.8) and (3A.9) the following conclusion can be drawn:

Conclusion 2. *The material point moves with respect to magnitude at constant velocity if forces* F_{τ} *and* R_{τ} *and mutually annul themselves, while the sum of the respective components of these forces and the inertia force on the principal pathway normal is equal to zero.*

With respect to other coordinate systems $(y, \boldsymbol{e})$ and $(x, \boldsymbol{g})$, equilibrium system (3A.1) is invariant and covariant. If forces $\boldsymbol{F}$ and $\boldsymbol{R}$ are resolved along coordinate vectors $\boldsymbol{e}$ or $\boldsymbol{g}$, that is,

$$\boldsymbol{F} = Y^i \boldsymbol{e}_i = X^j \boldsymbol{g}_j, \quad \boldsymbol{R} = R^i \boldsymbol{e}_i = R^j \boldsymbol{g}_j \tag{3A.11}$$

and are substituted, together with relation (1.38) in equation (3A.1), the invariance is obvious:

$$\left(I_y^i + Y^i + R_y^i\right) \boldsymbol{e}_i = \left(I_x^j + X^j + R_x^j\right) \boldsymbol{g}_j = 0. \tag{3A.12}$$

Scalar multiplication by coordinate vectors, according to relations (1.14), gives covariant differential equations of the point motion with respect to base system $(y, \boldsymbol{e})$:

$$m\ddot{y}_i = Y_i + R_i, \tag{3A.13}$$

or with respect to any other coordinate system $(x, \boldsymbol{g})$ that satisfies condition (1.15):

$$mg_{ij} \frac{Dv^i}{dt} = X_j + R_j, \tag{3A.14}$$

where

$$X_j = Y_i \frac{\partial y^i}{\partial x^j}, \quad R_j = R_i \frac{\partial y^i}{\partial x^j}$$

are projections of forces $\boldsymbol{F}$ and $\boldsymbol{R}$ upon coordinate directions $\boldsymbol{g}_j$.

Equilibrium principle (3A.1) or (3A.3) or (3A.12) can be written with respect to any coordinate system of coordinates $(x, \boldsymbol{g})$ in the covariant form:

$$g_{ij} \left(I^j + X^j + R^j\right) = I_i + X_i + R_i = 0, \tag{3A.15}$$

where $I_i = g_{ij} I^j$, $X_i = g_{ij} X^j$, $R_i = g_{ij} R^j$ are coordinates of the forces' covectors.

If a set of coordinates X^i of vector $\boldsymbol{F} - X^j \boldsymbol{g}_j$ is called a vector, then a set of projections $X_i = \boldsymbol{F} \cdot \boldsymbol{g}_i = \left(X^j \boldsymbol{g}_j\right) \boldsymbol{g}_i$ is called the *covariant vector coordinates*. That is why equations (3A.15) can also be called *covariant equations of the equilibrium principle* with respect to some system of coordinates $(x, \boldsymbol{g})$.

System of Material Points and Finite Constraints

It is dynamic equilibrium principle (3A.1) that refers to a multitude of material and dynamic points. All the relations from (3A.2) to (3A.13), derived only for one material point of mass m, stand for every ν-th material point of mass m_ν. Such a system of N material points will have N vector equations of the form (3A.1) or (3A.2)–(3A.4) and k constraints equations (2.6). Nothing more important than this changes. However, the manner of solving problems concerned with the system motion comprises some difficulties and innovations originabing from the limitations of the applied mathematical apparatus as well as from mutual constraint of the material points that generate forces of a complex mathematical structure.

The simplest and thus, the most widely used, way of describing is the one with respect to base coordinate system $(y, \boldsymbol{e})$.

It is assumed that there are N material points of mass m_ν $(\nu = 1, \dots, N)$ whose position vectors $\boldsymbol{r}_\nu = y^i_\nu \boldsymbol{e}_i$ $(i = 1, 2, 3)$ and that they are connected by k finite constraints

$$f_\mu(y^1_\nu, y^2_\nu, y^3_\nu) = f_\mu(y^1, \dots, y^{3N}) = 0, \tag{3A.16}$$

where the following notations are introduced

$$y^1_\nu =: y^{3\nu-2},\; y^2_\nu =: y^{3\nu-1},\; y^3_\nu =: y^{3\nu}, \tag{3A.17}$$

$$m_{3\nu-2} \equiv m_{3\nu-1} \equiv m_{3\nu}. \tag{3A.18}$$

The constraints (3A.16) must satisfy the velocities conditions

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^\alpha} \dot{y}^\alpha = 0, \quad (\alpha = 1, \dots, k, k+1, \dots, 3N), \tag{3A.19}$$

as well as the acceleration conditions

$$\ddot{f}_\mu = \frac{\partial^2 f_\mu}{\partial y^\beta \partial y^\alpha} \dot{y}^\alpha \dot{y}^\beta + \frac{\partial f_\mu}{\partial y^\alpha} \ddot{y}^\alpha = 0. \tag{3A.20}$$

These constraints are considered independent so that the determinant of the matrix $\left(\dfrac{\partial f_\mu}{\partial y^\alpha}\right)$ of the level k, is different from zero:

$$\left|\frac{\partial f_\mu}{\partial y^\alpha}\right| \neq 0. \tag{3A.21}$$

It follows from equations (3A.19) that the velocities vectors are perpendicular to the constraints gradients. This fact points to a possibility that the constraints forces (2.7) can be viewed as a sum of friction forces (2.9) and normal component $\boldsymbol{R}^N_\nu = \sum\limits_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu$, where $\boldsymbol{R}^\tau$ would be determined by the friction law unlike $\boldsymbol{R}^N$ determined by acceleration condition (3A.20).Thus all the constraints' forces $\boldsymbol{R}_\nu$, regarding the constraint and friction laws, can be written by means of the expression

$$\boldsymbol{R}_\nu = \boldsymbol{R}^\tau_\nu + \sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu \tag{3A.22}$$

where λ_μ are indefinite multipliers.

For the sake of brevity, friction forces λ_μ are taken as active forces $\boldsymbol{R}^\tau_\nu$, while the constraints are regarded as ideally smooth; likewise, the constraint force always has the gradient direction and magnitude:

$$R^N_{\nu\mu} = \left|\lambda_\mu grad_\nu f_\mu\right|. \tag{3A.23}$$

In view of all that has been said, it follows from the equilibrium principle that a system of recognizable differential equations is:

$$\boldsymbol{I}_\nu + \sum_s \boldsymbol{F}_{\nu s} + \boldsymbol{R}_\nu^N = 0 \tag{3A.24}$$

or, due to idealization of the constraints and the inertia force definition,

$$\left.\begin{aligned} m_\nu \frac{d\boldsymbol{v}_\nu}{dt} &= \boldsymbol{F}_\nu + \sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu, \\ f_\mu(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) &= 0; \end{aligned}\right\} \tag{3A.25}$$

in the scalar form, equations (3A.25) can be written, with respect to (3A.17) and (3A.18), in a brief form:

$$\begin{gathered} m\ddot{y} = Y + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y}, \\ f_\mu(y) = 0, \end{gathered} \tag{3A.26}$$

and there are $3N + k$ of them which is sufficient for an explicit determination of $3N$ coordinates y if Y functions are known or $3N$ coordinates of the force Y if motion $y(t)$ is known as well as k of the multiplier λ_μ.

By substituting $\ddot{y}$ from relation (1.32) in (3A.26) and subsequent multiplication of equation (3A.26), by matrix $\dfrac{\partial y}{\partial x}$, it is obtained, in accordance with relations (1.31)

$$\left.\begin{aligned} a_{rs} \frac{Dv^s}{dt} &= X_r + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^r}, \\ f_\mu(x) &= 0, \end{aligned}\right\} \quad (r, s = 1, \ldots, 3N) \tag{3A.27}$$

where tensor $a_{rs}(m_\nu, x)$ contains only one mass m_ν of particular material point and its coordinates $x_\nu^1, x_\nu^2, x_\nu^3$. These differential equations are suitable due to the possibility of reducing a great number of constraints in various coordinate systems x^i to a simple form $f_\mu = x_\mu^i - \text{const} = 0$, so that the constraint forces are reduced to $R_\mu = \lambda_\mu$. In any other case, equations (3A.27) are more complex and complicated. Another form of the same equations is denoted by number (3A.15).

Systems with Variable Constraints

In the case that finite constraints (3A.16) depend not only upon $y = (y^1, \ldots, y^{3N})$ coordinates, but also explicitly on time as well, velocity conditions (3A.19) and those of acceleration (3A.20) considerably change, since the number of addends is increasing under these conditions as is obvious in the following velocity conditions:

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^\alpha} \dot{y}^\alpha + \frac{\partial f_\mu}{\partial t} = \operatorname{grad}_\nu f_\mu \cdot \boldsymbol{v}_\nu + \frac{\partial f_\mu}{\partial t} = 0. \tag{3A.28}$$

It becomes clear that in the case of variable constraints in time, that is, of the constraints of the form $f_\mu(y,t)=0$, velocities vectors $\boldsymbol{v}$ are not orthogonal to the constraints gradient; thus, the velocities do not lie in the constraints' tangent planes; the pathway tangent does always not coincide with the constraints' tangents. For the sake of a clearer analysis of velocities conditions (3A.28), it should be noted that the mechanical constraints are not written in the form $f(y,t)=0$ since such a writing would comprise, for instance, the equation

$$f = t^2 + 2t + 3 = 0,$$

which loses the meaning of the mechanical constraints; it is in accordance neither with relation (2.2) nor with the constraints' law. The variable constraints in time must satisfy the dimension equation, that is, they have to be dimensionally homogeneous. In order to achieve this homogeneity between y coordinates of L dimension and time t of dimension T, it is necessary to connect these values by some parameter $\varkappa$ of the dimensions L and T. Therefore, time in mechanical constraints appears in the structure of the functions containing dimension parameters, so that variable or moveable constraints, in accordance with definition (2.6) are written in the form:

$$f_\mu(y,\tau) = 0 \quad (\mu = 1,\dots,k), \tag{3A.29}$$

where τ is some real time function with definite real coefficients having physical dimensions [49], [51]. For the sake of brevity, instead of function τ with definite coefficients, let's introduce an additional coordinate y^0, so that it satisfies the condition

$$f_0 = y^0(\varkappa, t) - \tau(t) = 0. \tag{3A.30}$$

Example 3. Let the motion of two material points be limited by means of three constraints, namely,

$$\begin{gathered} f_1 = (y_1-y_4)^2 + (y_2-y_5)^2 + (y_3-y_6)^2 - 4t^2 = 0 \\ f_2 = y_3 - 0.5t = 0, \\ f_3 = y_6 - 0.5t + 0.3 = 0. \end{gathered} \tag{E3.1}$$

Regarding the fact that the coordinates have a dimension of length, the coefficients 4 and 0.3 will also have a dimension of length L, while the coefficient 0.5 will have a dimension of velocity $\mathrm{L\,T^{-1}}$. By an appropriate choice of parameters of one or the other dimension μ, $\dim\mu = \mathrm{L\,T^{-1}}$, an auxiliary coordinate is introduced:

$$y^0(\mu,t) = 0.5t, \ \dim y^0 = \mathrm{L}\,.$$

Substituting time from this newly-introduced relation, $t = 2y_0$, in the given constraints, it can be written that

$$\begin{gathered} f_1 = (y_1-y_4)^2 + (y_2-y_5)^2 + (y_3-y_6)^2 - 16y_0^2 = 0, \\ f_2 = y_3 - y_0 = 0, \\ f_3 = y_6 - y_0 + 0.3 = 0. \end{gathered} \tag{E3.2}$$

With y^0 coordinate, constraints equations (3A.29) can be written in the form

$$f_\mu(\tilde{y}) = 0, \;\; \bar{y} = (y^0, \underbrace{y^1, \ldots, y^{3N}}_{y}) = (y^0, y) \tag{3A.31}$$

while the velocity and acceleration conditions in the form (3A.19) and (3A.2), that is,

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial \tilde{y}} \dot{\tilde{y}} = \frac{\partial f_\mu}{\partial y} \dot{y} + \frac{\partial f_\mu}{\partial y^0} \dot{y}^0 = 0, \tag{3A.32}$$

$$\begin{aligned} \ddot{f}_\mu &= \frac{\partial^2 f_\mu}{\partial \tilde{y} \partial \tilde{y}} \dot{\tilde{y}} \dot{\tilde{y}} + \frac{\partial f_\mu}{\partial \tilde{y}} \ddot{\tilde{y}} \\ &= \frac{\partial^2 f_\mu}{\partial y \partial y} \dot{y} \dot{y} + 2 \frac{\partial^2 f}{\partial y^0 \partial y} \dot{y} \dot{y}^0 + \frac{\partial^2 f}{\partial y^0 \partial y^0} \dot{y}^0 \dot{y}^0 + \frac{\partial f_\mu}{\partial y} \ddot{y} + \frac{\partial f_\mu}{\partial y^0} \ddot{y}^0 = 0. \end{aligned} \tag{3A.33}$$

The last acceleration relation can be written in a shorter form

$$\frac{\partial f_\mu}{\partial y} \ddot{y} + \frac{\partial f_\mu}{\partial y^0} \ddot{y}^0 = \Phi\left(\tilde{y}, \dot{\tilde{y}}\right) \tag{3A.34}$$

where the composition of function Φ is obvious.

If $\ddot{y}$ from equation (3A.26) is included in equation (3A.34), it is obtained that:

$$\frac{\partial f_\mu}{\partial y} \sum_{\sigma=1}^{k} \lambda_\sigma \frac{\partial f_\sigma}{\partial y} = m \left(\Phi - \frac{\partial f_\mu}{\partial y^0} \ddot{y}^0 \right) - \frac{\partial f_\mu}{\partial y} Y.$$

The solution with respect to unknown multipliers λ_σ shows that the reaction forces of variable constraints do not only depend upon $\tilde{y}$ coordinates and $\dot{\tilde{y}}$ velocities, but also on $\ddot{y}^0$, as well as on inertia force $-m\ddot{y}^0$ which emerges due to the constraints' change in time.

The constraints in equations (3A.31), and, especially in (3A.12), (3A.25), (3A.26) or (3A.27) can be written in the form:

$$\boldsymbol{r}_\nu = \boldsymbol{r}_\nu \left(q^0, q^1, \ldots, q^n\right), \;\; n = 3N - k \tag{3A.35}$$

where $q = \left(q^1, \ldots, q^n\right)$ are *independent generalized coordinates,* while q^0 is a *rheonomic coordinate* satisfying equation (3A.30), that is,

$$q^0 - \tau(t) = 0. \tag{3A.36}$$

By reducing the finite constraints to the geometrical form (3A.35) the number of differential equations for the constraints' number is also reduced; at the same time, constraints' forces $\boldsymbol{R}$ are eliminated which makes it considerably easier to solve the problem.

The velocities of ν-th material points, according to definition (1.1), can be written in the following form:

$$\boldsymbol{v}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \boldsymbol{r}_\nu}{\partial q^1}\dot{q}^1 + \cdots + \frac{\partial \boldsymbol{r}_\nu}{\partial q^n}\dot{q}^n = \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha \tag{3A.37}$$

where $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}(q)$ are coordinate vectors that will be marked by two-indices notation $\boldsymbol{g}_{\nu\alpha}$; index ν denotes the number of the material point, while index α denotes the number of independent coordinate q^α $(\alpha = 0, 1, \ldots, n)$.

For addition with respect to index ν, we use addition sign $\sum\limits_\nu$, while for addition with respect to the indices, coordinate α denotes iteration of the same letter in the same expression, as well as both the lower and the upper indices. Vector (3A.37), as can be seen, has $n+1$ independent elementary vectors. Accordingly, impulse vector (1.26) of the ν-th material point of mass m_ν of the observed system can also be represented by the formula

$$\boldsymbol{p}_\nu = m_\nu \boldsymbol{v}_\nu = m\frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha. \tag{3A.38}$$

Scalar multiplication by coordinate vectors $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^\beta}$ gives vector $\boldsymbol{p}_\nu$ projection upon the tangential direction of q^β coordinate of the ν-th material point. We will denote it by a two-indices letter:

$$\boldsymbol{p}_{\nu\beta} = m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^\beta} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha.$$

This is in accordance with the formula for impulse's coordinates (1.25) of one material point. Regarding the fact that $p_{\nu\beta}$ impulses are scalars, it is possible to sum them up:

$$p_\beta := \sum_{\nu=1}^{N} p_{\nu\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\beta}\dot{q}^\alpha = a_{\alpha\beta}\dot{q}^\alpha, \tag{3A.39}$$

from which it can be seen that $a_{\alpha\beta}$ is an inertia tensor of the whole system:

$$\begin{aligned} a_{\alpha\beta} &= \sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\beta} = \\ &= a_{\alpha\beta}\left(m_1, \ldots, m_N; q^0, q^1, \ldots, q^n\right). \end{aligned} \tag{3A.40}$$

If the masses are constant quantities, this tensor is written as a function of independent coordinates:

$$a_{\alpha\beta} = a_{\beta\alpha}\left(q^0, q^1, \ldots, q^n\right). \tag{3A.41}$$

By means of important relations (3A.39) the concept of generalized impulses of the material points' system is introduced. Therefore, *the sum of the material points' impulse vector projections upon the coordinate direction of the β-th generalized coordinate is considered as the generalized impulse p_β. The generalized impulses appear as linear homogeneous forms of the generalized velocities,* which is in accordance with the basic definition of impulse (1.24). Regarding the fact that the inertia tensor $a_{\alpha\beta}$ determinant is different from zero, it is possible to determine the generalized velocities $\dot{q}^\alpha$ as linear homogeneous combinations of the generalized impulses, namely:

$$\dot{q}^\alpha = a^{\alpha\beta} p_\beta, \tag{3A.42}$$

where $a^{\alpha\beta}$ is *countervariant inertia tensor.*

If the constraints do not explicitly depend upon the known functions of time τ, there is no rheonomic coordinate q^0, so that in all the expressions, from (3A.35) to (3A.34), coordinates $q^0, \dot{q}^0$ and p_0 vanish. The impulse form (3A.39) does not change, expect for the fact that indices $\alpha = 0, 1, \ldots, n$ do not assume values from 0 to n, but from 1 to n. In order to facilitate this distinction further on, let Greek indices $\alpha, \beta, \gamma, \delta$ assume values from 0 to n,($\alpha, \beta, \gamma, \delta = 0, 1, \ldots, n$), while the Latin ones take i, j, k, l from 1 to n ($i, j, k, l = 1, 2, \ldots, n$). Then it can be written [44]:

$$\boldsymbol{v}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\dot{q}^i \tag{3A.43}$$

$$p_i = a_{0i}\dot{q}^0 + a_{ij}\dot{q}^j \tag{3A.44}$$

$$p_0 = a_{00}\dot{q}^0 + a_{0j}\dot{q}^j \tag{3A.45}$$

$$\dot{q}^i = a^{i0}p_0 + a^{ij}p_j \tag{3A.46}$$

$$\dot{q}^0 = a^{00}p_o + a^{0j}p_j. \tag{3A.47}$$

Covariant Differential Equations of the System's Motion

If equations (3A.1) are successively multiplied scalarly by coordinate vectors $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}$ respective to index ν and if they are added with respect to index ν, the system of $n+1$ covariant equations of the equilibrium principle is obtained, namely,

$$\sum_\mu \mathsf{F}_{\nu\mu} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} = \boldsymbol{Q}_\alpha = 0 \tag{3A.48}$$

or, relative to equations (3A.3),

$$I_\alpha + Q_\alpha = 0 \tag{3A.49}$$

where now

$$I_\alpha = \sum_{\nu=1}^{N} \boldsymbol{I}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \tag{3A.50}$$

are *generalized inertia forces*, while

$$Q_\alpha = \sum_{\nu=1}^{N} \left(\sum_{\mu=1} \boldsymbol{F}_{\mu\nu} \right) \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \tag{3A.51}$$

are *generalized forces*.

Differential equations (3A.49) represent a system of $n+1$ differential covariant equations of motion which will be written in an extended form. In order to understand better their mechanical meaning, vector equations (3A.25) should be the ones to start from. Scalar multiplication of equation (3A.25) by vectors $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}$, followed by addition with respect to indices ν, gives

$$\sum_{\nu=1}^{N} m_\nu \frac{d\boldsymbol{v}_\nu}{dt} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} + \sum_{\nu=1}^{N} \sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}.$$

The ordinary notations are introduced:

$$\sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} =: Q_\alpha \tag{3A.52}$$

$$\sum_{\nu=1}^{N} \sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} = 0, \tag{3A.53}$$

since it is $\quad \operatorname{grad}_\nu f_\mu \cdot \dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i} = \dfrac{\partial f_\mu}{\partial q^i} \quad$ for $\quad i = 1, \ldots, n$;

$$\sum_{\nu=1}^{N} \sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^0} = -\sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial q^0} =: R_0; \tag{3A.54}$$

$$\begin{aligned} \sum_{\nu=1}^{N} m_\nu \frac{d\boldsymbol{v}_\nu}{dt} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} &= \sum_{\nu=1}^{N} m_\nu \frac{d}{dt} \left(\frac{\partial \boldsymbol{r}_\nu}{\partial q^\beta} \dot{q}^\beta \right) \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \\ &= \sum_{\nu} m_\nu \left(\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\gamma \partial q^\beta} \dot{q}^\beta \dot{q}^\gamma + \frac{\partial \boldsymbol{r}_\nu}{\partial q^\beta} \ddot{q}^\beta \right) \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}. \end{aligned}$$

If vector $\dfrac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\gamma \partial q^\beta}$ is resolved along coordinate vectors $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^\delta}$, namely,

$$\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\gamma \partial q^\beta} = \Gamma^\delta_{\gamma\beta} \frac{\partial \boldsymbol{r}_\nu}{\partial q^\delta}$$

it further follows that it is

$$\sum_{\nu=1}^{N} m_\nu \frac{d\boldsymbol{v}_\nu}{dt} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} = a_{\alpha\beta}\ddot{q}^\beta + a_{\alpha\delta}\Gamma^\delta_{\gamma\beta}\dot{q}^\gamma \dot{q}^\beta = \\ = a_{\alpha\beta}\left(\ddot{q}^\beta + \Gamma^\beta_{\gamma\delta}\dot{q}^\gamma\dot{q}^\delta\right) = a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt}. \tag{3A.55}$$

By substituting relations (3A.52), (3A.54) and (3A.55), regarding (3A.53), in equation (3A.4) the system of $n+1$ covariant differential equations of motion is obtained [38], [49], [51]:

$$a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt} = Q_\alpha, \quad (\alpha = 0, 1, \ldots, n) \tag{3A.56}$$

or

$$a_{i\beta}\frac{D\dot{q}^\beta}{dt} = Q_i \quad (i = 1, \ldots, n) \tag{3A.56a}$$

$$a_{0\beta}\frac{D\dot{q}^\beta}{dt} = Q^*_0 + R_0 =: Q_0 \tag{3A.56б}$$

In accordance with the definition of the material point impulse it follows that generalized system impulses $p_\beta = a_{\alpha\beta}\dot{q}^\alpha$ are linear combinations of generalized velocities where the inertia tensor of a more general and complex structure:

$$a_{\alpha\beta} = a_{\beta\alpha}\left(m_1, \ldots, m_N, q^0, q^1, \ldots, q^n\right).$$

The respective system of the covariant differential equations of motion (3A.56) can be written in the form:

$$I_\alpha + Q_\alpha = 0, \qquad (\alpha = 0, 1. \ldots, n)$$

The number of degrees of freedom can be made identical to the number of $n+1$ equations (3A.56).

In this way, by means of the dynamic equilibrium principles, the laws of dynamics and the basic definitions, the theory about motion of material points or bodies, as well as deformable medium, has been completely comprised, regarding the fact that the material point can represent both the body as an entity and its particular parts.

Example 4. A material point of mass $m = \text{const}$ is moving under the impact of the constraints:

$$f_1 = x - l_0 - \varkappa\tau(t) = 0, \quad \tau = \cos\Omega t,$$
$$f_2 = y = 0,$$
$$f_3 = z = 0,$$

where $\varkappa, l_0$ and Ω are concrete real numbers.

Determine the forces acting upon the body.

In this case, the system of differential equations (3A.26) is:

$$\begin{aligned} m\ddot{x} &= \lambda_1 \\ m\ddot{y} &= \lambda_2 \\ m\ddot{z} &= \lambda_3. \end{aligned}$$

From condition (3A.33) it follows $\lambda_2 = \lambda_3 = 0$, while it follows from equation $\ddot{f}_1 = \ddot{x} - \varkappa\ddot{\tau} = 0$

$$\frac{\lambda_1}{m} - \varkappa\ddot{\tau} = \frac{\lambda_1}{m} + \varkappa\Omega^2 \cos \Omega t = 0,$$

so, it is obtained that the acting force is

$$X = \lambda_1 = -m\varkappa\Omega^2 \cos \Omega t = -m\Omega^2(x - l_0).$$

Therefore, the force inducing the given motion is proportional to elongation $(x - l_0)$ where proportionality factor Ω^2 has the frequency dimension of T^{-2}.

Example 5. Three non-free points. Let the points $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$ and $C(x_C, y_C, 0)$ are connected in plane $z \equiv 0$ by constraints (as in Fig. 2):

$$\begin{aligned} f_1 &= \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2} - l_1 = 0, \\ f_2 &= \sqrt{(x_B - x_c)^2 + (y_B - y_c)^2} - l_2 = 0, \\ f_3 &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} - 2l = 0, \\ f_4 &= x_A = 0, \\ f_5 &= y_A = 0, \\ f_6 &= y_B = 0, \end{aligned}$$

where $l = l_0\tau(t)$, while $l_0 = \text{const}$ and $\tau(t)$ is a known time function of initial value $\tau(t_0 = 0) = 1$; force $F_B = (0, -G_B, 0)$ is acting at point B, while force $F_C = (0, -G, 0)$ is acting upon point C; $G = \text{const}$ (as in Fig. 2). Constraints' reactions $f_4 = 0$, $f_5 = 0$ and $f_6 = 0$ should be determined [67], [68], [71].

Fig. 2

There are nine equations (3A.24), that is, (3A.26) for three dynamic points. However, a set of the forces, projections upon z axis is empty, so that equations (3A.26) can be written in the form:

$$-\lambda_1 \frac{x_C - x_A}{l_1} - \lambda_3 \frac{x_B - x_A}{2l} + \lambda_4 = 0, \tag{E5.1}$$

$$-\lambda_1 \frac{y_C - y_A}{l_1} - \lambda_3 \frac{y_B - y_A}{2l} + \lambda_5 = 0, \tag{E5.2}$$

$$I_x(C) + \lambda_1 \frac{x_C - x_A}{l_1} - \lambda_2 \frac{x_B - x_C}{l_1} = 0; \tag{E5.3}$$

$$I_y(C) + \lambda_1 \frac{y_C - y_A}{l} - \lambda_2 \frac{y_B - y_C}{l} - G = 0, \tag{E5.4}$$

$$I_x(B) + \lambda_2 \frac{x_B - x_C}{l} + \lambda_3 \frac{x_B - x_A}{2l} = 0, \tag{E5.5}$$

$$\lambda_2 \frac{y_B - y_C}{l} + \lambda_3 \frac{y_B - y_A}{2l} + \lambda_6 - G_B = 0. \tag{E5.6}$$

It follows from (E5.3) and (E5.4) that it is:

$$\lambda_1 = \frac{l_1}{2y_C}\left(G - I_y(C) - I_x(C)\frac{y_C}{l}\right),$$
$$\lambda_2 = \frac{l_1}{2y_C}\left(G - I_y(C) + I_x(C)\frac{y_C}{l}\right),$$

and then, it is obtained from (E5.5):

$$\lambda_3 = \frac{l}{2y_C}\left(I_y(C) - G - \frac{I_x(C) + 2I_x(B)}{2}\right).$$

If quantities λ_1, λ_2 and λ_3 are introduced in equations (E5.1), (E5.2) and (E5.3), the required constraints' reactions are obtained $f_4 = 0$, $f_5 = 0$ and $f_6 = 0$, namely:

$$\lambda_4 = -I_x(C) - I_x(B) = R_{Ax},$$
$$\lambda_5 = \frac{G}{2} - \frac{I_y(C)}{2} - \frac{y_C}{2l} I_x(C) = R_{Ay},$$
$$\lambda_6 = \frac{G}{2} + G_B - \frac{I_y(C)}{2} + I_x(C)\frac{y_C}{2l} = R_{By}.$$

Since the inertia forces:

$$I_x(C) = -m\ddot{x}_C = -m\ddot{l}, \quad I_y(C) = -m\ddot{y}_C,$$
$$I_x(B) = -m_B\ddot{x}_B = -2m_B\ddot{l},$$

the found constraints' reactions obtain a concrete form:

$$\begin{aligned}
\lambda_4 &= R_{Ax} = (m + 2m_B)\ddot{l},\\
\lambda_5 &= R_{Ay} = \frac{G}{2} + \frac{m}{2}\left(\ddot{y}_C + \frac{y_C}{l}\ddot{l}\right),\\
\lambda_6 &= R_{By} = \frac{G}{2} + G_B + \frac{m}{2}\left(\ddot{y}_C - \frac{y_C}{l}\ddot{l}\right).
\end{aligned} \tag{E5.7}$$

Regarding the fact that line segments $\overline{AB}$, $\overline{AC}$, $\overline{CB}$ are given as known time functions, so ordinate $y_C = y_C(l, l_1)$ can be determined as depending on $\tau(t)$ like $\ddot{y}_C$ derivative. For degenerative system $C \in \overline{AB}$, $y_C = 0$, it follows from the previous solutions:

$$R_{Ax} = (m + 2m_B)\ddot{y}, \quad R_{Ay} = \frac{G}{2}, \quad R_{By} = \frac{G}{2} + G_B. \tag{E5.7a}$$

that $l = \text{const}$ and $G_B = 0$ it follows that $R_{Ax} = 0$, $R_{Ay} = R_{By} = \dfrac{G}{2}$.

Such a problem is solved in mechanics in a considerably simpler and shorter way by means of the "moment of force" and the "moment of the couples of forces". This statement will be explained in the following example.

Example 6. Coupled Points. Two material points of masses $m_1 = \text{const}$ and $m_2 = \text{const}$ are connected by a rigid constraint

$$f_1 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0,$$

$l = \text{const}$. The first point is acted upon by force $\boldsymbol{F}_1 = \boldsymbol{F}$, while the other is acted upon by a parallel, but inversely directed force $\boldsymbol{F}_2$; the magnitudes of these forces are equal, $F_1 = F = F_2$. By eliminating the constraint multiplier the dynamic equilibrium conditions of the material points should be determined.

Fig. 3

Differential equations (3A.26) are:

$$\begin{aligned} m_1\ddot{x}_1 &= X_1 - 2\lambda_1(x_2 - x_1)\\ m_1\ddot{y}_1 &= Y_1 - 2\lambda_1(y_2 - y_1)\\ m_1\ddot{z}_1 &= Z_1 - 2\lambda_1(z_2 - z_1) \end{aligned}$$

$$\begin{aligned} m_2\ddot{x}_2 &= X_2 + 2\lambda_1(x_2 - x_1)\\ m_2\ddot{y}_2 &= Y_2 + 2\lambda_1(y_2 - y_1)\\ m_2\ddot{z}_2 &= Z_2 + 2\lambda_1(z_2 - z_1). \end{aligned}$$

Elimination of the constraint multiplier is possible in two ways:

1. By summing up the respective projections which gives:

$$\begin{aligned} m_1\ddot{x}_1 + m_2\ddot{x}_2 &= X_1 + X_2,\\ m_1\ddot{y}_1 + m_2\ddot{y}_2 &= Y_1 + Y_2,\\ m_1\ddot{z}_1 + m_2\ddot{z}_2 &= Z_1 + Z_2. \end{aligned}$$

2. By identifying the obtained values

$$\lambda_1 = \frac{m_1\ddot{x}_1 - X_1}{2(x_1 - x_2)} = \frac{m_1\ddot{y}_1 - Y_1}{2(y_1 - y_2)} = \frac{m_1\ddot{z}_1 - Z_1}{2(z_1 - z_2)}$$

or

$$\lambda_1 = \frac{m_2\ddot{x}_2 - X_2}{2(x_2 - x_1)} = \frac{m_2\ddot{y}_2 - Y_2}{2(y_2 - y_1)} = \frac{m_2\ddot{z}_2 - Z_2}{2(z_2 - z_1)}$$

it is obtained that

$$\begin{aligned}
\mathfrak{M}_z^1 &= (m_1\ddot{x}_1 - X_1)(y_2 - y_1) - (m_1\ddot{y}_1 - Y_1)(x_2 - x_1) = 0,\\
\mathfrak{M}_x^1 &= (m_1\ddot{y}_1 - Y_1)(z_2 - z_1) - (m_1\ddot{z}_1 - Z_1)(y_2 - y_1) = 0,\\
\mathfrak{M}_y^1 &= (m_1\ddot{z}_1 - Z_1)(x_2 - x_1) - (m_1\ddot{x}_1 - X_1)(z_2 - z_1) = 0,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{M}_z^2 &= (m_2\ddot{x}_2 - X_2)(y_2 - y_1) - (m_2\ddot{y}_2 - Y_2)(x_2 - x_1) = 0,\\
\mathfrak{M}_x^2 &= (m_2\ddot{y}_2 - Y_2)(z_2 - z_1) - (m_2\ddot{z}_2 - Z_2)(y_2 - y_1) = 0,\\
\mathfrak{M}_y^2 &= (m_2\ddot{z}_2 - Z_2)(x_2 - x_1) - (m_2\ddot{x}_2 - X_2)(z_2 - z_1) = 0,
\end{aligned} \tag{3A.57}$$

where letter $\mathfrak{M}$ is introduced, for the time being, in order to make the notation shorter.

Summing up the respective relations with respect to the axes it is obtained that:

$$\begin{aligned}
\mathfrak{M}_z^1 + \mathfrak{M}_x^2 &= m_1\ddot{x}_1(y_2 - y_1) + m_2\ddot{x}_2(y_2 - y_1) -\\
&- [m_1\ddot{y}_1(x_2 - x_1) + m_2\ddot{y}_2(x_2 - x_1)] -\\
&- [Y_1(x_2 - x_1) - X_1(y_2 - y_1)] -\\
&- [X_2(y_2 - y_1) - Y_2(x_2 - x_1)] =\\
&= \mathfrak{M}_z(\boldsymbol{I}_1) + \mathfrak{M}_z(\boldsymbol{I}_2) + \mathfrak{M}_z(\boldsymbol{F}_1) + \mathfrak{M}_z(\boldsymbol{F}_2) =\\
&= \sum_{i=1} \mathfrak{M}_z(\boldsymbol{I}_i) + \sum_{i=1} \mathfrak{M}_z(\boldsymbol{F}_i) = 0,
\end{aligned} \tag{3A.57a}$$

where:

$$\mathfrak{M}_z(\boldsymbol{I}_i) := l_x I_{iy} - l_y I_{ix}, \tag{3A.58}$$

$$\mathfrak{M}_z(\boldsymbol{F}_i) := l_x Y_i - l_y X_i, \tag{3A.59}$$

$$I_{iy} := -m_i\ddot{y}_i, \quad I_{ix} := -m_i\ddot{x}_i, \quad l_x = x_2 - x_1, \quad l_y = y_2 - y_1.$$

It can be similarly proved that two more relations follow, namely:

$$\left.\begin{aligned}
\sum_{i=1}^{2} \mathfrak{M}_y(\boldsymbol{I}_i) + \sum_{i=1}^{2} \mathfrak{M}_y(\boldsymbol{F}_i) = 0,\\
\sum_{i=1}^{2} \mathfrak{M}_x(\boldsymbol{I}_i) + \sum_{i=1}^{2} \mathfrak{M}_x(\boldsymbol{F}_i) = 0.
\end{aligned}\right\} \tag{3A.60}$$

Moments of the Couples of Forces

For dynamic equilibrium of a system of points connected by various constraints, principle (3A.1), and, consequently, (3A.24) and (3A.26), produces other conditions as well such as (3A.56), (3A.57) and (3A.58). Quantity $\mathfrak{M}$ is qualitatively different from forces; its dimension is $\mathrm{M\,L^2\,T^{-2}}$. The values of this dimension are called *moments of forces* in mechanics. The moment of forces, including the inertia forces moment, represents an attribute of motion produced by constraints. It can easily be shown that generalized forces (3A.51) corresponding to the dimensionless generalized coordinates - angles, also have a dimension of the forces' moment. A system of two points and a line segment that couples two parallel forces of equal magnitude, but opposite sense, is called a *couple of forces*, while vectors

$$\mathfrak{M}_k(\boldsymbol{F}) = \left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{e}_i}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{e}_j} - \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{e}_j}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{e}_i}\right)\boldsymbol{k}, \quad i \neq j \neq k$$

are called *moments of the couples of forces*. Therefore, the moments of forces are derived concepts as special products of forces and lengths. The moment of the inertia forces' couple $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, $I_1 = -I_2$, as well as of other forces, can be written in the form:

$$\boldsymbol{\mathfrak{M}} = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{I} = \boldsymbol{\rho} \times \boldsymbol{I} = \boldsymbol{\rho} \times (-m\boldsymbol{a}), \tag{3A.61}$$

where $\boldsymbol{\rho} = \boldsymbol{r}_2 - \boldsymbol{r}_1$

In accordance with relations (3A.24) and (1.37), the moment of the inertia forces' couple can be represented in the following form:

$$\begin{aligned}\boldsymbol{\mathfrak{M}}\,(\boldsymbol{I}) = \boldsymbol{\rho} \times \left(-m\frac{d\boldsymbol{v}}{dt}\right) &= \boldsymbol{\rho} \times \boldsymbol{v}\frac{dm}{dt} - \boldsymbol{\rho} \times \frac{d}{dt}(m\boldsymbol{v}) \\ &= \boldsymbol{\rho} \times \boldsymbol{v}\dot{m} - \boldsymbol{\rho} \times \dot{\boldsymbol{p}}.\end{aligned} \tag{3A.62}$$

This could be written in another way: *the moment of the inertia forces' couple of the material points is equal to the difference of the moment of couple $\boldsymbol{\mathfrak{M}}(\boldsymbol{v}\dot{m})$ of reactive forces $\boldsymbol{v}_1\dot{m}_1$ and $\boldsymbol{v}_2\dot{m}_2$ and moment couple $\mathfrak{M}(\dot{\boldsymbol{p}})$ of vectors $\dot{\boldsymbol{p}}_1$ and $\dot{\boldsymbol{p}}_2$.*

If the mass is constant, the relation (3A.62) shows that the *moment of the inertia force's couple is equal to the couple's moment of the impulse change in time*

$$\boldsymbol{\mathfrak{M}}(I) = -(\boldsymbol{\rho} \times \dot{\boldsymbol{p}}), \tag{3A.63}$$

with a negative sign.

Since expression (3A.61) shows that the couple's moment $\boldsymbol{\mathfrak{M}}$ does not depend upon the choice of the position vector pole, it follows that $\boldsymbol{\mathfrak{M}}$ is a free vector, so that it can be summed up with other couples' moments. Therefore, *the forces' vectors, bound at dynamic points, can be "transmitted", in parallel way, to any other point, and thus, they can be added, if the sum of thus parallelly displaced forces is added the sum of the couples' moments of the respective forces.*

Example 7. A horizontal beam of variable length. The "beam" as a homogeneous body of rectilinear form and of constant cross-section should be modeled by a degenerative system of points (E5.7).

Regarding the above-mentioned consequential concept of the couple, it is expected that the same example could be solved by the couples' moments as well.

In this case, it is necessary to state that in $z = 0$ plane there are dynamic points $A(0,0,0)$, $B(2l,0,0)$, $C(l,0,0)$ at which forces' coordinates R_{Ax}, R_{Ay}; R_{By}, $-G_B$, $I_x(B) = -m_B\ddot{x}_B$, $I_x(C) = -m\ddot{x}_C$, $I_y(C) = -m\ddot{y}_C$, $-G$ are present instead of constraints.

The equilibrium equations are:

$$\sum_i X_i = R_{Ax} - m\ddot{x}_C - m_B\ddot{x}_B = 0,$$
$$\sum_i Y_i = R_{Ay} - G - G_B + R_{By} = 0,$$
$$\sum \mathfrak{M}_{iA} = -Gl - m\ddot{y}_C l + m\ddot{x}_C y_C + 2lR_{By} - 2lG_B = 0.$$

It follows from this that it is identical to the result of examples (E5.7) and (E5.7a).

The couple of forces' moment is here introduced by means of real constraints. This essential fact provides for the fact that the dynamic equilibrium principle is applied to the motion of bodies, either rigid or deformable, in a simple way.

The "rigid body" assumes an uncountable multitude of particles that are mutually linked by invariable real line segments. Out of a multitude of these particles, let's observe any four of them, mutually linked by means of six lengths of tetrahedron. The lengths of the tetrahedron sides are denoted by letters $l_{\nu\mu}$, where indices ν mark ordinal numbers of particles $1, 2, 3$, and $\mu = 1, \ldots, 6$ the number of independent constraints.

The constraints' equations are of the form

$$\delta_{ij}(y^i_{\nu+1} - y^i_\nu)(y^j_{\nu+1} - y^j_\nu) = l^2_\nu. \tag{3A.64}$$

By observing every new point of the body, whose position is explicitly determined by three new numbers, the number of new constraints increases for three. Thus the number of independent coordinates for determining the positions of the rigid body points does not change. The number of independent coordinates for determining motion of the rigid body's points is reduced to $4 \cdot 3 - 6 = 6$, which six equations of dynamic equilibrium correspond to. These equations will comprise, in addition to forces, the forces' couples' moments, including the moments of inertia forces' couples.

Example 8. Motion of Two Bodies. The motion of a system of two bodies, observed as material points, is known in the celestial mechanics as "the problem of two bodies". Kepler's laws as well as Newton's gravitational force, are the ones that relate to the motion of two bodies mutually attracting each other; this is a simple example of the system of two material points, but its reduction to the above-mentioned laws makes it a *significant problem* [69], [70].

Two bodies, observed as material points M_1 and M_2, whose masses are m_1 and m_2, are moving towards each other so that the distance between their inertia centers is a time function $\rho(t)$.

The condition that “the distance between the bodies is a time function $\rho(t)$”, that is,

$$f = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} - \rho(t) = 0. \tag{3A.64a}$$

is similar to relations (E2.5) (E3.2) or (E5); hence the motion in question can be considered in a similar way.

