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Malliavin Calculus for Chaos Expansions of Generalized Stochastic Processes with Applications to Some Classes of Differential Equations

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Предговор

Предмет истраживања ове докторске дисертације је теоријско разматрање главних својстава оператора Малиавеновог рачуна: Малиавеновог извода, Скороходовог интеграла и Орнштајн-Уленбековог оператора, дефинисаних на класи уопштених стохастичких процеса над просторима белог шума и фракционог белог шума, као и примена добијених резултата на решавање одређених класа стохастичких диференцијалних једначина.

У дисертацији су разматрани уопштени стохастички процеси који се могу развити у ред по бази Хилбертовог простора средње квадратно интеграбилних процеса на простору белог шума, изражeној у облику фамилије ортогоналних полинома. У првом делу дисертације су разматрани простори Гаусовог и Поасоновог белог шума, као и њихове одговарајуће фракционе верзије, где су везе између свака два простора успостављене преко унитарних пресликавања.

У другом делу дисертације је дато проширење дефиниција оператора Малиавеновог рачуна са класе квадратно интеграбилних случајних величина на класе уопштених стохастичких процеса. Истакнута је њихова интерпретација, као и веза са одговарајућим фракционим Малиавеновим операторима.

У завршном делу дисертације, метод хаос експанзија је примењен на решавање стохастичких диференцијалних једначина у којима фигурише Малиавенов извод и Орнштајн-Улембеков оператор. Између осталог, представљено је решење уопштеног проблема сопствених вредности за оператор Малиавеновог извода, као и решење стохастичког Дирихлеовог проблема са пертурбацијама генерисаним дејством Орнштајн-Улембековог оператора.

Ова докторска дисертација представља део резултата вишегодишњег истраживања под менторством професора Стевана Пилиповића и Доре Селеши.

Нови Сад, 25. децембар 2011. Тијана Левајковић

Preface

The main subject of this doctoral dissertation is the theoretical investigation of properties of the operators of Malliavin calculus: the Malliavin derivative, the Skorokhod integral and the Ornstein-Uhlenbeck operator, all defined on a class of generalized stochastic processes, which admit the chaos expansion representation form in terms of orthogonal polynomial basis in white noise framework, the interpretation and applications of obtained results to solving some classes of stochastic differential equations.

Generalized stochastic processes defined on white noise spaces, which have a series expansion representation form given by the Hilbert space orthogonal polynomials basis of square integrable processes, found place in the dissertation. The first part of the thesis is devoted to Gaussian and Poissonian white noise spaces together with their corresponding fractional versions, where any two of them can be identified through a unitary mapping.

In the second part of the dissertation, theorems which characterize the operators of Malliavin calculus, extended from the space of square integrable random variables to the space of generalized stochastic processes are obtained. Moreover the connections with the corresponding fractional versions of these operators are emphasized and proved.

The closing part of this dissertation contains several examples of stochastic differential equations involving the Malliavin derivative operator and the Ornstein-Uhlenbeck operator, all solved by use of the chaos expansion method. Particularly, the solutions of a generalized eigenvalue problem with the Malliavin derivative and the stochastic Dirichlet problem with a perturbation term driven by the Ornstein-Uhlenbeck operator are presented.

This dissertation is the result of several years of research guided and supervised by Professor Stevan Pilipović and Dora Seleši.

Novi Sad, 25 December 2011. Tijana Levajković

Introduction

The mathematical theory, known as the *Malliavin calculus* or the *stochastic calculus of variations* was first introduced by Paul Malliavin in [38] as an infinite dimensional integration by parts technique. The original motivation and important application of this theory is to provide a probabilistic proof of Hörmanders sum of squares theorem for hypoelliptic operators. Moreover, the theory is used when proving the results involving smoothness of densities of solutions of stochastic differential equations driven by Brownian motion. This deep and fascinating theory was further developed by Stroock, Bismut, Watanabe, Nualart, Øksendal, Rozovsky and others. It remained relatively unknown for some time until, in the recent years, the ideas became increasingly important in applications, for instance, in stochastic filtering and in financial mathematics to compute sensitivities of financial derivatives.

A crucial fact in this theory is the integration by parts formula, which relates the Malliavin derivative operator on the Wiener space and the divergence operator, called the Itô-Skorohod stochastic integral in white noise setting.

Generalized stochastic processes on white noise spaces have a series expansion form given by the Hilbert space basis of square integrable processes, i.e. processes with finite second moments, and depending on the stochastic measure this basis can be represented as a family of orthogonal polynomials defined on an infinite dimensional space. The classical Hida approach ([17], [18], [19]) suggests to start with a nuclear space E and its dual E', such that

$$E \subset L^2(\mathbb{R}) \subset E',$$

and then take the basic probability space to be $\Omega = E'$ endowed with the Borel sigma algebra of the weak topology and an appropriate probability measure P. Since Gaussian processes and Poissonian processes represent the two most important classes of Lévy processes, we will focus on these two types of measures.

In case of a Gaussian measure, the orthogonal basis of $L^2(P)$ can be constructed from any orthogonal basis of $L^2(\mathbb{R})$ that belongs to E and from the

Hermite polynomials, while in the case of a Poissonian measure the orthogonal basis of $L^2(P)$ is constructed using the Charlier polynomials together with the orthogonal basis of $L^2(\mathbb{R})$. We will focus on the case when E and E' are the Schwartz spaces of rapidly decreasing test functions $S(\mathbb{R})$ and tempered distributions $S'(\mathbb{R})$. In this case the orthogonal family of $L^2(\mathbb{R})$ can be represented by the Hermite functions. Following the idea of the construction of $S'(\mathbb{R})$ as an inductive limit space over $L^2(\mathbb{R})$ with appropriate weights, one can define stochastic generalized random variable spaces over $L^2(P)$ by adding certain weights in the convergence condition of the series expansion (also known as the Wiener-Itô chaos expansion) and thus weakening the topology of the L^2 norm. We will define several spaces of this type, weighted by a sequence q and denote them by $(Q)^P_{-\rho}$, for $\rho \in [0,1]$ and thus obtaining a Gel'fand triplet

$$(Q)^P_\rho \subset L^2(P) \subset (Q)^P_{-\rho}.$$

Recently, there have been made improvements in economics and financial modelling by replacing the Brownian motion with the fractional Brownian motion, and replacing white noise by fractional white noise (see [2], [3], [9]). In this dissertation we will define the fractional Poissonian process in a framework that will make it easy to link it to its regular version.

In [8] it was proved that there exists a unitary mapping between the Gaussian and the Poissonian white noise space, by mapping the Hermite polynomial basis into the Charlier polynomial basis. In [6] and [10] a unitary mapping was introduced between the Gaussian and the fractional Gaussian white noise space. We extend these ideas to define the fractional Poissonian white noise space itself and to connect it to the classical Poissonian white noise space. As a result we obtain four types of white noise spaces: Gaussian, Poissonian, fractional Gaussian and fractional Poissonian, where any two of them can be identified through a unitary mapping.

In white noise setting, the Skorokhod integral represents an extension of the Itô integral from a set of adapted processes to a set of nonanticipating processes. Its adjoint operator D is known as the Malliavin derivative. In spite of many similarities, there are important distinctions between interpretations of the Malliavin derivative in the Gaussian and the Poissonian case. On a space of Gaussian random variables the Malliavin derivative is interpreted as a directional derivative and on a set of Poissonian random variables the Malliavin derivative is interpreted as a difference operator.

Both operators, the Skorokhod integral and the Malliavin derivative, having an interpretation also in the Fock space sense as the annihilation and the creation operator, are widely used in solving stochastic differential equations (see [4], [13], [14], [16], [17]). Their composition is known as the Ornstein-Uhlenbeck operator, and it is a self-adjoint operator on $L^2(P)$ that has the elements of the orthogonal basis (Hermite or Charlier polynomials) as its eigenvalues.

The Malliavin derivative and its related operators are all defined on either of the four white noise spaces we are working on, and their domains are characterized in terms of convergence in a stochastic distribution space $(Q)^P_{-\rho}$ with special q-weights.

Furthermore, as the description of the chaos expansion method we provide some applications to solving several examples of stochastic differential equations involving the Malliavin derivative, the Ornstein-Uhlenbeck operator and their fractional versions. All equations we solved can be interpreted on all four types of white noise spaces. We provide a general method of solving, using the Wiener-Itô chaos decomposition form, also known as the propagator method (see [12], [13], [14], [21]). With this method we reduce a problem to an infinite system of deterministic equations. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial equation. Another type of equations investigated by the same method can be found in several papers: [26], [32], [33], [36], [34], [43], [37], [56].

The dissertation is organized in five chapters. Chapter 1, titled *Fundamental Theory Background*, is expository and it represents an overview of some basic concepts of fundamental theories, which are necessary to understand the methods used in the sequent chapters of the dissertation. Spaces of deterministic generalized functions, used in the sequel, are introduced. We summarize definitions and the most important properties and relations of tensor products and Fock spaces, deterministic fractional calculus, stochastic analysis and classical Malliavin differential theory, i.e. the stochastic calculus of variations on an abstract Wiener space.

Chapter 2, entitled *White Noise Analysis and Chaos Expansions*, contains introduction of four types of white noise spaces, Gaussian and Poissonian, classical and fractional, together with the unitary mappings which connect each two of them. Moreover we introduce the weighted stochastic distribution spaces and the Wick multiplication of their elements, together with definition of the generalized stochastic processes and present their chaos expansion representation forms.

Chapters 2, 4 and 5 contain the original parts of the dissertation. All the results have been achieved in joint work with Dora Seleši and Stevan Pilipović and are already published in [26], [28], [27], [29] and [30]. Some results have been partially presented on several international conferences and workshops.

Chapter 3, named *Malliavin Calculus in Chaos Expansions Framework for Square Integrable Processes*, is devoted to overview of the Malliavin calculus on sets of Gaussian and Poissonian square integrable random variables, represented in their chaos expansion forms. The fractional versions of the Malliavin operators are introduced on both, classical and fractional versions of Gaussian and Poissonian spaces, and some connections with the classical calculus are emphasized.

The definitions of the Malliavin derivative and the Skorokhod integral which are extensions of the definitions of these operators to a space of singular generalized stochastic processes are presented in Chapter 4, entitled *Operators of Malliavin Calculus For Singular Generalized Stochastic Processes*. We allow values in q-weighted spaces of generalized stochastic functionals and obtain larger domains of operators of Malliavin calculus then in the case of square integrable random variables described in Chapter 3. In addition, Chapter 4 contains the characterization of the fractional Malliavin operators in terms of the corresponding classical versions.

Chapter 5 is titled *Applications of the Chaos Expansion Method to Some Classes of Equations* and is devoted to solving some classes of stochastic differential equations which are driven by the Malliavin derivative operator and functionals of the Ornstein Uhlenbeck operator. In particular, we present and solve a first order equation and a generalized eigenvalue problem with the Malliavin derivative in a white noise space of general type (Theorem 5.1.1 and Theorem 5.1.4 respectively). In addition, we present the explicit forms of solutions of equations involving the Ornstein-Uhlenbeck operator and the exponential of the Ornstein-Uhlenbeck operator, belonging to a certain space of q-weighted generalized stochastic processes (Theorem 5.2.1 and Theorem 5.3.1 respectively). Chapter 5 also deals with the stochastic version of the Fredholm alternative considered in the framework of chaos expansion methods on white noise probability space. We apply the results to solve the Dirichlet problem generated by an elliptic second order differential operator with stochastic coefficients, stochastic input data and boundary conditions, and with the Ornstein-Uhlenbeck operator as a perturbation term. The stochastic Dirichlet problem has been previously studied in [57], [58], [67]. Solvability and uniqueness of the solution to the stochastic Dirichlet problem under assumptions made only on the expectation of L and certain conditions on the positivity of the perturbation term are stated and proven. Theorem 5.4.3 represents one of the main contributions of this dissertation to the Malliavin calculus of generalized stochastic processes within white noise theory. All solutions obtained in equations we considered in this chapter are singular generalized stochastic processes having values in a certain q-weighted space of stochastic distributions.

I would like to take this opportunity to express my huge thank to my family, my parents Zvezdana and Branko and my brother Aleksandar for their love, encouragement, kindness, patience and support during years of my studies and work on this dissertation.

I am truly thankful to my friends and colleagues for providing constant support and help during my work on this dissertation.

A great thank goes to Professor Michael Oberguggenberger for his hospitality and support during my stay at the Unit for Engineering Mathematics, Department of Civil Engineering in Innsbruck, Austria. In particular, I wish to thank him for his interesting Seminar on Stochastic Analysis and the Mathematics Colloquium and for his useful advices and ideas in our discussions during my stay in Innsbruck. In addition, I owe him a debt of gratitude for a careful reading of this text and providing a sequence of improvements.

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I want to express my deep appreciation for the constant guidance and help in many areas given to me by my supervisor Professor Stevan Pilipović. I am heartly grateful for his continuous encouragement and outstanding support, both personally and professionally, during the past years.

Last but not least, I am greatly obligated to my mentor Dora Seleši, for her constant support, patience, invaluable effort and supervision of this dissertation.

Contents

Chapter 1

Fundamental Theory Background

In this introductory chapter some basic concepts of fundamental theories, which are necessary to understand the methods used in the subsequent chapters of the dissertation are presented. We summarize definitions and the most important properties and relations of generalized functions theory, tensor products and Fock spaces, deterministic fractional calculus, stochastic analysis and theory of classical Malliavin calculus. Most of the material presented here is known and therefore given without proofs but with references for further reading.

Some basic notation we will use throughout the thesis is the following: Let V be a topological vector space, V' its dual space, and $\mathcal{L}(V, U)$ be the space of all linear continuous mappings from V into a topological vector space U. By $L^r(\mathbb{R})$, $r \geq 1$, we denote the space of r-integrable functions with respect to the Lebesgue measure λ, by $C^k(\mathbb{R})$ denote the space of k-times continuously differentiable functions, and by $C_0(\mathbb{R})$ the space of continuous functions with compact support.

1.1 Spaces of Deterministic Functions

At the beginning, we focus on a brief overview of some classes of deterministic generalized function spaces. We introduce the Schwartz space of generalized functions, the space of generalized functions of exponential growth and the Sobolev spaces.

1.1.1 Hermite functions

The *Hermite polynomial of order* n, $n \in \mathbb{N}_0$, is defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \quad x \in \mathbb{R}.$$

These polynomials are the coefficients of the expansion in powers of t of the generating function $F(x,t) = \exp(tx - \frac{t^2}{2})$. We have

$$\begin{aligned} F(x,t) &= \exp(\frac{x^2}{2} - \frac{1}{2}(x-t)^2) \\ &= e^{\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\frac{d^n}{dt^n}\, e^{-\frac{1}{2}(x-t)^2}) \,|_{t=0} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\, h_n(x) \end{aligned} \tag{1.1}$$

From the property (1.1) we have the relation:

$$\frac{d}{dx} h_n(x) = n h_{n-1}(x), \quad n \in \mathbb{N}. \tag{1.2}$$

It is well known that the family $\{\frac{1}{\sqrt{n!}} h_n : \ n \in \mathbb{N}_0\}$ forms an orthonormal basis of the space $L^2(\mathbb{R})$ with respect to the Gaussian measure $d\mu = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$.

The *Hermite function of order* $n+1$, $n \in \mathbb{N}_0$, is defined as

$$\xi_{n+1}(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{n!}} e^{-\frac{x^2}{2}} h_n(\sqrt{2}x), \quad x \in \mathbb{R}.$$

The family of Hermite functions $\{\xi_{n+1} : n \in \mathbb{N}_0\}$ constitutes a complete orthonormal system of $L^2(\mathbb{R})$ with respect to the Lebesque measure. Namely, every deterministic function $g \in L^2(\mathbb{R})$ has a series representation of the form

$$g(x) = \sum_{k \in \mathbb{N}} a_k\, \xi_k(x),$$

with coefficients $a_k = (g, \xi_k)_{L^2(\mathbb{R})} \in \mathbb{R}$ satisfying the convergence condition $\sum_{k \in \mathbb{N}} a_k^2 < \infty$.

Moreover

$$|\xi_n| \leq \begin{cases} C n^{-\frac{1}{12}}, & |x| \leq 2\sqrt{n} \\ C e^{-\gamma x^2}, & |x| > 2\sqrt{n} \end{cases},$$

hold for constants C and γ independent of n.

1.1.2 Schwartz spaces

The *Schwartz space of rapidly decreasing functions* is defined as

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}_0,\ \|f\|_{\alpha,\beta} < \infty\},$$

and the topology on $S(\mathbb{R})$ is given by the family of seminorms

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha D^\beta f(x)|, \quad \alpha, \beta \in \mathbb{N}_0.$$

The space $S(\mathbb{R})$ is a nuclear countable Hilbert space and the orthonormal basis of $S(\mathbb{R})$ is the family of Hermite functions $\{\xi_n\}_{n\in\mathbb{N}}$.

It is well known that the Schwartz space of rapidly decreasing functions can be constructed as the projective limit of the family of spaces $S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} S_l(\mathbb{R})$, where

$$S_l(\mathbb{R}) = \{\varphi = \sum_{k=1}^{\infty} a_k\, \xi_k \in L^2(\mathbb{R}) :\ \|\varphi\|_l^2 = \sum_{k=1}^{\infty} a_k^2 (2k)^l < \infty\},\ l \in \mathbb{N}_0.$$

The *Schwartz space of tempered distributions* $S'(\mathbb{R})$ is the dual space of the space of rapidly decreasing functions, equipped with the strong topology, which is equivalent to the inductive topology. Its elements are called *generalized functions* or *distributions*.

The Schwartz space of tempered distributions is isomorphic to the inductive limit of the family of spaces $S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$, where

$$S_{-l}(\mathbb{R}) = \{f = \sum_{k=1}^{\infty} b_k\, \xi_k :\ \|f\|_{-l}^2 = \sum_{k=1}^{\infty} b_k^2 (2k)^{-l} < \infty\},\ l \in \mathbb{N}_0.$$

The action of a generalized function $f = \sum_{k\in\mathbb{N}} b_k \xi_k \in S'(\mathbb{R})$ on a test function $\varphi = \sum_{k\in\mathbb{N}} a_k \xi_k \in S(\mathbb{R})$ is given by the dual paring

$$\langle f, \varphi\rangle = \sum_{k\in\mathbb{N}} a_k b_k.$$

Thus,

$$S(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq S'(\mathbb{R})$$

form a Gel'fand triple, with continuous inclusions.

The characterization of the Schwartz spaces of test functions and distributions in terms of the Hermite functions orthonormal basis gives us motivation to build on analogous type of spaces consisting of stochastic elements which allows the decomposition in terms of an orthogonal polynomial basis.

1.1.3 Deterministic spaces of exponential growth

In this thesis we also consider the test space of deterministic test functions of exponential growth rate $\exp S(\mathbb{R})$ and the corresponding space of deterministic distributions of exponential growth rate $\exp S'(\mathbb{R})$, introduced in [54] and [55].

The space of test functions of exponential growth rate, denoted by $\exp S(\mathbb{R})$, is constructed as the projective limit of the family of spaces $\exp S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} \exp S_l(\mathbb{R})$ where

$$\exp S_l(\mathbb{R}) = \{\varphi = \sum_{k=1}^{\infty} c_k\, \xi_k \in L^2(\mathbb{R}) :\ \|\varphi\|^2_{\exp,l} = \sum_{k=1}^{\infty} c_k^2\, e^{2kl} < \infty\},\ l \in \mathbb{N}_0.$$

The space of deterministic distributions of exponential growth rate is considered to be the inductive limit of the family of spaces $\exp S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} \exp S_{-l}(\mathbb{R})$, where

$$\exp S_{-l}(\mathbb{R}) = \{f = \sum_{k=1}^{\infty} d_k\, \xi_k :\ \|f\|^2_{\exp,-l} = \sum_{k=1}^{\infty} d_k^2\, e^{-2kl} < \infty\},\ l \in \mathbb{N}_0.$$

These spaces satisfy the relationship

$$\exp S(\mathbb{R}) \subseteq S(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq S'(\mathbb{R}) \subseteq \exp S'(\mathbb{R}),$$

where each inclusion mapping is compact.

1.1.4 Sobolev Spaces

Let I be an open subset of $\mathbb{R}$. The αth *weak derivative* of f, denoted by $D^\alpha f$ is given by the action

$$\int_I D^\alpha f(x)\varphi(x)dx = -\int_I f(x)D^\alpha\varphi(x)dx,$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.

Denote by $W^{k,p}(I)$ the space of weakly differentiable functions f such that $D^\alpha f \in L^p(I)$ for all $|\alpha| \le k$. We endow $W^{k,p}(I)$ with the norm

$$\|f\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^p(I)}.$$

Clearly, $W^{k,2}(I)$ is a Hilbert space.

Another important space we will consider is $W_0^{k,p}(I)$ defined as the closure of $C_0^\infty(I)$ in $W^{k,p}(I)$. The dual space of $W_0^{k,p}(I)$ will be denoted by $W^{-k,p}(I)$. An isomorphism between $W_0^{k,p}(I)$ and $W^{-k,p}(I)$ can be established via the Laplace operator. By its Hilbert structure, we also may identify $W_0^{k,2}(I)$ with $W^{-k,2}(I)$. Thus, we obtain a Gel'fand triple

$$W_0^{k,2}(I) \subseteq L^2(I) \subseteq W^{-k,2}(I).$$

For further notions and properties of Sobolev spaces we refer to [2].

1.2 Tensor Products and Fock Spaces

Now we summarize standard facts on tensor products of real Hilbert spaces.

Let H_1 and H_2 be two Hilbert spaces, equipped with the scalar products $(\cdot,\cdot)_{H_1}$ and $(\cdot,\cdot)_{H_2}$ respectively. Dual spaces are denoted by H_1' and H_2' and corresponding dual parings are denoted by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$.

Definition 1.2.1 *Let $f_1 \in H_1$ and $f_2 \in H_2$ be fixed. The* tensor product *$f_1 \otimes f_2$ is a bilinear form over $H_1' \times H_2'$ given by*

$$f_1 \otimes f_2(g_1, g_2) = \langle g_1, f_1\rangle_1 \langle g_2, f_2\rangle_2,$$

for $(g_1, g_2) \in H_1' \times H_2'$.

The *tensor product* of Hilbert spaces $H_1 \otimes H_2$ is defined to be a Hilbert space equipped with a bilinear map $H_1 \times H_2 \to H_1 \otimes H_2$, denoted by $(f_1, f_2) \mapsto f_1 \otimes f_2 \in H_1 \otimes H_2$, such that

$$(f_1 \otimes f_2, g_1 \otimes g_2)_{H_1 \otimes H_2} = (f_1, g_1)_{H_1} (f_2, g_2)_{H_2}.$$

The closed linear span of the range of this map equals to $H_1 \otimes H_2$.

Definition 1.2.2 *Let $n \in \mathbb{N}$. The n*th tensor power *of a Hilbert space H is defined by*

$$\mathcal{F}^{(n)}(H) = \underbrace{H \otimes \cdots \otimes H}_{n} = H^{\otimes n},$$

with $\mathcal{F}^{(0)}(H)$ equal to the space of scalars. The corresponding tensor norm is denoted by $\|\cdot\|_{\mathcal{F}^{(n)}(H)}$.

Definition 1.2.3 *Let $f_1, \ldots, f_n \in H$. The* symmetrization *of a tensor product is given by*

$$f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_n = \frac{1}{n!} \sum_{\pi \in Perm(n)} f_{\pi_1} \widehat{\otimes} \ldots \widehat{\otimes} f_{\pi_n} \tag{1.3}$$

where $Perm(n)$ denotes the group of permutations of first n natural numbers.

Definition 1.2.4 *For arbitrary $n \in \mathbb{N}$ we define the* nth symmetric tensor power *of a Hilbert space H*

$$\Gamma^{(n)}(H) = \underbrace{H\widehat{\otimes} \dots \widehat{\otimes}}_{n} = H^{\widehat{\otimes} n} \tag{1.4}$$

as a completion of symmetrized tensor products of elements in H with respect to the norm $\| \cdot \|_{\Gamma^{(n)}(H)}$ induced by the scalar product

$$(\widehat{\otimes}_{i=1}^{n} f_i, \widehat{\otimes}_{i=1}^{n} g_i)_{\Gamma^{(n)}(H)} = \sum_{\pi \in Perm(n)} (f_1, g_{\pi_1}) \dots (f_n, g_{\pi_n}).$$

Note that $\Gamma^{(0)}(H)$ is the one-dimensional space of scalars and $\Gamma^{(1)}(H) = H$. Thus $\Gamma^{(n)}(H)$ is also called the nth *homogeneous chaos* of a Hilbert space H. Moreover, $\Gamma^{(n)}(H)$ is a subspace of $\mathcal{F}^{(n)}(H)$ and

$$\| \cdot \|_{\Gamma^{(n)}(H)} = \sqrt{n!}\, \| \cdot \|_{\mathcal{F}^{(n)}(H)}$$

Definition 1.2.5 *The* Fock space *over a Hilbert space H is defined by*

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(H)$$

and the symmetric Fock space *over H is defined by*

$$\Gamma(H) = \bigoplus_{n=0}^{\infty} \Gamma^{(n)}(H).$$

Example 1.2.1 *Let H be a Hilbert space. Consider*

$$exp^{\widehat{\otimes}}(f) = \sum_{n=0}^{\infty} \frac{f^{\widehat{\otimes} n}}{n!}, \quad for\ f \in H.$$

These elements satisfy the property

$$\begin{aligned} \|exp^{\widehat{\otimes}}(f)\|^2_{\mathcal{F}(H)} &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \|f^{\widehat{\otimes} n}\|^2_{\Gamma^{(n)}} \\ &= \sum_{n=0}^{\infty} \frac{n!}{(n!)^2} \|f^{2n}\|^2_{H} \\ &= \exp(\|f\|^2_H). \end{aligned}$$

In particular, in a Gaussian Hilbert space, the exponentials $exp^{\widehat{\otimes}}(\xi)$ coincide with the normalized stochastic exponential, which will be defined later in Section 2.2.3.

1.2.1 Operators on the Fock space

Let $A : H \to H$ be a bounded linear operator on a Hilbert space H. The operator A induces linear operators between their symmetric tensor powers by

$$A^{\widehat{\otimes}n}(f_1 \widehat{\otimes} \dots \widehat{\otimes} f_n) = Af_1 \widehat{\otimes} \dots \widehat{\otimes} Af_n$$

and $\|A^{\widehat{\otimes}n}\| = \|A\|^n$.

Define now the *second quantization operator* of an operator A on the Fock space $\Gamma(H)$, i.e. the mapping

$$\Gamma(A) : \ \Gamma(H) \to \Gamma(H)$$

such that

$$\Gamma(A) \restriction_{H^{\widehat{\otimes}n}} = A^{\otimes n}, \quad n \in \mathbb{N},$$

and

$$\Gamma(A) \left(\sum_{n=0}^{\infty} X_n \right) = \sum_{n=0}^{\infty} A^{\otimes n} X_n, \quad X_n \in \Gamma^{(n)}(H).$$

Moreover, if A is a contraction, i.e. if $\|A\| \leq 1$, then the linear operator $\Gamma(A)$ is of a unit norm $\|\Gamma(A)\| = 1$.

Annihilation and creation operators

The annihilation and creation operators in quantum mechanics are constructed in the framework of Nelson's stochastic mechanics and have several applications in the study of quantum harmonic oscillators and particle systems.

Definition 1.2.6 *Let $F^{(n)} \in \Gamma^{(n)}(H)$ be of the form $F^{(n)} = \widehat{\otimes}_{i=1}^{n} f_i$, for $f_1, ..., f_n \in H$. The* annihilation *operator of a given vector $f \in H$ is the operator*

$$\partial(f) : \Gamma^{(n)}(H) \to \Gamma^{(n-1)}(H)$$

defined by

$$\partial(f) F^{(n)} = \sum_{j=1}^{n} (f, f_j) \, \widehat{\otimes}_{\substack{i=1 \\ i \neq j}}^{n} f_i.$$

The norm of the annihilation operator is represented by

$$\|\partial(f)\| = \sqrt{n} \, \|f\|_H.$$

Definition 1.2.7 *The adjoint operator ∂^* of the annihilation operator is called the* creation *operator.*

Theorem 1.2.1 *For $F^{(n)} \in \Gamma^{(n)}(H)$ the creation operator has the property*

$$\partial^*(f)F^{(n)} = f\widehat{\otimes}F^{(n)},$$

where $\partial^*(f)F^{(n)} \in \Gamma^{(n+1)}(H)$.

The annihilation operator lowers the number of particles in a given state by one and the creation operator increases the number of particles in a given state by one.

Theorem 1.2.2 *The annihilation and the creation operators satisfy the following canonical commutations:*

- $[\partial^*(f),\, \partial^*(g)] = [\partial(f), \partial(g)] = 0,$
- $[\partial(f), \partial^*(g)] = (f, g),$

where $[A, B]$ denotes the commutator defined by $[A, B] = AB - BA$.

The annihilation operator $\partial(f)$ is a derivation on a subspace of symmetrized Fock space $\Gamma(H)$ of sequences consisting of finitely many non zero elements

$$\partial(f)\,(F\,\widehat{\otimes}\,G) \;=\; \partial(f)F\,\widehat{\otimes}\,G + F\,\widehat{\otimes}\,\partial(f)G.$$

Further on in the following chapters, we will consider nuclear spaces and thus, by the notation $\otimes$ we will mean the π-completion, i.e. ε-completion of the tensor product space.

Number operator

Denote by Id the identity operator. Let now r be a real number with $|r| \leq 1$. Consider the operator $\Gamma(rId)$. It is a linear operator and

$$\Gamma(rId)(\sum_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} r^n X_n.$$

We can express this by

$$\Gamma(rId) = r^{\mathcal{N}}, \tag{1.5}$$

where $\mathcal{N}$ is an unbounded operator on $\Gamma(H)$ defined by

$$\mathcal{N}\left(\sum_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} n\, X_n, \qquad X_n \in \Gamma^{(n)}(H),$$

whenever the right-hand side converges.

$\mathcal{N}$ is a self-adjoint operator. In quantum field theory it is called the *number operator.*

The operators $\Gamma(e^{-t}Id) = e^{-t\mathcal{N}}$, $t \geq 0$ form an operator semigroup known as the Ornstein-Uhlenbeck semigroup. The operator $\Gamma(rId)$ can also be regarded as a generalization of the Mehler transform for real functions.

1.3 Deterministic Fractional Calculus

Fractional calculus is the field of functional analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order.

In recent years fractional operators of differentiation D^α and integration J^α, $\alpha \in \mathbb{R}$ are used in many applications in physics, control theory, heat conduction, electricity, mechanics, chaos and fractals. In evolution equations the time derivative is replaced with a derivative of fractional order. When modeling constitutive equations for viscoelastic bodies the relations between stress and strain involve linear fractional differential operators.

Depending on the definition, several types of fractional operators can be found in the literature. Here we focus our attention on basic definitions and properties of the Riemann-Liouville deterministic fractional operators of differentiation and integration and the Laplace transform.

In the framework of the Riemann-Liouville calculus, motivation for defining fractional integral of order α ($\alpha > 0$) is found in the Cauchy formula,

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau, \quad t > 0, \quad n \in \mathbb{N},$$

which reduces the calculation of n-fold primitive of a causal-function $f(t)$ (i.e. identically vanishing for $t < 0$) to a single integral of convolution type. In the natural way the above formula can be extended from integer values of the index to any positive real value by using the Gamma function and the property $\Gamma(n) = (n-1)!$.

Denoting by D^n, $n \in \mathbb{N}$, the operator of derivative of order n we note

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0,$$

and $D^n J^n = I$, $J^n D^n \neq I$, $n \in \mathbb{N}$, where I is the identity operator.

1.3.1 Fractional integral and fractional derivative

Denote by $\mathcal{D}(\mathbb{R})$ the space of compactly supported smooth functions in $\mathbb{R}$, by $\mathcal{D}'(\mathbb{R})$ its dual space, the space of Schwartz distributions and $\mathcal{D}'(\mathbb{R})$ its subspace consisting of distributions supported on $[0,\infty)$. Denote by $L^1_{loc^+}(\mathbb{R})$ the space of locally integrable functions u on $\mathbb{R}$ such that $u(t) = 0$ for $t < 0$. By D^k, $k \in \mathbb{N}$ is denoted the operator of differentiation $D^k = \frac{d^k}{dx^k}$.

Definition 1.3.1 *Consider α to be an arbitrary positive real number. Then for $u \in L^1_{loc^+}(\mathbb{R})$ the* left Riemann-Liouville fractional integral of order $\alpha > 0$

is defined by

$$J^{\alpha}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,u(\tau)\,d\tau, \quad t>0, \tag{1.6}$$

where $J^0 := I$ *is the identity operator and* Γ *is the Euler Gamma function* $\Gamma(a) := \int_0^{+\infty} e^{-x}x^{a-1}dx$ *having property* $\Gamma(a+1) = a\,\Gamma(a)$, *for* $Re\{a\} > 0$.

In particular, we have $Ju(t) = \int_0^t u(\tau)\,d\tau$, for $t>0$.

Fractional integration admits the semigroup property $J^{\alpha}J^{\beta} = J^{\alpha+\beta}$ and the commutative property $J^{\alpha}J^{\beta} = J^{\beta}J^{\alpha}$, for all $\alpha, \beta \in \mathbb{R}^+$. The effect of the operator J^{α} on power functions is given by

$$J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}t^{\alpha+\gamma}, \ \ \alpha>0, \ \gamma>-1, \ t>0. \tag{1.7}$$

In particular, its effect on a characteristic function is

$$J^{\alpha}\chi(0,t)(x) = \frac{1}{\Gamma(\alpha+1)}((t-x)^{\alpha} - (-x)^{\alpha}), \quad \alpha \neq 0,\, t\in\mathbb{R}^+.$$

The proofs of these properties are based on the properties of the two Eulerian integrals, the Gamma function Γ and the Beta function B. Recall, the Beta function is defined by $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$ and satisfies the property $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(b,a)$, for $Re\{a,b\} > 0$.

Lemma 1.3.1 *Let* $0<\alpha<1$ *and let* $f \in L^p(\mathbb{R})$, $1 \le p < \frac{1}{\alpha}$. *If* $J^{\alpha}f = 0$ *then* $f(x) = 0$ *for almost all* x.

Definition 1.3.2 *Let* $u \in L^1_{loc^+}(\mathbb{R})$ *and suppose that* u *belongs to the space of functions which have continuous derivatives on* $\mathbb{R}_+$ *up to the order* $k-1$, $k\in\mathbb{N}$ *and* k*th derivative is an integrable function on* $[0,a]$, *for every* $a>0$. *The* Riemann-Liouville fractional derivative of order $\alpha \geq 0$, $k-1 \le \alpha < k$ *for some* $k \in \mathbb{N}$ *is defined by*

$$D^{\alpha}u(t) := D^k J^{k-\alpha}u(t), \quad t>0. \tag{1.8}$$

If $\alpha = k \in \mathbb{N}$ then $D^{\alpha}u(t) = D^k u(t)$, for $t>0$.
Namely,

$$D^{\alpha}u(t) := \begin{cases} \frac{d^k}{dt^k}\left[\frac{1}{\Gamma(k-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-k}}\,d\tau\right], & k-1\le\alpha<k \\ \frac{d^k}{dt^k}\,u(t), & \alpha = k \end{cases}.$$

For $\alpha = 0$ one defines $D^0 = J^0 = I$. It follows that $D^\alpha u \in L^1_{loc^+}(\mathbb{R})$.

Note that $D^\alpha J^\alpha u = u$, for $u \in L^1_{loc^+}(\mathbb{R})$, $\alpha \geq 0$.

We have, for $\alpha \geq 0$ the relation

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \ \alpha \geq 0, \ \gamma > -1, \ t > 0.$$

Thus it it follows that $D^\alpha t^{\alpha-1} \equiv 0$, for $\alpha > 0$, $t > 0$, which implies that D^α is not right inverse to J^α. We have $J^\alpha D^\alpha t^{\alpha-1} \equiv 0$ but $D^\alpha J^\alpha t^{\alpha-1} \equiv t^{\alpha-1}$ for $t > 0$, $\alpha > 0$.

Note the remarkable fact that fractional derivative $D^\alpha u$ is not zero for the constant function $u(t) \equiv 1$ if α is not an integer number. In fact,

$$D^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad \alpha \geq 0, t > 0.$$

1.3.2 The Laplace convolution

In $\mathcal{D}'_+(\mathbb{R})$ we consider the *causal-function* $f_\alpha(\cdot)$ defined by

$$f_\alpha(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t), & \alpha > 0, \ t \in \mathbb{R} \\ \\ \frac{d^n}{dx^n} f_{\alpha+n}, & \alpha \leq 0, \ \alpha + n > 0, \ n \in \mathbb{N} \end{cases},$$

where H is the Heaviside function. It is clear that $f_0 = \delta$, $f_{-1} = \delta'$, etc, where δ is the Dirac δ-distribution and $H' = \delta$.

The causal-function f_α is locally absolutely integrable on $\mathbb{R}^+$ for all $\alpha > 0$ and f_α is vanishing for $t < 0$.

Definition 1.3.3 *The* Laplace convolution integral *of two causal-functions f_α and f_β, denoted by $f_\alpha * f_\beta$, is defined as*

$$f_\alpha(t) * f_\beta(t) := \int_0^t f_\alpha(t-\tau) f_\beta(\tau) d(\tau) = f_\beta(t) * f_\alpha(t).$$

Based on the properties of the Euler integrals, the composition rule

$$f_\alpha(t) * f_\beta(t) = f_{\alpha+\beta}(t), \quad \alpha, \beta > 0$$

is valid.

The Laplace convolution operator in $\mathcal{D}'_+(\mathbb{R})$ is the operator of fractional integration for $\alpha > 0$ and of fractional differentiation for $\alpha < 0$. Clearly, the fractional integral of order $\alpha > 0$ of a function $u \in L^1_{loc^+}(\mathbb{R})$ can be represented as follows

$$J^\alpha u(t) = f_\alpha(t) * u(t), \quad \alpha > 0.$$

For $\alpha < 0$ the causal-function f_α is called the *function of fractional differentiation.* If $0 \leq \alpha < k$, $k \in \mathbb{N}$ then we can use

$$\begin{aligned} D^\alpha u(t) &= f_{-\alpha} * u(t) \\ &= \frac{1}{\Gamma(-\alpha)} \int_{0^-}^{t^+} \frac{f(\tau)}{(t-\tau)^{1+\alpha}} d\tau. \end{aligned}$$

for the formal definition of the fractional derivative of order α.

The formal character follows from the fact that the kernel $f_{-\alpha}$ is not absolutely integrable and thus the integral is in general divergent. This is reflected through the non-commutative convolution property. Clearly, for $k \in \mathbb{N}$, $k-1 < \alpha < k$ we have

$$[f_{-k}(t) * f_{k-\alpha}(t)] * u(t) = f_{-k}(t) * [f_{k-\alpha}(t) * u(t)] = D^k J^{k-\alpha} u(t),$$

$$[f_{k-\alpha}(t) * f_{-k}(t)] * u(t) = f_{k-\alpha}(t) * [f_{-k}(t) * u(t)] = J^{k-\alpha} D^k u(t).$$

1.3.3 Laplace transform and Fourier transform

Now we give definitions of two integral transforms which will be used in the following chapters: the Laplace transform and the Fourier transform.

The *Laplace transform* of a function $f(t)$ is given by

$$\mathcal{L}\{f(t)\}(s) := \int_0^{+\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}.$$

The *Fourier transform* of a function f is defined by

$$\mathcal{F}(f)(y) = \widehat{f}(y) = \int_{\mathbb{R}} e^{-ixy} f(x)\, dx. \tag{1.9}$$

The Fourier transform is unitary on $L^2_{\mathbb{C}}(\mathbb{R}, dx)$. It is well known that the Hermite polynomials represent the sequence of eigenfunctions to the Fourier transform, i.e.

$$\mathcal{F}\left(h_n(\sqrt{2}x)\, e^{-\frac{x^2}{2}}\right) = i^n\, h_n(\sqrt{2}x)\, e^{-\frac{x^2}{2}}. \tag{1.10}$$

The Laplace transform is connected with the Fourier transform through the following relation

$$\mathcal{L}\{f(t)\} = \mathcal{F}[\, e^{-xt} f(t)](y) = \widehat{e^{-xy} f}(y), \quad s = x + iy.$$

The Laplace transform of the convolution integral of two functions coincides with product of the Laplace transforms of those two functions,

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}. \tag{1.11}$$

The same property stays also for the Fourier transform, i.e.

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

From the previous properties and rules $\mathcal{L}\{f_\alpha(t)\} = \frac{1}{s^\alpha}$ for $\alpha > 0$ and $Res > 0$ the important identity follows

$$\mathcal{L}\{J^\alpha f(t)\} = \mathcal{L}\{f_\alpha * u\} = \frac{1}{s^\alpha}\, \mathcal{L}\{u\}(s). \tag{1.12}$$

1.4 Basic Stochastic Analysis

Now we recall some basic results and concepts of probability theory, which can be understood as a mathematical model for the intuitive notion of uncertainty. Probability theory is used in many branches of pure mathematics, but also in modeling problems in physics, biology and economics. The modern period of probability theory is connected with names of Bernstein, Borel and Kolmogorov. Particularly, in 1933 Kolmogorov published his modern approach to probability theory, including the notion of a measurable space and a probability space.

We start this overview with definitions of probability spaces, random variables and classical stochastic processes on a given probability space, then we continue with some of the most important examples of classical stochastic processes which we will use in our work. We finish this section with presenting basic parts of stochastic integration, in particular we will introduce the Itô integral and the Itô-Poisson integral. For more information on basic stochastic analysis we refer to [14], [16], [39], [52], [60].

1.4.1 Probability space and random variables

Let Ω be a *sample space*, i.e. a non-empty set of all possible outcomes ω called *elementary events* or *states*, of a certain random experiment.

Definition 1.4.1 *A family $\mathcal{F}$ of subsets of a given sample space Ω is called a $\sigma-$algebra on Ω if:*

- $\Omega \in \mathcal{F}$,
- $A \in \mathcal{F}$ *implies* $A^c \in \mathcal{F}$, *where* $A^c = \Omega \setminus A$
- $A_1, A_2, ... \in \mathcal{F}$ *implies* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Elements of the $\sigma-$algebra $\mathcal{F}$ are $\mathcal{F}-$*measurable* sets and are called random events. The pair $(\Omega, \mathcal{F})$ is called the measurable space.

Definition 1.4.2 *A real function $P : \mathcal{F} \to [0, 1]$ which satisfies conditions:*

- $\sigma-$additivity, *i.e. if $A_1, A_2, ...$ are disjoint sets in $\mathcal{F}$ then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \textit{and}$$

- normalized condition $P(\Omega) = 1$

is called a probability measure.

The triplet $(\Omega, \mathcal{F}, P)$, where Ω is a space of elementary events, $\mathcal{F}$ a $\sigma-$algebra of events on Ω and P a probability measure on $\mathcal{F}$ is called a *probability space*. A probability space $(\Omega, \mathcal{F}, P)$ is called *complete* if $\mathcal{F}$ contains all subsets G of Ω with P measure zero, i.e. if $G \subseteq F$, $(F \in \mathcal{F})$ and $P(F) = 0$ implies $G \in \mathcal{F}$. A probability measure P_1 is called *absolutely continuous* with respect to measure P on a measurable space $(\Omega, \mathcal{F})$, if for every $A \in \mathcal{F}$ from $P(A) = 0$ it follows that $P_1(A) = 0$.

Theorem 1.4.1 (Radon-Nikodym theorem) *Let P and P_1 be two probability measures given on a measurable space $(\Omega, \mathcal{F})$ such that P_1 is absolutely continuous with respect to P. Then, there exists a unique non-negative measurable function $f : \Omega \to \mathbb{R}$ such that*

$$P_1(E) = \int_E f\, dP, \quad \textit{for every } E \in \mathcal{F}.$$

For any family $\mathcal{U}$ of subsets of Ω, the smallest $\sigma-$algebra $\mathcal{B}_{\mathcal{U}}$ containing $\mathcal{U}$, i.e. $\mathcal{B}_{\mathcal{U}} = \bigcap\{\mathcal{B} \,|\, \mathcal{B}$ is σ-algebra of subsets of $\Omega,\ \mathcal{U} \subset \mathcal{B}\}$ is the *$\sigma-$algebra generated by* $\mathcal{U}$.

In particular, we consider a minimal $\sigma-$algebra which contains all open sets on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, and call it the *Borel $\sigma-$algebra* on $\mathbb{R}^n$. The elements of $\mathcal{B}(\mathbb{R}^n)$ are called *Borel sets*.

Let $(\Omega, \mathcal{F}, P)$ be a given probability space, then a function $Y : \Omega \to \mathbb{R}^n$ is called *$\mathcal{F}$-measurable* if

$$Y^{-1}(A) = \{\, \omega \in \Omega \,|\, Y(\omega) \in A \,\} \in \mathcal{F},$$

for all Borel sets $A \in \mathbb{R}^n$.

Definition 1.4.3 *A $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$ from a complete probability space $(\Omega, \mathcal{F}, P)$ to a measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called a n-*dimensional random variable.

Lemma 1.4.1 *Let $X : \Omega \to \mathbb{R}^n$ be a random variable.*

$$\mathcal{U}(X) := \{X^{-1}(B) \,:\, B \in \mathcal{B}(\mathbb{R}^n)\}$$

is a σ-algebra, called the σ-algebra generated by X. This is the smallest sub-σ-algebra of $\mathcal{F}$ with respect to which X is measurable.