Differential equations of motion (3A.26) for these two material points, in the presence of “constraint” (3A.64a), can be reduced to the form:

$$\left.\begin{aligned} m_1\ddot{x}_1 &= \frac{\lambda}{\rho}(x_1 - x_2), \\ m_1\ddot{y}_1 &= \frac{\lambda}{\rho}(y_1 - y_2); \end{aligned}\right\} \tag{3A.65}$$

$$\left.\begin{aligned} m_2\ddot{x}_2 &= -\frac{\lambda}{\rho}(x_1 - x_2), \\ m_2\ddot{y}_2 &= -\frac{\lambda}{\rho}(y_1 - y_2). \end{aligned}\right\} \tag{3A.66}$$

It is obtained from acceleration condition (3A.33) that the multiplier is

$$\lambda = \frac{m_1 m_2}{m_1 + m_2}\frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho}$$

where

$$v_{or}^2 = (\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2.$$

If the letter χ denotes the expression

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2} \tag{3A.67}$$

and is substituted in equations (3A.65) and (3A.66), the following form of the differential equations of motion is obtained:

$$\left.\begin{aligned} m_1\ddot{x}_1 &= \chi\frac{m_1 m_2}{\rho^2}(x_1 - x_2), \\ m_1\ddot{y}_1 &= \chi\frac{m_1 m_2}{\rho^2}(y_1 - y_2); \end{aligned}\right\} \tag{3A.68}$$

$$\left.\begin{aligned} m_2\ddot{x}_2 &= -\chi\frac{m_1 m_2}{\rho^2}(x_1 - x_2), \\ m_2\ddot{y}_2 &= -\chi\frac{m_1 m_2}{\rho^2}(y_1 - y_2). \end{aligned}\right\} \tag{3A.69}$$

The right sides in equations (3A.68) represent the coordinates of vector $\boldsymbol{F}_1$ which acts upon the body of mass m_1, so that the magnitude of force $\boldsymbol{F}_1$ is equal to:

$$F_1 = \chi\frac{m_1 m_2}{\rho}. \tag{3A.70}$$

Force $\boldsymbol{F}_2$ is of the same size, but of opposite sense ($\boldsymbol{F}_2 = -\boldsymbol{F}_1$) and it acts upon the body of mass m_2, which indicates a changed sign. This force is identical to mutual attraction force (2.21).

The problem arises during the differential equations' integration if the structure of function $\chi(t, \dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2)$ is taken into consideration or when it is compared to Newton's gravitational law:

$$F = \varkappa \frac{m_1 m_2}{\rho^2}.$$

This comparison is worth further consideration.

Conjunction 1. For the distance $\rho = R = \text{const}$ it follows:

$$\chi = -\frac{v_r^2}{m_1 + m_2} \quad \text{and} \quad F = -\frac{m_1 m_2}{m_1 + m_2} \frac{v_{or}^2}{R}$$

Conjunction 2. Mass m_1 is mass of the planet, $m_1 = m$, while mass m_2 is the Sun mass, $m_2 = M$; R is the distance between the centers of the planet's and the Sun's masses; velocity R is equal to average velocity of the Earth's revolution around the Sun. The formula for the force could be written in the following general form:

$$\begin{aligned} F = -\frac{mM}{m+M}\frac{v_r^2}{R} &= -\frac{mMR^3}{m+M}\frac{4\pi^2}{T^2R^2} \\ &= -\frac{mM}{m+M}\frac{Rv^2}{R^2} = \varkappa^* \frac{mM}{R^2} \end{aligned} \tag{3A.71}$$

where

$$\varkappa^* = \frac{4\pi^2 R^3}{(m+M)T^2} \tag{3A.72}$$

while T is the period of the planet's revolution around the Sun. Under these assumptions, $\varkappa^*$ is constant. This formula (where R is a great semi-axis of elliptical pathway) is given in the book "Celestial Mechanics" by M. Milanković [14, p. 56]; further on, it says in the book:

"This equation expresses one important relation of parameters a andi T, which is not completely identical with the third Kepler's law. This law states that quotient $\frac{a^3}{T^2}$ for all the planets is always the same, but this would not be the case regarding the former equation since the presence of mass m in this equation changes the value of the above-mentioned quotient from one planet to another. Still, since the masses of the planets are very small comparing to the Sun mass, m in the above equation could be neglected beside M, and thus, the identity of the third Kepler's law with the laws of the celestial mechanics is obtained."

If mass m is neglected, constant (3A.72) is written by the expression [14, p. 38, formula (28)]. In expression (2.16) this constant $\varkappa$ is denoted, as usual, with letter k.

$$\varkappa = \frac{4\pi^2 a^3}{MT^2}$$

and is called *the universal gravitational constant* whose accepted numerical value is

$$\varkappa = 6.67 \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{gr}^{-1}\,\mathrm{sec}^{-2}\,.$$

The difference between $\varkappa^*$ and $\varkappa$ can be determined with great accuracy, regarding the fact that it is:

$$\frac{1}{m+M} = \frac{1}{M\left(1+\dfrac{m}{M}\right)} = \frac{1}{M}\left(1 - \frac{m}{M} + \left(\frac{m}{M}\right)^2 - \cdots\right).$$

Accordingly,

$$\varkappa^* = \varkappa - \varkappa\varepsilon + \varkappa\varepsilon^2 - \cdots$$

where $\varepsilon = \dfrac{m}{M}$. Since the relationship between the Earth's and the Sun's masses is

$$\frac{m}{M} = \frac{1}{333432} = 299.112263 \times 10^{-8}$$

at the first approximation it turns out to be $\varkappa^* = 0.999997\varkappa = 6.66997999 \times 10^{-8}$.

For Jupiter, it is $\frac{m}{M} = \frac{318{,}36 m_\oplus}{330000 m_\oplus} = 95479{,}7379 \times 10^{-8}$ so that it is

$$\varkappa^* = 0{,}999045202\varkappa = 6{,}663565264 \times 10^{-8}.$$

When all the previously mentioned conjunctions from the relation (3A.70) are taken into consideration, it is obtained that:

$$\varkappa^* = \frac{R v_{or}^2}{m+M} \tag{3A.73}$$

where R would be average distance of the planet's inertia center from the Sun's inertia center, while v_{or} is average orbital velocity of the planet's revolution around the Sun. For the Sun's mass M the following numerous values are found in literature: $M = 2 \times 10^{33}$ gr, $M = 333432 m_\oplus$ [14].

If quantity m is taken as the mass of a planet or a satellite, on the basis of the data presented in Table,[1] it is easy to compute quantity $\varkappa^*$ by means of formula (3A.73):

[1]Hames Alfen et all, Evolution of the Solar System, National Aeronautics and Space Administration (NASA), SP-345, 1976, p. 16

Sun Mass	2×10^{33}	$333432 m_{\oplus}$
Planets	$\varkappa^*$ $(10^{-8}\,\mathrm{cm}^3\,\mathrm{gr}^{-1}\,\mathrm{sec}^{-2})$	
Mercury	6.6423	6.6737
Venus	6.6528	6.6843
Earth	6.6603	6.6917
Mars	6.6762	6.7078
Jupiter	6.6993	6.7008
Saturn	6.6426	6.6739
Uranium	6.6547	6.6861
Neptune	6.6582	6.6897
Pluto	6.6559	6.6874
Earth – Moon Jupiter – Europe	6.63	
Average Values	6.6569	6.6864
Average Value	6.67	

Therefore, when the above-mentioned conjunctions about the motion of two bodies are taken into consideration, the numerical values can be obtained from formula (3A.73) which can be reduced, only by averaging, to one accepted gravitational constant.

On the basis of the obtained values $\varkappa^* = 6.6864$, the radius of the Earth $R = 6.38$[2] and of the Earth's mass $m_{\oplus} = 5.974$ [14, p. 197] is found by means of (3A.73) so that the square velocity of the body's revolution around the Earth (in immediate vicinity) would be

$$v_r^2 = \varkappa^* \frac{m + m_{\oplus}}{R}$$

and, consequently, the Earth's gravity acceleration

$$g = \frac{v_r^2}{R} = 6,6864 \frac{5,974 \times 10^{27}}{(6,38 \times 10^8)^2} 10^{-8} = 981,33 cm/sec.$$

All the above-mentioned numerous data, if the above-mentioned conjunctions are taken into consideration, show under what conditions the classical value for the gravitational force is obtained. However, formulae (3A.67) and (3A.80) indicate that the attraction force depends upon velocity and acceleration of the distance change between the bodies. In the case of a free fall of the body of mass m, $v_r = \dot{\rho} = (R + \zeta)^{\cdot} = \dot{\zeta}$; thus, it follows:

$$F_1 = \chi \frac{m M_{\oplus}}{R + \zeta} = \frac{(R + \zeta)\ddot{\zeta} m M_{\oplus}}{(m + M_{\oplus})(R + \zeta)} =$$

$$= \frac{m\ddot{\zeta}}{1 + \dfrac{m}{M_{\oplus}}} \approx m\ddot{\zeta}, \quad \frac{m}{M_{\oplus}} \approx 0.$$

[2] Hames Alfen et all, Evolution of the Solar System, NASA, SP-345, p. 17

As discovered a long time ago by Galileo who obtained by measurement that it is $\zeta = \frac{1}{2}gt^2$, it is more difficult to get $F_1 = mg$, for the magnitude of the Earth's gravitational force, as was expected.

The characteristic case is that of motion of two bodies having masses m_1 and m_2, whose distance ρ changes according to formula $\rho = A\cos(\Omega t + \alpha)$, where Ω and α are constants; $v_r = \dot{\rho}$.

By means of formula (3A.67) it is obtained that it is

$$\chi = -\frac{\rho^2\Omega^2}{m_1 + m_2}.$$

By substituting in differential equations of motion (3A.68) and (3A.69) it is obtained that:

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2(x_1 - x_2), \\ \ddot{y}_1 &= -\omega_1^2(y_1 - y_2); \\ \ddot{x}_2 &= \omega_2^2(x_1 - x_2), \\ \ddot{y}_2 &= \omega_2^2(y_1 - y_2); \end{aligned}$$

where, for the sake of brevity, the following notations are introduced:

$$\omega_1^2 = \frac{m_2\Omega^2}{m_1 + m_2} \quad \text{and} \quad \omega_2^2 = \frac{m_1\Omega^2}{m_1 + m_2}.$$

If we further state that $x = x_1 - x_2$, $y = y_1 - y_2$ and $z = z_1 - z_2$, the above-given system of equations can be reduced to three homogeneous linear differential equations:

$$\ddot{x} = -\Omega^2 x, \quad \ddot{y} = -\Omega^2 y, \quad \ddot{z} = -\Omega^2 z.$$

Their solutions, as is known,

$$\begin{aligned} x &= C_1\cos\Omega t + C_2\sin\Omega t \\ y &= C_3\cos\Omega t + C_4\sin\Omega t \\ z &= C_5\cos\Omega t + C_6\sin\Omega t. \end{aligned}$$

at various initial conditions, determine various trajectories such as, for instance:

a) For $t_0 = 0$ and $x(t_0) = x_0$, $y(t_0) = y_0$, $\dot{x}(t_0) = \dot{x}_0 = 0$, $\dot{y}(t_0) = \dot{y}_0 = 0$, $\dot{z}(t_0) = \dot{z}_0 = 0$ oscillation $x = x_0\cos\Omega t$, $y = y_0\cos\Omega t$, $z = z_0\cos\Omega t$ is obtained along the straight line

$$\frac{x}{x_0} = \frac{y}{y_0} = \frac{z}{z_0}.$$

b) For $y_0 = 0$, $\dot{x}_0 = 0$, $z_0 = z_{10} - z_{20} = 0$ and $\dot{z}_0 = 0$ and motion is determined by finite equations $x = x_0\cos\Omega t$ and $y = \frac{\dot{y}_0}{\Omega}\sin\Omega t$ along the ellipse

$$\frac{x^2}{x_0^2} + \frac{\Omega^2 y^2}{\dot{y}_0^2} = 1,$$

that is,

$$\frac{(x_1 - x_2)^2}{(x_1 - x_2)_0^2} + \Omega^2 \frac{(y_1 - y_2)^2}{(\dot{y}_1 - \dot{y}_2)_0^2} = 1.$$

æ

3B. WORK PRINCIPLE

The statement of this principle requires more than the concepts defined so far. It is, first of all, necessary to define the concept of work.

Definition 5. Work. *The integral*

$$W(\boldsymbol{F}) \stackrel{\text{def}}{=} \int_s \boldsymbol{F} \cdot \boldsymbol{ds} \tag{3B.1}$$

is the work of force $\boldsymbol{F}$ *upon real displacement* $\boldsymbol{ds} = d\boldsymbol{r}$ *along pathway* s.

Unlike the four basic definitions in which vector invariants (1.1), (1.24), (1.29) and (1.37) are introduced, this definition introduces a scalar invariant into dynamics. This eliminates the difficulties which arise in algebra as well as in the constrained vectors' analysis due to the parallel displacement of the vectors and their addition.

For the system of N dynamic points the forces' work is equal to the sum of all the forces' works:

$$W = \sum_{\nu=1}^{N} W_\nu = \int_s \sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot d\boldsymbol{r}_\nu. \tag{3B.2}$$

The physical dimension of work is

$$[\dim W] = \mathrm{M\,L\,T^{-2}\,L} = \mathrm{M\,L^2\,T^{-2}}\,.$$

Integral (3B.1) is curvilinear. In accordance with the introduced coordinate systems and their respective forces' vectors, formula (3B.1) can be written in the following forms:

$$\begin{aligned} W &= \int_s \boldsymbol{F} \cdot d\boldsymbol{r} = \int_s F_i dr^i = \\ &= \int_s Y_i dy^i = \int_s X_i dx^i = \int_s Q_\alpha dq^\alpha. \end{aligned} \tag{3B.3}$$

Integral (3B.2) is also reduced to the same invariant forms. In the general and final case work is a function of the dynamic point's position on the pathway as well as of kinematic and dynamic parameters that a family of trajectories depends upon. Subintegral functions are forces or coordinates of the forces' vectors that, in the general case, depend upon the given dynamic parameters, the position coordinates, the coordinates of the velocities vectors as well as upon the acceleration vectors' coordinates when the inertia forces, that is $X(\varkappa, x, \dot{x}, \ddot{x})$ are in question.

For some forces the integral (3B.3) can be integrated independently of the pathway; this is always the case when the form $\boldsymbol{F}\cdot d\boldsymbol{r}$ is a total differential of some function, let's say, U, namely,

$$X_i dx^i = dU. \tag{3B.4}$$

Regarding the fact that the second derivative $\ddot{x}$, due to the nature of forces X, can appear only in a linear form, $U(x, \dot{x})$ appears as a function of parameters $\varkappa$, positions x and velocities $\dot{x}$. This function is called the *function of the forces*, while the function of opposite sign is called the *function of energy* or, shorter, energy.

Work of Particular Forces

Work of Inertia Force (1.37)–(1.40) is represented by the relation

$$W(\boldsymbol{I}_F) = \int\limits_s \left(-m\frac{d\boldsymbol{v}}{dt}\right)\cdot d\boldsymbol{r} = -m\int\limits_{v_0}^{v} \boldsymbol{v}\cdot d\boldsymbol{v} = -\frac{m}{2}(v^2 - v_0^2). \tag{3B.5}$$

If it is assumed that $v_0 = 0$, negative work of inertia force can be expressed by the formula

$$E_k = \frac{m}{2}v^2, \tag{3B.6}$$

which is in classical literature known as kinetic energy of the material point having mass m.

Inertia forces' work of the system of N material points having masses $m_\nu =$ const $(\nu = 1, \ldots, N)$, according to (3B.2), represents the sum of all kinetic energies of all material points with a negative sign:

$$\begin{aligned} W(\boldsymbol{I}) &= -\int\limits_s \sum_\nu m_\nu \boldsymbol{a}_\nu \cdot d\boldsymbol{r}_\nu = -\sum_{\nu=1}^{N}\int\limits_0^{\nu} m_\nu \boldsymbol{v}_\nu\cdot d\boldsymbol{v}_\nu \\ &= -\sum_{\nu=1}^{N}\frac{m_\nu}{2}v_\nu^2 = -E_k. \end{aligned} \tag{3B.7}$$

For the points of constant mass, integral (3B.5) is easily obtained for expression (1.40) as well. Namely,

$$\begin{aligned} W(\boldsymbol{I}_F) &= -\int a_{ij}(x)\frac{Dv^j}{dt}dx^i = -\hat{\int} a_{ij}v^i Dv^j = \\ &= -\frac{1}{2}\hat{\int} D(a_{ij}v^i v^j) = \\ &= -\frac{1}{2}\left(a_{ij}v^i v^j - a_{ij}(x_0)v_0^i v_0^j\right) = -(E_k - E_{0k}). \end{aligned}$$

since it is for $m_\nu = \text{const}\ Da_{ij} = 0$ [36].

For $v_0 = 0$, regarding formula (3B.6), it is obtained that kinetic energy is equal to negative work of inertia force:

$$E_k = -W(\boldsymbol{I}) = \frac{1}{2} a_{ij} v^i v^j. \tag{3B.7a}$$

It follows from here that *inertia force work is equal to negative kinetic energy.*

Work of Newton's Gravitational Force (2.1) is represented by the expression

$$W(\boldsymbol{F}_{\nu\mu}) = -\varkappa m_\nu m_\mu \int_0^{r_{\nu\mu}} \frac{dr_{\nu\mu}}{r_{\nu\mu}^2} = \frac{\varkappa m_\nu m_\mu}{r_{\nu\mu}} = -\Pi$$

where $\Pi = -\dfrac{\varkappa m_\nu m_\mu}{r_{\nu\mu}}$ is gravitational potential energy.

Potential Energy. For all the forces having functions of forces $U(x)$, $X(x) = \operatorname{grad} U$, dependent upon the material point's position, namely, such functions that $dU(x) = X(x)dx$, *potential energy E_p, as negative work of forces $X(x)$ is the function of position x:*

$$\Pi(x) = E_p := -\int_{x_0}^{x} X(x)dx = -(U(x) + U(x_0)). \tag{3B.8}$$

Work of Constraints' Reaction Forces.

$$W(\boldsymbol{R}) = \int_s \left(\boldsymbol{R}^\tau + \sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad} f_\mu \right) \cdot d\boldsymbol{r}, \tag{3B.9}$$

demands the previous knowledge of the resistance force or friction as well as determination of Lagrange's multipliers λ_μ. Friction forces $\boldsymbol{R}^\tau$ are determined on the basis of the friction law. Integral

$$\int_s \boldsymbol{R}_\mu^\tau \cdot d\boldsymbol{r}$$

is more definite than integral (3B.3) only if it is known that $\boldsymbol{R}^\tau$ belongs to the tangential plane of constraint $f_\mu = 0$. It most often appears as a function of velocity, so that determination of this force's work requires the previous knowledge of finite equations of motion or other relations by which velocity can be determined as a function of the object's position.

Example 9. The work of force $\boldsymbol{R}^{\tau} = -\mu \boldsymbol{v}$, $\mu \in \mathbb{R}$, $\boldsymbol{v} \in \mathbf{R}_3$ that causes the material point of mass m to move from the initial position:

$$(y_0^1, y_0^2, y_0^3; \dot{y}_0^1, \dot{y}_0^2, \dot{y}_0^3).$$

It is obtained from the finite or differential equations

$$\dot{y}_i = \frac{\mu}{m}(y_{0i} - y_i) + \dot{y}_{0i},$$

so that the work of the given force:

$$W(\boldsymbol{R}_\tau) = \int\limits_s \boldsymbol{R}_\tau \cdot d\boldsymbol{r} = -\int\limits_s \mu \boldsymbol{v} \cdot d\boldsymbol{r} = -\mu \int\limits_s \dot{y}_i dy^i =$$

$$= -\mu \int\limits_{y_{0i}}^{y_i} \left[\frac{\mu}{m}(y_{0i} - y_i) dy^i + \dot{y}_{0i} dy^i \right] =$$

$$= -\frac{\mu^2}{m} \left(y_{0i} y^i - \frac{1}{2} y_i y^i \right) - \mu \dot{y}_{0i} y^i \Big|_{y_{i_0}}^{y_i} .$$

For $y_{0i} = 0$, as is most often possible to take,

$$W(\boldsymbol{R}_\tau) = \frac{\mu^2}{2m} \sum_{i=1}^{3} y_i^2 - \mu \dot{y}_{0i} y^i; \ \left(\mathrm{M\,L^2\,T^{-2}}\right).$$

The integral

$$\int\limits_s \sum_\mu \lambda_\mu \operatorname{grad} f_\mu \cdot d\boldsymbol{r}$$

is considerably simplified, namely:

a) If the constraints are geometrical and depend only upon position $f(\boldsymbol{r}) = 0$. Then from the velocity condition

$$\frac{df}{dt} = \operatorname{grad} f \cdot \frac{d\boldsymbol{r}}{dt} = 0$$

it follows that integral (3B.7) is equal to zero, so that components $\lambda_\mu \operatorname{grad} f_\mu$ of the reaction forces of constraints f do not produce work;

b) In the case that the constraints are variable in time, that is, that functions $f_\mu(\boldsymbol{r}, \tau)$ also depend, in addition to $\boldsymbol{r}$, upon some explicit time functions $\tau(\varkappa, t)$, for which the velocity conditions have the form:

$$\frac{df}{dt} = \operatorname{grad} f \cdot \frac{d\boldsymbol{r}}{dt} + \frac{\partial f}{\partial \tau}\frac{d\tau}{dt} = 0.$$

For this reason, the previous integral reduces to

$$-W_2 = \int_{\tau_0}^{\tau} \sum_{\mu} \lambda_\mu \frac{\partial f_\mu}{\partial \tau} d\tau =: \mathcal{P} \tag{3B.10}$$

where $\mathcal{P}$ is *rheonomic pseudopotential* [41], [49], that can be determined if generalized force (3A.54) is reduced to a function of τ or if

$$d\mathcal{P} = -\sum_{\mu} \lambda_\mu \frac{\partial f_\mu}{\partial \tau} d\tau$$

is total potential, which is possible in some cases.

In order to understand the following exposition much easier, attention should be also paid to the work of time-dependent forces $F = (Y_1(\varkappa,t),\ Y_2(\varkappa,t),\ Y_3(\varkappa,t))$. In this case, the integral can be solved

1) as curvilinear, along the trajectory; in this case, it is necessary to determine time $t = \tau(\varkappa, y)$ from $y = y(\varkappa, t)$ finite equations of motion, or,

2) by reducing curvilinear integral (3B.3) to a definite integral by introducing an independent time parameter, namely,

$$\begin{aligned} W &= \int_s Y_i(\varkappa,t) dy^i = \\ &= \int_{t_0}^{t_1} Y_i(\varkappa,t)\dot{y}^i(t) dt = \int_{t_0}^{t_1} P(\varkappa,t) dt. \end{aligned} \tag{3B.10a}$$

Regarding the preprinciple of existence, time t is an independent variable, so that in both of the cases the treatment of time as a function of a new independent parameter is excluded. The relation $t = t(\varkappa, y, \dot{y})$ is nothing else but a motion equation of the given system solved with respect to t.

Example 10. Motion of the material point whose pathway has the form of ellipse

$$y_1 = a\cos\omega t, \quad y_2 = b \sin\omega t,$$

can be written in the form:

$$t = \frac{1}{\omega} \operatorname{arctg} \frac{a y_2}{b y_1}.$$

More generally, curvilinear integrals (3B.3) can be reduced to ordinary integrals of the form

$$W = \int_s Y_i(y,\dot{y},\varkappa,t) dy^i = \int_{t_0}^{t} Y_i(\varkappa,t)\dot{y}^i(\varkappa,t) dt, \tag{3B.11}$$

that is,

$$W(\varkappa, t) = \int_{t_0}^{t} P(\varkappa, t)dt, \tag{3B.12}$$

since at real displacement dy there is velocity $\dot{y}$, so that it is $dy^i = \dot{y}^i(t)dt$.

The function

$$P(\boldsymbol{F}) = \boldsymbol{F} \cdot \boldsymbol{v} = Y_i \dot{y}^i = X_i \dot{x}^i = Q_\alpha \dot{q}^\alpha \tag{3B.13}$$

is known as power in mechanics.

It is most often written in the form

$$\frac{dW}{dt} = P = X_i \frac{dx^i}{dt}. \tag{3B.14}$$

Elementary Work

From relation (3B.14) or directly from (3B.11) or from (3B.13) it follows that even differentially small work

$$dW = \boldsymbol{F} \cdot d\boldsymbol{r} = Y_i dy^i = X_i dx^i = Q_\alpha dq^\alpha \tag{3B.15}$$

is a scalar invariant. This work is often considered as *elementary work of forces upon real displacement.* The expression "upon real displacement" emphasizes the difference from the other hypothetical and arbitrarily small work of these forces upon any *possible small displacement* $\Delta\boldsymbol{r}$,

$$\Delta W = \boldsymbol{F} \cdot \Delta\boldsymbol{r}. \tag{3B.16}$$

The concept of *possible displacement* implies any, no matter how small, diversion from the material point's real position that could be achieved by that point. This concept is even more general than differential $d\boldsymbol{r}$ or than variation $\delta\boldsymbol{r}$ of the position vector. In other words, this is any hypothetically feasible distance at possible displacement. In practice, it could be understood as a tentative, factual or mental displacement. The size of the smallness cannot be accurately determined; it is arbitrarily small ranging from negligibly small to some finite size. Analytically speaking, this concept can imply a difference between the position vector of the possible displacement of a possibly displaced point and vector $\boldsymbol{r}$ of the fixed or given position, namely, $\Delta\boldsymbol{r} := \boldsymbol{r}(x + \Delta x) - \boldsymbol{r}(x)$. Following the finite increments formulae, function vector $\Delta\boldsymbol{r}$ can be expressed in analytical form:

$$\Delta\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial y^i}(y^{*i} - y^i) = \frac{\partial \boldsymbol{r}}{\partial y^i}\Delta y^i \tag{3B.17}$$

as well as

$$\Delta\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial x^j}\frac{\partial x^j}{\partial y^i}\Delta y^i + \cdots = \frac{\partial \boldsymbol{r}}{\partial x^j}\Delta x^j + \cdots = \frac{\partial \boldsymbol{r}}{\partial q^\alpha}\Delta q^\alpha + \cdots \tag{3B.18}$$

where Δy, Δx, Δq are coordinates of possible displacement vector in various coordinate systems. Vector $\Delta\boldsymbol{r}$ coordinates are the ones that are most often called *possible displacements.*

By analogy with elementary work upon real displacement (3B.15), formula (3B.16) is called *work upon possible displacements.* smallskip Formula (3B.16) is a scalar invariant, as well as (3B.15), and thus, because of possible and real displacements, it satisfies the preprinciple of existence.

The invariant form

$$\boldsymbol{F}\Delta\boldsymbol{r} = Y_i \Delta y^i = X_i \Delta x^i = Q_\alpha \Delta q^\alpha \tag{3B.19}$$

satisfies the preprinciple of invariance, while relations (3B.18) and (3B.19) determine a degree of accurate determination; hence it also satisfies the preprinciple of casual definiteness. Regarding the fact that $\boldsymbol{F} \cdot d\boldsymbol{r}$ is a scalar value, the following addition is possible:

$$\sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \Delta\boldsymbol{r}_\nu = \sum_{k=1}^{3N} Y_k \Delta y^k = \sum_{\beta=1}^{n} Q_\beta \Delta q^\beta \tag{3B.20}$$

which makes up total work of all the forces $\boldsymbol{F}_\nu$ $(\nu = 1, \dots, N)$ upon possible displacement.

Beside elementary work (3B.15) upon real displacement $d\boldsymbol{r}$ and work (3B.16) upon possible displacement $\Delta\boldsymbol{r}$, *work upon variations* $\delta\boldsymbol{r}_\nu$ or $\delta y, \delta x, \delta q$, is considerably important and is formulated by the expression

$$\delta W := \boldsymbol{F} \cdot \delta\boldsymbol{r} \tag{3B.21}$$

or in another invariant form

$$\delta W := Y\delta y = X\delta x = Q\delta q. \tag{3B.22}$$

This work cannot be made equal with elementary work dA, regardless of the fact that relations (3B.15) and (3B.22) are similar. However, work (3B.21) can be regarded as elementary work (3B.18) upon possible displacement since δx variations can belong to a set of possible displacements Δx. Unlike the differential

$$dx = \frac{\partial x}{\partial t} dt = \frac{dx}{dt} dt = \dot{x} dt, \tag{3B.23}$$

Differential δx that is called the variation (See [5, pp. 27, 177] or [23]) shows the presence of some change of function $x(\alpha, t)$ due to increment of parameter $\alpha = \overline{\alpha} - \delta\alpha$.

Therefore, the concept of *function variation* $x = x(\alpha, t)$ *implies the product of the function derivative with respect to the parameter and small increment of the given parameter*, that is,

$$\delta x = \frac{\partial x}{\partial \alpha} \delta\alpha = \frac{dx}{d\alpha} \delta\alpha =: \frac{\delta x}{\delta \alpha} \delta\alpha \tag{3B.24}$$

This means that it is

$$\frac{\delta x}{\delta \alpha} := \lim_{\Delta\alpha \to 0} \frac{x(\alpha + \Delta\alpha, t) - x(\alpha, t)}{\Delta\alpha} \tag{3B.25}$$

In the same way, for the work written in the function form

$$W = \int_s X dx = W(x(t, \alpha)) \tag{3B.26}$$

or

$$W = \int_{t_0}^{t} X \dot{x} dt = W(t)_{t=t(\alpha,x)}, \tag{3B.27}$$

the differentiating operation is valid

$$\delta W = \frac{\partial W}{\partial \alpha}\delta\alpha = \frac{\partial W}{\partial x}\frac{\partial x}{\partial \alpha}\delta\alpha = \frac{\partial W}{\partial x}\delta x \tag{3B.28}$$

or, if x is expressed by means of time t, derived from motion $t = t(x)$:

$$\delta W = \frac{\partial W}{\partial \alpha}\delta\alpha = \frac{\partial W}{\partial t}\frac{\partial t}{\partial x}\frac{\partial x}{\partial \alpha}\delta\alpha = \frac{\partial W}{\partial t}\frac{\partial t}{\partial x}\delta x = \frac{\partial W}{\partial x}\delta x. \tag{3B.29}$$

Because of these characteristics of relations (3B.28) and (3B.29), *elementary work* upon possible variations is better to be called *work variations*.

Excerpta: Considering the fact that in the referential literature there is no unanimous understanding of the concepts of *real displacement* $d\boldsymbol{r}$, *possible displacement* $\Delta\boldsymbol{r}$ and *variations* $\delta\boldsymbol{r}$ and, consequently, no respective elementary works (3B.15), (3B.16) and (3B.21), it is necessary to note that:

1. real elementary displacement symbolized by differential $d\cdot$ refers to a change in time along the actual or given trajectory and it directly springs from definition (1.1),

2. possible displacement is any - no matter how small - displacement of indefinite smallness or any hypothetical deviation of the dynamic point from its position, provided by the constraints in its continuous medium; this displacement, that does not really take place, disregards the time factor or any other parameter except for boundaries established by the constraints,

3. the variation, symbolized by differential $\delta\cdot$, which is in direct relation with derivative (3B.25), is the points' deviation from the calculated or given trajectory due to insufficiently accurate casual definiteness or disturbance of some parameter contained within finite equations of motion or the trajectory equation in time t and for this reason, it is also a time function. If varying of parameter $\alpha + \delta\alpha$ is not definite, but hypothetical, it can be regarded that variation $\delta\boldsymbol{r}$ belongs to a set of possible displacements [46].

The above-given conclusions tend to emphasize that actual displacement is here identified with neither possible displacement nor variations.

Finally, it should be also noted that the work dimension, which also implies "elementary works", is equal to the dimension of the moment of force, that is,

$$\dim W = \dim E = \dim \mathfrak{M} = \mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-2}\,. \tag{3B.30}$$

For this reason, the elementary work upon possible displacements is sometimes called the *possible moment of forces* (see [12, p. 410]). These two concepts of work and moment of forces, due to the preprinciple of non-formality, are here regarded as different, since work is, by its definition, a scalar invariant, while the moment is a derived vector invariant or, simply, a vector.

Formulations of Work Principle

The essence of the *work principle* has been known in the literature (following to Galileo's postulate: "Quanto si guadagna di forza, tanto perdersi in velocita", Opere 2, p. 1830) as "the golden rule of mechanics" since the days of Aristotle, while later on, it has been known as "the principle of possible displacements", "the principle of possible variations", "the fundamental basic equation of mechanics", "the principle of virtual work", "the D'Alembert-Lagrange principle" and so on. One of the strictest mathematical analysts of classical mechanics, A. M. Ljapunov, has written the following:

"Principle of possible displacements has been known to Galileo; later on, it was used by Wallis and Ivan Bernoulli. But the first general proof of this principle was given only by Lagrange who built it into the foundation of his analytical mechanics. Afterwards, it was also proved by Poison, Cauchy and others; though, by its best proof, it remains Lagrange's."

In this approach to the theory of body motion, the principle is not being proved, but, as is written about the preprinciples or about the concept of the *principles of mechanics*, the principle is a true statement, in oral or written form, or both, and it is as much accurate as can be best stated on the basis of the existing knowledge. The formulation of the principle comprises its generality. Instead of proving it, its application to various systems is interpreted and proved. The work principle can be, in the shortest possible way, expressed by the following sentence:

The total work of all the forces upon possible displacements is worthless, while in the presence of unilateral constraints it is not positive.

The mathematical statement, regarding relation (3B.20), is even shorter:

$$\sum_{\nu=1} \widetilde{\boldsymbol{F}}_\nu \cdot \Delta \boldsymbol{r}_\nu \leq 0. \tag{3B.31}$$

The reader proficient in mathematics maybe finds the following formulation even more comprehensible:

The total work of all the forces upon all independent possible displacements is equal to zero, while for the system with unilateral constraints it is not positive.

Relation (3B.31) is very general, abut it is not directly operative. Its application requires a strict mathematical analysis, which implies, first of all, understanding of the elements it contains. The limited arbitrariness of possible displacements is described. Vectors $\widetilde{\boldsymbol{F}}_\nu$ comprise, as components, inertia force $\boldsymbol{I}_\nu$ of ν-th of this material point as well as the principal vectors of all other forces $\boldsymbol{F}_{\nu k}$ that exert their action upon the ν-th point, that is, $\boldsymbol{F}_\nu = \sum_k \boldsymbol{F}_{\nu k}$. Accordingly, without reducing the generality of relations (3B.31), this principle can be written in the form

$$\sum_{\nu=1}^{N} (\boldsymbol{I}_\nu + \boldsymbol{F}_\nu) \cdot \Delta \boldsymbol{r}_\nu \leq 0 \tag{3B.32}$$

The principle written in this way implies that vector $\boldsymbol{F}_\nu$ comprises, as pointed out, all the forces except for inertia ones; it also comprises the constraints reactions, according to the law of constraints. It implies that the relations of constraints μ are abstracted by the forces

$$\boldsymbol{R}_\nu = \sum_{\mu} \boldsymbol{R}_{\nu\mu}.$$

If the constraints' relations are not calculated a priori, as previously said, the relations describing the constraints should be added to relation (3B.32), namely,

$$\sum_{\nu=1}^{N} (\boldsymbol{I}_\nu + \boldsymbol{F}_\nu) \cdot \Delta \boldsymbol{r}_\nu = 0 \tag{3B.33}$$

$$f_\mu(\boldsymbol{r}, \boldsymbol{v}, \tau) \geq 0. \tag{3B.34}$$

Regarding the signs of equality and inequality, a difference is noticed between relations (3B.33) and (3B.32); the sign of inequality from (3B.32) is comprised by relations (3B.34). In the case of bilateral constraints abstracted by the forces, the relation of principle (3B.32) is written in the form (3B.33), while in the case that the constraints are not taken into consideration in relation (3B.32), writings (3B.33) and (3B.34) are given in the form:

$$\sum_{\nu=1}^{N} (\boldsymbol{I}_\nu + \boldsymbol{F}_\nu) \cdot \Delta \boldsymbol{r}_\nu = 0, \tag{3B.35}$$

$$f_\mu(\boldsymbol{r}, \boldsymbol{v}, \tau) = 0. \tag{3B.36}$$

Starting from the fact that the constraints are more frequently written in the coordinate form, as in relations (2.3)–(2.8), let's observe the principle's application to particular mechanical systems with respect to Descartes coordinate system $y := (y^1, y^2, y^3)$.

Static Systems

The concept of *static system* here implies N points of application M_ν ($\nu = 1, \dots, N$) of forces $\widetilde{\boldsymbol{F}}_\nu = \boldsymbol{F}_\nu = Y_\nu^i \boldsymbol{e}_i$,($i = 1, 2, 3$) connected by k finite constraints (2.5). These constraints are written more concretely as

$$f_\mu(y_1^1, y_1^2, y_1^3, \dots, y_N^1, y_N^2, y_N^3) = 0, \tag{3B.37}$$

or, by formalizing of indices

$$y_\nu^1 = y^{3\nu-2}, \;\; y_\nu^2 = y^{3\nu-1}, \;\; y_\nu^3 = y^{3\nu}, \tag{3B.38}$$

as

$$f_\mu(y^1, \dots, y^{3N}) = 0. \tag{3B.39}$$

For such a system $\boldsymbol{I}_\nu = 0$, so that relations (3B.35) and (3B.36) can be written in the following coordinate form:

$$Y_\alpha \Delta y^\alpha := Y_1 \Delta y^1 + \dots + Y_{3N} \Delta y^{3N} = 0, \tag{3B.40}$$

$$f_\mu := f_\mu(y^1, \dots, y^{3N}) = 0. \tag{3B.41}$$

Firstly, it has to be stated that the non-ideal constraint factor is abstracted by the force comprised within Y_α forces, while relations (3B.41) describe the constraints' idealization. Developing into the order with respect to possible displacements of these constraints in the neighborhood of the equilibrium positions of points $M_\nu(y = b)$, what is obtained, beside linear form (3B.40), are k linear forms with respect to Δy, namely:

$$f_\mu(y) - f_\mu(b) = a_{\mu\alpha} \Delta y^\alpha = a_{\mu 1} \Delta y^1 + \dots + a_{\mu 3N} y^{3N} = 0, \tag{3B.42}$$

where

$$a_{\mu\alpha} = \frac{\partial f_\mu}{\partial y^a}\Big|_{y^\alpha = b^\alpha}. \tag{3B.43}$$

Therefore, relations (3B.40) and (3B.41) are reduced to $k + 1$ linear equations

$$Y_\alpha \Delta y^\alpha = 0, \tag{3B.44}$$

$$a_{\mu\alpha} \Delta y^\alpha = 0, \;\; (\mu = 1, \dots, k < 3N), \tag{3B.45}$$

in which there are $3N$ mutually dependent possible displacements Δy^{3N}. Regarding the fact that relations (3B.44), according to the work principle formulation, should comprise independent possible displacements, this problem can further be solved in two ways in order to eliminate dependent possible displacements, namely:

a) by direct solving of equations (3B.45),

b) by introducing indefinite constraints' multipliers.

Solution with Respect to Dependent Displacements. If possible displacements are separated into dependent $\Delta y^1, \dots, \Delta y^k$ and independent $\Delta y^{k+1}, \dots, \Delta y^{3N}$, then in equations (3B.44) and (3B.45) the addends with dependent and independent possible displacements are separated:

$$Y_\nu \Delta y^\nu + Y_\beta \Delta y^\beta = 0,\ \nu = 1, \dots, k, \tag{3B.46}$$

$$a_{\mu\nu} \Delta y^\nu + a_{\mu\beta} y^\beta = 0,\ \beta = k+1, \dots, 3N. \tag{3B.47}$$

Substituting

$$\Delta y^\nu = -a^{\mu\nu} a_{\mu\beta} \Delta y^\beta == b^\nu_\beta,\ |a_{\mu\nu}| \neq 0 \tag{3B.48}$$

in equation (3B.46), where $a^{\mu\nu}$ is an inverse matrix $a_{\mu\nu}$, an equation with independent displacements is obtained, namely,

$$\left(Y_\beta - Y_\nu b^\nu_\beta\right) \Delta y^\beta = 0. \tag{3B.49}$$

Due to independence of displacement Δy^β it follows that the system of the observed forces in the presence of constraints (3B.41) will be in equilibrium if it satisfies the following system of $3N - k$ algebraic equations:

$$Y_\beta - Y_1 b^1_\beta - \dots - Y_k b^k_\beta = 0. \tag{3B.50}$$

As can be seen from this system of equations, it is possible to determine $3N-k$ coordinates of the forces' vector by means of the remaining k.

Indefinite Constraints' Multipliers. If each of equations (3B.42) is multiplied by its respective multiplier λ_μ and then added with respect to index μ, the systems of $k+1$ equations (3B.44) and (3B.45) are reduced to two equations:

$$\left.\begin{aligned} Y_\alpha \Delta y^\alpha &= 0 \\ \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} \Delta y^\alpha &= 0. \end{aligned}\right\} \tag{3B.51}$$

The sum of these two relations

$$\left(Y_\alpha + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha}\right) \Delta y^\alpha = 0, \tag{3B.52}$$

provides, just like in the previous method, for elimination of dependent possible displacements $\Delta y^1, \dots, \Delta y^k$. Regarding the fact that λ_μ are indefinite multipliers for the time being, it is permissible to separate the conditions that annul k multipliers λ_μ from equations (3B.52), so that it is

$$Y_\sigma + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\sigma} = 0, \quad \sigma = 1, \dots, k. \tag{3B.53}$$

k equations (3B.52) of $3N - k$ independent variations are left, namely:

$$\left(Y_\beta + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\beta} \right) \Delta y^\beta = 0. \tag{3B.54}$$

From this relation more $3N - k$ equations of the form (3B.53) are obtained. Thus, as a solution of the static problem, a system is obtained of $3N$ equations of the forces

$$Y_\alpha + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} = 0 \quad (\alpha = 1, \dots, 3N).$$

and k equations of the constraints $f_\mu(y^1, \dots, y^{3N}) = 0$.

Rheonomic Systems. As in the previously-discussed static system, the work principle is also applied to the mechanical system with variable constraints (2.8). For the sake of brevity, and without making it less general, it should be assumed that the constraints are given by the constraints equations

$$f_\mu(y^0; y^1, \dots, y^{3N}) = 0, \quad y^0 = \tau(t), \tag{3B.55}$$

where $\tau(t)$ is a known function of time.

Developing the function into a power series, as in (3B.42), it is shown that there are $3N + 1$ possible displacements $\Delta y^0, \dots, \Delta y^{3N}$. Therefore,

$$\Delta f_\mu = \frac{\partial f_\mu}{\partial y^0} \Delta y^0 + \frac{\partial f_\mu}{\partial y^i} \Delta y^i = 0, \ (i = 1, \dots, 3N).$$

The work principle comprises "all possible displacements" as well as the work done by particular forces upon these displacements. Therefore, in addition to works upon possible displacements $Y_i \Delta y^i$, the work upon possible displacement Δy^0, that is, $Y_0 \Delta y^0$ should be added here. Thus, in such a system with variable constraints (3B.55), instead of relations (3B.40) and (3B.41), there is a system of equations:

$$Y_\alpha \Delta y^\alpha = Y_0 \Delta y^0 + Y_i \Delta y^i = 0, \tag{3B.56}$$

$$f_\mu(y^0, y) = f_\mu(y^0, y^1, \dots, y^{3N}) = 0. \tag{3B.57}$$

From this system of equations, by the same procedure as from (3B.51) and (3B.55), another additional equation is obtained

$$Y_0 + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^0} = 0. \tag{3B.58}$$

The force

$$Y_0 = -\sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^0} \tag{3B.59}$$

is evident even in more general relations (3B.56) and (3B.54); [61].

System with Unilateral and Bilateral Constraints. The work principle, written by relation (3B.31), shows that the inequality sign refers to unilateral constraints. In the case of only unilateral constraints, the principle says that work upon possible displacements is less than zero, that is ($\widetilde{\boldsymbol{F}}_\nu = \boldsymbol{F}_\nu$),

$$\sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \Delta \boldsymbol{r}_\nu \leq 0 \tag{3B.60}$$

while in the case of bilateral constraints, as shown,

$$\sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \Delta \boldsymbol{r}_\nu = 0. \tag{3B.61}$$

Let's consider the simultaneous presence of bilateral constraints

$$f_\mu(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = 0 \quad \mu = 1, \ldots, k \tag{3B.62}$$

and unilateral ones

$$\varphi_\sigma(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \geq 0 \quad \sigma = 1, \ldots, l \tag{3B.63}$$

under the condition that it is $k + l < 3N$.

Let's choose again coordinate system $(y, \boldsymbol{e})$ and apply the method of the indefinite constraints' multipliers, while relations (3B.60) and (3B.61) are reduced, similarly to equations (3B.51), to the forms:

$$Y_\alpha \Delta y^\alpha = \Delta c, \;\; \Delta c \leq 0 \tag{3B.64}$$

$$\sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} \Delta y^\alpha = 0 \tag{3B.65}$$

$$\sum_{\sigma=1}^{l} \chi_\sigma \frac{\partial f_\mu}{\partial y^\alpha} \Delta y^\alpha = \sum \chi_\sigma \Delta c_\sigma, \tag{3B.66}$$

where they are either $\Delta c_\sigma \geq 0$ or $\Delta c_\sigma \leq 0$.

The sum of all the three equations

$$\sum_{\alpha=1}^{3N} \left(Y_\alpha + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^\alpha} + \sum_{\sigma=1}^{l} \chi_\sigma \frac{\partial f_\sigma}{\partial y^\alpha} \right) \Delta y^\alpha = \Delta c + \sum_{\sigma=1}^{l} \chi_\sigma \Delta c_\sigma \tag{3B.67}$$

gives a number of equations necessary and sufficient for problem-solving.

As in the case of bilateral constraints $k+l$ of displacement Δy^α it is possible to exclude the requirement that multipliers λ_μ and χ_σ should be such that the following equations are satisfied:

$$Y_i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i} + \sum_{\sigma=1}^{l} \chi_\sigma \frac{\partial f_\sigma}{\partial y^i} = 0, \qquad (3B.68)$$
$$i = 1, \dots, k; k+1, \dots, k+l.$$

Remaining $3N-(k+l)$ coefficients, in addition to possible probable displacements Δy^j $(j = k+l+1, \dots, 3N)$ will be also equal to zero, that is,

$$Y_j + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^j} + \sum_{\sigma=1}^{l} \chi_\sigma \frac{\partial f_\sigma}{\partial y^j} = 0 \qquad (3B.69)$$

so that, in accordance with the principle, there should be

$$\Delta c + \sum \chi_\sigma \Delta c_\sigma = 0.$$

However, since it is, in accordance with (3B.60) and (3B.61), $\Delta c < 0$, it follows

$$\sum_{\sigma=1}^{l} \chi_\sigma \Delta c_\sigma > 0. \qquad (3B.70)$$

If independence of the indefinite constraints' multipliers is taken into consideration, the following conditions of equations (3B.68) and (3B.69) follow, namely, that χ_σ and Δc_σ are of the same sign.

Kinetic Systems

Let's remember that vector functions F_ν, whose coordinates are Y, comprise all the active forces $\boldsymbol{F}$, including inertia force $\boldsymbol{I} = -m\dfrac{d\boldsymbol{v}}{dt}$ as well. Consequently, $3N$ differential equations of motion (3B.68) and (3B.69), as well as $k+l$ of finite constraints equations along with the conditions resulting from (3B.70), make up the total system of relations for solving motion of the observed system with finite unilateral and bilateral constraints

Non-holonomic Systems. A non-holonomic system implies a system of N material points, whose motion is, among other things, restricted by at least one differential non-integrable (non-holonomic) constraint. If the previous restriction is taken into consideration, let them be the constraints

$$\varphi_\mu := \varphi_\mu(y^1, \ldots, y^{3N}, \dot{y}^1, \ldots, \dot{y}^{3N}) = 0. \tag{3B.71}$$

Due to the difficulties arising while developing functions (3B.71) into a series in the vicinity of trajectory $C(y)$ as well as due to the complexity of these constraints' possible equations or their kinematic nature – and for the sake of brevity – the method of constraints' abstraction will be applied here, in accordance with the constraints' law, by respective forces – reactions of constraints $\boldsymbol{R}_{\nu\mu}$. In other words, each constraint (3B.71) acting upon the ν-th point is replaced by the resultant vector of the constraints' reaction as in (2.9), that is, $\boldsymbol{R}_{\nu\mu}$. Regarding the previously introduced notation, this can be written in a shorter way, as well as other forces' vectors, by means of a set of $3N$ coordinates $R_1, \ldots, R_{3N}$. In such a general approach to the work principle (3B.32) it can be written in the coordinate form

$$(I_\alpha + Y_\alpha + R_\alpha)\,\Delta y^\alpha = 0. \tag{3B.72}$$

The system of $3N$ differential equations of motion

$$I_\alpha + Y_\alpha + R_a = 0, \quad (\alpha = 1, \ldots, 3N) \tag{3B.73}$$

that is,

$$m_\alpha \ddot{y}_\alpha = Y_\alpha + R_\alpha, \tag{3B.73a}$$

comprises, among other things, $3N$ of unknown reactions of the constraints R_α which should satisfy the acceleration conditions

$$\dot{\varphi}_\mu(y, \dot{y}, \ddot{y}) = \frac{\partial \varphi_\mu}{\partial y^\alpha}\dot{y}^\alpha + \frac{\partial \varphi_\mu}{\partial \dot{y}^\alpha}\ddot{y}^\alpha = 0. \tag{3B.74}$$

Substituting $\ddot{y}_\alpha$ from equations (3B.73) in previous equations (3B.74), k linear equations with respect to R_α are obtained, namely:

$$\frac{\partial \varphi_\mu}{\partial y^\alpha}\dot{y}^\alpha + \frac{1}{m_\alpha}(Y_a + R_\alpha)\frac{\partial \varphi_\mu}{\partial \dot{y}^\alpha} = 0.$$

From these equations it is possible to determine k reactions:

$$R_i = R_i(m, y, \dot{y}, Y, R_{k+1}, \ldots, R_{3N}) \quad (i = 1, \ldots, k)$$

depending, among other things, upon $3N$ coordinates of force Y and $3N - k$ of reactions R_j $(j = k + 1, \ldots, 3N)$. Further on, by substituting R_i in equations (3B.73), that is, (3B.73a), in the system of $3N$ differential equations of motion, $3N - k$ unknown constraints' reactions still remain. They are, as such, possible to determine from this system depending on other functions in these equations or to look for new $3N - k$ conditions that define or determine the rest of $3N - k$ unknown reactions of differential constraints (3B.71). Many studies have been devoted to this problem that is still acute.

First Conclusion. By means of the work principle it is possible to derive and extend the dynamic equilibrium relations just like from the equilibrium principle; the work principle and the equilibrium principle are equivalent.

Invariant Writings of Work Principle

Expressions (3B.18), (3B.19) and (3B.20) point to the fact that relations (3B.31) or (3B.32) can be written in a similar form with respect to various coordinate systems. Let $(y, \boldsymbol{e})$ be still the Cartesian orthonormal stationary coordinate system, $(z, \boldsymbol{\ni})$ rectilinear coordinate system, $(x, \boldsymbol{g}_{(x)})$ curvilinear system of coordinates and $(q, \boldsymbol{g}_{(q)})$ a system of independent generalized coordinates. The same constraints, as can be seen from (2.2), are written by invariant expressions:

$$f_\mu(\boldsymbol{r}, \boldsymbol{v}, \tau) = 0 \to f_\mu(y, \dot{y}, \tau) = 0 \to f_\mu(z, \dot{z}, \tau) = 0 \to f_\mu(x, \dot{x}, \tau) = 0,$$
$$\tau = y^0 = z^0 = x^0, \ \mu = 1, \ldots, k,$$

or in the parametric form:

$$\boldsymbol{r}_\nu = \boldsymbol{r}_\nu(q^0, q^1, \ldots, q^n) =: \boldsymbol{r}_\nu(q), \tag{3B.75}$$
$$q^0 = \tau(t). \tag{3B.76}$$

According to relation (3B.18), possible displacements, on the basis of the choice of coordinate system, are written in the following forms:

$$\Delta \boldsymbol{r}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial y^i}\Delta y^i = \frac{\partial \boldsymbol{r}_\nu}{\partial z^i}\Delta z^i = \frac{\partial \boldsymbol{r}_\nu}{\partial x^i}\Delta x^i \tag{3B.77}$$

or

$$\Delta \boldsymbol{r}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\Delta q^\alpha =: \frac{\partial \boldsymbol{r}_\nu}{\partial q}\Delta q, \tag{3B.78}$$

where $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q}$ are coordinate vectors of the ν-th point upon the configurational multifoldness M. The number of possible displacements allows for possible changes of the constraints:

$$\left.\begin{aligned} &\Delta f_\mu = \frac{\partial f_\mu}{\partial y}\Delta y + \frac{\partial f_\mu}{\partial \tau}\Delta\tau = \frac{\partial f_\mu}{\partial x}\Delta x + \frac{\partial f_\mu}{\partial \tau}\Delta\tau = 0 \\ &\Delta f_0 = \Delta y^0 - \Delta\tau = 0. \end{aligned}\right\} \tag{3B.79}$$

If force R_0 is applied to abstract constraint (3B.76), as much existent as other constraints, possible changes of constraints (3B.79) show that there are $3N+1$ possible displacements, so that the indices in relations (3B.77) and (3B.78) take on values $i = 0, 1, \ldots, 3N$; $\alpha = 0, 1, \ldots, n$. For this reason, work formulation (3B.31) has the following invariants:

$$\begin{aligned} &\widetilde{Y}\Delta y = \widetilde{Z}\Delta z = \widetilde{X}\Delta x = \\ &\qquad = \widetilde{Y}_i\Delta y^i = \widetilde{Z}_i\Delta z^i = \widetilde{X}_i\Delta x^i \leq 0, \end{aligned} \tag{3B.80}$$

$$f_\mu \geq 0; \quad \mu = 1, \ldots, k < 3N, \ i = 0, 1, \ldots, 3N$$

as well as

$$\widetilde{Q}\Delta q := \widetilde{Q}_\alpha \Delta q^\alpha \leq 0, \quad (\alpha = 0, 1, \ldots, n). \tag{3B.81}$$

In the case that the constraints' functions do not explicitly depend on time, q^0 coordinate does not exist; therefore, in relations (3B.80) and (3B.81) there are no zero indices $i = 0$, either. The same invariance also refers to relation (3B.32). Regarding the fact that observed relations (3B.31) and (3B.32) have been previously extended with respect to rectilinear coordinates y, curvilinear coordinates x and generalized independent coordinates $q \in M$ will be used further on.

a) Work Principle in Curvilinear Coordinate Systems.

It has been shown in relations (1.40) that the inertia force vector's coordinates, with respect to curvilinear coordinate systems, are determined by the expressions:

$$I_i = -a_{ij}\frac{Dv^j}{dt}. \tag{3B.82}$$

As $\widetilde{X}_i$ in relation (3B.8) denotes a sum of active forces and inertia forces (3B.82), it is:

$$\left(X_i - a_{ij}\frac{Dv^j}{dt}\right)\Delta x^i \leq 0. \tag{3B.83}$$

This inequality directly follows from relation (3B.32), if it is kept in mind that possible displacements, as in (3B.18) are

$$\Delta \boldsymbol{r}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial x^s}\Delta x^s, \quad (r, s = 1, 2, 3). \tag{3B.84}$$

Substituting in relation (3B.32), it is obtained, regarding (1.38)

$$\begin{aligned}
&\sum_{\nu=1}^{N}\left(I_\nu^s \frac{\partial \boldsymbol{r}_\nu}{\partial x^s} + X_\nu^s \frac{\partial \boldsymbol{r}_\nu}{\partial x^s}\right)\cdot \frac{\partial \boldsymbol{r}_\nu}{\partial x^r}\Delta x^r = \\
&= \sum_{\nu=1}^{N}\left(X_\nu^s \frac{\partial \boldsymbol{r}_\nu}{\partial x^s} - m_\nu \frac{Dv_\nu^s}{dt}\frac{\partial \boldsymbol{r}_\nu}{\partial x^s}\right)\cdot \frac{\partial \boldsymbol{r}_\nu}{\partial x^r}\Delta x^r = \\
&= \sum_{\nu=1}^{N}\left(g_{(\nu)sr}X_\nu^s - a_{(\nu)rs}\frac{Dv_\nu^s}{dt}\right)\Delta x^r \leq 0.
\end{aligned}$$

If indices $i, j = 1, \ldots, 3N$, are introduced $m_{3k} = m_{3k-1} = m_{3k-2},\ i = 3\nu, 3\nu - 1, 3\nu - 2$, the relation follows:

$$\left(g_{ij}X^j - a_{ij}\frac{Dv^j}{dt}\right)\Delta x^i \leq 0, \tag{3B.85}$$

or relation (3B.83), regarding the fact that $X_i = g_{ij}X^j$.

If displacements are constrained by bilateral constraints

$$f_\mu(x^1, \ldots, x^{3N}, \tau), \quad \mu = 1, \ldots, k,$$

in relation (3B.83), a sign of inequality drops, while, through the constraint

$$f_0 = x^0 - \tau(t) = 0,$$

abstracted by force R_0, k homogeneous linear equations with respect to possible displacements are obtained:

$$\Delta f_\mu = \frac{\partial f_\mu}{\partial x^i}\Delta x^i + \frac{\partial f_\mu}{\partial x^o}\Delta x^0 = 0, \tag{3B.86}$$

$$\Delta f_0 = \Delta x^0 - \Delta \tau = 0. \quad (i = 1, \ldots, 3N).$$

Multiplying with respective indefinite multipliers λ_μ and λ_0 and summing up with

$$\left(X_i - a_{ij}\frac{Dv^j}{dt}\right)\Delta x^i = 0 \tag{3B.87}$$

it is obtained that:

$$\left(X_i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} - a_{ij} \frac{Dv^j}{dt}\right) \Delta x^i + \left(\lambda_0 + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^0}\right) \Delta x^0 = 0.$$

From this equation, $3N$ differential equations of motion follow

$$a_{ij} \frac{Dv^j}{dt} = X_i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i}, \tag{3B.88}$$

as well the force of the constraints' change

$$\lambda_0 = -\sum_{\mu=1}^{N} \lambda_\mu \frac{\partial f_\mu}{\partial x^0} =: X_0, \tag{3B.89}$$

to which k finite equations of the observed constraints $f_\mu = 0$ should be added.

b) Work Principle in Independent Generalized Coordinates

If all constraints (3B.57) are abstracted by respective constraints' reactions

$$R_i = \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i}, \quad (i = 1, \dots, 3N) \tag{3B.90}$$

and if additional constraint $y^0 - \tau = 0$ is abstracted by force R_0, then equation (3B.72) will have the following form:

$$\left(I_i + Y_i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i}\right) \Delta y^i + R_0 \Delta y^0 = 0. \tag{3B.91}$$

Substituting constraints' equations (3B.57) by the parametric form

$$y^i = y^i(q^0, q^1, \dots, q^n), \quad y^0 = q^0, \quad n = 3N - k,$$

and displacement Δy^i by independent possible displacements

$$\Delta y^i = \frac{\partial y^i}{\partial q^\alpha} \Delta q^\alpha \quad (\alpha = 0, 1, \dots, n),$$

equation (3B.91), regarding equations (3A.57), is reduced to a new invariant form:

$$(I_\alpha + Q_\alpha)\, \Delta q^\alpha + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i} \frac{\partial y^i}{\partial q^\alpha} \Delta q^\alpha = 0.$$

However, since it is $f_\mu(q^0, q^1, \dots, q^n) \equiv 0$, thus, it is

$$\Delta f_\mu(y(q)) = \frac{\partial f_\mu}{\partial y^i}\Delta y^i = \frac{\partial f_\mu}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\Delta q^\alpha = \frac{\partial f_\mu}{\partial q^\alpha}\Delta q^\alpha \equiv 0,$$

so that it follows for index $j = 1, \dots, n = 3N - k$, that the work principle, observed with respect to the generalized coordinates, has the following form:

$$(I_j + Q_j)\,\Delta q^j + (I_0 + Q_0^* + R_0)\,\Delta q^0 = 0. \tag{3B.92}$$

This equation is equivalent to the system of equations (3B.56) and (3B.57). Due to the described nature of possible independent generalized displacements Δq^α, beside producing equations (3B.56) and (3B.57), work principle (3B.92) also points to the mutual constraint of forces I_j, Q_j, displacement Δq^α and forces I_0, Q_0^* and R_0.

Second Conclusion. As can be seen from the previous statements, the work principle can be applied to any coordinate system, simultaneously preserving its linear invariant scalar form for all the coordinate systems, constraints systems and systems of forces.

Work Principle Upon Possible Variations

Relation (3B.22) has introduced the concept of work upon possible variations, along with the statement that variations (3B.24) can belong to the set of possible displacements. Consequently, for a particular subset of possible displacements, all the introduced work principle relations, namely, (3B.31), (3B.32), (3B.40), (3B.51) and (3B.52), (3B.56), (3B.64), (3B.72), (3B.81) and (3B.92) will have this very form, except for the fact that, instead of possible displacements $\Delta\cdot$, possible variations $\delta\cdot$ will be written. From this identity of the forms it cannot be concluded that the work principle upon possible variations $\Delta\boldsymbol{r}$ is the same as the principle upon possible variations (3B.24), since $\Delta\boldsymbol{r}$ and $\delta\boldsymbol{r}$ are not identical. æ

3C. PRINCIPLE OF ACTION

In the statements such as “under the action of the force” or “interaction of the bodies” or “action equals reaction”, the term *action* implies the presence of the forces and their inducement rather than some particular concept of action. On the other hand, in analytical mechanics, theoretical physics or even mathematics, the concept of *action* implies a more or less accurately determined functional whose definition makes no reference of force. For this reason, as in the case of work principle, it is necessary here to determine the concept of *action*.

Definition 6. Action. *Action of a force* $\boldsymbol{F}$ *of the mechanical system is an integral value*

$$A(\boldsymbol{F}) = \int_{t_0}^{t} W(\boldsymbol{F})dt. \tag{3C.1}$$

where $W(\boldsymbol{F})$ *is work of force* $\boldsymbol{F}$.

The physical dimension of action is, just like that of the impulse moment,

$$\dim A = \mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-1}\,. \tag{3C.2}$$

It is obvious that the subintegral expression of action is a scalar invariant that can be written, regarding relations ($3B.2$) or ($3B.3$), invariantly with respect to all the observed coordinate systems, as

$$\begin{aligned} A(\boldsymbol{F}) &= \int\limits_{t_0}^{t} W(\boldsymbol{F})dt = \int\limits_{t_0}^{t} W(Y)dt = \\ &= \int\limits_{t_0}^{t} W(X)dt = \int\limits_{t_0}^{t} W(Q)dt, \end{aligned} \tag{3C.3}$$

or, regarding expression ($3B.7$), in the form

$$A(\boldsymbol{I}) = \int_{t_0}^{t} E_k dt. \tag{3C.4}$$

This written form also points to the belief that action is an integral of product of the work of a force and the time interval.

As there are many invariant and equivalent forms of writing down action, the *principle of action* can be and is expressed by various, though equivalent sentences. The mathematical statement is important here:

Action variation (3C.3) *during time* $[t_0, t]$ *is equal, by the value, to variations of* (3C.4) *for the same amount of time; thus, if*

$$\delta A(\boldsymbol{F}) = \delta A(\boldsymbol{I}). \tag{3C.5}$$

According to relations (3B.22), the active forces' work upon possible variations is

$$\delta W(Y) = Y_j \delta y^j = X_j \delta x^j = Q_\alpha \delta q^\alpha = 0, \tag{3C.6}$$

while the action variation

$$\delta A(\boldsymbol{I}) = \delta \int\limits_{t_0}^{t} W(\boldsymbol{I})dt = \int\limits_{t_0}^{t} \delta W(\boldsymbol{I})dt = \int\limits_{t_0}^{t} \delta E_k dt. \tag{3C.7}$$

In order to harmonize $\delta\mathcal{A}$ and $\delta W(Y)$, let us multiply the expression of the principle of work (3B.35) by the differential of time $dt > 0$ and subsume under the sign of integral, i.e.

$$\int_{t_0}^{t} \sum_{\nu=1}^{N} (\boldsymbol{\mathcal{I}}_\nu + \boldsymbol{F}_\nu) \cdot \delta \boldsymbol{r}_\nu dt = \int_{t_0}^{t} \left[\delta W(\boldsymbol{\mathcal{I}}) + \delta W(\boldsymbol{F})\right] dt.$$

It is shown by certain transformations that

$$\int_{t_0}^{t} \delta E_k dt = -\sum_{\nu=1}^{N} \int_{t_0}^{t} m_\nu \boldsymbol{a}_\nu \cdot \delta \boldsymbol{r}_\nu dt = \\ = \int_{t_0}^{t} \sum_{\nu=1}^{N} (-m_\nu \boldsymbol{a}_\nu) \cdot \delta \boldsymbol{r}_\nu dt = \int_{t_0}^{t} \delta W(\boldsymbol{\mathcal{I}}) dt \tag{3C.8}$$

therefore the principle of action can be now operationalized using the relation

$$\int_{t_0}^{t} [\delta E_k + \delta W(\boldsymbol{F})]\, dt = \int_{t_0}^{t} (\delta E_k + Y_i \delta y^i) dt = 0. \tag{3C.8a}$$

A more complete and accurate determination of the relation of principle (3C.8) or (3C.8a) will be explained by its application to particular mechanical systems, from simpler to more complex ones.

Static Problem

a) The point $M(y)$ is attacked by force $\boldsymbol{F}_\nu = Y_\nu^i \boldsymbol{e}_i$ ($i = 1, 2, 3$; $\nu = 1, \ldots, N$). A point M belongs to constraints $f_\mu(y) = 0$, $\mu \leq 3$. What results from the principle of action?

According to expressions (3B.22), the active forces' work upon variations δy_i is

$$\delta W = Y_r^i \delta y_i \tag{3C.9}$$

where δy_i are coordinates of resultant force $Y_r^i = \sum_{\nu=1}^{N} Y_\nu^i$. Since the static problem implies that $\ddot{y} = 0$, inertia force is absent, and thus $W(I) = 0$ as well, so that action principle relation (3C.8) comes to

$$\int_{t_0}^{t} Y_r^i \delta y_i dt = 0. \tag{3C.10}$$

The constraints have to satisfy the relations' variation

$$\delta f_\mu = \frac{\partial f_\mu}{\partial y_i} \delta y_i = 0. \tag{3C.11}$$

Multiplying by indefinite coefficients λ_μ, summing up with respect to μ, integrating upon the interval $[t_0, t)$ and summing up with (3C.11), we have

$$\int_{t_0}^{t} \left(Y_r^i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y_i} \right) \delta y_i dt = 0, \tag{3C.12}$$

which is equivalent to relation (3C.10). By the indefinite multipliers' method, as from (3B.52) to (3B.53), three equations of the dynamic point equilibrium are obtained

$$\sum_{\nu=1}^{N} Y_{\nu}^{i} + \sum_{\mu=1}^{k} \lambda_{\mu} \frac{\partial f_{\mu}}{\partial y_{i}} = 0, \quad i = 1, 2, 3.$$

b) Static system of N dynamic points M_{ν} ($\nu = 1, \ldots, N$), connected by constraints $f_{\mu}(x^1, \ldots, x^{3N}) = 0$; forces $\boldsymbol{F}_{\nu} = X_{\nu}^{i} \boldsymbol{g}_{(\nu)}$ are functions of x coordinates of $M_{\nu}(x)$ points' positions.

The work of the given forces upon variations, according to definition (3B.22) and equations (3B.36), is

$$\delta W = \frac{\partial W}{\partial x^j} \delta x^j = X_j(x) \delta x^j, \tag{3C.13}$$

so that, similarly to relation (3C.8), the principle of action can be written in the form

$$\int_{t_0}^{t} \left(X_j + \sum_{\mu=1}^{k} \lambda_{\mu} \frac{\partial f_{\mu}}{\partial x^j} \right) \delta x^j dt = 0. \tag{3C.14}$$

or in the form

$$\int_{t_0}^{t} \left(\delta W(\boldsymbol{F}) + \sum_{\mu=1}^{k} \lambda_{\mu} \delta f_{\mu} \right) dt = 0. \tag{3C.15}$$

Kinetic Problem

Unlike the previous static problem, the number of forces F is here enlarged with inertia forces (1.37), (1.40) and (3A.50) and their work is being determined, that is, kinetic energy (3B.7):

$$E_k = \frac{1}{2} \sum_{\nu=1}^{N} m_{\nu} \boldsymbol{v}_{\nu} \cdot \boldsymbol{v}_{\nu} = \frac{1}{2} a_{ij}(m, y) \dot{y}^i \dot{y}^j = \frac{1}{2} a_{ij}(m, x) \dot{x}^i \dot{x}^j,$$
$$(i, j = 1, \ldots, 3N).$$

Then, according to (3C.8), relation (3C.15) becomes

$$\int_{t_0}^{t_1} \left(\delta E_k + Y_j \delta y^j + \sum_{\mu=1}^{k} \lambda_{\mu} \delta f_{\mu} \right) dt = 0. \tag{3C.16}$$

Therefore, the principle of action emerges here in the form of equivalent relations (3C.5), (3C.7), (3C.8) and (3C.10). The essential difference comparing to the work principle lies in the fact that the principle of action is used in the study of motion by means of the kinetic energy functions.

Example 11. All forces, except for inertia ones, exerted upon a material point of constant mass m mutually annul themselves, that is, the resultant of these forces equals zero.

It results from relations (3C.16), (3C.12) and from definition (3C.1) that there is only one action

$$A(\boldsymbol{I}) = \int_{t_0}^{t_1} E_k dt \tag{3C.17}$$

so that in this case the principle is written down as

$$\delta \int_{t_0}^{t} E_k dt = 0 \tag{3C.18}$$

The significance of formula (3C.17) is also stressed by the fact that it is called *action function*, while relation (3C.18) is called the *principle of least action* [50] that has been formulated and elaborated by the most distinguished and deserving theorists of analytical mechanics[3] Jacobi even wrote that the principle of least action is the "mother" of the entire analytical mechanics. Relation (3C.18), derived here from a simple example, can also be obtained from much more general observation. If the active forces' work variation is equal to zero, then it is

$$\int_{t_0}^{t_1} \delta E_k dt = 0 \quad \text{and} \quad \int_{t_0}^{t_1} \delta W dt = 0,$$

as well as vice versa. Hence relation (3C.5) can be replaced by a much more precise formulation

$$\delta A = 0 \Leftrightarrow \delta W = 0. \tag{3C.19}$$

For a system of kinetic energy (3B.5) relation (3C.8) is reduced to the form

$$\int_{t_0}^{t_1} (\delta E_k + \delta W(x))\, dt = \int_{t_0}^{t_1} (\delta E_k - \delta E_p)\, dt = \delta \int_{t_0}^{t_1} L dt = 0 \tag{3C.20}$$

where the function

$$L := E_k - E_p, \tag{3C.21}$$

[3]Wolff (1726), Maupertius (1746), Euler (1748), Lagrange (1760) and others.

is known as the *Lagrange's function, Lagrangian* or *kinetic potential.* Relation (3C.20) is known as the Hamilton's principle or *the principle of stationary action*, whose action function

$$A = \int_{t_0}^{t_1} L dt \tag{3C.22}$$

most often called *Hamilton's action*, is most widely used in analytical dynamics, despite the fact that it relates only to the mechanical systems with potential forces. Relation (3C.8a) from which, as can be seen, (3C.18) and (3C.20) follow, is called the *principle Hamilton-Ostrogradsky.* Regarding that all three relations are shown here, namely (3C.8), (3C.15) and (3C.2), in a more general and modified form, the author has chosen the principle of action as the term. While applying Hamilton's principle (3C.20) the physical meaning of function (3C.21) for which the principle has been set is often disregarded, so that for function L known as Lagrangian any function dependent on the used independent coordinate x, its derivatives x and time t is accepted. Such an approach has led do some results inconsistent with the preprinciples of mechanics, and, consequently, with the real motion as well. This happens especially when the principle is applied to the systems of manifolds. In order to make comparison of the assertions made here with the standards of the classical analytical mechanics much easier, we will show the application of action principle (3C.8) or (3C.16), in slightly more details when the configurational manifolds are taken into consideration.

Configurational Manifolds

Let's observe N material points of mass m_ν $(\nu = 1, \ldots, N)$. With respect to a arbitrarily chosen pole O and orthonormal coordinate system $(y, \boldsymbol{e})$, the position of the ν-th point shall be determined by vector $\boldsymbol{r}_\nu = y^i_\nu \boldsymbol{e}_i$. Let's motion of the point be limited by $k \leq 3N$ of bilateral constraints which can be represented, according to the laws of constraints, by vectors $\boldsymbol{R}^\tau_\nu$ (of resistance, friction, etc.) as well as by means of independent equations:

$$f_\mu(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, \tau(t)) = 0 \quad (\mu = 1, \ldots, k) \tag{3C.23}$$

or,

$$f_\mu(y^1_1, y^2_1, y^3_1; \ldots; y^1_N, y^2_N, y^3_N, \tau(t)) = 0, \tag{3C.24}$$

that is,

$$f_\mu(y^1, \ldots, y^{3N}, y^0) = 0, \quad y^0 = \tau(t). \tag{3C.25}$$

Functions f_μ are ideally smooth and regular in the area of constraining the material points.

The condition for the constraints' independence is, in the simplest way, reflected in the velocity conditions at the constraints:

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^i} \dot{y}^i + \frac{\partial f_\mu}{\partial y^0} \dot{y}^0 = 0. \tag{3C.26}$$

These equations will be written in the following form:

$$\begin{aligned}&\frac{\partial f_\mu}{\partial y^1}\dot{y}^1+\cdots+\frac{\partial f_\mu}{\partial y^k}\dot{y}^k=\\&=-\left(\frac{\partial f_\mu}{\partial y^{k+1}}\dot{y}^{k+1}+\cdots+\frac{\partial f_\mu}{\partial y^{3N}}\dot{y}^{3N}+\frac{\partial f_\mu}{\partial y^0}\dot{y}^0\right).\end{aligned} \tag{3C.27}$$

From this system, linear with respect to velocities $\dot{y}$, it is possible to determine k of velocities $\dot{y}^1,\ldots\dot{y}^k$ by means of remaining $3N-k+1$ velocities $\dot{y}^{k+1},\ldots,\dot{y}^{3N}$ under the condition that the determinant is

$$\left\|\frac{\partial f_\mu}{\partial y^m}\right\|_k^k\neq 0 \quad (\mu,m=1,\ldots,k). \tag{3C.28}$$

A multitude of ways, or, briefly, a manifold choice of sets of coordinates q^α, by means of which the position or configuration of the system's points in a moment of time is determined, suggests that a set of independent coordinates $q:=(q^0,q^1,\ldots,q^n)\in M^{n+1}$ should be called configurational manifolds. Accordingly, a set of coordinates q and velocity $\dot{q}=(\dot{q}^0,\dot{q}^1,\ldots,\dot{q}^n)^T$ should be called tangential manifolds TM^{n+1}. The pencil of all the velocities vectors at the point q will consequently be denoted as $\boldsymbol{T}_qM^{n+1}$ which implies $n+1$ base vectors $\dfrac{\partial \boldsymbol{r}}{\partial q^\alpha}$ at each point upon manifolds M^{n+1}. Hence, we will further on consider two sets, namely M^{n+1} and TM^{n+1}, as well as pencil $\boldsymbol{T}_qM^{n+1}$ of the linear vectors. For the sake of brevity, the following notations are introduced:

$$\mathcal{N}:=M^{n+1},\ M:=M^n,$$

$$T\mathcal{N}:=TM^{n+1},\ TM:=TM^n,$$

and, accordingly, $\boldsymbol{T}_qM$, $\boldsymbol{T}_q\mathcal{N}$ as well.

Considering this condition as well as the above-stated properties of functions f_μ, it is possible, according to the implicit functions theory, to determine, from equations (3C.25), k dependent coordinates $y^1,\ldots,y^k$ by means of remaining $3N-k+1$ coordinates $y^1,\ldots,y^{3N},y^0$. The choice of dependent and independent coordinates is arbitrary, along with a special choice of q^0 coordinates, so that each of coordinates $y^1,\ldots,y^{3N}$ can itself be expressed as function $3N-k+1$ of coordinates y. Since, as needed, constraints (3C.25) can be expressed - as in (2.2) - in curvilinear coordinate systems, the possibility of selecting independent coordinates is enlarged. If the independent generalized coordinates are denoted by letters $q^0,q^1,\ldots,q^n$, it follows that constraints (3C.25) can be written down in the parametric form:

$$y^i=y^i(q^0,q^1,\ldots,q^n),\quad q^0=\tau(t), \tag{3C.29}$$

and thus, also as

$$\boldsymbol{r}_\nu=\boldsymbol{r}_\nu(q^0,q^1,\ldots,q^n). \tag{3C.30}$$

Velocity conditions (3C.26) are thus substituted, according to definition (1.1), by relations (3A.37), that is

$$\boldsymbol{v}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \boldsymbol{r}_\nu}{\partial q^1}\dot{q}^1 + \cdots + \frac{\partial \boldsymbol{r}_\nu}{\partial q^n}\dot{q}^n =: \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha. \tag{3C.31}$$

In this topology, action principle is

$$\int_{t_0}^{t_1} [Q_\alpha \delta q^\alpha + \delta W(I)]\, dt = 0 \tag{3C.32}$$

where

$$Q_\alpha = Y_i \frac{\partial y^i}{\partial q^\alpha}; \quad i = 0, 1, \ldots, 3N, \quad \alpha = 0, 1, \ldots, n.$$

are generalized forces. The inertia forces' work, for $m_\nu = \text{const}$, is determined in relation (3B.7) as negative kinetic energy; hence, regarding expression (3C.31) it is

$$\begin{aligned} W = -E_k &= -\sum_{\nu=1}^{N} \frac{m_\nu}{2} \boldsymbol{v}_\nu^2 \cdot \boldsymbol{v}_\nu^2 = \\ &= -\frac{1}{2}\sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\beta}\dot{q}^\alpha \dot{q}^\beta = \\ &= -\frac{1}{2} a_{\alpha\beta}(m_\nu, q)\dot{q}^\alpha \dot{q}^\beta, \quad \dot{q} \in T\mathcal{N}. \end{aligned} \tag{3C.33}$$

Thus, action principle relation (3C.32) has an invariant form (3C.8a) in the generalized coordinates [10]

$$\int_{t_0}^{t_1} (\delta E_k + Q_\alpha \delta q^\alpha)\, dt = 0, \quad q \in \mathcal{N}. \tag{3C.34}$$

Let us also show that the relation (3C.16) comes up to equation (3C.34). Again y^0 is taken as an auxiliary coordinate $y^0 = \tau(\varkappa, t)$. Then, in the case of constraints (3C.24), expression (3C.16) can be written in its extended form as:

$$\int_{t_0}^{t_1} \left(\delta E_k + Y_i \delta y^i + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y^i}\delta y^i + R_0 \delta y^0 \right) dt = 0 \tag{3C.35}$$

Variations of equations (3B.29) are:

$$\delta y^i = \frac{\partial y^i}{\partial q^\alpha}\delta q^\alpha, \quad \delta y^0 = \delta q^0 \quad (\alpha = 0, 1, \ldots, n).$$

Substituting in relation (3B.35) it follows that is

$$\int_{t_0}^{t_1}\left(\delta E_k + Y_i\frac{\partial y^i}{\partial q^\alpha}\delta q^\alpha + \sum_{\mu=1}^{k}\lambda_\mu\frac{\partial f_\mu}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\delta q^\alpha + R_0\delta q^0\right)dt = 0$$

i.e.,

$$\int_{t_0}^{t_1}(\delta E_k + Q_\alpha\delta q^\alpha)\,dt = 0, \tag{3C.36}$$

regarding the fact that

$$\frac{\partial f_\mu}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\delta q^\alpha \equiv 0$$

and

$$Q_0 = R_0 + Y_i\frac{\partial y^i}{\partial q^0} = R_0 + Q_0^* \tag{3C.37}$$

as in relation (3A.56b).

The kinetic energy is a homogeneous quadratic form:

$$E_k = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta, \quad (\alpha,\beta = 0,1,\ldots,n). \tag{3C.38}$$

Following classical variation calculation, after varying

$$\delta E_k = \frac{\partial E_k}{\partial q^\alpha}\delta q^\alpha + \frac{\partial E_k}{\partial \dot{q}^\alpha}\delta\dot{q}^\alpha$$

and integrating (3C.36), it is obtained that:

$$\frac{\partial E_k}{\partial \dot{q}^\alpha}\delta q^\alpha\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\left(\frac{\partial E_k}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^\alpha} + Q_\alpha\right)\delta q^\alpha dt = 0 \tag{3C.39}$$

This is always satisfied when the differential equations of motion are dealt with

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^\alpha} - \frac{\partial E_k}{\partial q^\alpha} = Q_\alpha, \quad \alpha = 0,1,\ldots,n; \tag{3C.40}$$

and boundary conditions

$$\delta q^\alpha(t_1) = 0, \quad \delta q^\alpha(t_0) = 0.$$

Differential equations (3C.40) which amount to $n+1$, that is,

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^i} - \frac{\partial E_k}{\partial q^i} = Q_i, \quad i = 1,\ldots,n; \tag{3C.40a}$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^0} - \frac{\partial E_k}{\partial q^0} = Q_0 = Q_0^* + R_0, \tag{3C.40b}$$

are reduced to the differential equations in the extended form since it is clear that the kinetic energy is easy to set up $a_{\alpha\beta}$ known inertia tensor.

What is obviously adequate, for the invariable constraints' systems in which all the generalized forces are equal to zero, is the principle of least action (3C.18) which can be, on the basis of (3B.5), brought into agreement with the preprinciple of existence.

For the systems with invariable constraints and with such potential energy E_p that the active forces are

$$Q_i = -\frac{\partial E_p}{\partial q^i}, \tag{3C.41}$$

equation (3C.40b) is non-existent, while equations (3C.40a) are reduced to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad (i = 1, \ldots, n). \tag{3C.42}$$

They are equivalent to principle (3C.20).

If the *natural Lagrange function*

$$L = E_k - E_p(q^0, q^1, \ldots, q^n)$$

is added the function

$$\mathcal{P}(q^0) = -\int R_0(q^0)dq^0, \tag{3C.43}$$

which appears at constraints (3C.23) or (3C.29), that is, [44]

$$\mathcal{L} = E_k - (E_p + \mathcal{P}), \tag{3C.44}$$

equations (3C.4) are reduced to

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathcal{L}}{\partial q^\alpha} = 0 \tag{3C.45}$$

that are equivalent to the principle

$$\delta\int_{t_0}^{t_1} \mathcal{L}dt = 0. \tag{3C.46}$$

This relation produces, in addition to equations (3B.42), one more equation, namely:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^0} - \frac{\partial L}{\partial q^0} = R_0. \tag{3C.47}$$

Conclusion 1. The principle of action provides for the consideration of the mechanical systems' motion by means of the energy functions if the non-potential forces are absent.

Principle of Action upon $T^*\mathcal{N}$

The notation $T^*\mathcal{N}$ here implies $2n+2$ dimensional manifolds which form $n+1$ generalized coordinates $q = (q^0, q^1, \dots, q^n)$ and $n+1$ generalized impulses $p = (p_0, p_1, \dots, p_n)$, meaning (1.25), that is (3A.39). Regarding the fact that $p\&q \in T^*\mathcal{N}$ denotes the tangent manifolds, then the symbol $T^*\mathcal{N}$ is called the cotangent manifolds. In the literature other terms can be sometimes found such as "phase space", "state space", "Hamilton's variables", or "cotangential spaces". If the starting point is the fact that the motion state is characterized by the position coordinates of point q as well as the coordinates of impulse p, then it could be said that $T^*\mathcal{N}$ is the state of the system's motion or state manifolds. Since $\mathcal{N} := M^{n+1}$, $T^*\mathcal{N}$ can also be called the extended manifolds if it is necessary to stress its difference from configurational manifolds M^n and its respective cotangent manifolds T^*M [45], [49], [51], [59], [63].

What is even more important than the term itself is the understanding and acceptance that $p_0, p_1, \dots, p_n$ are the impulses whose essence is determined by definition 2, that is, derived by formulae (1.25). At the same time, as can be seen from relations (3A.39) and (3A.42), there is a mutually linear combination between generalized impulses p_α and generalized velocities $\dot{q}^\alpha$:

$$p_\alpha = a_{\alpha\beta}\dot{q}^\beta \quad \Leftrightarrow \quad \dot{q}^\alpha = a^{\alpha\beta}p_\beta. \tag{3C.48}$$

The next step in considering the action principle upon $T^*\mathcal{N}$ implies the substitution of velocities $\dot{q}^\alpha$ in the above-discussed relations by means of generalized impulses p_β.