Thus, the probability measure $P_X = P \circ X^{-1}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, induced by a n-dimensional random variable X, is defined by

$$P_X(B) = P(X^{-1}(B)), \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^n),$$

and is called the *law* or the *distribution* of X. In probabilistic terms, the essential fact is that σ-algebra $\mathcal{U}(X)$ can be interpreted as the set which contains all relevant information about the random variable X.
For measure P_X there exists a unique function $F_X : \mathbb{R}^n \to [0, 1]$ such that

$$\begin{aligned} F_X(x) &= F_X(x_1, ..., x_n) \\ &= P\{X_1 \le x_1, ..., X_n \le x_n\} \\ &= P\{\omega \in \Omega : X(\omega) \le x\}. \end{aligned}$$

Function F_X is called the *distribution function* of a random variable X. Note that the structure of a probability space $(\Omega, \mathcal{F}, P)$ is transferred onto the space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_X)$.

Definition 1.4.4 *A random variable is called* discrete *if there exists a countable set S in $\mathbb{R}^n$ satisfying $P_X(S) = 1$.*

A discrete random variable X has the following expression

$$X(\omega) = \sum_{i=1}^{\infty} x_i I_{A_i}(\omega), \quad x_i \in S,$$

where $I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$ is the *indicator function* of an event $A \in \mathcal{F}$, such that $\bigcup_{i=1}^{\infty} A_i$ is a disjoint decomposition of Ω.

Definition 1.4.5 *A random variable X with measure P_X that is absolutely continuous with respect to the Lebesgue measure is called an* absolutely continuous random variable.

For such a random variable X, there exists a non-negative function $g(x)$, $x \in \mathbb{R}^n$, measurable with respect to the Borel σ-algebra satisfying

$$P_X(M) = \int_M g(x)\, dx, \quad \text{for} \ \ M \subset \mathbb{R}^n,$$

called the *probability density* function of a random variable X.

If $\int_\Omega |X(\omega)|\, dP(\omega) < \infty$, then the number

$$E_P(X) := \int_\Omega X(\omega)\, dP(\omega) = \int_{\mathbb{R}^n} x\, dP_X(x)$$

is called the *expectation* of X with respect to the measure P. Further on in this text, we will omit writing P in index of notation of the expectation value whenever it is clear under which probability measure P is expected value taken.

The *covariance matrix* of an n-dimensional random vector $X = (X_1, ..., X_n)$ is given by $B_X = [Cov(X_i, X_j)]_{1 \le i,j \le n}$, where

$$Cov(X_i, X_j) \; = \; E_P(X_i X_j) - E_P(X_i)\, E_P(X_j), \quad 1 \le i, j \le n.$$

In particular, $Cov(X_i, X_i) = Var(X_i)$, $1 \le i \le n$ and is called the *variance* of an element X_i. Equivalently, variance can be also calculated from

$$Var(X) \; = \; \int_\Omega \left[X - E_P(X)\right]^2 \, dP.$$

Theorem 1.4.2 *Matrix B is the covariance matrix of some random process if and only if it is symmetric and non-negative definite, i.e.*

$$\sum_{i,j=1}^{n} B_X(i,j)\, a_i\, a_j \; \ge \; 0, \quad \textit{for all} \quad a_1, ..., a_n \in \mathbb{R}.$$

A random variable X is said to have a finite moment of order $p \geq 1$, provided $E(|X|^p) < \infty$. In this case, the pth moment of X is defined by $E(X^p)$. The set of all random variables with finite pth moment is denoted by $L^p(P) = L^p(\Omega, \mathcal{F}, P)$.

It is convenient now to introduce an integral transform, which will later provide us with a useful means to identify normal random variables.

Definition 1.4.6 Characteristic function *of an n-dimensional random variable $X = (X_1, ..., X_n)$ is the function $f : \mathbb{R}^n \to \mathbb{C}$ defined by relation*

$$\begin{aligned} C_X(t_1, t_2, \cdots, t_n) &= E(e^{i(t,X)}) \\ &= E(e^{i\sum_{k=1}^{n} t_k X_k}), \quad t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n, \end{aligned}$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Characteristic function of an absolutely continuous random variable X represents the Fourier transform of its probability density function g. Namely,

$$C_X(t) = \int_{\mathbb{R}^n} e^{i(x,t)}\, g\, dP = \int_{\mathbb{R}^n} e^{i\sum_{k=1}^{n} t_k x_k}\, g(x_1, ..., x_n)\, dx_1 ... dx_n.$$

Characteristic function of every random variable X exists and uniquely determines the distribution of X. Namely, if X_1 and X_2 are random variables such that $C_{X_1}(t) = C_{X_2}(t)$ for all t, then their distribution functions also coincide $F_{X_1}(x) = F_{X_2}(x)$, for all x.

Theorem 1.4.3 (Properties of the characteristic function)

- $C_X(0) = 1,\ |C_X(t)| \leq 1,\ C_X(-t) = C_X(t),$
- *If $Y = \alpha_1 X + \alpha_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}$, then $C_Y(t) = C_X(\alpha_1 t)\, e^{it\alpha_2}$,*
- *If $E(X^n)$ exists, then the moments of a random variable can be computed from the derivatives of the characteristic function at the origin, i.e. we have*

$$EX^n = \frac{1}{i^n} C_X^{(n)}(t)\,|_{t=0}, \quad n = 1, 2, 3, ...$$

Theorem 1.4.4 (The Bochner-Minlos theorem) *Function C is the characteristic function of a random variable X if and only if C has the following properties:*

1. $C(0) = 1$,
2. $|C(t)| \leq 1$,
3. *C is continuous and*
4. *C is non-negative definite, i.e. for any set of real numbers $t_1, t_2, \cdots, t_n \in \mathbb{R}$ and complex numbers $z_1, z_2, \cdots, z_n \in \mathbb{C}$, $n \in \mathbb{N}$:*

$$\sum_{j,k=1}^{n} f(t_j - t_k)\, z_j\, \overline{z_k} \geq 0.$$

For the proof and more details we refer to [17], [19].

Theorem 1.4.5 *Let the characteristic function $C(t)$ of a given random variable X be an absolutely continuous function. Then its distribution function $F(x)$ is an absolutely continuous function and its corresponding probability density function $g(x)$ is continuous. Moreover,*

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx}\, C(t) dt.$$

Note that density g is obtained as inverse Fourier transform of the characteristic function C. Important property of characteristic functions, which is often applied in probability theory, is described by the following theorem.

Theorem 1.4.6 (Multiplication rule for characteristic functions) *If $X_1, ..., X_n$ are independent random variables then the characteristic function of their sum is equal to the product of characteristic functions, i.e.*

$$C_{X_1+\ldots+X_n}(t) = \prod_{k=1}^{n} C_{X_k}(t), \quad t \in \mathbb{R}^n.$$

We continue with the notion of Hilbert space of random variables.
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by

$$L^2(P) = L^2(\Omega, \mathcal{F}, P)$$

the space of square integrable random variables, i.e. the space of random variables which have the second moment finite $E(X^2) < \infty$. It is a Hilbert space with the norm $\|X\|^2_{L^2(P)} = E(X^2)$ induced by the scalar product

$$(X, Y)_{L^2(P)} = E_P(XY), \quad X, Y \in L^2(P).$$

Convergence of random variables in $L^2(P)$ is the mean square convergence. Namely, if we assume that X_n is a sequence of random variables in $L^2(P)$ for all $n \in \mathbb{N}$, then $X_n \overset{\mathrm{L}^2}{\to} X$, if $E|X_n - X|^2 \to 0$, when $n \to \infty$.

We now state the Bochner Minlos theorem for $S'(\mathbb{R})$.

Theorem 1.4.7 (The Bochner-Minlos theorem for infinite dimensional case) *A necessary and sufficient condition for the existence of a probability measure P on $S'(\mathbb{R})$ and a functional g on $S(\mathbb{R})$ such that*

$$g(\phi) = \int_{S'(\mathbb{R})} e^{i\langle\omega,\phi\rangle}\, dP(\omega), \quad \phi \in S(\mathbb{R})$$

is that g satisfies:

1. *$g(0) = 1$,*
2. *g is positive definite and*
3. *g is continuous in the Fréchet topology.*

This important theorem will be used in Chapter 2, when defining white noise probability measure. Proof of the previous theorem can be found in [19].

Gaussian random variable

We say that a random variable X is a one-dimensional *Gaussian* (*normal*) random variable with parameters m and σ^2, and write $X : \mathcal{N}(m, \sigma^2)$, if its density function is of the form

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad \text{for } x \in \mathbb{R}.$$

The expectation of a Gaussian random variable is $EX = m$ and variance $Var(X) = \sigma^2$.

An n-dimensional random vector $X = (X_1, ..., X_n)$ has a multi-dimensional *Gaussian* (*normal*) law $X : \mathcal{N}(m, B)$, with parameters m i B, where $m = (m_1, ...m_n) \in \mathbb{R}^n$ and B is a symmetric, regular, positive definite matrix with the inverse matrix A, if X has the density function

$$g_X(x_1, ..., x_n) = \sqrt{\frac{detB}{(2\pi)^n}}\, e^{-\frac{1}{2}(x-m)^T A (x-m)},$$

for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

Thus if $X = (X_1, ..., X_n) : \mathcal{N}(m, B)$ is an n-dimensional Gaussian random vector then $E(X_i) = m_i$, $i = 1, ..., n$, where $m = (m_1, ..., m_n)$ and the matrix B is the covariance matrix of a random vector X.

The characteristic function of an n-dimensional Gaussian random vector $X : \mathcal{N}(m, B)$ is given by

$$C_X(t_1, ..., t_n) = e^{i(t,m)-\frac{1}{2}t^T Bt}, \quad \text{where} \quad t = (t_1, ..., t_n). \tag{1.13}$$

Theorem 1.4.8 *Let $X_1, ..., X_n : \Omega \to \mathbb{R}$ be one-dimensional random variables. Then, an n-dimensional random variable $X = (X_1, ..., X_n)$ is Gaussian if and only if the random variable $Y = \lambda_1 X_1 + ... + \lambda_n X_n$ is Gaussian for every $\lambda_1, ..., \lambda_n \in \mathbb{R}$.*

Theorem 1.4.9 *Let $\{X_n\}_{n\in N}$ be a sequence of Gaussian random variables. If $X_n \stackrel{L^2}{\to} X$ then the mean square limit X is also a Gaussian random variable.*

Poisson random variable

A random variable X is said to be *Poisson random variable* of parameter $\lambda > 0$ if its density law is of the form

$$p_\lambda(k) = P\{X = k\} = e^{-\lambda}\,\frac{\lambda^k}{k!}, \quad k = 1, 2, ... \tag{1.14}$$

It is a discrete random variable with values in $\mathbb{N}_0$ and with $E(X) = \lambda$ and $Var(X) = \lambda$.

The characteristic function of Poisson distribution is

$$C_X(t) = \exp(\lambda(e^{it} - 1)). \tag{1.15}$$

The Poisson distribution is used, for example, to model stochastic processes with a continuous time parameter and jumps: the probability that the process jumps k times between the time-points s and t with $0 \leq s < t < 1$ is equal to $p_{\lambda(t-s)}(k)$.

Compound Poisson random variable

Let $Z(n)$, $n \in \mathbb{N}$ be a sequence of i.i.d. random variables with values in $\mathbb{R}$ with common law and let N be a Poisson random variable that is independent of all $Z(n)$. The *compound Poisson random variable* X is defined by $X = Z(1) + \ldots Z(N)$. One can think of X as a random walk with a random number of steps, which are controlled by the Poisson random variable N.

Exponential random variable

A random variable $X : \Omega \to (0, +\infty)$ has *exponential distribution of parameter* $\lambda > 0$ if

$$P\{X > t\} = e^{-\lambda t}, \quad \text{for all}\, t \geq 0. \tag{1.16}$$

Then X has a density function

$$g_X(t) = \lambda\, e^{-\lambda t} \chi_{(0,+\infty)}(t).$$

The expected value of an exponential random variable X is given by $E(X) = \frac{1}{\lambda}$ and its variance is $Var(X) = \frac{1}{\lambda^2}$. The exponential distribution plays a fundamental role in continuous time Markov processes because of the following result.

Lemma 1.4.2 (Memoryless property) *A continuous random variable $X : \Omega \to (0, +\infty)$ has an exponential distribution if and only if it has the memoryless property*

$$P\{X > s + t \,|\, X > s\} = P\{X > t\}, \quad \textit{for all } s, t > 0. \tag{1.17}$$

Conditional expectation

Definition 1.4.7 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : \Omega \to \mathbb{R}^n$ an n-dimensional random variable such that $E|X| < \infty$ and $\mathcal{A} \subset \mathcal{F}$ a $\sigma-$algebra. The unique function $E(X|\mathcal{A}) : \Omega \to \mathbb{R}^n$ which satisfies the following conditions:*

- *$E(X|\mathcal{A})$ is $\mathcal{A}$-measurable and*
- *$\int_A E(X|\mathcal{A})\, dP = \int_A X\, dP$, for all $A \in \mathcal{A}$*

is called the conditional expectation *of a random variable X with respect to $\sigma-$algebra $\mathcal{A}$.*

The existence and uniqueness of the conditional expectation follow from the Radon-Nikodym theorem. Let

$$P(M|\mathcal{A}) = E(I_M|\mathcal{A}), \quad \text{for} \quad M \in \mathcal{F}.$$

Then $P(\cdot|\mathcal{A})$ is called the *conditional probability* on a $\sigma-$algebra $\mathcal{F}$ with respect to $\sigma-$algebra $\mathcal{A}$. In the special case, when $\sigma-$algebra $\mathcal{A}$ is generated by a random variable Y, then the conditional expectation is denoted by $E(X|Y)$.

Theorem 1.4.10 *The main* properties od conditional expectation *are:*

- $E(E(X|\mathcal{A})) = E(X)$,
- *If* X *is* $\mathcal{A}$*-measurable, then* $E(X|\mathcal{A}) = X$ *a.s.,*
- *If* X *is independent of* $\mathcal{A}$*, then* $E(X|\mathcal{A}) = X$*,*
- *If* $X \geq 0$ *then is also* $E(X|\mathcal{A}) \geq 0$ *a.s.,*
- *If* $\mathcal{A} = \{\emptyset, \Omega\}$ *is trivial* $\sigma-$*algebra, then* $E(X|\mathcal{A}) \overset{\text{a.s.}}{=} E(X)$*,*
- *If* Y *is* $\mathcal{A}$*-measurable and* $E|XY| < \infty$*, then* $E(Y \cdot X|\mathcal{A}) = Y \cdot E(X|\mathcal{A})$*, where* $\cdot$ *represents the scalar product in* $\mathbb{R}^n$*.*

Theorem 1.4.11 *Let* $X_i : \Omega \to \mathbb{R}^n$*,* $i = 1, 2$ *be two random variables such that* $E|X_i| < \infty$*,* $i = 1, 2$*. Then*

$$E(aX_1 + bX_2|\mathcal{A}) \overset{\text{a.s.}}{=} aE(X_1|\mathcal{A}) + bE(X_2|\mathcal{A}), \quad a, b \in \mathbb{R}.$$

Theorem 1.4.12 *Let* $\mathcal{G}, \mathcal{A}$ *be* $\sigma-$*algebras such that* $\mathcal{G} \subset \mathcal{A}$*. Then the following is valid*

$$E(X|\mathcal{G}) = E(E(X|\mathcal{A})|\mathcal{G}).$$

Theorem 1.4.13 (Theorem of dominated convergence) *Let* $\{X_n\}$ *be a sequence of random variables which converges* $X_n \overset{\text{a.s.}}{\to} X$*. If there exists* $Y \in L^1(\Omega)$ *such that for all* $n \in \mathbb{N}$ $|X_n| \leq Y$ *a.s. then*

$$E(|X_n - X|\,|\mathcal{A}) \overset{\text{a.s}}{\to} 0.$$

Theorem 1.4.14 *Let* X_n *be non-negative random variables for all* $n \in \mathbb{N}$*. Then we have*

$$E(\sum_{n=1}^{\infty} X_n\,|\mathcal{A}) = \sum_{n=1}^{\infty} E(X_n\,|\mathcal{A}) \text{ a.s.}$$

If a random variable X is square integrable, but not necessarily measurable with respect to $\mathcal{A}$, then the conditional expectation $Y = E(X|\mathcal{A})$ represents the best approximation (in context of least squares) of X upon the class of all measurable functions with respect to Y. Moreover, if $\widetilde{Y}$ is $\mathcal{A}$-measurable then

$$E(\widetilde{Y} - X)^2 \geq E(Y - X)^2.$$

Hence it represents the *orthogonal projection* of a random variable X onto a closed convex subset of a Hilbert space.

1.4.2 Classical stochastic processes

This subsection is devoted to classical stochastic processes, their definitions, main properties and important examples. In particular we will focus on two special types of Lévy processes, the Wiener process (Brownian motion) and the Poisson process.

A classical stochastic process $X_t(\omega) = X(t, \omega)$, $t \in T \subseteq \mathbb{R}$, $\omega \in \Omega$ can be defined in three equivalent ways. It can be regarded either as a family of random variables $X_t(\cdot)$, $t \in T$, as a family of trajectories $X_\cdot(\omega)$, $\omega \in \Omega$, or as a family of functions $X : T \times \Omega \to \mathbb{R}$ such that for each fixed $t \in T$, $X(t, \cdot)$ is an $\mathbb{R}$-valued random variable and for each fixed $\omega \in \Omega$, $X(\cdot, \omega)$ is an $\mathbb{R}$-valued deterministic function, called a trajectory.

Using basic properties of stochastic processes, later on in Section 2.7, we will generalize the definition of a classical stochastic process and define generalized stochastic processes with respect to Gaussian and Poissonian measures.

Definition 1.4.8 *A real-valued* stochastic process *is a parameterized collection of random variables* $\{X_t\}_{t\in T}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$*, taking values in* $\mathbb{R}$.

The parameter space T is also called the *index set.* If $T = \mathbb{N}$ then the process is said to be a *discrete parameter process* and if T is not countable, the process is said to have a *continuous parameter.* Here we will usually consider T to be the halfline $[0, +\infty)$.

We may regard the stochastic process $X_t = X_t(\omega) = X(t, \omega)$ as a function of two variables $t \in T$ and $\omega \in \Omega$. Note that for each $t \in T$ fixed we obtain a $\mathcal{F}$-measurable function, i.e. a random variable

$$\omega \to X_t(\omega), \quad \omega \in \Omega$$

and for each $\omega \in \Omega$ fixed we can consider the function $X(\cdot, \omega) : T \to \mathbb{R}$, given by

$$t \to X_t(\omega), \quad t \in T,$$

called a *path, trajectory* or *realization* of a stochastic process X_t.

Definition 1.4.9 Finite-dimensional marginal distributions *of a stochastic process* $\{X_t\}_{t\in T}$ *are given by*

$$F_{t_1,...t_n}(x_1, ...x_n) = P\{ X_{t_1} \leq x_1, ..., X_{t_n} \leq x_n \},$$

where $t_1, ..., t_n \in T$ *and* $x_1, ..., x_n \in \mathbb{R}$, $n \in \mathbb{N}$.

A famous Kolmogorov theorem states that it is possible to construct a stochastic process having finite-dimensional marginal distribution functions equal to a given finite family of measures.

Theorem 1.4.15 *The family of finite-dimensional marginal distributions of a random process satisfies conditions:*

- consistency

$$F_{t_1,...,t_k,t_{k+1},...,t_n}(x_1, ..., x_k, +\infty, ..., +\infty) = F_{t_1,...t_k}(x_1, ...x_k) \qquad (1.18)$$

for every $k < n$ *and* $t_1, ...t_n \in T$, $k, n \in \mathbb{N}$ *and*

- symmetry

$$F_{t_1,...,t_n}(x_1, ..., x_n) = F_{t_{\sigma_1},...,t_{\sigma_n}}(x_{\sigma_1}, ..., x_{\sigma_n}) \qquad (1.19)$$

for all $n \in \mathbb{N}$ *and all permutations* σ *on* $\{1, 2, ..., n\}$.

Theorem 1.4.16 (The Kolmogorov extension theorem)
Let $\{F(t_1, .., t_n, x_1, ..., x_n),$ *for finite* $\{t_1, ..t_n\} \subset T\}$ *be a family of functions which satisfy the consistency condition* (1.18) *and the symmetry condition* (1.19)*. Then there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and a stochastic process* $\{X_t\}_{t\in T}$ *defined on* Ω *such that all finite-dimensional marginal distributions of* X_t *are equal to the given family of functions* F*.*

The mean and the covariance function of a second order process $\{X_t\}_{t\in T}$, i.e. process with $E(X_t^2) < \infty$ for all $t \in T$, are defined by $m_X(t) = E(X_t)$ and $B_X(t, s) = Cov(X_t, X_s)$.

A stochastic process having all finite-dimensional marginal distributions invariant with respect to translation of the time component, i.e. for all $t_i, t_i + h \in T$, $i \in \mathbb{N}$ and every $h > 0$

$$F_{t_1+h,...,t_n+h}(x_1, ..., x_n) = F_{t_1,...,t_n}(x_1, ..., x_n),$$

is called a *stationary* stochastic process. A stationary process X_t has finite second moments and the corresponding covariance function is of the form

$$Cov(X_t, X_s) \;=\; B_X(t, s) \;=\; B(t - s), \quad \text{for} \;\; t \geq s \geq 0.$$

Suppose that $\{X_t\}$ and $\{Y_t\}$ are stochastic processes on same probability space $(\Omega, \mathcal{F}, P)$. Then, $\{X_t\}$ is called a *version* or *modification* of $\{Y_t\}$ if

$$P\{\omega \,|\, X_t(\omega) = Y_t(\omega)\} = 1 \quad \text{for all } t.$$

It is clear that if $\{X_t\}$ is a version of $\{Y_t\}$, then X_t and Y_t have the same finite-dimensional distributions. Although two processes of such a type are the same, their path properties may be different.

Theorem 1.4.17 (The Kolmogorov continuity criterion) *Let* $X = \{X_t\}_{t \in T}$ *be a stochastic process which satisfies the condition: for all* $T > 0$ *there exist positive constants* α, β, D *such that*

$$E\left(|X_t - X_s|^{\alpha}\right) \;\leq\; D \cdot |t - s|^{1+\beta}, \quad 0 \leq s, t \leq T. \tag{1.20}$$

Then, there exists a continuous version of a process X*.*

Let H be a Hilbert space of random variables which have finite second moments and zero mean value. Let $X = \{X_t\}_{t \in T}$ be a stochastic L^2-process. Let $H(X)$ consist of all finite linear combinations of the form

$$a_1 X_{t_1} + a_2 X_{t_2} + ... + a_n X_{t_n}, \qquad \text{for all} \;\; t_1, t_2, ..., t_n \in T$$

and mean-square limits of such linear combinations. Subspace $H(X)$ in H is the *Hilbert space of the stochastic process* X_t. A stochastic process can be regarded as a function in the Hilbert space H, i.e. as a curve $c(X)$ in H. Then, $H(X)$ is the minimal subspace of H which contains the curve $c(X)$.

Martingales

Now, we focus ourselves on a brief review of definition and some properties of a martingale.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family of sub-$\sigma-$algebras $\{\mathcal{F}_t\}$ of σ-algebra $\mathcal{F}$ is called a *filtration* if for every $s < t$ it follows that $\mathcal{F}_s \subset \mathcal{F}_t$. A stochastic process $\{X_t\}_{t \in T}$ is called *adapted* to the filtration $\{\mathcal{F}_t\}$ if for every $t \in T$ random variable $X_t(\omega)$ is $\mathcal{F}_t$-measurable.

Definition 1.4.10 *A stochastic process* $\{M_t\}$ *is a* martingale *with respect to the filtration* $\{\mathcal{F}_t\}$ *if the following is valid:*

- M_t *is* $\{\mathcal{F}_t\}$*-measurable for every* t (adaptivity property),
- $E_P(|M_t|) < \infty$*, for every* $t \in T$ *and*
- $E_P(M_t \,|\, \mathcal{F}_s) = M_s$*, for every* $s \leq t$.

Definition 1.4.11 *A stochastic process* $\{X_t\}$ *is a* Markov *process if for every* $t > s$ *and every Borel set* $B \in \mathbb{R}$ *follows*

$$P\{X_t \in B \,|\, \mathcal{F}_s\} = P\{X_t \in B \,|\, X_s\}, \ a.s. \tag{1.21}$$

For Markov process, the parameter t is interpreted as time and values X_t describe the rate of change of evolution stages of the a certain stochastic physical system during time.

1.4.3 Important examples of classical processes

a) Gaussian process

Gaussian processes form a class of stochastic processes widely used in pure and in applied mathematics. Among all Gaussian processes, Brownian motion and fractional Brownian motion are explored the most. Some typical examples in applications can be found in modeling of telecommunication traffic, where the fractional Brownian motion is used. In real analysis the Laplace operator is directly connected to the Brownian motion, and in the theory of stochastic processes many processes can be represented and investigated as transformations of the Brownian motion.

Definition 1.4.12 *A real-valued stochastic process* $\{X_t\}_{t\in T}$ *is said to be a* Gaussian (normal) process *if each of its finite-dimensional marginal distributions is a multi-dimensional Gaussian random variable.*

Recall, every Gaussian process is uniquely determined by its mean function and the covariance function.

b) Brownian motion

Botanist Robert Brown in 1826 observed the irregular motion of pollen particles suspended in water and noted that the path of a given particle is very irregular, having a tangent at no point, and the motions of two distinct particles appear to be independent. In 1900 Bachelier described fluctuations in stock prices mathematically and essentially discovered first certain results later extended by Einstein in 1905. Einstein suggested that the main characteristics of this motion were randomness, its independent increments, its Gaussian distribution and its continuity.

Definition 1.4.13 *A real valued stochastic process* $\{B_t : t \in [0, \infty)\}$ *is called a one-dimensional* Brownian motion *with a parameter* σ^2 *if:*

- $B_0 = 0$ *a.s,*
- *increments are independent and*
- $B_t - B_s$ *has distribution* $\mathcal{N}(0, (t-s)\,\sigma^2)$ *for all* $0 \le s \le t$.

A stochastic process $X_t(\omega)$ can be interpreted as a realization of a certain experiment ω at the moment of time t, thus the Brownian motion B_t is interpreted as a position of a pollen particle in the moment t.

One-dimensional density function of Brownian motion is

$$g_1(t,x) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{x^2}{2\sigma^2 t}}.$$

Notice that Brownian motion is a centered Gaussian process with the covariance function

$$Cov(B_t, B_s) = \sigma^2 \min\{t, s\}.$$

In particular $Var(B_t) = \sigma^2\, t$, $t > 0$.

Brownian motion satisfies the Kolmogorov continuity condition (1.20) with constants $\alpha = 4$, $D = n(n+2)$ and $\beta = 1$ (for proof see for example [52] and references therein) and therefore it has a continuous version. From now we will assume B_t is such a continuous version.

In Definition 1.4.13, we have assumed that Brownian motion is defined on an arbitrary probability space $(\Omega, \mathcal{F}, P)$.

The mapping

$$\Omega \to C([0, +\infty), \mathbb{R})$$

defined by $\omega \mapsto B_\cdot(\omega)$ induces a probability measure $P_B = P \circ B^{-1}$, called the *Wiener measure*, on the space of continuous functions $C = C([0, +\infty), \mathbb{R})$ equipped with its Borel σ-field $\mathcal{B}_C$. Then we can take as canonical probability space for the Brownian motion the space $(C, \mathcal{B}_C, P_B)$. All random variables in this canonical space are the evaluation mappings $X_t(\omega) = \omega(t)$.

Brownian motion admits the Markov property described by (1.21) i.e. a distribution function of difference $B_t - B_s$ on an interval (s, t) does not depend on the past. Clearly, for $t_1 < t_2 < \cdots < t_n$, the difference $B_{t_n} - B_{t_1}$ can be represent as a sum of independent random variables

$$B_{t_n} - B_{t_1} = [B_{t_n} - B_{t_{n-1}}] + \cdots + [B_{t_3} - B_{t_2}] + [B_{t_2} - B_{t_1}],$$

thus for $0 < t_1 < \cdots < t_n$ the *n-dimensional density function* is defined by

$$g_n(t_1, \cdots, t_n; x_1, \cdots, x_n) = g_1(t_1, x_1) g_1(t_2 - t_1, x_2 - x_1) \cdots g_1(t_n - t_{n-1}, x_n - x_{n-1}).$$

Brownian motion is a Gaussian process almost all whose trajectories are continuous, but nowhere differentiable functions. This statement means that classical stochastic process which is equal to the first derivative of Brownian motion does not exist. We overcome this problem by defining the generalized derivative of Brownian motion called the *white noise*.

The sum of square of differences of Brownian motion, denoted by

$$\sum_{k=1}^{n} (\triangle B_k)^2 = \sum_{k=1}^{n} [B_{t_k} - B_{t_{k-1}}]^2$$

converges in mean square to the length of the interval, as the norm of the subdivision tends to zero.

Theorem 1.4.18 (The Kolmogorov theorem) *Let $a = t_0 < t_1 < \cdots t_n = b$. Then*

$$\sum_{k=1}^{n} (\triangle B_k)^2 \to \sigma^2 (b - a), \quad \max(\triangle t_k) \to 0.$$

On the other hand, the total variation is infinite with probability one. The trajectories of Brownian motion B_t have infinite variation on any finite interval. i.e. for any $A \in \mathbb{R}$ stays

$$P\{\sum_{k=1}^{\infty} |B_{t_k} - B_{t_{k-1}}| > A\} \to 1, \quad \max(\triangle t_k) \to 0.$$

For any $a > 0$ the process $\{\frac{1}{\sqrt{a}} B_{at}\}_{t \geq 0}$ is a Brownian motion (this property is called the *self-similarity* property).

Note one more important property: Brownian motion B_t is a martingal with respect to σ-algebra $\mathcal{F}$ generated by $\{B_s : s \leq t\}$.

Example 1.4.1 (Stochastic processes related to Brownian motion)

- Brownian motion with drift *is the process*

$$X_t = B_t + at, \quad t \geq 0,$$

$a \in \mathbb{R}$ is a constant. It is a Gaussian process with $E(X_t) = at$ and $B_X(s,t) = \sigma^2 \min\{s,t\}$.

- Geometric Brownian motion *is the exponential of a Brownian motion with drift*

$$X_t = e^{B_t + at}, \quad t \geq 0,$$

where $a \in \mathbb{R}$. It is not a Gaussian process and the probability distribution of X_t is lognormal. This process is proposed by Black, Scholes and Merton as model for the curve of prices of financial assets.

c) Poisson process

Definition 1.4.14 *A stochastic process $\{N_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a* Poisson process of intensity λ *if it satisfies the following:*

- $N_t = 0$,
- *for any $n \geq 1$ and any $0 \leq t_1 \leq ... \leq t_n$ the increments $N_{t_n} - N_{t_{n-1}}, ..., N_{t_2} - N_{t_1}$ are independent random variables,*
- *for any $0 \leq s < t$, the increment $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t-s)$*

$$P\{N_t - N_s = k\} \ = \ e^{-\lambda(t-s)} \, \frac{[\lambda(t-s)]^k}{k!}, \quad k = 0, 1, 2, ..., \tag{1.22}$$

where $\lambda > 0$ is a fixed constant.

Increments of a Poisson process are independent and stationary. Poisson process can be constructed from a sequence X_n, $n \geq 1$ of independent random variables with exponential law of parameter $\lambda > 0$, defined by (1.16). Clearly, if we set $L_0 = 0$ and $L_n = X_1 + X_2 + ... + X_n$ for $n \geq 1$, then $\lim\limits_{n\to\infty} L_n = \infty$ a.s. The process $\{N_t\}_t \geq 0$, which represents the arrival process associated with the interarrival times X_n

$$N_t = \sum_{n=1}^{\infty} n \, \chi_{L_n \leq t < L_{n+1}}.$$

is a Poisson process with parameter λ. Notice that $E(N_t) = \lambda t$. Thus λ is the expected number of arrivals in an interval of unit length, or in another words, λ is the *arrival rate*. The expected time until a new arrival is $\frac{1}{\lambda}$. Moreover, $Var(N_t) = \lambda t$.

Sample paths of a Poisson process are discontinuous with jumps of size 1. However, a Poisson process is continuous in mean of square:

$$E\left[(N_t - N_s)^2\right] \;=\; \lambda(t-s) + [\lambda(t-s)]^2 \to\; 0, \quad s \to t.$$

Notice that we cannot apply here the Kolmogorov continuity criterion (1.20).

Definition 1.4.15 *A process* $\{M_t\}_{t\geq 0}$ *defined by*

$$M_t := N_t - \lambda t \tag{1.23}$$

is called compensated Poisson process with intensity $\lambda > 0$.

It is clear that $E(M_t) = 0$ and $Var(M_t) = \lambda t$ for all t.

A compensated Poisson process is a càdlàg (has continuous paths from the right with left-sided limits) martingale with respect to the σ-algebra $\mathcal{F}_t^N$ generated by N_s, $0 \leq s \leq t$, i.e. $E(M_t - M_s \mid \mathcal{F}_s^N) = 0$.

Let $\{Z(n),\, n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}$ with common law and let N be a Poisson process of intensity λ that is independent of all the $Z(n)$. The *compound Poisson process* Y is defined by

$$Y(t) = Z(1) + \cdots + Z(N(t)),$$

for all $t \geq 0$, so each $Y(t)$ is a Poisson process of intensity λt.

The Charlier polynomials

Consider now the generating function

$$F_\lambda(k,t) := e^{-\lambda t}\,(1+\lambda)^k$$

of the Poisson density $p_k(t) = \frac{(\lambda t)^k}{k!}\,e^{-\lambda t}$.

One can prove that for $\lambda \in (-1,1)$ it holds

$$F_\lambda(k,t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,C_n(k,t), \tag{1.24}$$

where $C_n(k,t)$ are the *Charlier polynomials* of order $n \in \mathbb{N}$ and of parameter $t \geq 0$ defined by

$$C_0(k,t) := 1,$$

$$C_1(k,t) := k - t, \quad k \in \mathbb{R},\, t \geq 0$$

and by the induction relation

$$C_{n+1}(k,t) := (k - n - t)\, C_n(k,t) - n\, C_{n-1}(k,t), \quad n \in \mathbb{N}.$$

The Charlier polynomials represent orthogonal polynomials for the Poisson distribution with parameter $t \geq 0$. The equivalent definition is

$$C_n(k,t) = \sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j}\, t^{-j}\ (k)_j,$$

where $(k)_j$ denotes the descending factorial $k(k-1)...(k-j+1)$ with $(k)_0$ interpreted as 1.

d) Lévy process

In this part of thesis we deal with a more general class of processes, the Lévy processes. First we have introduced the special cases of Lévy processes, Brownian motion, described in Section 1.4.3 and Poisson process presented in Section 1.4.3. For theory of Levý processes we refer to [3], [18], [19], [64].

Definition 1.4.16 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. A real valued stochastic process* $\eta_t = \eta(t,\omega)$, $t \in [0,\infty)$, $\omega \in \Omega$ *is called an one-dimensional* Lévy process *if it satisfies the conditions:*

- $\eta(0) = 0$ *a.s.*
- η *has independent increments*
- η *has stationary increments*
- η *is stochastically continuous and*
- η *has càdlaàg paths, i.e. paths of* η *are continuous from the right (continue à droite) with left-sided limits (limites à gauche).*

Note that both the Brownian motion and the Poisson process are temporary homogeneous Lévy processes, meaning the probability distribution of the increment $X_{t+h} - X_t$, for $h > 0$ is independent of t. A Lévy process has stationary, independent increments like Brownian motion, but it differs from B_t in that it does not necessarily have continuous paths. Almost all sample functions of a Brownian motion are continuous while those of a Poisson process are discontinuous, and they increases only by jumps of unit magnitude.

The reproducing property is satisfied by both the Gaussian and the Poisson distributions. Namely, a given distribution has the reproducing property if for independent random variables X and Y having the common distribution law it follows that their sum $X+Y$ also has the same distribution law.

The jump of η at time t is defined by

$$\triangle\eta(t) = \eta(t) - \eta(t^-).$$

Put $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the topology of all Borel sets $S \subseteq \mathbb{R}$ such that $\overline{S} \subset \mathbb{R}_0$. If $S \in \mathcal{B}(\mathbb{R}_0)$ and $t > 0$ we define $N(t,S)$ to be the number of jumps of $\eta(\cdot)$ of size $\triangle\eta(s) \in S$, $s \leq t$. Since the paths are càdlaàg then $N(t,S) < \infty$ for all $t > 0$, $S \in \mathcal{B}(\mathbb{R}_0)$. Then for all $\omega \in \Omega$ the function

$$(a,b) \times S \mapsto N(b,S) - N(a,S), \quad 0 \leq a < b < \infty,\ S \in \mathcal{B}(\mathbb{R}_0)$$

defines a measure on $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R}_0)$, called the *Poisson* random measure of η. The differential form of this measure is denoted by $N(dt,dz)$. The *Lévy measure* ν of $\eta(\cdot)$ is defined by

$$\nu(S) = E[N(1,S)], \quad S \in \mathcal{B}(\mathbb{R}_0).$$

The Lévy measute ν determines the law of $\eta(\cdot)$.

We continue with the Lévy-Khintchine formula.

Theorem 1.4.19 (The Lévy-Khintchine formula) *Let η be a Lévy process with the Lévy measure ν. Then*

$$\int_{\mathbb{R}} \min\{1, z^2\}\, \nu(dz)\ < \infty \tag{1.25}$$

and

$$E(e^{iu\eta}) = e^{t\psi(u)}, \quad u \in \mathbb{R} \tag{1.26}$$

where

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\alpha u \int_{|z|<1} (e^{iuz} - 1 - iuz)\, \nu(dz) + \int_{|z|\geq 1} (e^{iuz} - 1)\nu(dz) \tag{1.27}$$

for some constants $\alpha, \sigma \in \mathbb{R}$. Conversely, given constants $\alpha, \sigma \in \mathbb{R}$ and a measure ν such that (1.25) *is satisfied, there exists a unique Lévy process η such that* (1.26) *and* (1.27) *hold.*

A complete description of a Lévy process is given by the following theorem.

Theorem 1.4.20 (The Itô-Lévy decomposition theorem) *Let η be a Lévy process. Then η can be written in the form*

$$\eta_t = a_1 t + \sigma B_t + \int_{|z|<1} z\widetilde{N}(t,dz) + \int_{|z|\geq 1} zN(t,dz) \tag{1.28}$$

where a_1, σ are constants, B is a Brownian motion, $N(\cdot,\cdot)$ is the Poisson random jump measure of η, $\widetilde{N}(ds,dz) = N(ds,dz) - \nu(dz)ds$ is the compensated Poisson random measure of η and $\eta(dz)$ the Lévy measure of η.

We conclude that every Lévy process can be decomposed into the sum of three terms. The first term, seen as a continuous part of a Lévy process is represented by a Brownian motion with drift. The second term is the process $\int_{|z|<1} z\widetilde{N}(t,dz)$ which represents a compensated sum of small jumps and the third, given by the process $\int_{|z|\geq 1} zN(t,dz)$ that describes the large jumps in (1.28) is a compound Poisson process.

e) Fractional Brownian motion

Fractional Brownian motion represents a natural one-parameter extension of a standard Brownian motion, represented by the Hurst parameter H. The parameter H is called after the climatologist Hurst, who developed statistical analysis of the early water run-offs of the river Nile. The Hurst index H allows values in interval $(0,1)$ and in particular, for $H = \frac{1}{2}$ a fractional Brownian motion coincides with a standard Brownian motion.

Fractional Brownian motion is a processes with dependent increments which have long-range dependence and self-similarity properties. Many problems in hydrology, telecommunications, queueing theory and mathematical finance gave motivation to input noises without independent increments which have long-range dependence and self-similarity properties to appropriate models. If $H > \frac{1}{2}$ then fractional Brownian motion has a certain memory feature and this property has been used, for example, in the modeling of weather derivatives, the temperature at a specific place as a function of time, in the modeling the water level in a river as a function of time, when describing the widths of consecutive annual rings of a tree or when describing the values of the log returns of a stock. In addition, if $H < \frac{1}{2}$ then fractional Brownian motion has a certain turbulence feature and this property found applications, for example in mathematical finance in the modeling of financial turbulence, i.e. empirical volatility of a stock or in modeling the prices of electricity in a liberated Nordic electricity market.

Fractional Brownian motion was first introduced within a Hilbert spaces framework by Kolmogorov in 1940, where it was called the Wiener Spirals. He was the first to consider continuous Gaussian process with stationary increments and with self-similarity property. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who in paper [41] from 1968, provided a stochastic integral representation of this process in terms of a standard Brownian motion on an infinite interval.

Definition 1.4.17 *Fractional Brownian motion with the Hurst index* $H \in (0,1)$ *on a probability space* $(\Omega, \mathcal{F}, P)$ *is defined to be a Gaussian process* $B^{(H)} = \{B_t^{(H)}(\cdot),\ t \in \mathbb{R}\}$ *having properties:*

- $B_0^{(H)} = 0$ *a.s.,*
- *zero expectation* $E[B_t^{(H)}] = 0$ *for all* $t \in \mathbb{R}$, *and*
- *the covariance function*

$$E[B_s^{(H)} B_t^{(H)}] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}, \ \ s, t \in \mathbb{R}. \tag{1.29}$$

Fractional Brownian motion is a centered Gaussian process with non-independent stationary increments and its dependence structure is modified by the Hurst parameter $H \in (0,1)$.

For $H = \frac{1}{2}$ the covariance function can be written in the form

$$E(B_t^{(\frac{1}{2})} B_s^{(\frac{1}{2})}) = \min\{s, t\}$$

and the process $B_t^{(\frac{1}{2})}$ becomes a standard Brownian motion and it has independent increments. From (1.29) it follows that

$$E(B_t^{(H)} - B_s^{(H)})^2 = |t-s|^{2H}$$

and according to the Kolmogorov continuity criterion (1.20), stated in Theorem 1.4.17, with values $\alpha = 2$, $D = 1$ and $\beta = 1$, we conclude that fractional Brownian motion $B^{(H)}$ has a continuous modification. From now on we assume for $B^{(H)}$ to be that continuous version.

Furthermore, for all $n \in \mathbb{N}$ it holds

$$E(B_t^{(H)} - B_s^{(H)})^n = \sqrt{\frac{2^n}{\pi}}\Gamma(\frac{n+1}{2})|t-s|^{nH}.$$

The parameter H controls the regularity of trajectories.

The characteristic function has the form

$$\varphi_\lambda(t) := Ee^{i\sum\limits_{k=1}^{n}\lambda_k B_{t_k}^{(H)}} = e^{-\frac{1}{2}(C_t\lambda,\lambda)},$$

where $C_t = (EB_{t_k}^{(H)}B_{t_i}^{(H)})_{1\le k\le n}$ and $(\cdot,\cdot)$ is the scalar product on $\mathbb{R}^n$.

The covariance function (1.29) is homogeneous of order $2H$, thus fractional Brownian motion $B^{(H)}$ is an H self-similar process, i.e.

$$B_{\alpha t}^{(H)} = \alpha^H B_t^{(H)}, \quad \alpha > 0.$$

Hence, self-similarity can be considered as a fractal property in probability.

For $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ and $t_1 < t_2 < t_3 < t_4$ it follows that

$$E(B_{t_4}^{(H)} - B_{t_3}^{(H)})(B_{t_2}^{(H)} - B_{t_1}^{(H)}) = H(2H-1)\int_{t_1}^{t_2}\int_{t_3}^{t_4}(u-v)^{2H-2}\,dudv.$$

Therefore, the increments are positively correlated for $H \in (\frac{1}{2},1)$ and negatively correlated for $H \in (0,\frac{1}{2})$. For any $n \in \mathbb{Z}$, $n \neq 0$ the autocovariance function is given by

$$\begin{aligned} r(n) := E[B_1^{(H)}(B_{n+1}^{(H)} - B_n^{(H)})] &= H(2H-1)\int_0^1\int_n^{n+1}(u-v)^{2H-2}dudv \\ &\sim H(2H-1)|n|^{2H-1}, \quad \text{when } |n| \to \infty. \end{aligned}$$

For $H \in (\frac{1}{2},1)$ fractional Brownian motion has the long-range dependence property $\sum_{n=1}^{\infty} r(n) = \infty$ and for $H \in (0,\frac{1}{2})$ the short-range property $\sum_{n=1}^{\infty}|r(n)| < \infty$. For $H \in (\frac{1}{2},1)$ the difference sequence $B_{n+1}^{(H)} - B_n^{(H)}$, $n \ge 0$ presents an aggregation behavior which can be used to describe cluster phenomena and for $H \in (0,\frac{1}{2})$ this sequence can be used to model sequence with intermittency. For more details in applications we refer to [44], [47].

Furthermore, note that fractional Brownian motion is neither a semimartingale (except for $H = \frac{1}{2}$) nor a Markov process.

Because of these properties fractional Brownian motion has been suggested as a useful tool in modeling in finance and physics. More details on fractional Brownian motion, modeling and applications can be also found in [7], [8], [13], [20], [26], [41], [46], [62].

1.4.4 Stochastic integration

The problem of defining a stochastic integral with respect to Brownian motion in the Riemann-Stieltjes sense arises from the fact that the total variation of the path of Brownian motion is infinite almost surely. Moreover the paths of Brownian motion are nowhere differentiable almost surely. Two the most common concepts to overcome this problem are the concept of the Itô integral and the Stratonovich integral (difference is a consequence of the choice of the partition points of an integration interval). Here we will present the construction of the Itô integral. First we will define the stochastic integral for simple processes and then extend the definition to an appropriate class of processes, i.e. to the class of predictable processes. The reason of defining the stochastic integrals for integrands which are predictable processes is the usage of martingale theory for such a construction. Our aim is to define an integral of the form

$$\int_0^t f(s,\omega)\, dB_s(\omega), \quad t \in [S,T]$$

where B_t is an one-dimensional Brownian motion. First, a given function $f(t,\omega)$, i.e. a stochastic process which satisfy certain initial assumptions (for process to be predictable), will be approximated by the sum

$$\sum_{j=0}^{2^n-1} f(t_j^*,\omega)\, \chi_{[t_j,t_{j+1})}, \quad \text{for} \quad t_j^* \in [t_j,t_{j+1}).$$

Then definition of the stochastic integral $\int_S^T f(t,\omega)\, dB_t(\omega)$ is obtained by taking limits of sums

$$\sum_{j=0}^{2^n-1} f(t_j^*,\omega)\, [B_{t_{j+1}} - B_{t_j}](\omega),$$

when the maximal lenght of partition intervals $[t_j,t_{j+1})$ tends to zero. The choice of t_j^* leads us to several different types of integrals. In particular, if we choose the left end point $t_j^* = t_j$, we obtain the *Itô integral* and if we choose the mid-point, $t_j^* = \frac{1}{2}(t_j + t_{j+1})$, we obtain the *Stratonovich* integral.