Action (3C.4) has, in its turn, just been defined by means of impulses,

$$A(\boldsymbol{I}) = -\frac{1}{2}\int_{t_0}^{t} p_\alpha dq^\alpha = -\frac{1}{2}\int_{t_0}^{t} p_\alpha \dot{q}^\alpha dt.$$

because kinetic energy has the following forms:

$$E_k = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = \frac{1}{2}p_\beta\dot{q}^\beta = \frac{1}{2}a^{\beta\gamma}p_\beta p_\gamma. \tag{3C.49}$$

Hamilton's action is expressed by the relation:

$$\begin{aligned} A &= \int_{t_0}^{t} L dt = \int_{t_0}^{t} (E_k - E_p) dt = \\ &= \int_{t_0}^{t} (2E_k - (E_k + E_p)) dt = \\ &= \int_{t_0}^{t} (p_\alpha \dot{q}^\alpha - H) dt, \end{aligned} \tag{3C.50}$$

where

$$H := E_k + E_p = \frac{1}{2} a^{\beta\gamma} p_\beta p_\gamma + E_p(q). \tag{3C.51}$$

If generalized forces Q_α are separated into potential and non-potential Q^*_α, so that it is

$$Q_\alpha = -\frac{\partial E_p}{\partial q^\alpha} + Q^*_\alpha \tag{3C.52}$$

and if they are substituted in relation (3C.36), it is obtained that

$$\int_{t_0}^{t} \left[\delta\left(p_\alpha \dot{q}^\alpha - H\right) + Q^*_\alpha \delta q^\alpha\right] dt = 0 \tag{3C.53}$$

Further, it is

$$\int_{t_0}^{t_1} \delta\left(p_\alpha \dot{q}^\alpha - H\right) dt = \tag{3C.54}$$

$$= \int_{t_0}^{t_1} \left[\delta p_\alpha \dot{q}^\alpha + p_\alpha \delta \dot{q}^\alpha - \left(\frac{\partial H}{\partial p_\alpha} \delta p_\alpha + \frac{\partial H}{\partial q^\alpha} \delta q^\alpha\right)\right] dt =$$

$$= p_\alpha \delta q^a\big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\left(\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha}\right) \delta p_\alpha - \left(\dot{p}_\alpha + \frac{\partial H}{\partial q^\alpha}\right) \delta q^\alpha\right] dt$$

Substituting in relation (3C.53), it follows

$$p_\alpha \delta q^a\big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\left(\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha}\right) \delta p_\alpha + \left(Q^*_\alpha - \dot{p}_\alpha - \frac{\partial H}{\partial q^\alpha}\right) \delta q^\alpha\right] dt = 0 \tag{3C.55}$$

It can be seen from formula (3C.51) that it is

$$\frac{\partial H}{\partial p_\alpha} = a^{\alpha\beta} p_\beta,$$

so that, due to linear combinations (3C.48)

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}. \tag{3C.56}$$

Consequently, relation (3C.55) is reduced to

$$p_\alpha \delta q^\alpha\big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(Q^*_\alpha - \dot{p}_\alpha - \frac{\partial H}{\partial q^\alpha}\right) \delta q^\alpha dt = 0, \tag{3C.57}$$

which is equivalent to relation (3C.39). Under conditions (3C.41), relation (3C.56) is satisfied if it is

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} + Q^*_\alpha, \quad (\alpha = 0, 1, \ldots, n) \tag{3C.58}$$

these being differential equations of the system's motion; they, along with transformations (3C.56), form the system of $2n + 2$ differential equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + Q^*_i, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{3C.59}$$

$$\dot{p}_0 = -\frac{\partial H}{\partial q^0} + Q^*_0, \quad \dot{q}^0 = \frac{\partial H}{\partial p_0}, \tag{3C.60}$$

where, as in (3C.40b), $Q^*_0 = Q^{**}_0 + R_0$.

In the case that $P_i = 0$ and $P^*_0 = 0$ occurs, function (3C.51) can be extended to the total mechanical energy

$$E = H + \mathcal{P}, \tag{3C.61}$$

so that the system of equations (3C.56) and (3C.58), as well as (3C.59) and (3C.60), can be written in the canonical form:

$$\left.\begin{aligned} \dot{p}_\alpha &= -\frac{\partial E}{\partial q^\alpha}, \\ \dot{q}^\alpha &= \frac{\partial E}{\partial p_\alpha}. \end{aligned}\right\} \quad \alpha = 0, 1, \ldots, n. \tag{3C.62}$$

In the case of the system's invariable constraints, when there is no rheonomic coordinate q^0, equations (3C.60) vanish, while in equations (3C.62) indices range from 1 to n.

Conclusion 2. The principle of action provides for direct consideration of the mechanical system's motion upon $T^*\mathcal{N}$, as upon T^*M, by the relations of the same type.

It is the action principle upon which the analytical mechanics, known also as Lagrange and Hamilton's mechanics, has been developed.

Example 12. Motion of a material point of mass m upon a vertical constraining smooth circular line of radius r, revolving at angular velocity ω around the central vertical axis [3].

Fig. 4

Let $(y_1, y_2, y_3)^T \in E^3$ be Cartesian coordinates starting from the center of circumference 0, while axis Oy_3 is directed vertically upwards. The spherical system of coordinates ρ, φ, θ is also introduced and placed in such a way that the circumference plane and plane $y_2 = 0$ form an angle φ. It follows from the problem that the material point is constrained by two constraints, namely,

$$f_1 = \rho - r = 0, \quad f_2 = \varphi - \omega t = 0.$$

Let's denote $q := \theta$, $q_0 = \omega t$, while the respective impulses are p and p_0. Manifold M is a circumference, while $\mathcal{N}$ is a sphere.

Kinetic energy has the forms

$$\begin{aligned} E_k &= \frac{m}{2}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) = \\ &= \frac{mr^2}{2}\left(\dot{q}^2 + \dot{q}_0^2 \sin^2 q\right) = \frac{1}{2mr^2}\left(p^2 + \frac{p_0^2}{\sin^2 q}\right), \end{aligned}$$

since it is $p = mr^2\dot{q}$ and $p_0 = mr^2\dot{q}_0 \sin^2 q$.

Potential energy is

$$E_p = -mgr(1 - \cos q).$$

Function H, accordingly, has the form

$$H = \frac{1}{2mr^2}\left(p^2 + \frac{p_0^2}{\sin^2 q}\right) - mgr(1 - \cos q).$$

Differential equations of motion (3C.59) and (3C.60) are:

$$\begin{aligned} \dot{p} &= \frac{p_0^2}{2mr^2}\frac{\cos q}{\sin^3 q} - mgr \sin q, & \dot{q} &= \frac{p}{mr^2}, \\ \dot{p}_0 &= R_0, & \dot{q}_0 &= \frac{p_0}{mr^2 \sin^2 q}. \end{aligned}$$

æ

3D. PRINCIPLE OF COMPULSION

In the related literature the principle of compulsion is also known as Gauss' principle though the author himself did not considered it as a principle. The analytical form of the Gauss' principle has been considered by many well-known scientists of classical mechanics. Without going into historical analysis, the concept of compulsion will be the first to determine here [17], [43], [48].

Definition 7. Compulsion *Compulsion is a semi-sum of the products of mass m_ν and squared acceleration difference*

$$Z \stackrel{\text{def}}{=} \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\frac{d\boldsymbol{v}_\nu}{dt} - \frac{\boldsymbol{F}_\nu}{m_\nu}\right)^2 . \tag{3D.1}$$

The previous formula for compulsion can be written in the form

$$Z = \sum m_\nu a_\nu^2 \tag{3D.2}$$

where

$$\boldsymbol{a}_\nu := \frac{\boldsymbol{I}_\nu + \boldsymbol{F}_\nu}{m_\nu} = \frac{\mathsf{F}_\nu}{m_\nu} \tag{3D.3}$$

accelerations of the ν-th material points caused by resultant force F_ν.

Formulae (1.37), (2.12), (2.15) and (2.17) show that F_ν are functions of position vector $\boldsymbol{r}$, velocity $\dot{\boldsymbol{r}}$ and acceleration $\ddot{\boldsymbol{r}}$. Since in finite equations of motion the position vector always appears as a function of parameter $\varkappa$ and time, it also follows that function Z indirectly depends on these parameters and time, that is,

$$Z = Z(a(\varkappa, t)) \tag{3D.4}$$

The physical dimension of compulsion is

$$\dim Z = \mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-4} . \tag{3D.5}$$

Function Z satisfies the preprinciple of existence regarding the fact that mass, distance and time are existent, as well as the laws of dynamics and definition (1.37) which determine the existence of the forces. The preprinciple of casual definiteness is satisfied with as much accuracy as parameters $\varkappa$ in function (3D.3) are accurately measured. Regarding the fact that Z, as can be seen from (3D.1), is a homogeneous quadratic form of acceleration, the invariance of compulsion under any regular coordinate transformation cannot be doubted. Consequently, there is no impediment from the aspect of the invariance preprinciple, either. The difficulties that arise in that sense should be looked for in mathematical skill. If at least metrics and coordinate systems, considered together with definitions 1 and 3 are taken into consideration, function Z can be written in the following forms:

$$\begin{aligned}2Z &= \sum_{\nu} m_\nu \boldsymbol{a}_\nu \cdot \boldsymbol{a}_\nu = \sum_{\nu} m_\nu a_\nu^k \boldsymbol{e}_k \cdot a_\nu^l \boldsymbol{e}_l = \\ &= \sum_{\nu}^{N} m_\nu \delta_{kl} a_\nu^k a_\nu^l = a_{ij} a^i a^j .\end{aligned} \tag{3D.6}$$

With respect to the natural trihedron, forces F_ν can be resolved as, for example (1.40a)

$$\mathsf{F}_\nu = \mathcal{F}_\nu^\tau \boldsymbol{\tau} + \mathcal{F}_\nu^n \boldsymbol{n} + \mathcal{F}_\nu^b \boldsymbol{b} \tag{3D.7}$$

where $\boldsymbol{\tau}$, $\boldsymbol{n}$, and $\boldsymbol{b}$, are orthogonal unit vectors controlling the tangent, normal and binormal. Thus compulsion again emerges as a sum of the squares

$$2Z = \mathcal{F}_\tau^2 + \mathcal{F}_n^2 + \mathcal{F}_b^2 \tag{3D.8}$$

where

$$\mathcal{F}_\tau^2 = \sum_\nu m_\nu \left(\mathcal{F}_\nu^\tau\right)^2, \quad \mathcal{F}_n^2 = \sum_\nu m_\nu \left(\mathcal{F}_\nu^n\right)^2, \quad \mathcal{F}_b^2 = \sum_\nu m_\nu \left(\mathcal{F}_\nu^b\right)^2 .$$

Analogously, each ν-th vector F_ν can be resolved by means of tangent pencil $\boldsymbol{T}_\nu M$,

$$\frac{\partial \boldsymbol{r}}{\partial q^\alpha} \in \boldsymbol{T}_\nu M$$

and respective vectors $\boldsymbol{n}_{(\nu)\alpha}$, perpendicular to $\boldsymbol{T}_\nu M$:

$$\mathsf{F}_\nu = \mathcal{F}^\alpha \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} + \mathcal{N}^\alpha \boldsymbol{n}_{\nu\alpha}. \tag{3D.9}$$

By substituting expression (3D.6), after scalar multiplication at which it is

$$\frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \cdot \boldsymbol{n}_{\nu\alpha} = 0,$$

and by taking into consideration (3D.3), it is obtained that

$$2Z = a_{\alpha\beta} a^\alpha a^\beta + b_{\alpha\beta} \mathcal{N}^\alpha \mathcal{N}^\beta \tag{3D.10}$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are coefficients of the given quadratic forms. All the above-given expressions for compulsion are represented by homogeneous quadratic forms of the coordinates of vector a which has a dimension of acceleration. In order to be applied in mechanics, beside stressing the statement that Z is a homogeneous quadratic form of $\boldsymbol{a}$, and, that through acceleration it appears as $\boldsymbol{a}$ function of parameter $\varkappa$

and time, it is necessary to reduce defined compulsion (3D.1) to clearer coordinate forms. Therefore, three descriptions should be distinguished by means of:

1. orthonormal rectilinear coordinate system $(y, \boldsymbol{e})$,
2. curvilinear coordinate system $(x, \boldsymbol{g})$, and,
3. configurational manifolds M.

In all the three cases $\boldsymbol{F}_\nu$ in formula (3D.1) is regarded to be a resultant vector of all the forces acting upon the ν-th point, except for inertia force $\boldsymbol{I}_\nu = -m_\nu \dfrac{d\boldsymbol{v}_\nu}{dt}$, which is set apart by definition 4.

1. Compulsion in Coordinate System $(y, \boldsymbol{e})$

With respect to coordinate system (y, e), acceleration vectors are $\boldsymbol{a}_\nu = \ddot{y}_\nu^k \boldsymbol{e}_k$, while forces are $\boldsymbol{F}_\nu = Y_\nu^l \boldsymbol{e}_l$. $(k, l = 1, 2, 3;\ \nu = 1, \ldots, N)$.

Substituting in formula (3D.1) it follows

$$\begin{aligned} 2Z &= \sum_{\nu=1}^{N} m_\nu \left(\ddot{y}_\nu^k \boldsymbol{e}_k - \frac{Y_\nu^k \boldsymbol{e}_k}{m_\nu} \right) \cdot \left(\ddot{y}_\nu^l \boldsymbol{e}_l - \frac{Y_\nu^l \boldsymbol{e}_l}{m_\nu} \right) = \\ &= \sum_{\nu=1} m_\nu \delta_{kl} \left(\ddot{y}_\nu^k \ddot{y}_\nu^l - 2\frac{\ddot{y}_\nu^k Y_\nu^l}{m_\nu} + \frac{Y_\nu^k Y_\nu^l}{m_\nu \cdot m_\nu} \right). \end{aligned} \tag{3D.11}$$

If the notations are introduced: $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$; $i, j = 3\nu - 2, 3\nu - 1, 3\nu = 1, 2, \ldots, 3N$ as well as $\bar{Y}_\nu^i := \dfrac{Y_\nu^i}{m_\nu}$, it is obtained

$$\begin{aligned} Z &= \frac{1}{2}\bar{\delta}_{ij} \left(\ddot{y}^i - \bar{Y}^i\right) \left(\ddot{y}^j - \bar{Y}^j\right) \\ &= \frac{1}{2}\bar{\delta}_{ij}\ddot{y}^i\ddot{y}^j - \bar{\delta}_{ij}\ddot{y}^i\bar{Y}^j + \frac{1}{2}\bar{\delta}_{ij}\bar{Y}^i\bar{Y}^j, \end{aligned} \tag{3D.12}$$

where $\bar{\delta}_{ij} = m_i \delta_{ij}$.

2. Compulsion in Curvilinear Coordinate Systems

With respect to curvilinear coordinate systems that are in uniform correspondence with $(y, \boldsymbol{e})$, that is,

$$y^k = y^k(x^1, x^2, x^3), \quad \boldsymbol{e}_k = \frac{\partial y^l}{\partial x^k}\boldsymbol{g}_l, \quad \left|\frac{\partial y^l}{\partial x^k}\right| \neq 0,$$

and their simple substitution in relations (3D.11) or (3D.12), what would be obtained is:

$$Z = \frac{1}{2}a_{ij} \left(\frac{D\dot{x}^i}{dt}\frac{D\dot{x}^j}{dt} - 2\bar{X}^j \frac{D\dot{x}^i}{dt} + \bar{X}^i \bar{X}^j \right), \tag{3D.13}$$

since

$$\ddot{y}^i = \frac{\partial y^i}{\partial x^j}\frac{D\dot{x}^j}{dt}, \quad a_{ij} = \sum_{\nu=1}^{N} m_\nu \delta_{kl}\frac{\partial y_\nu^k}{\partial x^i}\frac{\partial y_\nu^l}{\partial x^j}.$$

In order to understand better the subsequent particularities, relation (3D.13) will be directly derived from definition (3D.1). According to expression (1.30) it can be written:

$$\frac{d\boldsymbol{v}_\nu}{dt} = \frac{D\dot{x}_\nu^k}{dt}\boldsymbol{g}_{\nu k}, \quad \frac{\boldsymbol{F}_\nu}{m_\nu} = \bar{X}^k \boldsymbol{g}_{\nu k}.$$

Substituting in formula (3D.1), it follows

$$2Z = \sum_{\nu=1}^{N} m_\nu \left(\frac{D\dot{x}_\nu^k}{dt} - \bar{X}_\nu^k\right)\boldsymbol{g}_{\nu k}\cdot\left(\frac{D\dot{x}_\nu^l}{dt} - \bar{X}_\nu^l\right)\boldsymbol{g}_{\nu l} =$$
$$= \sum_{\nu=1}^{N} m_\nu \boldsymbol{g}_{\nu k}\cdot\boldsymbol{g}_{\nu l}\left(\frac{D\dot{x}_\nu^k}{dt} - \bar{X}_\nu^k\right)\left(\frac{D\dot{x}_\nu^l}{dt} - \bar{X}_\nu^l\right),$$

or if the indices are used, as from (3D.11) to (3D.12), relation (3D.13) is obtained where

$$a_{ij} := \sum_{\nu=1}^{N} m_\nu \boldsymbol{g}_{\nu i}\cdot\boldsymbol{g}_{\nu j} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial x^i}\cdot\frac{\partial \boldsymbol{r}_\nu}{\partial x^j}.$$

Therefore, if the motion of the system of N material points is observed, in which the constraints are abstracted by the forces so that every point is viewed as "free" - at which every vector can be resolved into three components, compulsion Z is described by forms from (3D.12) or (3D.13), each with $3N$ quadratic addends of acceleration $(\ddot{y} - \bar{Y})$ or $\left(\frac{D\dot{x}}{dt} - \bar{X}\right)$.

3. Compulsion in Generalized Systems of Coordinates

With respect to independent generalized coordinates $q \in M^n$, $n \leq 3N$, the acceleration

$$\boldsymbol{a}_\nu = \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha\right) = \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\beta \partial q^\alpha}\dot{q}^\alpha\dot{q}^\beta + \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\ddot{q}^\alpha \tag{3D.14}$$

has a complex coordinate structure. Tangential pencil $\boldsymbol{T}_\nu M$ of vector on the basis $\boldsymbol{g}_{(\nu)\alpha} := \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}$, is not sufficient enough to be used as a means of resolving the acceleration vector. The vector

$$\frac{\partial \boldsymbol{g}_{(\nu)\alpha}}{\partial q^\beta} = \Gamma_{\alpha\beta}^\gamma \boldsymbol{g}_{(\nu)\gamma} + b_{(\nu)}\alpha\beta\boldsymbol{n}_{(\nu)} \tag{3D.15}$$

in the general case, it does not belong, as a whole, to pencil $\boldsymbol{T}_\nu M$; instead, at every ν-th point, it also possesses a component perpendicular to $\boldsymbol{T}_\nu M$; $\boldsymbol{g}_{(\nu)} \perp \boldsymbol{n}_{(\nu)}$.

If vector $\boldsymbol{n}_\nu$ is also resolved as $\boldsymbol{n}_\nu = \varkappa^\gamma_{(\nu)}\boldsymbol{\eta}_{(\nu)\gamma}$. Substituting expression (3D.15) in (3D.14), it is obtained that

$$\begin{aligned}\boldsymbol{a}_\nu &= \left(\ddot{q}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta\right)\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} + b_{(\nu)\alpha\beta}\dot{q}^\alpha\dot{q}^\beta\boldsymbol{n}_n u \\ &= \frac{D\dot{q}^\gamma}{dt}\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} + b_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta\varkappa^\gamma_{(\nu)}\boldsymbol{n}_{(\nu)\gamma}.\end{aligned} \tag{3D.16}$$

By means of base vectors $\boldsymbol{g}_{(\nu)\gamma}$ and $\boldsymbol{\eta}_{(\nu)}$, vector $\dfrac{F_\nu}{m_n u}$, should also be resolved:

$$\frac{\boldsymbol{F}_\nu}{m_\nu} = Q^\gamma\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} + F_N\boldsymbol{n}_\nu = Q^\gamma\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} + F_N\varkappa^\gamma_{(\nu)}\boldsymbol{n}_{(\nu)\gamma}. \tag{3D.17}$$

This provides for writing compulsion in the coordinate form. Namely, by substituting relations (3D.17) and (3D.16) in (3D.1), it follows:

$$\begin{aligned}Z &= \frac{1}{2}\sum_{\nu=1}^{N} m_\nu\left[\left(\frac{D\dot{q}^\gamma}{dt} - Q^\gamma\right)\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} + \left(b_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - F_N\right)\varkappa^\gamma_{(\nu)}\boldsymbol{n}_{(\nu)\gamma}\right]^2 = \\ &= \frac{1}{2}\sum_{\nu=1}^{N}\left[m_\nu\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma}\frac{\partial \boldsymbol{r}_\nu}{\partial q^\delta}\left(\frac{D\dot{q}^\gamma}{dt} - Q^\gamma\right)\left(\frac{D\dot{q}^\delta}{dt} - Q^\delta\right) + \right. \\ &\quad \left. + \sum_{\nu=1}^{N} m_\nu\varkappa^\gamma_{(\nu)}\varkappa^\delta_{(\nu)}\boldsymbol{n}_{(\nu)\gamma}\cdot\boldsymbol{n}_{(\nu)\delta}\left(b_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - F_N\right)^2 = \right. \\ &= \frac{1}{2}a_{\gamma\delta}\left(\frac{D\dot{q}^\gamma}{dt} - Q^\gamma\right)\left(\frac{D\dot{q}^\delta}{dt} - Q^\delta\right) + \frac{1}{2}a_N^2,\end{aligned} \tag{3D.18}$$

where

$$a_{\gamma\delta} = \sum m_\nu\frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma}\cdot\frac{\partial \boldsymbol{r}_\nu}{\partial q^\delta}, \tag{3D.19}$$

inertia tensor

$$a_N^2 = \varkappa^2\left(b_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - F_N\right)^2, \tag{3D.20}$$

while

$$\begin{aligned}\varkappa^2 &= \sum_{\nu=1}^{N} m_{(\nu)}\varkappa^\gamma_{(\nu)}\varkappa^\delta_{(\nu)}\boldsymbol{n}_{(\nu)\gamma}\cdot\boldsymbol{n}_{(\nu)\delta} = \\ &= \sum_{\nu=1}^{N} m_{(\nu)}\varkappa^\gamma_{(\nu)}\varkappa^\delta_{(\nu)}\delta_{(\nu)\gamma\delta} = \sum_{\nu=1}^{N} m_{(\nu)}\varkappa^2_{(\nu)}.\end{aligned} \tag{3D.21}$$

By comparison to relations (3D.13) and (3D.12), it can be noticed that the formal side of compulsion is:

$$\begin{aligned}Z_M &= \frac{1}{2}a_{\alpha\beta}\left(\frac{D\dot{q}^\alpha}{dt} - Q^\alpha\right)\left(\frac{D\dot{q}^\beta}{dt} - Q^\beta\right) = \\ &= \frac{1}{2}a_{\alpha\beta}\frac{D\dot{q}^\alpha}{dt}\frac{D\dot{q}^\beta}{dt} - Q_\beta\frac{D\dot{q}^\beta}{dt} + \frac{1}{2}a_{\alpha\beta}Q^\alpha Q^\beta.\end{aligned} \tag{3D.22}$$

This is exactly *compulsion upon manifolds* M and as such, it is sufficient to consider motion in the same way as described by means of energy (3C.38) upon M^n and M^{n+1}.

Compulsion (3D.18) has both a quality and a quantity more

$$Z_N = \frac{1}{2}a_N^2 = \frac{\varkappa^2}{2}\left(b_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - F_N\right)^2, \tag{3D.23}$$

then compulsion (3D.22) that can be interpreted as compulsion of motion upon TM and TM^{n+1}.

Compulsion $Z = Z_M + Z_N$ is equivalent to compulsions (3D.13) and (3D.12). But, if it is desirable to describe it only at configurational manifoldness, it is sufficient to consider respective function (3D.22). Due to the difficulties with determining factors $\varkappa_\nu$ in expression (3D.21), the determination of compulsion Z_N should be done in a somewhat more accessible way for particular forms of constraints.

Formulation of Compulsion Principle

Compulsion upon real motion is the least.

In other words, function (3D.4) has the least value at differentially small changes of parameter $\varkappa$. The mathematical relation of this principle is very simple, namely:

$$\delta Z = 0, \tag{3D.24}$$

or, concerning function (3D.4), as well as (3D.2), and similarly to

$$\delta Z = \sum_\nu \frac{\partial Z}{\partial \boldsymbol{a}_\nu}\delta\boldsymbol{a}_\nu = 0. \tag{3D.25}$$

Consequently, the compulsion principle can be also formulated by the following sentence:

The first variation of compulsion with respect to acceleration is equal to zero.

Relation (3D.25) is satisfied for

$$\frac{\partial Z}{\partial \boldsymbol{a}_\nu} = 0 \tag{3D.26}$$

The same equations follow from the expression "compulsion is the least". Namely, it can be seen from function (3D.2) that Z is a positively definite quadratic coordinate form which is the least and equal to zero only if $\boldsymbol{a}_\nu = 0$ for every ν. The same is obtained from equations (3D.26), while the opposite is obtained for all $\boldsymbol{a}_\nu = 0 \longrightarrow Z = 0$.

On the basis of this principle of mechanics as well as the three previous ones, it is possible to develop the whole theory about motion of a system of material points. This can be shown by the relations recognizable from the previous discussion of the other principles.

Relation of Compulsion Principle with Respect to Coordinate Systems

Coordinate System $(y, \boldsymbol{e})$**.** With respect to orthonormal coordinate system $(y, \boldsymbol{e})$, motion of N material points of mass m under the action of active forces $\boldsymbol{F}_\nu$ and k geometrically ideally smooth bilateral constraints is observed:

$$f_\mu(y_\nu^1,\, y_\nu^2,\, y_\nu^3) = 0, \quad (\nu = 1, \ldots, N). \tag{3D.27}$$

Compulsion has the form (3D.11).

The accelerations existing in the expression for compulsion are conditioned by the equations:

$$\ddot{f}_\mu = \sum_{\nu=1}^{N} \left(\frac{\partial^2 f_\mu}{\partial y_\nu^k \partial y_\nu^l} \dot{y}_\nu^k \dot{y}_\nu^l + \frac{\partial f_\mu}{\partial y_\nu^k} \ddot{y}_\nu^k \right) = 0. \tag{3D.27a}$$

In accordance with the principle, the first variation with respect to acceleration $\ddot{y}_\nu^k$ of compulsion (3DG.11) is:

$$\delta Z = \sum_{\nu}^{N} \frac{\partial Z}{\partial \ddot{y}_\nu^k} \delta \ddot{y}_\nu^k = \sum_{\nu=1}^{N} m_\nu \delta_{kl} \left(\ddot{y}_\nu^l - \frac{Y_\nu^l}{m_\nu} \right) \delta \ddot{y}_\nu^k = 0, \tag{3D.28}$$

while respective constraint variations (3D.27) are:

$$\delta \ddot{f}_\mu = \sum_{\nu=1}^{N} \frac{\partial f_\mu}{\partial y_\nu^k} \delta \ddot{y}_\nu^k = 0. \tag{3D.29}$$

By introducing k indefinite multipliers of constraints λ_μ, it follows:

$$\delta Z = \sum_{\nu=1}^{N} \delta_{kl} \left(m_\nu \ddot{y}_\nu^l - Y_\nu^l - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y_{\nu l}} \right) \delta \ddot{y}_\nu^k = 0. \tag{3D.30}$$

It is equivalent to equations (3D.25) or (3D.26). Relation (3D.30) can be written in a shorter form:

$$\sum_{\nu=1}^{N} \left(\frac{\partial Z}{\partial \ddot{y}_\nu^k} - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y_\nu^k} \right) \delta \ddot{y}_\nu^k = 0. \tag{3D.31}$$

Consequently, the differential equations of the observed system are:

$$\frac{\partial Z}{\partial \ddot{y}_\nu^k} - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial y_\nu^k} = 0. \tag{3D.32}$$

Together with k given constraints $f_\mu = 0$ are necessary and sufficient for problem-solving. The same result would be achieved if the given constraints were abstracted by constraints' reactions (2.9), (3.22). In that case, compulsion would be

$$Z^* = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\frac{d\boldsymbol{v}_\nu}{dt} - \frac{\boldsymbol{F}_\nu + \boldsymbol{R}_\nu}{m_\nu}\right)^2, \tag{3D.33}$$

while relation (3D.3)

$$\boldsymbol{a}_\nu = \frac{\boldsymbol{I}_\nu + \boldsymbol{F}_\nu + \boldsymbol{R}_\nu}{m_\nu}.$$

Equations (3D.26) would be reduced to (3D.32) while care should be taken about the difference between Z and Z^*.

As for mechanical systems that, in addition to holonomic constraints $f_\mu = 0$, also comprise *non-integrable (non-holonomic) differential constraints*

$$\varphi_\sigma(y_\nu, \dot{y}_\nu) = 0, \quad \sigma = 1, 2, \ldots, l \leq 3N - k. \tag{3D.34}$$

The problem may be observed as in relation (3D.33) while additional knowledge about reactions of constraints

$$\boldsymbol{R}_\nu(\varphi) = \sum_{\mu=1}^{l} \boldsymbol{R}_{\nu\mu}(\varphi)$$

is needed.

Let, as in the previous consideration of compulsion functions (3D.11), the variation of binomic constraints' acceleration conditions be (3D.29).

The conditions for acceleration of constraints $\varphi_\sigma = 0$ are

$$\dot{\varphi}_\sigma = \sum_{\nu=1}^{N}\left(\frac{\partial \varphi_\sigma}{\partial y_\nu^k}\dot{y}_\nu^k + \frac{\partial \varphi_\sigma}{\partial \dot{y}_\nu^k}\ddot{y}_\nu^k = 0\right), \tag{3D.35}$$

while the respective variations are

$$\delta\dot{\varphi}_\sigma = \sum_{\nu=1}^{N} \frac{\partial \dot{\varphi}_\sigma}{\partial \ddot{y}_\nu^k}\delta\ddot{y}_\nu^k = \sum_{\nu=1}^{N} \frac{\partial \varphi_\sigma}{\partial \dot{y}_\nu^k}\delta\ddot{y}_\nu^k = 0. \tag{3D.36}$$

Following the previous method of indefinite multipliers $\bar{\lambda}_\sigma$, an extended relation (3D.31) will be obtained, namely,

$$\sum_{\nu=1}^{N}\left(\frac{\partial Z}{\partial \ddot{y}_\nu^k} - \sum_{\mu=1}^{k}\lambda_\mu \frac{\partial f_\mu}{\partial y_\nu^k} - \sum_{\sigma=1}^{l}\bar{\lambda}_\sigma \frac{\partial \varphi_\sigma}{\partial \dot{y}_\nu^k}\right)\delta\ddot{y}_\nu^k = 0. \tag{3D.37}$$

Curvilinear Coordinate System $(x, \boldsymbol{g})$. First of all, it should be stressed once again that our initial or base coordinate system is system $(y, \boldsymbol{e})$ for which equations

(1.12) are valid. Let's also repeat that coordinates $y^1_\nu, y^2_\nu, y^3_\nu$ of the ν-th point can be substituted by coordinates $x^1_\nu, x^2_\nu, x^3_\nu$ of some other curvilinear coordinate system. Due to a multitude of possible coordinate systems these $3N$ coordinates $x = (x^1_\nu, x^2_\nu, x^3_\nu) = (x^1, x^2, \ldots, x^{3N})$ form configurational manifolds M^{3N}; each coordinate x^k_ν is directed by coordinate vector $\bar{g}_{(\nu)k}(x)$ whose pencils at particular points are denoted by $\boldsymbol{T}_x M^{3N}$.

By substituting $y = y(x)$ in $Z(\ddot{y})$ the compulsion of the system with N material points is reduced to formulae (3D.13). By the same substitution, constraint $f_\mu(y) = 0$ is transformed into invariant form

$$f_\mu(y)_{y=y(x)} = f_\mu(x^1_\nu, x^2_\nu, x^3_\nu) = 0$$

while relations (3D.27a), regarding (1.30) and (1.32) are reduced to the form:

$$\ddot{f}_\mu = \sum_{\nu=1}^{N} \left(\frac{\partial^2 f_\mu}{\partial y^k_\nu \partial y^l_\nu} \frac{\partial y^k_\nu}{\partial x^r_\nu} \frac{\partial y^l_\nu}{\partial x^s_\nu} \dot{x}^r_\nu \dot{x}^s_\nu + \frac{\partial f_\mu}{\partial y^k_\nu} \frac{\partial y^k_\nu}{\partial x^r_\nu} \frac{D\dot{x}^r}{dt} \right) = 0,$$

or, by means of the changed indices, to the form

$$\ddot{f}_\mu = A_{(\mu)ij} \dot{x}^i \dot{x}^j + \frac{\partial f_\mu}{\partial x^i} \frac{D\dot{x}^i}{dt} = 0 \tag{3D.38}$$

The variation with respect to acceleration is

$$\delta \ddot{f}_\mu = \frac{\partial \ddot{f}_\mu}{\partial a^i} \delta a^i = \frac{\partial \ddot{f}_\mu}{\partial a^i} \delta \frac{D\dot{x}^i}{dt} = 0. \tag{3D.39}$$

These relations are equivalent to relations (3D.29) since it is

$$\delta \ddot{y}^j = \delta \left(\frac{\partial y^j}{\partial x^i} \frac{D\dot{x}^i}{dt} \right) = \frac{\partial y^j}{\partial x^i} \delta \frac{D\dot{x}^i}{dt} = \frac{\partial y^j}{\partial x^i} \delta a^i.$$

In order to eliminate variables of acceleration in the equations (3D.39), indefinite multipliers λ_μ should be used as in relations (3C.51) to (3C.54).

The compulsion variation (3D.13) with respect to accelerations has the form:

$$\frac{\partial Z}{\partial a^i} \delta a^i = g_{ij} \left(\frac{D\dot{x}^j}{dt} - \bar{X}^j \right) \delta \frac{D\dot{x}^i}{dt} = 0; \tag{3D.40}$$

$$\left(g_{ij} \frac{D\dot{x}^j}{dt} - X_i - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} \right) \delta \frac{D\dot{x}^i}{dt} = 0, \tag{3D.41}$$

or

$$\left(\frac{\partial Z}{\partial a^i} - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} \right) \delta a^i = 0. \tag{3D.42}$$

A view of relations (3A.27) and (3D.41) points to the conclusion that the differential equations of motion (3D.32) can be written in the curvilinear coordinate system in the same form:

$$\frac{\partial Z}{\partial a^i} - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} = 0 \tag{3D.43}$$

where $a^i = \dfrac{D\dot{x}^i}{dt}$ are coordinates of the material points' acceleration.

Under action of non-holonomic constraints (3D.34), beside constraints $f_\mu(x) = 0$, it is necessary to substitute y coordinates in equations (3D.35) or their variations (3D.36) by means of x coordinates so that it follows that:

$$\sum_{\nu=1}^{N} \frac{\partial \varphi_\sigma}{\partial \dot{y}_\nu^k} \frac{\partial y_\nu^k}{\partial x_\nu^l} \delta \frac{D\dot{x}_\nu^l}{dt} = \frac{\partial \varphi_\sigma}{\partial \dot{y}^j} \frac{\partial y^j}{\partial x^i} \delta \frac{D\dot{x}^i}{dt} = \tag{3D.44}$$

$$= b_{\sigma i} \delta \left(\frac{D\dot{x}^i}{dt} \right) = b_{\sigma i} \delta a^i = 0$$

Multiplying by $\bar{\lambda}_\sigma$, summing up with respect to σ and making equal the same sides of equations (3D.44), (3D.42), it is obtained:

$$\left(\frac{\partial Z}{\partial a^i} - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} - \sum_{\sigma=1}^{l} \bar{\lambda}_\sigma b_{\sigma i} \right) \delta a^i = 0. \tag{3D.45}$$

This relation is always satisfied for the motion whose differential equations are:

$$\frac{\partial Z}{\partial a^i} = \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^i} + \sum_{\sigma=1}^{l} \bar{\lambda}_\sigma b_{\sigma i} \tag{3D.46}$$

Principle of Compulsion Generalized Systems of Coordinates. With respect to the generalized independent coordinates, compulsion (3D.1) is reduced to the form (3D.18), that is,

$$Z = \frac{1}{2} a_{\alpha\beta} \left(a^\alpha - Q^\alpha \right) \left(a^\beta - Q^\beta \right) + \frac{1}{2} a_N^2, \tag{3D.47}$$

where $a_{\alpha\beta}(m,q)$ are inertia tensors upon M, a^{α} are acceleration vector coordinates, Q^{α} are generalized countervariants of the forces' coordinates, while a_N^2 is double compulsion Z_N determined by expression (3D.20); the indices denote ordinal numbers of the generalized coordinates.

In order to describe motion upon manifoldnesses M^n, M^{n+1}; TM^n and TM^{n+1}, as is described by work principle (3C.92) or action principle (3C.34), it is enough to observe compulsion upon manifoldnesses (3D.22), that is

$$\begin{aligned} Z_M &= \frac{1}{2}a_{\alpha\beta}\left(a^{\alpha}-Q^{\alpha}\right)\left(a^{\beta}-Q^{\beta}\right) = \\ &= \frac{1}{2}a_{\alpha\beta}a^{\alpha}a^{\beta} - a_{\alpha\beta}a^{\alpha}Q^{\beta} + \frac{1}{2}a_{\alpha\beta}Q^{\alpha}Q^{\beta}. \end{aligned} \tag{3D.48}$$

where the indices go from 1 to n if the geometrical constraints are invariable, while they go from 0,1, ... ,n, if the constraints are explicitly time-dependent.

The compulsion law is simple in this case:

$$\delta Z_M = \frac{\partial Z_M}{\partial a^{\alpha}}\delta a^{\alpha} = a_{\alpha\beta}(a^{\beta}-Q^{\beta})\delta a^{\alpha} = 0 \tag{3D.49}$$

Regarding the fact that all variations δa^{α} are independent, coefficients with δa^{α} are equal to zero, that is,

$$\frac{\partial Z_M}{\partial a^{\alpha}} = 0$$

or

$$a_{\alpha\beta}(a^{\beta}-Q^{\beta}) = a_{\alpha\beta}\frac{D\dot{q}^{\beta}}{dt} - Q_{\alpha} = 0, \tag{3D.50}$$

and these are differential equations of system's motion (3D.56).

Accordingly, the compulsion principle produces a new form of differential equations of the holonomic systems' motion upon TM by means of compulsion

$$\frac{\partial Z}{\partial a^{\alpha}} = 0. \tag{3D.51}$$

As for the systems with constraints (3D.34), what is important for this principle are the acceleration constraint conditioned by equations (3D.35) so that the change of principle (3D.49) requires the substitution of $y^i = y^i(q)$ by means of generalized independent coordinates q in equations (3D.35). Therefore,

$$\frac{\partial\varphi_{\sigma}}{\partial y^i|_{y(q)}}\frac{\partial y^i}{\partial q^{\alpha}}\dot{q}^{\alpha} + \frac{\partial\varphi_{\sigma}}{\partial\dot{y}^i|_{y(q)}}\frac{\partial y^i}{\partial q^{\alpha}}a^{\alpha} = 0.$$

Since the expression with accelerations is important, these equations should be written in the form

$$\Phi_{\sigma}(q,\dot{q}) + C_{\sigma\alpha}a^{\alpha} = 0 \tag{3D.52}$$

where the notations are introduced

$$\begin{aligned}\Phi_\sigma &:= \frac{\partial \varphi_\sigma}{\partial y^i}\frac{\partial y^i}{\partial q^\alpha}\dot{q}^\alpha,\\ C_{\sigma\alpha} &= \frac{\partial \varphi_\sigma}{\partial \dot{y}^i\big|_{y(q)}}\,\frac{\partial y^i}{\partial q^\alpha}.\end{aligned} \tag{3D.53}$$

Hence, variation (3D.52) with respect to the acceleration variations has the form

$$C_{\sigma\alpha}\delta a^\alpha = 0, \quad \sigma = 1,\dots,l. \tag{3D.54}$$

The same acceleration variations exist in equation (3D.49) as an expression of the compulsion principle. The dependent variations from equations (3D.52) are eliminated in two ways, namely,

1. by means of indefinite multipliers of constraints λ_σ,
2. by substituting dependent variations from equations (3D.54) into (3D.52).

By the indefinite multipliers' method the equations (3D.54) and (3D.49) are reduced to

$$\left(\frac{\partial Z}{\partial a^\alpha} - \sum_{\sigma=1}^{l}\lambda_\sigma C_{\sigma\alpha}\right)\delta a^\alpha = 0. \tag{3D.55}$$

In that case, the motion equations on the right side obtain, instead of zero, generalized reactions

$$R_\alpha = \sum_{\sigma=1}^{l}\lambda_\sigma C_{\sigma\alpha}, \tag{3D.56}$$

so that, together with l, equations (3D.52) provide for solving the given problem.

The method of substituting the dependent variations by independent acceleration ones is not, in essence, more complicated. Equations (3D.54) are extended, for the sake of greater clarity, in the following way:

$$\begin{aligned} C_{10}\delta a^0 + C_{11}\delta a^1 + \cdots + C_{1k}\delta a^k &= -C_{1\alpha'}\delta a^{\alpha'}\\ &\vdots\\ C_{k0}\delta a^0 + C_{k1}\delta a^1 + \cdots + C_{kk}\delta a^k &= -C_{k\alpha'}\delta a^{\alpha'}\\ \alpha' = k+1,\dots,n \end{aligned} \tag{3D.57}$$

or, even shorter,

$$C_{\sigma\alpha''}\delta a^{\alpha''} = -C_{\sigma\alpha'}\delta a^{\alpha'}. \tag{3D.57a}$$

For $\|C_{\sigma\alpha''}\| \neq 0$ it is obtained

$$\delta a^{\gamma''} = -C^{\sigma\gamma''}C_{\sigma\alpha'}\delta a^{\alpha'} = -B^{\gamma''}_{\alpha'}\delta a^{\alpha'} \tag{3D.58}$$

where $C^{\sigma\gamma''}$ is inverse matrix $C_{\sigma\alpha'}$.

If these solutions are substituted in (3D.49), it is obtained that:

$$\frac{\partial Z}{\partial a^{\alpha'}}\delta a^{\alpha'} + \frac{\partial Z}{\partial a^{\alpha''}}\delta a^{\alpha''} = \left(\frac{\partial Z}{\partial a^{\alpha'}} - B^{\alpha''}_{\alpha'}\frac{\partial Z}{\partial a^{\alpha''}}\right)\delta a^{\alpha'} = 0.$$

Regarding the fact that $n+1-l$ variations $\delta a^{\alpha'}$, are independent $n+1-l$ differential equations of motion, freed from the constraints' multipliers, arise that is,

$$\frac{\partial Z}{\partial a^{\alpha'}} - B^{\alpha''}_{\alpha'}\frac{\partial Z}{\partial a^{\alpha''}} = 0, \tag{3D.59}$$

$$(\alpha' = l+1, l+2, \ldots, n+1-l;\ \alpha'' = 1, \ldots, l).$$

There are $n+1-l$ of these equations for the rheonomic system and $n-l$ for the scleronomic system. On the other hand, that there are $n+1+l$ equations

$$\frac{\partial Z}{\partial a^{\alpha}} - \sum \lambda_\sigma C_{\sigma\alpha}, \tag{3D.60}$$

originated from (3D.55) for rheonomic and $n+l$ for scleronomic system. A system of l constraints' equations (3D.34) should be also added to the system of equations (3D.60).

Equations (3D.59) in the extended form, with no compulsion functions (3D.48) or (3D.47), are easily reduced to the recognizable form:

$$\begin{aligned} &a_{\alpha'\beta}\left(a^\beta - Q^\beta\right) - B^{\alpha''}_{\alpha'}a_{\alpha''\beta}\left(a^\beta - Q^\beta\right) = \\ &= a_{\alpha'\beta}\frac{D\dot{q}^\beta}{dt} - Q_{\alpha'} - B^{\alpha''}_{\alpha'}\left(a_{\alpha''\beta}\frac{D\dot{q}^\beta}{dt} - Q_{\alpha''}\right) = 0. \end{aligned} \tag{3D.61}$$

Conclusion 1. It is sufficiently clear from the motion equations of mechanical systems (3D.60), (3D.59), (3D.51), (3D.46), (3D.43) and (3D.32) that the compulsion principle is not less operative in the coordinate description of motion than other principles of mechanics - or even more operative in its application to the non-holonomic constraints' systems.