In general Stratonovich integral has advantage of leading to ordinary chain rule formulas under a transformation (change of variable), i.e. there are no second order terms in Stratonovich analogue of the Itô transformation formula (1.35). This property makes the Stratonovich integral natural to use, for example, in connection with stochastic differential equations on manifolds.

Also the Stratonovich integral interpretation is the most frequently used interpretation within the physical sciences. On the other hand, Stratonovich integrals are not martingales as Itô integrals are. This gives the Itô integral an important computational advantage in many real-world applications, such as in financial mathematics for modeling stock prices, or in biology. In following we describe both concepts of stochastic integrals but in further chapters we will base our discussion on the Itô type of integrals.

More information on the Stratonovich type of integral can be found in [60], and on the Itô integral in [14], [19], [52].

The Itô integral

Assume that $B_t = B_t(\omega)$ is a one-dimensional Brownian motion given on a probability space $(\Omega, \mathcal{F}, P)$, starting at zero. For $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra generated by the random variables $\{B_s\}_{s\leq t}$. Then for $0 \leq t < s$ we have an increasing family $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}$.

Let $0 \leq S < T$. The class of functions $f(t,\omega) : [0,+\infty) \times \Omega \to \mathbb{R}$ for which the Itô integral will be defined is denoted by $\mathcal{L} = \mathcal{L}(S,T)$. It is the class of $\mathcal{F}_t$-adapted functions (meaning that $f(t,\cdot)$ is $\mathcal{F}_t$-measurable), such that mapping $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable and the condition

$$E\left(\int_S^T f^2 dt\right) < \infty$$

is satisfied.

A stochastic process $\phi = \{\phi_t\}_{t\geq 0} \in \mathcal{L}(S,T)$ is called a *simple* or *elementary* process if it is of the form

$$\phi(t,\omega) = \phi_t(\omega) = \sum_{j=0}^{2^n-1} e_j(\omega) \cdot \chi_{[j\cdot 2^{-n},\,(j+1)\cdot 2^{-n})}(t),$$

where χ is the characteristic function, $n \in \mathbb{N}$ and e_j is $\mathcal{F}_{t_j}$-measurable provided $E(e_j)^2 < \infty$.

Definition 1.4.18 *The* Itô integral *of an elementary function $\phi \in \mathcal{L}(S,T)$ on the interval (S,T), denoted by $I(\phi)$, is defined by*

$$I(\phi) := \int_S^T \phi(t,\omega)\, dB_t(\omega) = \sum_{j=0}^{2^n-1} e_j(\omega)\, [B_{t_{j+1}} - B_{t_j}](\omega),$$

with points

$$t_j = t_j^{(n)} = \begin{cases} j\cdot 2^{-n}, & S \leq j\cdot 2^{-n} \leq T \\ S, & j\cdot 2^{-n} < S \\ T, & j\cdot 2^{-n} > T \end{cases}$$

If elementary function $\phi \in \mathcal{L}(S,T)$ is bounded then the Itô isometry holds

$$E\left[\left(\int_0^T \phi(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_0^T \phi^2(t,\omega)dt\right]. \tag{1.30}$$

We use the isometry (1.30) to extend the definition from elementary functions to functions in $\mathcal{L}(S,T)$.

Definition 1.4.19 *The* Itô integral *of a function $f \in \mathcal{L}(S,T)$, denoted by $I(f)$, is defined by*

$$I(f) := \int_S^T f(t,\omega)\,dB_t(\omega) \;=\; \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\,dB_t(\omega), \tag{1.31}$$

where limit is taken in $L^2(\Omega,\mathcal{F},P)$ and $\{\phi_n\}$ is a sequence of elementary processes such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2\,dt\right] \to 0, \;\; n \to \infty. \tag{1.32}$$

Note, the sequence $\{\phi_n\}$ exists and (1.32) implies that $\lim\limits_{n\to\infty} I(\phi_n)$ exists in $L^2(\Omega,\mathcal{F},P)$.

Remark 1.4.1 *The Itô isometry*

$$E\left[\left(\int_S^T f(t,\omega)\,dB_t(\omega)\right)^2\right] = E\left[\int_S^T f^2(t,\omega)dt\right]$$

holds for all functions $f \in \mathcal{L}(S,T)$.

Let $0 \le S < T$. For given functions $f,g \in \mathcal{L}(0,T)$ and constants $a,b \in \mathbb{R}$ the Itô integral satisfies the following properties:

- linearity $\int_0^T (af + bg)dB_t = a\int_0^T f dB_t + b\int_0^T g dB_t$,
- zero expectation $E(\int_0^T f dB_t) = 0$,
- variance $Var I(f) = E\left[(\int_0^T f dB_t)^2\right] = E\left(\int_0^T f^2 dt\right)$,
- polarization formula $E(\int_0^T f dB_t \cdot \int_0^T g dB_t) = E(\int_0^T f g\, dt)$,
- Itô integral $\int_0^T f dB_t$ is $\mathcal{F}$-measurable,

- additivity with respect to the integration interval
$\int_0^T f dB_t = \int_0^S f dB_t + \int_S^T f dB_t$, for almost all ω,

- for all $0 \leq t \leq T$, the Itô integral process

$$M_t := \int_0^t f(s,\omega) dB_s(\omega)$$

as a function of the upper limit t, has a continuous version,

- Itô integral is a martingale, i.e. for $s \leq t$ we have

$$E\left(\int_0^T f(u,\omega)\, dB_u(\omega)\,|\,\mathcal{F}_s\right) = \int_0^s f(u,\omega) dB_u(\omega).$$

Moreover, $P\{\sup\limits_{0\leq t\leq T} |M_t| \geq \lambda\} \leq \frac{1}{\lambda^2} \cdot E\left(\int_0^T f(s,\omega)^2 ds\right)$, for $\lambda, T > 0$.

The Itô formula

Instead of using Definition 1.4.19, in concrete situations we usually apply the *Itô formula* to compute the Itô integral.

Definition 1.4.20 *Let B_t be an one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. The one-dimensional* Itô process *is a stochastic process $X_t = X_t(\omega)$ on $(\Omega, \mathcal{F}, P)$ of the form*

$$X_r = X_s + \int_s^r f dt + \int_s^r g dB_t \tag{1.33}$$

for $f \in L^1(0,T), g \in L^2(0,T)$ and $0 \leq s \leq r \leq T$.

From (1.33) it is clear that in the representation of an Itô process two integrals appear, one Itô integral and one Lebesgue integral. If a process X_t is an Itô process then the expression (1.33) can be replaced with the corresponding stochastic differential form

$$dX_t = f\, dt + g\, dB_t, \quad \text{for } 0 \leq t \leq T. \tag{1.34}$$

Theorem 1.4.21 (The Itô formula for an one-dimensional Itô process) *Let X_t be an Itô process, represented in the differential notation* (1.34). *If a function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous together with its partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$, then the process*

$$Y_t := u(X_t, t)$$

is again an Itô process described explicitly by the **Itô formula**

$$\begin{aligned} dY_t &= \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dX_t \,+\, \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,(dX_t)^2 \\ &= \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,f \,+\, \frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\,g^2\right)dt \,+\, \frac{\partial u}{\partial x}\,g\,dB_t, \end{aligned} \tag{1.35}$$

where $(dX_t)^2 = dX_t \cdot dX_t$ *is computed using the multiplication rules:*

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0,\; dB_t \cdot dB_t = dt.$$

Theorem 1.4.22 (Itô multiplication rule) *Let*

$$\begin{cases} dX_1 = f_1\,dt + g_1\,dB_t \\ dX_2 = f_2\,dt + g_2\,dB_t \end{cases}, \qquad 0 \le t \le T$$

for $f_1, f_2 \in L^1(0,T)$ *and* $g_1, g_2 \in L^2(0,T)$*, then we have the multiplication rule for Itô processes given in the differential form*

$$d(X_1 X_2) = X_2\,dX_1 + X_1\,dX_2 + g_1 g_2\,dt. \tag{1.36}$$

The last term on the right-hand side of (1.36) represents so-called *Itô correction term.* Integral version of such multiplication rule for two Itô processes given in differential forms is called the Itô partial integration formula

$$\int_s^t X_2\,dX_1 = X_1(r)X_2(r) - X_1(s)X_2(s) - \int_s^t X_1 dX_2 - \int_s^t g_1 g_2 dt.$$

In particular, for a continuous function f which is of bounded variation on $[0,t]$, such that $f(s,\omega) = f_s$ is independent of ω, then the expression of the Itô integral is given by

$$\int_0^t f_s\,dB_s = f_t\,B_t - \int_0^t B_s\,df_s.$$

Recall, the property (1.2) describes a nice behavior of the first derivative of nth Hermite polynomial h_n, $n \in \mathbb{N}_0$ and its connection with $(n-1)$th Hermite polynomial h_{n-1}. Now, we state an important theorem which connects the nth normalized Hermite polynomial of parameter t, defined by

$$h_n(x,t) := (-1)^n\,\frac{t^n}{n!}\,e^{\frac{x^2}{2t}}\,\frac{d^n}{dx^n}(e^{-\frac{x^2}{2t}}), \quad x \in \mathbb{R},\; n \in \mathbb{N}_0$$

with the Itô integral.

Theorem 1.4.23 *Let* $h_n(x,t)$*,* $n \in \mathbb{N}_0$*,* $x \in \mathbb{R}$ *be the* n*th normalized Hermite polynomial of parameter* t*. Then for* $t \geq 0$ *and* $n \in \mathbb{N}_0$*, the following*

$$\int_0^t h_n(B_s, s)\, dB_s = h_{n+1}(B_t, t) \tag{1.37}$$

holds. Therefore, equality (1.37) *can be rewritten as*

$$dh_{n+1}(B_t, t) = h_n(B_t, t)\, dB_t.$$

The following important theorem, due to Itô, states that any random variable can be represented in terms of a unique adapted stochastic process.

Theorem 1.4.24 (The Itô representation theorem) *For any random variable* $F \in L^2(\Omega, \mathcal{F}, P)$*, which is* $\mathcal{F}$*-measurable there exists a unique adapted stochastic process* $\varphi(t,\omega)$ *such that*

$$F(\omega) = E(F) + \int_0^T \varphi_t(\omega)\, dB_t(\omega). \tag{1.38}$$

We have to point out here that from the Clark-Ocone formula follows the explicit form of such adapted process

$$\varphi_t(\omega) = E(DF \,|\, \mathcal{F}_t),$$

i.e. it is represented as a conditional expectation of the Malliavin derivative D of a given function $F \in L^2(\Omega, \mathcal{F}, P)$ with respect to the filtration $\mathcal{F}_t$. The Clark-Ocone formula represents an important result in applications in finance when obtaining explicit formula for replicating portfolios of contingent claims in complete markets. For more information see [10].

Now we focus to the Girsanov theorem, which states that a Brownian motion with drift can be seen as a Brownian motion without drift, with a change of probability.

Theorem 1.4.25 (Girsanov theorem) *Let* Y_t *be an Itô process given in the stochastic differential form*

$$dY_t = u_t(\omega)dt + dB_t, \quad 0 \leq t \leq T, \quad Y_0 = 0,$$

where T *is a given constant and* B_t *is a Brownian motion. Denote*

$$M_t = e^{-\int_0^t u_s(\omega)\, dB_s - \frac{1}{2}\int_0^t u_s^2(\omega) ds}, \qquad t \leq T \tag{1.39}$$

and denote by Q the measure on $(\Omega, \mathcal{F}_T)$ such that

$$dQ(\omega) = M_T(\omega)\, dP(\omega).$$

If we assume that stochastic process $u_t(\omega)$ satisfies the Novikov condition

$$E(e^{\frac{1}{2}\int_0^t u_s^2(\omega)\, ds}) < \infty. \tag{1.40}$$

then the stochastic process Y_t represents a Brownian motion with respect to the probability law Q, for $t \leq T$.

Probability transformation $P \to Q$ is called the *Girsanov transformation of measures*. Furthermore, if the Novikov condition (1.40) is replaced with assumption that $\{M_t\}_{t \leq T}$ is a martingale with respect to filtration $\mathcal{F}_t$ and measure P, then the Girsanov theorem is still valid.

In particular we have the following result.

Theorem 1.4.26 *Let $dX_t(\omega) = u_t(\omega)\, dt + dB_t(\omega)$ be an Itô process and let u be a bounded function. If*

$$Y_t(\omega) = X_t(\omega)\, M_t, \quad t \leq T$$

for M_t given in the form (1.39)*, then the stochastic process Y_t is a $\mathcal{F}$-martingale.*

In particular, for any bounded function $g : \mathbb{R} \to \mathbb{R}$ and all $t \leq T$ we have

$$E_Q[g(Y_t)] = E_P[g(B_t)].$$

If $u_t(\omega) = u_t = u(t)$ is a deterministic function and if we assume that $u_t = 0$ for $t > T$, then we can write

$$\exp\left[-\int_0^T u(t)\, dB_t - \frac{1}{2}\int_0^T u^2(t)\, dt\right] = \exp^\diamond[-\langle \omega, u(t)\rangle], \tag{1.41}$$

where $\exp^\diamond$ is for the stochastic Wick exponential, which will be defined latter in Example 2.39. Thus,

$$\int_\Omega g(B_t)\, \exp^\diamond[\langle \omega, u(t)\rangle]\, dQ(\omega) = \int_\Omega g\left(B_t + \int_0^t u(s)ds\right)\, dQ(\omega).$$

For the proof and more details on the Itô integral and its applications we refer to [10], [18], [19], [21], [39], [52].

The Itô-Poisson integral

Providing a construction analogous to the one from the previous subsection, when defining the Itô integral, one can define a stochastic integration with respect to a compound Poisson process. Integral obtained is called the *Itô-Poisson integral* and is denoted by

$$I^P(f) = \int_{\mathbb{R}} f_t(\omega)\, dP_t(\omega).$$

It is defined for a class of $\mathcal{F}_t$ adapted functions, where $\mathcal{F}_t$ is the $\sigma-$algebra generated by compensated Poisson random variables $\{P_s\}_{s\leq t}$.

1.5 Classical Malliavin Calculus

The *Malliavin calculus* or *the stochastic calculus of variations* is an infinite-dimensional differential calculus on the Wiener space. It was originally created by Paul Malliavin in work [38] in 70ies as a tool for finding a proof of smoothness for densities of solutions of stochastic differential equations. The original motivation, and the most important application of this theory, is to provide a probabilistic proof of Hörmander's sum of squares theorem. Originally, the Malliavin calculus is a Gaussian calculus, i.e. a calculus with respect to a Gaussian process. Nowadays the theory has found many applications which include numerical methods, stochastic control, not only for systems driven by Brownian motion, but also for systems driven by Lévy processes. Malliavin calculus has been developed by Stroock, Bismut, Watanabe, Nualart, Øksendal, Rozovsky and others. The integration-by-parts formula, which relates the Malliavin derivative operator on the Wiener space and the divergence operator, called the Itô-Skorohod stochastic integral in white noise setting, represents a crucial fact in this theory.

There are many ways of introducing the Malliavin derivative. The original construction was given on the Wiener space. In this section we present the stochastic calculus of variations in the framework of an abstract Wiener space and focus our attention on the notions and results that depend only on the covariance operator or the associated Hilbert space. We follow [15], [16], [46] for the case of an abstract Wiener space.

A survey of the different approaches to the Malliavin calculus can be found in [32], [34], [35], [36], [44], [46], [51], [52].

In Chapter 3, we will return again to notions of the Malliavin calculus and consider the operators of the Malliavin calculus within the white noise analysis approach, give representations of their domains in terms of chaos expansions and prove several important properties. Further on, in Chapter 4 we will define the generalizations of these operators on the space of singular generalized stochastic processes and thus in Chapter 5 introduce and solve several stochastic differential equations involving generalized versions of such operators.

1.5.1 The Wiener space

At the beginning we define the abstract Wiener space and introduce its chaos expansion decomposition. Then we will introduce a notion of the derivative DF of a square integrable random variable $F : \Omega \to \mathbb{R}$, defined in a weak sense on an abstract Wiener space, without assuming topological structure of Ω. The idea is to differentiate F with respect to the random parameter $\omega \in \Omega$.

The idea of the abstract Wiener space or Gaussian Hilbert space is to use a Hilbert space with an underlying Gaussian structure. In the following we assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space and H is a real separable Hilbert space with the scalar product $(\cdot, \cdot)_H$.

Recall that a family of random variables $G_i : \Omega \to \mathbb{R}$, $i \in \mathbb{N}$ is Gaussian provided that all finite linear combinations

$$\sum_{j=1}^{n} c_j \, G_{i_j} \; : \; \Omega \to \mathbb{R}$$

are Gaussian random variables for all $n \in \mathbb{N}$ and $c_1, c_2, ..., c_n \in \mathbb{R}$.

Definition 1.5.1 (Abstract Wiener space)

(i) The family of Gaussian random variables $G = \{G_h, \, h \in H\}$, where $G_h : \Omega \to \mathbb{R}$ is called isonormal *provided that G is centered with the covariance function of the form*

$$E(G_h G_k) = (h, k)_H, \quad \textit{for all} \quad h, k \in H.$$

(ii) If $\mathcal{F}$ is the completion of $\sigma\{G\}$, then the space $L^2(\Omega, \mathcal{F}, P) = L^2(P)$ is called the Wiener space associated with the Gaussian family *G or* Gaussian Hilbert space.

Thus G is a Gaussian process indexed by functions in a Hilbert space which describes the covariance of G. The standard Brownian motion, from Definition 1.4.13, fits in the setting of isonormal Gaussian process in the sense of Definition 1.5.1.

Example 1.5.1

- *Let* $H = L^2(\mathbb{R}, dt)$ *be the space of deterministic functions* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\int_{\mathbb{R}} f^2(t)dt < \infty$. *We define*

$$G_h := \int_{\mathbb{R}} h(t)\, dB_t, \quad h \in H, \tag{1.42}$$

where the stochastic integral is the Itô integral. Then, G_h *is an isonormal Gaussian process and*

$$G_{\chi[0,t]} \;=\; \int_{\mathbb{R}} \chi[0,t](s)\, dB_s \;=\; B_t, \quad t \geq 0.$$

Moreover, from this representation one can recover the covariance function of the standard Brownian motion

$$E(B_t\, B_s) \;=\; E(G_{\chi[0,t]}\, G_{\chi[0,s]}) = \langle \chi[0,t], \chi[0,s] \rangle = \min\{t, s\}.$$

According to Wiener, Brownian motion is a certain Gaussian measure $\mathcal{W}$, *now called the Wiener measure, on the space of continuous paths, called the Wiener space. Cameron and Martin discovered that Wiener measure is translation invariant measure in infinite dimensions. They showed that if* H *is the Hilbert subspace of Wiener space, whose elements* h *are absolutely continuous and have square integrable derivative, then translation of* $\mathcal{W}$ *by an* $h \in H$ *results in a measure* $\mathcal{W}_h$ *that is absolutely continuous with respect to* $\mathcal{W}$ *and has simple Radon-Nikodym* R_h *all of whose powers are integrable.*

- *A fractional Brownian motion can be seen as an isonormal Gaussian process. Consider the set of step functions* $\mathcal{S}$ *on* $[0, T]$ *and the Hilbert space* H *which is the closure of* $\mathcal{S}$ *with respect to the scalar product*

$$\langle \chi[0,t], \chi[0,s] \rangle_H \;:=\; \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad t, s \in [0, T].$$

We define the family of random variables

$$G^H_{\chi[0,t]} := B^H_t, \quad \textit{for every } t,$$

which constitutes the isonormal Gaussian process associated to the fractional Brownian motion B^H.

For any separable Hilbert space H we can construct an isonormal Gaussian family and a Wiener space associated with the family G. Clearly, we assume an orthogonal basis $\{e_n,\, n \in \mathbb{N}\}$ of infinite dimensional Hilbert space H and a sequence of i.i.d. random variables $\{\varphi_n : n \in \mathbb{N}\} \sim \mathcal{N}(0,1)$ on $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the completion of the σ-algebra generated by $\{\varphi_n : n \in \mathbb{N}\}$. If we let

$$G_h := \sum_{n\in\mathbb{N}} (h, e_n)_H \, \varphi_n \ \in \ L^2(P),$$

then $G = \{G_h,\, h \in H\}$ is an isonormal family of random variables.

Note that from Definition 1.5.1 it follows that $h \mapsto G_h$, $h \in H$ is a linear isometry from H into $L^2(P)$.

Definition 1.5.2 *Let*

$$\mathcal{H}_0 := \{f : \Omega \to \mathbb{R} \,:\, f \equiv c \text{ a.s. for some } c \in \mathbb{R}\}, \quad \text{and}$$
$$\mathcal{H}_n := \overline{\text{span}\,\{h_n(G_h) \,:\, h \in H,\, \|h\| = 1\}}, \quad n \in \mathbb{N},$$

for the Hermite polynomials h_n. The closed linear subspace $\mathcal{H}_n \subseteq L^2(P)$ is called the Wiener chaos of order n.

The following fundamental theorem of the decomposition of $L^2(P)$ space is due to Wiener 1938, Itô 1951 and Segal 1956.

Theorem 1.5.1 (The Wiener chaos expansion) *It holds that the space $L^2(P)$ can be decomposed in the following way*

$$L^2(P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where the sum is an orthogonal sum in $L^2(P)$, i.e. the following is valid

1. *$\mathcal{H}_n \perp \mathcal{H}_m$ for $n \neq m$, i.e. $E(fg) = 0$, for $f \in \mathcal{H}_n$, $g \in \mathcal{H}_m$.*
2. *For all $f \in L^2(P)$ there exist unique $f_n \in \mathcal{H}_n$ such that*

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in} \quad L^2(P), \tag{1.43}$$

such that the condition

$$\sum_{n=0}^{\infty} \|f_n\|^2_{L^2} = \|f\|^2_{L^2(P)}$$

is fulfilled.

The proof of the previous theorem is based on properties of the *orthogonal projection* $P_n : L^2(P) \to \mathcal{H}_n \subseteq L^2(P)$ of $L^2(P)$ onto the nth Wiener chaos. Recall, if P_n is the orthogonal projection, then P_n is linear and $P_n(L^2) \subseteq \mathcal{H}_n$. Moreover, we have $P_n f = f$ and $(P_n f, f - P_n f)_{L^2(P)} = 0$ valid for all $f \in L^2(P)$. From the property $(P_n f, g)_{L^2(P)} = (f, P_n g)_{L^2(P)}$, valid for $f, g \in L^2(P)$, it follows that $P_n^2 = P_n$. Then

$$\sum_{n=0}^{\infty} \|P_n f\|^2_{L^2(P)} \leq \|f\|^2_{L^2(P)}$$

so that $\sum_{n=0}^{\infty} P_n f$ converges in $L^2(P)$ and

$$f = \sum_{n=0}^{\infty} P_n f \quad \text{in } \; L^2(P).$$

Thus, every square integrable random variable with respect to a Gaussian random field could be written as a sum of elements in the Wiener chaos.

The closure of the set $\mathcal{P}_n$ of all polynomials $p = p_n(G_{h_1}, ..., G_{h_k})$ of k variables of degree less then or equal to n satisfies $\mathcal{P}_n = \mathcal{H}_0 \bigoplus ... \bigoplus \mathcal{H}_n$. Namely, the set if polynomials $\mathcal{P}_n$ is a dense subspace of $L^2(P)$.

Moreover, the set of finite linear combinations of the exponentials

$$\{\exp G_h, \;\; h \in H\}$$

is also a dense subspace of $L^2(P)$ and the projections P_n preserve the space of polynomial variables.

Moreover, one can show that the elements of the form $h_n(G_h)$, $h \in H$ can be considered as multiple integrals of order n. In particular, for illustration in the Brownian motion case see Theorem 1.4.23.

For more details on abstract Wiener space we refer to [15], [16].

1.5.2 The Malliavin derivative operator

In this section we assume that we have an abstract Wiener space $(\Omega, \mathcal{F}, P)$ based on a Gaussian structure $(G_h)_{h \in H}$. Following [7], [20] and [46] we will introduce a notion of the differential DF of a smooth square integrable random variable $F : \Omega \to \mathbb{R}$, defined in a weak sense on abstract Wiener space, without assuming topological structure of Ω. The aim is to differentiate F with respect to the random parameter $\omega \in \Omega$.

Let $G = \{G_h,\ h \in H\}$ denote an isonormal Gaussian process defined in a complete probability space $(\Omega, \mathcal{F}, P)$ associated with the Hilbert space H. Thus, G is a centered Gaussian family of random variables such that

$$E(G_{h_1} G_{h_2}) = (h_1, h_2)_H, \qquad \text{for all } h_1, h_2 \in H.$$

We assume that $\mathcal{F}$ is generated by this Gaussian family G.

Denote by $\mathcal{E}$ the class of smooth random variables of the form

$$F = f(G_{h_1}, ..., G_{h_n}), \tag{1.44}$$

where $f \in C^\infty(\mathbb{R}^n)$ such that f and all its partial derivatives have polynomial growth and $h_1, ..., h_n \in H$, $n \in \mathbb{N}$. These random variables are called *elementary.*

Definition 1.5.3 *The* derivative D *of an elementary random variable* $F \in \mathcal{E}$ *of the form (1.44) is the* H*-valued random variable defined by*

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_{h_1}, ..., G_{h_n}) \cdot h_i. \tag{1.45}$$

The derivative D is also called the *stochastic gradient* or the *Malliavin derivative* of an elementary random variable F.

The scalar product $(DF, h)_H$ is the derivative at $\varepsilon = 0$ of the random variable F composed with shifted process $\{G_g + \varepsilon(g, h)_H,\ g \in H\}$. Therefore,

$$(DF, h)_H = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(G_{h_1} + \varepsilon(h_1, h)_H, ..., G_{h_n} + \varepsilon(h_n, h)_H) - f(G_{h_1}, ..., G_{h_n}) \right]$$

for all $h_1, ..., h_n, h \in H$. The derivative operator D is interpreted as a directional derivative in direction h.

From Definition 1.5.3 it follows that

$$DG_h = h, \qquad \text{for all } \ h \in H.$$

In particular, if G_h is a Brownian motion from (1.42), then the stochastic gradient is an inverse operator of the Itô integral.

Now we give some properties and state theorems without proofs, which can be found, for example, in [15], [16], [20], [46].

By definition of the gradient operator for smooth random variables of the form (1.45), we have

$$D(FG) = F\, DG + G\, DF.$$

We continue with the integration by parts formula.

Theorem 1.5.2 (Integration by parts formula I) *For $F \in \mathcal{E}$ and $h \in H$ we have the integration by parts formula*

$$E\left((DF, h)_H\right) = E(FG_h).$$

Theorem 1.5.3 (Integration by parts formula II) *For elementary smooth random variables $F_1, F_2 \in \mathcal{E}$ and $h \in H$ we have that*

$$E[F_1(DF_2, h)_H] = -E[F_2(DF_1, h)_H] + E[F_1 F_2 G_h].$$

Definition 1.5.4 *Let X and Y be Banach spaces, $\mathcal{E} \subseteq X$ be a linear subspace and $D : \mathcal{E} \to Y$ a linear operator. Operator D is called* closable *provided that $\mathcal{E} \ni x_n \to 0$ and $Dx_n \to y \in Y$ imply $y = 0$.*

The Malliavin derivative D is a closable operator from the space of elementary functions $\mathcal{E}$ into the space of H-valued random variables.

Theorem 1.5.4 *Let X and Y be Banach spaces and $\mathcal{E} \subseteq X$ be a subspace and $D : \mathcal{E} \to Y$ be a closable operator and let*

$$Dom(D) := \left\{ x \in X : \begin{array}{l} \exists\, x_n \in \mathcal{E} \text{ such that } x_n \to x \\ \exists\, y \in Y \text{ such that } Dx_n \to y \end{array} \right\}.$$

Given $x \in Dom(D)$ define

$$\widetilde{D}x := y, \quad \text{for } x_n \to x \text{ and } Dx_n \to y.$$

Thus, we have the following assertions:

- *it holds $\mathcal{E} \subseteq Dom(D) \subseteq X$,*
- *the operator $\widetilde{D}$ is an extension of D to $Dom(D)$,*
- *$Dom(D)$ is a Banach space under*

$$\|x\|^2_{Dom(D)} := \|x\|^2_X + \|\widetilde{D}x\|^2_Y,$$

- *the operator $\widetilde{D} : Dom(D) \to Y$ is continuous.*

Definition 1.5.5 *Consider the space of elementary smooth functions $\mathcal{E} \subseteq L^2(P)$. The domain of the extension of the Malliavin derivative $D : \mathcal{E} \to L^2(P, H)$ is denoted by $\mathcal{D}^{1,2}$ and called the* Malliavin Sobolev space*.*

The extension of D to $\mathcal{D}^{1,2} \to L^2(P,H)$ is also denoted by D. Thus we have

$$Dom(D) = \mathcal{D}^{1,2}.$$

In particular, this means that the space $\mathcal{D}^{1,2}$ is the closure of the class of elementary random variables $\mathcal{E}$ with respect to the norm

$$\|F\|^2_{\mathcal{D}^{1,2}} = E|F|^2 + E\|DF\|^2_H. \tag{1.46}$$

Characterization of the domain of the Malliavin derivative in terms of the orthogonal projection of a random variable is given by the following theorem.

Theorem 1.5.5 *Let $P_n : L^2(P) \to \mathcal{H}_n \subseteq L^2(P)$ be the orthogonal projection onto nth Wiener chaos. Then*

$$\mathcal{D}^{1,2} = \left\{ F \in L^2(P) \,:\, \sum_{n=0}^{\infty} (n+1)\, \|P_n F\|^2_{L^2(P)} < \infty \right\}$$

with

$$\|F\|^2_{\mathcal{D}^{1,2}} = \sum_{n=0}^{\infty} (n+1)\, \|P_n F\|^2_{L^2(P)} < \infty.$$

Theorem 1.5.6 (Chain rule) *Let $F \in \mathcal{D}^{1,2}$ and $f \in C^1(\mathbb{R})$ such that $f(F)$ is differentiable in Malliavin sense. Then the chain rule follows*

$$D(f(F)) \;=\; f'(F)\, DF, \quad a.s.$$

1.5.3 The divergence operator

In this section we consider the divergence operator defined as the adjoint operator of the Malliavin derivative operator in the framework of abstract Wiener space. In particular, if the underlying Hilbert space H is $L^2(\mathbb{R})$ space, we will interpret the divergence operator as a stochastic integral and we will call it the Skorokhod integral, because in the Brownian motion case it coincides with the generalization of the Itô stochastic integral to anticipating integrands. This will be the subject of Chapter 3, where we will focus on the interpretation of the operators of Malliavin calculus in the framework of white noise analysis. Furthermore we will deduce the expression of the Skorokhod integral in terms of the Wiener-Itô chaos expansion.

Definition 1.5.6 (Divergence operator) *Denote by δ the adjoint of the Malliavin derivative operator D. Then, δ is an unbounded operator defined on $L^2(P,H)$ with values in $L^2(P)$ such that:*

- *The domain of* δ, *denoted by* $Dom(\delta)$, *is the set of* H*-valued integrable random variables* $u \in L^2(P,H)$ *such that*

$$|E[(DF,u)_H]| \leq c\,\|F\|_{L^2(P)} \tag{1.47}$$

for all $F \in \mathcal{D}^{1,2}$, *where* c *is some constant depending on* u.

- *If* $u \in Dom(\delta)$, *then the unique element* $\delta(u) \in L^2(P)$ *such that*

$$E[F\,\delta(u)] = E[(DF,u)_H] \tag{1.48}$$

for all $F \in \mathcal{D}^{1,2}$ *is called the* divergence operator *of* u.

Thus the divergence operator $\delta : Dom(\delta) \to L^2(P)$ is closed as the adjoint of an unbounded and densely defined operator. It is a linear operator.

Remark 1.5.1 *Taking* $F = 1$ *in* (1.48) *we obtain*

$$E\delta(u) = 0, \qquad \text{for } u \in Dom(\delta).$$

Note that the divergence operator can be decomposed into two parts, one part that can be considered as a path-wise integral and another that involves the derivative operator. Thus it is possible to factor out a scalar random variable in a divergence.

Theorem 1.5.7 *Let* $F \in \mathcal{D}^{1,2}$ *and* $u \in Dom(\delta)$ *such that* $Fu \in L^2(P,H)$. *Then* $Fu \in Dom(\delta)$ *and*

$$\delta(Fu) = F\delta(u) - (DF,u)_H, \tag{1.49}$$

provided the right-hand side is square integrable.

Denote by $\mathcal{D}^{1,2}(H)$ the space of H-valued random variables F whose Malliavin derivative DF is a square integrable random variable with values in the Hilbert space $H \otimes H$. Then $\mathcal{D}^{1,2}(H)$ is included in the domain of δ. For $u, v \in \mathcal{D}^{1,2}(H)$ the following nice property is valid

$$E[\delta(u)\,\delta(v)] = E[(u,v)_H] + E[Tr(Du \circ Dv)].$$

1.5.4 The Ornstein-Uhlenbeck operator

Now we define the third important operator of the Malliavin calculus in the framework of abstract Wiener space, the Ornstein-Uhlenbeck operator.

Consider a square integrable random variable $F \in L^2(P)$ and the orthogonal projection $P_n : L^2(P) \to \mathcal{H}_n \subset L^2(P)$. Then, following the Wiener chaos expansion theorem, Theorem 1.5.1, F has representation of the form

$$F = \sum_{n=0}^{\infty} P_n(F).$$

Definition 1.5.7 *The* Ornstein-Uhlenbeck operator $\mathcal{R}$ *is defined by*

$$\mathcal{R}F = \sum_{n=0}^{\infty} nP_n(F), \tag{1.50}$$

provided the series converges in $L^2(P)$.

Thus the domain of the operator $\mathcal{R}$ is

$$Dom(\mathcal{R}) = \{F = \sum_{n=0}^{\infty} P_n(F) \in L^2(\Omega) : \sum_{n=1}^{\infty} n^2 \|P_n(F)\|^2_{L^2(P)} < \infty\}.$$

In particular, one can prove

$$Dom(\mathcal{R}) \subset \mathcal{D}^{1,2}.$$

Operator $\mathcal{R}$ is a linear, unbounded and symmetric operator on $L^2(P)$. That is,

$$E(G\mathcal{R}F) = E(F\mathcal{R}G),$$

for $F, G \in Dom(\mathcal{R})$. Operator $\mathcal{R}$ is a self-adjoint operator, hence closed and it coincides with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T_t,\, t \geq 0\}$, defined by

$$T_t(F) = \sum_{n=0}^{\infty} e^{nt} P_n(F), \quad \text{for } F \in L^2(P).$$

The relationship between three operators, D, δ and $\mathcal{R}$ of the classical Malliavin calculus is given in the following theorem.

Theorem 1.5.8 *Let* $F \in \mathcal{D}^{1,2}$ *and* $DF \in Dom(\delta)$. *Then a random variable* F *belongs to the domain of the operator* $\mathcal{R}$ *and*

$$\delta D\, F \;=\; \mathcal{R}\, F. \tag{1.51}$$

The Ornstein-Uhlenbeck operator can be considered as the composition of the divergence operator and the Malliavin derivative operator.
Operator $\mathcal{R}$ is a second order differential operator when it acts on smooth random variables.

Theorem 1.5.9 *Let* $F \in \mathcal{E}$ *be of the form* (1.44). *Then* $F \in Dom(\mathcal{R})$ *and*

$$\mathcal{R}F = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(G_{h_1}, ..., G_{h_n})(h_i, h_j)_H - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(G_{h_1}, ..., G_{h_n}) G(h_i).$$

Notice that $Dom\mathcal{R} = \mathcal{D}^{2,2}$.

Similarly, the Malliavin Sobolev space $\mathcal{D}^{1,2}$ can be characterized as the domain in $L^2(P)$ of the operator $C = -\sqrt{\mathcal{R}}$ defined by

$$CF = \sum_{n=0}^{\infty} -\sqrt{n} P_n(F). \tag{1.52}$$

One can show that operator C is the infinitesimal generator of the *Cauchy semigroup* of operators given by

$$Q_t F = \sum_{n=0}^{\infty} e^{-\sqrt{n}t} P_n(F).$$

Note that $DomC = \mathcal{D}^{1,2}$ and for any $F \in DomC$ we have

$$E(CF)^2 = \sum_{n=1}^{\infty} n \|P_n(F)\|^2_{L^2(P)} = E(\|DF\|^2_H).$$

Remark 1.5.2 *The operators of the Malliavin calculus and their chaos expansion representations have been used in different frameworks. In the general context of the Fock space, in particular in applications in quantum probability, the derivative operator D is interpreted as the annihilation operator. Also the divergence operator δ is interpreted as the creation operator and the Ornstein-Uhlenbeck operator $\mathcal{R}$ corresponds to the number operator in the Fock space setting (see Section 1.2).*

Remark 1.5.3 *An abstract Wiener space is built on a complete probability space $L^2(\Omega, \mathcal{F}, P)$, where P is a measure. In particular, if the Gaussian family G_h, given by Definition 1.5.1, is a classical Brownian motion then the Malliavin derivative is denoted by D and we consider whole calculus to be the classical Malliavin calculus. On the other hand, the differential operator with respect to a fractional Brownian motion is called the* fractional Malliavin derivative *and will be defined either on the space $L^2(P)$ or on the space $L^2(P_H)$, given by* (3.29)*. Thus, the fractional Malliavin derivative, depending on which underlying space is considered, is denoted by $D^{(H)}$ or $\widetilde{D}^{(H)}$ for $H \in (0,1)$. The corresponding* fractional divergence operator *and the* fractional Ornstein-Uhlenbeck operator *are denoted by $\delta^{(H)}$ and $\mathcal{R}^{(H)}$ respectively. Specificaly, for $H = \frac{1}{2}$ the fractional operators become the corresponding classical operators. The fractional Malliavin calculus on $L^2(P)$ is the subject of Section 3.2.1 and the fractional Malliavin operators are considered in Section 3.2.3.*

Chapter 2

White Noise Analysis and Chaos Expansions

White noise analysis, introduced by Hida in [17] and further developed by many authors (see for example [18], [19], [31], [46] and references therein), as a discipline of infinite dimensional analysis has found applications in solving stochastic differential equations and thus in the modeling of stochastic dynamical phenomena arising in physics, economy, biology. We mention some [34], [45], [37], [56].

The chaos expansion of stochastic processes provides a series decomposition of square integrable processes in a Hilbert space orthogonal basis built upon a class of special functions, Hermite polynomials and functions, in the framework of white noise analysis. In order to build spaces of stochastic test and generalized functions, one has to use series decompositions via orthogonal functions as a basis, with certain weight sequences.

We follow the classical Hida approach, which suggests to start with a nuclear space E and its dual E', such that

$$E \ \subset \ L^2(\mathbb{R}) \ \subset \ E',$$

and then take the basic probability space to be $\Omega = E'$ endowed with the Borel sigma algebra of the weak topology and an appropriate probability measure P. Since Gaussian processes and Poissonian processes represent the two most important classes of Lévy processes, in this chapter of the dissertation we are focused on these two types of measures. Some of the results presented in this chapter have been achieved in collaboration with Dora Seleši and represent an original part of the thesis. The results are already published in [29] and [30].

In case of a Gaussian measure, the orthogonal basis of $L^2(P)$ can be constructed from any orthogonal basis of $L^2(\mathbb{R})$ that belongs to E and from the Hermite polynomials, while in the case of a Poissonian measure the orthogonal basis of $L^2(P)$ is constructed using the Charlier polynomials together with the orthogonal basis of $L^2(\mathbb{R})$. We will focus on the case when E and E' are the Schwartz spaces of rapidly decreasing test functions $S(\mathbb{R})$ and tempered distributions $S'(\mathbb{R})$. In this case the orthogonal family of $L^2(\mathbb{R})$ can be represented by the Hermite functions.

The first part of this chapter is devoted to constructions of the Gaussian and Poissonian white noise spaces. We deal with chaos expansion representations of the corresponding random variables. There exists unitary mapping which connects the elements of these two spaces.

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions. They have no point value for $\omega \in \Omega$, only an average value with respect to a test random variable. For more details we refer to [17], [19], [25]. Several spaces of stochastic distributions, weighted by a sequence q will be introduced in this chapter. We denote them by $(Q)^P_{-\rho}$, $\rho \in [0,1]$ and thus obtain a Gel'fand triplet

$$(Q)^P_\rho \ \subset \ L^2(P) \ \subset \ (Q)^P_{-\rho}.$$

A class of generalized stochastic processes, defined as measurable mappings from $\mathbb{R}$ into some q-weighted space of generalized stochastic random variables $(Q)^P_{-\rho}$, will be introduced in this chapter. The chaos expansion of generalized stochastic processes will be given together with the main properties of the Wick calculus and stochastic integration.

We close this chapter with introduction of the fractional white noise spaces, by use of the fractional transform mapping, for all values of the Hurst index $H \in (0,1)$. As a result, we will define the fractional Poissonian white noise space and through composition of unitary mappings connect it with other white noise spaces we are working on, a Gaussian, a Poissonian and a fractional Gaussian space. Moreover, we will extend the action of the fractional transform operator to a class of generalized stochastic processes.

2.1 White Noise Space

Consider the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$, its dual space, the space of tempered distributions $S'(\mathbb{R})$, the Borel sigma-algebra $\mathcal{B}$ generated by the weak topology on $S'(\mathbb{R})$ and a given characteristic function C. Recall, a mapping $C : S(\mathbb{R}) \to \mathbb{C}$ given on a nuclear space $S(\mathbb{R})$ is called a characteristic function if it is continuous, positive definite, i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} z_i\overline{z}_j\, C(\varphi_i-\varphi_j)\geq 0,$$

for all $\varphi_1,...,\varphi_n \in S(\mathbb{R})$ and $z_1,...,z_n \in \mathbb{C}$, and if it satisfies $C(0) = 1$. Then by the Bochner-Minlos theorem, Theorem 1.4.7, there exists a unique probability measure P on $(S'(\mathbb{R}),\mathcal{B})$ such that for all $\varphi\in S(\mathbb{R})$ the relation

$$E_P(e^{i\langle\omega,\varphi\rangle}) = C(\varphi)$$

holds. Here E_P denotes the expectation with respect to the measure P, i.e.

$$E_P(f) = \int_{S'(\mathbb{R})} f(\omega)\,dP(\omega), \quad \text{for } f\in S'(\mathbb{R})$$

and $\langle\omega,\varphi\rangle$ denotes the usual dual paring between a tempered distribution $\omega\in S'(\mathbb{R})$ and a rapidly decreasing function $\varphi\in S(\mathbb{R})$. Thus,

$$\int_{S'(\mathbb{R})} e^{i\langle\omega,\varphi\rangle} dP(\omega) = C(\varphi), \qquad \varphi\in S(\mathbb{R}). \tag{2.1}$$

The triplet $(S'(\mathbb{R}),\mathcal{B},P)$ is called the *white noise probability space* and the measure P is called the *white noise probability measure.*

However, for different choices of positive definite functionals $C(\varphi)$ in (2.1) one can obtain different white noise probabilistic measures, which then correspond to such functionals. In particular, if $C(\varphi)$ is the characteristic function of the normal random variable then the corresponding white noise measure is the Gaussian white noise measure (which is described in Section 2.2), if $C(\varphi)$ is the characteristic function of the compound Poisson random variable then the corresponding white noise measure is the Poissonian white noise measure (which is the objective of Section 2.3). In the article [45] written by Mura and Mainardi the characteristic function $C(\varphi)$ was replaced by a completely monotonic function defined by the Mittag-Leffler function of order $0<\beta\leq 1$ and the measure obtained is the gray noise measure, which is generalization of the white noise measure. With a similar construction, one can also obtain the Lévy white noise measure, as it was done in [10].

In this dissertation we study the Gaussian and Poissonian measures and properties of functions defined on the related white noise spaces. In Section 2.8 we will introduce their fractional versions and give the connections between these four white noise spaces. In recent years many papers were published on this subject. We mention here some [10], [23], [32], [43].

From now on we suppose that the basic probability space $(\Omega, \mathcal{F}, P)$ is the space $(S'(\mathbb{R}), \mathcal{B}, P)$. If we put $L^2(P) = L^2(S'(\mathbb{R}), \mathcal{B}, P)$, then the space $L^2(P)$ is the Hilbert space of square integrable functions on $S'(\mathbb{R})$ with respect to the measure P, equipped with the norm induced by the inner product

$$(F,G)_{L^2(P)} \;=\; E_P(FG), \quad \text{for } F,G \in L^2(P).$$

2.1.1 Wiener-Itô chaos expansion of random variables

Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of sequences of non-negative integers which have finitely many nonzero components $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0 \ldots)$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2, ..., m$, $m \in \mathbb{N}$. The kth unit vector $\varepsilon^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, $k \in \mathbb{N}$ is the sequence of zeros with the number 1 as the kth component.