Besides, this principle points to existence of compulsion (3D.23) which is perpendicular to the tangential manifoldness and thus to the accelerations in TM as well as $T\mathcal{N}$. This is easy to show. Namely, principle (3D.25), applied to compulsion (3D.18), gives, beside equations (3D.51), another equation:

$$\frac{\partial Z}{\partial a_N} = a_N = \varkappa(b_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - F_N) = 0. \tag{3D.62}$$

Final Commentary On Compulsion Principle. It has been noticed that all the given equations, obtained by partial differentiating of function Z with respect to a^i, can be obtained in the same way by partial differentiating of compulsion Z

with respect to countervariant coordinates $\bar{Y}^i$, $\bar{X}^i$ or Q^α. This is here brought into accord with equations (3D.26), since, as can be seen from (3D.3) and (3D.17), Q^α are generalized accelerations. Because of this, a more strict application of relation (3D.25) to compulsion (3D.48) reduces to:

$$\begin{aligned}\delta Z = &\frac{1}{2}a_{\alpha\beta}\left(a^\beta - Q^\beta\right)\delta a^\alpha + \frac{1}{2}a_{\alpha\beta}\left(a^\alpha - Q^\alpha\right)\delta a^\beta - \\ &- \frac{1}{2}a_{\alpha\beta}\left(a^\beta - Q^\beta\right)\delta Q^\alpha - \frac{1}{2}a_{\alpha\beta}\left(a^\alpha - Q^\alpha\right)\delta Q^\beta = \\ &= a_{\alpha\beta}\left(a^\beta - Q^\beta\right)\left(\delta a^\alpha - \delta Q^\alpha\right) = 0.\end{aligned}$$

Any discussion that excludes the case that it is

$$\delta(\frac{D\dot{q}^\alpha}{dt}) = \delta Q^\alpha, \quad \text{or} \quad \delta\ddot{y} = \delta Y$$

leads to the previous results, as has been done.

Excerpta about the Principles of Mechanics

The principles of mechanics are statements of general significance, formed by means of the introduced concepts and definitions of mechanics, whose truthfulness is not subjected to verification. Each principle on its own can serve as the basis for developing the whole theory of mechanics.

Principle of Equilibrium. *The sums of all the forces acting at particular dynamic points are equal to zero*:

$$\sum_{\mu=1}\mathbf{F}_{\nu\mu} = 0.$$

Principle of Work. *The total work of all the forces upon possible displacements is equal to zero, while, for the system of unilateral constraints, it is equal or less than zero*:

$$\sum \mathbf{F}_\nu \cdot \Delta\boldsymbol{r}_\nu \leq 0.$$

Principle of Action. *The integral of the sum $\delta E_k + \delta A(\boldsymbol{F})$, calculated on real motion for the time $[t_0, t_1]$, equals zero, i.e.* :

$$\int_{t_0}^{t_1}(\delta E_k + \delta A(\boldsymbol{F}))dt = 0.$$

Principle of Compulsion. *The compulsion variation with respect to accelerations is equal to zero*:

$$\delta Z = \frac{\partial Z}{\partial a^\alpha}\delta a^\alpha = 0.$$

æ

IV. THEOREMS OF MECHANICS

The concept of *theorems of mechanics* here implies a mathematical assertion of general significance about material systems' motion whose truthfulness is proved on the basis of preprinciples, principles of mechanics, basic and consequent definitions and laws of dynamics.

The theorems of mechanics are used to effectuate the principles of mechanics.

The theorems, as consequential assertions, should satisfy the preprinciples.

Theorem on Motion Impulse Change

The natural derivative with respect to time of the generalized impulses of the mechanical system of constant mass are equal to the generalized forces:

$$\frac{Dp_\alpha}{dt} = Q_\alpha \tag{4.1}$$

Proof: By the basic definition 2 and relation (3A.39), the generalized impulses are defined. The differential equations (3C.56) follow from the equilibrium principle. Since it is

$$\frac{Da_{\alpha\beta}}{dt} = 0$$

for the mechanical system of constant masses, it is

$$a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt} = \frac{D}{dt}\left(a_{\alpha\beta}\dot{q}^\beta\right) = \frac{Dp_\alpha}{dt} = Q_\alpha,$$

as claimed by the theorem.

Lemma 1. *The natural derivative of the impulse of the rotary motion constant mass system, measured by angular change, is equal to the moment of forces.*

Proof 1. From the elementary work definition (3B.15) it follows that the generalized forces for dimensionless and angular coordinates have the dimension of the moment of forces

$$\mathrm{M\,L^2\,T^{-2}} = \dim Q$$

so that Theorem on motion impuls change confirms the lemma.

Proof 2. On the basis of the equilibrium principle, the concept of the moment of force is derived (3A.63). Respectively, for the rotary motion of the ν-th particle of mass Δm_ν, firmly attached to some fixed point 0 that belongs to eigen axis u of the rotary motion, the moment of inertia force is derived. Following expression (3A.63), the moment of inertia force

$$\boldsymbol{I}_\nu = -\Delta m_\nu \frac{d\boldsymbol{v}_\nu}{dt}$$

has the form:

$$-\mathfrak{M}(\boldsymbol{I}_\nu) = \boldsymbol{r}_\nu \times \Delta m_\nu \frac{d\boldsymbol{v}_\nu}{dt} = \boldsymbol{r}_\nu \times \frac{d}{dt}(\Delta m_\nu \boldsymbol{v}_\nu) = \frac{d}{dt}(\boldsymbol{r}_\nu \times \Delta m_\nu \boldsymbol{v}_\nu).$$

On the other hand, for $|r_\nu| = \text{const}$, $\boldsymbol{v}_\nu = \boldsymbol{\omega} \times \boldsymbol{r}_\nu$, it further follows that it is:

$$\begin{aligned} -\boldsymbol{\mathfrak{M}}_\nu(\boldsymbol{I}_\nu) &= \frac{d}{dt}\left[\Delta m_\nu(\boldsymbol{r}_\nu \times (\boldsymbol{\omega} \times \boldsymbol{r}_\nu))\right] = \\ &= \frac{d}{dt}\left[\Delta m_\nu\left[(\boldsymbol{\omega}(\boldsymbol{r}_\nu \cdot \boldsymbol{r}_\nu) - \boldsymbol{r}_\nu(\boldsymbol{r}_\nu \cdot \boldsymbol{\omega}))\right]\right] = \\ &= \frac{d}{dt}\left[\Delta m_\nu\left[r_\nu^2 \omega^j \boldsymbol{e}_j - y_\nu^k \boldsymbol{e}_k(\delta_{ij} y_\nu^i \omega^j)\right]\right] \end{aligned} \tag{4.2}$$

since it is $\boldsymbol{r}_\nu \cdot \boldsymbol{\omega} = y_\nu^i \boldsymbol{e}_i \cdot \omega^j \boldsymbol{e}_j = \delta_{ij} y_\nu^i \omega^j$.

If vector (4.2) is projected upon the coordinate axes, it will be obtained that:

$$\begin{aligned} -\boldsymbol{\mathfrak{M}}_\nu \cdot \boldsymbol{e}_i &=: \mathfrak{M}_{(\nu)i} \\ &= \frac{d}{dt}\left[\Delta m_\nu\left(r_\nu^2 \delta_{ij}\omega^j - y_{\nu i} y_{\nu j}\omega^j\right)\right] = \\ &= \frac{d}{dt}\left(\mathcal{I}_{(\nu)ij}\omega^j\right) = \frac{d}{dt} p_{\nu i} \end{aligned} \tag{4.3}$$

where

$$p_{\nu i} = \mathcal{I}_{(\nu)ij}\omega^j \tag{4.4}$$

are impulses of motion of the ν-th particle with respect to orthonormal coordinate system (y, e).

Returning projections (4.3) into equation (3A.58), Lemma 1 is obtained, by which it is:

$$\frac{dp_{(\nu)i}}{dt} = \mathfrak{M}_i(\boldsymbol{F}_\nu). \tag{4.5}$$

Example 13. The lemma's application to the description of the rotary motion of a rigid body of constant mass around a fixed point.

Fig. 5

For each ν-th tiny part of the body equations (4.5) and (4.4) are valid, where indices $i, j = 1, 2, 3$, $\nu = 1, 2, \ldots$. If the part's impulses (4.4) are added as parameters for the same i-th axis, it is obtained that

$$p_i = \sum_\nu p_{\nu i} = \sum_\nu \Delta m_\nu (r_\nu^2 \delta_{ij} - y_{\nu i} y_{\nu j}) \omega^j = \mathcal{I}_{ij} \omega^j \tag{4.6}$$

where

$$\mathcal{I}_{ij} = \sum_\nu \Delta m_\nu (r_\nu^2 \delta_{ij} - y_{\nu i} y_{\nu j}) \tag{4.7}$$

is *inertia tensor*.

On the other hand, the principal moments of all the active forces with respect to the coordinate axes are

$$\mathfrak{M}_i := \sum_\nu \mathfrak{M}_i(\boldsymbol{F}_\nu)$$

so that, for the observed rigid body, it is obtained

$$\frac{dp_i}{dt} = \frac{d}{dt} \left(\mathcal{I}_{ij} \omega^j \right) = \mathfrak{M}_i; \tag{4.8}$$

These are differential equations of a body's rotary motion around a fixed point with respect to the fixed coordinate system.

Further extension of these equations, at first glance, opens up a question of derivatives with respect to time of inertia tensor (4.7), since the previous equations

$$\frac{d\mathcal{I}_{ij}}{dt} \omega^j + \mathcal{I}_{ij} \frac{d\omega^j}{dt} = \mathfrak{M}_i \tag{4.9}$$

obviously comprise derivatives with respect to time $\dot{\mathcal{I}}_{ij}$. Let's find their analytical meaning. For now, it is assumed that masses are constant. In the sum

$$\frac{d\mathcal{I}_{1j}}{dt} \omega^j = \dot{\mathcal{I}}_{11} \omega^1 + \dot{\mathcal{I}}_{12} \omega^2 + \dot{\mathcal{I}}_{13} \omega^3 \tag{4.10}$$

the derivatives of the inertia tensor coordinates are:

$$\begin{aligned}
\dot{\mathcal{I}}_{11} &= \left(\sum_\nu \Delta m_\nu (y_{\nu 2}^2 + y_{\nu 3}^2)\right)^{\cdot} = 2\sum_\nu \Delta m_\nu (y_{\nu 2}\dot{y}_{\nu 2} + y_{\nu 3}\dot{y}_{\nu 3}) = \\
&= 2\sum_\nu \Delta m_\nu \left[y_{\nu 2}\left(y_{\nu 1}\omega^3 - y_{\nu 3}\omega^1\right) + y_{\nu 3}\left(y_{\nu 2}\omega^1 - y_{\nu 1}\omega^2\right)\right] = \\
&= 2I_{21}\omega^3 - 2I_{31}\omega^2; \\
\dot{\mathcal{I}}_{12} &= -\sum_\nu \Delta m_\nu \left(y_{\nu 1}y_{\nu 2}\right)^{\cdot} = \\
&= -\sum_\nu \Delta m_\nu \left[\left(y_{\nu 3}\omega^2 - y_{\nu 2}\omega^3\right) y_{\nu 2} + y_{\nu 1}\left(y_{\nu 1}\omega^3 - y_{\nu 3}\omega^1\right)\right] = \\
&= -\sum_\nu \Delta m_\nu \left(y_{\nu 1}^2\omega^3 - y_{\nu 2}^2\omega^3\right) + \sum_\nu \Delta m_\nu \left(y_{\nu 3}y_{\nu 2}\omega^2 - y_{\nu 1}y_{\nu 3}\omega^1\right) \\
\dot{\mathcal{I}}_{13} &= -\sum_\nu \Delta m_\nu \left(y_{\nu 1}y_{\nu 3}\right)^{\cdot} = \\
&= -\sum_\nu \Delta m_\nu \left[\left(y_{\nu 3}\omega^2 - y_{\nu 2}\omega^3\right) y_{\nu 3} + y_{\nu 1}\left(y_{\nu 2}\omega^1 - y_{\nu 1}\omega^2\right)\right] = \\
&= -\sum_\nu \Delta m_\nu \left(y_{\nu 3}^2\omega^2 - y_{\nu 1}^2\omega^2\right) + \sum_\nu \Delta m_\nu \left(y_{\nu 1}y_{\nu 2}\omega^1 - y_{\nu 2}y_{\nu 3}\omega^3\right)
\end{aligned} \tag{4.11}$$

By the cyclic change of indices $1,2,3 \to 2,3,1 \to 3,1,2$ it is easy to obtain derivatives of the other coordinates of inertia tensors $\mathcal{I}_{2j}$ and $\mathcal{I}_{3j}$.

Substituting in (4.9) equations (4.8) obtain the form:

$$\begin{aligned}
\frac{Dp_1}{dt} &= \mathcal{I}_{1j}\dot{\omega}^j + (\mathcal{I}_{33} - \mathcal{I}_{22})\omega^2\omega^3 + \mathcal{I}_{21}\omega^1\omega^3 - \\
&\quad - \mathcal{I}_{31}\omega^2\omega^1 + \mathcal{I}_{32}\omega^2\omega^2 - \mathcal{I}_{23}\omega^3\omega^3 = \mathfrak{M}_1, \\
\frac{Dp_2}{dt} &= \mathcal{I}_{2j}\dot{\omega}^j + (\mathcal{I}_{11} - \mathcal{I}_{33})\omega^3\omega^1 + \mathcal{I}_{32}\omega^2\omega^1 - \\
&\quad - \mathcal{I}_{12}\omega^3\omega^2 + \mathcal{I}_{13}\omega^3\omega^3 - \mathcal{I}_{31}\omega^1\omega^1 = \mathfrak{M}_2, \\
\frac{Dp_3}{dt} &= \mathcal{I}_{3j}\dot{\omega}^j + (\mathcal{I}_{22} - \mathcal{I}_{11})\omega^1\omega^2 + \mathcal{I}_{13}\omega^3\omega^2 - \\
&\quad - \mathcal{I}_{23}\omega^1\omega^3 + \mathcal{I}_{21}\omega^1\omega^1 - \mathcal{I}_{12}\omega^2\omega^2 = \mathfrak{M}_3.
\end{aligned} \tag{4.12}$$

When these equations are applied to engineering practice, namely, when they are applied to the models of body's rotary motion, it is important to consider the fact that there are derivatives of inertia tensor coordinates (4.7), $\mathcal{I}_{ij}(y(t))$, with respect to time (4.11).

The differential equations of the body's rotary motion are considerably simplified if it is possible to attach a moveable coordinate system $(z, \mathbf{э})$ to this body. With respect to this coordinate system the inertia tensor coordinates are constant. By choosing the coordinate origin in the inertia center, and by orienting the coordinate axes along the inertia axes the values $\mathcal{I}_{ij}$ vanish for $i \neq j$, while $\mathcal{I}_{ii}$, is reduced to the central and principal inertia moments $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ If in the moveable coordinate system the angular velocity is denoted $\boldsymbol{\Omega} = \Omega^i \boldsymbol{\ni}_i$, the differential equations of the body's rotary motion around the inertia center is reduced to well-known Euler dynamic equations:

$$\begin{aligned} \frac{Dp_1}{dt} &= \mathcal{I}_1 \dot{\Omega}^1 + (\mathcal{I}_3 - \mathcal{I}_2)\Omega^2\Omega^3 = \mathfrak{M}_1, \\ \frac{Dp_2}{dt} &= \mathcal{I}_2 \dot{\Omega}^2 + (\mathcal{I}_1 - \mathcal{I}_3)\Omega^3\Omega^1 = \mathfrak{M}_2, \\ \frac{Dp_3}{dt} &= \mathcal{I}_3 \dot{\Omega}^3 + (\mathcal{I}_2 - \mathcal{I}_1)\Omega^1\Omega^2 = \mathfrak{M}_3. \end{aligned} \tag{4.13}$$

At the same time, it is assumed that there is an explicit constraint between the coordinates

$$y^i = \gamma^i_j z^j \quad \Leftrightarrow \quad z^i = \overline{\gamma}^i_j y^j \tag{4.14}$$

where coefficients γ^i_j appear as time functions.

Theorem on Kinetic Energy Change

Definition 5 has introduced the concept of work (3B.1), while concept of *kinetic energy* is consequently introduced as negative work of inertia forces (3B.6) and (3B.7a) of the material point of constant mass upon given motion. Work change with respect to time (3B.14) is called power. These concepts are sufficient for formulating the theorem on kinetic energy change with respect to time. The phrase "change with respect to time" mathematically represents a natural derivative with respect to independent variable t; [44], [64].

Theorem. *Kinetic energy change of the system of material points with constant masses that forces Q_α act upon, with respect to time, is equal to power P of these forces*, that is,

$$\frac{dE_k}{dt} = P = Q_\alpha \dot{q}^\alpha. \tag{4.15}$$

The same theorem can be expressed in the following way: *The natural derivative of kinetic energy of the system of material points with constant mass, with respect to time, is equal to power.*

Proof 1. Multiplying differential equations of motion (3C.40) with $\dot{q}^\alpha$ and summing up with respect to index α, it is obtained that

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\alpha}\dot{q}^\alpha\right) - \frac{\partial E_k}{\partial \dot{q}^\alpha}\ddot{q}^\alpha - \frac{\partial E_k}{\partial q^\alpha}\dot{q}^\alpha = Q_\alpha \dot{q}^\alpha.$$

Since

$$\frac{\partial E_k}{\partial \dot{q}^\alpha}\dot{q}^\alpha = 2E_k, \quad \frac{\partial E_k}{\partial \dot{q}^\alpha}\ddot{q}^\alpha + \frac{\partial E_k}{\partial q^\alpha}\dot{q}^\alpha = \frac{dE_k}{dt},$$

while

$$Q_\alpha \dot{q}^\alpha = P,$$

the theorem on kinetic energy change (4.15) is proved.

Proof 2. Kinetic energy of the mechanical system is determined by one of formulae (3B.7)

$$E_k = \frac{1}{2}\sum_\nu m_\nu \boldsymbol{v}_\nu^2. \tag{4.16}$$

The derivative with respect to time, regarding relation (3A.4) is

$$\frac{dE_k}{dt} = \sum_\nu m_\nu \boldsymbol{v}_\nu \cdot \frac{d\boldsymbol{v}_\nu}{dt} = \sum_\nu (\boldsymbol{F}_\nu + \boldsymbol{R}_\nu) \cdot \boldsymbol{v}_\nu = P$$

that was to be proved.

Proof 3. Kinetic energy is represented by formula (3C.49) that differential equations of motion (3C.58) and (3C.56) correspond to. Generalized forces (3C.52) are formed by a sum of the generalized potential and non-potential forces. Multiplying equations (3C.58) by generalized velocities $\dot{q}^\alpha$, while equation (3C.56) is multiplied by the derivatives of impulse $\dot{p}_\alpha$, and by summing up with respect to α it is obtained that:

$$\dot{p}_\alpha \dot{q}^\alpha = \dot{p}_\alpha \frac{\partial H}{\partial p_\alpha} \tag{4.17}$$

$$\dot{p}_\alpha \dot{q}^\alpha = -\frac{\partial H}{\partial q^\alpha}\dot{q}^\alpha + Q^*_\alpha \dot{q}^\alpha \tag{4.18}$$

The difference between equations (4.17) and (4.18) is

$$\frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha + \frac{\partial H}{\partial q^\alpha}\dot{q}^\alpha - Q^*_\alpha \dot{q}^\alpha = 0 \tag{4.19}$$

If expression (3C.51) is taken into consideration, that is,

$$H = E_k(q,p) + E_p(q,t) \tag{4.20}$$

as well as (3C.52), it follows

$$\frac{\partial E_k}{\partial p_\alpha}\dot{p}_\alpha + \frac{\partial E_k}{\partial q^\alpha}\dot{q}^\alpha = \left(Q^*_\alpha - \frac{\partial E_p}{\partial q^\alpha}\right)\dot{q}^\alpha \tag{4.21}$$

or

$$\frac{dE_k}{dt} = Q_\alpha \dot{q}^\alpha = P$$

as desired.

Change of Hamilton's function

Lemma 2. *If potential forces do not explicitly depend upon time, the derivative of Hamilton's function $H(q^0, q^1, \ldots, q^n; p_0, p_1, \ldots, p_n)$ with respect to time t is equal to the power of non-potential forces* [62].

Proof. Equation (4.19) confirms the previous statement since it is

$$\frac{dH}{dt} = \frac{\partial H}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha = Q^*_\alpha \dot{q}^\alpha = P(Q^*). \tag{4.22}$$

Lemma 3. *Change with respect to time of Hamilton's function H of the potential system with variable constraints is equal to the power of constraints R_0,* that is,

$$\frac{dH}{dt} = R_0 \tag{4.23}$$

Proof 1. In accordance with equations (3C.40b) and (3C.60) for potential forces, it is $Q^*_i = 0$, $i = 1, \ldots, n$, while $Q^*_0 = Q^{**}_0 + R_0 = R_0$, since it is

$$Q^{**} = \sum \boldsymbol{F}_\nu \cdot \frac{\partial \boldsymbol{r}}{\partial q^0} = 0,$$

so that the right side of equation (4.22) degenerates into

$$Q^*_\alpha \dot{q}^\alpha = Q^*_0 \dot{q}^0,$$

that is, when for the rheonomic coordinate it is taken that $q^0 = t$,

$$Q^*_\alpha \dot{q}^\alpha = R_0,$$

which proves Lemma 3.

Proof 2. If each equation (3A.25) is multiplied by respective velocity vector $\boldsymbol{v}_\nu$ and summed up with respect to index ν, it is obtained:

$$\sum_\nu m_\nu \boldsymbol{v}_\nu \cdot \frac{d\boldsymbol{v}_\nu}{dt} = \sum_\nu \boldsymbol{F}_\nu \cdot \boldsymbol{v}_\nu + \sum_\nu \sum_\mu \lambda_\mu \operatorname{grad}_\nu f_\mu \cdot \boldsymbol{v}_\nu$$

For potential forces $\boldsymbol{F}_\nu = -\operatorname{grad}_\nu E_p(\boldsymbol{r}_\nu)$ and the velocity conditions upon the constraints

$$\frac{d}{dt}\left(\sum_\nu m_\nu \boldsymbol{v}_\nu \cdot \boldsymbol{v}_\nu\right) + \operatorname{grad}_\nu E_p \cdot \boldsymbol{v}_\nu = -\sum_\mu \lambda_\mu \frac{\partial f_\mu}{\partial t}.$$

the previous equation is reduced to

$$\frac{d}{dt}\left(\frac{1}{2}\sum_\nu m_\nu \boldsymbol{v}_\nu \cdot \boldsymbol{v}_\nu\right) + \operatorname{grad}_\nu E_p \cdot \boldsymbol{v}_\nu = -\sum_\mu \lambda_\mu \frac{\partial f_\mu}{\partial t}.$$

On the basis of relations (3A.54) and (3B.7) for $q^0 = t$, it follows from the previous equation that it is

$$\frac{d}{dt}(E_k + E_p) = \frac{dH}{dt} = R_0, \tag{4.24}$$

which is exactly Lemma 3.

Theorem on Mechanical Energy Change

By formulae (3C.43), (3C.51) and (3C.61) the total mechanical energy of the potential rheonomic holonomic system is defined by the formula

$$E = E_k(q,p) + E_p(q) + \mathcal{P}(q_0). \tag{4.25}$$

Theorem. *Change with respect to time of the total mechanical energy* (4.25) *of the system with constant masses is equal to the power of non-potential forces* $\boldsymbol{F}_\nu^*$, *that is*,

$$\frac{dE}{dt} = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu^* \cdot \frac{d\boldsymbol{r}_\nu}{dt} = Q_\alpha^* \dot{q}^\alpha. \tag{4.26}$$

Proof. For $Q_\alpha^* \neq 0$ and formula (3C.43), according to which it is

$$R_0 = -\frac{\partial \mathcal{P}}{\partial q^0} \tag{4.27}$$

differential equations (3C.62) comprise additional forces Q_α^*,

$$\dot{p}_\alpha = -\frac{\partial E}{\partial q^\alpha} + Q_\alpha^* \tag{4.28}$$

$$\dot{q}^\alpha = \frac{\partial E}{\partial p_\alpha}. \tag{4.29}$$

If equations (4.28) are multiplied by velocities $\dot{q}^\alpha$, while equations (4.29) are multiplied by $\dot{p}_\alpha$ and added with respect to indices α, the following two equations are obtained:

$$\dot{p}_\alpha \dot{q}^\alpha = -\frac{\partial E}{\partial q^\alpha}\dot{q}^\alpha + Q_\alpha^* \dot{q}^\alpha$$

$$\dot{q}^\alpha \dot{p}_\alpha = \frac{\partial E}{\partial p_\alpha}\dot{p}_\alpha$$

Their difference is

$$\frac{\partial E}{\partial p_\alpha}\dot{p}_\alpha + \frac{\partial E}{\partial q^\alpha}\dot{q}^\alpha - Q_\alpha^* \dot{q}^\alpha = 0$$

that is

$$\frac{dE}{dt} = Q^*_\alpha \dot{q}^\alpha \tag{4.30}$$

which proves Theorem (4.26).

Corollary. In the systems of constraints that do not change in time, all indices range from 1 to n ($i = 1, \ldots, n$), instead of from 0 to n ($\alpha = 0, 1, \ldots, n$); at the same time, additional coordinate q^0 and respective force or power R_0, as well as rheonomic pseudopotential $\mathcal{P}$ vanish.

Theorem on Controllability of Motion

The concept of *controllability of motion* implies the possibility that the mechanical system motion is realized according to a given program under the compulsion of special generalized forces. Motion-controlling forces U_α are here considered as *generalized forces of controllability.* The phrase "motion control" implies the *process of realizing a given or programmed motion.* Programs can be of great variety. This study includes the program of pathways and the program of velocities. In setting up and making a motion program, the coordinate system that mechanics is based on should be the starting point. For the motion upon the derived manifolds it is also necessary to know and take into consideration the relations of their generation. The motion upon $(2n + 2)$-dimensional tangent manifolds $T\mathcal{N}$ or their equivalent manifolds of the state $T^*\mathcal{N}$ will be observed further on.

The pathway program upon $\mathcal{N} = M^{n+1}$ can be given by one or many functions

$$f(q^0, q^1, \ldots, q^n) = 0, \tag{4.31}$$

while the velocities program upon $T\mathcal{N}$

$$\varphi(q^0, q^1, \ldots, q^n, \dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n) = 0, \tag{4.32}$$

or the impulse program on $T^*\mathcal{N}$ which is equivalent to them

$$\varphi^*(q^0, q^1, \ldots, q^n; p_0, p_1, \ldots, p_n) = 0. \tag{4.33}$$

As can be seen, the program relations are similar or equal to the ideal constraints' relations. Therefore, further problem-solving can be considered as conditioned ("constrained") motion of mechanical systems.

A system of material points of constant masses would be forced to move according to program (4.31) and (4.32) upon the manifolds $T\mathcal{N}$ whose inertia tensor is $a_{\alpha\beta}(q^0, q^1, \ldots, q^n)$; the system is acted upon by natural active generalized forces Q_α.

By using any of the previously mentioned principles, in addition to conditions (4.31) and (4.32), the differential equations of motion will be achieved

$$a_{\alpha\beta} \frac{D\dot{q}\beta}{dt} = Q_\alpha + U_\alpha \tag{4.34}$$

if programs (4.31) and (4.32) are abstracted by forces $U_0, U_1, \dots, U_m$, $m \le n$; or

$$a_{\alpha\beta}\frac{D\dot q^\beta}{dt} = Q_\alpha + \lambda\frac{\partial f}{\partial q^\alpha} + \mu\frac{\partial\varphi}{\partial\dot q^\alpha},$$
$$f(q^0, q^1, \dots, q^n) = 0, \tag{4.35}$$
$$\varphi(q^0, q^1, \dots, q^n, \dot q^0, \dot q^1, \dots, \dot q^n) = 0,$$

if the system of equations includes equations (4.31) and (4.32).

Condition $f = 0$ necessarily satisfies all the conditions of velocity and acceleration which means that the first and second natural derivatives of scalar function f are equal to zero:

$$\frac{Df}{dt} = \frac{df}{dt} = \dot f = \frac{\partial f}{\partial q^\alpha}\dot q^\alpha = 0$$
$$\ddot f = \frac{\partial^2 f}{\partial q^\alpha\partial\dot q^\beta}\dot q^\alpha\dot q^\beta + \frac{\partial\dot f}{\partial\dot q^\beta}\ddot q^\beta = \frac{\partial^2 f}{\partial q^\beta\partial\dot q^\alpha}\dot q^\alpha\dot q^\beta + \frac{\partial f}{\partial q^\beta}\ddot q^\beta = 0$$

or

$$\begin{aligned}\frac{\partial f}{\partial q^\beta}\frac{D\dot q^\beta}{dt} &= \frac{\partial f}{\partial q^\beta}\Gamma^\beta_{\alpha\gamma}\dot q^\alpha\dot q^\beta - \frac{\partial^2 f}{\partial q^\beta\partial q^\alpha}\dot q^\beta\dot q^a = \\ &= \left(\frac{\partial f}{\partial q^\beta}\Gamma^\beta_{\alpha\gamma} - \frac{\partial^2 f}{\partial q^\alpha\partial q^\beta}\right)\dot q^\alpha\dot q^\beta = G_{\alpha\gamma}\dot q^\alpha\dot q^\gamma\end{aligned} \tag{4.36}$$

where $G_{\alpha\gamma}$ is the expression in brackets.

After determining λ and μ, their substitutions in relation (4.35) and comparison with (4.34), the controlling forces are obtained.

Velocity program (4.32) should satisfy the acceleration condition

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial\dot q^\alpha}\frac{D\dot q^\alpha}{dt} - E_{\alpha\gamma}\dot q^\alpha\dot q^\gamma = 0 \tag{4.36a}$$

where $E_{\alpha\gamma} = \dfrac{\partial\varphi}{\partial\dot q^\beta}\Gamma^\beta_{\alpha\gamma}$.

Substituting

$$\frac{D\dot q^\gamma}{dt} = a^{\alpha\gamma}\left(Q_\alpha + \lambda\frac{\partial f}{\partial q^\alpha} + \mu\frac{\partial\varphi}{\partial\dot q^\alpha}\right)$$

From relations (4.35) in (4.36) and (4.36a), it follows

$$\begin{aligned}\frac{\partial f}{\partial q^\beta}a^{\alpha\beta}\left(Q_\alpha + \lambda\frac{\partial f}{\partial q^\alpha} + \mu\frac{\partial\varphi}{\partial\dot q^\alpha}\right) &= G_{\alpha\gamma}\dot q^\alpha\dot q^\beta \\ \frac{\partial\varphi}{\partial\dot q^\beta}a^{\alpha\beta}\left(Q_\alpha + \lambda\frac{\partial f}{\partial q^\alpha} + \mu\frac{\partial\varphi}{\partial\dot q^\alpha}\right) &= E_{\alpha\gamma}\dot q^\alpha\dot q^\beta.\end{aligned} \tag{4.37}$$

There are as many equations as unknown multipliers λ and μ; thus, they make up a solvable linear system:

$$\left.\begin{aligned} B_{11}\lambda + B_{12}\mu = C_1 \\ B_{21}\lambda + B_{22}\mu = C_2 \end{aligned}\right\} \tag{4.37a}$$

so that it is obtained

$$U_\alpha = \lambda(q,\dot{q})\frac{\partial f}{\partial q^\alpha} + \mu(q,\dot{q})\frac{\partial \varphi}{\partial \dot{q}^\alpha}. \tag{4.38}$$

Therefore, in order to make the system move upon manifold N according to program (4.31) and (4.32), the controlling forces should primarily satisfy relations (4.38); then, force U_α with respect to quantity $\|U_\alpha\|$ should be greater than natural generalized forces $\|Q_\alpha\|$, that is

$$\|U_\alpha\| \geq \|Q_\alpha\|. \tag{4.39}$$

if forces U_α control the motion opposite to the motion direction under the influence of Q_α.

In the way analogous to determination of the constraints' reactions, controlling forces U_α can be determined in the function of positions q and $\dot{q}$ if the program is given by means of relations (4.31) or (4.32).

In this way the needed controllability conditions of some mechanical system are derived. In other words, it is possible to calculate, with respect to formulae (4.37) and (4.38), what force - and how large – is needed to make a body move along a given pathway or to increase or reduce its motion velocity. What is, therefore, needed to control motion, in addition to conditions (4.31), (4.38) and (4.39), is the existence of sufficiently large controlling forces U in order to make the system controllable. The above-given assertion is expressed by the following controllability theorem.

Theorem. *The mechanical system motion is controllable according to a program given in advance if there are such controlling forces of such magnitude, dependent upon the program parameters, which are by their absolute value greater than other respective active forces if the controlling forces direct the motion opposite to the motion direction under the influence of other forces* Q_α.

Recognizable Example 14. Let's determine force U that can control the motion a weighty material point of mass m in vertical plane $z = 0$ according to the program:

$$f(x,y) = y - \frac{g}{2}x^2 - h = 0$$
$$\varphi(\dot{x}) = \dot{x} - c = 0$$

where g is the Earth's gravitational force acceleration, while h and c are given constants.

The differential equations of motion are:

$$m\ddot{x} = -\lambda gx + \mu = U_x$$
$$m\ddot{y} = -mg + \lambda = -mg + U_y.$$

Equations (4.37) for the given example are:

$$-g + \frac{\lambda}{m} - g\dot{x}^2 + \frac{gx}{m}(\lambda gx - \mu) = 0$$
$$-\lambda gx + \mu = 0 \longrightarrow \mu = \lambda gx,$$
$$\lambda = mg(1 + \dot{x}^2) = mg(1 + c^2).$$

According to relation (4.38), it is further obtained that:

$$U_x = -\lambda gx + \mu = 0,$$
$$U_y = \lambda \frac{\partial f}{\partial y} = mg(1 + c^2)$$

Therefore, force $U_y = mg(1 + c^2)$ can be used for realizing the given motion.

A simpler problem states that, instead of the program, the controlling forces are given in an analytical form, without limiting their magnitudes.

Limited sets of controlling forces are more often present in engineering than unlimited ones.

Example 15. Let's put into motion and direct a material point of mass m from the rest state by controlling force $\boldsymbol{U} = (U_1, U_2, U_3)$,

$$U_1 = U\cos\alpha_1, \quad U_2 = U\cos\alpha_2$$

$$U_3 = U\cos\alpha_3, \quad |U| = 1,$$

along the pathway to which point $M(1, 2, 3)$ belongs.

For $0 \leq \alpha_3 < \frac{\pi}{2}$ the controlling forces are larger than the active ones. The differential equations of motion are:

$$m\ddot{y}_i = \cos\alpha_i.$$

If the initial position from the rest state is taken to be the pole of coordinate system $(Oy_1y_2y_3)$, the finite equations of motion are:

$$y_i = \frac{\cos\alpha_i}{2m}t^2,$$

thus the pathway equations are

$$\frac{y_1}{\cos\alpha_1} = \frac{y_2}{\cos\alpha_2} = \frac{y_3}{\cos\alpha_3},$$

while, at the same time, it is necessary to satisfy the conditions:

$$\frac{1}{\cos\alpha_1} = \frac{2}{\cos\alpha_2} = \frac{3}{\cos\alpha_3}, \quad 0 \le \alpha_3 < \frac{\pi}{2},$$

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1.$$

Lemma. *The motion $T^*\mathcal{N}$ of the mechanical system with constant masses is controllable according to a given program if there are such controlling forces of such magnitude, dependent upon the program parameters of the motion state, which are by their absolute value greater than other respective active generalized forces if the controlling forces direct the motion opposite to the motion direction under the influence of other generalized forces Q_α.*

Proof. Differential equations of motion (4.34) of the system of material points with constant masses upon T^*N are

$$\frac{Dp_\alpha}{dt} = Q_\alpha + U_\alpha, \quad \dot{q}^\alpha = a^{\alpha\beta} p_\beta. \tag{4.40}$$

Equations (4.35) are reduced to:

$$\left.\begin{aligned} &\frac{Dp_\alpha}{dt} = Q_\alpha + \lambda\frac{\partial f}{\partial q^\alpha} + \mu a_{\alpha\beta}\frac{\partial \varphi^*}{\partial p_\beta} \\ &f(q^0, q^1, \ldots, q^n) = 0 \\ &\varphi^*(q^0, q^1, \ldots, q^n; p_0, p_1, \ldots, p_n) = 0 \end{aligned}\right\} \tag{4.41}$$

since it is

$$\left.\frac{\partial \varphi}{\partial \dot{q}^\alpha}\right|_{\dot{q}=\dot{q}(p)} = \frac{\partial \varphi^*}{\partial p_\beta}\frac{\partial p_\beta}{\partial \dot{q}^\alpha} = a_{\alpha\beta}\frac{\partial \varphi^*}{\partial p_\beta}.$$

Condition (4.36) is transformed into

$$\begin{gathered} \frac{\partial f}{\partial q^\beta} a^{\beta\xi}\frac{Dp_\xi}{dt} = G_{\alpha\gamma} a^{\alpha\xi} p_\xi a^{\gamma\eta} p_\eta = G^{*\xi\eta} p_\eta p_\xi, \\ (\xi, \eta = 0, 1, \ldots, n). \end{gathered} \tag{4.42}$$

Velocity condition (4.36a) is transformed into the conditions for constraining the impulse:

$$\frac{\partial \varphi^*}{\partial p_\beta} a_{\alpha\beta} a^{\alpha\xi}\frac{Dp_\xi}{dt} = E_{\alpha\gamma} a^{\alpha\xi} a^{\gamma\eta} p_\xi p_\eta,$$

that is,

$$\frac{\partial \varphi^*}{\partial p_\beta}\frac{Dp_\beta}{dt} = E^{*\xi\eta} p_\xi p_\eta, \tag{4.43}$$

where the substitutions are obvious.

Substituting $\dfrac{Dp_\alpha}{dt}$ from equation (4.41) into (4.42) and (4.43) a system of structure equations (4.37a) is obtained; thus, the lemma is proved.

Example 16. Translate a linear oscillator from the initial state $p = p_0 =$ const, $q = 0$, into equilibrium state $p = 0$, $q = 0$ by means of controlling force U, $|U| \leq 1$.

The differential equations of motion are:

$$\dot{p} = -q + U, \quad -1 \leq U \leq 1 \tag{E16.1}$$

$$\dot{q} = p, \tag{E16.2}$$

under the conjunction that inertia coefficient $a = 1$ and restitution coefficient $c = 1$. Eliminating time differential dt differential equations of the phase trajectory are obtained:

$$\frac{dp}{dq} = -\frac{(q \mp u)}{p}.$$

From a multitude of the curves

$$p^2 + (q \mp 1)^2 = c_{1,2} \quad \text{for} \quad u = \pm 1,$$

the ones satisfying the boundary points should be selected, that is, the first initial $(0, p_0)$ and the second final $(0, 0)$ ones. For the point $(0, p_0)$ this is a circumference

$$p^2 + (q \mp 1)^2 = p_0^2 + 1,$$

while for point $(0, 0)$ it is a circumference of smaller radius

$$p^2 + (q \mp 1)^2 = 1.$$

It is obvious that the circumferences of the same sign of forces U do not solve the problem; the solution of the equations with different signs should be looked for. According to the lemma, force U can be a controlled one only for $\|U\| \geq \|q\|$. The point of the circumferences' section $p^2 + (q+1)^2 = p_0^2 + 1$ and $p^2 + (q-1)^2 = 1$ will exist for $4q = p_0^2$, that is, for $p_0^2 \leq 8$, and this at the contacting point $p = 0, q = 2$.

Fig. 6

For smaller initial values of the impulse p_0, let's say $p_0 = 2$, it is possible to translate the oscillator to the rest state by a smaller force $|U| < 1$ with no collision. Namely, substituting in the first equation $p_0^2 = 4$, $p = 0$, it is obtained that $q = \sqrt{5} - 1$, and thus,

$$U = (\sqrt{5} - 1)/2 \approx 0.618$$

is brought into the position $p = 0, q = 0$ along the phase trajectory

$$p^2 + q^2 - (\sqrt{5} - 1)q = 0.$$

Theorem on Optimal Motion of Controllable Systems

The concept of *optimal motion* implies here motion of the mechanical systems whose particular attributes have extreme values with respect to some dynamic parameters. These are all the systems of least action and of least compulsion, described in the section five 3C (*Action Principle*) and in the section 3D (*Compulsion Principle*). Regarding the fact that both action and compulsion are of such nature that they reach extreme values at virtual motion, it can be said that the differential equations of the mechanical systems' motion describe extreme lines of action and compulsion. However, though they are indeed optimal motions, they are not usually considered as optimal in the referential literature. Only when, besides the above-given attributes, extreme values of particular and specific dynamic or kinetic properties (such as force, energy, impulse, mass) of controllable motions are looked for, the concept of optimal motion can be used. For this reason, a more specific phrase is used in this study, namely, *optimal motion of controllable mechanical systems* [52], [65], [66].

All the above-mentioned properties of motion for which extreme values will be looked for are set by the functional

$$J = \int_{t_0}^{t_1} \mathcal{F}(p, q, u, t) dt \tag{4.44}$$

which is most often called "criterion of quality" or, simply, "quality" in the literature about controllability theory. Function $\mathcal{F}$ is known and continuous together with the derivatives

$$\frac{\partial \mathcal{F}}{\partial p}, \quad \frac{\partial \mathcal{F}}{\partial q}, \quad \frac{\partial \mathcal{F}}{\partial u}$$

for every point $(p, q) \in T^*\mathcal{N}$ and all the values $u_1, \ldots, u_k$. Besides, $\mathcal{F}(p, q, u, t) \geq a\|Q^*\|^{\mathrm{p}}$ where $a > 0$, $\mathrm{p} > 1$ and Q^* are non-potential forces including the controlling forces.

Out of a great number of forms of differential motion equations let's choose equations (3C.59) and (3C.60) in the form (4.28) and (4.29), that is,

$$\dot{p}_\alpha = -\frac{\partial E}{\partial q^\alpha} + Q^*_\alpha(p, q, u) \tag{4.45}$$

$$\dot{q}^\alpha = \frac{\partial E}{\partial p_\alpha}, \quad (\alpha = 0, 1, \ldots, n), \tag{4.46}$$

where a multitude of forces Q^*_α also includes the controlling forces constrained together with the partial derivatives

$$\frac{\partial \mathcal{F}_\alpha}{\partial p_\beta}, \ \frac{\partial \mathcal{F}_\alpha}{\partial q^\beta}, \frac{\partial \mathcal{F}_\alpha}{\partial u^i}, \quad i = 0, 1, \ldots, k \leq n.$$

Controlling forces Q^*_α, as well as all controlling parameters $u(t) \in L_{\mathrm{p}}$, are available upon finite time interval $[t_0, t_1]$.

For the sake of an explicit understanding of the previous introduction let's accept the following determinations.

Determination. Motion of the controllable mechanical system described by differential equations (4.45) and (4.46) in the presence of the controlling forces will be considered as optimal if, according to the action principle,

$$\int_{t_0}^{t_1} [\delta(p_\alpha \dot{q}^\alpha - E) + Q^*_\alpha \delta q^\alpha] \, dt = 0 \tag{4.47}$$

functional (4.44) achieves its extreme value.

Determination. Generalized direction forces Q^*_α, by which optimal control of motion is realized, will be called optimal control forces, while control parameters $u_0, u_1, \ldots, u_k$ will be considered as optimal control parameters.

The problem of optimal controllable motion is to find dynamic parameters, that is, those forces that translate a controllable mechanical system from state $(p(t_0), q(t_0))$ to state $(p(t_1), q(t_1))$ so that functional (4.44) achieves its extreme value.

Theorem: *Functional* (4.44) *achieves its extreme value at direct motion of the controllable mechanical system from one point* $p(t_0)\&q(t_0) \in T^*\mathcal{N}$ *to the other* $p(t_1)\&q(t_1) \in T^*\mathcal{N}$ *upon:*

- a non-empty multitude of solutions $2n+2$ *of differential equations of motion* (4.45) *and* (4.46),

- a non-empty multitude of solutions of the system $2n+2$ *of differential equations of the variational problem*

$$\begin{aligned}(\delta q^\alpha)^\cdot = \frac{\partial^2 E}{\partial q^\beta \partial p_\alpha}\delta q^\beta + \frac{\partial^2 E}{\partial p_\beta \partial p_\alpha}\delta p_\beta - \\ - \frac{\partial P_\beta}{\partial p_\alpha}\delta q^\beta + \gamma\frac{\partial \mathcal{F}}{\partial p_\alpha},\end{aligned} \tag{4.48}$$

$$\begin{aligned}(\delta p_\alpha)^\cdot = -\frac{\partial^2 E}{\partial q^\alpha \partial q^\beta}\delta q^\beta - \frac{\partial^2 E}{\partial p_\beta \partial q^\alpha}\delta p_\beta + \\ + \frac{\partial P_\beta}{\partial q^\alpha}\delta q^\beta - \gamma\frac{\partial \mathcal{F}}{\partial q^\alpha},\end{aligned} \tag{4.49}$$

- a non-empty set of solutions out of a multitude of k *equations*,

$$\frac{\partial^2 E}{\partial u_r \partial p_\alpha}\delta p_\alpha + \frac{\partial^2 E}{\partial u_r \partial q^\alpha}\delta q^\alpha - \frac{\partial Q^*_\alpha}{\partial u_r}\delta q^\alpha + \gamma\frac{\partial \mathcal{F}}{\partial u_r} = 0 \tag{4.50}$$

$r = 0, 1, \ldots, m-1, m+1, \ldots, k \leq n$ *for* $2n+2$ *conditions*

$$\delta q^\alpha(t_0) = 0,\ \delta q^\alpha(t_1) = 0, \tag{4.51}$$

and

$$\frac{\partial^2 E}{\partial u_m \partial p_\alpha}\delta p_\alpha + \frac{\partial^2 E}{\partial u_m \partial q^\alpha}\delta q^\alpha - \frac{\partial^2 Q^*_\alpha}{\partial u_m}\delta q^\alpha + \gamma\frac{\partial \mathcal{F}}{\partial u_m} < 0, \tag{4.52}$$

Proof. On a non-empty multitude of solutions $p(t)$ and $q(t)$ of differential equations of direct controllable motion (4.45) and (4.46), the action principle is satisfied (4.47).

Functional (4.44) achieves its extreme value at the given motion if for some multiplier $\gamma \in R$

$$\gamma\delta\int_{t_0}^{t_1} \mathcal{F}(p, q, u, t)dt \leq 0 \tag{4.53}$$

and this being minimum for $\gamma < 0$, while maximum for $\gamma > 0$, at condition (4.44), (4.51) in the extended form:

$$\int_{t_0}^{t_1} \left[\left(\dot{q}^\alpha - \frac{\partial E}{\partial p_\alpha} \right) \eta_\alpha - \left(\dot{p}_\alpha + \frac{\partial E}{\partial q^\alpha} - Q_\alpha^* \right) \xi^\alpha \right] dt = 0. \tag{4.54}$$

Let's vary this condition in the following way:

$$\begin{aligned} \int_{t_0}^{t_1} \Bigg[& \left(\delta \left(\dot{q}^\alpha - \frac{\partial E}{\partial p_\alpha} \right) \right) \delta p_\alpha + \left(\dot{q}^\alpha - \frac{\partial E}{\partial p_\alpha} \right) \delta^2 p_\alpha - \\ & - \left(\delta \left(\dot{p}_\alpha + \frac{\partial E}{\partial q^\alpha} - P_\alpha \right) \right) \delta q^\alpha - \\ & - \left(\dot{p}_\alpha + \frac{\partial E}{\partial q^\alpha} - P_\alpha \right) \delta^2 q^\alpha \Bigg] dt = 0. \end{aligned} \tag{4.55}$$

Due to equations (4.45) and (4.46) the members with other variations $\delta^2 p$ and $\delta^2 q$ are dropped. According to conditions (4.51) it follows

$$\int_{t_0}^{t_1} \delta \dot{q}^\alpha \delta p_\alpha dt = \int_{t_0}^{t_1} \delta p_\alpha d\delta q^\alpha = \delta p_\alpha \delta q^\alpha |_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta q^\alpha d\delta p_\alpha =$$

$$= - \int_{t_0}^{t_1} (\delta p_\alpha)^{\cdot} \, \delta q^\alpha dt,$$

$$\int_{t_0}^{t_1} \delta \dot{p}_\alpha \delta q^\alpha dt = \int_{t_0}^{t_1} \delta q^\alpha d\delta p_\alpha = - \int_{t_0}^{t_1} \left(\delta q^\beta \right)^{\cdot} \delta p_\beta dt.$$

Therefore, relation (4.55) is further reduced to:

$$\begin{aligned} \int_{t_0}^{t_1} \Bigg\{ & \left[(\delta q^\beta)^{\cdot} - \frac{\partial^2 E}{\partial p_\beta \partial p_\alpha} \delta p_\alpha - \frac{\partial^2 E}{\partial p_\beta \partial q^\alpha} \delta q^\alpha + \frac{\partial Q_\alpha^*}{\partial p_\beta} \delta q^\alpha \right] \delta p_\beta \\ & - \left[(\delta p_\beta)^{\cdot} + \frac{\partial^2 E}{\partial q^\beta \partial p_\alpha} \delta p_\alpha + \frac{\partial^2 E}{\partial q^\beta \partial q^\alpha} \delta q^\alpha - \frac{\partial Q_\alpha^*}{\partial q^\beta} \delta q^\alpha \right] \delta q^\beta \\ & - \left(\frac{\partial^2 E}{\partial u_r \partial q^\alpha} \delta q^\alpha + \frac{\partial^2 E}{\partial u_r \partial p_\alpha} \delta p_\alpha - \frac{\partial Q_\alpha^*}{\partial u_r} \delta q^\alpha \right) \delta u_r \Bigg\} dt = 0. \end{aligned} \tag{4.56}$$

Relation (4.53) in its extended form

$$\gamma \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{F}}{\partial p_\beta} \delta p_\beta + \frac{\partial \mathcal{F}}{\partial q^\beta} \delta q^\beta + \frac{\partial \mathcal{F}}{\partial u_r} \delta u^r \right) dt \leq 0 \tag{4.57}$$

shows that relation (4.56) also contains the same variations δp_β, δq^β, δu_r as in the condition of achieving extreme values of quality functional (4.53). Due to indefiniteness and arbitrariness, of multiplier γ, by comparison (4.56) and (4.57), it is obtained:

$$\int_{t_0}^{t_1} \Bigg\{ \left[\left(\delta q^\beta\right)^{\cdot} - \frac{\partial^2 E}{\partial p_\beta \partial p_\alpha}\delta p_\alpha - \frac{\partial^2 E}{\partial p_\beta \partial q^\alpha}\delta q^\alpha + \frac{\partial Q^*_\alpha}{\partial p_\beta}\delta q^\alpha - \gamma\frac{\partial \mathcal{F}}{\partial p_\beta}\right]\delta p_\beta$$
$$- \left[(\delta p_\beta)^{\cdot} + \frac{\partial^2 E}{\partial q^\beta \partial p_\alpha}\delta p_\alpha + \frac{\partial^2 E}{\partial q^\beta \partial q^\alpha}\delta q^\alpha - \frac{\partial Q^*_\alpha}{\partial q^\beta}\delta q^\alpha + \gamma\frac{\partial \mathcal{F}}{\partial q^\beta}\right]\delta q^\beta$$
$$- \left(\frac{\partial^2 E}{\partial u_r \partial q^\alpha}\delta q^\alpha + \frac{\partial^2 E}{\partial u_r \partial p_\alpha}\delta p_\alpha - \frac{\partial Q^*_\alpha}{\partial u_r}\delta q^\alpha + \gamma\frac{\partial \mathcal{F}}{\partial u_r}\right)\Bigg\} dt = 0$$

On the non-empty set of solutions of equations (4.48) and (4.49) the previous integral variational relation is reduced to:

$$\int_{t_0}^{t_1} \left(\frac{\partial^2 E}{\partial u_r \partial q^\alpha}\delta q^\alpha + \frac{\partial^2 E}{\partial u_r \partial p_\alpha}\delta p_\alpha - \frac{\partial Q^*_\alpha}{\partial u_r}\delta q^\alpha + \gamma\frac{\partial \mathcal{F}}{\partial u_r}\right)\delta u_r dt \le 0 \tag{4.58}$$

Conditions (4.52) also follow from this relation; hence the theorem is proved.

Corollary 1. If the system constraints do not depend upon time, the number of differential equations (4.45), (4.46), (4.48), (4.49), (4.50) and (4.51) is reduced for one since there is no rheonomic coordinate q^0; neither is there respective impulse p_0, so that indices α and β are given values from 1 to n $(\alpha, \beta = 1, 2, \ldots, n)$.

Lemma. *Functional* (4.44) *achieves its extreme value at the mechanical system's motion directed by unconstrained forces* u_α, *from one motion state* $p(t_0)\&q(t_0)$ *to the other* $p(t_1)\&q(t_1)$ *upon a non-empty multitude of solutions of the equations*:

$$\dot{q}^\alpha = a^{\alpha\beta}, \quad \dot{p}_\alpha = -\frac{\partial E}{\partial q^\alpha} + Q^*_\alpha + U_\alpha \tag{4.59}$$

$$(\delta q^\alpha)^{\cdot} = \frac{\partial^2 E}{\partial q^\beta \partial p_\alpha}\delta q^\beta + a^{\alpha\beta}\delta p_\beta + \gamma\frac{\partial \mathcal{F}}{\partial p_\alpha} - \frac{\partial Q^*_\beta}{\partial p_\alpha}\delta q^\beta \tag{4.60}$$

$$(\delta p_\alpha)^{\cdot} = -\frac{\partial^2 E}{\partial q^\beta \partial q^\alpha}\delta q^\beta - \frac{\partial^2 E}{\partial p_\beta \partial q^\alpha}\delta p_\beta - \gamma\frac{\partial \mathcal{F}}{\partial q^\alpha} + \frac{\partial Q^*_\beta}{\partial q^\alpha}\delta q^\beta \tag{4.61}$$

$$\delta q^\alpha + \gamma\frac{\partial \mathcal{F}}{\partial U_\alpha} = 0, \tag{4.62}$$

and inequalities

$$\delta q^m + \gamma\frac{\partial \mathcal{F}}{\partial u_m} < 0, \quad \delta u_m > 0, \tag{4.63}$$

under the conditions

$$\delta q^\alpha(t_0) = 0, \quad \delta q^\alpha(t_1) = 0. \tag{4.64}$$

Proof. During the control of forces U_α, energy E and force Q^*_α do not depend upon U_α, so that the partial derivatives with respect to U_α vanish from equations and inequations (4.48)–(4.52).

Corollary. For scleronomic systems the number of equations (4.59)–(4.64) is reduced since there is no rheonomic coordinate q^0 so that the indices are $\alpha, \beta = 1, \dots, n$.

Theorem on Optimal Motion Control

The theory of optimal control is based upon more specific classifications of functions and sets of functions or controlling parameters than it is usually the case in standard mechanics. Further on, some concepts and parameters of the previously observed controllable optimal motion state upon $T^*\mathcal{N}$ are more precisely defined.

The phrase *optimal motion control* of the mechanical system implies functions $u(t) \in U$ (U region covers that of manifolds $T^*\mathcal{N}$) which, as generalized forces $Q^* = (Q^*_0, Q^*_1, \dots, Q^*_n)$ or their elements, realize optimal motion at which functional (4.44) reaches its extreme value.

Function $\mathcal{F}$ in functional (4.44) is convex and

$$\mathcal{F}(p, q, u) \geq a\|Q^*\|^{\mathrm{p}}, \quad a > 0, \mathrm{p} > 1. \tag{4.65}$$

All the controls $u(t)$ from L_{p} are accessible at given finite interval $[t_0, t_1]$, that, together with solutions $(p(t), q(t)$ of differential equations of motion (4.45) and (4.46) give finite value to functional (4.44).

Theorem. *For motion of the mechanical system upon* $T^*\mathcal{N}$ *which has*

- *a non-empty set of solutions of the system* $2n + 2$ *of differential motion equations* (4.45) *and* (4.46)*,*

- *a non-empty set of solutions of the system* $2n + 2$ *of equations* (4.48) *and* (4.49) *of the variational problem,*

- *a non-empty set of solutions of the system* (4.5) *at* $2n + 2$ *conditions* (4.51) *and* (4.52)*, as well as the conditions*

$$\|(q^T(t), p^T(t))^T\| \leq B\left(\int\limits_{t_0}^{t_1} \|Q^*(p, q, u, t)\|\, dt\right) < \infty, \tag{4.66}$$

where B *monotonously increases along with the multiplied integral, there are optimal controlling forces* $\tilde{Q}^*_\alpha = Q^*_\alpha(p, q, u^*)$ *for which functional* (4.44) *achieves its extreme value.*

Proof. It follows from relation (4.66) that the motion $(p(t), q(t))$ corresponds to the set of controlling forces constrained by

$$B\left(\int\limits_{t_0}^{t_1} \|Q^*\| dt\right),$$

Due to equations (4.65), that is, because $J \geq 0$, it follows that there is lower boundary of the value of functional J. Let $Q_\alpha^{(k)}$ be the succession of the functions that correspond to accessible kth control $u^{(k)}(t)$ for which the respective successive consequential significance $J(u^{(k)})$ tends towards boundary m; it follows that $J(u^{(k)}) \leq m+1$. For sufficiently great numbers k it further follows that

$$a \int_{t_0}^{t_1} \left\| Q^{(k)}(\cdot, s) \right\|^p ds \leq m+1.$$

That is why such $u^{(k)}$ can be chosen that will weakly tend towards boundary u^* from $L_p(t_0, t_1)$, so that it is

$$a \int_{t_0}^{t_1} \left\| Q^{(k)}(\cdot, s) \right\|^p ds \leq \frac{m+1}{a}.$$

Accordingly, it is

$$\int_{t_0}^{t_1} \| Q^*(p, q, u(t)) \| \, dt \leq \left(\frac{m+1}{a} \right)^{1/p} (t_1 - t_0)^{1/q}$$

where $1/p + 1/q = 1$. According to this and to relation (4.66), all solutions (p_α, q^α) of differential equations of motion (4.45) and (4.46) are uniformly constrained, that is

$$\left\| \left(q^T(t), p^T(t) \right)^T \right\| \leq B \left[\left(\frac{m+1}{a} \right)^{1/\mathrm{p}} (t - t_0)^{1/\mathrm{q}} \right].$$

Uniformly constrained solutions $(q^k(t),\, p^k(t))$ are continuous to the same degree upon interval $t_0 \leq t \leq t_1$, since for every two moments t' and t'' $(t_0 \leq t' \leq t'' \leq t_1)$ there are constants C and D, for which it is:

$$\begin{aligned}
&\| q^{(k)}(t') - q^{(k)}(t'') \| \leq C |t'' - t''| \\
&\| p^{(k)}(t') - p^{(k)}(t'') \| \leq D |t'' - t'| + \\
&\qquad D \left(\int_{t'}^{t''} \left\| Q^*(p^{(k)}(s), q^{(k)}(s), u^{(k)}(s) \right\|^{\mathrm{p}} ds \right)^{1/\mathrm{p}} (t'' - t')^{1/\mathrm{q}} \\
&\qquad \leq D |t'' - t'| + D \left(\frac{m+1}{a} \right)^{1/\mathrm{p}} |t'' - t'|^{1/\mathrm{q}}.
\end{aligned}$$

It is possible to choose such graduality $(q^{(k)(t)}, p^{(k)}(t))$ that it is

$$\lim_{k \to m} (q^{(k)}(t), p^{(k)}(t)) = (\bar{q}(t), \bar{p}(t)), \quad \forall t \in [t_0, t_1].$$

It remains to be proved, from the mechanical standpoint, a clear statement that motion state $(\bar{q}(t), \bar{p}(t))$ corresponds to the right sides of differential equations of motion (4.45) and (4.46), so that it is $Q^*_\alpha = Q_\alpha(p, q, u^*)$, that is

$$\bar{q} = q + \lim_{k\to\infty} \int_{t_0}^{t_1} \frac{\partial E(q^{(k)}, p^{(k)})}{\partial p^{(k)}} ds;$$

$$\bar{p} = p + \lim_{k\to\infty} \int_{t_0}^{t_1} \left[-\frac{\partial E(p^{(k)}, q^{(k)})}{\partial q^{(k)}} + Q^*(p^{(k)}(s), q^{(k)}(s), u^{(k)}(s)) \right] ds.$$

Due to the boundary relations:

$$\lim_{k\to\infty} \int_{t_0}^{t_1} \left\| \frac{\partial E(q^{(k)}, p^{(k)})}{\partial p^{(k)}} - \frac{\partial E(\bar{p}(s), \bar{q}(s))}{\partial \bar{p}} \right\| ds = 0,$$

$$\lim_{k\to\infty} \int_{t_0}^{t_1} \left\| \frac{\partial E(p^{(k)}, q^{(k)})}{\partial q^{(k)}} - \frac{\partial E(\bar{p}(s), \bar{q}(s))}{\partial \bar{q}} \right\| ds = 0, \tag{4.67}$$

$$\lim_{k\to\infty} \int_{t_0}^{t_1} \left\| Q^*(p^{(k)}, q^{(k)}, u^{(k)}(s)) - Q^*(\bar{p}(s), \bar{q}(s), u^*(s)) \right\| ds = 0,$$

that are always uniformly satisfied, except upon some multitude S of arbitrary small measure, as well as due to the inequality

$$\int_S \left\| Q^{(k)}(p^{(k)}, q^{(k)}, u^{(k)}) \right\| ds \leq \left(\int_{t_0}^{t_1} \left\| Q^{(k)}(p^{(k)}, q^{(k)}, u^{(k)}) \right\|^{\mathrm{p}} ds \right)^{1/\mathrm{p}} |s|^{1/\mathrm{q}},$$

which is equivalent to relation (4.67), it follows

$$\bar{q}(t) = q + \lim_{k\to\infty} \int_{t_0}^{t_1} \frac{\partial E(\bar{p}(s), \bar{q}(s))}{\partial \bar{p}} ds,$$

$$\bar{p}(t) = p + \lim_{k\to\infty} \int_{t_0}^{t_1} \left[Q^*(\bar{p}(s), \bar{q}(s), u^*(s)) - \frac{\partial E(\bar{p}(s), \bar{q}(s))}{\partial \bar{q}} \right] ds.$$

This proves that to controlling forces $Q^*(p, q, u^*)$ solutions $\bar{p}_\alpha(t)$ and $\bar{q}^\alpha(t)$ of the system of $2n + 2$ differential equations of motion (4.45) and (4.46) correspond to.

Part of the proof referring to the extreme value of functional (4.44) is identical to the one in proof (4.53) and (4.56) since it is, due to additional statement that F is a convex function of u, so that

$$J(u^*) \leq \lim_{k\to\infty} I(u^k),$$

which shows that $Q^*(p, q, u^*)$ is an optimal controlling force which optimizes the functional to its finite value.

Example 17. The differential equations of motion are:

$$\dot{q} = \frac{p}{a}, \quad \dot{p} = U(t), \quad a = \text{const}, \quad \dim a = \text{M} \tag{E17.1}$$

The optimal controlling force should be determined for which the functional

$$J = \int_{t_0}^{t_1} U^2(\varkappa t)dt, \quad \varkappa \in \mathbb{R}$$

achieves minimum at the transition from motion state $q(t_0 = 0) = 1\,\text{L}$, $p(t_0 = 0) = 2\,\text{M}\,\text{L}\,\text{T}^{-1}$ to rest state $q(t_1 = 1) = 0$, $p(t_1 = 1) = 0$.

Energy $E = \dfrac{p^2}{2a}$, force $Q^*_\alpha = 0$, $F = U^2$, so that equations (4.60)–(4.62) have a simple form:

$$(\delta p)^{\cdot} = 0 \quad \longrightarrow \quad \delta p = c_1 = \text{const} \tag{E17.2}$$

$$(\delta q)^{\cdot} = \frac{\delta p}{a} = \frac{c_1}{a} \quad \longrightarrow \quad \delta q = \frac{c_1}{a}t + c_2$$

$$\delta q + 2\gamma U \leq 0 \quad \longrightarrow \quad U_0 = -\frac{\delta q}{2\gamma} \tag{E17.3}$$

From conditions (4.64) for t_1 it follows

$$c_2 = -\frac{c_1}{a}t_1,$$

so that it is obtained $U = -\dfrac{c_1}{2a\gamma}(t - t_1)$.

Substituting in (E17.1) and integrating thus obtained differential equation

$$\dot{p} = -\frac{c_1}{2a\gamma}(t - t_1)$$

it follows

$$p(t) = -\frac{c_1}{2a\gamma}\left(\frac{t^2}{2} - tt_1\right) + c_3 = -\frac{c_1}{2a\gamma}\left(\frac{t^2}{2} - tt_1\right) + p_0$$

$$q(t) = -\frac{c_1}{2a^2\gamma}\left(\frac{t^3}{6} - \frac{t^2}{2}t_1\right) + \frac{p_0 t}{a} + q_0,$$

$$p_0 = p(t_0), \; q_0 = q(t_0).$$

At the last above given condition $q(t_1) = 0$, $p(t_1) = 0$ it follows

$$c_1 = -\frac{6a^2\gamma\left(\frac{p_0}{a}t_1 + q_0\right)}{t_1^3}.$$

Since it follows from relations (E17.3) that it is $\dim\gamma = \mathrm{M}^{-1}\,\mathrm{T}^2$, so that it is obtained for c_1 constant dimension

$$\dim c_1 = \frac{\mathrm{M}^2\,\mathrm{M}^{-1}\,\mathrm{T}^2 \times (\mathrm{L}+\mathrm{L})}{\mathrm{T}^3} = \mathrm{M\,L\,T}^{-1}$$

which is in accordance with relation (E17.2), so it can be written for $\gamma = -1$, $t_1 = 1$, $p_0 = 2$ and $q_0 = 1$,

$$c_1 = 6(ap_0 + a^2 q_0) = 6a(2+a)(\mathrm{M\,L\,T}^{-1}).$$

Accordingly,

$$U = 3(2+a)(t - t_1),$$

while the functional

$$J = \int_0^1 U^2 dt = 9(2+a)^2\left(\frac{t^3}{3} - t^2 t_1 + t_1^2 t\right) = 3(2+a)^2(\mathrm{M}^2\,\mathrm{L}^2\,\mathrm{T}^{-3})$$

If the inertia coefficient a is taken for unit ($a = 1$), it is obtained

$$J_{\min} = 27(\mathrm{M}^2\,\mathrm{L}^2\,\mathrm{T}^{-3}).$$

Coupling Function

For the sake of writing the system of differential equations of motion (4.45), (4.46), (4.48), (4.49), (4.50) and inequality (4.52) more briefly, as well as their consequential equations (4.59)–(4.63), the function is introduced

$$\mathcal{H} = \frac{\partial E}{\partial p_\alpha}\delta p_\alpha + \frac{\partial E}{\partial q^\alpha}\delta q^\alpha - Q^*_\alpha \delta q^\alpha + \gamma\mathcal{F} \tag{4.68}$$

This function has, as can be seen, a dimension of energy or work

$$\dim\mathcal{H} = \dim E = \mathrm{M\,L}^2\,\mathrm{T}^{-2}.$$

A possible doubt in the possibility of summing up small values

$$\delta E = \frac{\partial E}{\partial p_\alpha}\delta p_\alpha + \frac{\partial E}{\partial q^\alpha}\delta q^\alpha$$

and

$$\delta W = Q^*_\alpha \delta q^\alpha$$

with finite value $\mathcal{F}$ can be eliminated since multiplier γ can be regarded as an arbitrary small parameter or arbitrary small unit concrete (dimensional) number.

Function $\mathcal{H}$ can even be written in a much shorter form as

$$\mathcal{H} = \delta E - Q^*_\alpha \delta q^\alpha + \gamma \mathcal{F} \tag{4.68a}$$

where Q^*_α comprise all non-potential and controlling forces.

For systems with invariable (scleronomic) constraints, energy E is equal to the sum of kinetic energy E_k and potential energy E_p, as well as Hamilton function $H = E = E_k + E_p$, so that the coupling function can be written in the form

$$\mathcal{H} = \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial q^i}\delta q^i - Q^*_i \delta q^i + \gamma \mathcal{F} \tag{4.69}$$

or

$$\mathcal{H} = \delta H - Q^*_i \delta q^i + \gamma \mathcal{F}.$$

If the differential equations of motion are written by means of kinetic energy E_k of generalized forces Q_α, as (3B.40), the coupling function has the same significance as the previous ones, but another form:

$$\mathcal{H} = \frac{\partial E_k}{\partial p_\alpha}\delta p_\alpha + \frac{\partial E_k}{\partial q^\alpha}\delta q^\alpha - Q_\alpha \delta q^\alpha + \gamma \mathcal{F} \tag{4.70}$$

The attribute *coupling* imposes itself since function $\mathcal{H}$ couples the differential equations of the system's motion with the variational problem of optimal motion. By means of this function the above-given equations (4.45)–(4.46) are written in a shorter form:

$$\dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial(\delta q^\alpha)}, \quad \dot{q}^\alpha = \frac{\partial \mathcal{H}}{\partial(\delta p_\alpha)}, \tag{4.71}$$

$$(\delta p)^{\cdot} = -\frac{\partial \mathcal{H}}{\partial q^\alpha}, \quad (\delta q^\alpha)^{\cdot} = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \tag{4.72}$$

$$\frac{\partial \mathcal{H}}{\partial u_r} = 0; \tag{4.73}$$

$$\frac{\partial \mathcal{H}}{\partial u_m} < 0, \quad \forall \delta u_m > 0. \tag{4.74}$$

Accordingly, the theorem and the lemma on optimal control are expressed by means of relations (4.71)–(4.74), that is, by means of the coupling functions.

Example 18. The coupling function from the previous example is

$$\mathcal{H} = \frac{\partial E_k}{\partial p}\delta p - U\delta q + \gamma U^2 = \frac{p}{a}\delta p - U\delta q + \gamma U^2.$$

On Theorems. Theoretical mechanics, as other mathematical sciences, comprises more theorems than it is given here, especially in the control theory, the oscillation theory or the theory about motion stability. Such assertions – theorems – are meaningful as parts of some shorter paper, outside a comprehensive study about body motion and only if, in particular, assertions separated in this way are not proved on the basis of other theorems.

In this study, as it is stated in the beginning of this chapter, the theorem implies an assertion of general significance about body motion whose truthfulness is proved on the basis of definitions and principles of mechanics .

V. ON DETERMINING MOTION
(Analisis and Solutions of Relation of Motion)

The integration of differential equations or of a system of differential equations of motion and of analyses of the solutions obtained for known parameters at some moment of time represents the knowledge about mechanical objects' motion. Very few real motions of the body and, especially, systems of bodies, can be described by finite general analytical solutions of differential equations. Many system models described in the related textbooks do not reflect faithfully the real motion of objects. Still, with great accuracy and with a fairly proper estimate of the error size, mechanics successfully solves problems of all mechanical motions accessible to human eye or even more than that. Many books have been written about it; besides, solutions of new problems are daily published. Still, only a few statements are considered here, namely, those based upon the previous study, especially upon the preprinciples.

On Rectilinear Motion

Two conclusions that follow from differential equation (3A.4) and represent the starting points of Newton's mechanics have to be verified in accordance with the preprinciple of existence.

a) Material points, such as celestial bodies, ballistic projectiles or a thrown body are acted upon by the gravitational force, so that relation (3A.5), according to the present knowledge about forces, does not satisfy the preprinciple of existence; therefore, it cannot be claimed that the bodies move uniformly along straight lines.

If the force of universal gravitation of all the celestial bodies were known at every moment and in every position; and if the reactive motor could instantaneously produce opposite forces, the projectile would move along a straight line with respect to the hypothetical coordinate system $(y, \boldsymbol{e})$.

b) Ships can move on quiet ocean waters at constant velocity, but not in a straight line.

c) It can be arranged locally, on the Earth, with respect to the technical system of measurement related to the Earth, that the reaction and other forces compel a body to move at constant velocity; still, this does not lead to the conclusion about the pathway having a shape of straight line.

A straight line as a concept of plane geometry is not accessible to logical-physical experiment; therefore, it is not necessary to base mechanics upon it, especially since the whole theory can be extended without the principle of rectilinear motion.

Integrals of Material Point's Motion Impulse

For the material point of constant mass and the condition

$$\boldsymbol{F} + \boldsymbol{R} = 0 \tag{5.1}$$

it is obtained from equation (3A.4) that the motion impulse vector is constant, that is,

$$\boldsymbol{p} = m\boldsymbol{v}(t) = \boldsymbol{c} = \mathbf{const} = m\boldsymbol{v}(t_0) = \boldsymbol{p}_0. \tag{5.2}$$

At first sight, it seems to be the simplest first vector integral by which the problem of determining motion is solved:

$$\boldsymbol{r}(t) = \boldsymbol{v}(t_0)t + \boldsymbol{r}(t_0). \tag{5.3}$$

However, a view of relations (1.24) and (1.25), and especially of (3A.39) or (4.6), as well as disagreement about the impulse coordinates, both require that this essential meaning should be much more clarified. Integral (5.2) satisfies and best explains the preprinciple of casual definiteness. With as much accuracy as mass and velocity are known at some moment t_0, motion impulse $\boldsymbol{p}(t)$ can be determined under condition (5.1) at any other moment.

The preprinciple of invariance must be satisfied so that integral (5.2) - essential impulse $\boldsymbol{p}(t)$ - could be sustained in this theory. If vector (5.2) is resolved in coordinate system $(y, \boldsymbol{e})$, as in (1.24), that is

$$\boldsymbol{p} = m\boldsymbol{v} = m\dot{y}^i\boldsymbol{e}_i = c^i\boldsymbol{e}_i = m\dot{y}_0^i\boldsymbol{e}_i$$

and if it is scalarly multiplied by vector $\boldsymbol{e}_j$, it is obtained that

$$p_j(t) = m\dot{y}_j = m\dot{y}_j(t_0) = p_j(t_0). \tag{5.4}$$

Allowing for parallel displacement of base vectors $\boldsymbol{e}_i$, and thus of coordinate vectors $\boldsymbol{g}_k = \partial y^i/\partial x^k\boldsymbol{e}_i$ for free displacement of the point, vector (1.24), that is,

$$\boldsymbol{p} = m\dot{x}^k\boldsymbol{g}_k(x) = m\dot{x}^K(t_0)\boldsymbol{g}_K(x_0) \tag{5.4a}$$

can be scalarly multiplied by vector $\boldsymbol{g}(x)$. That is how projections of integral (5.2) upon coordinate directions $\boldsymbol{g}_l(x)$ are obtained in the form

$$p_l(x,\dot{x}) = a_{kl}(x)\dot{x}^k = a_{kl}(x_0,x)\dot{x}^k(t_0) = a_{Kl}a^{KL}p_L = a_l^L p_L, \tag{5.5}$$

where capital letters in the index denote respective value at the initial moment of time, while the tensor

$$a_{Kl} = m \left(\frac{\partial y}{\partial x^k} \right)_0 \frac{\partial y}{\partial x^l} = m g_{Kl} = m \boldsymbol{g}_K(x_0) \cdot \boldsymbol{g}_l(x) \tag{5.6}$$

represents a *bipoint inertia tensor*. In the referential literature, tensor g_{Kl} can be found as “the tensor of parallel displacement”.

In order to satisfy the preprinciple of invariance, integrals (5.4) and (5.5) should be directly obtained from the coordinate forms of motion equations (3A.13) and (3A.14).

According to the preprinciple of invariance, this relation should also be valid with respect to the curvilinear coordinate system. This is confirmed by integrating the equations (3.14) for $X_j + R_j = 0$. The covariant integral [36], [42] is

$$\hat{\int} a_{ij} D v^j = \hat{\int} D(a_{ij} v^j) = a_{ij} v^j - \mathcal{A}_i = 0, \tag{5.7}$$

where $\mathcal{A}_i$ is covariantly constant covector $\mathcal{A}_i = g_i^K p_K(t_0)$. Accordingly, integral (5.7) is integral (5.5)

$$p_i(t) = a_{ij} \dot{x}^j = a_{iJ} \dot{x}^J = a_{iJ} a^{JK} p_K = g_i^K p_K(t_0), \tag{5.8}$$

Without pointing to the possibility of parallel displacement of covector $\boldsymbol{g}_i$, impulses (5.6) can be translated from the system of y coordinates into x curvilinear coordinates. If x coordinates are denoted by indices $k, l = 1, 2, 3$, it will follow

$$p_j(t) = p_k \frac{\partial x^k}{\partial y^j} = p_j(t_0) = p_K(t_0) \frac{\partial x^K}{\partial y^j}.$$

Multiplying by matrix $\dfrac{\partial y^j}{\partial x^l}$ it is obtained that

$$p_j(t) \frac{\partial y^j}{\partial x^l} = p_K(t_0) \frac{\partial x^K}{\partial y^j} \cdot \frac{\partial y^j}{\partial x^l} = g_l^K p_K(t_0) = p_l(t),$$

since it is

$$g_l^K = \frac{\partial x^K}{\partial y^j} \frac{\partial y^j}{\partial x^l}.$$

Though the covariant integrals satisfy all the three preprinciples, such integration is not widely spread in mechanics due to the “difficulties” in determining tensor g_l^K. That is why the ordinary first integrals reduced to constants are looked for, instead of covariantly-constant integrals.

Let the differential equations of motion (3A.14) be written in the extended form:

$$a_{ij} \frac{D\dot{x}^j}{dt} = \frac{D a_{ij} \dot{x}^j}{dt} = \frac{D p_i}{dt} = \frac{d p_i}{dt} - p_k \Gamma_{ij}^k \frac{d x^j}{dt} = X_i + R_i. \tag{5.9}$$

For the conditions

$$X_i + R_i + p_k \Gamma^k_{ij} \dot{x}^i = 0, \tag{5.10}$$

that are different from conditions (5.1) the ordinary first integrals are obtained

$$p_j(t) = \text{const} = p_j(t_0) \tag{5.11}$$

with respect to coordinate system $(x, \boldsymbol{g})$. Therefore, it is the same as in the case of integral (5.4) in base coordinate system $(y, \boldsymbol{e})$. These integrals considerably differ from integral (5.8) and, therefore, from (5.4). That is why integrals (5.4) and (5.8) will be called covariant integrals. These ordinary integrals (5.11) destroy the tensor nature of the observed objects.

Example 19 (See [36, pp. 47 and 49]). Let's observe the material point's motion with respect to both rectilinear system y^1, y^2, y^3 and cylindrical coordinate system $x^1 := r$, $x^2 := \varphi$, $x^3 := z$.

It is known that [36]

$$y^1 = r\cos\varphi,\ y^2 = r\sin\varphi,\ y^3 = z$$

$$a_{ij} = m \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Bmatrix},$$

$$g_{iK} = \begin{Bmatrix} \cos(\varphi - \varphi_0) & r_0 \sin(\varphi - \varphi_0) & 0 \\ -r\sin(\varphi - \varphi_0) & r r_0 \cos(\varphi - \varphi_0) & 0 \\ 0 & 0 & 1 \end{Bmatrix}.$$

The differential equations of motion and the integrals for

$$Y + R_Y = 0 \quad \Rightarrow \quad X + R_X = 0$$

are

$$\left.\begin{matrix} m\ddot{y}^i = 0 \\ (i = 1,2,3) \end{matrix}\right\} \quad \overset{\tau}{\Leftrightarrow} \quad \begin{cases} m\left[\ddot{x}^1 - x^1\left(\dot{x}^2\right)^2\right] = 0, \\ m\left[x^1\ddot{x}^2 + 2\dot{x}^1\dot{x}^2\right] = 0, \\ m\ddot{x}^3 = 0 \end{cases}$$

$$\downarrow\ \uparrow \qquad\qquad \overset{*}{\downarrow}\ \underset{*}{\uparrow}$$

$$\dot{y}^i = \dot{y}^i_0 \qquad \underset{\tau}{\Leftarrow|\Rightarrow} \qquad \begin{cases} \dot{x}^1 = \sqrt{\left(\dot{x}^1_0\right)^2 + (x^1_0\dot{x}^2_0)^2\left[1 - \left(\dfrac{x^1_0}{x^1}\right)^2\right]}, \\ \dot{x}^2 = \dot{x}^2_0\left(\dfrac{x^1_0}{x^1}\right)^2, \\ \dot{x}^3 = \dot{x}^3_0. \end{cases}$$

By covariant differentiation and covariant integration the equivalence is established at one and the same transformation:

$$\frac{d\dot{y}^i}{dt} = \frac{D\dot{y}^i}{dt} = 0 \quad \Leftrightarrow \quad \frac{D\dot{x}^i}{dt} = 0$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$\dot{y}^i = \dot{y}^i_0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}^1 = \dot{x}^1_0 \cos(x^2 - x^2_0) + \\ \quad + x^1_0 \dot{x}^2_0 \sin(x^2 - x^2_0) \\ \dot{x}^2 = \dfrac{x^1}{(x^1)^2} \left[x^1_0 \dot{x}^2_0 \cos(x^2 - x^2_0) - \right. \\ \quad \left. - \dot{x}^1_0 \sin(x^2 - x^2_0) \right], \\ \dot{x}^3 = \dot{x}^3_0. \end{cases}$$

A shorter, clearer, more general and important difference of the first integrals of the impulses $p_i = c_i$ and the covariant integrals $p_i = \mathcal{A}_i$ shows integration of differential equations (5.4) under the condition that the generalized forces are $Q_i = 0$. Let it be, for the time being, once again motion of one material point in curvilinear system of coordinates x^1, x^2, x^3, that is,

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}^i} - \frac{\partial E_k}{\partial x^i} = 0, \quad (i = 1, 2, 3). \tag{5.12}$$

These equations can be written in the form

$$\frac{D}{dt}\frac{\partial E_k}{\partial \dot{x}^i} = 0. \tag{5.13}$$

From equations (5.12) for

$$\frac{\partial E_k}{\partial x^i} = 0,$$

integrals (5.11) are obtained, while from equations (5.13) covariant integrals (5.8) are obtained, since it is

$$\frac{\partial E_k}{\partial \dot{x}^i} = p_i.$$

Canonical equations (3C.59), as can be seen from

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} + X_i, \quad (i = 1, 2, 3)$$

usually produce integral impulses of the type (5.11) under the condition that the right sides of these equations are equal to zero.

The distribution of the ordinary integral and of integral (5.11) is greater comparing to covariant integrals (5.8). The reason for this mostly lies in insufficiently developed calculation with vectors, that is, tensors. The advantage of ordinary integration is also reflected in the fact that, at smaller number of integral impulses than that of impulse coordinates, constants can be determined depending on the given initial values of the observed impulse, for example,

$$p_1(t) = c_1 = p_1(t_0) \quad \text{and} \quad p_3(t) = c_3 = p_3(t_0);$$
$$p_2 \neq \text{const}\,.$$

This advantage becomes prominent with the system of material points with constraints, and especially upon manifolds TM. Accuracy of both of them is proved, though at various conditions. The covariant integration is invariant with respect to the linear homogeneous transformations of the coordinate systems; thus, it reflects the tensor nature of the integrals. This is not the case with ordinary integration; neither is it in accordance with the preprinciple of non-formality which points to the fact that the final results of the synthesis should be verified by comparing them to the respective results in coordinate systems $(y, \boldsymbol{e})$.

Example 20. Motion impulse integrals along the surface. The differential equations of the material point's motion along surface (3A.29)

$$f(y_1, y_2, y_3, y_0) = 0, \quad f_0 = y_0 - \tau(t) = 0 \tag{5.14}$$

are of form (3A.26) and (3B.53), that is,

$$m\ddot{y}_i = Y_i + \lambda \frac{\partial f}{\partial y^i} \tag{5.15}$$

and

$$\lambda_0 \frac{\partial f_0}{\partial y_0} + \lambda \frac{\partial f}{\partial y_0} = 0. \tag{5.16}$$

From acceleration conditions (3A.34), that is, in the concrete case $(y_i = y^i)$

$$\ddot{f} = \frac{\partial^2 f}{\partial y^k \partial y^l} \dot{y}^k \dot{y}^l + \frac{\partial f}{\partial y_i} \ddot{y}^i + \frac{\partial f}{\partial y_0} \ddot{y}_0 = 0, \tag{5.17}$$

it follows

$$\lambda = -\frac{m(\Phi + \frac{\partial f}{\partial y_0}\ddot{y}_0) + \frac{\partial f}{\partial y_i} Y_i}{\frac{\partial f}{\partial y_i}\frac{\partial f}{\partial y^i}} \tag{5.18}$$

where

$$\Phi = \frac{\partial^2 f}{\partial y_i \partial y_j} \dot{y}_i \dot{y}_j + 2 \frac{\partial^2 f}{\partial y_i \partial y_0} \dot{y}_i \dot{y}_0 + \frac{\partial^2 f}{\partial y_0 \partial y_0} \dot{y}_0 \dot{y}_0. \tag{5.19}$$

It becomes obvious that in the right sides of the differential equations of motion (5.15) there exists inertia force $-m\ddot{y}_0$ in the case that the surface equation (5.14) comprises a time function to the degree different from one, while in the case of the first degree there exists constant velocity $\dot{y}^0$. That is why it is necessary, before integrating differential equations (5.15), to take into consideration this fact in order to obtain accurate motion impulse integrals. In order to stress this important statement, nothing will be lost concerning the general proof if the absence of the resultant of active forces Y_i. ($Y_i = 0$) is assumed. If multiplier (5.18) is also assumed to be equal to zero, there would be motion impulse integrals (5.4). Moreover, if it is assumed that the surface did not change in time, that is, that equation (5.14) is of the form $f(y_1, y_2, y_3) = 0$, it would follow from relation (5.18)

$$\lambda = -m \frac{\frac{\partial^2 f}{\partial y_i \partial y_j} \dot{y}_i \dot{y}_j}{\frac{\partial f}{\partial y_i} \frac{\partial f}{\partial y^i}} = 0, \tag{5.20}$$

which brings us back to considering motion along a double-sided fixed surface. If, however, this surface here changed, it would follow from relations (5.18) and (5.19)

$$\lambda = -m \frac{\frac{\partial f}{\partial y_0} \ddot{y}_0 + 2 \frac{\partial^2 f}{\partial y_i \partial y_0} \dot{y}_i \dot{y}_0 + \frac{\partial^2 f}{\partial y_0 \partial y_0} \dot{y}_0 \dot{y}_0 + \frac{\partial^2 f}{\partial y_i \partial y_j} \dot{y}_i \dot{y}_j}{\frac{\partial f}{\partial y_i} \frac{\partial f}{\partial y^i}} \tag{5.21}$$

while its equalizing with zero would lead to the conclusion that motion impulses are constant at the material point's motion along the surface which moves uniformly and translatory in the absence of forces. However, all the given conjunctions contradict to the preprinciple of existence, to Galileo's laws as well as to the general gravitation law.