Throughout this thesis we will use notation $\alpha^\beta = \alpha_1^{\beta_1}\alpha_2^{\beta_2}\cdots$ for given multi-indices $\alpha, \beta \in \mathcal{I}$. The length of a multi-index $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{I}$ is defined as $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$ and $\alpha! = \prod_{k=1}^{\infty} \alpha_k!$. Let $(2\mathbb{N})^\alpha = \prod_{k=1}^{\infty}(2k)^{\alpha_k}$. Then,

- $\sum\limits_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$ if and only if $p > 1$, and
- $\sum\limits_{\alpha\in\mathcal{I}} e^{-p(2\mathbb{N})^\alpha} < \infty$ if and only if $p > 0$.

Let K_α, $\alpha \in \mathcal{I}$ be the orthogonal polynomial basis of a Hilbert space $L^2(P)$, which produces a Wiener chaos. Throughout the thesis we will consider only two special measures P, the Gaussian and the Poissonian measures, which produce the Wiener chaos.

The space spanned by $\{K_\alpha : |\alpha| = k\}$ is called the *Wiener chaos of order* k and is denoted by $\mathcal{H}_k$, $k \in \mathbb{N}_0$. Then, $\mathcal{H}_0$ is the set of constant random variables, i.e. for $\alpha = (0, 0, ...)$ we obtain the expectation of a random variable. The space $\mathcal{H}_1$ consists of linear combinations of elements $\langle\omega, \cdot\rangle$ (for example Brownian motion lives in the first order chaos) and the space $\bigoplus_{j=0}^{k} \mathcal{H}_j$ is the set of random variables of the form $p(\langle\omega, \cdot\rangle)$, where p is a polynomial of degree $n \leq k$ with real coefficients. This implies that each $\mathcal{H}_k$ is a finite-dimensional subspace of $L^2(P)$. Moreover,

$$L^2(P) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

where the sum is an orthogonal sum.

Guided by the well known fact that the Hermite polynomials form an orthogonal basis in $L^2(\mathbb{R})$, Wiener showed that there exists an analogous orthogonal basis for the Wiener or Gaussian measure on the space of trajectories. More precisely, just as in $L^2(\mathbb{R})$, it is best to group together all the Hermite polynomials of a fixed degree k and to consider the subspace spanned by them (subspaces consisting of functions that have homogeneous kth order randomness). Wiener looked at the spaces $\mathcal{H}_k$ that are obtained by closing in $L^2(P)$ the linear span of the kth order Hermite polynomials. In other words, Wiener described a spectral decomposition of $L^2(P)$ in which the spectral parameter is randomness.

We can now formulate the Wiener-Itô chaos expansion theorem for random variables in $L^2(P)$.

Theorem 2.1.1 (The Wiener-Itô chaos expansion theorem) *For each element $F \in L^2(P)$ there exists a unique family of real constants $\{c_\alpha\}_{\alpha\in\mathcal{J}}$ such that F has a representation of the form*

$$F(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha(\omega), \quad c_\alpha \in \mathbb{R}, \tag{2.2}$$

where $c_\alpha = (F, K_\alpha)_{L^2(P)}$. Moreover,

$$\|F\|^2_{L^2(P)} = \sum_{\alpha\in\mathcal{J}} c_\alpha^2 \|K_\alpha\|^2_{L^2(P)} < \infty. \tag{2.3}$$

In terms of the previous theorem, the Wiener chaos of order k is given as the set

$$\mathcal{H}_k = span\{F \in L^2(P)\,;\, F = \sum_{\alpha\in\mathcal{J}} a_\alpha\, K_\alpha,\, |\alpha| = k\}, \quad k \in \mathbb{N}_0.$$

Thus,

$$\begin{aligned} F &= \sum_{\alpha\in\mathcal{J}} c_\alpha\, K_\alpha(\omega) \\ &= \sum_{k=0}^{\infty} [\underbrace{\sum_{\substack{\alpha\in\mathcal{J},\\ |\alpha|=k}} c_\alpha K_\alpha(\omega)}_{\text{elements of the } k\text{th Wiener chaos}}], \end{aligned}$$

for every $F \in L^2(P)$.

Two important special cases of probability measures will be considered in this thesis, when the measure P is the Gaussian measure and the Poissonian measure. In these cases K_α can be taken as families of Hermite and Charlier polynomials respectively, defined on an infinite-dimensional space.

2.2 Gaussian White Noise Space

If we choose in (2.1) the characteristic function of a Gaussian random variable

$$C(\varphi) = \exp\left[-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}\right], \quad \varphi \in S(\mathbb{R}), \tag{2.4}$$

then the corresponding unique measure P from the Bochner-Minlos theorem is called the *Gaussian white noise measure* and is denoted by μ. The triplet $(S'(\mathbb{R}), \mathcal{B}, \mu)$ is called the *Gaussian white noise probability space* and $L^2(\mu)$ is the Hilbert space of square integrable random variables on $S'(\mathbb{R})$ with respect to the Gaussian measure μ.

Thus, from (2.1) and (2.4) it follows that

$$\int_{S'(\mathbb{R})} e^{i\langle\omega,\varphi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}}, \qquad \varphi \in S(\mathbb{R}), \tag{2.5}$$

where $\langle\omega, \varphi\rangle$ denotes the usual dual paring between a tempered distribution $\omega \in S'(\mathbb{R})$ and a rapidly decreasing function $\varphi \in S(\mathbb{R})$.

Note that from (2.5) it follows that the random element $\langle\omega, \varphi\rangle$ has a zero expectation $E_\mu(\langle\omega, \varphi\rangle) = 0$ and variance (the isometry)

$$Var(\langle\omega, \varphi\rangle) = E_\mu(\langle\omega, \varphi\rangle^2) = \|\varphi\|^2_{L^2(\mathbb{R})}, \quad \text{for } \varphi \in S(\mathbb{R}).$$

Moreover, by the formula

$$E_\mu(\langle\omega, f\rangle \langle\omega, g\rangle) = (f, g)_{L^2(\mathbb{R})}$$

holds for all $f, g \in S(\mathbb{R})$. Thus, the element $\langle\omega, \varphi\rangle$, with $f \in S(\mathbb{R})$ and $\omega \in S'(\mathbb{R})$ is a centered Gaussian square integrable random variable which belongs to $L^2(\mu)$.

The map

$$J_1 : \varphi \to \langle\omega, \varphi\rangle, \quad \varphi \in S(\mathbb{R})$$

can be extended to an isometry from $L^2(\mathbb{R})$ to $L^2(\mu)$.

2.2.1 Brownian motion

By extending the action of a distribution $\omega \in S'(\mathbb{R})$ not only onto test functions from $S(\mathbb{R})$ but also onto elements of $L^2(\mathbb{R})$ we obtain Brownian motion with respect to the measure μ in the form

$$B_t(\omega) := J_1(\chi[0,t]) \;=\; \langle\omega, \chi[0,t]\rangle, \quad \omega \in S'(\mathbb{R}),$$

where $\chi[0,t]$ represents the characteristic function of interval $[0,t]$, $t \in \mathbb{R}$. To be precise, $\langle \omega, \chi[0,t] \rangle$ is a well defined element of $L^2(\mu)$ for all t, defined by $\lim_{n\to\infty} \langle \omega, \varphi_n \rangle$, where $\varphi_n \to \chi[0,t]$, $n \to \infty$ in $L^2(\mathbb{R})$. It has a zero expectation value and its covariance function is

$$E_\mu\left(\langle \omega, \chi[0,t] \rangle \langle \omega, \chi[0,s] \rangle\right) = \min\{t,s\}, \quad t,s > 0.$$

Recall, in Section 1.4.3 we summarize definition and basic properties of the Brownian motion. Now, we connect them with the Gaussian probability measure. Recall an important property, that is Brownian motion is a Gaussian process almost all whose trajectories are continuous but nowhere differentiable functions.

2.2.2 The Itô integral

In Section 1.4.4 we defined the Itô integral on a set of adapted stochastic processes. Furthermore, the Itô integral of a deterministic function $f \in L^2(\mathbb{R})$ is also represented by

$$I(f) = \langle \omega, f \rangle = \int_{\mathbb{R}} f(t)\, dB_t(\omega).$$

Then $E_\mu(I(f)) = 0$ and the Itô isometry $\|I(f)\|_{L^2(\mu)} = \|f\|_{L^2(\mathbb{R})}$ holds for all $f \in L^2(\mathbb{R})$. In Section 2.2.6 the notion of the Itô integral is extended for processes which are not necessarily adapted.

2.2.3 Chaos expansion for Gaussian random variables

A crucial role in the chaos decomposition of $L^2(\mu)$ elements have the Fourier-Hermite polynomials. The family of Fourier-Hermite polynomials represents an analogous orthogonal basis for $L^2(\mu)$ to a standard Hermite polynomial orthogonal basis in $L^2(\mathbb{R})$.

Definition 2.2.1 *For a given* $\alpha \in \mathcal{I}$ *the* αth Fourier-Hermite polynomial *is defined by*

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, \xi_k \rangle), \quad \alpha \in \mathcal{I}, \tag{2.6}$$

where ξ_k are the Hermite functions of order k, $k \in \mathbb{N}$.

The Fourier-Hermite polynomials can be obtained by differentiating the *normalized stochastic exponential*

$$\varepsilon_h = \exp\left(\langle \omega, h \rangle - \frac{1}{2}\|h\|^2_{L^2(\mathbb{R})} \right), \quad h \in S(\mathbb{R}). \tag{2.7}$$

The stochastic exponential have equivalent representation in terms of the Wick exponentials given in Example 2.5.1.

The family of Fourier-Hermite polynomials $\{H_\alpha;\ \alpha \in \mathcal{I}\}$ forms an orthogonal basis of the space $L^2(\mu)$, where $\|H_\alpha\|^2_{L^2(\mu)} = \alpha!$.

In particular, for the kth unit vector $\varepsilon^{(k)}$ we have

$$H_{\varepsilon^{(k)}}(\omega) = \langle \omega, \xi_k \rangle = \int_{\mathbb{R}} \xi_k(t)\, dB_t(\omega) = I(\xi_k), \quad k \in \mathbb{N}.$$

From the Wiener-Itô chaos expansion theorem, Theorem 2.1.1, it follows that each element $F \in L^2(\mu)$ has a unique chaos expansion representation of the form

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha(\omega), \tag{2.8}$$

where the coefficients $c_\alpha = \frac{1}{\alpha!} E_\mu(F H_\alpha)$ satisfy the convergence condition

$$\|F\|^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{I}} c_\alpha^2\, \alpha! < \infty, \tag{2.9}$$

which correspond to the condition (2.3).

This dissertation is based on this chaos expansion construction of random elements. Note that definitions behind (2.8) are rather complicated.

Example 2.2.1 1. *Let $\varphi \in S(\mathbb{R})$ be fixed. The element*

$$\omega \to \langle \omega, \varphi \rangle, \qquad \omega \in S'(\mathbb{R})$$

is called the one-dimensional smoothed white noise. *Recall from (2.5), it is a zero-mean Gaussian random variable with the variance $E_\mu(\langle \omega, \varphi \rangle^2) = \|\varphi\|^2_{L^2(\mathbb{R})}$, for $\varphi \in S(\mathbb{R})$. Chaos expansion of an one-dimensional smoothed white noise is given by*

$$\begin{aligned} \langle \omega, \varphi \rangle &= \sum_{k=1}^{\infty} (\varphi, \xi_k)_{L^2(\mathbb{R})} \langle \omega, \xi_k \rangle \\ &= \sum_{k=1}^{\infty} (\varphi, \xi_k)_{L^2(\mathbb{R})} H_{\varepsilon^{(k)}}(\omega), \end{aligned}$$

where $\varphi = \sum_{k=1}^{\infty} (\varphi, \xi_k)_{L^2(\mathbb{R})} \xi_k \in S(\mathbb{R})$ is the decomposition of $\varphi \in S(\mathbb{R})$ in the Hermite orthonormal basis $\{\xi_k\}_{k \in \mathbb{N}}$.

2. *Function*

$$f(\omega) = e^{i\langle \omega, \varphi \rangle}, \quad \varphi \in S(\mathbb{R}) \tag{2.10}$$

is called the stochastic exponent.

3. *The* stochastic exponential ε_h, *defined by* (2.7) *belongs to the Kondratiev space* $(S)_1^\mu$ *as long as* $\|h\|_{L^2(\mathbb{R})}$ *is sufficiently small. The chaos expansion of the stochastic exponential is given by*

$$\varepsilon_h = \sum_{\alpha\in\mathcal{J}} h^\alpha\, H_\alpha(\omega),$$

where $h = \sum_{k=1}^{\infty} h_k\,\xi_k \in S(\mathbb{R})$.

We refer to [19] and [31] for more details.

We continue with an alternative formulation of the Wiener-Itô chaos expansion theorem in terms of iterated Itô integrals. Although this formulation will not play the central role in our presentation, we give a brief review of it. Moreover, we will need this version when defining the Skorokhod integral and Wick multiplication of generalized random variables.

2.2.4 Iterated Itô integral

Let $B_t = B_t(\omega), t \geq 0, \omega \in \Omega$ be a one-dimensional Wiener process (Brownian motion) on probability space $(\Omega, \mathcal{F}, \mu)$ such that $B_0(\omega) = 0$ a.s. μ. For $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra generated by $B_s(\cdot)$, $0 \leq s < t$. Let $\widehat{L^2}(\mathbb{R}^n)$ be the set of symmetric deterministic square integrable functions on $\mathbb{R}^n$, i.e. $f \in \widehat{L^2}(\mathbb{R}^n)$ if $f(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) = f(x_1, x_2, \cdots, x_n)$ for all permutations σ of $\{1, 2, \cdots, n\}$ and

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f^2(x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n < \infty.$$

If f is a real function defined on $\mathbb{R}^n$ then the symmetrization $\widetilde{f}$ of f is defined by

$$\widetilde{f}(x_1, x_2, \cdots, x_n) = \frac{1}{n!} \sum_{\sigma} f(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}),$$

where the sum is taken over all permutations σ of the set $\{1, 2, \cdots, n\}$.

Definition 2.2.2 *The n-fold iterated Itô integral of a symmetric deterministic function* $f \in \widehat{L^2}(\mathbb{R}^n)$ *is defined by*

$$\begin{aligned} I_n(f) &:= \int_{\mathbb{R}^n} f(t_1, t_2, \cdots, t_n) dB_t^{\otimes n} \\ &= n! \int_{-\infty}^{+\infty} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} f(t_1, \cdots, t_n)\, dB_{t_1} \cdots dB_{t_n}, \end{aligned} \tag{2.11}$$

where the integral on the right-hand consists of n iterated Itô integrals of the first order.

In each Itô integration with respect to dB_{t_i}, $1 \leq i \leq n$ the integrand is $\mathcal{F}_t$ adapted and square integrable with respect to $d\mu \times dt_i$, $1 \leq i \leq n$.

The family of iterated Itô integrals $\{I_n\}_{n\geq 0}$ forms an orthogonal zero-mean family of linear operators

$$I_n : \widehat{L^2}(\mathbb{R}^n) \rightarrow L^2(\mu),$$

satisfying

$$\|I_n(f)\|^2_{L^2(\mu)} = n!\,\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{2.12}$$

for $f \in \widehat{L^2}(\mathbb{R}^n)$.

In particular, for all $f, g \in \widehat{L^2}(\mathbb{R}^n)$ we can formulate these results as follows

$$E_\mu(I_n(f)\, I_m(g)) = \begin{cases} 0 & , n \neq m \\ n!\,(f,g)_{L^2(\mathbb{R}^n)} & , n = m, \end{cases} \;, \;\; n, m \in \mathbb{N}_0,$$

where $(f,g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}} ... \int_{\mathbb{R}} f(x_1, ..., x_n)\, g(x_1, ..., x_n)\, dx_1...dx_n$ denotes the inner product in $\mathbb{R}^n$.

The map I_0 is the identity, where the real scalars are embedded naturally in $L^2(\mu)$ as the constant random variables. Thus, every element F of the Hilbert space $L^2(\mu)$ can be represented in terms of iterated Itô integrals, i.e. another formulation of Wiener-Itô chaos expansion theorem holds.

2.2.5 Chaos expansion in terms of multiple Itô integrals

Now we formulate an alternative statement of the Wiener-Itô chaos expansion Theorem 2.1.1 in the one-dimensional case, which gives the chaos expansion decomposition of a Gaussian random variable in terms of multiple Itô integrals defined by (2.11).

Theorem 2.2.1 (The Wiener-Itô chaos expansion theorem)
Let $F \in L^2(\mu)$. Then there exists a unique family of symmetric functions $f_n \in \widehat{L^2}(\mathbb{R}^n)$ such that F has the chaos representation form

$$\begin{aligned} F(\omega) &= \sum_{n=0}^{\infty} I_n(f_n) \\ &= E_\mu(F) + \sum_{n=1}^{\infty} I_n(f_n), \quad f_n \in \widehat{L^2}(\mathbb{R}^n),\; n \in \mathbb{N}. \end{aligned} \tag{2.13}$$

Moreover, the isometry

$$\|F\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n!\,\|f_n\|^2_{L^2(\mathbb{R}^n)}$$

holds.

Remark 2.2.1 *For* $\alpha = (\alpha_1, \cdots, \alpha_m, 0, 0, \cdots) \in \mathcal{I}$, $|\alpha| = \alpha_1 + ... + \alpha_m = n$ *and the Hermite functions* $\{\xi_k,\, k \in \mathbb{N}\}$ *let* $\xi^{\widehat{\otimes}\alpha} := \xi_1^{\otimes\alpha_1}\widehat{\otimes}\ldots\widehat{\otimes}\xi_m^{\otimes\alpha_m}$ *be the symmetrized tensor product with factors* $\xi_1, ...\xi_m$ *each* ξ_i *being taken* α_i *times. In [21] Itô proved*

$$H_\alpha(\omega) = \int_{\mathbb{R}^{|\alpha|}} \xi^{\widehat{\otimes}\alpha}(t)\, dB_t^{\otimes|\alpha|}(\omega). \tag{2.14}$$

Thus the connection between two chaos expansion theorems (2.8) *and* (2.13) *is given by* $f_n = \sum_{|\alpha|=n} c_\alpha\, \xi_n^{\hat{\otimes}\alpha}$.

Clearly, the statement follows from

$$\begin{aligned} F(\omega) &= \sum_{\alpha\in\mathcal{I}} c_\alpha\, H_\alpha(\omega) \\ &= \sum_{n\in\mathbb{N}_0} \sum_{|\alpha|=n} c_\alpha \int_{\mathbb{R}^{|\alpha|}} \xi^{\widehat{\otimes}\alpha}(t)\, dB_t^{\otimes|\alpha|}(\omega) \\ &= \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}^n} \underbrace{\left(\sum_{|\alpha|=n} c_\alpha\, \xi^{\widehat{\otimes}\alpha}(t) \right)}_{f_n} dB_t^{\otimes n}(\omega) \\ &= \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}^n} f_n\, dB_t^{\otimes n}(\omega), \end{aligned} \tag{2.15}$$

where $f_n = \sum_{|\alpha|=n} c_\alpha\, \xi_n^{\hat{\otimes}\alpha}$ *are symmetric functions in* $\widehat{L^2}(\mathbb{R}^n)$. *Note that* f_n *belongs to the nth Wiener chaos* $\mathcal{H}_n$.

Moreover, from (2.9) *and* (2.12) *the isometry follows*

$$\|F\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n!\,\|f_n\|^2_{L^2(\mathbb{R}^n)}.$$

Furthermore, the *Itô representation theorem* states that if $F \in L^2(\mu)$ is $\mathcal{F}_t$-measurable, then there exists a unique $\mathcal{F}_t$-adapted process $\varphi(t,\omega)$ such that

$$F(\omega) = E_\mu(F) + \int_{\mathbb{R}} \varphi(t,\omega)\, dB_t(\omega).$$

By the *Clark-Ocone formula*, under some extra conditions, the integrand $\varphi(t,\omega)$ is given explicitly by

$$\varphi(t,\omega) = E_\mu[DF|\mathcal{F}_t](\omega),$$

where DF is the Malliavin derivative of F and conditional expectation is taken with respect to the filtration $\mathcal{F}_t$ (up the moment t). This formula is found to be very valuable in economic, when computing parameters of sensitivity of financial derivatives called Greeks. For applications we recommend papers [10], [13], [52], [31]. This theorem will be stated in next section.

Now we give the multiplication formula for Itô integrals

$$I_n(f)I_m(g) = \sum_{k=0}^{n\wedge m} k! \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} I_{m+n-2k}(f \otimes_k g)$$

which holds whenever $f \in \widehat{L^2}(\mathbb{R}^n)$ and $g \in \widehat{L^2}(\mathbb{R}^m)$.

As a consequence of the previous formula we have

$$I_n(u^{\otimes n}) \;=\; h_n(\langle \omega, u\rangle), \quad u \in L^2(\mathbb{R}),$$

where h_n, $n \in N_0$ are the Hermite polynomials of order n. This very useful result, which connects Itô integrals and the Hermite polynomials can be also stated as:

Theorem 2.2.2 *Let $g \in L^2(\mathbb{R})$. Then*

$$n! \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(t_1) \cdots g(t_n)\, dB_{t_1} \cdots dB_{t_n} \;=\; \|g\|^n h_n\left(\frac{\theta}{\|g\|}\right), \qquad (2.16)$$

where $\|g\| = \|g\|_{L^2(\mathbb{R})}$ and $\theta = \int_{\mathbb{R}} g(t)\, dB_t = \langle \omega, g\rangle$.

2.2.6 The Skorokhod integral

The Skorokhod integral is an extension of the Itô integral, for integrands which are not necessarily $\mathcal{F}_t$-adapted. Also, it is connected to the Malliavin derivative. We now give a brief overview of the definition and the most important properties of the Skorokhod integral. For more details we refer to [19], [46].

Let $u(t,\omega) = u_t(\omega)$, $\omega \in \Omega$, $t \in \mathbb{R}$ be a stochastic process such that $u_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}$ and $E_\mu(u_t^2(\omega)) < \infty$, for all $t \in \mathbb{R}$. Then, applying the Wiener-Itô chaos expansion theorem, Theorem 2.2.1 we obtain that there exist functions $f_{n,t}(t_1, \cdots, t_n) \in \widehat{L^2}(\mathbb{R}^n)$ such that

$$u_t(\omega) = \sum_{n=0}^{\infty} I_n(f_{n,t}(\cdot)). \tag{2.17}$$

The functions $f_{n,t}(\cdot)$ are functions of $n+1$-variables since they depend on parameter t, so we write $f_{n,t}(t_1, \cdots, t_n) = f_n(t_1, \cdots, t_n, t)$. The symmetrization of a function $f_{n,t}$ is denoted by $\widetilde{f}_{n,t}$ and is given by $\widetilde{f}_{n,t} = \widetilde{f}_n(t_1, ..., t_n, t) = \frac{1}{n+1}[f_n(t_1, ..., t_n, t) + ... + f_n(t_1, ..., t_{i-1}, t_{i+1}, ..., t_n, t, t_i) + ... + f_n(t_2, ..., t_n, t, t_1)]$, where we only sum over those permutations σ of $\{1, 2, ..., n, n+1\}$ which interchange only the last component with one of the others and leave the rest in place.

Let u be a square integrable $\mathcal{F}_t$-measurable stochastic process, for all $t \in \mathbb{R}$ represented in the form (2.17). Then u is $\mathcal{F}$-adapted if and only if

$$f_{n,t} = f_n(t_1, ..., t_n, t) = 0, \quad \text{for} \quad t < \max_{1 \le i \le n} t_i.$$

Definition 2.2.3 *The Skorokhod integral of a stochastic process u represented by chaos expansion (2.17) is defined by*

$$\begin{aligned} \delta(u) &:= \int_{\mathbb{R}} u_t(\omega)\, \delta B_t(\omega) \\ &= \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_{n,t}). \end{aligned} \tag{2.18}$$

We say that a process u is *integrable in the Skorokhod sense* and write $u \in Dom(\delta)$ if the series in (2.18) converges in $L^2(\mu)$. This occurs if and only if

$$\begin{aligned} \|\delta(u)\|^2_{L^2(\mu)} &= E_\mu(\delta(u)^2) \\ &= \sum_{n=0}^{\infty} (n+1)!\, \|\widetilde{f}_{n,t}\|^2_{L^2(\mathbb{R}^{n+1})} < \infty. \end{aligned} \tag{2.19}$$

From the Wiener-Itô chaos expansion theorem the isometry condition

$$\|u_t\|^2_{L^2(\mu \times \lambda)} = \sum_{n=0}^{\infty} n!\, \|\widetilde{f}_{n,t}\|^2_{L^2(\mathbb{R}^n)}$$

holds. Thus, $Dom(\delta)$ is included in $L^2(\mu \times \lambda)$. Moreover, the Skorokhod integral (2.18) is a linear operator

$$u \in Dom(\delta) \subseteq L^2(\mu \times \lambda) \Rightarrow \delta(u) \in L^2(\mu).$$

Lemma 2.2.1 *For any $u \in Dom(\delta)$ the Skorokhod integral has zero expectation, i.e.*

$$E_\mu \, \delta(u) \; = \; 0.$$

Proof. The assertion follows from the fact that Itô integrals and thus also iterated Itô integrals have zero expectation. □

The following important theorem states that the Itô integral and the Skorokhod integral, given by the formal definition (2.18), coincide on the set of adapted processes. Further on we will call these integrals Itô-Skorokhod integrals.

Theorem 2.2.3 (The Skorokhod integral as an extension of the Itô integral) *Let $u(t,\omega) = u_t(\omega)$ be a $\mathcal{F}_t$-adapted stochastic process for all $t \in \mathbb{R}$ such that*

$$\int_{\mathbb{R}} E_\mu[u_t^2(\omega)] \, dt < \infty.$$

Then $u \in Dom(\delta)$ and

$$\int_{\mathbb{R}} u_t(\omega) \, \delta B_t(\omega) \; = \; \int_{\mathbb{R}} u_t(\omega) \, dB_t(\omega).$$

Proof. We will give the version of the proof found in [19]. Assume that process u_t has the chaos expansion $u_t = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(x,t) \, dB^{\otimes n}(x)$, for $f_n(\cdot,t) \in \widehat{L^2}(\mathbb{R}^n)$ for all $n \in \mathbb{N}$. Thus, $f_n(x_1,...,x_n,t) = 0$ if $\max\limits_{1 \le i \le n} x_i > t$. The symmetrization $\widetilde{f}_n(x_1,...,x_n,t)$ of $f_n(x_1,...,x_n,t)$ is given by

$$\widetilde{f}_n(x_1,...,x_n,t) = \frac{1}{n+1} f(y_1,...,y_n, \max_{1 \le i \le n} x_i).$$

Hence, the Itô integral of u_t is

$$\begin{aligned}
\int_{\mathbb{R}} u_t \, dB_t \;&=\; \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} f_n(x_1,...,x_n,t) \, dB^{\otimes n} \right) dB_t \\
&=\; \sum_{n=0}^{\infty} n! \, (n+1) \int_{\mathbb{R}} \int_{-\infty}^{x_{n+1}} \ldots \int_{-\infty}^{x_2} \widehat{f}_n(x_1,...,x_n,x_{n+1}) \, dB_{x_1} \ldots dB_{x_{n+1}} \\
&=\; \int_{\mathbb{R}} u_t \, \delta B_t
\end{aligned}$$

as it is claimed. □

In Section 2.7.5 we will give a chaos expansion representation of the Skorokhod integral of a stochastic process in terms of the Fourier-Hermite orthogonal polynomials basis and state the integrability conditions for such representation. In similar manner we will decompose the Itô-Skorokhod integral of a $S'(\mathbb{R})$-valued singular generalized stochastic process in terms of a family of orthogonal polynomials.

So far analysis has been exclusively with Gaussian white noise, starting with the Bochner-Minlos theorem (2.1). One could replace the characteristic function C of a standardized Gaussian random variable by other positive definite functionals and obtain a different measure. An important case is the case of Poisson white noise which is objective on next subsection. Thus now we introduce the Poissonian white noise probability space and later on we will construct the corresponding fractional white noise spaces. In this settings we represent square integrable functionals in terms of orthogonal polynomial basis.

2.3 Poissonian White Noise Space

Recently, there have been made improvements in economics and financial modeling by replacing Brownian motion with more general processes and Gaussian white noise with more general white noise. In particular, fractional Brownian motion and Lévy processes are used as driving processes in many applications. It has been pointed out that certain classes of processes based on Lévy processes fit the stock prices data better then classical models based on Brownian motion. In this section we will restrict our research to the special, the Poissonian process.

From a modeling point of view one can investigate physical phenomena where the underlying basic probability measure is not only the Gaussian measure. For example, in [3], [19] the stochastic models for pollution growth when the rate of increase of the concentration is a Poissonian noise were discussed.

Now we introduce the Poissonian white noise space, state chaos expansion theorem for Poisson random variables and define the iterated Itô-Poisson integral.

If we choose in (2.1) the characteristic function of a compensated Poisson random variable

$$C(\varphi) = \exp\left[\int_{\mathbb{R}} (e^{i\varphi(x)} - 1)dx\right], \quad \varphi \in S(\mathbb{R}) \tag{2.20}$$

then the corresponding unique measure P from the Bochner-Minlos theorem is called the *Poisonian white noise measure* ν and the triplet $(S'(\mathbb{R}), \mathcal{B}, \nu)$ is called the *Poissonian white noise probability space*. The Hilbert space of square integrable random variables on $S'(\mathbb{R})$ with respect to the Poissonian measure ν is denoted by $L^2(\nu)$.

Thus, from (2.1) and (2.20) we take

$$\int_{S'(\mathbb{R})} e^{i\langle\omega,\varphi\rangle} d\nu(\omega) = \exp\left[\int_{\mathbb{R}} (e^{i\varphi(x)} - 1)dx\right], \quad \varphi \in S(\mathbb{R}) \tag{2.21}$$

for the definition of the Poissonian white noise space.

From (2.21) it follows that an element $\langle\omega, \varphi\rangle$ has a non-zero expectation

$$E_\nu(\langle\omega,\varphi\rangle) = \int_{\mathbb{R}} \varphi(x)dx \qquad \text{and}$$

$$E_\nu(\langle\omega,\varphi\rangle^2) = \|\varphi\|^2_{L^2(\mathbb{R})} + (\int_{\mathbb{R}} \varphi(x)dx)^2$$

i.e. its variance is $Var(\langle\omega,\varphi\rangle) = \|\varphi\|^2_{L^2(\mathbb{R})}$, for all $\varphi \in S(\mathbb{R})$.

Hence the map

$$J_2 : \varphi \mapsto \langle\omega,\varphi\rangle - \int_{\mathbb{R}} \varphi(x)dx, \quad \varphi \in S(\mathbb{R})$$

can be extended to an isometry from $L^2(\mathbb{R})$ into $L^2(\nu)$. Then $E_\nu(J_2(\varphi)) = 0$ and $\|J_2(\varphi)\|^2_{L^2(\nu)} = \|\varphi\|^2_{L^2(\mathbb{R})}$, for all $\varphi \in L^2(\mathbb{R})$. The formula

$$E_\nu(J_2(\phi)J_2(\varphi)) = (\phi,\varphi)_{L^2(\mathbb{R})}$$

holds for all $\phi, \varphi \in L^2(\mathbb{R})$.

2.3.1 Compensated Poisson process

A right continuous integer valued version of the process

$$P_t(\omega) = J_2(\chi[0,t]) = \langle\omega, \chi[0,t]\rangle - t, \quad \omega \in S'(\mathbb{R})$$

belongs to $L^2(\nu)$ and is called the compensated *one-parameter Poisson process*. Process $P_t(\cdot), t \in \mathbb{R}$ has independent increments. Moreover P_t is a martingale, so it is possible to define the stochastic integral in the same way as we did in the Gaussian case.

2.3.2 Chaos expansion of Poissonian random variables

An essential role in chaos decomposition of $L^2(\nu)$ elements have the Charlier polynomials, which are built by use of the Hermite functions.

Definition 2.3.1 *For a given multi-index* $\alpha = (\alpha_1, ..., \alpha_m, 0, 0, ...) \in \mathcal{I}$, *the* αth Charlier polynomial *is defined as*

$$C_\alpha(\omega) = C_{|\alpha|}(\omega; \underbrace{\xi_1, ..., \xi_1}_{\alpha_1}, ..., \underbrace{\xi_m, ..., \xi_m}_{\alpha_m}), \tag{2.22}$$

where ξ_k *are the Hermite functions and*

$$C_k(\omega; \varphi_1, ..., \varphi_k) = \frac{\partial^k}{\partial u_1 .... \partial u_k} \exp\Big[\ \langle \omega, \log\Big(1 + \sum_{j=1}^k u_j \varphi_j\Big) - $$

$$ - \sum_{j=1}^k u_j \int_{\mathbb{R}} \varphi_j(y) dy \rangle \ \Big] \Big|_{u_1 = ... = u_k = 0},$$

for $k \in \mathbb{N}$ *and* $\varphi_j \in S(\mathbb{R})$.

In particular, for $\omega \in S'(\mathbb{R}), k, j \in \mathbb{N}$ we have

$$C_0(\omega) = 1, \tag{2.23}$$

$$C_{\varepsilon^{(k)}}(\omega) = C_1(\omega, \xi_k) = \langle \omega, \xi_k \rangle - \int_{\mathbb{R}} \xi_k(x) dx = J_2(\xi_k), \tag{2.24}$$

and

$$\begin{aligned} C_{\varepsilon^{(k)} + \varepsilon^{(j)}}(\omega) &= \langle \omega, \xi_k \rangle \langle \omega, \xi_j \rangle - \langle \omega, \xi_k \xi_j \rangle - \langle \omega, \xi_k \rangle \int_{\mathbb{R}} \xi_j(x) dx - \\ &- \langle \omega, \xi_j \rangle \int_{\mathbb{R}} \xi_k(x) dx + \int_{\mathbb{R}} \xi_k(x) dx \int_{\mathbb{R}} \xi_j(x) dx. \end{aligned} \tag{2.25}$$

It is a familiar fact that the family of Charlier polynomial functionals $\{C_\alpha; \alpha \in \mathcal{I}\}$ forms an orthogonal basis of the space of Poissonian square integrable random variables $L^2(\nu)$ and $\|C_\alpha\|^2_{L^2(\nu)} = \alpha!$. For more information we refer to [10], [19], [59].

From the Wiener-Itô chaos expansion theorem, Theorem 2.1.1 it follows that every element $G \in L^2(\nu)$ is given in a unique form

$$G(\omega) = \sum_{\alpha \in \mathcal{I}} b_\alpha C_\alpha(\omega), \quad b_\alpha \in \mathbb{R}, \tag{2.26}$$

where $\|G\|^2_{L^2(\nu)} = \sum_{\alpha \in \mathcal{I}} \alpha! \, b_\alpha^2$ is finite.

2.3.3 Stochastic integrals with respect to the Poissonian measure

The iterated Itô integral with respect to the Poissonian measure is defined in an analogous way as iterated Itô integral in Gaussian case. In the following we will call it the *iterated Itô-Poisson integral.*

We assume to have the compensated one-parameter Poisson process $P_t(\cdot), t \in \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$. For $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra generated by $P_s(\cdot)$, $0 \leq s < t$. Let $\widehat{L^2}(\mathbb{R}^n)$ be the set of symmetric deterministic square integrable functions on $\mathbb{R}^n$. Then the *n-fold iterated Itô-Poisson integral* of a function $f \in \widehat{L^2}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} I_n^\nu(g) &= \int_{\mathbb{R}^n} g(t_1, t_2, \cdots, t_n) dP_t^{\otimes n} \\ &= n! \int_{-\infty}^{+\infty} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} g(t_1, \cdots, t_n)\, dP_{t_1} \cdots dP_{t_n}, \end{aligned}$$

where the integral on the right-hand of equality consist of n-iterated Itô-Poisson integrals of the first order.

Thus, every element F of the Hilbert space $L^2(\nu)$ can be represented in terms of iterated Itô integrals, i.e. another formulation of Wiener-Itô chaos expansion theorem holds.

2.3.4 Chaos expansion in terms of multiple Itô-Poisson integrals

Now we formulate an alternative statement of the Wiener-Itô chaos expansion theorem 2.1.1 in the one-dimensional case, which gives the chaos expansion decomposition of a Poissonian random variable in terms of multiple Itô-Poisson integrals.

Theorem 2.3.1 (The Wiener-Itô chaos expansion theorem) *For every Poissonian random variable $G \in L^2(\nu)$ there exists a unique family of symmetrized functions $g_n \in \widehat{L^2}(\mathbb{R})$, $n \in \mathbb{N}$ such that*

$$G(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n(t_1, ..., t_n)\, dP_t^{\otimes n}(\omega). \tag{2.27}$$

Moreover, the isometry $\|G\|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} n! \|g_n\|^2_{L^2(\mathbb{R})}$ holds.

Similar as in Gaussian case, there is connection between chaos expansions (2.26) and (2.27). Multiple integrals with respect to P_t are expressed in terms of the Charlier polynomials in the following way

$$\begin{aligned} C_\alpha(\omega) &= \int_{\mathbb{R}^{|\alpha|}} \xi_1^{\hat{\otimes}\alpha_1} \hat{\otimes} ... \hat{\otimes} \xi_k^{\hat{\otimes}\alpha_k} \, dP_t^{\otimes|\alpha|} \\ &= \int_{\mathbb{R}^{|\alpha|}} \xi^{\hat{\otimes}\alpha} \, dP_t^{\otimes|\alpha|}, \qquad \alpha = (\alpha_1, ..., \alpha_k, 0, 0, ...) \in \mathcal{J}. \end{aligned}$$

Clearly, we have

$$\begin{aligned} G(\omega) &= \sum_{\alpha\in\mathcal{J}} b_\alpha \, C_\alpha(\omega) \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} b_\alpha \int_{\mathbb{R}^n} \xi^{\hat{\otimes} n} \, dP_t^{\otimes n} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n \, dP_t^{\otimes n}, \end{aligned}$$

with the sequence of symmetrized functions in $\widehat{L^2}(\mathbb{R}^n)$

$$g_n = \sum_{|\alpha|=n} b_\alpha \xi^{\hat{\otimes} n}. \tag{2.28}$$

Moreover we have the isometry

$$\|G\|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} n! \, \|g_n\|^2_{\widehat{L^2}(\mathbb{R}^n)}.$$

For more details on Poissonian processes, Itô-Poisson integrals and the Charlier polynomials we refer to [3], [19], [64].

2.4 Unitary Mapping $\mathcal{U}$

The following important theorem, proved by Benth and Gjerde in [11], states the existence of a unitary correspondence between the Gaussian and the Poissonian spaces of random variables.

Theorem 2.4.1 ([11]) *The map $\mathcal{U} : L^2(\mu) \to L^2(\nu)$ defined by*

$$\mathcal{U}\left(\sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha(\omega)\right) = \sum_{\alpha\in\mathcal{J}} b_\alpha C_\alpha(\omega), \qquad b_\alpha \in \mathbb{R}, \alpha \in \mathcal{J} \tag{2.29}$$

is unitary i.e. it is surjective and the isometry $\|\mathcal{U}(F)\|_{L^2(\nu)} = \|F\|_{L^2(\mu)}$ holds.

Using the isometry $\mathcal{U}$ all results obtained in the Gaussian case can be carried over to the Poissonian case. The Fourier-Hermite orthogonal basis $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ of the space of Gaussian random variables just has to be replaced with the corresponding elements of the Charlier polynomials orthogonal basis $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of the space of Poissonian random variables. In [29], [30] we used this isometry to interpret stochastic differential equations with the Malliavin derivative and their solutions obtained in Gaussian versions of q-weighted spaces with their corresponding Poissonian versions.

For more details on Gaussian white noise spaces, Poissonian white noise spaces, Hermite and Charlier polynomials we refer to [3], [8], [17], [19].

2.5 Spaces of Generalized Random Variables (q-weighted Stochastic Spaces)

Following the ideas introduced in [17], [19], [31], [34] we define q-weighted stochastic spaces of test functions $(Q)_\rho^P$ and stochastic generalized functions $(Q)_{-\rho}^P$, with respect to the measure P, which represent the stochastic analogue of the deterministic spaces $S(\mathbb{R})$, $S'(\mathbb{R})$, $\exp S(\mathbb{R})$ and $\exp S'(\mathbb{R})$ for $l \in \mathbb{N}_0$. These q-weighted stochastic spaces of test functions and distributions will constitute our spaces of smooth and generalized random variables respectively. The choice of the weight q depends on a concrete problem which is studied. The proceeding characterization of q-weighted spaces is taken from our papers [29] and [30].

Let $q_\alpha > 1$, $\alpha \in \mathcal{I}$ and let $\rho \in [0,1]$.

The *space of q-weighted P-stochastic test functions (test random variables)*, denoted by $(Q)_\rho^P$, consists of elements $f = \sum\limits_{\alpha\in\mathcal{I}} b_\alpha K_\alpha \in L^2(P)$, $b_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that

$$\|f\|^2_{(Q)^P_{\rho,p}} = \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1+\rho}\, b_\alpha^2\, q_\alpha^p < \infty, \quad \text{for all } p \in \mathbb{N}_0.$$

The *space of q-weighted P-stochastic generalized functions (generalized random variables)*, denoted by $(Q)_{-\rho}^P$, consists of formal expansions of the form $F = \sum\limits_{\alpha\in\mathcal{I}} c_\alpha K_\alpha$, $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that

$$\|F\|^2_{(Q)^P_{-\rho,-p}} = \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1-\rho}\, c_\alpha^2\, q_\alpha^{-p} < \infty, \quad \text{for some } p \in \mathbb{N}_0.$$

The action of a generalized function $F \in (Q)_{-\rho}^P$ onto a test function

$f \in (Q)^P_\rho$ is given by

$$\ll F, f \gg = \sum_{\alpha\in\mathcal{I}} \alpha!\, c_\alpha b_\alpha.$$

The *generalized expectation* of F is defined as $E_P(F) = \ll F, 1 \gg = b_0$. It is considered to be the zero coefficient in the chaos expansion of a generalized function F in orthogonal basis $\{K_\alpha\}_{\alpha\in\mathcal{I}}$. In particular, if $F \in L^2(P)$ it coincides with usual expectation.

Note that the space $(Q)^P_\rho$ can also be constructed as the projective limit of the family $(Q)^P_{\rho,p} = \{f = \sum_{\alpha\in\mathcal{I}} b_\alpha K_\alpha \in L^2(P) : \|f\|^2_{(Q)^P_{\rho,p}} < \infty\}$, $p \in \mathbb{N}_0$, i.e.

$$(Q)^P_\rho = \bigcap_{p\in\mathbb{N}_0} (Q)^P_{\rho,p}.$$

Space $(Q)^P_{-\rho}$ can also be constructed as the inductive limit of the family $(Q)^P_{-\rho,-p} = \{F = \sum_{\alpha\in\mathcal{I}} c_\alpha K_\alpha : \|F\|^2_{(Q)^P_{-\rho,-p}} < \infty\}$, $p \in \mathbb{N}_0$, i.e.

$$(Q)^P_{-\rho} = \bigcup_{p\in\mathbb{N}_0} (Q)^P_{-\rho,-p}.$$

Three important special cases will be given by weights of the form:

- $q_\alpha = (2\mathbb{N})^\alpha$,
- $q_\alpha = e^{(2\mathbb{N})^\alpha}$,
- $q_\alpha = a^\alpha (2\mathbb{N})^\alpha$.

For weights of the form $q_\alpha = (2\mathbb{N})^\alpha$ we obtain the *Kondratiev spaces* of P-stochastic test functions and P-stochastic generalized functions, denoted by $(S)^P_\rho$ and $(S)^P_{-\rho}$, respectively. In particular, for $\rho = 0$ the Kondratiev spaces are called the *Hida spaces* of test and generalized stochastic functions, denoted by (S) and $(S)^*$ respectively.

For $q_\alpha = e^{(2\mathbb{N})^\alpha}$ we obtain the *exponential growth spaces* of P-stochastic test functions and P-stochastic generalized functions, denoted by $\exp(S)^P_\rho$ and $\exp(S)^P_{-\rho}$ respectively. It holds that

$$\exp(S)^P_\rho \subseteq (S)^P_\rho \subseteq L^2(P) \subseteq (S)^P_{-\rho} \subseteq \exp(S)^P_{-\rho}, \qquad (2.30)$$

with continuous inclusions.

Particulary, for $P = \mu$ the spaces in (2.30) become Gaussian q-weighted spaces and in that case relation (2.30) was proven in [56]. For $P = \nu$ we obtain the Poissonian q-weighted spaces. For more details on the Kondratiev spaces we refer to [19] and references therein and on spaces of exponential growth to [56].

The largest spaces of q-weighted P-stochastic distributions $(Q)_{-1}^P$ are obtained for $\rho = 1$.

For weights of the form $q_\alpha = a^\alpha (2\mathbb{N})^\alpha$ we obtain the *Kondratiev spaces* of P-stochastic test functions and P-stochastic generalized functions *modified* by the given sequence $a = (a_k)_{k\in\mathbb{N}}$, $a_k \geq 1$, and denoted by $(Sa)_{\rho,p}$ and $(Sa)_{-\rho,-p}$ respectively. We use notation $a^\alpha = \prod_{k=1}^\infty a_k^{\alpha_k}$, $\frac{a^\alpha}{\alpha!} = \prod_{k=1}^\infty \frac{a_k^{\alpha_k}}{\alpha_k!}$ and $(2\mathbb{N}a)^\alpha = \prod_{k=1}^\infty (2k\, a_k)^{\alpha_k}$.

The result, proven by Zhang in [69], which states that

$$\sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty \quad \text{if and only if} \quad p > 1$$

is used to verify the statement $\sum_{\alpha\in\mathcal{I}} (2\mathbb{N}a)^{-p\alpha} < \infty$ if and only if $p > 1$.