Conjunction (5.1) is possible, but, in that case, the multiplier of constraints (5.2) and (5.21) points to a considerable difference between the material point's motion along a fixed, that is, a moveable surface.

With respect to the curvilinear systems of coordinates $(x, \boldsymbol{g})$, constraint equation (5.14) is transformed into

$$f(x^1, x^2, x^3, x^0) = 0, \quad x^0 = \tau(t) \tag{5.22}$$

while the differential equations of motion are transformed into form (3A.14). It is from these equations - along with the assumed conditions - that covariant impulse integrals (5.8) are obtained, while under conditions (5.10) the first integrals of form (5.11) will be obtained. If the observed motion along surface (5.22) is determined by means of equations (3B.40) in which kinetic energy

$$E_k = \frac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta, \quad (\alpha, \beta = 0, 1, 2),$$

while the impulses are $p_0 = a_{0\beta}\dot{q}^\beta$, $p_1 = a_{1\beta}\dot{q}^\beta$, $p_2 = a_{2\beta}\dot{q}^\beta$, three covariant integrals will be obtained, namely,

$$p_\alpha(t) = A_\alpha\left(p(t_0), q(t)\right)$$

under the condition that the generalized forces are equal to zero $Q_\alpha = 0$ or that the first three integrals

$$p_\alpha(t) = c_\alpha = p_\alpha(t_0) \tag{5.23}$$

along with the conditions

$$Q_\alpha + \frac{\partial E_k}{\partial q^\alpha} = 0.$$

In the case that constraints (5.22) do not explicitly depend on time, coordinate q^0 and its respective impulse vanish. Only two impulses (5.23) exist in that case.

Integrals of System Motion Impulse

For an arbitrary system of material points, it is from the theorem on points' impulse change (4.11), that the covariant impulse integrals are obtained

$$p_\alpha = \mathcal{A}_\alpha \left(p(t_0), q(t)\right)$$

where $\mathcal{A}_\alpha$ are covariantly constant vectors if the generalized forces are equal to zero. Since the covariant integrals upon $T\mathcal{N}$ are not elaborated, the first integrals $p_\alpha(t) = c_\alpha = p_\alpha(t_0)$ are looked for; they are obtained in the simplest way from differential equations (3C.58) where from it becomes obvious that there are also first integrals, along with the conditions

$$Q^*_\alpha - \frac{\partial H}{\partial q^\alpha} = 0, \quad (\alpha = 0, 1, \ldots, n).$$

For $\alpha = 0$ from these equations, as well as from (3C.60), it is proved that $p_0 \neq -H$.

Integrals of Body's Rotary Motion Impulse

On the basis of relations (4.5) and (4.8) it follows that there are integrals of the rotary motion impulse of the body with constant mass around a fixed point and with respect to fixed coordinate orthonormal system $(y, \boldsymbol{e})$:

$$p_i = \mathcal{I}_{ij}(t)\omega^j(t) = \mathcal{A}_i = \mathcal{I}_{ij}(t_0)\omega^j(t_0). \tag{5.24}$$

if the moments of forces are $\mathfrak{M}_i = 0$, $(i, j = 1, 2, 3)$

Similarly, from differential equations (4.13) for $\mathfrak{M}_i = 0$ it is obtained that

$$\left.\begin{aligned} p_1 &= \mathcal{I}_{11}\Omega^1 = \mathcal{A}_1 = c_1, \\ p_2 &= \mathcal{I}_{22}\Omega^2 = \mathcal{A}_2 = c_2, \\ p_3 &= \mathcal{I}_{33}\Omega^3 = \mathcal{A}_2 = c_3, \end{aligned}\right\} \tag{5.25}$$

where $c_i = \text{const}$.

By squaring these equations and by summing up it is obtained that (See, for example, [4, p. 74])

$$\left(\mathcal{I}_{11}\Omega^1\right)^2 + \left(\mathcal{I}_{22}\Omega^2\right)^2 + \left(\mathcal{I}_{33}\Omega^3\right)^2 = c^2 \tag{5.26}$$

where c=const; [4].

Energy Integral

The theorem on kinetic energy change (4.15) shows that E_k is equal to the integral

$$E_k = \int P dt + c_1, \tag{5.27}$$

so that it is constant only if the system power P is equal to zero.

Relation (4.30) shows that the total mechanical energy (4.25) is constant, that is,

$$E_k + E_p + \mathcal{P}(q^0) = c_2 \tag{5.28}$$

if the power of non-potential forces is equal to zero. Regarding constraints (4.32), the same integral can be written in the form

$$E_k + E_p = \int R_0(q^0) dq^0 + c_2. \tag{5.29}$$

If it happens that the constraints are invariable, the right side integral (5.29) vanishes; thus, the known integral about energy “preservation” is obtained:

$$E_k + E_p = h = \text{const}\,. \tag{5.30}$$

The difference between integrals (5.29) and (5.30) is comprehensively and clearly presented in References [54]–[64].

Tentative Integrals of the Canonical Differential Equations of Motion

Every function $f_\mu(q^0, \ldots, q^n; p_0, \ldots, p_n)$ or

$$f_\mu(q^0, \ldots, q^n; p_0, \ldots, p_n) = c_\mu = \text{const} \tag{5.31}$$

is an integral of the equations

$$\begin{aligned} \dot{q}^\alpha &= \frac{\partial E}{\partial p_\alpha}, \quad (\alpha = 0, 1, \ldots, n) \\ \dot{p}_\alpha &= -\frac{\partial E}{\partial q^\alpha} + Q^*_\alpha \end{aligned} \tag{5.32}$$

if the derivative with respect to time of function f_μ is equal to zero along the system's phase trajectory, that is,

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial q^\alpha}\frac{\partial E}{\partial p_\alpha} - \frac{\partial f_\mu}{\partial p_\alpha}\frac{\partial E}{\partial q^\alpha} + Q^*_\alpha \frac{\partial f_\mu}{\partial p_\alpha} = 0 \tag{5.33}$$

or

$$(f_\mu, E) + Q^*_\alpha \frac{\partial f_\mu}{\partial p_\alpha} = 0, \tag{5.34}$$

where (f_μ, E) are Poisson's brackets for $T^*\mathcal{N}$.

Example 21. The gyroscopic forces are given by the formula

$$Q^*_\alpha = G_{\alpha\beta}\dot{q}^\beta, \quad G_{\alpha\beta} = -G_{\beta\alpha}.$$

Let's verify if E is an integral of differential equations (5.32). Since it is $(E, E) \equiv 0$ and

$$G_{\alpha\beta}\dot{q}^\beta \frac{\partial E}{\partial p_\alpha} = G_{\alpha\beta}\dot{q}^\beta \dot{q}^\alpha \equiv 0$$

it follows that there is an integral

$$E = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta + E_p(q^0, q^1, \dots, q^n) + \int R_0(q^0) dq^0 = c.$$

Similarly, the existence of the energy integral in the presence of the non-holonomic constraints of form $\varphi_\sigma = b_{\sigma\alpha}(q^0, q^1, \dots, q^n)\dot{q}^\alpha = 0$ can be proved.

Example 22. Hamilton's function $H(p_1, \dots, p_n; q^1, \dots, q^n)$ is not an integral of differential equations (3C.62) in the general case, since it is $(H, E) \neq 0$. Namely, if relations (3C.51) and (3C.61) are kept in mind, it is obtained that:

$$\begin{aligned}(H, H + \mathcal{P}) = (H, H) + (H, \mathcal{P}) = (H, \mathcal{P}) &= \frac{\partial H}{\partial q^\alpha}\frac{\partial \mathcal{P}}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial \mathcal{P}}{\partial q^\alpha} \\ &= \frac{\partial H}{\partial q^i}\frac{\partial \mathcal{P}}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial \mathcal{P}}{\partial q^i} + \frac{\partial H}{\partial q^0}\frac{\partial \mathcal{P}}{\partial p_0} - \frac{\partial H}{\partial p_0}\frac{\partial \mathcal{P}}{\partial q^0} \\ &= -\frac{\partial H}{\partial p_0}\frac{\partial \mathcal{P}}{\partial q^0} = \dot{q}^0 R_0 \neq 0.\end{aligned}$$

Only in the case that the constraints do not depend upon time or that it is $R_0 = 0$, the Hamilton's function appears as an integral of the potential mechanical system.

Example 23. By the composition of differential equations (4.13) with Ω^i, for $I_{ik} = 0$, $(i \neq k)$, $M_i = 0$ or by gradual multiplication of equations (4.15) by respective angular velocities $\Omega^1, \Omega^2, \Omega^3$,), by addition and integration, energy integral is obtained

$$2E_k = I_{11}(\Omega^1)^2 + I_{22}(\Omega^2)^2 + I_{33}(\Omega^3)^2 = h = \text{const}$$

of the body's rotation around the inertia center.

Integration and Preprinciples

In the course of developing the theory of mechanics on the basis of particular principles of mechanics, it has been shown that one and the same kind of motion of one and the same mechanical system can be described by different differential equations with respect to the same or different coordinate systems. For all the given systems of differential equations of motion it has been shown that they are in accordance with the preprinciples. The preprinciple of invariance could be involved in very complex systems of the differential equations of motion due to the developed theory of differential geometry upon manifolds and invariance of the natural ("covariant" or "absolute") derivative of the vector with respect to time.

However, in the calculus and its application to mechanics almost no attention seems to be paid to the question of invariance of the differential expression's integration, namely the differential equations among which the differential equations of motion are most frequent. It has been said that ordinary integration destroys the tensor character of geometrical and mechanical objects; this is not in accordance with the preprinciples, especially those of casual determinacy and invariance. The vector generalization as an ordered set of functions over the vector base which is, in its turn, made up of functions, does not lead to determining the motion attributes in mechanics either by means of differentiating or by means of integration; thus, it is not possible to bring into accord the deduced theory with the preprinciple of casual determinacy on the basis of this generalization. More general theories of knowledge belong to upper levels of mathematics. Example 13 clearly shows the difficulties that are also encountered in dealing with the preprinciple of invariance if the vector base is not definite and known. The still present "truths" are such as "acceleration is not a vector (in tensor sense)", "acceleration vector is not a vector" or "inertia tensor is not a tensor". Such theses have no place in the theory that starts from the preprinciples introduced here, namely, from the preprinciples of existence, casual determinacy and the preprinciple of invariance. There is no one single general configurational ordering in mechanics - namely, there is no one generally ordered set of bodies and their mutual distances; instead, there are many sets and subsets whose motion problems are not solved in one single way, i. d., uniformly, but in many equivalent ways, that is, in polifold or manifold ways. Therefore, the statement *differentiation and integration of tensor on manifolds* is meaningful so long as it is clearly stated what particular manifolds are referred to or if valid proofs are given about invariance of differentiation and integration upon manifolds. The generality speaks about a multitude of variety, so that, regarding the preprinciple of casual determinacy, solutions of general accuracy can be also looked for; they also require definite and general knowledge about the given problem.

A simple integral, for instance

$$f(x) = \int x dx = \frac{1}{2}x^2 + c, \quad c = \text{const}$$

is *indefinite* or *definite to the constant since*, unless some other knowledge about the function $f(x)$ is possessed, the particular curve (pathway, force, energy and so on) cannot be determined from a continuous multitude of curves for each $c \in \mathbb{R}$. Not before one more data about $f(x)$ at any point is known, for instance $f(2) = 2$, can the particular line be known.

Something similar refers to the covariant integrals upon metrical differential manifolds that are, as has been seen, present in mechanics. For the integral [36]:

$$\hat{f} = \hat{\int} g_{ij}(x) v^i(x) dv^j(x) = \frac{1}{2} g_{ij}(x) v^i(x) v^j(x) + \mathcal{A} \tag{5.35}$$

or, much simpler,

$$\hat{f} = \hat{\int} g_{ij}(x) dv^j(x) = g_{ij}(x) v^j + \mathcal{A}_i \tag{5.36}$$

can be said that it is indefinite or definite up to the covariantly constant tensor ($\mathcal{A}_i$ - vector, $\mathcal{A}$ - constant). The required integral can be determined only to the degree of knowledge about manifolds, which also implies that of the metrical tensor $g_{ij}(x)$ and covariantly constant tensor $\mathcal{A}$ at some particular point. Integral (5.36) is of energy integral type (5.30), while integral (5.36) is of impulse type (5.24).

Example 24. A system of N material points of constant masses m_ν ($\nu = 1, \ldots, N$) and $3N-2$ of finite constraints $f_\mu(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = 0$ has a two-dimensional manifolds M^2 whose metrical, or, more precisely, inertia tensor

$$a_{ij} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^j} = a_{ji}(q^1, q^2). \tag{5.37}$$

Differential equations of motion (3B.40a) for $Q_1 = 0,\ Q_2 = 0$ are

$$\frac{D}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^i}\right) = 0,$$

or, regarding that it is

$$\frac{\partial E_k}{\partial \dot{q}^i} = p_i = a_{ij}\dot{q}^j,$$

$$Dp_i = D(a_{ij}\dot{q}^j) = 0.$$

Covariant integral is of the form (5.7), that is,

$$\hat{\int} D(a_{ij}(q)\dot{q}^j) = a_{ij}\dot{q}^j - \mathcal{A}_i = 0 \tag{5.38}$$

where $\mathcal{A}_i$ is a covariantly constant vector, that is

$$D\mathcal{A}_i = d\mathcal{A}_i - \mathcal{A}_k \Gamma^k_{ij} dq^j = 0. \tag{5.39}$$

If M^2 is Euclidean manifolds, the covariantly constant vector can be determined by means of initial conditions $q(t_0)$, $\dot{q}(t_0)$ and autoparallel displacement operator g_i^K, that is, $\mathcal{A}_i = g_i^K p_K(t_0)$. For manifolds of more complex structure, where the method of parallel transition (5.39) induces additional difficulties, other much simpler ways of determining $\mathcal{A}_i$ are required [42]. The difficulties in determining the boundary and initial conditions in solving partial differential equations are not the reason to conclude that "the integral is not correct" or that it is "impossible". By simplifying the example let's observe the point's motion along rotary surface $z = f(\rho)$ with respect to cylindrical coordinate system $(\rho, \varphi, z; \boldsymbol{g})$. In that case, the inertia tensor is

$$a_{ij} = mg_{ij} = m \begin{Bmatrix} 1 + \dfrac{df}{d\rho} & 0 \\ 0 & \rho^2 \end{Bmatrix}.$$

Regarding the fact that the coordinates of this tensor do not depend upon $\varphi =: q^2$, the coordinate $\mathcal{A}_2$ will be constant and determined from the initial conditions. Other coordinate $\mathcal{A}_1$ can be looked for and determined by means of the observed surface's metrics

$$ds^2 = g_{ij} dq^i dq^j \quad \rightarrow \quad g_{ij} \frac{dq^i}{ds} \frac{dq^j}{ds} = 1,$$

that is,

$$a_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} = a^{ij} p_i p_j = h = \text{const}\,.$$

Substituting $\mathcal{A}_2 = c = a_{2j} \dot{q}^j = p_2(t_0)$ and $\mathcal{A}_1 = a_{1j} \dot{q}^j = p_1$ in the previous relation it follows

$$a^{11} (\mathcal{A}_1)^2 + a^{22} (c)^2 = h$$

which determines coordinate $\mathcal{A}_1$, since it is

$$a^{11} = \frac{m}{1 + \left(\dfrac{df}{d\rho} \right)^2}, \quad a^{22} = \frac{m}{\rho^2}.$$

The difficulties in integration of the vector differential equations' system - as the differential equations of the mechanical systems' motion are in essence, regardless of the coordinate form in which they are written - are best expressed by the known Jacobi integral

$$\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L = \text{const}\,. \quad (i = 1, \ldots, N) \tag{5.40}$$

where L is Lagrange's function (3C.21). This integral satisfies the system of differential equations of motion (3C.40a) if kinetic energy does not explicitly depend on time, that is,

$$E_k = \frac{1}{2} a_{ij} (q^1, \ldots, q^n) \dot{q}^i \dot{q}^j$$

while forces $Q_i = -\dfrac{\partial E_p}{\partial q^i}$ are conservative, that is, these forces' work does not depend on the pathway. Then it is

$$\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i = a_{ij}\dot{q}^j\dot{q}^i = 2E_k,$$

so that integral (5.40) is reduced to the energy preservation integral:

$$E_k(t) + E_p(t) = h = E_k(t_0) + E_p(t_0). \tag{5.41}$$

However, integral (5.40) is accepted; while considering the systems whose kinetic energy is of the form

$$E_k = \underbrace{\frac{1}{2}a_{ij}(q)\dot{q}^i\dot{q}^j}_{T_2} + \underbrace{b_i(q)\dot{q}^i}_{T_1} + \underbrace{c(q)}_{T_0},$$

when, instead of (5.41), a different integral is obtained:

$$T_2 - T_0 + E_p = h_1 = \text{const}\,. \tag{5.42}$$

This integral does not follow directly from Theorem (4.15) on kinetic energy change in the mechanical systems whose constraints change in time. This discrepancy occurs due to overlooking equations (3C.40b), which was possible only in the case that it is

$$\frac{\partial E}{\partial q^0} = 0 \quad \text{and} \quad \frac{\partial E_k}{\partial \dot{q}^0} = \text{const} = p_0.$$

Accordingly, this special integral of the rheonomic system is far from the energy preservation integral of the rheonomic system. From Theorem (4.15) or from Lemma (4.23) it follows that for the rheonomic system

$$H = E_k + E_p = \int R_0 dq^0 + C.$$

or

$$E = \frac{1}{2}a_{\alpha\beta}(q^0, q^1, \ldots, q^n)\dot{q}^\alpha\dot{q}^\beta + E_p(q^0, q^1, \ldots, q^n) + \mathcal{P}(q^0) = h_1.$$

It has been proved that n standard Lagrange differential equations (3C.40a) of the second kind are not equivalent to the system of differential equations of the first kind (3A.25); in order to make these two systems compatible, it is necessary to add equation (3C.40b) to the system of standard equations (3C.40a).

On the basis of n differential equations (3C.40a) or their respective $2n$ differential equations (3C.59) the theorem on kinetic energy change (4.15) cannot be proved. In order to prove it, it is necessary to take into consideration equation (3C.40b), that is, relations (3C.60) equivalent to it. In view of this, it is natural to expect that the integrals of different differential equations' systems should be different. This is of great importance for mechanics, since overlooking or neglecting of particular parameters, let alone equations, does not give truthful information about motion. Similar confusion in mechanics is caused by various substitutions of coordinates while integrating or transforming differential equations into simpler forms if the basic conjunctions of the motion theory are not taken into consideration. It is difficult to find any work on mechanics which does not comprise a linear differential equation

$$\ddot{x} + \omega^2 x = 0 \tag{5.43}$$

describing the "harmonic oscillator". Such system of equations, as it is known, describes periodical motion with respect to various ellipsoidal phase pathways, though, in essence, this can only be the mapped motion with respect to phase spirals whose initial differential equations of motion

$$\ddot{z} + 2b\dot{z} + cz = \pm G,$$

where b is a coefficient of the medium resistance, while G is the ratio between dry friction force and mass. Namely, with two substitutions of the coordinates

$$\bar{z} = z \mp \frac{c}{k^2} \quad \text{and} \quad \bar{z} = xe^{-bt}$$

the previous equation comes up to equation (5.43), but not to harmonious motion. Truly, we could say "harmonious oscillations" with respect to function $x(t) = \bar{z}e^{bt}$, but, as can be concluded from function $\bar{z}$ and resistance coefficient b, such motion of the mechanical object is not harmonious. Mechanics is not only based upon mathematical relations but also upon the source of these relations that satisfy its preprinciples. Even the most careful attempt to determine accurate solutions of the differential equations is liable to impermissible errors. The general solution of the differential equations of motion at different initial conditions determine different trajectories. It is for this reason that the solutions of the differential equations should be subjected to verification that, theoretically, mostly comprise a qualitative analysis of the solution or the theory of motion stability, beside practical models used in practice.

I strove to present Lyapunov's results without false modernization.

N. G. Chetaev

VI. ON STABILITY OF MOTION AND REST

Introductory Remarks

"Dynamics is a science about real equilibrium and motions of material systems. Galileo and Newton have discovered its principles and proved their veracity by experimenting with the heavy bodies' fall and by explaining the planets' motion. However, every state of the mechanical system that corresponds to mathematically strict solutions of both the rest equations and the differential equations of motion is not being observed in reality."

"The general principle for choosing solutions that correspond to stable states in mechanics has not been given; instead, the character of science about idealized systems has been accepted and for every strict application to our nature – every time, on principle – solutions of the stability problems were looked for."

"The general problem of motion stability in the classical study was solved by Lyapunov ..." [6]

The above-quoted statements of Nicolaj Gurevich Chetaev are completely in accordance with the preprinciples of this mechanics, whereas the work of V.V. Rumyantsev and A.S. Oziraner [21] best precede a short discussion of the mechanical systems' stability of motion presented in this study.

Differential Equations of Motion

In order to comprise all the mechanical systems, $2n+2$ differential equations (3C.59) and (3C.60) are observed, that is,

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} + Q^*_\alpha \tag{6.1}$$

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad (\alpha = 0, 1, \ldots, n), \tag{6.2}$$

where $H(p_0, p_1, \ldots, p_n; q^0, q^1, \ldots, q^n)$ is the function determined by formula (3C.51), namely,

$$H = \frac{1}{2} a^{\beta\gamma}(q^0, q^1, \ldots, q^n) p_\beta p_\gamma + E_p(q^0, q^1, \ldots, q^n, t) \tag{6.3}$$

In the system of equations (6.1) and (6.2) there are $n+1$ unknown impulses

$$p_\alpha = a_{\alpha\beta}(q^0, q^1, \ldots, q^n)\dot{q}^\beta, \tag{6.4}$$

n unknown and independent generalized coordinates $q^1(t), \ldots, q^n(t)$ and, to the solution of differential equations (6.1), the unknown force of the constraints' change $R_0(q^0)$. Coordinate $q^0(\varkappa, t)$ is given in advance to the accuracy of the chosen parameter.

Inertia matrix $a_{\alpha\beta}$ is positively definite and has rank $n+1$. This is easily proved by means of positively definite function of kinetic energy E_k. If starting from determinants (3B.5), (3B.7) and (3C.31), it can be seen, as in (3C.33) and (3C.49), that kinetic energy

$$E_k = \frac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta \geq 0 \tag{6.5}$$

is a homogeneous quadratic form of generalized velocities $\dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n$ or generalized impulses $p_0, p_1, \ldots, p_n$; it is positive for every $\dot{q}^\alpha \neq 0$, while it is equal to zero only in the case of rest, that is, for $\dot{q}^\alpha = 0$ $(\alpha = 0, 1, \ldots, n)$ or $p_\alpha = 0$. Accordingly, both matrix $a_{\alpha\beta}$ and its inverse matrix $a^{\alpha\beta}$, are positively definite.

Equation (3C.60), that is,

$$\dot{p}_0 = -\frac{\partial H}{\partial q^0} + Q_0^{**} + R_0, \tag{6.6}$$

which, as the only one of the whole system (6.1), comprises function R_0, can be passed over by observing only the system of $2n$ differential equations of motion (6.1). Such a system of differential equations is not complete, namely, it does not completely describe motion of the mechanical system with variable constraints, so that it can be called the system of differential equations of motion with respect to a part of the variables. By excluding additional coordinate q, function (6.5) loses the degree of homogeneity 2, which is not in accordance with the preprinciple of invariance.

When the mechanical systems of material points with the time-independent constraints are dealt with, relations (3B.60) vanish due to the absence of auxiliary coordinate q^0, so that equations (6.1) and (6.2) satisfy the same form:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + Q_i^* \tag{6.7}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad (i = 1, \ldots, n), \tag{6.8}$$

where the function

$$H = \frac{1}{2} a_{ij}(q^1, \ldots, q^n) \dot{q}^i \dot{q}^j + E_p(q^1, \ldots, q^n) \tag{6.9}$$

comprises positively definite matrix $a_{ij} = a_{ji}$ of the rank n.

With the systems with variable masses m the inertia matrix depends, through masses $m_\nu(t)$, on time t as well, as can be seen from relations (1.26), (3A.40) and (3C.33).

Equilibrium State and Position

The concept of *the system's equilibrium state* implies rest of the observed bodies in particular position $q^\alpha = q_0^\alpha = \text{const}$; all the generalized velocities are equal to zero so that, with respect to relation (6.4), generalized impulses are also $p_a = 0$.

The equilibrium state equations, consequently, originate from equations (6.7), that is,

$$\left(Q_\alpha^* - \frac{\partial E_p}{\partial q^\alpha}\right)_{p_\alpha=0} = 0, \tag{6.10}$$

or, in accordance to motion equations (3B.40) and (3B.52),

$$Q_\alpha(\dot{q}, q)\big|_{\dot{q}=0} = 0 \tag{6.11}$$

so that the solutions of equations (6.10) or (6.11) determine the equilibrium state $q_0^\alpha = \text{const}$ of the material system.

Determination 1. *The equilibrium state of the mechanical system implies a set of solutions* $q_0^\alpha \in N$ *of equations* (6.11) *and* $\dot{q}_\alpha(t) = 0$ *or* $\dot{p}_\alpha(t) = 0$.

Determination 2. *The equilibrium position of the mechanical system implies position* $q^\alpha = q_0^\alpha$ *on the coordinate manifolds whose coordinates satisfy equations* (6.11).

Example 25. On a rotary ellipsoid whose equation is in coordinate system $(y, \boldsymbol{e})$

$$f(y,t) = c^2(t)(y_1^2 + y_2^2) + a^2(t)y_3^2 - a^2(t)c^2(t) = 0$$

or with respect to generalized coordinates $q^1 = \varphi$, $q^2 = \theta$, $q^0 = a(t)$,

$$\begin{aligned} y^1 &= q^0 \cos\theta \sin\varphi, \\ y^2 &= q^0 \sin\theta \sin\varphi, \\ y^3 &= c(q^0)\cos\varphi, \end{aligned}$$

there is a point of weight G; c axis of the ellipsoid is vertical, as well as y^3 coordinate.

The equilibrium state of the observed point is determined by $2+1$ equation (6.11), namely:

$$\begin{aligned} Q_1 &= Y_i \frac{\partial y^i}{\partial q^1} = -G\frac{\partial y^3}{\partial \varphi} = Gc\sin\varphi = 0, \\ Q_2 &= 0, \\ Q_0 &= -G\frac{\partial c}{\partial q^0}\cos\varphi + R_0 = 0. \end{aligned}$$

It follows that the equilibrium positions at the given variable constraint

$$\varphi = k\pi \quad (k = 0, 1, 2, \ldots, n)$$

under the condition

$$R_0 = \pm G \frac{\partial c}{\partial q^0}, \quad \text{or} \quad \frac{\partial c}{\partial q^0} = 0 \to R_0 = 0,$$

so that the ellipsoid axis along which the force G is acting does not change.

Deviations from solutions $q^\alpha = q_0^\alpha$ and $p_\alpha = 0$ that can be called undisturbed or given equilibrium state are described by differential equations of motion (6.7) and (6.8) and thus they can be called *differential equations of the disturbed equilibrium state* and, according to (4.1), they can be written in the covariant form

$$\frac{Dp_\alpha}{dt} = Q_\alpha \tag{6.12}$$

$$\dot{q}^\alpha = a^{\alpha\beta} p_\beta \tag{6.13}$$

where it is assumed that the disturbances belong to the medium of the equilibrium state in $T^*\mathcal{N}$, while at the equilibrium state point $q = q_0$, $p = 0$, the right sides of the previous equations are equal to zero:

$$Q_\alpha(q_0, 0, \ldots, 0) = 0, \tag{6.14}$$

$$a^{\alpha\beta} p_\beta = 0. \tag{6.15}$$

That is why previous motion equations (6.7) and (6.8) differ from disturbed equilibrium state equations (6.12)–(6.15).

The equations of the disturbed equilibrium state $\bar{q}^\alpha = q_0^\alpha =: b^\alpha = \text{const}$ can be interpreted with approximate accuracy by means of equations (6.11). For some other values $\bar{q} = b + \Delta q$ and $\dot{\bar{q}} = 0$ forces $\bar{Q}_\alpha$ will not satisfy equations (6.11). The first degree of accuracy

$$\bar{Q}(q, \dot{q})\big|_{\dot{q}=0} = Q(b + \Delta q, 0) = Q(q, 0) + \frac{\partial Q}{\partial q}\Big|_{\substack{q=b,\\ \dot{q}=0}} \Delta q + \cdots$$

due to equations (6.11) is reduced to

$$\overline{Q}_\alpha = \frac{\partial Q_\alpha}{\partial q^\beta}\Big|_{q=b} \Delta q^\beta. \tag{6.16}$$

By analyzing these expressions for solutions $\|\Delta q^\alpha\| \neq 0$, in the sense of the derivatives $\left\| \frac{\partial Q_\alpha}{\partial q^\beta} \right\|_b$, some required conditions about the equilibrium position $q = b$ of the system and its stability can be reached; since they are not reliable enough, much more strict criteria of stability are looked for.

Differential Equations of Disturbed Motion

In the referential literature about bodies' motion the differential equations of disturbed motion do not always imply the same thing, regardless of the fact that the term is general. In the general theory of planet disturbances, these are, in the most general sense, differential equations of motion (See, for instance, [15, p. 53])

$$m_\nu \ddot{\boldsymbol{r}}_\nu = \boldsymbol{F}_\nu + \boldsymbol{G}_\nu \tag{6.17}$$

which the disturbance forces are added to. While describing the system's motion by means of equations (3B.59), when forces Q_i^* are absent, the equations of disturbed motion are found in the form of variation:

$$\begin{aligned} \frac{d}{dt}\delta p_i &= -\frac{\partial^2 H}{\partial q^j \partial q^i}\delta q^j - \frac{\partial^2 H}{\partial p_j \partial q^i}\delta p_j, \\ \frac{d}{dt}\delta q^i &= \frac{\partial^2 H}{\partial q^j \partial p_i}\delta q^j + \frac{\partial^2 H}{\partial p_j \partial p_i}\delta p_j. \end{aligned} \tag{6.18}$$

While attempting to derive the equations of disturbed motion described by covariant equations (6.12), the tensor variation equations[1] have been derived

$$\frac{D^2\xi^i}{dt^2} + R^i_{jkl}\dot{q}^j\xi^k\dot{q}^l = \nabla_l Q^i \xi^l \tag{6.19}$$

that, due to their complex non-linear structure, are not considerably present in the stability theory.[2] Differential equations (6.19) are equivalent to differential equations (6.18) in which $\xi^i := \delta q^i$, while Q^i are generalized forces dependent on position q and time [31, p. 41–47].

In the motion stability theory, the differential equations of disturbed motion are reduced to the general form:

$$\frac{d\xi}{dt} = f(t,\xi), \quad \xi \in R^n \tag{6.20}$$

Equations (6.17) essentially differ from the other given ones; they serve as the basis for elaborating the whole theory of the planet disturbances. All the other above-given systems of differential equations of disturbances are formed of the basic differential equations of motion by being developed into the degree order or by varying the functions and their derivatives.

In [31] it has been proved that the vector projection variation is not equal to the variation vector projection; thus, instead of equations (6.19) *the covariant differential equations of disturbance* are derived in the form

$$\frac{D\eta_\alpha}{dt} = \psi_\alpha(t,\eta,\xi) \tag{6.21}$$

$$\frac{D\xi^\beta}{dt} = a^{\alpha\beta}\eta_\alpha. \tag{6.22}$$

[1] Syng J.L., TENSORIAL METHODS IN DYNAMICS, Toronto, 1936.

[2] Equations (6.19), (6.21), (6.22) are often called the perturbed equations

For the sake of further clarification and estimation of the preprinciples' satisfaction, let's derive the previous equations starting from the basic equations of the dynamic equilibrium (3A.3) or from the theorem on impulse change (4.1), that is,

$$\frac{d}{dt}\left(m_\nu \boldsymbol{v}_\nu\right) = \boldsymbol{F}_\nu(\boldsymbol{r}, \boldsymbol{v}, t). \tag{6.23}$$

The solutions of the undisturbed motion are:

$$\boldsymbol{v}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha \quad \text{and} \quad \boldsymbol{r} = \boldsymbol{r}(q(t)). \tag{6.24}$$

To every other (disturbed) solution

$$\boldsymbol{r}^*_\nu = \boldsymbol{r}_\nu + \xi^\alpha \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha}, \tag{6.25}$$

the corresponding impulse is

$$m_\nu \boldsymbol{v}^*_\nu = m_\nu \frac{d\boldsymbol{r}^*_\nu}{dt} = m_\nu \left(\boldsymbol{v}_\nu + \dot{\xi}^\alpha \frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} + \xi^\alpha \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\beta \partial q^\alpha}\dot{q}^\beta\right)$$

so that the impulse disturbances, according to (1.25) and (3A.39), will be:

$$\begin{aligned} p^*_\gamma - p_\gamma =: \eta_\gamma &= \sum_{\nu=1}^{N} m_\nu \left(\boldsymbol{v}^*_\nu - \boldsymbol{v}_\nu\right) \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} = \\ &= \sum_{\nu=1}^{N} m_\nu \left(\frac{\partial \boldsymbol{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma}\dot{\xi}^\alpha + \xi^\alpha \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\beta \partial q^\alpha} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma}\dot{q}^\beta\right). \end{aligned}$$

However, since there is connection

$$\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^\beta \partial q^\alpha} = \Gamma^\delta_{\alpha\beta}\frac{\partial \boldsymbol{r}_\nu}{\partial q^\delta}, \tag{6.26}$$

regarding relations (1.26) and (3A.41), it follows

$$\begin{aligned} \eta_\gamma &= a_{\alpha\gamma}\dot{\xi}^\alpha + a_{\gamma\delta}\Gamma^\delta_{\alpha\beta}\xi^\alpha \dot{q}^\beta = \\ &= a_{\alpha\gamma}(\dot{\xi}^\alpha + \Gamma^\alpha_{\delta\beta}\xi^\delta \dot{q}^\beta) = a_{\alpha\gamma}\frac{D\xi^\alpha}{dt} \end{aligned} \tag{6.27}$$

or

$$\frac{D\xi^\alpha}{dt} = a^{\alpha\gamma}\eta_\gamma. \tag{6.28}$$

For solutions (6.25) differential equations of motion (6.23) are:

$$\begin{aligned} \frac{d}{dt}(m_\nu \boldsymbol{v}^*_\nu) &= m_\nu(\partial_{\beta\alpha}\boldsymbol{r}_\nu \dot{\xi}^\alpha \dot{q}^\beta + \partial_\alpha r_\nu \ddot{\xi}^\alpha \\ &\quad + \partial_{\delta\alpha\beta}\boldsymbol{r}_\nu \xi^\alpha \dot{q}^\beta \dot{q}^\delta + \partial_{\alpha\beta}\boldsymbol{r}_\nu \dot{\xi}^\alpha \dot{q}^\beta + \partial_{\alpha\beta}\boldsymbol{r}_\nu \xi^\alpha \ddot{q}^\beta) \\ &= F^*_\nu(\boldsymbol{r}_\nu + \boldsymbol{\rho}_\nu, \boldsymbol{v}_\nu + \dot{\boldsymbol{\rho}}_\nu, t), \end{aligned}$$

where

$$\partial_\alpha := \frac{\partial \cdot}{\partial q^\alpha}, \quad \partial_{\alpha\beta} = \frac{\partial^2 \cdot}{\partial q^\alpha \partial q^\beta}.$$

After scalar multiplication of these equations and equations (6.23) by coordinate vectors $\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma}$, as well as after addition with respect to index ν, it is obtained:

$$\begin{aligned}\sum_{\nu=1}^{N} & m_\nu(\partial_\alpha \boldsymbol{r}_\nu \cdot \partial_\gamma \boldsymbol{r}_\nu \ddot{\xi}^\alpha + 2\partial_\gamma \boldsymbol{r}_\nu \cdot \partial_{\alpha\beta} \boldsymbol{r}_\nu \xi^\alpha \dot{q}^\beta + \\ & + \partial_\gamma \boldsymbol{r}_\nu \cdot \partial_{\delta\alpha\beta} \boldsymbol{r}_\nu \xi^\alpha \dot{q}^\beta \dot{q}^\delta + \partial_\gamma \boldsymbol{r}_\nu \cdot \partial_{\alpha\beta} \boldsymbol{r}_\nu \xi^\alpha \ddot{q}^\beta) = \\ & = \sum_{\nu=1}^{N} (\boldsymbol{F}_\nu^* - \boldsymbol{F}_\nu) \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma}.\end{aligned} \tag{6.29}$$

Partial derivatives $\partial_{\delta\alpha\beta}\boldsymbol{r}_\nu$ that exist in the previous relation can be reduced, by means of relation (6.26), to:

$$\begin{aligned}\partial_{\delta\alpha\beta}\boldsymbol{r}_\nu &= \partial_\delta(\partial_{\alpha\beta}\boldsymbol{r}_\nu) = \partial_\delta(\Gamma^\lambda_{\alpha\beta}\partial_\lambda \boldsymbol{r}_\nu) = \\ &= \partial_\lambda \boldsymbol{r}_\nu \partial_\delta \Gamma^\lambda_{\alpha\beta} + \Gamma^\lambda_{\alpha\beta}\partial_{\delta\lambda}\boldsymbol{r}_\nu = \\ &= \partial_\lambda \boldsymbol{r}_\nu \partial_\delta \Gamma^\lambda_{\alpha\beta} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\delta\lambda}\partial_\mu \boldsymbol{r}_\nu.\end{aligned}$$

If these derivatives are taken into consideration, as well as relation (6.26) and inertia tensor (3A.14), equation (6.29) is reduced to the form:

$$\begin{aligned}a_{\alpha\gamma}\ddot{\xi}^\alpha &+ a_{\gamma\lambda}\Gamma^\lambda_{\alpha\delta}\dot{\xi}^\alpha \dot{q}^\delta + a_{\gamma\lambda}\Gamma^\lambda_{\alpha\beta}\dot{q}^\beta \dot{\xi}^\alpha + \\ &+ (a_{\gamma\lambda}\partial_\delta \Gamma^\lambda_{\alpha\beta} + a_{\gamma\mu}\Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\delta\lambda})\xi^\alpha \dot{q}^\beta \dot{q}^\delta + \\ &+ a_{\gamma\lambda}\Gamma^\lambda_{\alpha\beta}\xi^\alpha \ddot{q}^\beta = \Psi_\gamma,\end{aligned} \tag{6.30}$$

where

$$\Psi_\gamma := \sum_{\nu=1}^{N} (\boldsymbol{F}_\nu^* - \boldsymbol{F}_\nu) \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^\gamma} = \Psi_\gamma(\xi, \eta, t). \tag{6.31}$$

Equations (6.30) can be further reduced to a shorter form:

$$a_{\alpha\gamma}\frac{d}{dt}\left(\dot{\xi}^\alpha + \Gamma^\alpha_{\sigma\delta}\xi^\sigma \dot{q}^\delta\right) + a_{\gamma\mu}\Gamma^\mu_{\sigma\beta}\left(\dot{\xi}^\sigma + \Gamma^\sigma_{\alpha\delta}\xi^\alpha \dot{q}^\delta\right)\dot{q}^\beta = \Psi_\gamma$$

or, if equations (6.27) are considered, to the form

$$a_{\alpha\gamma}\frac{d}{dt}\left(\frac{D\xi^\alpha}{dt}\right) + a_{\gamma\mu}\Gamma^\mu_{\sigma\beta}\left(\frac{D\xi^\sigma}{dt}\right)\dot{q}^\beta = \Psi_\gamma,$$

or

$$a_{\alpha\gamma}\frac{D}{dt}\left(\frac{D\xi^\alpha}{dt}\right) = \frac{D}{dt}\left(a_{\alpha\gamma}\frac{D\xi^\alpha}{dt}\right) = \frac{D\eta_\gamma}{dt} = \Psi_\gamma$$

and this, together with equations (6.28), makes up $2n+2$ differential equations of disturbance (6.21) and (6.22) and explains the function vector in them.

Stability of Equilibrium State and Position

The concept of *stability of equilibrium state and position* is not explicit, regardless of the fact that the concepts of equilibrium state and position have been previously defined. The concept of stability is necessarily preceded by explicit determinations [26], [30].

Determination 3. *At any given positive real numbers A^i and B_i – regardless of how small they are not – some positive numbers λ_i and $\overline{\lambda}_i$ can be chosen for all numerical values of the coordinates of equilibrium state $q^i = q_0^i$, $p_i = 0$, that are liable to the constraint*

$$\left|q^i(t_0) - q_0^i\right| \leq \lambda_i, \quad |p_i(t_0)| \leq \overline{\lambda}_i, \tag{6.32}$$

and for every time $t > t_0$ satisfy the inequalities

$$\left|q^i(t) - q_0^i\right| < A^i, \quad |p_i(t)| < B_i \tag{6.33}$$

equilibrium state $(q^i = q_0^i; p_i = 0)$ of the system is stable with respect to disturbances $q^i \neq q_0^i$ and $p_i \neq 0$; otherwise, it is unstable.

The previous determination 1 can be formulated in other words or other relations, but the meaning of the disturbance's constraints (6.32) and (6.33) should remain the same. By an appropriate choice of the coordinate system origin in equilibrium state, the equilibrium state can be represented by the zero point upon manifoldness $T^*\mathcal{N}$, that is $q^\alpha = 0$, $p_\alpha = 0$; then equation (6.14) is reduced to

$$Q_\alpha(0, \ldots, 0, t) = 0 \tag{6.34}$$

Determination 4. *If at any randomly given number $A > 0$, regardless of how small it is not, such a real number λ can be chosen for which all the initial disturbances are constrained by the relation*

$$\delta_{\alpha\beta} q^\alpha(t_0) q^\beta(t_0) + \delta^{\alpha\beta} p_\alpha(t_0) p_\beta(t_0) \leq \lambda, \tag{6.35}$$

and for every $t \geq t_0$ the inequality is satisfied

$$\delta_{\alpha\beta} q^\alpha q^\beta + \delta^{\alpha\beta} p_\alpha p_\beta < A, \tag{6.36}$$

the undisturbed equilibrium state $p_\alpha = 0$, $q^\alpha = 0$ is stable; otherwise, it is unstable.

As in the previous proposition, $\delta_{\alpha\beta}$ and $\delta^{\alpha\beta}$ are Kronecker's symbols.

If the stability of the equilibrium state or of the undisturbed motion is regarded only with respect to part of $2m$ of variables $q^1, \ldots, q^m, p_1, \ldots, p_m$, $m < n$, the stability condition (6.36) is reduced to the observed variables:

$$\delta_{kl} q^k q^l + \delta^{kl} p_k p_l < A \qquad (k, l = 1, \ldots, m) \tag{6.37}$$

Stability Criterion

If for the differential equations of motion of the scleronomic system (6.12) *and* (6.13) *the positively definite function* $W(t, q^1, \ldots, q^n)$ *could be found, such that it is*

$$\frac{\partial W}{\partial t} + \left(Q_i + \frac{\partial W}{\partial q^i}\right)\dot{q}^i \leq 0 \quad (i = 1, \ldots, n) \tag{6.38}$$

the equilibrium state $q = q_0$, $p = 0$ *or* $q = 0$, $\dot{q} = 0$ *is stable.*

Proof. With the conjunction that there is function W, the function

$$V = \frac{1}{2}a^{ij}(q^1, \ldots, q^n)p_i p_j + W(q^1, \ldots, q^n, t) \tag{6.39}$$

is positively definite since kinetic energy

$$E_k = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j = \frac{1}{2}a^{ij}p_i p_j$$

is, by its definition, positively definite.

The derivative with respect to time of function (6.39) is since it is

$$\dot{V} = a^{ij}\frac{Dp_i}{dt}p_j + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial q^i}\frac{dq^i}{dt},$$

while

$$\dot{V} = \frac{dV}{dt} = \frac{DV}{dt},$$

If equations (6.12) and (6.13) are kept in mind, the previous derivative is reduced to the form

$$\begin{aligned} \frac{\partial W}{\partial t} + a^{ij}Q_i p_j + \frac{\partial W}{\partial q^i}a^{ij}p_j &= \frac{\partial W}{\partial t} + a^{ij}\left(Q_i + \frac{\partial W}{\partial q^i}\right)p_j \\ &= \frac{\partial W}{\partial t} + \left(Q_i + \frac{\partial W}{\partial q^i}\right)\dot{q}^i, \end{aligned} \tag{6.40}$$

and the criterion is proved by this.

Corollaries

1. If the system is autonomous, function W should be looked for only depending upon the coordinates, so that condition (6.38) is reduced to

$$\left(Q_i + \frac{\partial W}{\partial q^i}\right)\dot{q}^i \leq 0. \tag{6.41}$$

That is what the conservative mechanical systems for which there exists potential energy $E_p(q^1, \ldots, q^n)$ are like. The choice of this very energy, if it is positively definite, for function W, $W = E_p$, shows that it is

$$a^{ij}\left(-\frac{\partial E_p}{\partial q^i} + \frac{\partial W}{\partial q^i}\right)p_j = \left(-\frac{\partial W}{\partial q^i} + \frac{\partial W}{\partial q^i}\right)\dot{q}^i \equiv 0$$

namely, that the equilibrium state of the system is stable.

2. If the generalized forces consist of conservative and any other forces $Q_i^*(q,\dot{q})$, that is,

$$Q_i = -\frac{\partial E_p}{\partial q^i} + Q_i^*(q,\dot{q})$$

by the repeated choice $W = E_p$, as the condition of the system's equilibrium state stability, it is obtained as

$$a^{ij} Q_i^* p_j = Q_i^* \dot{q}^i \leq 0. \tag{6.42}$$

3. If the system is acted upon by gyroscopic forces

$$Q_i^* = G_{ij}\dot{q}^j = -G_{ji}\dot{q}^j \tag{6.43}$$

the equilibrium state stability condition (6.41) is satisfied since it is

$$Q_i^* \dot{q}^i = G_{ij}\dot{q}^j \dot{q}^i \equiv 0$$

4. For dissipating forces $Q_i^* = b_{ij}\dot{q}^j$ condition (6.41) is reduced to the fact that the quadratic function of energy dissipation $\mathcal{R} = -b_{ij}\dot{q}^i\dot{q}^j$ should be either greater or equal to zero.

Generalization of the Criterion. The previous theorem is also valid for mechanical systems with rheonomic constraints. Condition (6.38) changes only if indices $i, j = 1, \ldots, n$ take on values $\alpha, \beta = 0, 1, \ldots, n$. Therefore, three additional addends are obtained:

$$\begin{aligned}\frac{\partial W}{\partial t} + a^{\alpha\beta}\left(Q_\alpha + \frac{\partial W}{\partial q^\alpha}\right)p_\beta &= \frac{\partial W}{\partial t} + a^{ij}\left(Q_i + \frac{\partial W}{\partial q^i}\right)p_j + \\ &+ a^{i0}\left(Q_i + \frac{\partial W}{\partial q^i}\right)p_0 + a^{0j}\left(Q_0 + \frac{\partial W}{\partial q^0}\right)p_j \\ &+ a^{00}\left(Q_0 + \frac{\partial W}{\partial q^0}\right)p_0 \leq 0\end{aligned} \tag{6.44}$$

The proof is identical to the previous one, except for the fact that the indices in equations (6.12) and (6.13) remain in the range $0, 1 \ldots, n$.

When the system of forces (3C.52) is considered, where potential energy $E_p = E_p(q^0, q^1, \ldots, q^n)$, and $Q_0^* = Q_0^* + R_0$, function $W = E_p$ can be chosen if E_p is a positively definite function of $q^0, q^1, \ldots, q^n$, so that expression (6.43) is reduced to

$$a^{\alpha\beta} Q_\alpha^* p_\beta = Q_\alpha^* \dot{q}^\alpha = Q_i^* \dot{q}^i + (Q_0^{**} + R_0)\dot{q}^0 \leq 0.$$

Corollaries

1. Expressions (6.38)–(6.43) appear as consequences of relations (6.44) in the case that the constraints are scleronomic since auxiliary coordinate q^0 vanishes.

2. The classical (standard) way of examining stability of the rheonomic system's equilibrium state with respect to variables $q^1, \dots, q^n; p_1, \dots, p_n$ can be regarded as stability with respect to a part of the variables.

Necessary Additional Commentary

While verifying criteria (6.38) or (6.44) the starting point has been the fact that function (6.39), that is

$$V = E_k + W(t, q^0, q^1, \dots, q^n) \tag{6.45}$$

is a positively definite function. Since the starting conjunction is that W is a positively definite function, while E_k is kinetic energy, there should be no disagreement about the casual definiteness of function V. Still, the question is asked concerning casual definiteness of kinetic energy. In order to prove this, the starting point is, firstly, the preprinciple of invariance which states that motion attributes do not depend upon formal mathematical description, and, secondly, from the expression for the system's kinetic energy

$$2E_k = m_1 v_1^2 + m_2 v_2^2 + \dots + m_N v_N^2 = \sum_{\nu=1}^{N} m_\nu \boldsymbol{v}_\nu \cdot \boldsymbol{v}_\nu. \tag{6.46}$$

All masses m_ν are positive concrete real numbers, so that it cannot be refuted that E_k is a positive function of $\boldsymbol{v}_\nu$ which is equal to zero only if all the velocities, that is, functions $\boldsymbol{v}_\nu(t)$ are equal to zero. Therefore, it is true that:

$$2E_k = \sum_{\nu=1}^{N} m_\nu \boldsymbol{v}_\nu \cdot \boldsymbol{v}_\nu \geq 0. \tag{6.47}$$

With respect to the orthonormal coordinate system, regarding expression (3B.7), it also follows

$$2E_k = \sum_{\nu=1}^{N} m_\nu (\dot{y}_{\nu 1}^2 + \dot{y}_{\nu 2}^2 + \dot{y}_{\nu 3}^2) \geq 0. \tag{6.48}$$

Nothing is going to change if other notations are introduced:

$$m_{3i} = m_{3i-1} = m_{3i-2}; \quad i = 3\nu - 2,\ 3\nu - 1,\ 3\nu$$

since relation (6.48) will be

$$2E_k = \sum_{i=1}^{3N} m_i \dot{y}_i^2 \geq 0. \tag{6.48a}$$

In other coordinate systems, let's say $(z, \boldsymbol{e})$ or $(x, \boldsymbol{g})$ between which there are explicit dotted mappings $y^i = y^i(z^1, \dots, z^{3N})$, $y^i = y^i(x^1, \dots, x^{3N})$ or constraints

$y^i = y^i(q^0, q^1, \ldots, q^n; n < 3N)$, the quadratic homogeneous form (6.48) will remain what it is in the forms:

$$\begin{aligned}\sum_{i=1}^{3N} m_i \dot{y}_i^2 &= \sum_{i=1}^{3N} m_i \frac{\partial y_i}{\partial z^k}\frac{\partial y_i}{\partial z^l}\dot{z}^k \dot{z}^l = \sum_{i=1}^{3N} m_i \frac{\partial y_i}{\partial x^k}\frac{\partial y_i}{\partial x^l}\dot{x}^k \dot{x}^l \\ &= \sum_{i=1}^{3N} m_i \frac{\partial y^i}{\partial q^\alpha}\frac{\partial y^i}{\partial q^\alpha}\dot{q}^\alpha \dot{q}^\beta \\ &= ə_{kl}\dot{z}^k\dot{z}^l = a_{kl}(m,x)\dot{x}^k\dot{x}^l = a_{\alpha\beta}(m,q)\dot{q}^\alpha\dot{q}^\beta \geq 0,\end{aligned}$$

where $ə_{kl}$, a_{kl}, $a_{\alpha\beta}$ are positively definitive matrices. “The deviation from the matrices' casual definiteness” for particular values x or q does not spring from the kinetic energy's nature, but from irregularity of the transformation matrices (6.20) $\left(\dfrac{\partial y_i}{\partial x^k}\right)$ or $\left(\dfrac{\partial y_i}{\partial q^k}\right)$ during the transition from one to other coordinates. For those values of coordinates x for which transformations $\dot{y}^i = \dfrac{\partial y^i}{\partial x^\alpha}\dot{x}^\alpha$ are irregular (therefore, non-existent), neither the casual definiteness of the matrix a_{ij} nor the kinetic energy's coordinate forms can be estimated.

Example 26. Kinetic motion energy of the point of mass m in the plane can be written with respect to cylindrical coordinate system ρ, θ, z, for which there are relations

$$y_1 = \rho\cos\theta,\ y_2 = \rho\sin\theta,\ y_3 = z,$$

under the condition $\rho \neq 0$, in the form

$$E_k = \frac{m}{2}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) \geq 0 \tag{6.49}$$

or on the plane $z = c = \text{const}$,

$$E_k = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right).$$

All the three expressions for E_k are equal to zero only if the velocities are equal to zero since $\rho = 0$ cannot be taken into consideration in view of the fact that $\rho = 0$ is excluded from consideration during the transformation between the observed coordinate systems.

Invariant Criterion of Motion Stability

The concept of the *invariant criterion* implies general measurement standard in all the coordinate systems for estimating stability of some undisturbed mechanical system's motion. As such, it comprises stability of the equilibrium position and state, stability of stationary motions and, in general, of motion of mechanical systems whose disturbance equations are of coordinate shape (6.21) and (6.22).

If for the differential equations of disturbance (6.21)–(6.22) *there is such a positively definitive function* W *of disturbance* $\xi^0, \dots, \xi^n$ *and time* t *that the expression is*

$$\frac{\partial W}{\partial t} + a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \leq 0 \tag{6.50}$$

smaller or equal to zero, the undisturbed state of the mechanical system's motion is stable.

Proof. As can be seen from equation (6.31), functions Ψ_α for undisturbed motion $\xi^\alpha = 0$, $\eta_\alpha = 0$ are equal to zero, that is, $\Psi_\alpha(0, 0, t) = 0$.

The function

$$V = \frac{1}{2}a^{\alpha\beta}\eta_\alpha\eta_\beta + W(\xi, t) \tag{6.51}$$

is positively definite, since it is

$$a^{\alpha\beta}(q^0(t), q^1(t), \dots, q^n(t))$$

a positively definite matrix of the functions upon M^{n+1}, while $W(\xi, t)$ is a positively definite function of disturbance ξ^α. As a scalar invariant, V is a tensor of zero order. That is why ordinary derivative $\frac{dV}{dt}$ is equal to the natural derivative

$$\frac{DV}{dt} = a^{\alpha\beta}\frac{D\eta_\alpha}{dt}\eta_\beta + \frac{\partial W}{\partial \xi^\alpha}\frac{D\xi^\alpha}{dt} + \frac{\partial W}{\partial t} \tag{6.52}$$

which necessarily has to be smaller or identical to zero. By substitution of the natural derivatives from equations (6.21) and (6.22) in (6.52) it is obtained that

$$\frac{DV}{dt} = \frac{\partial W}{\partial t} + a^{\alpha\beta}\Psi_\alpha\eta_\beta + \frac{\partial W}{\partial \xi^\alpha}a^{\alpha\beta}\eta_\beta,$$

and this, along with the criterion requirement, is reduced to

$$\frac{\partial W}{\partial t} + a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \leq 0. \tag{6.53}$$

Therefore, the stability criterion is proved [47].

If neither forces $\boldsymbol{F}_\nu^*$ and $\boldsymbol{F}$ from relations (6.31) nor differences $\boldsymbol{F}_\nu^* - \boldsymbol{F}_\nu$ depend of time t on position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$, function Ψ_γ will also be explicitly independent of t. Then function W should also be looked for only in its dependence on disturbances $\xi^0, \xi^1, \dots, \xi^n$, that is, $W = W(\xi^0, \xi^1, \dots, \xi^n)$, so that expressions (6.50) and (6.53) are reduced to

$$a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \leq 0. \tag{6.54}$$

If the mechanical system's constraints do not depend on time, $q^0, \xi^0, \eta_0, \Psi_0$, vanish, so that expression (6.50), that is (6.53), is reduced to

$$\frac{\partial W}{\partial t} + a^{ij}\left(\Psi_i + \frac{\partial W}{\partial \xi^i}\right)\eta_j \leq 0, \tag{6.55}$$

while expression (6.54) is reduced to

$$a^{ij}\left(\Psi_i + \frac{\partial W}{\partial \xi^i}\right)\eta_j \leq 0 \tag{6.56}$$

where Ψ_i and W do not depend on ξ^0 and η^0.

All the expressions of the previously given criterion for the equilibrium state stability appear as consequences of expression (6.53) if ξ and η are regarded as disturbances of equilibrium state q and p.

On Integrals of Covariant Equations of Disturbance

Covariant equations of motion (6.12) or differential equations of disturbance respective to them (6.21) in their extended form and in the general case have a very complex structure what makes their integration difficult. However, by applying the covariant integration some first covariantly constant integrals are obtained as a means of assessing the equilibrium state stability as well as undisturbed motion. As an addition to this assertion, the two recognizable and acceptable examples are presented here.

1. Let generalized forces Q_α in equations (6.12) have a function of force $U(q^0, q^1, \ldots, q^n)$. Let's multiply each equation (6.12) by respective differential from equation (6.13) and add in the following way,

$$a^{\alpha\beta} p_\beta D p_\alpha = Q_\alpha dq^\alpha = \frac{\partial U}{\partial q^\alpha} dq^\alpha.$$

Since $Da^{\alpha\beta} = 0$, it is

$$\frac{1}{2} D\left(a^{\alpha\beta} p_\beta p_\alpha\right) = dU$$

and further

$$\frac{1}{2} a^{\alpha\beta} p_\beta p_\alpha - U = C = \text{const}\,.$$

2. Let the right sides of covariant equations (6.21) be linear forms of disturbance from $\xi^1, \ldots, \xi^n$, that is,

$$\Psi_i = -g_{ij}(q^1(t), \ldots, q^n(t))\xi^j$$

where g_{ij} as well as $a^{ij}(q^1, \ldots, q^n)$ are covariantly constant tensor. For the given disturbances, equations (6.21) and (6.22) can be written in the covariant form:

$$\frac{D\eta_i}{dt} = g_{ij}\xi^j, \frac{D\xi^i}{dt} = a^{ij}\eta_j.$$

By mutual complete multiplication and addition with respect to index i, as in the previous example with respect to α, it follows

$$a^{ij}\eta_j D\eta_i = -g_{ij}\xi^j D\xi^i.$$

The covariant integration gives

$$\frac{1}{2}(a^{ij}\eta_j\eta_i - g_{ij}\xi^j\xi^i) = \mathcal{A}$$

where $\mathcal{A}$ is a constant, $D\mathcal{A} = d\mathcal{A} = 0$.

Therefore, by covariant or ordinary integration and the solution analysis or directly by applying criterion (6.50) or (6.38), the stability of undisturbed motion $\xi = 0$, $\eta = 0$ or that of the equilibrium state of system $q = q_0$, $p = 0$ can be assessed.

Along with additional conditions, the initial definitions can be used to speak about *asymptotic, uniform, equiasymptotic or any similar stability on the whole or regarding part of the variables.* As for stability of motion or equilibrium of the mechanical systems it is more important to note whether disturbances in the disturbances equations are caused by some error in calculation or they result from some newly-induced change of forces, namely, of inertia force due to inertia tensor change, or of active forces due to approximate accuracy of dynamic parameters, instead of ideally accurate laws of dynamics from which formulae of particular forces such as (2.16), (2.11), (2.13) and (2.14) are derived. In this study, the laws of dynamics are formulated on the basis of stable processes in the sense of the above-given definitions about stability; more precisely, it means that they are formulated accurately to the point of the boundary value of the given number, regardless of how small it is not. In differential equations (6.12) and, especially (6.21) every deviation of functions or their parameters (no matter how small they are) from the real ones can but it does not have to affect stability or instability of the observed motion. That is why the mechanical systems' stability with respect to forces is of enormous importance.

AFTERWORD

The title of this monograph as well as the selection of the given contents for each of its sections can be regarded as an introduction to more comprehensive works in the field of mechanics. What is, mostly and briefly, given here, though, in the author's opinion, are only essential assertions of one theory of motion and interaction of bodies. Not a priori assertion, but inherited, existing and acquired knowledge was the starting point. The acquired knowledge suppressed some inherited and existing logical and mathematical standards thus making relative the accuracy of the most accurate natural science in the mathematical sense. This can particularly be seen in the following relations with variable constraints:

Standard	Modification
Velocity	
$\boldsymbol{v} = \dfrac{\partial \boldsymbol{r}}{\partial q}\dot{q} + \dfrac{\partial \boldsymbol{r}}{\partial t}$ $\dot{q} = (\dot{q}^1, \ldots, \dot{q}^n)^T$	$\boldsymbol{v} = \dfrac{\partial \boldsymbol{r}}{\partial q}\dot{q} + \dfrac{\partial \boldsymbol{r}}{\partial q^0}\dot{q}^0$ $\dot{q} = (\dot{q}^0, \dot{q}^1, \ldots, \dot{q}^n)^T$
Motion impulse	
$p_i = a_{ij}\dot{q}^j + b_i$ $p_0 =: -H$	$p_i = a_{ij}\dot{q}^j + a_{i0}\dot{q}^0$ $p_0 = a_{0j}\dot{q}^j + a_{00}\dot{q}^0$
Acceleration	
$a^i = \dfrac{D\dot{q}^i}{dt}$	$a^i = \dfrac{D\dot{q}^i}{dt},\ a^0 = \dfrac{D\dot{q}^0}{dt}$
Forces	
Q	$Q,\ Q_0$
Work	
$W(Q) = \int\limits_s Q dq$	$W = \int\limits_s (Q dq + Q_0 dq^0)$
Variational principles	
$\delta \int\limits_{t_0}^{t_1} E_k dt = 0,\ \delta \int\limits_{t_0}^{t_1} L dt = 0$	$\int_{t_0}^{t_1} (\delta E_k + \delta A(\boldsymbol{F}))dt = 0.$

Kinetic energy

$$E_k = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j + b_i\dot{q}^i + c \qquad\qquad E_k := -W(I) = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta$$

$$|a_{ij}|_n^n \qquad\qquad |a_{\alpha\beta}|_{n+1}^{n+1}$$

Dieferential equations of motion

$$a_{ij}\frac{D\dot{q}^j}{dt} = 0 \qquad\qquad a_{ij}\frac{D\dot{q}^j}{dt} + a_{i0}\frac{D\dot{q}^0}{dt} = Q_i$$

$$a_{0j}\frac{D\dot{q}^j}{dt} + a_{00}\frac{D\dot{q}^0}{dt} = Q_0$$

or

$$\frac{D}{dt}\left(\frac{\partial L}{\partial\dot{q}^i}\right) = 0 \qquad\qquad \frac{D}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) = 0$$

$$\frac{D}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^0}\right) = 0$$

or

$$\dot{q}^i = \frac{\partial H}{\partial p_i},\ \dot{q}^0 = 1 \qquad\qquad \dot{q}^i = \frac{\partial E}{\partial p_i},\ \dot{q}^0 = \frac{\partial E}{\partial p_0}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i},\ \dot{p}_0 = -\frac{\partial H}{\partial t} \qquad\qquad \dot{p}_i = \frac{\partial E}{\partial q^i},\ \dot{p}_0 = \frac{\partial E}{\partial q^0}$$

$$p_0 = -H \qquad\qquad p_0 \neq -H \neq E$$

Regarding these and similar comparisons the author has been posed some important and logical questions at various scientific conferences, namely questions like “Do you find assertions made so far in the standard mechanics erroneous?” or “Assuming that your assertions are correct, how do you explain that they have not been noticed in practice?”

Avoiding the word “erroneous” the author has replied that the assertions made in this theory are better and more thorough. From Aristotle and Galileo, that is, Newton, the contention has been accepted that the body moves uniformly under the action of constant force. When Newton wrote his first axiom or law that the body moves uniformly or rectilinearly in absence of forces, philosophy considered and assessed that Aristotle’s view was erroneous. Such a rough assessment was not given by Newton; neither did Einstein state that the proposition about rectilinear motion was erroneous; instead, Einstein found a more complete and finer statement that “rectilinear motion does not spring from experience either logically or experimentally”. Example (E5.7) simply gives an answer to the second question - though such object of mechanics is taken into consideration without including axial forces; therefore, there is always a possibility of displacing one end in practice; in other words, it has been more than just noticed in practice. In this monograph the mathematical knowledge that can be applied to the theory about motion of body is extended; and thus, some other views of particular attributes of motion appear. The innovations with respect to describing known and accepted relations are stressed in more details. Thus, for instance, the concept of the *material point* is differentiated in details from the concept of the *particle* or *covariant integration* from *standard integration* of differential equations of the rigid body's rotary motion. It has been shown that the model of material point can be used to develop the theory applicable to all mechanical objects.

The section on **Preprinciples** that precedes the core of the book gives an explicit determination of the starting conjunction in mechanics as well as its basic concepts such as *mass, distance* and *time*; this defines its domain of research by means of three disjunctive sets of real numbers and pencils of three oriented vectors; the concept of geometrical spaces is abandoned, unlike that of the body volume; the possibility of two particle's coincidence is excluded, namely, the fact that, in the geometrical sense, differs the concept of the particle from both the material and the geometrical point while, at the same time, makes the "law of non-penetration" redundant. The possibility of determining motion is accepted in advance, while the accuracy is made relative by available knowledge of the relevant natural parameters about some moment of rest. The knowledge about motion and rest of the body in mechanics, described by mathematical relations in various coordinate systems, is made relative - by the precondition of invariance that the natural attributes of motion do not depend upon the formal way of description. Therefore, the preprinciples objectify the subject of the theoretical mechanics while, at the same time, they make relative its general knowledge; they are accompanying corrector and verificator of all the assertions of the body motion theory.

The first section dealing with the **Basic Definitions** introduces and defines only four concepts by means of which it is possible to elaborate further one theory of the body motion. In accordance with the preprinciples, it was necessary in the beginning to open up the problem of selecting base oriented vectors, invariable in time. Unlike the velocity definition by means of the boundary values of distances, what is avoided in the velocity definition is the boundary transfer of one vector to another and thus the standard definition of velocity is accepted as a natural derivative of velocity with respect to time. In describing *motion impulse* the importance of the inertia tensor and of its difference from the geometrical metric tensor is especially stressed. This definition, just like the others, remains in the whole later theory which excludes from the present discussion the motion impulse as negative energy (Hamilton's function), that is, work of the forces. The term "motion impulse" is used instead of "impulse" in order to stress its difference from the forces' impulse. The definition of the *inertia force* determines a dimension of the force in general which later becomes prominent at the introduction and dimensioning of various dynamic parameters, as well as formulating the laws of dynamics.

The second section of the **Laws of Dynamics** gives to the concept of the "law" a unique meaning of the force's determinant; this makes it considerably different from the concept of Newton's laws; it is due to it that the concept of *law* in mechanics is strictly differentiated from the concepts of *principles* and *theorems*. The dominant place in this section belongs to the law of constraints by which it is stressed that the constraint between material points or particles can be abstracted by forces, that is, that the constraints are sources of the forces' origins, so that the mathematical or mental relation implied in the concept of the constraint should necessarily be distinguished from the motion of mechanically and objectively existing constraint.

In the part entitled *On Mutual Attraction Force* formula (2.21) is derived, from which the Newton's law of gravity follows for some particular conjunctions. By dropping determinants of other forces, that is, of the laws of dynamics (for the sake of brevity), the newly-introduced concept of the law of dynamics is not brought into doubt.

The third section entitled **Principles of Mechanics** comprises four principles on the basis of which (meaning, of each of them) it is possible to develop the whole theory about the body motion. The *equilibrium principle* is most comprehensively described with the good arguments, though it is based on the least number of definitions and consequential determinations. It is sufficient enough to comprise all the body motions coupled with any constraints in any coordinate systems. The consequential effect of the coupled forces' moment at the system of material or dynamic points subdued to the constraints is shown. From this principle the necessity to generalize the formula of the gravitational force has followed or the need to doubt the validity of the differential equations of motion with the constraints' multipliers.

By introducing an additional definition of the concept of work the *work principle* is formulated. Unlike the vector invariant of the equilibrium principle, the work principle is expressed by means of the scalar invariant thus avoiding the difficulties in summing up the constrainted vectors. As a consequence, beside potential energy, "rheonomic pseudopotential" also appears as negative work of the constraint-changing force; that is why it is shown that kinetic energy is a negative work of inertia force. In a unique way *elementary works* upon real displacements, possible displacements as well as work upon possible variations are characterized. By introducing an additional coordinate - rheonomic coordinate - the principle of the rheonomic constraints' solidification is abandoned, so that the work principle relation is extended for one adequate addend. This was preceded by modification of the constraints' variations, as well as work of the mechanical system with rheonomic constraints.

The concept of action is defined by means of the concept of work; the concept of action is the object of the general integral variational principle called the *principle of action.* Therefore, the statement of the action principle required six basic definitions. For such formulation of the principle and with the unique concept of variation, the classical integral variational principles appear as corollaries. Since by the preprinciple of existence time is taken as an invariable, it does not vary as such; thus, this integral principle shows itself to be invariant upon the extended configurational manifolds TM^n and T^*M^n as well as for scleronomic systems upon TM^{n+1} and T^*M^{n+1}; in other words, on the relations which are of the same shape for autonomous and non-autonomous systems. A more essential meaning of this principle is expressed in the section IV which proves the theorem on optimal control of motion.

On the basis of the first four definitions and the compulsion definition the differential variational *principle of compulsion* is expressed that, in essence, scalarizes the vector invariant of the equilibrium principle. By describing compulsion as a homogeneous quadratic form of the acceleration vector coordinate over the inertia tensor the possibility of its transformation into any coordinate system has been proved. From the principle's requirement that compulsion has the least value on actual motion, it is easy to arrive at simple scalar differential equations of motion expressed by the compulsion function.

The section on **Theorems of Mechanics**, states clearly, first of all, what is implied by the "theorem" in mechanics. By means of the natural derivative with respect to time the theorem on motion impulse change and the theorem on kinetic energy change are proved; both theorems, in accordance with to the preprinciples, have invariant sense and they differ from the accepted assertions of the analytical mechanics. This becomes obvious when using the example of the change of impulse of the rigid body's rotary motion by which the derivatives with respect to time of the inertia tensor coordinates are developed. The theorem on controllable motion and optimal motion control that comprise all the mechanical systems connect the control theory with its basic roots of the analytical mechanics.

The fifth section, namely, **Motion Determination** by Analisis and Solutions of the Relations of Motion is mostly devoted to unextended covariant integration, to the first integrals and to the covariant integrals; Poisson's' brackets are extended for rheonomic systems. A brief, but sufficiently clear description of energy integral modification is given.

The final part is the sixth section entitled **Stability of Motion and Rest** by which accuracy and validity of the differential equations of motion are assessed depending on the observed dynamic or kinetic parameters. A special emphasis is paid to the thoughts of the highly distinguished Professor Nicolai Gurevich Chetaev concerning *false modernization*, namely the thoughts that are no less actual today; besides, not only general but covariant differential equations of disturbances are presented as well as the author's general criterion of stability of the equilibrium state and of the mechanical system motion.

The book is properly referred to as a monograph since it presents one theoretical entirety based on the authors' results published in scientific journals and monographs listed in References. This theory comprises all the mechanical systems which also include rigid and deformable bodies. The author's concept of the rheonomic coordinate's application to deformable bodies has been left out. It has been shown [67], [71] that deformable bodies can be represented as a system of material points with rheonomic constraints, so that deformable medium can be modeled by (3+1)-dimensional manifolds. Such mechanics would develop upon the derived deformation tensor

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \varepsilon_{03} \\ \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{20} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{30} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{03} \end{pmatrix}$$

and metrics

$$ds^2 = \varepsilon_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 0, 1, 2, 3.$$

This metrics has invited quoting of the examples (E7) and (E8). Even more than that, it refutes, at the end of this book, any argument trying to prove that mechanics, as a science about motion of bodies, accomplished itself a long time ago; on the contrary, it stimulates new knowledge about motion and interaction of bodies.

BIBLIOGRAPHY

[1] Anđelić P.T., Vujičić A. V., *On the Poisson-Jacobi's method in non-autonomous systems*, Bulletin de l'Académie Serbe des Sciences, Classe des sci. math. et nat., N. 18, pp. 7–12, 1992.

[2] Alfven, H., Arrhenius, G., *EVOLUTION OF THE SOLAR SYSTEM*, National Aeronautics and Space administration (NASA), SP–345, Washington, 1976.

[3] Arnold V.I., *MATHEMATICAL METHODS OF CLASSICAL MECHANICS*, Second Edition, Springer-Verlag, 1989.

[4] Bilimović A., (Bilimovich A.), *ДИНАМИКА ЧВРСТОГ ТЕЛА*, Beograd, 1955.

[5] Kartan, E., (Cartan E.), *РИМАНОВА ГЕОМЕТРИЯ В ОРТОГОНАЛЬНОМ РЕПЕРЕ*, MGU, Moskva, 1960.

[6] Четаев Н.Г., (Chetaev N.G.),*УСТОЙЧИВОСТЬ ДВИЖЕНИЯ*, "Готехисдат", Москва, 1955.

[7] Gradić S., *Dissertationes physico-mathematicae quatuor*, Amstelodami, 1680; See (Жарко Дадић: *Стјепан Градић о проблемима гибања*, Мат. весник 5 (20), 1968.).

[8] Харламов П.В., (Harlamov P.V.), *РАЗНОМЫСЛИЕ В МЕХАНИКЕ*, Академия наук Украини, Донецк, 1993.

[9] Kittel Ch., Knight D., Ruderman M., *MECHANICS*, New-Yourk, 1965.

[10] Kozlov V.V., Vujičić V.A., *A contribution to the theory of rheonomic systems*, Bulletin T. CXI de l'Académie Serbe des Sciences et Arts – classe des Sciences Mathematiques et naturelles, Sciences Mathematiques, No 21, pp. 85–91, 1996.

[11] Lagrange J.L., *MECANIQUE ALANITIQUE*, Paris, 1788.

[12] Ляпунов А.М., (Lyapunov A.M.), *ЛЕКЦИИ ПО ТЕОРЕТИЧЕСКОЙ МЕХАНИКЕ*, Киев, 1982.

[13] Матросов В.М., (Matrosov V.M.), *Принцип сравнения с векторфункцией Ляпунова*, I, II, III, IV, Диференциалкние уравнения, т. 4, No 10; т. 5, No 7, No 12.

[14] Миланковић М., (Milankovich M.), *НЕБЕСКА МЕХАНИКА*, Београд, 1935.

[15] Миланковић М., (Milankovich M.), *ОСНОВИ НЕБЕСКЕ МЕХАНИКЕ*, "Просвета", Београд, 1947.

[16] Mušicki Đ., *A parametric formulation of the mechanics of rheonomic systems*, Theoretical and Applied Mechanics, 18, pp. 103–117, 1992.

[17] Mušicki Đ., *Gauss's principle in a parametric formulation of mechanics*, Theoretical and Applied Mechanics, 22, pp. 61–73, 1996.

[18] Newton Is., *PHILOSOPHIAE NATURALIS PRINCIPIA MATEMATICA*, Londoni, Anno, MDCLXXXVII.

[19] Newton Is., *MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY*, (translated into English by Robert Thorp, MA), London, 1969.

[20] Полак Л.С., (Polak L.S.), *Примечания – ВАРИАЦИОННЫЕ ПРИНЦИПЫ МЕХАНИКИ*, сборник статей, изд. "МИР", стр. 888–891, Москва, 1959.

[21] Румянцев В.В., (Rumyantsev V.V.) Озиранер А., *УСТОЙЧИВОСТЬ И СТАБИЛИЗАЦИЯ ДВИЖЕНИЯ ПО ОТНОШЕНИЮ К ЧАСТИ ПЕРЕМЕННЫХ*, "Наука", Москва, 1987.

[22] Седов Л.И., (Sedov L.I.), *МЕХАНИКА СПЛОШНОЙ СРЕДЫ*, том I, изд. второе, "Наука", Москва, 1973.

[23] Смирнов В.И., (Smirnov V.I.), *КУРС ВИШЕЙ МАТЕМАТИКИ*, том IV, изд. 3, "Готехиздат", Москва, 1957.

[24] Вујановић Б.Д., Јонес С. Е., *METHODS IN NON-CONSERVATIVE PHENOMENA*, Ацадемиц Пресс, 1989.

[25] Vujičić A.V., *On the covariant differential equations of motions of dynamical systems with variable mass*, Tensor, Vol. 18, N. 2, pp. 181–183, Fujisawa, 1967.

[26] Vujičić A.V., *General conditions of stability of the state of equilibrium of the dynamic sistem of a variable mass*, Tensor, N. S., Vol. 19, pp. 314–316, 1968.

[27] Vujičić A.V., *Über die stabilität der stationären Bevegungen*, ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik, GAMM – Tagung, Akademie – Verlag – Berlin, Band 48, pp. 291–293, 1968.

[28] Вујичић А.В., (Vujichich A.V.), *Общее следствие прямого метода Ляпунова об устойчивости (The general corollary of the direct Lyapunov's method of stability)*, Publ. Inst. Math. 9 (23), pp. 139–142, 1969.

[29] Вујичић А.В., *Абсолютный интеграл тензора (The absolute integral of a tensor)*, Publ. Math. Inst. N. S., t. 10 (24), pp. 199–202, 1970.

[30] Вујичић А.В., *Общее утверждение об устойчивости движения и состояния равновесия механических систем (The general theorem of stability of motion and equilibrium of mechanical systems)*, Publ. Inst. Math., t. 11 (25), pp. 33–41, 1971.

[31] Vujičić A.V., *Covariant equations of disturbed motion of mechanical systems*, Tensor, N. S., Vol. 22, pp. 41–47, Fujisawa, 1971.

[32] Вујичић А.В., *Некоторые общие интегралы нелинейных механических систем движения (Certain general integrals of nonlinear mechanical systems)* (in Russian, Polish and English summaries), Nonlinear vibration problems, Polish Academy of Sciences, t. 14, pp. 369–377, 1973.

[33] Вујичић А.В., *Ковариантные интегралы в механике*, ПММ – прикладная математика и механика, т. 40, Вып. 2, АН СССР, стр. 346–351, Москва, 1976.

[34] Vujičić A.V., *Covariant integrals in mechanics*, PMM – Journal of App. Mech. 40, pp. 320–326, 1976 (translation of the Soviet journal Прикладная математика и механика), Publ. Pergamon - press, Oxford, Toronto.

[35] Vujičić A.V., *Optimal control of motion of a holonomic system*, Bulletin de l'Académie Serbe des Sciences, LXXVI, N. 11, pp. 1–10, 1981.

[36] Vujičić A.V., *COVARIANT DYNAMICS* (in Serbian: *KOVARIJANTNA DINAMIKA*) (English summary), Mat. Inst. Serbian Acad. Sci., Editions spéciales, 14, pp. 136, Beograd, 1981.

[37] Vujičić A.V., *On the absolute integral in n-dimensional configuration space*, Topics in differential geometry, Vol. I, II, pp. 1297–1308 (1984) Debrecen; Colloq. Math. Soc. János Bolyai, 46, North Holland (1988), New York.

[38] Vujičić A.V., *Invariance of D'Alembert's principle*, Bulletins for Applied Mathematics (BAM) 218/84, Budapest, 1984.

[39] Вујичић А.В., *К теории интегрирования тензорных дифференциальных уравнений (On the theory of integration of tensorial differential equations),* Ves. Moscow University; ser. math. mech., 1985.

[40] Vujičić A.V., *Stability of steady motions of systems with general potential* (in Serbian, English summary), Glas de l'Académie Serbe des Science, 346, 50, pp. 1–7, 1986.

[41] Vujičić A.V., *The potential and power of rheonomous constraints*, Bulletins for Applied Mathematics, BAM 355–381/86, Budapest.

[42] Vujičić A.V., *The covariant integration on manifolds*, Tensor, Vol. 43, N. 3, Chigasaki, 1986.

[43] Вујичић А.В., *Одно следствые инвариантности принципа Гаусса*, Прикладная математика и механика, т. 51, вып. 5, стр. 735–740, Москва, 1987.

[44] Vujičić A.V., *The modification of analytical dynamics of rheonomic systems*, Tensor, N. S., N. 46, pp. 418–432, Chigasaki, 1987.

[45] Vujičić A.V., *On Hamilton's principle for the rheonomous systems*, Bulletin de l'Académie Serbe des Sciences, Classe des sci. math. et nat., N. 16, pp. 37–50, 1988.

[46] Вујичић А.В., *К принципу возможных перемещеный для реономых систем*, Прикладная механика, Т. 24, Н. 2, пп. 125–127, Киев, 1990.

[47] Vujičić A.V., *On the stability of motion of rheonomic sustems*, Proc. of the

International Conference on Applied Mechanics, Vol. 1, pp. 143–148 (1989), Beijing, China.

[48] Vujičić A.V., *A consequence of the invariance of the Gauss principle*, PMM Journal of applied mathematics and mechanics (translation of the Soviet journal Прикладная мат. и мех.), Vol. 51, N. 5, pp. 573–578; Printed in Great Britain, Pergamon Press, 1989.

[49] Вујичић А.В., Козлов В.В, *Об устайчивости в непотенциальном сыловом поле (On equilibrium stability in a nonpotential force field)*, Teo. prim. meh., N. 15, pp. 139–145, 1989.

[50] Vujičić A.V., *DYNAMICS OF RHEONOMIC SYSTEMS*, Mat. Inst. Serbian Acad. Sci., éditions spéciales, pp. 96, Beograd, 1990.

[51] Вујичић А.В., *О принципе наименьшего действия для реономных систем (The principle of least action for rheonomous systems)*, Прикладная механика, т. 26, N. 7, пп. 114–116, Киев, 1990.

[52] Вујичић А.В., *НЕКОТОРЫЕ ЗАДАЧИ МЕХАНИКИ НЕАУТОНОМНЫХ СИСТЕМ*, Математички институт САНУ Београд – Институт за механику АН Украјине, Кијев, стр. 109, 1991.

[53] Vujičić A.V., *A theorem in the problem of optimal control of mechanical systems*, Comptes rendus de l'Academie des Sciences, Paris, t. 313, série II, N. 1, pp. 1–5, 1991.

[79] Вујичић А.В., Козлов В.В., *К задаче Ляпунова об устойчивости по отношению к заданим функциям состояния*, Прикладная мат. и мех., т. 55, вып. 4, стр. 555–559, Москва, 1991.

[54] Vujičić A.V., *The cocyclic energy integral*, European Journal of Mechanics, A 10, N. 1, pp. 41–44, Paris, 1991.

[55] Vujičić A.V., Kozlov V.V., *On the Lyapunov's problem of stability in comparations with functions of state* (coauthor V. V. Kozlov), Journal of applied math. and mech. (translation of the PMM), Vol. 51, N. 5, 1991.

[56] Vujičić A.V., *A general theorem of optimal control of motion of mechanical non-autonomous systems*, XVIII International Congress of Theoretical and Applied Mechanics, pp. 154, Haifa, 1992.

[57] Вујичић А.В., *О практической устойчивости равновесия и движения механических систем (On the practical stability of equilibrium and motion of mechanical systems)*, Прикладная механика, т. 28, N. 11, стр. 64–69, Киев, 1992.

[58] Вујичић А.В., *О практической устойчивости голономных систем (Practical stability of holonomic systems)*, Докл. Акад. наук Украини, N. 9, стр. 21–23, Киев, 1992.

[59] Vujičić A.V., *A universal integral invariant of non-autonomous dynamical systems*, Proc. of the First World Congres of nonlinear analysis (WCNA.92), Tampa, Florida, 1992.

[60] Vujičić A.V., *The principle of optimal motion*, Proc. of the 2^{nd} International Conference on Nonlinear Mechanics, pp. 737–740, Beijing, China, 1993.

[61] Vujičić A.V., Hedrih K., *The rheonomic constraints force*, Facta Universitatis, Vol. 1, N. 3, pp. 313–322, 1993.

[62] Vujičić A.V., *Optimal control of rheonomic mechanical system*, Facta Universitatis, Vol. 1, N. 4, pp. 413–423, 1994.

[63] Вујичић А.В., *Об однородном формализме класической динамики реономних систем*, Електроное моделирование, т. 17, No 4, стр. 10–14, 1995.

[64] Vujičić A.V., *Energy exchange theorems in systems with time-dependent constraints*, teo. prim. mehanika, 21, pp. 105–121, 1955.

[65] Вујичић А.В., Мартиньк А.А., *К аналитической динамике управляемых систем*, Прикладная механика, 32, No 9, стр. 88, 1966.

[66] Vujičić A.V., Martinyuk A.A., *Applied Mechanics (Прикладная механика)*, 32 (9), pp. 88, 1996.

[67] Вујичић А.В., *Прилог механици деформабилних тела*, Техника, година LI, стр. 11–12, НГ 1–6, 1996.

[68] Vujičić A.V., *A Deformation Body as a Model of Rheonomic System*, Contemporary problems of fluid mechanics, Beogradski univerzitet, 1996.

[69] Vujičić A.V., *A Possible Reconsideration of Newton Gravitation Law* , Scientific Review, Number 24, pp. 61-68, 1997.

[70] Вујичић А.В., *О кретању два тела и закону гравитације (One approash to problem of two and more bodies).* А – *Општа маханика*, Зборник радова, YUCTAM 1997.

[71] Vujičić A.V., *Some comments on the theory of deformation bodies*, Solid Mechanics – Proceeding of Second Serbian – Greek Symposium on Solid Mechanics, 1996, (Scientific Meetings of the Serbian Academy of Sciences and Arts, Vol. LXXXVII, Department of Technical Sciences, Book 3), pp. 77–86, 1997.

INDEX OF SUBJECTS

CONTENTS