The space of *Kondratiev P-stochastic test functions modified by the sequence a*, denoted by $(Sa)_\rho^P = \bigcap_{p\in\mathbb{N}_0} (Sa)_{\rho,p}^P$, $p \in \mathbb{N}_0$, is the projective limit of spaces

$$(Sa)_{\rho,p}^P = \{f = \sum_{\alpha\in\mathcal{I}} b_\alpha K_\alpha \in L^2(P) : \; \|f\|^2_{(Sa)_{\rho,p}^P} = \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1+\rho}\, b_\alpha^2\, (2\mathbb{N}\, a)^{p\alpha} < \infty\}.$$

The space of *Kondratiev P-stochastic generalized functions modified by the sequence a*, denoted by $(Sa)_{-\rho}^P = \bigcup_{p\in\mathbb{N}_0} (Sa)_{-\rho,-p}^P$, $p \in \mathbb{N}_0$, is the inductive limit of the spaces

$$(Sa)_{-\rho,-p}^P = \{F = \sum_{\alpha\in\mathcal{I}} c_\alpha K_\alpha : \; \|F\|^2_{(Sa)_{-\rho,-p}^P} = \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1-\rho}\, c_\alpha^2 (2\mathbb{N}\, a)^{-p\alpha} < \infty\}.$$

For $a_k = 1$, $k \in \mathbb{N}$ these spaces reduce to the spaces of Kondratiev P-stochastic test functions $(S)_\rho^P$ and the Kondratiev P-stochastic generalized functions $(S)_{-\rho}^P$ respectively. For all $\rho \in [0,1]$ we have a Gel'fand triplet

$$(Sa)_\rho^P \subseteq L^2(P) \subseteq (Sa)_{-\rho}^P.$$

In particular, the largest space of the Kondratiev P-stochastic distributions modified by the sequence a is obtained for $\rho = 1$ and is denoted by $(Sa)_{-1}^P$. For $P = \mu$ these spaces are called Gaussian and for $P = \nu$ Poissonian Kondratiev spaces modified by the sequence a. In [27] we introduced the Gaussian type of these spaces and solve equations related to them.

We will return to the notion of q-weighted spaces and recall some properties when presenting applications of the chaos expansion method for solving stochastic differential equations. Equations presented and solved in Chapter 5 have solutions which are generalized stochastic processes with values in some space of q-weighted stochastic distributions.

2.5.1 Unitary mapping $\mathcal{U}$ of q-weighted stochastic distributions

We can extend the unitary mapping $\mathcal{U}$, given in the Theorem 2.4.1, into a linear and isometric mapping on q-weighted spaces by defining

$$\mathcal{U}: \ (Q)^{\mu}_{-\rho} \to (Q)^{\nu}_{-\rho}$$

such that

$$\mathcal{U}\left[\sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega)\right] = \sum_{\alpha\in\mathcal{I}} a_\alpha C_\alpha(\omega), \quad a_\alpha \in \mathbb{R}, \tag{2.31}$$

for elements $F = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega) \in (Q)^{\mu}_{-\rho,-p_0}$. Furthermore, the isometry

$$\|\mathcal{U}(F)\|_{(Q)^{\nu}_{-\rho,-p}} = \|F\|_{(Q)^{\mu}_{-\rho,-p}}$$

holds for all $p \geq p_0$. More details can be found in [11], [19] and [29].

2.5.2 Wick product for stochastic distributions

In the framework of white noise analysis, the problem of pointwise multiplication of generalized functions is overcome by introducing the Wick product. Historically, the Wick product first arose in quantum physics, as a renormalization operation and is close connected to the $\mathcal{S}$-transform. The most important property of the Wick multiplication is its relation to the Itô-Skorokhod integration. For more details we refer to [18],[19], [24], [31], [35].

The Wick product is well defined in the Hida and Kondratiev spaces of test and generalized stochastic functions; see for example [17], [19], [25]. In [56] it is defined for stochastic test functions and distributions of exponential growth. In this subsection we give a generalization of the Wick multiplication of random variables belonging to spaces of q-weighted test functions and distributions.

The Wick product can be defined in a very simple manner:

Definition 2.5.1 *Let $\rho \in [0,1]$ and let $F, G \in (Q)^{P}_{-\rho}$ be given by their chaos expansions $F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha K_\alpha(\omega)$, $G(\omega) = \sum_{\beta\in\mathcal{I}} g_\beta K_\beta(\omega)$, for unique $f_\alpha, g_\beta \in \mathbb{R}$. The P-*Wick product *of F and G is the element denoted by $F\lozenge^P G$ and defined by*

$$F\lozenge^P G(\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta\right) K_\gamma(\omega). \tag{2.32}$$

The same definition is provided for the Wick product of test q-weighted stochastic functions belonging to $(Q)^P_\rho$.

Consider now the special case, the spaces obtained for $\rho = 1$. Providing the additional condition (2.33), we prove that the space $(Q)^P_{-1}$ is closed under the P-Wick multiplication.

Theorem 2.5.1 *Let $F, G \in (Q)^P_{-1}$. Assume that, for some $C > 0$ weights q_α satisfy the property*

$$q_{\alpha+\beta} \geq C\, q_\alpha\, q_\beta, \qquad \alpha, \beta \in \mathcal{I}. \tag{2.33}$$

Then the element $F \Diamond^P G$, defined by (2.32), belongs to $(Q)^P_{-1}$.

Proof. Let $F, G \in (Q)^P_{-1}$. Then there exist $p_1 \geq 0$ such that

$$\sum_{\alpha \in \mathcal{I}} f_\alpha^2\, q_\alpha^{-p_1} < \infty \quad \text{and} \quad \sum_{\beta \in \mathcal{I}} f_\beta^2\, q_\beta^{-p_1} < \infty.$$

The P-Wick product is given by

$$F \Diamond^P G(\omega) = \sum_{\gamma \in \mathcal{I}} c_\gamma K_\gamma(\omega), \quad \text{for} \quad c_\gamma = \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta.$$

Then there exists $k > 0$ such that for $p = p_1 + k$ we have

$$\begin{aligned}
\sum_{\gamma \in \mathcal{I}} c_\gamma^2\, q_\gamma^{-p} &= \sum_{\gamma \in \mathcal{I}} \Big(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \Big)^2 q_\gamma^{-p_1}\, q_\gamma^{-k} \\
&\leq C \sum_{\gamma \in \mathcal{I}} q_\gamma^{-k} \Big(\sum_{\alpha+\beta=\gamma} f_\alpha^2 \Big) \Big(\sum_{\alpha+\beta=\gamma} g_\beta^2 \Big) q_\alpha^{-p_1}\, q_\beta^{-p_1} \\
&\leq C\, m \cdot \Big(\sum_{\alpha \in \mathcal{I}} f_\alpha^2\, q_\alpha^{-p_1} \Big) \Big(\sum_{\beta \in \mathcal{I}} g_\beta^2\, q_\beta^{-p_1} \Big) < \infty,
\end{aligned}$$

for $m = \sum_{\gamma \in \mathcal{I}} q_\gamma^{-k} < \infty$. □

Theorem 2.5.2 *Let $F, G \in (Q)^P_1$. Assume that, for some $C > 0$ weights q_α satisfy the following property*

$$q_{\alpha+\beta} \leq C\, q_\alpha\, q_\beta, \qquad \alpha, \beta \in \mathcal{I}. \tag{2.34}$$

Then the element $F \Diamond^P G$, defined by (2.32), belongs to $(Q)^P_1$.

In particular, if we focus on two special types of weights, $q_\alpha = (2\mathbb{N})^\alpha$ and $q_\alpha = e^{(2\mathbb{N})^\alpha}$, we verify that, in both cases, both conditions (2.33) and (2.34) are satisfied since $q_{\alpha+\beta} = q_\alpha\, q_\beta$, thus the corresponding spaces of stochastic distributions $(S)^P_{\pm 1}$ and $\exp(S)^P_{\pm 1}$ are closed under the Wick multiplication. The Wick product for $F, G \in \exp(S)^\mu_{-1}$ was first introduced in [56]. Moreover it is also known that the spaces (S), $(S)^*$, $(S)^P_1$, $\exp(S)^P_1$ are closed under the P-Wick multiplication while the space $L^2(P)$ is not closed under the P-Wick multiplication.

Further on we will write $\lozenge$ for the P-Wick multiplication $\lozenge^P$ and E for the expectation E_P with respect to P, whenever the underlying measure P is understood.

The Wick product is a commutative, associative operation, distributive with respect to addition. In particular, for the orthogonal polynomial basis of $L^2(P)$, in both cases $P = \mu$ and $P = \nu$, we have

$$K_\alpha \lozenge K_\beta = K_{\alpha+\beta}, \quad \text{for} \quad \alpha, \beta \in \mathcal{I}. \tag{2.35}$$

Whenever F, G and $F \lozenge G$ are P-integrable, the following equality

$$E(F \lozenge G) \;=\; E(F) \cdot E(G)$$

holds. Here E denotes the generalized expectation. Note that independence of F and G is not required.

The Wick powers of element $F \in (Q)^P_{-1}$ are defined inductively by

$$\begin{cases} F^{\lozenge\, 0} = 1, \\ F^{\lozenge k} = F \lozenge\, F^{\lozenge\,(k-1)}, \quad k \in \mathbb{N}. \end{cases} \tag{2.36}$$

More generally, if $p(x) = \sum_{k=0}^m a_k\, x^k$, $a_k \in \mathbb{R}$, $x \in \mathbb{R}$ is a polynomial of degree m with real coefficients, then its Wick version $p^\lozenge : (Q)^P_{-1} \to (Q)^P_{-1}$ is defined by

$$p^\lozenge(F) = \sum_{k=0}^m a_k\, F^{\lozenge k}, \quad \text{for} \quad F \in (Q)^P_{-1}. \tag{2.37}$$

The *Wick exponential* of $X \in (Q)^P_{-1}$ is defined as a formal sum

$$\exp^\lozenge X = \sum_{n=0}^\infty \frac{X^{\lozenge n}}{n!}. \tag{2.38}$$

In view of the properties mentioned above, for $F, G \in (Q)^P_{-1}$ we have

$$(F+G)^{\lozenge\, 2} \;=\; F^{\lozenge 2} + 2F \lozenge G + G^{\lozenge 2} \quad \text{and}$$

$$\exp^{\lozenge}(F+G) \;=\; \exp^{\lozenge} F \lozenge \exp^{\lozenge} G.$$

By induction it follows that

$$E(\exp^{\lozenge} F) = \exp(EF), \qquad F \in (Q)^{P}_{-1}.$$

Example 2.5.1 (Normalized stochastic exponential) *Consider now a special case of the Wick exponential defined by* (2.38) *in $P=\mu$ case. We let $X = \langle \omega, \varphi\rangle \in (S)^{\mu}_{-1}$ to be a one-dimensional smoothed white noise, defined in Example 2.2.1, for $\varphi \in S(\mathbb{R})$ and $\omega \in S'(\mathbb{R})$. One can show that this element coincides with the normalized stochastic exponential, defined in* (2.7)*. In particular we have*

$$\begin{aligned} \exp^{\lozenge}(\langle \omega, \varphi\rangle) &= \exp\left(\langle \omega, \varphi\rangle - \frac{1}{2}\,\|\varphi\|^2_{L^2(\mathbb{R})}\right) \\ &= \varepsilon_{\varphi}, \qquad\qquad \varphi \in S(\mathbb{R}). \end{aligned} \tag{2.39}$$

The identity (2.39) *follows from the chaos expansion theorem and the generating property* (1.1) *for the Hermite polynomials, i.e.*

$$\begin{aligned} \exp^{\lozenge}(\langle \omega, \varphi\rangle) &= \sum_{n=0}^{\infty} \frac{1}{n!}\,\langle \omega, \varphi\rangle^{\lozenge n} \\ &\underset{\varphi=\lambda\xi_1}{=} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\langle \omega, \xi_1\rangle^{\lozenge n} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,H_{n\varepsilon^{(1)}}(\omega) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,h_n(\langle \omega, \xi_1\rangle) \\ &= \exp\left(\lambda\langle \omega, \xi_1\rangle - \frac{1}{2}\lambda^2\right) \\ &= \exp\left(\langle \omega, \varphi\rangle - \frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}\right), \\ &= \varepsilon_{\varphi}, \qquad \varphi \in S(\mathbb{R}). \end{aligned}$$

Hence, combining with (2.16) *we have the decomposition*

$$\varepsilon_{\varphi} = \exp\left(\langle \omega, \varphi\rangle - \frac{1}{2}\,\|\varphi\|^2_{L^2(\mathbb{R})}\right) = \sum_{n=0}^{\infty} \frac{\|\varphi\|^n_{L^2(\mathbb{R})}}{n!}\,h_n\left(\frac{\langle \omega, \varphi\rangle}{\|\varphi\|_{L^2(\mathbb{R})}}\right), \quad \varphi \in S(\mathbb{R}).$$

Note that from Theorem 2.2.2 we can conclude

$$\langle \omega, g\rangle^{\lozenge n} = \|g\|^n_{L^2(\mathbb{R})}\,h_n\left(\frac{\langle \omega, g\rangle}{\|g\|_{L^2(\mathbb{R})}}\right)$$

and thus (2.39) *is obtained directly.*

The set of all combinations of functions of the form $\exp^{\lozenge}(\langle\omega,\varphi\rangle)$ *is dense in both spaces of random variables* $(S)_1^{\mu}$ *and* $(S)_{-1}^{\mu}$. *Moreover they are normalized in the sense that the expectation is* $E\exp^{\lozenge}(\langle\omega,\varphi\rangle)=1$ *for all* φ. *For more details we refer to [19], [35].*

Definition of the P-Wick multiplication based on chaos expansion in terms of the orthogonal polynomials basis $\{K_\alpha\}_{\alpha\in\mathcal{I}}$ is wide enough to include also the P-singular white noise (particularly, for $P=\mu$ the Gaussian singular white noise W_t and for $P=\nu$ the Poissonian white noise V_t). It is also important to know how to express the P-Wick product in terms of multiple Itô integrals in $L^2(\mu)$, respectively the multiple Itô-Poisson integrals in $L^2(\nu)$.

Further on we denote by

$$I_n(f_n)=\int_{\mathbb{R}} f_n\, d\mathcal{Q}_t^{\otimes n}$$

the n-fold iterated stochastic integral of a symmetrized sequence of functions $f_n\in\widehat{L^2}(\mathbb{R}^n)$, for all $n\in\mathbb{N}$, with respect to $\mathcal{Q}_t$, where $\mathcal{Q}_t$ denotes either Brownian motion B_t or compensated Poisson process P_t. Thus for $P=\mu$ the integral I_n represents the nth Itô integral and for $P=\nu$ the integral I_n is the nth Itô-Poisson integral.

Theorem 2.5.3 *Let* $X=\sum_{n=0}^{\infty}I_n(f_n)$, $f_n\in\widehat{L^2}(\mathbb{R}^n)$, $n\in\mathbb{N}$ *and* $Y=\sum_{m=0}^{\infty}I_m(g_m)$, $g_m\in\widehat{L^2}(\mathbb{R}^m)$, $m\in\mathbb{N}$ *belong to the Kondratiev space of generalized functions* $(S)_{-1}$. *Then the Wick product* $X\lozenge Y$ *of* X *and* Y *can be expressed by*

$$X\lozenge Y=\sum_{n,m=0}^{\infty}I_{n+m}(f_n\widehat{\otimes}g_m)=\sum_{k=0}^{\infty}(\sum_{n+m=k}^{\infty}I_k(f_n\widehat{\otimes}g_m)).$$

Remark 2.5.1 *In* $L^2(\mu)$ *case, the property* $I_1(f)^{\lozenge n}=I_n(f^{\widehat{\otimes}n})$ *for* $f\in\widehat{L^2}(\mathbb{R})$, *which is similar to the Fubini theorem, follows from the previous theorem. If* $f\in L^2(\mathbb{R})$ *then this result becomes*

$$(\int_{\mathbb{R}} f(t)dB_t)^{\lozenge n}=n!\int_{\mathbb{R}^n} f^{\otimes n}(x_1,...,x_n)dB^{\otimes n}.$$

In particular, the partial integration formula holds

$$I_1(f)\lozenge I_1(g)=I_1(f)\cdot I_1(g)-(f,g)_{L^2(\mathbb{R})}$$

for deterministic $f, g \in L^2(\mathbb{R})$. This identity gives connection between the Wick and ordinary multiplication by the correction term $(f,g)_{L^2(\mathbb{R})}$.

Note that the Wick multiplication and ordinary multiplication coincide when at least one of the multiplication terms is deterministic, i.e.

$$F \lozenge G = F \cdot G, \quad F \in \mathbb{R}.$$

$\mathcal{S}$-transform

An equivalent definition of the Wick product can be formulated in terms of the $\mathcal{S}$-transform. In [19], [24], [35] the $\mathcal{S}$-transform is considered on the Kondratiev space of generalized stochastic random variables $(S)_{-\rho}$, for $\rho \in [0,1]$.

Definition 2.5.2 *The* $\mathcal{S}$-transform *of an element* $F \in (S)_{-\rho}$ *is defined by*

$$\mathcal{S}F(h) := \ll F, \varepsilon_h \gg, \tag{2.40}$$

where $h \in S_p(\mathbb{R})$ with $\|h\|_p^2 < 1$.

Recall ε_h is the normalized stochastic exponential defined by (2.7) and $\ll \cdot, \cdot \gg$ denotes the duality paring between $(S)_\rho$ and $(S)_{-\rho}$. Following Definition 2.5.2 the $\mathcal{S}$-transform of an element $F = \sum_{\alpha \in \mathcal{I}} f_\alpha K_\alpha$ from $(S)_{-\rho}$ is given by the chaos expansion

$$\mathcal{S}F(h) = \sum_{\alpha \in \mathcal{I}} h^\alpha f_\alpha, \tag{2.41}$$

where $h = \sum_{k \in \mathbb{N}} h_k \xi_k \in S(\mathbb{R})$ and $h^\alpha = \prod_{j \in \mathbb{N}} (h_j)^{\alpha_j}$.

Therefore, if $\rho < 1$ then $\mathcal{S}F(h)$ is well-defined for all $h \in S(\mathbb{R})$ and if $\rho = 1$, the $\mathcal{S}F(h)$ is well-defined for h with sufficiently small $L^2(\mathbb{R})$ norm. The $\mathcal{S}$-transform is a bijection onto a space of so-called $\mathcal{U}$-functionals. For detailed construction of the $\mathcal{S}$-transform and its properties we refer to [17],[24], [35], [37].

Definition 2.5.3 *The Wick product $\lozenge$ of two Kondratiev stochastic distributions $F, G \in (S)_{-\rho}$, $\rho \in [0,1]$ is the unique element whose $\mathcal{S}$-transform is $\mathcal{S}F \cdot \mathcal{S}G$.*

If $\mathcal{S}^{-1}$ is the inverse $\mathcal{S}$-transform then

$$F \lozenge G = \mathcal{S}^{-1}(\mathcal{S}F \cdot \mathcal{S}G). \tag{2.42}$$

Now, the singular white noise W_t on $\mathbb{R}$ can be defined as the unique element of the Hida space $(S)_{-0} = (S)^*$ whose $\mathcal{S}$-transform satisfies $\mathcal{S}W_h = h$.

If $G \in L^2(\mu)$ then $G \in (S)^{\mu}_{-0}$ and the Fourier transform is defined

$$\mathcal{F}(G)(h) = \int_{S'(\mathbb{R})} e^{i\langle\omega,h\rangle} G(\omega) d\mu(\omega).$$

Thus for a random variable G the $\mathcal{S}$-transform is given by

$$\mathcal{S}(G)(ih) = \mathcal{F}G(h)\, e^{\frac{1}{2}\|h\|^2_{L^2(\mathbb{R})}}.$$

As a result, we conclude that the Wick product can be interpreted as a convolution on the infinite-dimensional space $(S)_{-\rho}$.

The definition (2.40) of the $\mathcal{S}$-transform can be extended on an analogous way, from the Kondratiev space $(S)_{-\rho}$, $\rho \in [0,1]$ to all q-weighted stochastic distributions $(Q)_{-\rho}$.

2.6 Hilbert Space Valued q-weighted Generalized Random Variables

In this subsection, by H we mean a separable Hilbert space with the orthonormal basis $\{\eta_i\}_{i\in\mathbb{N}}$ and the inner product $(\cdot,\cdot)_H$. We will treat H as the state space. Recall that the basic probability space is $(S'(\mathbb{R}), \mathcal{B}, P)$. We denote by $L^2(P,H)$ the space of functions on Ω with values in H, which are square integrable with respect to the white noise measure P. It is a Hilbert space equipped with the inner product

$$\ll F, G \gg_{L^2(P,H)} = E_P\left((F,G)_H\right), \quad \text{for all } F, G \in L^2(P,H).$$

The family of functions $\{\frac{1}{\sqrt{\alpha!}} K_\alpha\, \eta_i\}_{i\in\mathbb{N},\alpha\in\mathcal{I}}$ forms an orthonormal basis of the Hilbert space $L^2(P,H)$.

Now we define *H-valued q-weighted generalized random variables* of growth rate determined by the sequence q_α, over $L^2(P,H)$.

Let $q_\alpha > 1$, $\alpha \in \mathcal{I}$ and let $\rho \in [0,1]$.

The *space of H-valued q-weighted P-stochastic test random variables* $Q(H)^P_\rho$ consists of functions $f \in L^2(P,H)$, with the expansion $f(\omega) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k}\, \eta_k\, K_\alpha(\omega)$, $a_{\alpha,k} \in \mathbb{R}$, such that

$$\begin{aligned} \|f\|^2_{Q^P(H)_\rho,p} &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha!^{1+\rho}\, a^2_{\alpha,k}\, q^p_\alpha \\ &= \sum_{k\in\mathbb{N}} \sum_{\alpha\in\mathcal{I}} \alpha!^{1+\rho}\, a^2_{\alpha,k}\, q^p_\alpha < \infty, \quad \text{for all } p \in \mathbb{N}_0. \end{aligned}$$

Note, that $f(\omega)$ can be expressed in several ways

$$\begin{aligned} f(\omega) &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} a_{\alpha,k}\,\eta_k\,K_\alpha(\omega) \\ &= \sum_{\alpha\in\mathcal{I}} a_\alpha\,K_\alpha(\omega) \\ &= \sum_{k\in\mathbb{N}} a_k(\omega)\,\eta_k, \end{aligned}$$

where

$$a_\alpha = (f, K_\alpha)_{L^2(P)} = \sum_{k\in\mathbb{N}} a_{\alpha,k}\,\eta_k \;\in H$$

and

$$a_k(\omega) = (f,\eta_k)_H = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k}\,K_\alpha(\omega) \in (Q)_\rho^P,$$

with $a_{\alpha,k} = \ll f, \eta_k\,K_\alpha \gg_{L^2(P,H)} \in \mathbb{R}$ for $k\in\mathbb{N}, \alpha\in\mathcal{I}$.

The corresponding *space of q-weighted P-stochastic generalized functions (generalized random variables)* $Q(H)^P_{-\rho}$ consists of formal expansions of the form $F(\omega) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} b_{\alpha,k}\,\eta_k\,K_\alpha(\omega)$, $b_{\alpha,k}\in\mathbb{R}$, such that

$$\begin{aligned} \|F\|^2_{Q(H)^P_{-\rho},-p} &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha!^{1-\rho}\,b^2_{\alpha,k}\,q_\alpha^{-p} \\ &= \sum_{k\in\mathbb{N}}\sum_{\alpha\in\mathcal{I}} \alpha!^{1-\rho}\,b^2_{\alpha,k}\,q_\alpha^{-p} < \infty, \quad \text{for some } p\in\mathbb{N}_0. \end{aligned}$$

It is clear that $F(\omega)$ can be expressed in several ways

$$\begin{aligned} F(\omega) &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} b_{\alpha,k}\,\eta_k\,K_\alpha(\omega) \\ &= \sum_{\alpha\in\mathcal{I}} b_\alpha\,K_\alpha(\omega) \\ &= \sum_{k\in\mathbb{N}} b_k(\omega)\,\eta_k, \end{aligned}$$

where $b_\alpha = \sum_{k\in\mathbb{N}} b_{\alpha,k}\,\eta_k \;\in H$, and $b_k(\omega) = \sum_{\alpha\in\mathcal{I}} b_{\alpha,k}\,K_\alpha(\omega) \in (Q)^P_{-\rho}$, with a unique $b_{\alpha,k}\in\mathbb{R}$ for $k\in\mathbb{N}$ and $\alpha\in\mathcal{I}$.

The action of F onto f is given by

$$\ll F, f \gg = \sum_{\alpha\in\mathcal{I}} \alpha!\,(b_\alpha, a_\alpha)_H.$$

When we choose one of the two special types of weights, either weights of the form $q_\alpha = (2\mathbb{N})^\alpha$ to obtain the H-valued Kondratiev type of P-stochastic test functions and distributions, denoted by $S(H)_\rho^P$ and $S(H)_{-\rho}^P$ respectively, or weights of the form $q_\alpha = e^{(2\mathbb{N})^\alpha}$ to obtain the H-valued exponential growth type stochastic test functions and distributions, denoted by $\exp S(H)_\rho^P$ and $\exp S(H)_{-\rho}^P$ respectively, the following important results are valid

$$S(H)_{-\rho} \cong (S)_{-\rho} \otimes H \quad \text{and} \quad \exp S(H)_{-\rho} \cong \exp(S)_{-\rho} \otimes H. \tag{2.43}$$

An isomorphism (2.43) with tensor product spaces, the property similar as for the Schwartz spaces in the deterministic case, is based on nuclear structure of q-weighted stochastic spaces $(S)_\rho^P$ and $\exp(S)_\rho^P$.

Both $S(H)_\rho$ and $\exp S(H)_\rho$ are countably Hilbert spaces and

$$\exp S(H)_\rho^P \subseteq S(H)_\rho^P \subseteq L^2(P,H) \subseteq S(H)_{-\rho}^P \subseteq \exp S(H)_{-\rho}^P.$$

An important example arises when the separable Hilbert space H is the space $L^2(\mathbb{R})$ with the Hermite functions orthonormal basis $\{\xi_i\}_{i\in\mathbb{N}}$.

2.7 Generalized Stochastic Processes

Generalized stochastic processes can be defined in several ways depending on whether the author regards them as a family of random variables or as a family of trajectories, but also depending on the type of continuity implied onto this family.

In Section 1.4.2 we pointed out that a classical stochastic process $X_t(\omega) = X(t,\omega)$, $t \in T \subseteq \mathbb{R}$, $\omega \in \Omega$ can be defined in three equivalent ways. It can be regarded either as a family of random variables $X_t(\cdot)$, $t \in T$, as a family of trajectories $X_.(\omega)$, $\omega \in \Omega$, or as a family of functions $X : T \times \Omega \to \mathbb{R}$ such that for each fixed $t \in T$, $X(t,\cdot)$ is an $\mathbb{R}$-valued random variable and for each fixed $\omega \in \Omega$, $X(\cdot,\omega)$ is an $\mathbb{R}$-valued deterministic function, called a trajectory.

By replacing the space of trajectories with some space of deterministic generalized functions, or by replacing the space of random variables with some space of generalized random variables, different types of generalized stochastic processes can be obtained. In this manner, we can obtain processes generalized with respect to the t argument, processes generalized with respect to the ω argument and also processes generalized with respect to both arguments, t and ω argument.

The classification of generalized stochastic processes by various conditions of continuity, their structural theorems and series expansions, is subject of various articles. Here we mention [19], [56], [70] with references therein. Detailed survey on generalization of classical stochastic processes is given in [56], where several classes of generalized stochastic processes were distinguish and represented in appropriate chaotic expansions. In this dissertation we follow the classification from [56] and focus only on two classes of such generalized processes. Definition and main properties of the first class, the class of *generalized stochastic processes of type (O)*, here named *generalized stochastic processes*, are subject of Section 2.7.1. Study of the second class, the class of *generalized stochastic processes of type (I)*, here called the *singular generalized stochastic processes* we will present in Section 4.1.

2.7.1 Generalized stochastic processes

A very general concept of generalized stochastic processes, based on chaos expansions was developed in [17], [19], [52], [56], etc.

In [19] generalized stochastic processes are defined as measurable mappings $T \to (S)_{-1}^{\mu}$, where $(S)_{-1}^{\mu}$ denotes the Kondratiev space for the Gaussian measure, but one can consider also other spaces of generalized random variables instead of it. Thus, they are pointwisely defined with respect to the parameter $t \in T$ and generalized with respect to $\omega \in \Omega$.

In this dissertation we will consider a class of generalized stochastic process wider than in [19]. We follow [29], [56], [65] and [66] to define such processes and give their chaos expansion representations in terms of orthogonal polynomial basis.

Let $\rho \in [0,1]$.

Definition 2.7.1 Generalized stochastic processes *are measurable mappings from $\mathbb{R}$ into some q-weighted space of generalized functions i.e. measurable mappings $\mathbb{R} \to (Q)_{-\rho}^{P}$.*

From definition it follows that for every fixed t we obtain generalized random variable $F_t(\cdot)$ from q-weighted space $(Q)_{-\rho}^{P}$.

2.7.2 Chaos expansion of generalized stochastic processes

We let $\rho \in [0,1]$. Since generalized stochastic processes with values in $(Q)_{-\rho}^{P}$ are defined pointwisely with respect to the parameter $t \in \mathbb{R}$, their chaos expansion representation follows directly from the Wiener-Itô chaos expansion theorem, Theorem 2.1.1.

Theorem 2.7.1 (Chaos expansion theorem for generalized stochastic process) *Let $F : \mathbb{R} \to (Q)^P_{-\rho}$ be a generalized stochastic process with respect to measure P. Then it is given by the formal expansion*

$$F_t(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha(t)\, K_\alpha(\omega), \quad t \in \mathbb{R} \tag{2.44}$$

where $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha \in \mathcal{I}$ are measurable functions and there exists $p \in \mathbb{N}_0$ such that for all $t \in \mathbb{R}$

$$\|F_t\|^2_{(Q)^P_{-\rho,-p}} = \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho}\, |f_\alpha(t)|^2\, q_\alpha^{-p} < \infty. \tag{2.45}$$

For different choices of measure P, generalized stochastic processes are expressed in terms of corresponding orthogonal basis $\{K_\alpha\}_{\alpha \in \mathcal{I}}$. In particular, for $P = \mu$ orthogonal basis is $K_\alpha = H_\alpha$ and in case $P = \nu$ is $K_\alpha = C_\alpha$, $\alpha \in \mathcal{I}$. On the other hand, if generalized stochastic processes have values in a certain type of q-weighed space, the convergence condition (2.45) modifies.

2.7.3 Pettis intregral of generalized stochastic processes

Now let the q-weighted stochastic spaces be either the Kondratiev spaces or the stochastic spaces of exponential growth. We extend the definition of the Pettis integral with values in the Hida space of stochastic distributions, given for the Gaussian case in [17] and [19]. Consider now a special case, the space $(Q)^P_{-1}$ for $\rho = 1$.

Suppose $Y : \mathbb{R} \to (Q)^P_{-1}$ is a given q-weighted stochastic function, i.e. generalized stochastic process, such that $\ll Y_t, F \gg \in L^1(\mathbb{R})$ for all functions $F \in (Q)^P_1$ from the corresponding q-weighted test space. Then q-weighted integral of Y_t, denoted by $\int_{\mathbb{R}} Y_t\, dt$, is defined to be the unique element of $(Q)^P_{-1}$ such that

$$\ll \int_{\mathbb{R}} Y_t dt, F \gg = \int_{\mathbb{R}} \ll Y_t, F \gg\, dt, \quad F \in (Q)^P_1. \tag{2.46}$$

We also say that generalized stochastic process Y_t is *Pettis integrable* in $(Q)^P_{-1}$ or *q-Pettis integrable.*

Theorem 2.7.2 *Assume that a generalized stochastic process $Y_t \in (Q)^P_{-1}$, for $t \in \mathbb{R}$ has a chaos expansion of the form $Y_t(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha(t)\, K_\alpha(\omega)$,*

$\omega \in S'(\mathbb{R})$, where coefficients $f_\alpha : \mathbb{R} \to \mathbb{R}$ satisfy the following convergence condition

$$\sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_{L^1(\mathbb{R})} q^{-p\alpha} \|K_\alpha\|^2_{L^2(P)} < \infty \quad \textit{for some } p \geq 0. \tag{2.47}$$

Then generalized stochastic process Y_t is said to be q-Pettis integrable *and*

$$\int_{\mathbb{R}} Y_t(\omega)\, dt = \sum_{\alpha \in \mathcal{I}} \left(\int_{\mathbb{R}} f_\alpha(t) dt \right) K_\alpha(\omega). \tag{2.48}$$

Proof. Assume that $F = \sum_{\alpha \in \mathcal{I}} a_\alpha K_\alpha \in (Q)_1^P$ for $a_\alpha \in \mathbb{R}$. Recall the fact $\|K_\alpha\|_{L^2(P)} = \sqrt{\alpha!}$, for all $\alpha \in \mathcal{I}$ in both cases, when $P = \mu$ and $P = \nu$. Thus, the sum

$$\|F\|^2_{(Q)_1^P} = \sum_{\alpha \in \mathcal{I}} \|a_\alpha\|^2 \alpha!^2\, q^{p\alpha}$$

is finite for all $p \in \mathbb{N}_0$. The q-Pettis integrability of a generalized stochastic process Y_t follows from (2.47). Clearly,

$$\begin{aligned}
\int_{\mathbb{R}} | \ll Y_t, F \gg |\, dt &= \int_{\mathbb{R}} |\sum_{\alpha \in \mathcal{I}} f_\alpha(t) \alpha!\, a_\alpha| dt \\
&\leq \sum_{\alpha \in \mathcal{I}} \|f_\alpha(t)\|_{L^1(\mathbb{R})} \alpha!\, |a_\alpha| dt \leq \\
&\leq \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_{L^1(\mathbb{R})}\, q^{-\frac{p\alpha}{2}}\, |a_\alpha| \alpha!\, q^{\frac{p\alpha}{2}} \\
&\leq \left(\sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_{L^1(\mathbb{R})}\, q^{-p\alpha} \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \mathcal{I}} \|a_\alpha\|^2 \alpha!^2\, q^{p\alpha} \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Then, the statement of the theorem is completed with

$$\begin{aligned}
\int_{\mathbb{R}} \ll Y_t, F \gg\, dt &= \int_{\mathbb{R}} \ll \sum_{\alpha \in \mathcal{I}} f_\alpha(t) K_\alpha, \sum_{\beta \in \mathcal{I}} a_\beta K_\beta \gg dt = \\
&= \int_{\mathbb{R}} \sum_{\alpha \in \mathcal{I}} f_\alpha(t) a_\alpha \|K_\alpha\|^2_{L^2(P)} dt \\
&= \sum_{\alpha \in \mathcal{I}} \left(\int_{\mathbb{R}} f_\alpha(t) dt \right) a_\alpha \|K_\alpha\|^2_{L^2(P)} \\
&= \ll \sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}} f_\alpha(t) dt\, K_\alpha, \sum_{\beta \in \mathcal{I}} a_\beta K_\beta \gg \\
&= \ll \int_{\mathbb{R}} Y_t\, dt, F \gg .
\end{aligned}$$

□

The proof for a Gaussian case in the Hida space of stochastic distributions $(S)^*$ can be found in [19].

We will return to this topic again when proving the chaos expansion formula for the Skorokhod integral of singular generalized stochastic processes in Chapter 4.

The expansion theorems for singular generalized stochastic processes, in [56] also called the generalized stochastic processes of type (I), defined in Section 4.1.1 as linear and continuous mappings from a certain space of deterministic functions $\mathcal{T}$ into the space of q-weighted generalized functions $(Q)_{-1}^{P}$ i.e. elements of $\mathcal{L}(\mathcal{T},(Q)_{-1}^{P})$, will give an extension of the expansion of the q-Pettis integrable generalized stochastic processes in the sense that the coefficients $f_\alpha(t), \alpha \in \mathcal{J}$ in (2.44) will be generalized functions, for example from the Schwartz space of tempered distributions $S'(\mathbb{R})$.

2.7.4 Unitary mapping $\mathcal{U}$ of generalized stochastic processes

We extend the unitary mapping $\mathcal{U}$, defined by (2.29) in the Theorem 2.4.1, to the class of generalized stochastic processes in a similar way as we did in (2.31). Consider $\mathcal{U}$ to be a linear and isometric mapping on the space of generalized stochastic processes with values in q-weighted μ-measured space such that for all $t \in \mathbb{R}$

$$\mathcal{U}\left[\sum_{\alpha\in\mathcal{J}} f_\alpha(t)H_\alpha(\omega)\right] = \sum_{\alpha\in\mathcal{J}} f_\alpha(t)C_\alpha(\omega), \quad f_\alpha \in \mathbb{R}, \tag{2.49}$$

for generalized stochastic process $F:\mathbb{R}\to(Q)_{-\rho}^{\mu}$, given in the form $F_t = \sum_{\alpha\in\mathcal{J}} f_\alpha(t)H_\alpha(\omega)$. Furthermore, for every $t\in\mathbb{R}$ the isometry

$$\|\mathcal{U}(F_t)\|_{(Q)_{-\rho,-p}^{\nu}} = \|F_t\|_{(Q)_{-\rho,-p}^{\mu}}$$

holds for all $p \geq p_0$.

Examples of generalized stochastic processes are Brownian motion and singular white noise, given by their chaos expansions in Example 2.7.1, and compensated Poisson process and Poissonian compensated white noise, described in Example 2.7.2.

Example 2.7.1 *Brownian motion is given by the chaos expansion*

$$B_t(\omega) = \sum_{k=1}^{\infty}\left(\int_0^t \xi_k(s)ds\right) H_{\varepsilon^{(k)}}(\omega) \tag{2.50}$$

and it is an element of $L^2(\mu)$.

Singular white noise $W_t(\cdot)$ *is defined by the chaos expansion*

$$W_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t) H_{\varepsilon^{(k)}}(\omega), \tag{2.51}$$

and it is an element of the space $(S)^{\mu}_{-1}$, *for all* $t \in \mathbb{R}$. *It is integrable and the relation*

$$\frac{d}{dt} B_t = W_t$$

holds in the $(S)^{\mu}_{-1}$ *sense, see* (2.46).

Example 2.7.2 *The chaos expansion of compensated Poisson process* $P_t(\omega) \in L^2(\nu)$ *is given by*

$$P_t(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s) ds \right) C_{\varepsilon^{(k)}}(\omega). \tag{2.52}$$

The Poissonian compensated white noise $V_t(\cdot)$ *is defined by the chaos expansion*

$$V_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t)\, C_{\varepsilon^{(k)}}(\omega), \tag{2.53}$$

and it is an element of the space $(S)^{\nu}_{-1}$ *for all* $t \in \mathbb{R}$. *It is integrable and the relation*

$$\frac{dP_t}{dt} = V_t$$

holds in $(S)^{\nu}_{-1}$ *sense. Note that*

$$P_t(\omega) = \mathcal{U}(B_t(\omega)) \quad \text{and} \quad V_t(\omega) = \mathcal{U}(W_t(\omega)),$$

which is consistent with (2.31).

In Chapter 4 we will study another class of generalized stochastic processes, the class of singular generalized stochastic processes. We regard them as elements of tensor product of a certain topological space X onto $(Q)^{P}_{-\rho}$. Thus, Brownian motion and singular white noise can be considered as elements of spaces $C^{\infty}(\mathbb{R}) \otimes L^2(\mu)$ and $C^{\infty}(\mathbb{R}) \otimes (S)^{\mu}_{-1,-p}$, for $p > \frac{5}{12}$, respectively. Same stays for the corresponding analogs in the Poisson case, i.e. a compensated Poisson process is considered to be an element of $C^{\infty}(\mathbb{R}) \otimes L^2(\nu)$ and Poissonian compensated white noise the process belonging to $C^{\infty}(\mathbb{R}) \otimes (S)^{\nu}_{-1}$ (see Example 4.1.1).

2.7.5 Itô-Skorokhod integral of generalized stochastic processes

We assume now that the measure P is either the Gaussian measure μ or the Poissonian measure ν. The Kondratiev space $(S)_{-1}^{P}$ is denoted now by $(S)_{-1}$ and the Wick product $\Diamond^P$ is denoted by $\Diamond$. We state a fundamental property of the P-Wick product, which is the following relation to Itô-Skorokhod integration. The proof of the theorem is based on the chaos expansion form of a generalized stochastic process and will be presented here in an analogous form to one given in [19]. Denote by

$$\delta(Y_t) = \int_{\mathbb{R}} Y_t \; \delta \mathcal{Q}_t(\omega)$$

the P-Itô-Skorokhod stochastic integral of a process Y_t, and $\mathcal{Z}_t$ a generalized stochastic process satisfying $\frac{d}{dt}\mathcal{Z}_t = \mathcal{Q}_t$ for a.a. t, in sense of the relation (2.46). For the Gaussian measure, $\mathcal{Q}_t = B_t$ is a Brownian motion and $\mathcal{Z}_t = W_t$ is a singular white noise. In Poissonian case, $\mathcal{Q}_t = P_t$ denotes a compensated Poisson process and $\mathcal{Z}_t = V_t$ a Poissonian compensated white noise.

Theorem 2.7.3 *Let $Y_t(\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ be a Skorohod-integrable stochastic process. Then $Y_t(\omega) \Diamond \mathcal{Z}_t$ is dt-integrable in $(S)_{-1}$ and*

$$\int_{\mathbb{R}} Y_t(\omega)\, \delta \mathcal{Q}_t(\omega) = \int_{\mathbb{R}} Y_t \Diamond \mathcal{Z}_t dt. \tag{2.54}$$

The left-hand side of (2.54) *denotes the Skorohod integral of the stochastic process $Y = Y_t(\omega)$ which coincides with the Itô integral if Y is adapted. Integral on the right-hand side of* (2.54) *is interpreted as $(S)_{-1}$-valued Pettis integral. This generalization we will call the Itô-Skorokhod integral.*

Proof. First we compute the left-hand side of (2.54) explicitly. Assume that a Skorokhod integrable process Y_t is given by the chaos expansion forms $Y_t = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(u_1, ..., u_n, t) d\mathcal{Q}^{\otimes n}(u_1, ..., u_n)$, for symmetric functions f_n and equivalently $Y_t = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) K_\alpha$, for measurable c_α. Then by (2.15) and (2.28) we have $f_n = \sum\limits_{|\alpha|=n} c_\alpha\, \xi_n^{\hat{\otimes}\alpha}$ the symmetric functions in $\widehat{L}^2(\mathbb{R}^n)$ and

$$\begin{aligned} Y_t &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \sum_{|\alpha|=n} c_\alpha\, \xi_n^{\hat{\otimes}\alpha}(u_1, ..., u_n)\, d\mathcal{Q}^{\otimes n}(u_1, ..., u_n) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \sum_{|\alpha|=n} \sum_{k \in \mathbb{N}} (c_\alpha, \xi_k) \xi_k\, \xi_n^{\hat{\otimes}\alpha}(u_1, ..., u_n)\, d\mathcal{Q}^{\otimes n}(u_1, ..., u_n) \end{aligned} \tag{2.55}$$

Due to the symmetrization $\xi^{\widehat{\otimes}(\alpha+\varepsilon^{(k)})}$ and (2.14) and (2.2.6), the Skorokhod integral of Y_t becomes

$$\begin{aligned}\int_{\mathbb{R}} Y_t\,\delta \mathcal{Q}_t &= \sum_{n=0}^{\infty}\int_{\mathbb{R}^{n+1}}\sum_{|\alpha|=n}\sum_{k\in\mathbb{N}}(c_\alpha,\xi_k)\xi^{\widehat{\otimes}(\alpha+\varepsilon^{(k)})}\,\mathcal{Q}^{\otimes(n+1)}\\ &= \sum_{n=0}^{\infty}\sum_{|\alpha|=n}\sum_{k\in\mathbb{N}}(c_\alpha,\xi_k)K_{\alpha+\varepsilon^{(k)}}\\ &= \sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}}(c_\alpha,\xi_k)K_{\alpha+\varepsilon^{(k)}}. \end{aligned} \tag{2.56}$$

Providing direct computation of the right-hand side of (2.54) we obtain

$$\begin{aligned}\int_{\mathbb{R}} Y_t \lozenge \mathcal{Z}_t\,dt &= \int_{\mathbb{R}}\left(\sum_{\alpha\in\mathcal{J}} c_\alpha(t)K_\alpha\right)\lozenge\left(\sum_{k\in\mathbb{N}}\xi_k(t)\,K_{\varepsilon^{(k)}}\right)dt\\ &= \int_{\mathbb{R}}\left(\sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} c_\alpha(t)\,\xi_k(t)K_{\alpha+\varepsilon^{(k)}}\right)dt\\ &= \sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}}\left(\int_{\mathbb{R}} c_\alpha(t)\,\xi_k(t)dt\right)K_{\alpha+\varepsilon^{(k)}}\\ &= \sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}}(c_\alpha,\xi_k)K_{\alpha+\varepsilon^{(k)}}, \end{aligned} \tag{2.57}$$

because

$$\begin{aligned}\|\sum_{\substack{\alpha,k\\ \alpha+\varepsilon^{(k)}=\beta}} c_\alpha(t)\,\xi_k(t)\|^2_{L^1(\mathbb{R})} &\le [\sum_{\substack{\alpha,k\\ \alpha+\varepsilon^{(k)}=\beta}}\|c_\alpha(t)\|_{L^1(\mathbb{R})}\,\xi_k(t)]^2\\ &\le C^2|\beta|^2\sum_{\substack{\alpha,k\\ \alpha+\varepsilon^{(k)}=\beta}}\|c_\alpha\|^2_{L^1(\mathbb{R})}\end{aligned}$$

and

$$\sum_{\beta\in\mathcal{J}}\beta!\|\sum_{\substack{\alpha,k\\ \alpha+\varepsilon^{(k)}=\beta}} c_\alpha(t)\,\xi_k(t)\|^2_{L^1(\mathbb{R})}(2\mathbb{N})^{-p\beta}<\infty \tag{2.58}$$

for some $p\in\mathbb{N}_0$. With this statement we complete the proof. □

The relation (2.54) is an important and a very useful property in applications, when solving stochastic differential equations. That means the Wick calculus with ordinary differential calculus rules is equivalent to the Itô calculus governed by the Itô formula and ordinary multiplication. For more information we refer to [19], [49].

Moreover the previous theorem can be extended to the case of q-weighted generalized stochastic processes.

Now we state and prove the main theorem of this overview, which we will use as the starting point when defining the Itô-Skorokhod integral of singular generalized stochastic processes. A similar proof of the following theorem, for $P = \mu$ is given in [19].

Theorem 2.7.4 *Let $Y_t : \mathbb{R} \to (S)_{-1}$ be a Skorokhod integrable generalized stochastic process given by the chaos expansion*

$$Y_t(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha(t)\, K_\alpha(\omega),\ t \in \mathbb{R},$$

with the coefficients $c_\alpha \in L^2(\mathbb{R})$ for all $\alpha \in \mathcal{I}$. Then the chaos expansion of its Skorokhod integral is given in the form

$$\int_{\mathbb{R}} Y_t(\omega)\, \delta \mathcal{Q}_t(\omega) \;=\; \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} c_{\alpha,k}\, K_{\alpha + \varepsilon^{(k)}}(\omega), \tag{2.59}$$

where the real numbers

$$c_{\alpha,k} = (c_\alpha, \xi_k)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} c_\alpha(t)\, \xi_k(t)\, dt, \quad k \in \mathbb{N}$$

represent the Fourier coefficients of c_α, $\alpha \in \mathcal{I}$, provided that the right-hand of equality (2.59) *converges in $(S)_{-1}$.*

If we assume, in addition, that $\int_{\mathbb{R}} Y_t(\omega)\, \delta \mathcal{Q}_t(\omega) \in L^2(P)$, then

$$E_P \left[\int_{\mathbb{R}} Y_t(\omega)\, \delta \mathcal{Q}_t(\omega) \right] = 0. \tag{2.60}$$

Proof. The proof of this theorem directly follows from (2.56), (2.57) and (2.58). Recall that expectation of a $L^2(P)$ element $\int_{\mathbb{R}} Y_t(\omega)\, \delta \mathcal{Q}_t(\omega)$ is equal to the zero coefficient in its chaos expansion. Thus, from (2.59) we obtain and $c_{(0,0,0,\dots)} = 0$ the assertion (2.60) is verified. □

Note that the expansion (2.59) is not necessarily orthogonal, since it may happen that $\alpha + \varepsilon^{(k)} = \beta + \varepsilon^{(j)}$ for some $\alpha, \beta \in \mathcal{I}$, $\alpha \neq \beta$, $k, j \in \mathbb{N}$.

2.8 Fractional White Noise Spaces

Recall, in Section 1.4.3 e) we introduced a fractional Brownian motion as a one-parameter extension of a standard Brownian motion and presented the main properties of such a Gaussian process with respect to values of the Hurst parameter $H \in (0,1)$.

Fractional Brownian motion, as a process with independent increments which have a long-range dependence and self-similarity properties found many applications when modeling a wide range of problems in hydrology, telecommunications, queueing theory and mathematical finance.

This section is devoted to a specific construction of a stochastic integral with respect to a fractional Brownian motion defined for all possible values $H \in (0,1)$, introduced by Elliot and van der Hoek in [13]. Several different definitions of stochastic integration for fractional Brownian motion appear in literature. We refer reader to [8], [13], [52], [44], [48] for illustration.

We focus here on defining the fractional white noise spaces by use of the fractional transform mapping for all values of $H \in (0,1)$. We extend the action of the fractional transform operator to a class of generalized stochastic processes. The main properties of the fractional transform operator and the connection of a fractional Brownian motion with a classical Brownian motion on the classical white noise space will be stated. Moreover, we will define the fractional Poissonian process in a framework that will make it easy to link it to its regular version.

In [19] it was proved that there exists a unitary mapping between the Gaussian and the Poissonian white noise space, by mapping the Hermite polynomial basis into the Charlier polynomial basis. In [13] and [29] a unitary mapping was introduced between the Gaussian and the fractional Gaussian white noise space. We extend these ideas to define the fractional Poissonian white noise space itself and to connect it to the classical Poissonian white noise space. As a result we obtain four types of white noise spaces: Gaussian, Poissonian, fractional Gaussian and fractional Poissonian, where any two of them can be identified through a unitary mapping. The construction of fractional Poissonian space and the structural properties of the aforementioned four types of white noise spaces and operators defined on them are published in [29] and [30] and represent an original part of this thesis.

2.8.1 Fractional transform operator $M^{(H)}$

Further on we follow the ideas represented by Elliot and van der Hoek in [13], where fractional white noise theory for Hurst parameter $H \in (0,1)$ was developed. In [13] the fractional transform operator $M = M^{(H)}$ was introduced, which connects fractional Brownian motion $B_t^{(H)}$ and classical Brownian motion B_t on the white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$.

Definition 2.8.1 ([13]) *Let* $H \in (0,1)$. *The* fractional transform operator $M = M^{(H)} : S(\mathbb{R}) \to L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ *is defined by*

$$\widehat{Mf}(y) = |y|^{\frac{1}{2}-H}\, \widehat{f}(y), \quad y \in \mathbb{R},\ f \in S(\mathbb{R}), \tag{2.61}$$

where $\widehat{f}(y) := \int_{\mathbb{R}} e^{-ixy} f(x)dx$ *is the Fourier transform of* f.

An equivalent definition of the operator $M = M^{(H)}$ is given by

$$Mf(x) = -\frac{d}{dx}\frac{C_H}{H-\frac{1}{2}} \int_{\mathbb{R}} (t-x)\, \frac{f(t)}{|t-x|^{H-\frac{3}{2}}}\, dt, \quad f \in S(\mathbb{R}), \tag{2.62}$$

where $C_H = [2\Gamma(H-\frac{1}{2}) \cos(\frac{\pi}{2}(H-\frac{1}{2}))]^{-1}[\Gamma(2H+1)\sin(\pi H)]^{\frac{1}{2}}$ is a normalizing constant and Γ is the Gamma function. This definition can be restated as follows

$$M^{(H)}f(x) = \begin{cases} C_H \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}}\, dt, & H \in (0, \frac{1}{2}) \\ f(x), & H = \frac{1}{2} \\ C_H \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}}\, dt, & H \in (\frac{1}{2}, 1) \end{cases}.$$

Note that the operator $M = M^{(H)}$ has the structure of a convolution operator (we recall (1.12) from the Section 1.3.3).

The form of the *inverse operator* $M^{-1} = M^{(1-H)}$ follows from (2.61), i.e. for all $H \in (0,1)$

$$M^{(H)} \circ M^{(1-H)}(f) = f, \quad f \in S(\mathbb{R}). \tag{2.63}$$

Definition 2.8.2 *The* inverse fractional transform operator M^{-1} *is defined by*

$$\widehat{M^{-1}f}(y) = |y|^{H-\frac{1}{2}}\, \widehat{f}(y), \quad y \in \mathbb{R},\ f \in S(\mathbb{R}). \tag{2.64}$$

Following the work of [61], the fractional transform operator $M = M^{(H)}$, for $H \in (\frac{1}{2}, 1)$ can be interpreted as the Riesz potential

$$I^\alpha \varphi \triangleq \frac{1}{2\Gamma(\alpha)\cos(\frac{\alpha\pi}{2})} \int_{\mathbb{R}} \frac{\varphi(t)}{|t-x|^{1-\alpha}}\, dt$$

for $Re\{\alpha\} > 0$, $\alpha \neq 1, 3, 5, ...$ and $\varphi \in L^p(\mathbb{R})$, $1 \leq p < \frac{1}{Re\alpha}$, if we chose $\alpha = H - \frac{1}{2}$. The corresponding inverse operator of the operator I^α is

$$(I^\alpha)^{-1} f(x) = \frac{\cos(\frac{\alpha\pi}{2})}{2\Gamma(-\alpha)} \int_{\mathbb{R}} \frac{f(t-x) - f(x)}{|t|^{1+\alpha}}\, dt,$$

for $f \in I^\alpha(L^p(\mathbb{R}))$. We conclude that for $H \in (\frac{1}{2}, 1)$ the fractional operator $M = M^{(H)}$ corresponds to the Riesz potential $M^{(H)}\varphi = I^{H-\frac{1}{2}}\varphi$ and for $H \in (0, \frac{1}{2})$ to the inverse of Riesz potential $(I^{H-\frac{1}{2}})^{-1} f = M^{(1-H)} f$.

From (2.62) follows that the operator M can be interpreted as the αth Riemann-Liouville fractional integral of f, where $\alpha = \frac{1}{2} - H$. Recall, basic definitions and properties of the theory of deterministic fractional derivatives and integrals are given in Section 1.3. For more details we refer to [61].

Let

$$L^2_H(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R};\ M^{(H)} f(x) \in L^2(\mathbb{R})\}.$$

The space $L^2_H(\mathbb{R})$ is the closure of $S(\mathbb{R})$ with respect to the norm $\|f\|_{L^2_H(\mathbb{R})} = \|Mf\|_{L^2(\mathbb{R})}$, $f \in S(\mathbb{R})$ induced by the inner product

$$(f, g)_{L^2_H(\mathbb{R})} = (Mf, Mg)_{L^2(\mathbb{R})}.$$

The operator $M = M^{(H)}$ is self-adjoint and for $f, g \in L^2(\mathbb{R}) \cap L^2_H(\mathbb{R})$ we have

$$(f, Mg)_{L^2_H(\mathbb{R})} = (Mf, g)_{L^2_H(\mathbb{R})}.$$

Let $H \in (\frac{1}{2}, 1)$ be fixed. Define $\phi(s,t) = H(2H-1)|s-t|^{2H-2}$, $s, t \in \mathbb{R}$. Then,

$$\int_{\mathbb{R}} (M^{(H)} f(x))^2 dx = c_H \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s,t) ds dt, \tag{2.65}$$

where c_H is constant.

In [13], [20], [26] and [41], the classical white noise calculus was adapted to the fractional white noise by the use of property (2.65). For that purpose the fractional white noise and stochastic integral as an element of the fractional stochastic distributions spaces were defined. We proved in [26] that generalized stochastic processes with values in these spaces have a series expansion, and different Wick products were discussed. Analogous theorems of the fractional Itô-Skorokhod calculus to Theorem 2.7.1, Theorem 2.7.2 and Theorem 2.7.4 were obtained. Here we just mention these results without further detailed presentation.

Example 2.8.1 *Let $H \in (0,1)$. The characteristic function $\chi[0,t](\cdot)$, for fixed $t \in \mathbb{R}$ belongs to the Hilbert space $L^2_H(\mathbb{R})$. Moreover,*

$$M\chi[0,t](x) = \frac{1}{2\Gamma(H+\frac{1}{2})\cos(\frac{\pi}{2}(H+\frac{1}{2}))}\left(\frac{t-x}{|t-x|^{\frac{3}{2}-H}} + \frac{1}{|x|^{\frac{1}{2}-H}}\right).$$

From the Parseval theorem and $\widehat{\chi[0,t]}(y) = -\frac{1}{iy}(e^{-ity}-1)$ it follows that

$$\int_{\mathbb{R}} (M\chi[0,t](x))^2\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} |y|^{1-2H}\frac{|e^{-ity}-1|^2}{|y|^2}dy = \frac{1}{\sin(\pi H)(2H+1)}t^{2H}.$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}} M\chi[0,t](x)\,M\chi[0,s](x)\,dx &= \frac{1}{2}\left(|t^{2H}| + |s|^{2H} - |t-s|^{2H}\right)\\ &= (\chi[0,t](x), \chi[0,s](x))_{L^2_H(\mathbb{R})}\end{aligned}$$

holds for arbitrary $t, s \in \mathbb{R}$.

The following important theorem gives the orthonormal basis for the fractional version of $L^2(\mathbb{R})$. The proof can be found in [7] and [13].

Theorem 2.8.1 ([13]) *Let $M : L^2_H(\mathbb{R}) \to L^2(\mathbb{R})$ defined by (2.61) be the extension of the operator M from Definition 2.8.1. Then, M is an* isometry *between the two Hilbert spaces $L^2(\mathbb{R})$ and $L^2_H(\mathbb{R})$. The functions*

$$e_n(x) = M^{-1}\xi_n(x), \quad n \in \mathbb{N}, \tag{2.66}$$

belong to $S(\mathbb{R})$ and form an orthonormal basis in $L^2_H(\mathbb{R})$.

Following [7] and [13] we extend M onto $S'(\mathbb{R})$ and define the fractional operator $M : S'(\mathbb{R}) \to S'(\mathbb{R})$ by

$$\langle M\omega, f\rangle = \langle \omega, Mf\rangle, \quad f \in S(\mathbb{R}), \omega \in S'(\mathbb{R}). \tag{2.67}$$

Example 2.8.2 *For all $t \in \mathbb{R}$ define the process*

$$B^{(H)}_t(\omega) := \langle \omega, M\chi[0,t](\cdot)\rangle, \quad \omega \in S'(\mathbb{R}).$$

It is a Gaussian process with zero expectation and the covariance function

$$\begin{aligned} E[B_s^{(H)} B_t^{(H)}] &= (M\chi[0,s], M\chi[0,t])_{L^2(\mathbb{R})} = (\chi[0,s], \chi[0,t])_{L_H^2(\mathbb{R})} \\ &= \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}, \qquad s,t \in \mathbb{R}. \end{aligned}$$

The t-continuous version of the process $B_t^{(H)}$ is an element of $L^2(\mu)$ and is called fractional Brownian motion.

The relationships between classical Brownian motion B_t and fractional Brownian motion $B_t^{(H)}$, as elements of $L^2(\mu)$ are given by

$$B_t^{(H)}(\omega) = \int_{\mathbb{R}} M^{(H)} \chi[0,t]\, dB_t(\omega) \qquad \text{and}$$

$$B_t(\omega) = \int_{\mathbb{R}} M^{(1-H)} \chi[0,t]\, dB_t^{(H)}(\omega), \quad \omega \in S'(\mathbb{R}).$$

Fractional Itô integral

The *fractional Itô integral* of a deterministic function $f \in L_H^2(\mathbb{R})$ is defined by

$$\begin{aligned} I^{(H)}(f) &= \int_{\mathbb{R}} f(t)\, dB_t^{(H)}(\omega) \\ &= \int_{\mathbb{R}} Mf(t)\, dB_t(\omega) = I(Mf), \end{aligned} \tag{2.68}$$

which implies isometries

$$\|I(Mf)\|_{L^2(\mu)} = \|Mf\|_{L^2(\mathbb{R})} = \|f\|_{L_H^2(\mathbb{R})}.$$

Furthermore, the Itô integral of a deterministic function $f \in L_{1-H}^2(\mathbb{R})$ can be expressed in terms of the fractional Itô integral and the inverse $M^{-1} = M^{(1-H)}$ of the fractional transform operator $M^{(H)}$ by

$$\begin{aligned} I^{(H)}(M^{(1-H)} f) &= \int_{\mathbb{R}} M^{(1-H)} f(t) dB_t^{(H)}(\omega) \\ &= \int_{\mathbb{R}} f(t) dB_t(\omega) = I(f). \end{aligned}$$

For more details on this subject we refer to [7], [8], [13] and [19].

2.8.2 Fractional Gaussian white noise space

Let $H \in (0,1)$. Now we extend the action of the operator M from $S'(\mathbb{R})$ onto $L^2(\mu)$ and define the stochastic analogue of $L^2_H(\mathbb{R})$. Denote by

$$\begin{aligned} L^2(\mu_H) &= L^2(\mu \circ M^{-1}) = L^2(\mu \circ M^{(1-H)}) \\ &= \{G : \Omega \to \mathbb{R}\,;\, G \circ M \in L^2(\mu)\}. \end{aligned} \tag{2.69}$$

It is the space of square integrable functions on $S'(\mathbb{R})$ with respect to *fractional Gaussian white noise measure* μ_H. Since $G \in L^2(\mu_H)$ if and only if $G \circ M \in L^2(\mu)$, it follows that G has an expansion of the form

$$\begin{aligned} G(M\omega) &= \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega) \\ &= \sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{i=1}^{\infty} h_{\alpha_i}(\langle\omega,\xi_i\rangle) \\ &= \sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{i=1}^{\infty} h_{\alpha_i}(\langle\omega,Me_i\rangle) \\ &= \sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{i=1}^{\infty} h_{\alpha_i}(\langle M\omega,e_i\rangle). \end{aligned}$$

Definition 2.8.3 *Define the family of Fourier-Hermite polynomials by*

$$\widetilde{\mathcal{H}}_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle\omega,e_k\rangle), \quad \alpha\in\mathcal{I}. \tag{2.70}$$

Now, it follows that the family $\{\widetilde{\mathcal{H}}_\alpha;\ \alpha\in\mathcal{I}\}$ forms an orthogonal basis of $L^2(\mu_H)$, with $\|\widetilde{\mathcal{H}}_\alpha\|^2_{L^2(\mu_H)} = \alpha!$, $\alpha\in\mathcal{I}$. Thus $G \in L^2(\mu_H)$ has a chaos expansion representation of the form

$$G(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha \widetilde{\mathcal{H}}_\alpha(\omega), \quad c_\alpha \in \mathbb{R}$$

such that $\|G\|^2_{L^2(\mu_H)} = \sum\limits_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha!$. Moreover, $c_\alpha = \frac{1}{\alpha!} E_{\mu_H}(G\widetilde{\mathcal{H}}_\alpha(\omega))$ and $\|G\|_{L^2(\mu_H)} = \|G\circ M\|_{L^2(\mu)}$.

Definition 2.8.4 ([29]) *Let* $\mathcal{M} : L^2(\mu_H) \to L^2(\mu)$ *be defined by* $\mathcal{M}(\widetilde{\mathcal{H}}_\alpha) = H_\alpha$ *and extend it by linearity and continuity to*

$$\mathcal{M}\left(\sum_{\alpha\in\mathcal{I}} c_\alpha \widetilde{\mathcal{H}}_\alpha\right) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha \tag{2.71}$$

for $G = \sum_{\alpha\in\mathcal{I}} c_\alpha \widetilde{\mathcal{H}}_\alpha \in L^2(\mu_H)$.

Note that from (2.6), (2.70) and (2.71) it follows that

$$\mathcal{M}(\widetilde{\mathcal{H}}_\alpha(\omega)) = \widetilde{\mathcal{H}}_\alpha(M\omega) = H_\alpha(\omega), \quad \omega \in S'(\mathbb{R}),\, \alpha \in \mathcal{I}.$$

It holds that

$$\|\mathcal{M}(\widetilde{\mathcal{H}}_\alpha)\|_{L^2(\mu)} = \|H_\alpha\|_{L^2(\mu)} = \alpha! = \|\widetilde{\mathcal{H}}_\alpha\|_{L^2(\mu_H)},$$

thus the operator $\mathcal{M}$ is an *isometry* between spaces of classical Gaussian and fractional Gaussian random variables and its action can be seen as a transformation of the corresponding elements of the orthogonal basis $\{\widetilde{\mathcal{H}}_\alpha\}_{\alpha\in\mathcal{I}}$ into $\{H_\alpha\}_{\alpha\in\mathcal{I}}$. The connection between the two bases is given by $H_\alpha(\omega) = \mathcal{M}\widetilde{\mathcal{H}}_\alpha(\omega)$ and $\widetilde{\mathcal{H}}_\alpha(\omega) = \mathcal{M}^{-1}H_\alpha(\omega)$, $\omega \in S'(\mathbb{R}), \alpha \in \mathcal{I}$. Thus, every element $F \in L^2(\mu_H)$ can be represented as the image of a unique $f(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega) \in L^2(\mu)$ such that $F = \mathcal{M}^{-1}f$. Then, F is of the form $F(\omega) = \sum\limits_{\alpha\in\mathcal{I}} c_\alpha\, \widetilde{\mathcal{H}}(\omega)$.

For $P = \mu_H$ the spaces in (2.30) reduce to fractional q-weighted spaces of stochastic test functions and stochastic generalized functions. In [26] we considered the following inclusions

$$\exp(S)_1^{\mu_H} \subseteq (S)_1^{\mu_H} \subseteq L^2(\mu_H) \subseteq (S)_{-1}^{\mu_H} \subseteq \exp(S)_{-1}^{\mu_H}.$$

The action of the operator $\mathcal{M}$ can be extended to q-weighted spaces by defining $\mathcal{M} : (Q)_{-1}^{\mu_H} \to (Q)_{-1}^{\mu}$ given by

$$\mathcal{M}\left[\sum_{\alpha\in\mathcal{I}} a_\alpha \widetilde{\mathcal{H}}_\alpha(\omega)\right] = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega), \quad a_\alpha \in \mathbb{R}. \tag{2.72}$$

This extension is well defined since there exists $p \in \mathbb{N}$ such that

$$\sum_{\alpha\in\mathcal{I}} a_\alpha^2 q_\alpha^{-p} < \infty.$$

In an analogous way the action of the operator $\mathcal{M}$ can be extended to generalized stochastic processes.

Example 2.8.3 *Fractional Brownian motion $B_t^{(H)}(\omega)$ as an element of $L^2(\mu)$ is defined by the chaos expansion*

$$\begin{aligned} B_t^{(H)}(\omega) &= \langle \omega, M\chi[0,t]\rangle = \langle M\omega, \chi[0,t]\rangle \\ &= \sum_{k=1}^{\infty} (\chi[0,t], e_k)_{L_H^2(\mathbb{R})} \langle M\omega, e_k\rangle = \sum_{k=1}^{\infty} (M\chi[0,t], Me_k)_{L^2(\mathbb{R})} \langle \omega, Me_k\rangle \\ &= \sum_{k=1}^{\infty} (\chi[0,t], M\xi_k)_{L^2(\mathbb{R})} \langle \omega, \xi_k\rangle = \sum_{k=1}^{\infty} \left(\int_0^t M\xi_k(s)ds\right) H_{\varepsilon^{(k)}}(\omega). \end{aligned} \tag{2.73}$$

Applying the map $\mathcal{M}^{-1} = \mathcal{M}^{(1-H)}$ *we obtain the chaos decomposition form of fractional Brownian motion in* $L^2(\mu_H)$*:*

$$B_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t M\xi_k(s)ds \right) \widetilde{\mathcal{H}}_{\varepsilon^{(k)}}(\omega). \tag{2.74}$$

On the other hand,

$$\begin{aligned} B_t^{(H)}(\omega) &= \sum_{k=1}^{\infty} \langle \chi[0,t], \xi_k \rangle_{L^2(\mathbb{R}^n)} \langle \omega, M\xi_k \rangle \\ &= \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s)ds \right) \langle \omega, M\xi_k \rangle. \end{aligned} \tag{2.75}$$

Note that for a fixed Hurst parameter $H \in (0,1)$ *we have* $M = M^{(H)}$ *and due to* (2.63) $M^{-1} = M^{(1-H)}$, *thus* $e_k^{(H)} = M^{(1-H)}\xi_k$ *implies* $M^{(H)}\xi_k = e_k^{(1-H)}$ *and we may consider* (2.75) *to be the chaos decomposition of fractional Brownian motion in* $L^2(\mu_{(1-H)}) = L^2(\mu \circ M^{(H)})$ *by the orthogonal basis*

$$\widetilde{\mathcal{H}}_{\varepsilon^{(k)}}(\omega) = \langle \omega, e_k^{(1-H)} \rangle.$$

In other words, fractional Brownian motion with Hurst parameter $H \in (0,1)$ *is the image of classical Brownian motion under the mapping* $\mathcal{M} = \mathcal{M}^{(H)}$ *in the fractional white noise space* $L^2(\mu_{(1-H)})$*. Thus, we can consider fractional Brownian motion as an element of three different spaces, as defined in* (2.73), (2.74) *and* (2.75)*.*

Example 2.8.4 *Fractional white noise* $W_t^{(H)}(\cdot)$ *is defined by the chaos expansions*

$$\begin{aligned} W_t^{(H)}(\omega) &= \sum_{k=1}^{\infty} M\xi_k(t) H_{\varepsilon^{(k)}}(\omega) \\ &= \sum_{k=1}^{\infty} \xi_k(t)\ \widetilde{\mathcal{H}}_{\varepsilon^{(k)}}(\omega) \end{aligned} \tag{2.76}$$

in the spaces $(S)_{-1}^{\mu}$ *and* $(S)_{-1}^{\mu_{(1-H)}}$ *respectively. It is integrable and the relation* $\frac{d}{dt}B_t^{(H)} = W_t^{(H)}$ *holds in the* $(S)_{-1}^{\mu}$ *sense.*

2.8.3 Fractional Poissonian white noise space

In this subsection we follow [29] and use the same idea as in the Gaussian case and apply the isomorphism $M = M^{(H)}$ to the elements of the Poissonian white noise space to obtain their corresponding fractional versions. Let $H \in (0,1)$ and recall the mapping $J : L^2(\mathbb{R}) \to L^2(\nu)$, defined in the Section 2.3 by

$$J(f) = \langle \omega, f \rangle - \int_{\mathbb{R}} f(x)dx.$$

Now we define

$$J^{(H)} := J \circ M$$

as the mapping $L^2(\mathbb{R}) \to L^2(\nu)$,

$$f \mapsto \langle \omega, Mf \rangle - \int_{\mathbb{R}} Mf(x)dx.$$

Then $\|J^{(H)}(f)\|_{L^2(\nu)} = \|J(Mf)\|_{L^2(\nu)} = \|Mf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ holds for all $f \in L^2(\mathbb{R})$. Similarly as in (2.69) we let

$$\begin{aligned} L^2(\nu_H) &= L^2(\nu \circ M^{-1}) \\ &= \{F : \Omega \to \mathbb{R};\, G \circ M \in L^2(\nu)\} \end{aligned} \tag{2.77}$$

be the space of square integrable functions on $S'(\mathbb{R})$ with respect to the *fractional Poissonian white noise measure* ν_H.

Definition 2.8.5 *Define the family of Charlier polynomials*

$$\widetilde{\mathcal{C}}_\alpha(\omega) = C_{|\alpha|}(\omega; \underbrace{e_1, ..., e_1}_{\alpha_1}, ..., \underbrace{e_m, ..., e_m}_{\alpha_m}), \quad \alpha = (\alpha_1, ..., \alpha_m, 0, 0, ...) \in \mathcal{I},$$

where $\mathcal{C}_k$ *are defined by* (2.22) *and the family* $\{e_k\}_{k \in \mathbb{N}}$ *by* (2.66).

The family of Charlier polynomials forms the orthogonal basis of the Hilbert space of fractional Poissonian random variables i.e. $L^2(\nu_H)$ consists of elements $F = \sum_{\alpha \in \mathcal{I}} a_\alpha \widetilde{\mathcal{C}}_\alpha(\omega)$, $a_\alpha \in \mathbb{R}$ such that $\|F\|^2_{L^2(\nu_H)} = \sum_{\alpha \in \mathcal{I}} a_\alpha^2 \alpha! < \infty$.

Definition 2.8.6 *The mapping* $\mathcal{M}^{-1} : L^2(\nu) \to L^2(\nu_H)$ *defined by*

$$\widetilde{\mathcal{C}}_\alpha(\omega) = \mathcal{M}^{-1} C_\alpha(\omega), \qquad \alpha \in \mathcal{I}, \tag{2.78}$$

extends by linearity and continuity to $L^2(\nu)$.

Thus, every element $G \in L^2(\nu_H)$ can be represented as an inverse image of a unique $g = \sum_{\alpha\in\mathcal{J}} a_\alpha C_\alpha(\omega) \in L^2(\nu)$ such that

$$G(\omega) = \mathcal{M}^{-1} g(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha\, C_\alpha(\omega) \in L^2(\nu).$$

Example 2.8.5 *A right t-continuous version of the process*

$$P_t^{(H)}(\omega) = J^{(H)}\chi[0,t](\omega) = J(M\chi[0,t])(\omega), \quad \omega \in S'(\mathbb{R})$$

belongs to $L^2(\nu_H)$ and is called the fractional compensated Poisson process. It is given by the chaos expansions

$$P_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t M\xi_k(s)ds \right) C_{\varepsilon^{(k)}}(\omega) \quad \text{in } \; L^2(\nu), \tag{2.79}$$

$$P_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s)ds \right) \widetilde{\mathcal{C}}_{\varepsilon^{(k)}}(\omega) \quad \text{in } \; L^2(\nu_{(1-H)}). \tag{2.80}$$

Example 2.8.6 *Fractional compensated Poissonian noise is defined by the chaos expansions*

$$V_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \xi_k(t)\; \widetilde{\mathcal{C}}_{\varepsilon^{(k)}}(\omega) = \sum_{k=1}^{\infty} M\xi_k(t) C_{\varepsilon^{(k)}}(\omega), \tag{2.81}$$

in the spaces $(S)_{-1}^{\nu_{(1-H)}}$ and $(S)_{-1}^{\nu}$ respectively.

Theorem 2.8.2 ([29]) *Let $\mathcal{U} : L^2(\mu) \to L^2(\nu)$ and $\mathcal{M} :\, L^2(\mu_H) \to L^2(\mu)$ be the isometries defined by* (2.29) *and* (2.71) *respectively. Then $\mathcal{U} \circ \mathcal{M}^{-1}$ is well defined and we have*

$$\mathcal{M}^{-1} \circ \mathcal{U} \;=\; \mathcal{U} \circ \mathcal{M}^{-1}$$

i.e. the following diagram is commutative.

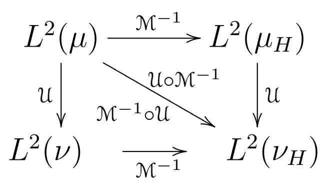

Diagram 1.

Proof. Let $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \widetilde{\mathcal{H}}_\alpha \in L^2(\mu_H)$. Then by applying the composition $\mathcal{M}\circ\mathcal{U}\circ\mathcal{M}^{-1}$ to F we obtain an element in $L^2(\nu_H)$ given in the form

$$\mathcal{M}\circ\mathcal{U}\circ\mathcal{M}^{-1}(F) = \mathcal{M}\circ\mathcal{U}\left(\sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha\right) = \mathcal{M}\left(\sum_{\alpha\in\mathcal{I}} f_\alpha C_\alpha\right) = \sum_{\alpha\in\mathcal{I}} f_\alpha \widetilde{\mathcal{C}}_\alpha.$$

On the other hand, when applying $\mathcal{U}$ to F we obtain the same element, i.e.

$$\mathcal{U}(F) = \mathcal{U}\left(\sum_{\alpha\in\mathcal{I}} f_\alpha \widetilde{\mathcal{H}}_\alpha\right) = \sum_{\alpha\in\mathcal{I}} f_\alpha \widetilde{\mathcal{C}}_\alpha.$$

This follows from the uniqueness of chaos expansion in orthogonal basis. □

2.8.4 Summary

- Since there exists an isomorphism $\mathcal{M}$ between the classical white noise spaces (Gaussian or Poissonian) and their corresponding fractional white noise spaces; and also there exists the isomorphism $\mathcal{U}$ between Gaussian and Poissonian white noise spaces (classical or fractional), all results obtained, for example, in the classical Gaussian case can be interpreted in all other spaces.

  In this manner, the space $L^2(\nu_H)$ can be obtained from the fractional Gaussian white noise space by $L^2(\nu_H) = \mathcal{U}[L^2(\mu_H)]$ or directly from the Gaussian white noise space $L^2(\nu_H) = \mathcal{U}[\mathcal{M}^{-1}[L^2(\mu)]]$ or $L^2(\nu_H) = \mathcal{M}^{-1}[\mathcal{U}[L^2(\mu)]]$. All connections are described in the Diagram 1, given in Theorem 2.8.2.

- Denote by $\mathfrak{e}_k$, $k\in\mathbb{N}$ the orthonormal basis of $L^2_H(\mathbb{R})$, i.e. the orthonormal fractional basis $e_k = M^{-1}\xi_k$, $k\in\mathbb{N}$ for $H\in(0,1)$, which reduces to the orthonormal Hermite basis ξ_k, $k\in\mathbb{N}$ for $H=\frac{1}{2}$. The orthogonal basis K_α, $\alpha\in\mathcal{I}$ of the four white noise spaces $L^2(P)$, built on the white noise space $(S'(\mathbb{R}),\mathcal{B},P)$, is thus obtained in the manner described in the following table:

white noise space	classical		fractional	
	Gaussian	Poissonian	Gaussian	Poissonian
measure P	μ	ν	μ_H	ν_H
basis K_α	H_α	C_α	$\widetilde{\mathcal{H}}_\alpha$	$\widetilde{\mathcal{C}}_\alpha$
basis $\mathfrak{e}_k$	ξ_k	ξ_k	e_k	e_k

Table 1.

Chapter 3

Malliavin Calculus in Chaos Expansions Framework for Square Integrable Processes

In this chapter we return to an infinite-dimensional differential calculus of variations, called the Malliavin stochastic calculus, in the white noise space. We summarize the most important results within this theory involving the operators of Malliavin calculus acting on appropriate random variables, altogether with their fractional versions. Recall that the main operators of the Malliavin calculus: the Malliavin derivative operator D, the divergence operator i.e. Itô-Skorokhod integral δ and the Ornstein-Uhlenbeck operator $\mathcal{R} = \delta D$ and their chaos expansion forms have been used in different frameworks. In the general context of a Fock space the Malliavin derivative D coincides with the annihilation operator, the divergence operator δ coincides with the creation operator and their composition, the Ornstein-Uhlenbeck operator $\mathcal{R}$, with the number operator studied in quantum probability.

The chaos expansion method provides us with a unified approach, valid for both, the continuous and discontinuous measures and can be carried out naturally to the Lévy processes setting.

Following [7], [8], [20], [31], [46], [63], [59] we first introduce the Malliavin derivative, the Skorokhod integral and the Ornstein-Uhlenbeck operator as operators with values in $L^2(P)$. In addition, in Chapter 4 we will allow the operators of Malliavin calculus to take values in some q-weighted distribution space and thus obtain a larger domain for all operators. Moreover, in Chapter 5 we will extend actions of the Malliavin operators to a class of singular generalized stochastic processes and apply them to some classes of stochastic differential equations.

The main theorems and properties of Malliavin operators in this chapter will be stated for both, the Gaussian and the Poissonian white noise spaces due to the unitary mapping $\mathcal{U} : L^2(\mu) \to L^2(\nu)$. We end this chapter with introducing the fractional operators of Malliavin calculus defined on $L^2(P)$ and $L^2(P_H)$ spaces for all $H \in (0,1)$. Throughout this chapter we will use the notation $L^2(P,H)$ for the set of H-valued random variables $L^2(\Omega, H)$.

3.1 Classical Malliavin calculus

The first part of this chapter is devoted to definitions of stochastic derivatives considered separately for random variables belonging to the space of Gaussian square integrable random variables $L^2(\mu)$ and to the space of Poissonian square integrable random variables $L^2(\nu)$. In spite of many similarities, there are important distinctions between the Gaussian and the Poissonian case. In particular, in the Gaussian case definitions of the directional derivative and the Malliavin derivative are equivalent and both, the ordinary chain rule and the Wick chain rule are valid. In the Poissonian case, when the stochastic gradient is defined as the directional derivative the ordinary chain rule is satisfied, but the Wick chain rule is not. Thus in the Poissonian case we have to abandon the Malliavin derivative based on the directional derivative and define it in terms of chaos expansions in the Charlier polynomials orthogonal basis of $L^2(\nu)$, analogously to the definition in general Gaussian case. With this restriction we continue. For more information we refer to [1], [10], [22], [49], [59].

3.1.1 The derivative operator in $L^2(\mu)$

We assume that the basic probability space is the Gaussian white noise space.

Following [7], [19], [20] and [46] we introduce a notion of the stochastic derivative DF of a square integrable random variable $F : \Omega \to \mathbb{R}$, defined on a Gaussian white noise space $(\Omega, \mathcal{F}, \mu)$, where $\Omega = S'(\mathbb{R})$, and we recall the main relations between integral and differential calculus. Suppose now that the separable Hilbert space H is $L^2(\mathbb{R})$ space equipped with the Lebesque measure λ on $\mathbb{R}$.

In the theory of Gaussian Hilbert spaces, introduced in Section 1.5, the scalar product $(DF, h)_H$ is interpreted as a directional derivative in direction h. We consider now only directions which belong to a subspace of Ω called the *Cameron-Martin space* $\mathbb{H}$. An absolutely continuous function with respect to the Lebesque measure, i.e. a trajectory $\gamma \in \Omega$ belongs to the

Cameron-Martin space $\mathbb{H}$ if it can be represented as

$$\gamma(t) = \int_0^t g(s)\, ds, \tag{3.1}$$

for some $g \in L^2(\mathbb{R})$. Thus $\mathbb{H}$ is a Hilbert space with respect to the inner product

$$(\gamma_1, \gamma_2)_{\mathbb{H}} = \int_0^t g_1(s)\, g_2(s)\, ds$$

and is isomorphic to $L^2(\mathbb{R})$.

Elementary random variables $F \in \mathcal{E}$ are defined in Section 1.5. In this particular case they are of the form

$$F = f(\langle \omega, y_1 \rangle, ..., \langle \omega, y_n \rangle), \tag{3.2}$$

for a smooth function $f \in C^\infty(\mathbb{R}^n)$ of polynomial growth with and all its partial derivatives, where $y_1, ..., y_n \in L^2(\mathbb{R})$ are deterministic functions and $n \in \mathbb{N}$. We formulate a version of Definition 1.5.3 for this specific case.

Definition 3.1.1 *The* stochastic derivative D *of an elementary random variable* $F \in \mathcal{E} \subseteq L^2(\mu)$ *of the form* (3.2) *is the* $L^2(\mathbb{R})$*-valued random variable defined by*

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\langle \omega, y_1 \rangle, ..., \langle \omega, y_n \rangle) \cdot y_i. \tag{3.3}$$

In particular, when the underlying isonormal family is the family of Brownian motion $B_t = \langle \omega, \chi[0,t] \rangle$, $t \geq 0$ it it follows that

$$\begin{aligned} (DF, \gamma)_{L^2(\mathbb{R})} &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, ..., B_{t_n}) \cdot \int_0^{t_i} g(s)ds \\ &= \frac{d}{d\varepsilon} F(\omega + \varepsilon\gamma) \mid_{\varepsilon=0} \end{aligned}$$

As a result, we can use the following definition for the Malliavin derivative operator in the Gaussian white noise space.

Definition 3.1.2 *Let* $F \in L^2(\mu)$ *and* $\gamma \in L^2(\mathbb{R})$ *be of the form* (3.1). *Then the* directional derivative of F in direction γ *is defined by*

$$\begin{aligned} D_\gamma F(\omega) &= \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} \\ &= \frac{d}{d\varepsilon} F(\omega + \varepsilon\gamma) \mid_{\varepsilon=0}, \end{aligned} \tag{3.4}$$

provided that the limit exists in $L^2(\mu)$.

Assume that there exists a function $\psi : \mathbb{R} \to L^2(\mu)$ *such that*

$$D_\gamma F(\omega) = \int_{\mathbb{R}} \psi_t(\omega)\, \gamma(t)\, dt, \quad \textit{for all} \quad \gamma \in L^2(\mathbb{R}).$$

Then we say that F *is* differentiable *and we call*

$$DF(\omega) := \psi_t(\omega), \quad t \in \mathbb{R} \tag{3.5}$$

the Malliavin derivative *or the* stochastic gradient *of a random variable* F.

The Malliavin derivative DF is not a derivative with respect to t, but a kind of derivative with respect to $\omega \in \Omega$.

The domain of extension of $D : \mathcal{E} \to L^2(\mu, L^2(\mathbb{R}))$ onto $L^2(\mu)$ is called the *Malliavin Sobolev space* and is denoted by $\mathcal{D}^{1,2}$. That is the set of all Malliavin differentiable random variables. Note that when considering the Gaussian white noise case definitions of the stochastic gradient of a random variable Definition 3.1.1 and Definition 3.1.2 are equivalent, i.e. the Malliavin derivative in Brownian motion case can be interpreted as a stochastic gradient.

Example 3.1.1 *Let* $F(\omega) = \langle \omega, f \rangle = \int_{\mathbb{R}} f(s) dB_t(\omega)$, *for some deterministic function* $f \in L^2(\mathbb{R})$ *and* $\omega \in \Omega$. *Then by linearity*

$$\begin{aligned} D_\gamma F(\omega) &= \lim_{\varepsilon \to 0} \frac{\langle \omega + \varepsilon\gamma, f \rangle - \langle \omega, f \rangle}{\varepsilon} \\ &= \int_{\mathbb{R}} f(t)\gamma(t) dt \\ &= (f, \gamma)_{L^2(\mathbb{R})}, \quad \textit{for all } \gamma \in L^2(\mathbb{R}). \end{aligned}$$

Thus the random variable F *is differentiable in the sense of the Definition 3.1.2 and its derivative is given by*

$$D\left(\int_{\mathbb{R}} f(s) dB_s \right) = DI(f) = f(t), \ \textit{for almost all } t. \tag{3.6}$$

In particular, for $F(\omega) = B_t(\omega)$ *we obtain*

$$DB_t = \chi[0, t], \quad \textit{for } t \in \mathbb{R}.$$

On the other hand, in the sense of Definition 3.1.1 it is clear that we also obtain $DF(\omega) = D\langle \omega, f \rangle = f(t)$, $t \in \mathbb{R}$.

Some of the basic properties of the calculus, such as chain rule, follow easily from Definition 3.1.2. The stochastic gradient satisfies the chain rule, with respect to both, the ordinary product and the Wick product. Proofs can be found in [10] and [46].

Theorem 3.1.1 (Chain rule) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function, i.e. there exists $C > 0$ such that*

$$|f(x) - f(y)| \leq C|x - y|, \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Let $X = (X_1, ..., X_n)$ be an n-dimensional random variable where each component $X_i : \Omega \to \mathbb{R}^n$ is differentiable. Then $f(X)$ is differentiable and

$$Df(X) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(X) \cdot DX_k. \tag{3.7}$$

We use now the definition (2.37) for the Wick version $f^\lozenge$ of a function f and assume the sum converges in the Hida space of distributions $(S)^*$. We state the Wick chain rule theorem, which will also remain true for q-weighted stochastic distributions.

Theorem 3.1.2 (Wick chain rule) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function and let $X = (X_1, ..., X_n)$ be an n-dimensional vector such that $X_i \in (S)^*$ for every $i = 1, ..., n$. Then, if $f^\lozenge(X) \in (S)^*$ then it is differentiable and*

$$Df^\lozenge(X) = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}\right)^\lozenge (X) \lozenge DX_k. \tag{3.8}$$

Alternatively, the Malliavin derivative can be also introduced by means of the Wiener-Itô chaos expansion of a random variable. Recall, every random variable $F \in L^2(\mu)$ has a chaos representation form either expressed in terms of iterated stochastic integrals or in terms of orthogonal Fourier-Hermite basis. The following theorems are describing the domain of the Malliavin derivative of F given in both representation forms.

Let $F \in L^2(\mu)$ be represented in terms of the Wiener-Itô chaos expansion as a series of iterated Itô integrals of symmetric functions $f_n \in L^2(\mathbb{R}^n)$ with respect to Brownian motion $F = \sum_{n=0}^{\infty} I_n(f_n)$. Then the domain $\mathcal{D}^{1,2}$ of the derivative operator D is characterized by

$$E(\|DF\|_H^2) = \sum_{n=1}^{\infty} n\|I_n(f_n)\|^2_{L^2(\mu)} < \infty.$$

In this setting the Malliavin derivative gets the following chaos expansion representation form.

Theorem 3.1.3 *Let* $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mu)$ *for a family of symmetrized functions* $f_n \in \widehat{L^2}(\mathbb{R}^n)$*. Then,* F *is differentiable if and only if condition*

$$\sum_{n=1}^{\infty} n\, n!\, \|f_n\|^2_{L^2(\mathbb{R}^n)} < \infty \tag{3.9}$$

is satisfied. Thus the Malliavin derivative of F *has the chaos expansion*

$$DF = \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t)), \tag{3.10}$$

where $I_{n-1}(f_n(\cdot,t))$ *represents the* $(n-1)-$*iterated first order Itô integrals with respect to the first* $n-1$ *variables* $t_1, ..., t_{n-1}$ *of a function* $f_n(t_1, ..., t_{n-1}, t)$.

Here we omit the proof and refer the reader to [8] and [46].

The characterization of the Malliavin differentiable square integrable random variables, represented in the chaos expansion form of the Fourier-Hermite orthogonal basis is stated in the following theorem.

Theorem 3.1.4 *Let* $F \in L^2(\mu)$ *have a chaos expansion representation of the form* $F(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha(\omega)$, $c_\alpha \in \mathbb{R}$*. A random variable* F *is differentiable in the Malliavin sense if and only if the condition*

$$\sum_{\alpha \in \mathcal{I}} |\alpha|\, \alpha!\, c_\alpha^2 < \infty \tag{3.11}$$

is satisfied, with $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$ *the length of multi-index* $\alpha = (\alpha_1, \alpha_2, \cdots) \in \mathcal{I}$*. Then the stochastic gradient of* F *has the chaos expansion of the form*

$$DF(\omega) = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} c_\alpha \alpha_k H_{\alpha - \varepsilon^{(k)}}(\omega) \xi_k(t), \quad \textit{for a.a. } t \in \mathbb{R}. \tag{3.12}$$

Proof. Assume $F \in L^2(\mu)$ is differential in the Malliavin sense. The expression (3.12) follows directly by use of the linearity property, chain rule (3.7), result (3.6) from Example 3.1.1 and property (1.2) for the Hermite polynomials

$$\begin{aligned}
DF &= D\left[\sum_{\alpha\in\mathcal{I}} c_\alpha \prod_{k=1}^{\infty} h_{\alpha_k}(\langle\omega,\xi_k\rangle)\right] \\
&= \sum_{\alpha\in\mathcal{I}} c_\alpha \sum_{i\in\mathbb{N}} \left[h'_{\alpha_i}(\langle\omega,\xi_i\rangle)\prod_{k\neq i} h_{\alpha_k}(\langle\omega,\xi_k\rangle)\right] \\
&= \sum_{\alpha\in\mathcal{I}} c_\alpha \sum_{i\in\mathbb{N}} \left[\alpha_i h_{\alpha_i-1}\,\xi_i(t)\prod_{i\neq k} h_{\alpha_k}(\langle\omega,\xi_k\rangle)\right] \\
&= \sum_{\alpha\in\mathcal{I}} c_\alpha \sum_{i\in\mathbb{N}} \alpha_i\,\xi_i(t)\left[h_{\alpha_i-1}\cdot\prod_{i\neq k} h_{\alpha_k}(\langle\omega,\xi_k\rangle)\right] \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{i\in\mathbb{N}} \alpha_i\; c_\alpha \xi_i(t) H_{\alpha-\varepsilon^{(i)}}(\omega)
\end{aligned}$$

with $\alpha-\varepsilon^{(k)} = (\alpha_1,...,\alpha_{k-1},\alpha_k-1,\alpha_{k+1},...)$. Operator D maps the domain $\mathcal{D}^{1,2}$, represented through the condition (3.11), into $L^2(\mu, L^2(\mathbb{R}))$, i.e. we have to prove now that $\|DF\|^2_{L^2(\mu,L^2(\mathbb{R}))}$ is finite. Recall, the family of the Hermite functions $\{\xi_k\}_{k\in\mathbb{N}}$ constitutes an orthonormal basis in $L^2(\mathbb{R})$ and the family of the Fourier-Hermite polynomials is an orthogonal basis in $L^2(\mu)$, therefore we have $(\xi_k,\xi_j)_{L^2(\mathbb{R})} = \delta_{k,j}$, for all $k,j\in\mathbb{N}$ and $E_\mu(H_\alpha H_\beta) = \alpha!\delta_{\alpha\beta}$, for all $\alpha,\beta\in\mathcal{I}$, with δ denotes the Kronecker delta symbol. Thus

$$\begin{aligned}
\|DF\|^2_{L^2(\mu,L^2(\mathbb{R}))} &= E_\mu(DF,DF)_{L^2(\mathbb{R})} = \\
&= \sum_{\alpha,\beta\in\mathcal{I}}\sum_{k,j\in\mathbb{N}} \alpha_k\beta_j c_\alpha c_\beta\, E_\mu(H_{\alpha-\varepsilon^{(k)}} H_{\beta-\varepsilon^{(j)}})(\xi_k,\xi_j)_{L^2(\mathbb{R})} \\
&= \sum_{\alpha,\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k^2\, c_\alpha^2\, E_\mu(H_{\alpha-\varepsilon^{(k)}} H_{\beta-\varepsilon^{(k)}}) \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k^2\,(\alpha-\varepsilon^{(k)})!\, c_\alpha^2 \\
&= \sum_{\alpha\in\mathcal{I}}(\sum_{k\in\mathbb{N}} \alpha_k)\,\alpha!\, c_\alpha^2 \\
&= \sum_{\alpha\in\mathcal{I}} |\alpha|\,\alpha!\, c_\alpha^2 \;<\; \infty
\end{aligned}$$

by assumption, provided $(\alpha-\varepsilon^{(k)})! = \frac{\alpha!}{\alpha_k}$, $k\in\mathbb{N}$. □

Operator D is continuous from $\mathcal{D}^{1,2}$ into $L^2(\mu, L^2(\mathbb{R}))$.

By linearity and continuity the definition of the Malliavin derivative in terms of the chaos expansions can be extended to wider classes of random variables, in particular, to q-weighted random variables.

3.1.2 The derivative operator in $L^2(\nu)$

Now we focus on the Poissonian white noise space and consider the Malliavin derivative of square integrable Poissonian random variables discussed in Section 2.3.

Every Poissonian random variable $F \in L^2(\nu)$ admits two types of chaos expansion representations, first as series of iterated Itô-Poisson integral of symmetric functions with respect to the compensated Poisson process and second in terms of the Charlier polynomials orthogonal basis of $L^2(\nu)$. We define the Malliavin derivative of a Poissonian random variable analogously to both definitions, Definition 3.1.1 and Definition 3.1.2, in the Gaussian case. In the Poissonian case these two definitions of the Malliavin derivative are *not* equivalent and the Malliavin derivative cannot be interpreted as a directional derivative. Actually in discontinuous case it is interpreted as a difference operator.

In particular, if we maintain definitions (3.3) and (3.5) in the Poissonian case, than as in the Gaussian case the ordinary chain rule (3.7) holds but the Wick chain rule (3.8) is no more valid. To see this clearly we present the example from [1].

Example 3.1.2 *Let* $F(\omega) = \langle \omega, \xi_i \rangle^{\lozenge^\nu 2}$. *Then from properties* (2.24) *and* (2.25) *of the Charlier polynomials and* $C_\alpha \lozenge^\nu C_\beta = C_{\alpha+\beta}$, $\alpha, \beta \in \mathcal{I}$ *it follows that*

$$C_{2\varepsilon^{(i)}} = \langle \omega, \xi_i \rangle^2 - \langle \omega, \xi_i^2 \rangle - 2\langle \omega, \xi_i \rangle \int_{\mathbb{R}} \xi_i(t)\, dt + (\int_{\mathbb{R}} \xi_i(t)\, dt)^2, \quad i \in \mathbb{N}$$

and thus we have

$$\begin{aligned} F(\omega) &= \left(C_{\varepsilon^{(i)}}(\omega) + \int_{\mathbb{R}} \xi_i(t)\, dt \right)^{\lozenge^\nu 2} \\ &= C_{2\varepsilon^{(i)}}(\omega) + 2C_{\varepsilon^{(i)}}(\omega) \int_{\mathbb{R}} \xi_i(t)\, dt + (\int_{\mathbb{R}} \xi_i(t)\, dt)^2 \\ &= \langle \omega, \xi_i \rangle^2 - \langle \omega, \xi_i^2 \rangle. \end{aligned}$$

Therefore, by (3.7), *we obtain*

$$\begin{aligned} DF(\omega) &= D\left(\langle \omega, \xi_i \rangle^2 - \langle \omega, \xi_i^2 \rangle \right) \\ &= 2\langle \omega, \xi_i \rangle \xi_i(t) - \xi_i^2(t) \\ &\neq 2\langle \omega, \xi_i \rangle\, \xi_i(t). \end{aligned}$$

Equivalently we have

$$\begin{aligned} D(C_{2\varepsilon^{(i)}}(\omega)) &= D\left(\langle \omega, \xi_i \rangle^{\diamond^\nu 2} - 2C_{2\varepsilon^{(i)}} \int_{\mathbb{R}} \xi_i(t)\, dt + (\int_{\mathbb{R}} \xi_i(t)\, dt)^2\right) \\ &= 2C_{\varepsilon^{(i)}}(\omega)\xi_i(t) - \xi_i^2(t) \\ &\neq 2C_{2\varepsilon^{(i)}}(\omega)\xi_i(t). \end{aligned}$$

Having in mind the previous example, we must abandon in the Poissonian case the Malliavin derivative based on the directional derivative in Definition 3.1.2. We orientate on stating the formal definition of a stochastic derivative, i.e the Malliavin derivative in terms of chaos expansions via the Charlier polynomials in an analogous way as in a general Gaussian case.

Definition 3.1.3 *Let* $F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha\, C_\alpha(\omega) \in L^2(\nu)$, $f_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$. *We say that a random variable* F *is Malliavin differentiable if the condition*

$$\sum_{\alpha\in\mathcal{I}} |\alpha|\, \alpha!\, f_\alpha^2 < \infty$$

is satisfied and write $F \in \mathcal{D}_\nu^{1,2}$. *Then the chaos expansion form of the Malliavin derivative of* F *is given by*

$$D^\nu F(\omega) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} f_\alpha\, \alpha_k\, C_{\alpha-\varepsilon^{(k)}}(\omega)\, \xi_k(t), \quad \textit{for a.a. } t\in\mathbb{R}. \tag{3.13}$$

With this definition the ordinary chain rule does not hold.

Consider again $F(\omega) = \langle \omega, \xi_i \rangle^{\diamond^\nu 2}$ from Example 3.1.2. From (3.13) and the Wick chain rule (3.8) for the Poissonian case, we obtain

$$\begin{aligned} D^\nu\left(\langle \omega, \xi_i \rangle^2\right) &= D^\nu\left(\langle \omega, \xi_i \rangle^{\diamond^\nu 2} + \langle \omega, \xi_i^2 \rangle\right) \\ &= 2\langle \omega, \xi_i \rangle\, \xi_i(t) + \xi_i^2(t) \\ &\neq 2\langle \omega, \xi_i \rangle\, \xi_i(t). \end{aligned} \tag{3.14}$$

More precisely, D^ν is equivalent to a finite *difference operator* and we have

$$D^\nu F(\omega) = F(\omega + \delta_t) - F(\omega), \quad \text{a.a } t, \omega,$$

if F is in $L^2(\nu)$ domain of D^ν, where $\delta_t \in S'(\mathbb{R})$ is the Dirac measure at t. As a consequence, the non-classical derivation property holds

$$D^\nu(FG) = F\, D^\nu G + G\, D^\nu F + D^\nu F\, D^\nu G, \quad \text{for} \quad F, G, FG \in \mathcal{D}_\nu^{1,2}.$$

For more information we refer to [1], [10], [22], [49], [59].
One can prove that Theorem 3.1.3 is also valid in the Poissonian case.

3.1.3 The divergence operator in $L^2(P, L^2(\mathbb{R}))$

In the framework of abstract Wiener space, the divergence operator is defined as adjoint operator of the Malliavin derivative, see Section 1.5.3. In particular, if the underlying Hilbert space H is $L^2(\mathbb{R})$ space, we interpret the divergence operator as a stochastic integral and we call it the Skorokhod integral, because in the Brownian motion case it coincides with the generalization of the Itô stochastic integral to anticipating integrands. In Section 2.2.6 we defined the Skorokhod integral on a set of stochastic process $u(t,\omega) = u_t(\omega)$, $\omega \in \Omega$, $t \in \mathbb{R}$ which are $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}$ and $E_\mu u_t^2(\omega) < \infty$, for all $t \in \mathbb{R}$ and represented them in chaos expansion form in terms of multiple Itô integrals.

Now we follow the general notions from Section 2.7.5 and denote by

$$\delta(u_t) = \int_{\mathbb{R}} u_t \; \delta\mathcal{Q}_t(\omega)$$

the P-Itô-Skorokhod stochastic integral of a process u_t with respect to $\mathcal{Q}_t$. If we choose $P = \mu$, the Gaussian measure, then $\mathcal{Q}_t = B_t$ is a Brownian motion and if we choose $P = \nu$, the Poissonian measure, then $\mathcal{Q}_t = P_t$ denotes a compensated Poisson process. The stochastic derivative, defined in terms of orthogonal polynomial basis of $L^2(P)$, is denoted by D and its domain by $\mathcal{D}^{1,2}$.

In white noise setting, the domain $Dom(\delta)$ of the divergence operator of a process u is the set of $L^2(\mathbb{R})$-valued integrable random variables $u \in L^2(P, L^2(\mathbb{R}))$ such that

$$|E[(DF, u)_{L^2(\mathbb{R})}]| \leq c \, \|F\|_{L^2(P)}, \quad \text{for all } F \in \mathcal{D}^{1,2},$$

where c is some constant depending on u. If $u \in Dom(\delta)$, then the unique element $\delta(u) \in L^2(P)$ is obtained from

$$E[F\,\delta(u)] \; = \; E[(DF, u)_{L^2(\mathbb{R})}], \quad \text{for all } F \in \mathcal{D}^{1,2}.$$

Two types of conditions which characterize the domain of the Skorokhod integrable random variables are distinguished. In Section 2.2.6 we defined the Skorokhod integral in terms of multiple Itô integrals by relation (2.18) and thus characterized its domain $Dom(\delta)$ by condition (2.19), i.e. by

$$\sum_{n=0}^{\infty} (n+1)! \, \|\widetilde{f}_{n,t}\|^2_{L^2(\mathbb{R}^{n+1})} < \infty.$$

The same condition is valid for the Itô-Poissonian integrals.

On the other hand, the operator δ can be computed in terms of chaos expansion of a $L^2(\mathbb{R})$-valued random variable u, by the orthogonal basis

$$\{\, \xi_k(t)\, K_\alpha(\omega)\,\}_{\alpha\in\mathcal{J},\, k\in\mathbb{N}},$$

i.e. we have the relation

$$\delta\,[\,\xi_k(t)\,K_\alpha(\omega)\,] \;=\; K_{\alpha+\varepsilon^{(k)}}(\omega), \quad \alpha\in\mathcal{J},\; k\in\mathbb{N}.$$

This is a consequence of Theorem 2.7.4. Recall that the chaos expansion form of the Skorokhod integral of a generalized stochastic process $F_t = \sum\limits_{\alpha\in\mathcal{J}} f_\alpha(t)\,K_\alpha$, with the coefficients $f_\alpha = \sum\limits_{k\in\mathbb{N}} f_{\alpha,k}\,\xi_k \in L^2(\mathbb{R})$ for all $\alpha\in\mathcal{J}$ is given by

$$\begin{aligned}\int_{\mathbb{R}} F_t(\omega)\,\delta\mathcal{Q}_t(\omega) \;&=\; \int_{\mathbb{R}}\left(\sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f_{\alpha,k}\,\xi_k(t)\,K_\alpha(\omega)\right)\delta\mathcal{Q}_t(\omega)\\ &=\; \sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f_{\alpha,k}\,K_{\alpha+\varepsilon^{(k)}}(\omega),\end{aligned}$$

An equivalent characterization of the domain $Dom(\delta)$, stated by the criterion (2.19), is given in the following theorem.

Theorem 3.1.5 *Let $F_t:\mathbb{R}\to L^2(P)$, $t\in\mathbb{R}$ be a generalized stochastic process given in the form $F_t = \sum_{\alpha\in\mathcal{J}} f_\alpha(t)\,K_\alpha$, with the coefficients $f_\alpha = \sum_{k\in\mathbb{N}} f_{\alpha,k}\,\xi_k \in L^2(\mathbb{R})$ for all $\alpha\in\mathcal{J}$. If the condition*

$$\sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f^2_{\alpha,k}\;|\alpha|\;\alpha!\;<\;\infty \tag{3.15}$$

is satisfied then the stochastic process F_t is Skorokhod integrable, *i.e. $F_t\in Dom(\delta)$.*

Proof. Due to the condition (3.15) we have

$$\begin{aligned}\|\delta(F_t)\|^2_{L^2(P)} \;&=\; \|\sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f_{\alpha,k}\,\xi_k(t)\,K_\alpha\|^2_{L^2(P)}\\ &=\; \sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f^2_{\alpha,k}\,(\alpha+\varepsilon^{(k)})!\\ &=\; \sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f^2_{\alpha,k}\;(\alpha_k+1)\;\alpha!\\ &\le\; C\sum_{\alpha\in\mathcal{J}}\sum_{k\in\mathbb{N}} f^2_{\alpha,k}\,|\alpha|\;\alpha! < \infty.\end{aligned}$$

□

Further on we consider the Skorokhod integral of stochastic generalized processes and state some of the most important results of stochastic differential calculus. For proofs and more details we refer to [19], [20], [32], [34], [35], [37], [46].

The connection between the Malliavin derivative and the Itô-Poisson-Skorokhod integral is given by relation (3.16) in the following theorem.

Theorem 3.1.6 (The fundamental theorem of stochastic calculus) *Let F_t be a stochastic process such that $E_P[\int_{\mathbb{R}} F_t^2\, dt] < \infty$. Assume*

- $F_t \in \mathcal{D}^{1,2}$, *for all* $t \in \mathbb{R}$,
- $DF_t \in Dom(\delta)$ *for all* $t \in \mathbb{R}$ *and*
- $E[\int_{\mathbb{R}} DF_t\, dt]^2 < \infty$.

Then $\int_{\mathbb{R}} F_t\, \delta \mathcal{Q}_t$ is a well defined element from $\mathcal{D}^{1,2}$ and

$$D\left(\int_{\mathbb{R}} F_t(\omega)\, \delta\mathcal{Q}_t(\omega)\right) = \int_{\mathbb{R}} DY_t(\omega)\, \delta\mathcal{Q}_t(\omega) + Y_t(\omega). \tag{3.16}$$

Proof. Suppose $F_t = \sum_{\alpha\in\mathcal{I}} f_\alpha(t)\, K_\alpha(\omega)$. Then we apply the Malliavin derivative D, given by (3.12) and (3.13), to the chaos expansion of the Skorokhod integral of F, represented by the property (2.59) and obtain

$$\begin{aligned} D\left(\int_{\mathbb{R}} Y_t\, \delta\mathcal{Q}_t(\omega)\right) &= D\left(\sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} f_{\alpha,k}\, K_{\alpha+\varepsilon^{(k)}}(\omega)\right) \\ &= \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} f_{\alpha,k} \sum_{i\in\mathbb{N}} (\alpha+\varepsilon^{(k)})_i\, K_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}\, \xi_i(t) \\ &= \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} f_{\alpha,k} \left[(\alpha_k+1)K_\alpha\xi_k + \sum_{i\in\mathbb{N},i\neq k} \alpha_i K_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}\xi_i\right]. \end{aligned}$$

On the other side, by applying the formula (2.59) to process DY_t we obtain

$$D\left(\int_{\mathbb{R}} Y_t\, \delta\mathcal{Q}_t(\omega)\right) = \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} (f_\alpha, \xi_k)_{L^2(\mathbb{R})} \left[(\alpha_k+1)K_\alpha\xi_k + \sum_{i\in\mathbb{N},i\neq k} \alpha_i K_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}\xi_i\right].$$

Thus the equality (3.16) follows. □

Relation between the ordinary product and the Wick product is stated by the following theorem.

Theorem 3.1.7 *Let $g \in L^2(\mathbb{R})$ be a deterministic function and $F \in L^2(P)$ be a random variable. Then*

$$F \lozenge \int_{\mathbb{R}} g d\mathcal{Q}_t = F \cdot \int_{\mathbb{R}} g\, d\mathcal{Q}_t - \int_{\mathbb{R}} g\, DF\, dt.$$

Theorem 3.1.8 (Integration by parts formula) *Let $F \in L^2(P)$ and assume that $Y : \mathbb{R} \times \Omega \to \mathbb{R}$ is Skorokhod integrable with $\int_{\mathbb{R}} Y_t(\omega)\delta\mathcal{Q}_t(\omega) \in L^2(P)$. Then*

$$F \cdot \int_{\mathbb{R}} Y_t\, \delta\mathcal{Q}_t = \int_{\mathbb{R}} FY_t\, \delta\mathcal{Q}_t + \int_{\mathbb{R}} Y_t\, DF\, dt, \tag{3.17}$$

provided that the integral $\int_{\mathbb{R}} Y_t\, DF\, dt$ converges in $L^2(P)$. Moreover, the Itô-Skorokhod isometry holds

$$E_P\left(\int_{\mathbb{R}} Y_t \delta\mathcal{Q}_t\right)^2 = E_P\left(\int_{\mathbb{R}} Y_t^2\, dt\right) + E_P\left(\int_{\mathbb{R}}\int_{\mathbb{R}} DY_s\, DY_t\, dsdt\right).$$

Taking expectations on the both sides in (3.17), we obtain the *duality formula*

$$E_P\left(F \cdot \int_{\mathbb{R}} Y_t\, \delta\mathcal{Q}_t\right) = E_P\left(\int_{\mathbb{R}} Y_t\, DF\, dt\right), \tag{3.18}$$

i.e. the Skorokhod integral is the dual operator of the Malliavin derivative. This important relationship between these two operators we will use when defining the Malliavin operators of generalized stochastic processes in the following chapter. As a consequence of the duality formula (3.18) it follows that the Skorokhod integral is a closed operator.

Stochastic integral representation of Wiener functionals

We have already seen that any random variable F, which is measurable with respect to a one-dimensional Brownian motion B_t, can be written as

$$F(\omega) = E(F) + \int_{\mathbb{R}} \varphi_t(\omega)\, dB_t(\omega),$$

where the process φ is an adapted square integrable process. One of the main contributions of the Malliavin calculus is the famous Clark-Ocone formula which gives an explicit representation of process φ in terms of the Malliavin derivative. In particular, when $\varphi \in \mathcal{D}^{1,2}$ then by the Clark-Ocone representation formula the process φ is represented as a conditional expectation of the Malliavin derivative D of a given function $F \in L^2(\Omega, \mathcal{F}, \mu)$ with respect to the filtration $\mathcal{F}_t$. Moreover, the process φ can be identified as the orthogonal projection of the derivative of F.

Theorem 3.1.9 (The Clark-Ocone formula) *Let $F \in \mathcal{D}^{1,2}$ be $\mathcal{F}_t$-measurable. Then*

$$F(\omega) = E(F) + \int_{\mathbb{R}} E(DF \,|\, \mathcal{F}_t)\; dB_t(\omega).$$

The proof of the theorem, provided by the chaos expansion approach, can be found, for example in [10] and [46]. The formula (3.19) can only be applied to random variables in $\mathcal{D}^{1,2}$. Extensions beyond this domain to the whole $L^2(\mu)$ are possible in the white noise framework (see [10]).

The corresponding version of the Clark-Ocone formula for the Poissonin case is given in [1] and [49].

3.1.4 The Ornstein-Uhlenbeck operator

Let $F \in L^2(P)$ be a square integrable random variable with respect to the measure P. We let P to be either the Gaussian measure μ or the Poissonian measure ν. Let F be represented in the form

$$F = \sum_{\alpha \in \mathcal{I}} a_\alpha K_\alpha, \quad a_\alpha \in \mathbb{R},$$

where K_α, $\alpha \in \mathcal{I}$ is notation for the Fourier-Hermite polynomial basis in $L^2(\mu)$ or respectively for the Charlier polynomial basis in $L^2(\nu)$. Consider general results from the Section 1.5.

Definition 3.1.4 *The operator defined as a composition of the Malliavin derivative and the Skorokhod integral*

$$\mathcal{R} = \delta \circ D$$

is called the Ornstein-Uhlenbeck operator*.*

The Ornstein-Uhlenbeck operator is defined on the set $\mathcal{D}^{1,2}$ of Malliavin differentiable square integrable random variables. We obtain

$$\begin{aligned} \mathcal{R}\left(\sum_{\alpha \in \mathcal{I}} a_\alpha K_\alpha\right) &= \delta\left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_k a_\alpha K_{\alpha - \varepsilon^{(k)}}\right) \\ &= \sum_{\alpha \in \mathcal{I}} \left(\sum_{k \in \mathbb{N}} \alpha_k\right) a_\alpha K_\alpha \\ &= \sum_{\alpha \in \mathcal{I}} |\alpha|\, a_\alpha\, K_\alpha, \end{aligned} \tag{3.19}$$

provided that the sum converges in $L^2(P)$. The orthogonal polynomials K_α represent eigenfunctions of the operator $\mathcal{R}$ and the corresponding eigenvalues are lengths of multi-indices $|\alpha|$.

The domain of $\mathcal{R}$ is denoted by $Dom(\mathcal{R})$ and is obtained from the condition

$$\|\mathcal{R}F\|^2_{L^2(P)} < \infty.$$

Thus in terms of the chaos expansions the domain $Dom(\mathcal{R})$ is characterized by the following condition

$$\sum_{\alpha\in\mathcal{I}} a_\alpha^2\, |\alpha|^2 \alpha! < \infty. \tag{3.20}$$

3.2 Fractional Malliavin Calculus

In this section we study fractional versions of Malliavin operators: the fractional Malliavin derivative, the fractional Itô-Skorokhod integral and the fractional Ornstein-Uhlenbeck operator, for any value of the Hurst parameter $H \in (0,1)$ in the space of square integrable random variables $L^2(P)$. Using the white noise analysis approach we define fractional Malliavin operators within chaos expansions in the classical white noise space. Recall that the fractional transform operator $M = M^{(H)}$, introduced in Section 2.8.1, connects fractional Brownian motion $B_t^{(H)}$ and classical Brownian motion B_t on the Gaussian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$. It also connects fractional compensated Poisson process $P_t^{(H)}$ and classical compensated Poissonian process P_t on the Poissonian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \nu)$. Due to the isometry provided by the fractional transform operator, introduction of the fractional white noise theory is not required. Definition of the fractional Malliavin derivative is connected to definition of directional derivative and is an element of q-weighted distributional space. The fractional stochastic integral, as an element of classical q-weighted distributional space, is the adjoint operator of the fractional Malliavin derivative. All the results stated in this section are obtained analogously to the results from the classical Malliavin calculus and are given in the general case, i.e. they are valid in both cases Gaussian and Poissonian. Moreover, we define the fractional operators of the Malliavin calculus on fractional space $L^2(P_H) = L^2(P \circ M^{-1})$.

The main references used in this section are [7], [8], [13], [20], [41], [46].

The second approach in fractional Malliavin calculus is based on studying fractional versions of the Malliavin derivative and the stochastic integral for $H \in (0, \frac{1}{2})$ and for $H \in (\frac{1}{2}, 1)$ separately. For all $H \neq \frac{1}{2}$ fractional

Brownian motion is not a semimartingale. Semimartingales are the natural class of processes for which a stochastic calculus can be developed and they can be expressed as the sum of a bounded variation process and a local martingale which has finite quadratic variation. In the case $H < \frac{1}{2}$ the quadratic variation is infinite and if $H > \frac{1}{2}$ the quadratic variation is zero and the 1-variation is infinite. For $H \neq \frac{1}{2}$ it is necessary to define fractional white noise space which differ from the classical one. Stochastic integral in $H < \frac{1}{2}$ case is of the Stratonovich type and for $H > \frac{1}{2}$ is of the Itô-Skorokhod type. For introduction to the classical and fractional Malliavin calculus in these two concepts and connections between we refer to [7], [8], [20], [26], [47].

3.2.1 The fractional Malliavin derivative in $L^2(\mu)$

Let $H \in (0,1)$. Following [7], [13] and [20] we introduce a notion of the fractional Malliavin derivative $D^{(H)}F$ of a square integrable random variable $F \in L^2(\mu)$. This survey on fractional stochastic gradient for the Gaussian case provides us a motivation for defining the fractional Malliavin derivative in general case $L^2(P)$. Let $M = M^{(H)}$ be the fractional transform operator defined by (2.61).

Definition 3.2.1 *Let $F \in L^2(\mu)$ and $\gamma \in \Omega$ be of the form* (3.1)*. Then F has a* directional derivative in the direction γ *if*

$$D_\gamma^{(H)} F(\omega) \quad = \quad \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\, M\gamma) - F(\omega)}{\varepsilon} \tag{3.21}$$

the limit exists in $L^2(\mu)$.

Assume that there exists function $\psi : \mathbb{R} \to L^2(\mu)$ such that

$$D_\gamma^{(H)} F(\omega) = \int_{\mathbb{R}} M\psi_t\, M\gamma(t)\, dt, \quad \textit{for all} \quad \gamma \in L_H^2(\mathbb{R}).$$

Then we say that F is Malliavin differentiable *and*

$$D^{(H)} F(\omega) := \psi_t(\omega), \qquad t \in \mathbb{R} \tag{3.22}$$

is called the fractional stochastic gradient *or the* fractional Malliavin derivative *of a random variable F.*

The fractional stochastic gradient in $L^2(\mu)$ satisfies a chain rule both with respect to the ordinary product and with respect to the Wick product of random variables

$$D^{(H)} \left(\langle \omega, Mf \rangle^n \right) = n \langle \omega, Mf \rangle^{n-1} f(t), \tag{3.23}$$

and

$$D^{(H)}\left(\langle\omega, Mf\rangle^{\Diamond n}\right) = n\langle\omega, Mf\rangle^{\Diamond(n-1)} f(t), \tag{3.24}$$

for a.a. $t \in \mathbb{R}$. In particular,

$$D^{(H)}(\int_{\mathbb{R}} f\, dB_t^{(H)}) = f(t) \quad \text{for a.a. } t \in \mathbb{R}.$$

The set of all differentiable random variables is denoted by $\mathcal{D}_{1,2}^{(H)}$ and is called the *fractional Malliavin Sobolev space*. The fractional Malliavin derivative $D^{(H)}$ is a continuous mapping from $\mathcal{D}_{1,2}^{(H)} \subseteq L^2(\mu)$ onto $L^2(\mu, L_H^2(\mathbb{R}))$.

Example 3.2.1 *From* (3.23) *and* (3.24) *we obtain*

$$D^{(H)} H_\alpha(\omega) = \sum_{k\in\mathbb{N}} c_\alpha\, \alpha_k\, H_{\alpha-\varepsilon^{(k)}}(\omega)\, e_k(t), \tag{3.25}$$

where $e_k = M^{-1}\xi_k$, $k \in \mathbb{N}$ *are the elements of an orthonormal basis of* $L_H^2(\mathbb{R})$.

This property is used as a motivation for stating an equivalent definition of the fractional Malliavin derivative in $L^2(\mu)$.

Definition 3.2.2 (Fractional Malliavin Sobolev spaces) *Let* $\mathcal{D}_{1,2}^{(H)}$ *be the set of all random variables*

$$F(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha\, H_\alpha(\omega) \in L^2(\mu), \quad c_\alpha \in \mathbb{R}$$

which satisfy the condition

$$\sum_{\alpha\in\mathcal{J}} |\alpha|\, \alpha!\, c_\alpha^2 < \infty. \tag{3.26}$$

Then, a random variable $F \in \mathcal{D}_{1,2}^{(H)}$ *is called* Malliavin differentiable *and the* fractional Malliavin derivative *of* F *has the chaos expansion of the form*

$$D^{(H)} F(\omega) = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha\, \alpha_k\, H_{\alpha-\varepsilon^{(k)}}(\omega)\, e_k(t), \quad \textit{for a.a. } t \in \mathbb{R}. \tag{3.27}$$

3.2.2 The fractional Malliavin derivative in $L^2(\nu)$

Due to Definition 3.2.2 and results stated in Section 3.2.1 for the Gaussian case, we give the formal definition of the fractional Malliavin derivative in the space of square integrable Poissonian random variables.

Definition 3.2.3 *Let $\mathcal{D}^{(H)}_{1,2}$ be the set of all random variables*

$$F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha\, C_\alpha(\omega) \;\in L^2(\nu), \quad c_\alpha \in \mathbb{R}$$

which satisfy the condition

$$\sum_{\alpha\in\mathcal{I}} |\alpha|\, \alpha!\, c_\alpha^2 \; < \infty. \tag{3.28}$$

Then the fractional Malliavin derivative *of F has the chaos expansion of the form*

$$D^{(H)}F(\omega) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} c_\alpha\, \alpha_k\, C_{\alpha-\varepsilon^{(k)}}(\omega)\, e_k(t), \quad \text{for a.a. } t\in\mathbb{R}.$$

3.2.3 Relation with the standard Malliavin derivative in $(Q)^P_{-\rho}$

We continue with presentation of properties of the fractional Malliavin derivative in the general white noise space $(\Omega, \mathcal{F}, P)$, where P is either Gaussian measure μ or Poissonian measure ν. Let K_α, $\alpha\in\mathcal{I}$ denote the orthogonal basis of $L^2(P)$, i.e. the Fourier-Hermite polynomials H_α in $L^2(\mu)$ and the Charlier polynomials C_α in $L^2(\nu)$. Let $M = M^{(H)}$ be the fractional transform operator defined by (2.61). Denote by

$$L^2(P_H) = L^2(P\circ M^{-1}) \tag{3.29}$$

the space of fractional random variables, where P_H is the fractional measure corresponding to the P, i.e. the fractional Gaussian measure μ_H or fractional Poissonian measure ν_H. Let $\widetilde{\mathcal{K}}_\alpha$, $\alpha\in\mathcal{I}$ denote the orthogonal polynomials basis of $L^2(P_H)$, i.e. the fractional Fourier-Hermite polynomials $\widetilde{\mathcal{H}}_\alpha$ and the fractional Charlier polynomials $\widetilde{\mathcal{C}}_\alpha$. Due to the Wiener-Itô chaos expansion theorem, every $F\in L^2(P_H)$ is of the form

$$F(\omega) = \sum_{\alpha\in\mathcal{I}} a_\alpha \widetilde{\mathcal{K}}_\alpha(\omega),$$

for a unique family of constants $a_\alpha \in \mathbb{R}$. Then the operator M induces the mapping $\mathcal{M} : L^2(P_H) \to L^2(P)$ defined by

$$\mathcal{M}\left[\sum_{\alpha\in\mathcal{J}} a_\alpha \widetilde{\mathcal{K}}_\alpha(\omega)\right] = \sum_{\alpha\in\mathcal{J}} a_\alpha K_\alpha(\omega),$$

which follows from Definition 2.8.4 in Section 2.8.2 and also Definition 2.8.6 in Section 2.8.3.

Denote by D the Malliavin derivative and $D^{(H)}$ the fractional Malliavin derivative on $L^2(P)$. Thus for a random variable $F = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha \in L^2(P)$, $c_\alpha \in \mathbb{R}$, the chaos expansion forms of its Malliavin derivatives, classical and fractional, are given respectively by

- $DF(\omega) = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha \, \alpha_k \, K_{\alpha-\varepsilon^{(k)}}(\omega) \, \xi_k(t)$, and
- $D^{(H)}F(\omega) = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha \, \alpha_k \, K_{\alpha-\varepsilon^{(k)}}(\omega) \, e_k(t)$, for a.a. $t \in \mathbb{R}$.

On an analogous way, we define the Malliavin derivative and the fractional Malliavin derivative on fractional space $L^2(P_H)$ and denote them by $\widetilde{D}$ respectively $\widetilde{D}^{(H)}$. In particular, for $F = \sum_{\alpha\in\mathcal{J}} c_\alpha \widetilde{\mathcal{K}}_\alpha \in L^2(P_H)$, $c_\alpha \in \mathbb{R}$ these operators on $L^2(P_H)$ are represented in the following chaos expansion forms

- $\widetilde{D}F(\omega) = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha \, \alpha_k \, \widetilde{\mathcal{K}}_{\alpha-\varepsilon^{(k)}}(\omega) \, e_k(t)$, and
- $\widetilde{D}^{(H)}F(\omega) = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha \, \alpha_k \, \widetilde{\mathcal{K}}_{\alpha-\varepsilon^{(k)}}(\omega) \, M^{-1}e_k(t)$, for a.a. $t \in \mathbb{R}$.

The extension of the fractional Malliavin derivative, stated in Definition 3.2.2, from the space $L^2(P)$ to the space of q-weighted stochastic distributions $(Q)_{-\rho}^P$, for $\rho \in [0, 1]$ is characterized by (3.27) and also denoted by D.

Relation between the fractional Malliavin derivative and the classical Malliavin derivative of elements from $L^2(P)$, respectively from $L^2(P_H)$, are given through the mapping $\mathbf{M_1}$.

Definition 3.2.4 *Let $F : \mathbb{R} \to (Q)_{-\rho}^P$ be a generalized stochastic processes with respect to the measure P given in the form* (2.44). *We define the mapping $\mathbf{M_1}$ of F by the following*

$$\mathbf{M_1} F_t(\omega) = \mathbf{M_1}\left(\sum_{\alpha\in\mathcal{J}} f_\alpha(t)\, K_\alpha(\omega)\right) = \sum_{\alpha\in\mathcal{J}} M f_\alpha(t)\, K_\alpha(\omega), \tag{3.30}$$

for coefficients $f_\alpha \in L^2_H(\mathbb{R})$, *which are measurable functions, satisfying the convergence condition* (2.45) *for some* $p > 0$.

Theorem 3.2.1 *Let* $F \in (Q)^P_{-\rho}$. *Then*

$$DF = \mathbf{M_1} \circ D^{(H)} F, \quad \text{for all } t \in \mathbb{R}. \tag{3.31}$$

Proof. *Let* F *have the chaos expansion representation*

$$F(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha \, K_\alpha(\omega), \qquad c_\alpha \in \mathbb{R}.$$

Then by (3.27) *and* (3.30) *we obtain*

$$\begin{aligned} \mathbf{M_1} D^{(H)} F &= \mathbf{M_1} \left(\sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha \, \alpha_k \, e_k(t) \, K_{\alpha-\varepsilon^{(k)}}(\omega) \right) \\ &= \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} c_\alpha \, \alpha_k \, \xi_k(t) \, K_{\alpha-\varepsilon^{(k)}}(\omega) \\ &= DF. \end{aligned}$$

□

- Similarly, for $F \in (Q)^{P_H}_{-\rho}$ we have

$$\widetilde{D}\, F = \mathbf{M_1} \circ \widetilde{D}^{(H)} F. \tag{3.32}$$

Theorem 3.2.2 *Let* $F \in (Q)^{P_H}_{-\rho}$ *for* $\rho \in [0,1]$. *Then the following is true*

$$\widetilde{D} F = \mathcal{M}^{-1} \circ D^{(H)} \circ \mathcal{M} \, F. \tag{3.33}$$

Proof. Assume $F = \sum_{\alpha\in\mathcal{J}} f_\alpha \, \widetilde{\mathcal{K}}_\alpha \in (Q)^{P_H}_{-\rho}$ such that the convergence condition (2.45) in $(Q)^{P_H}_{-\rho}$ is satisfied for some $p > 0$. Then we obtain

$$\begin{aligned} \mathcal{M}^{-1} \circ D^{(H)} \circ \mathcal{M}\, F &= \mathcal{M}^{-1} \circ D^{(H)} \left(\sum_{\alpha\in\mathcal{J}} f_\alpha \, K_\alpha \right) \\ &= \mathcal{M}^{-1} \left(\sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} f_\alpha \, \alpha_k \, K_{\alpha-\varepsilon^{(k)}}(\omega) \, e_k(t) \right) \\ &= \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} f_\alpha \, \alpha_k \, \widetilde{\mathcal{K}}_{\alpha-\varepsilon^{(k)}}(\omega) \, e_k(t) \\ &= \widetilde{D}\, F. \end{aligned}$$

□

- Moreover, for $F \in (Q)^{P_H}_{-\rho}$, $\rho \in [0,1]$ we have:

$$\widetilde{D}^{(H)} F = \mathbf{M_1}^{-1} \circ \mathcal{M}^{-1} \circ D^{(H)} \circ \mathcal{M}\, F. \tag{3.34}$$

3.2.4 The fractional Wick Itô-Skorokhod integral

In this section we follow the notations from Section 2.7.5. Let $\mathcal{Z}_t$ be a P-white noise, i.e. a generalized stochastic process such that $\frac{d}{dt}\,\mathcal{Z}_t = \mathcal{Q}_t$ for a.a. $t \in \mathbb{R}$, in sense of the relation (2.46) and let $\mathcal{Z}_t^{(H)}$ be the corresponding fractional generalized stochastic process such that $\frac{d}{dt}\,\mathcal{Z}_t^{(H)} = \mathcal{Q}_t^{(H)}$, $t \in \mathbb{R}$. In particular, in the Gaussian case $\mathcal{Q}_t = B_t$ is a Brownian motion and $\mathcal{Z}_t = W_t$ is a singular white noise, defined by (2.50) and (2.51) respectively, and $\mathcal{Q}_t^{(H)} = B_t^{(H)}$ is a fractional Brownian motion and $\mathcal{Z}_t^{(H)} = W_t^{(H)}$ is a fractional singular white noise, defined respectively by chaos expansion forms (2.73) and (2.76). In the Poissonian case, $\mathcal{Q}_t = P_t$ denotes a compensated Poisson process and $\mathcal{Z}_t = V_t$ a Poissonian compensated white noise, defined respectively by (2.52) and (2.53) with the corresponding fractional versions, a fractional compensated Poisson process $P_t^{(H)}$ defined by (2.79) and a fractional Poissonian compensated white noise $V_t^{(H)}$ defined (2.81). Then, the chaos expansion of P-white noise is given by

$$\mathcal{Z}_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t)\, K_{\varepsilon^{(k)}}, \quad t \in \mathbb{R}$$

and the chaos expansion form of the fractional P-white noise is given by

$$\mathcal{Z}_t^{(H)}(\omega) = \sum_{k=1}^{\infty} M\xi_k(t)\, K_{\varepsilon^{(k)}}, \quad t \in \mathbb{R}.$$

Definition 3.2.5 (The Wick Itô-Skorokhod integral) *Let $Y : \mathbb{R} \to (S)_{-1}$ be a stochastic process such that $Y_t \lozenge \mathcal{Z}_t^{(H)}$ is P-Pettis integrable in $(S)_{-1}$. Then Y is integrable in the Itô-Skorokhod sense and the* Wick Itô-Skorokhod integral *of $Y = Y_t(\omega)$ is defined by*

$$\begin{aligned} \delta^{(H)}(Y_t) &= \int_{\mathbb{R}} Y_t(\omega)\, d\mathcal{Q}_t^{(H)} \\ &= \int_{\mathbb{R}} Y_t \lozenge \mathcal{Z}_t^{(H)}\, dt \end{aligned} \tag{3.35}$$

where $\lozenge$ denotes the P-Wick product.

Consider now a special case, when the process $Y = f$ is a deterministic function belonging to $L^2_H(\mathbb{R})$. Then from the chaos expansion form of the fractional P-white noise it follows that the previous definition of the Wick Itô-Skorokhod integral coincides with

$$\int_{\mathbb{R}} f(t)\, d\mathcal{Q}_t^{(H)} = \int_{\mathbb{R}} Mf(t)\, d\mathcal{Q}_t.$$

Clearly, we have

$$\begin{aligned}
\int_{\mathbb{R}} f(t) \lozenge \mathcal{Z}_t^{(H)}\, dt &= \sum_{k=1}^{\infty} \left[\int_{\mathbb{R}} f(t)\, M\xi_k(t)\, dt \right] K_{\varepsilon^{(k)}}(\omega) \\
&= \sum_{k=1}^{\infty} (f, M\xi_k)_{L^2(\mathbb{R})}\, K_{\varepsilon^{(k)}}(\omega) \\
&= \sum_{k=1}^{\infty} (Mf, \xi_k)_{L^2(\mathbb{R})}\, K_{\varepsilon^{(k)}}(\omega) \\
&= \int_{\mathbb{R}} Mf \lozenge \mathcal{Z}_t\, dt \\
&= \int_{\mathbb{R}} Mf(t)\; d\mathcal{Q}_t.
\end{aligned}$$

Example 3.2.2 *The* fractional normalized stochastic exponential *is defined by*

$$\begin{aligned}
\varepsilon_{Mh} &= \exp^{\lozenge}[\langle \omega, Mh \rangle] \\
&= \exp\left(\langle \omega, Mh \rangle - \frac{1}{2} \|Mh\|^2_{L^2(\mathbb{R})} \right), \qquad (3.36)
\end{aligned}$$

for $h \in L^2_H(\mathbb{R})$.

Theorem 3.2.3 *Let a generalized stochastic process* $Y_t = \sum_{\alpha \in \mathcal{J}} f_\alpha(t)\, K_\alpha(\omega)$, $t \in \mathbb{R}$ *be integrable in the Wick Itô-Skorokhod sense. Then the chaos expansion of its Wick Itô-Skorokhod integral is given by*

$$\int_{\mathbb{R}} Y_t\, d\mathcal{Q}_t^{(H)} = \sum_{\alpha \in \mathcal{J}} \sum_{k \in \mathbb{N}} (f_\alpha, M\xi_k)_{L^2(\mathbb{R})}\, K_{\alpha + \varepsilon^{(k)}}. \qquad (3.37)$$

Moreover, if $\int_{\mathbb{R}} Y_t\, d\mathcal{Q}_t^{(H)} \in L^2(P)$ *then*

$$E_P \left[\int_{\mathbb{R}} Y_t\, d\mathcal{Q}_t^{(H)} \right] = 0. \qquad (3.38)$$

Proof. Using the chaos expansion method, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} Y_t \, d\mathcal{Q}_t^{(H)} &= \int_{\mathbb{R}} Y_t(\omega) \lozenge \mathcal{Z}_t^{(H)}(\omega) \\
&= \int_{\mathbb{R}} \left(\sum_{\alpha \in \mathcal{I}} f_\alpha(t) \, K_\alpha(\omega) \right) \lozenge \left(\sum_{k \in \mathbb{N}} M\xi_k \, K_{\varepsilon^{(k)}}(\omega) \right) dt \\
&= \int_{\mathbb{R}} \left(\sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} f_\alpha(t) \, M\xi_k \, K_{\alpha+\varepsilon^{(k)}} \right) dt \\
&= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (f_\alpha, M\xi_k)_{L^2(\mathbb{R})} \, K_{\alpha+\varepsilon^{(k)}}.
\end{aligned}$$

Moreover, the expectation of a $L^2(P)$ element $\int_{\mathbb{R}} Y_t(\omega) \, d\mathcal{Q}_t^{(H)}(\omega)$ is equal to the zero coefficient in its chaos expansion, thus we obtain $f_{(0,0,0,\ldots)} = 0$ and the assertion (3.38) is proved. □

The corresponding result for the chaos expansion representation of generalized stochastic process in the classical Malliavin calculus is already stated in Theorem 2.7.4.

Theorem 3.2.4 (Fractional integration) *Suppose* $Y : \mathbb{R} \to (S)_{-1}$ *is integrable in the sense of Definition 3.2.5 and* (3.30) *converges in* $(S)_{-1}$. *Then the Wick Itô-Skorokhod integral with respect to process* $\mathcal{Q}_t^{(H)}$ *coincides with the Skorokhod integral with respect to filtration* $\mathcal{Q}_t$, $t > 0$, *i.e. we have*

$$\int_{\mathbb{R}} Y_t \, d\mathcal{Q}_t^{(H)}(\omega) = \int_{\mathbb{R}} MY_t \, \delta\mathcal{Q}_t(\omega). \tag{3.39}$$

Proof. Let $Y_t(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha(t) \, K_\alpha(\omega)$ be a generalized stochastic process, such that $f_\alpha \in L_H^2(\mathbb{R})$, $\alpha \in \mathcal{I}$. Then the chaos expansion of its fractional Wick Itô-Skorokhod integral is unique and represented by (3.37). Moreover, the fractional integral coincides with the Skorokhod integral with respect to $\mathcal{Q}_t$. Clearly,

$$\begin{aligned}
\int_{\mathbb{R}} Y_t \, dQ_t^{(H)} &= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (f_\alpha, M\xi_k)_{L^2(\mathbb{R})} \, K_{\alpha+\varepsilon^{(k)}} \\
&= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (Mf_\alpha, \xi_k)_{L^2(\mathbb{R})} \, K_{\alpha+\varepsilon^{(k)}} \\
&= \int_{\mathbb{R}} MY_t \, \delta\mathcal{Q}_t^{(H)}.
\end{aligned}$$

□

The following theorem is an analogue to the fundamental theorem of classical stochastic calculus, Theorem 3.1.6. The proof in the Gaussian case can be found in [8] and [46].

Theorem 3.2.5 *Let $Y : \mathbb{R} \to (S)_{-1}$ be a generalized process, such that $Y_t \in \mathcal{D}_{1,2}^{(H)}$. Assume thet Y and $D^{(H)}Y : \mathbb{R} \to (S)_{-1}$ are Wick Itô- Skorokhod integrable. Then we have*

$$D^{(H)}\left(\int_{\mathbb{R}} Y_t \, d\mathcal{Q}_t^{(H)}\right) = \int_{\mathbb{R}} D^{(H)} Y_t \, dQ_t^{(H)} + Y_t. \tag{3.40}$$

Proof. If $Y_t = \sum_{\alpha\in\mathcal{I}} f_\alpha(t)\, K_\alpha(\omega)$ then by (3.37) and (3.27) we obtain

$$\begin{aligned} D^{(H)}\left(\int_{\mathbb{R}} Y_t \, d\mathcal{Q}_t^{(H)}\right) &= D^{(H)}\left(\sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} (f_\alpha, M\xi_k)_{L^2(\mathbb{R})}\, K_{\alpha+\varepsilon^{(k)}}(\omega)\right) \\ &= \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} (f_\alpha, M\xi_k)_{L^2(\mathbb{R})} \sum_{i\in\mathbb{N}} (\alpha+\varepsilon^{(k)})_i\, K_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}\, e_i(t) \\ &= \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} (f_\alpha, M\xi_k)_{L^2(\mathbb{R})}\, (\alpha_k+1)\, K_\alpha\, e_k \\ &\quad + \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} \sum_{i\in\mathbb{N}, i\neq k} \alpha_i\, (f_\alpha, M\xi_k)_{L^2(\mathbb{R})}\, K_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} e_i. \end{aligned}$$

On the other side, when applying the differentiation formula (3.27) for $D^{(H)}Y_t$ we obtain the right-hand side of (3.40). This follows because the fractional operator M is a self adjoint operator

$$(f_\alpha, e_k)_{L_H^2(\mathbb{R})} = (Mf_\alpha, Me_k)_{L^2(\mathbb{R})} = (Mf_\alpha, \xi_k)_{L^2(\mathbb{R})} = (f_\alpha, M\xi_k)_{L^2(\mathbb{R})}. \qquad \square$$

3.2.5 The fractional Ornstein-Uhlenbeck operator

The fractional Ornstein-Uhlenbeck operator $\mathcal{R}^{(H)}$ of a random variable $F \in L^2(P)$ is defined as the composition of the fractional Wick Itô-Skorokhod integral and the fractional Malliavin derivative, i.e.

$$\mathcal{R}^{(H)}\, F = \delta^{(H)} \circ D^{(H)}\, F, \quad F \in \mathcal{D}_{1,2}^{(H)} \subseteq L^2(P).$$

Let $F = \sum_{\alpha\in\mathcal{I}} f_\alpha\, K_\alpha$, for $f_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$. Then we obtain

$$\begin{aligned} \mathcal{R}^{(H)}\, F &= \sum_{\alpha\in\mathcal{I}} f_\alpha\, K_\alpha \\ &= \mathcal{R}\, F. \end{aligned} \tag{3.41}$$

Thus the fractional Ornstein-Uhlenbeck operator and the standard Ornstein-Uhlenbeck operator coincide on the set of Malliavin differentiable random variables.

Chapter 4

Operators of Malliavin Calculus For Singular Generalized Stochastic Processes

Recall that the Malliavin derivative appears as the adjoint operator of the Skorokhod integral which is an extension of the stochastic Itô integral of anticipating processes to the class of non-anticipating processes. Moreover the composition of these two operators, called the Ornstein-Uhlenbeck operator, is a linear, unbounded and self-adjoint operator.

We give now the definitions of the Malliavin derivative and the Skorokhod integral which are extension of the definitions of these operators to a space of singular generalized stochastic processes. We allow now values in q-weighted spaces of generalized stochastic functionals and obtain larger domains of operators of Malliavin calculus then in the $L^2(P)$-case described in the previous chapter.

This chapter represents an original part of the dissertation and all the results presented are obtained in collaboration with Professor Stevan Pilipović and Dora Seleši and are already published in [27], [28], [29] and [30].

The Malliavin derivative, further on denoted by $\mathbb{D}$, and its related operators, δ and $\mathcal{R}$, are all defined on either of the four white noise spaces we are working on, and their domains are characterized in terms of convergence in a stochastic distribution space $(Q)_{-1}^{P}$ with special q-weights.

In the following, we denote by $\mathfrak{e}_k$, $k \in \mathbb{N}$ the orthonormal basis of $L^2_H(\mathbb{R})$, i.e. $\mathfrak{e}_k$ is the orthonormal fractional basis $e_k = M^{-1}\xi_k$, $k \in \mathbb{N}$, for all $H \in (0,1)$, which reduces to the orthonormal Hermite basis ξ_k, $k \in \mathbb{N}$ when $H = \frac{1}{2}$. Note, $\|\mathfrak{e}_k\|^2_{-l} = (2k)^{-l}$ and $\|\mathfrak{e}_k\|^2_{\exp,-l} = e^{-2kl}$ for all $k, l \in \mathbb{N}$. We denote by K_α, $\alpha \in \mathcal{I}$ the orthogonal basis of the space of square integrable random variables $L^2(P)$ on the white noise space $(S'(\mathbb{R}), \mathcal{B}, P)$.

4.1 Singular Generalized Stochastic Processes

In Section 2.7 we presented a survey on generalization of stochastic processes and categorized the generalized stochastic processes (also known as generalized stochastic processes of type (O)) as measurable mappings from $\mathbb{R}$ into some q-weighted space of generalized functions i.e. measurable mappings $\mathbb{R} \to (Q)_{-1}^P$ and provided a version of chaos expansion representation of such processes. These processes are generalized in ω but not in t.

Since generalized stochastic processes with values in $(Q)_{-1}^P$ are defined pointwise with respect to the parameter $t \in \mathbb{R}$, their chaos expansion is given by

$$F_t(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha(t) K_\alpha(\omega), \quad t \in \mathbb{R}$$

where $f_\alpha : \mathbb{R} \to \mathbb{R}, \alpha \in \mathcal{I}$ are measurable functions, such that there exists $p \in \mathbb{N}_0$ such that for all $t \in \mathbb{R}$

$$\|F_t\|^2_{-\rho,-p} = \sum_{\alpha \in \mathcal{I}} |f_\alpha(t)|^2 q_\alpha^{-p} < \infty.$$

Now, we define *singular generalized stochastic processes* as linear and continuous mappings from some deterministic space of distributions into the space of q-weighted generalized functions $(Q)_{-1}^P$.

4.1.1 Chaos expansion of singular generalized stochastic process

Let X be a topological vector space and X' its dual. The most common examples used in applications are Schwartz spaces $S(\mathbb{R})$ and $S'(\mathbb{R})$, distributions with compact support $X = \mathcal{E}(\mathbb{R})$ and $X' = \mathcal{E}'(\mathbb{R})$, the Sobolev spaces $X = W_0^{1,2}(\mathbb{R})$ and $X' = W^{-1,2}(\mathbb{R})$, and essentially bounded functions $X = L^\infty(I)$, where $I \subseteq \mathbb{R}$ has finite Lebesgue measure.

We extend the Wiener-Itô chaos expansion theorem to the class of singular generalized stochastic processes as it was done in [56]. The definition 2.7.1 of generalized stochastic processes is now generalized in the sense that coefficients in (4.1) can be also deterministic generalized functions. Processes of such type are generalized by both arguments, t and ω, and they do not have values in fixed points. We can only see their action on appropriate test functions.

Definition 4.1.1 Singular generalized stochastic processes *are linear and continuous mappings from* X *into the space of q-weighted generalized functions* $(Q)_{-1}^P$ *i.e. elements of* $\mathcal{L}(X, (Q)_{-1}^P)$.

If at least one of the spaces X or $(Q)_{-1}^P$ is nuclear, then

$$\mathcal{L}(X, (Q)_{-1}^P) \cong X' \otimes (Q)_{-1}^P, \tag{4.1}$$

and thus one can consider singular generalized processes as elements of the space $X' \otimes (Q)_{-1}^P$. The Kondratiev space $(S)_{-1}$ and the space of stochastic distributions of exponential growth $\exp(S)_{-1}$ are nuclear and consequently in these cases we have isomorphisms $(X \otimes (S)_{-1})' \cong X' \otimes (S)_{-1}$ and $(X \otimes \exp(S)_{-1})' \cong X' \otimes \exp(S)_{-1}$. Thus one can consider stochastic processes also as elements of the spaces $X \otimes (S)_{-1}$ and $X \otimes \exp(S)_{-1}$ respectively.

The chaos expansion theorems for a class of generalized stochastic processes which belong to $X \otimes (Q)_{-1}^P$ are given by the following statements, proved in [56].

Theorem 4.1.1 ([56]) *Let* X *be a Banach space endowed with* $\|\cdot\|_X$. *Singular generalized stochastic processes as elements of* $X \otimes (Q)_{-1}^P$ *have a chaos expansion of the form*

$$u = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes K_\alpha, \quad f_\alpha \in X, \alpha \in \mathcal{I} \tag{4.2}$$

and there exists $p \in \mathbb{N}_0$ *such that*

$$||u||^2_{X \otimes (Q)^P_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X q_\alpha^{-p} < \infty. \tag{4.3}$$

Example 4.1.1 *Brownian motion defined in* (2.50) *and fractional Brownian motion defined in* (2.73), *as well as the Poissonian process defined in* (2.52) *and fractional Poissonian process* (2.79) *are regular generalized stochastic processes, i.e. elements of the space* $X \otimes (Q)_{-1}^P$ *where* $X = C^\infty([0, +\infty))$. *White noise* (2.51), *fractional white noise* (2.76), *Poissonian noise* (2.53) *and fractional Poissonian noise* (2.81) *are singular generalized stochastic processes, i.e. elements of the space* $X \otimes (Q)_{-1}^P$ *where* $X = S'(\mathbb{R})$.

Theorem 4.1.2 ([56]) *Let $X = \bigcap_{k=0}^{\infty} X_k$ be a nuclear space endowed with a family of seminorms $\{\|\cdot\|_k; k \in \mathbb{N}_0\}$ and let $X' = \bigcup_{k=0}^{\infty} X_{-k}$ be its topological dual. Singular generalized stochastic processes as elements of $X' \otimes (Q)_{-1}^{P}$ have a chaos expansion of the form*

$$u = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes K_\alpha, \quad f_\alpha \in X_{-k},\ \alpha \in \mathcal{I}, \tag{4.4}$$

where $k \in \mathbb{N}_0$ does not depend on $\alpha \in \mathcal{I}$, and there exists $p \in \mathbb{N}_0$ such that

$$||u||^2_{X' \otimes (Q)^P_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_{-k} q_\alpha^{-p} < \infty.$$

The action of a singular generalized stochastic process u, represented in the form (4.4), on a test function $\varphi \in X$ gives a generalized random variable from q-weighed space $(Q)_{-1}^{P}$

$$\ll u, \varphi \gg = \sum_{\alpha \in \mathcal{I}} \langle f_\alpha, \varphi \rangle\, K_\alpha \in (Q)_{-1}^{P}$$

and the action of such process u onto a test q-weighted random variable $\theta \in (Q)_1^P$ gives a generalized deterministic function in X'

$$\langle u, \theta \rangle = \sum_{\alpha \in \mathcal{I}} \ll K_\alpha, \theta \gg f_\alpha \in X'.$$

In particular, if $X = S(\mathbb{R})$ then for $\theta = \sum_{\beta \in \mathcal{I}} \theta_\beta K_\beta \in S(\mathbb{R})$ the action of process u on θ is given by

$$\begin{aligned}
\langle u, \theta \rangle &= \sum_{\alpha \in \mathcal{I}} f_\alpha \ll K_\alpha, \sum_{\beta \in \mathcal{I}} \theta_\beta K_\beta \gg \\
&= \sum_{\alpha \in \mathcal{I}} \theta_\alpha\, f_\alpha\, \alpha! \\
&= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha!\, \theta_\alpha\, f_{\alpha,k}\, \xi_k(t) \\
&= \sum_{k \in \mathbb{N}} \left(\sum_{\alpha \in \mathcal{I}} \alpha!\, \theta_\alpha\, f_{\alpha,k} \right) \xi_k(t),
\end{aligned}$$

for $f_\alpha = \sum\limits_{k \in \mathbb{N}} f_{\alpha,k} \xi_k(t) \in S'(\mathbb{R})$.

With the same notation as in (4.2) we will denote by $E(u) = f_{(0,0,0,\dots)}$ the *generalized expectation* of the singular process u.

An important case is when $X = L^\infty(I)$, $I \subseteq \mathbb{R}$, $\lambda(I) < \infty$ and $q_\alpha = (2\mathbb{N})^\alpha$, $\alpha \in \mathcal{I}$ in Theorem 4.1.1.

4.1.2 Extension of operators $\mathcal{U}$ and $\mathcal{M}$

We extend the action of the operator $\mathcal{U}$ given by (2.31) and also the action of the operator $\mathcal{M}$ given by (2.72) to the class of singular generalized stochastic processes.

We define $\mathcal{U} : X \otimes (Q)_{-1}^{\mu} \to X \otimes (Q)_{-1}^{\nu}$ such that for every singular generalized stochastic process $\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \in X \otimes (Q)_{-1}^{\mu}$

$$\mathcal{U}\left[\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha\right] = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes C_\alpha, \quad u_\alpha \in X,\, \alpha\in\mathcal{I}. \tag{4.5}$$

For all processes in $X \otimes (Q)_{-1}^{\mu_H}$, represented in the form $\sum_{\alpha\in\mathcal{I}} v_\alpha \otimes \widetilde{\mathcal{H}}_\alpha(\omega)$ we define the operator $\mathcal{M} : X \otimes (Q)_{-1}^{\mu_H} \to X \otimes (Q)_{-1}^{\mu}$ by

$$\mathcal{M}\left[\sum_{\alpha\in\mathcal{I}} v_\alpha \otimes \widetilde{\mathcal{H}}_\alpha\right] = \sum_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha, \quad v_\alpha \in X,\, \alpha\in\mathcal{I}. \tag{4.6}$$

Remark 4.1.1 *Note that* $\mathcal{U}\circ\mathcal{M}^{-1} : X \otimes (Q)_{-1}^{\mu} \to X \otimes (Q)_{-1}^{\nu_H}$ *such that*

$$\mathcal{U}\circ\mathcal{M}^{-1}\left[\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha\right] = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes \widetilde{\mathcal{C}}_\alpha, \quad u_\alpha \in X,\, \alpha\in\mathcal{I}.$$

The same is obtained by action of the operator $\mathcal{M}^{-1}\circ\mathcal{U}$*, which follows from the commutative property* $\mathcal{U}\circ\mathcal{M}^{-1} = \mathcal{M}^{-1}\circ\mathcal{U}$ *(see the Diagram 1).*

4.1.3 Wick product of singular generalized stochastic processes

We generalize the definition of the Wick product of random variables (Definition 2.5.1 in Section 2.5.2) and give the corresponding statement for special type of singular generalized stochastic processes in the way as it is done in [28] and [57].

Definition 4.1.2 *Let* $\rho\in[0,1]$*. Let* $F, G \in X\otimes(Q)_{-\rho}^{P}$ *be singular generalized processes given in chaos expansions of the form* (4.2)*. Assume* X *to be a space closed under the multiplication* $f_\alpha g_\beta$*, for* $f_\alpha, g_\beta \in X$*. Then the* Wick product $F\lozenge G$ *of processes* F *and* G *is defined by*

$$F\lozenge G = \sum_{\gamma\in\mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta\right)\otimes K_\gamma.$$

An important example appears when considering essentially bounded processes. In particular, an essentially bounded singular generalized stochastic process $F \in L^\infty(I) \otimes (S)^P_{-1}$ has an expansion $F(x,\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha(x) \otimes K_\alpha(\omega)$ such that for all $\alpha \in \mathcal{I}$, $f_\alpha \in L^\infty(I)$ and there exists $q \in \mathbb{N}_0$ such that

$$\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^\infty(I)} (2\mathbb{N})^{-q\alpha} < \infty.$$

We denote by T_q the mapping $T_q : L^\infty(I) \otimes (S)^P_{-1,-(q-4)} \to L^\infty(I) \otimes (S)^P_{-1,-\frac{q}{2}}$ defined by

$$T_q(F) = \tilde{F} = \sum_{\alpha\in\mathcal{I}} \sqrt{|f_\alpha(x)|} \otimes K_\alpha(\omega).$$

Due to the nuclearity of the Kondratiev spaces, T_q is a continuous mapping from $L^\infty(I) \otimes (S)^P_{-1,-(q-4)}$ to $L^\infty(I) \otimes (S)^P_{-1,-\frac{q}{2}}$ (for a proof see [57]).

The following lemma, proven in [57], shows that the Wick product is well defined, and that for fixed F the mapping $G \mapsto F\lozenge G$ is continuous. Here we omit the proof.

Lemma 4.1.1 ([57]) *If* $F \in L^\infty(I)\otimes(S)^P_{-1,-(p-4)}$ *and if* $G \in L^2(I)\otimes(S)^P_{-1,-p}$, *then their* Wick product $F\lozenge G$ *given by*

$$F\lozenge G(x,\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha(x) g_\beta(x) \right) \otimes K_\gamma(\omega)$$

is an element of $L^2(I) \otimes (S)_{-1,-p}$. *Moreover, there exists* $C > 0$ *such that:*

$$\|F\lozenge G\|_{L^2(I)\otimes(S)^P_{-1,-p}} \leq C \|\tilde{F}\|^2_{L^\infty(I)\otimes(S)^P_{-1,-\frac{p}{2}}} \|G\|_{L^2(I)\otimes(S)^P_{-1,-p}}.$$

4.1.4 S'-valued singular generalized stochastic process

In [65] and [66] we provided a general setting of vector-valued singular generalized stochastic processes. $S'(\mathbb{R})$-valued generalized random processes are elements of $\widetilde{X} \otimes (Q)^P_{-1}$, where $\widetilde{X} = X \otimes S'(\mathbb{R})$, and are given by chaos expansions of the form

$$\begin{aligned} f &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \mathfrak{e}_k \otimes K_\alpha \\ &= \sum_{\alpha\in\mathcal{I}} b_\alpha \otimes K_\alpha \\ &= \sum_{k\in\mathbb{N}} c_k \otimes \mathfrak{e}_k, \end{aligned} \tag{4.7}$$

where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \mathfrak{e}_k \in X \otimes S'(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{J}} a_{\alpha,k} \otimes K_\alpha \in X \otimes (Q)_{-1}^P$ and $a_{\alpha,k} \in X$. Thus, for some $p, l \in \mathbb{N}_0$,

$$\|f\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(Q)^P_{-1-p}} = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} \|a_{\alpha,k}\|^2_X (2k)^{-l} q_\alpha^{-p} < \infty.$$

In a similar manner one can also consider $\exp S'(\mathbb{R})$-valued singular generalized stochastic processes as elements of $X \otimes \exp S'(\mathbb{R}) \otimes (Q)^P_{-1}$ given by a chaos expansion of the form (4.7), with the convergence condition

$$\|f\|^2_{X\otimes \exp S_{-l}(\mathbb{R})\otimes(Q)^P_{-1-p}} = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} \|a_{\alpha,k}\|^2_X e^{-2kl} q_\alpha^{-p} < \infty,$$

for some $p, l \in \mathbb{N}_0$.

4.2 Malliavin Calculus for Singular Generalized Stochastic Processes

In this part of the thesis we present original results in exploring the properties of generalized operators of Malliavin calculus, the Malliavin derivative, the Skorokhod integral and the Ornstein-Uhlenbeck operator on the set of singular generalized stochastic processes, their chaos expansion representations and applications in some classes of equations. These results are published in [27], [28], [29] and [30] and are achieved in collaboration with Professor Stevan Pilipović and Dora Seleši.

From this section and further on we will consider only the Kondratiev-type spaces $(S)^P_{-\rho}$ and spaces of exponential growing rate $\exp(S)^P_{-\rho}$, $\rho \in [0,1]$, defined by the weights $q_\alpha = (2\mathbb{N})^\alpha$ and $q_\alpha = e^{(2\mathbb{N})^\alpha}$ respectively. We will omit writing the measure P, and denote these spaces $(S)_{-\rho}$ and $\exp(S)_{-\rho}$, since there exist unitary mappings between all four white noise spaces (Diagram 1).

We give now the definitions of the Malliavin derivative and the Skorokhod integral which are slightly more general than in [10], [46], [48], [51]. Instead of setting the domain in a way that the Malliavin derivative and the Skorokhod integral take values in $L^2(P)$, we allow values in $(S)_{-\rho}$ and $\exp(S)_{-\rho}$ and thus obtain a larger domain for both operators.

4.2.1 The Malliavin derivative

Recall, $\varepsilon^{(k)}$ is the kth unit vector, the sequence of zeros with the number 1 as the kth component, for $k \in \mathbb{N}$. Denote by ι the multi-index $\iota = \sum_{k=1}^{\infty} \varepsilon^{(k)} = (1,1,1,\ldots)$. Note that $\iota \notin \mathcal{I}$, but we will use the following convention: for $\alpha \in \mathcal{I}$, define $\alpha - \iota$ as the multi-index with kth component

$$(\alpha - \iota)_k = \begin{cases} \alpha_k - 1, & \alpha_k \geq 2 \\ 0, & \alpha_k \in \{0,1\} \end{cases}.$$

Thus, $\alpha - \iota \in \mathcal{I}$, for all $\alpha \in \mathcal{I}$.

Definition 4.2.1 *Let $u \in X \otimes (S)_{-1}$ be of the form* (4.2)*. If there exists $p \in \mathbb{N}_0$ such that*

$$\sum_{\alpha \in \mathcal{I}} |\alpha|^2 \, \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty, \tag{4.8}$$

then the Malliavin derivative *of u is defined by*

$$\mathbb{D}u = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_k \, f_\alpha \otimes \mathfrak{e}_k \otimes K_{\alpha - \varepsilon^{(k)}}. \tag{4.9}$$

The operator $\mathbb{D}$ is also called the *stochastic gradient* of a singular generalized stochastic process u. The set of processes u such that (4.8) is satisfied is the domain of the Malliavin derivative, which will be denoted by $Dom(\mathbb{D})$. All processes which belong to $Dom(\mathbb{D})$ are called *differentiable* in Malliavin sense.

We characterize separately the domains of the Malliavin derivative of singular generalized processes which are elements of spaces $X \otimes (S)_{-1}$ and $X \otimes \exp(S)_{-1}$. The following theorem describes the domain of $\mathbb{D}$ in $X \otimes (S)_{-1}$ as it was done in [27].

Theorem 4.2.1 ([27]) *Let $u \in X \otimes (S)_{-1}$ be a Malliavin differentiable singular process. Then the Malliavin derivative $\mathbb{D}$ is a linear and continuous mapping*

$$\mathbb{D}: \; Dom(\mathbb{D}) \subseteq X \otimes (S)_{-1,-p} \to X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p},$$

for some $p \in \mathbb{N}_0$ and $l > p+1$, $l \in \mathbb{N}$.

Proof. Assume that a process u is of the form (4.2) satisfying the condition (4.9). Note $(2\mathbb{N})^{\varepsilon^{(k)}} = (2k)$ and $\|\mathfrak{e}_k\|^2_{-l} = (2k)^{-l}$. Thus we have

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k=1}^{\infty} \alpha_k f_\alpha \otimes \mathfrak{e}_k\|^2_{X\otimes S_{-l}} (2\mathbb{N})^{-p(\alpha-\varepsilon^{(k)})}\\
&\leq \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} \alpha_k^2 \cdot \sum_{k\in\mathbb{N}} \|\mathfrak{e}_k\|^2_{-l}\,(2k)^p \right) \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha}\\
&\leq \sum_{\alpha\in\mathcal{I}} |\alpha|^2 (\sum_{k\in\mathbb{N}} (2k)^{-l+p})\, \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha}\\
&\leq C \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty,
\end{aligned}$$

where $\sum_{k\in\mathbb{N}} (2k)^{-l+p} = C$ for $l > p+1$. □

Following [29] we define the Malliavin derivative of singular generalized processes from $X \otimes \exp(S)_{-1}$ and characterize the domain of such operator.

Definition 4.2.2 *Let a singular process $u \in X \otimes \exp(S)_{-1}$ be of the form* (4.2). *If there exists $p \in \mathbb{N}_0$ such that*

$$\sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|f_\alpha\|^2_X\, e^{-p(2\mathbb{N})^{\alpha-\iota}} < \infty \tag{4.10}$$

then process u is differentiable in Malliavin sense *i.e. $u \in Dom_{exp}(\mathbb{D})$ and the Malliavin derivative of u is defined by* (4.9).

Theorem 4.2.2 ([29]) *Consider a process $u \in X \otimes \exp(S)_{-1}$. Then the Malliavin derivative of u is a linear and continuous mapping*

$$\mathbb{D}: \; Dom_{exp}(\mathbb{D}) \subseteq X \otimes \exp(S)_{-1,-p} \to X \otimes \exp S_{-l}(\mathbb{R}) \otimes \exp(S)_{-1,-p},$$

for all $l \in \mathbb{N}_0$.

Proof. Clearly, from $\|\mathfrak{e}_k\|^2_{\exp,-l} = e^{-2kl}$ for all $k, l \in \mathbb{N}$ and (4.10) we have

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes\exp S_{-l}(\mathbb{R})\otimes\exp(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k=1}^{\infty} \alpha_k f_\alpha \otimes \mathfrak{e}_k\|^2_{X\otimes\exp S_{-l}(\mathbb{R}^n)} e^{-p(2\mathbb{N})^{(\alpha-\varepsilon^{(k)})}}\\
&\leq \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k^2\, e^{-2kl}\, \|f_\alpha\|^2_X e^{-p(2\mathbb{N})^{\alpha-\iota}}\\
&\leq \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|f_\alpha\|^2_X e^{-p(2\mathbb{N})^{\alpha-\iota}} < \infty.
\end{aligned}$$

□

Remark 4.2.1 *Note that $Dom_{exp}(\mathbb{D}) \supseteq Dom(\mathbb{D})$.*

Let now $\rho \in [0,1]$. Consider the Kondratiev spaces $(S)_{-\rho}$ and also the spaces of exponential growth $\exp(S)_{-\rho}$ defined in the Section 2.5. Recall that inclusion $(S)_{-\rho} \subseteq \exp(S)_{-\rho}$ is continuous.

Definition 4.2.3 *Let a singular generalized stochastic process $u \in X \otimes (S)_{-\rho}$ be of the form* (4.2). *We say that u belongs to $Dom(\mathbb{D})_{-\rho,-p}$ if there exists $p \in \mathbb{N}_0$ such that*

$$\sum_{\alpha \in \mathcal{I}} |\alpha|^{1+\rho} (\alpha!)^{1-\rho} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{4.11}$$

Then the process u is Malliavin differentiable *and its Malliavin derivative is given by* (4.9).

Note that if a process $u \in X \otimes (S)_{-\rho}$ then there exists $p \in \mathbb{N}_0$ such that

$$\|u\|^2_{X\otimes(S)_{-\rho,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_X^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-p\alpha}$$

is finite. We proceed with proving the statement that the Malliavin derivative is a continuous operator on the set of processes from $X \otimes (S)_{-\rho}$.

Theorem 4.2.3 *The Malliavin derivative of a process $u \in X \otimes (S)_{-\rho}$ is a linear and continuous mapping*

$$\mathbb{D}: \ Dom(\mathbb{D})_{-\rho,-p} \subseteq X \otimes (S)_{-\rho,-p} \to X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p},$$

for $l > p+1$ and $p \in \mathbb{N}_0$.

Proof. We use the property $(\alpha - \varepsilon^{(k)})! = \frac{\alpha!}{\alpha_k}$, for $k \in \mathbb{N}$ in the proof of this theorem. Assume that a singular process is of the form (4.2) such that it satisfies (4.11) for some $p \geq 0$. Then we have

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}} &= \sum_{\alpha\in\mathcal{I}} \| \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \otimes \mathfrak{e}_k \|^2_{X\otimes(S)_{-\rho,-p}} (2\mathbb{N})^{-p\alpha+p\varepsilon^{(k)}} \\
&\leq \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} \alpha_k^2 (\alpha - \varepsilon^{(k)})!^{1-\rho} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p(\alpha-\varepsilon^{(k)})} (2k)^{-l} \\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} \alpha_k^2 \left(\frac{\alpha!}{\alpha_k}\right)^{1-\rho} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} (2k)^{-(l-p)} \\
&\leq C \sum_{\alpha\in\mathcal{I}} \left(\sum_{k=1}^{\infty} \alpha_k\right)^{1+\rho} (\alpha!)^{1-\rho} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} \\
&= C \sum_{\alpha\in\mathcal{I}} |\alpha|^{1+\rho} (\alpha!)^{1-\rho} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty,
\end{aligned}$$

where $C = \sum_{k=1}^{\infty}(2k)^{-(l-p)} < \infty$ for $l > p+1$. □

It is clear that when $\rho = 1$ the result of the previous theorem reduces to the corresponding one in Theorem 4.2.1. We formulate now an analogue theorem for a class of singular generalized processes belonging to $X \otimes \exp(S)_{-\rho}$. Recall, if a process $u \in X \otimes \exp(S)_{-\rho}$ then it can be decomposed in the way (4.2) such that the condition

$$\sum_{\alpha\in\mathcal{I}}(\alpha!)^{1-\rho}\|f_\alpha\|_X^2\, e^{-p(2\mathbb{N})^\alpha} < \infty \tag{4.12}$$

is fulfilled for some $p \in \mathbb{N}_0$.

Definition 4.2.4 *We say that a given singular generalized stochastic process $u \in X \otimes \exp(S)_{-\rho}$ is Malliavin differentiable and write $u \in Dom_{\exp}(\mathbb{D})_{-\rho}$ if it satisfies the condition*

$$\sum_{\alpha\in\mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\|f_\alpha\|_X^2\, e^{-p(2\mathbb{N})^{\alpha-\iota}} < \infty \tag{4.13}$$

for some $p \in \mathbb{N}_0$. Thus the chaos expansion of its Malliavin derivative is given by (4.9).

Theorem 4.2.4 *The Malliavin derivative of a singular generalized stochastic process $u \in X \otimes \exp(S)_{-\rho,-p}$, $p \in \mathbb{N}_0$ is a linear and continuous mapping*

$$\mathbb{D}:\ X \otimes \exp(S)_{-\rho,-p} \to X \otimes \exp S_{-l}(\mathbb{R}) \otimes \exp(S)_{-\rho,-p},$$

for all $l \in \mathbb{N}$.

Proof. Clearly, from $\|\mathfrak{e}_k\|^2_{\exp,-l} = e^{-2kl}$ for all $k, l \in \mathbb{N}$ and (4.13) it follows that

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes\exp S_{-l}(\mathbb{R})\otimes\exp(S)_{-\rho,-p}} &\le \sum_{\alpha\in\mathcal{I}}\sum_{k=1}^{\infty}\alpha_k^2(\alpha-\varepsilon^{(k)})!^{1-\rho}\|f_\alpha\|_X^2 e^{-p(2\mathbb{N})^{\alpha-\varepsilon^{(k)}}}e^{-2kl}\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k=1}^{\infty}\alpha_k^2\left(\frac{\alpha!}{\alpha_k}\right)^{1-\rho}\|f_\alpha\|_X^2 e^{-p(2\mathbb{N})^{\alpha-\varepsilon^{(k)}}}e^{-2kl}\\
&\le C\sum_{\alpha\in\mathcal{I}}\left(\sum_{k=1}^{\infty}\alpha_k\right)^{1+\rho}(\alpha!)^{1-\rho}\|f_\alpha\|_X^2 e^{-p(2\mathbb{N})^{\alpha-\varepsilon^{(k)}}}\\
&\le C\sum_{\alpha\in\mathcal{I}}|\alpha|^{1+\rho}(\alpha!)^{1-\rho}\|f_\alpha\|_X^2 e^{-p(2\mathbb{N})^{\alpha-\iota}} < \infty,
\end{aligned}$$

where $C = \sum\limits_{k=1}^{\infty} e^{-2kl} < \infty$ for all $l \in \mathbb{N}$. □

When comparing the families of the Malliavin Sobolev type of spaces $Dom(\mathbb{D})_{-\rho}$ and $Dom_{\exp}(\mathbb{D})_{-\rho}$ for different values $\rho \in [0,1]$ the following properties arise:

- If $p < q$ then $Dom(\mathbb{D})_{-\rho,-p} \subseteq Dom(\mathbb{D})_{-\rho,-q}$.
- The inclusion $Dom(\mathbb{D})_{-\rho} \subseteq Dom_{\exp}(\mathbb{D})_{-\rho}$ is satisfied for all $\rho \in [0,1]$.
- Note $Dom(\mathbb{D})_{-1} = Dom(\mathbb{D})$ and $Dom_{\exp}(\mathbb{D})_{-1} = Dom_{\exp}(\mathbb{D})$.
- For all $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{I}$ we have

$$|\alpha| = \sum_{k \in \mathbb{N}} \alpha_k \ < \ \alpha! = \prod_{k \in \mathbb{N}} \alpha_k, \quad \alpha_k \in \mathbb{N}.$$

  Thus, the smallest domains of the spaces $Dom(\mathbb{D})_{-\rho}$ and $Dom_{\exp}(\mathbb{D})_{-\rho}$ are obtained for $\rho = 0$ and the largest domains are obtained for $\rho = 1$. In particular we have inclusions

$$Dom(\mathbb{D})_{-0} \subset Dom(\mathbb{D})_{-1} \subseteq Dom_{\exp}(\mathbb{D})_{-0} \subset Dom_{\exp}(\mathbb{D})_{-1}.$$

4.2.2 The Skorokhod integral

Motivated by the identity (2.59) for the Skorokhod integral of an H-valued generalized random variables, we extend the definition of the Skorokhod integral to the class of singular generalized processes. As an adjoint operator of the Malliavin derivative the Skorokhod integral is defined as follows.

Definition 4.2.5 *Let $F = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes v_\alpha \otimes K_\alpha \in X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-p}$, $p \in \mathbb{N}_0$ be a singular generalized $S_{-p}(\mathbb{R})$-valued stochastic process and let $v_\alpha \in S_{-p}(\mathbb{R})$ be given by the expansion $v_\alpha = \sum_{k \in \mathbb{N}} v_{\alpha,k}\, \mathfrak{e}_k$, $v_{\alpha,k} \in \mathbb{R}$. Then the process F is* integrable in the Skorokhod sense *and the chaos expansion of its stochastic integral is given by*

$$\delta(F) = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} v_{\alpha,k}\, f_\alpha \otimes K_{\alpha+\varepsilon^{(k)}}. \tag{4.14}$$

Next theorem is proved in [27].

Theorem 4.2.5 ([27]) *The Skorokhod integral* δ *of a* $S_{-p}(\mathbb{R})$*-valued singular generalized stochastic process is a linear and continuous mapping*

$$\delta : X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-p} \to X \otimes (S)_{-1,-p}.$$

Proof. Clearly,

$$\begin{aligned} \|\delta(F)\|^2_{X\otimes(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} v_{\alpha,k}\, f_\alpha\|^2_X (2\mathbb{N})^{-p(\alpha+\varepsilon^{(k)})} \\ &\leq \sum_{\alpha\in\mathcal{I}} (\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\,(2k)^{-p})\|f_\alpha\|^2_X(2\mathbb{N})^{-p\alpha} \\ &= \sum_{\alpha\in\mathcal{I}} \|v_\alpha\|^2_{-p}\,\|f_\alpha\|^2_X\,(2\mathbb{N})^{-p\alpha} < \infty, \end{aligned}$$

because $F \in X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-p}$, $p \in \mathbb{N}_0$. □

Definition 4.2.6 *Let a singular generalized* $\exp S_{-p}(\mathbb{R})$*-valued stochastic process be of the form*

$$F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes v_\alpha \otimes K_\alpha \;\in\; X \otimes \exp S_{-p}(\mathbb{R}) \otimes \exp(S)_{-1,-p},$$

for some $p \in \mathbb{N}_0$ *and let* $v_\alpha \in \exp S_{-p}(\mathbb{R})$ *be given by the expansion* $v_\alpha = \sum_{k\in\mathbb{N}} v_{\alpha,k}\,\mathfrak{e}_k$, $v_{\alpha,k} \in \mathbb{R}$. *Then the process* u *is integrable in the Skorokhod sense and* $\delta(F)$ *is defined by* (4.14).

The proof of the following theorem can be found in [29].

Theorem 4.2.6 ([29]) *Let* $F \in X \otimes \exp S_{-p}(\mathbb{R}) \otimes \exp(S)_{-1,-p}$ *be an* $\exp S_{-p}(\mathbb{R})$*-valued singular generalized process for some* $p > 0$. *Then the Skorokhod integral* δ *of* F *is a linear and continuous mapping*

$$\delta : X \otimes \exp S_{-p}(\mathbb{R}) \otimes \exp(S)_{-1,-p} \to X \otimes \exp(S)_{-1,-p}.$$

Proof. This assertion follows from the inequality

$$e^{-p(2\mathbb{N})^\alpha\,(2k)} \leq e^{-2kp} \cdot e^{-p(2\mathbb{N})^\alpha} \tag{4.15}$$

valid for $\alpha \in \mathcal{I}$ and $k, p \geq 0$. Clearly,

$$\begin{aligned} \|\delta(F)\|^2_{X\otimes\exp(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} v_{\alpha,k}\, f_\alpha\|^2_X e^{-p(2\mathbb{N})^{\alpha+\varepsilon^{(k)}}} \\ &\leq \sum_{\alpha\in\mathcal{I}} (\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\, e^{-2kp})\|f_\alpha\|^2_X e^{-p(2\mathbb{N})^\alpha} \\ &= \sum_{\alpha\in\mathcal{I}} \|v_\alpha\|^2_{\exp,-p}\,\|f_\alpha\|^2_X\, e^{-p(2\mathbb{N})^\alpha} < \infty, \end{aligned}$$

since $F \in X \otimes \exp S_{-p}(\mathbb{R}) \otimes \exp(S)_{-1,-p}$, for $p \in \mathbb{N}_0$. □

From $2(2\mathbb{N})^{\alpha} \leq (2\mathbb{N})^{2\alpha}$ we conclude that the image of the Malliavin derivative is included in the domain of the Skorokhod integral and thus we can define their composition.

Both theorems, Theorem 4.2.5 and Theorem 4.2.6 can be stated for singular generalized stochastic processes which have values in the Kondratiev space $(S)_{-\rho}$ respectively in the space with exponential growing rate $\exp(S)_{-\rho}$, for any $\rho \in [0,1]$.

Theorem 4.2.7 *Let $\rho \in [0,1]$. The Skorokhod integral δ of a $S_{-q}(\mathbb{R})$-valued singular generalized stochastic process is a linear and continuous mapping*

$$\delta : X \otimes S_{-q}(\mathbb{R}) \otimes (S)_{-\rho,-p} \to X \otimes (S)_{-\rho,-(q+1-\rho)}, \quad \text{for } q-p>1.$$

Proof. This statement follows from inequalities $(\alpha_k+1) \leq |\alpha+\varepsilon^{(k)}| \leq (2\mathbb{N})^{\alpha+\varepsilon^{(k)}}$, when $\alpha \in \mathcal{I}$, $k \in \mathbb{N}$. Clearly, we have

$$\begin{aligned}
\|\delta(F)\|^2_{X\otimes(S)_{-\rho,-p}} &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\, \|f_\alpha\|^2_X (2\mathbb{N})^{-(q+1-\rho)(\alpha+\varepsilon^{(k)})}\, (\alpha+\varepsilon^{(k)})!^{1-\rho} \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\, \|f_\alpha\|^2_X (2\mathbb{N})^{-(q+1-\rho)(\alpha+\varepsilon^{(k)})}\, (\alpha_k+1)^{1-\rho}\, \alpha!^{1-\rho} \\
&\leq \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\, \|f_\alpha\|^2_X (2\mathbb{N})^{-(q+1-\rho)(\alpha+\varepsilon^{(k)})}\, (2\mathbb{N})^{(1-\rho)(\alpha+\varepsilon^{(k)})}\, \alpha!^{1-\rho} \\
&\leq \sum_{\alpha\in\mathcal{I}}\Big(\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\, (2k)^{-q}\Big)\|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha}\alpha!^{1-\rho}\, (2\mathbb{N})^{-(q-p)\alpha} \\
&\leq C \sum_{\alpha\in\mathcal{I}} \|v_\alpha\|^2_{-q}\, \|f_\alpha\|^2_X\, (2\mathbb{N})^{-p\alpha}\, \alpha!^{1-\rho} < \infty,
\end{aligned}$$

because $F \in X \otimes S_{-q}(\mathbb{R}) \otimes (S)_{-\rho,-p}$ and $C = \sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-(q-p)\alpha}$ is a finite constant for $q-p>1$. □

Theorem 4.2.8 *Let $F \in X \otimes \exp S_{-p}(\mathbb{R}) \otimes \exp(S)_{-\rho,-q}$ be an $\exp S_{-p}(\mathbb{R})$-valued singular generalized process for some $p>0$ and $q>0$. Then the Skorokhod integral δ of F is a linear and continuous mapping*

$$\delta : X \otimes \exp S_{-p}(\mathbb{R}) \otimes \exp(S)_{-\rho,-q} \to X \otimes \exp(S)_{-\rho,-l},$$

when $l-q>1-\rho$ and $l-p>1-\rho$.

Proof. From (4.15) and inequality $(\alpha_k+1)\leq e^{2k}\,e^{(2\mathbb{N})^{\alpha}}$, $\alpha\in\mathcal{I}$, $k\in\mathbb{N}$ it follows that

$$\begin{aligned}
\|\delta(F)\|^2_{X\otimes\exp(S)_{-\rho,-l}} &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} v^2_{\alpha,k}\,\|f_\alpha\|^2_X\,(\alpha+\varepsilon^{(k)})!^{1-\rho}\,e^{-l(2\mathbb{N})^{\alpha+\varepsilon^{(k)}}}\\
&\leq \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \|v_\alpha\|^2_{\exp,-p}\,e^{2kp}\,\|f_\alpha\|^2_X e^{-(l+\rho-1)(2\mathbb{N})^{\alpha}}(\alpha!)^{1-\rho}\,e^{-2k(l+\rho-1)}\\
&\leq C_1\,C_2\sum_{\alpha\in\mathcal{I}}\|v_\alpha\|^2_{\exp,-p}\,\|f_\alpha\|^2_X\,e^{-q(2\mathbb{N})^{\alpha}}\,(\alpha!)^{1-\rho}<\infty,
\end{aligned}$$

for finite constants $C_1=\sum_{k\in\mathbb{N}}e^{-2k(l+\rho-1-p)}<\infty$, when $1-\rho>l-p$ and $C_2=\sum_{\alpha\in\mathcal{I}}e^{-(l+\rho-1-q)(2\mathbb{N})^{\alpha}}<\infty$ when $l-q>1-\rho$. □

4.2.3 The Ornstein-Uhlenbeck operator

Definition 4.2.7 *The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathcal{R}=\delta\circ\mathbb{D}$ and called the Ornstein-Uhlenbeck operator.*

The Fourier-Hermite i.e. the Charlier polynomials are eigenfunctions of $\mathcal{R}$ and the corresponding eigenvalues are $|\alpha|$, $\alpha\in\mathcal{I}$, i.e.

$$\mathcal{R}(K_\alpha)=|\alpha|K_\alpha.$$

Moreover, if we apply the previous identity k times successively, we obtain

$$\mathcal{R}^k(K_\alpha)=|\alpha|^kK_\alpha,\quad k\in\mathbb{N},\ \text{for }\alpha\in\mathcal{I}.$$

Theorem 4.2.9 *Let a singular generalized stochastic process $u\in Dom(\mathbb{D})$ be given by the chaos expansion $u=\sum_{\alpha\in\mathcal{I}}u_\alpha\otimes K_\alpha$, $u_\alpha\in X$. Then*

$$\mathcal{R}u=\sum_{\alpha\in\mathcal{I}}|\alpha|u_\alpha\otimes K_\alpha. \tag{4.16}$$

Denote by

$$Dom(\mathcal{R})=\{u\in X\otimes(S)_{-1}:\ \exists p\in\mathbb{N}_0,\ \sum_{\alpha\in\mathcal{I}}|\alpha|^2\|u_\alpha\|^2_X(2\mathbb{N})^{-p\alpha}<\infty\}$$

and

$$Dom_{exp}(\mathcal{R})=\{u\in X\otimes\exp(S)_{-1}:\ \exists p\in\mathbb{N}_0,\ \sum_{\alpha\in\mathcal{I}}|\alpha|^2\|u_\alpha\|^2_Xe^{-p(2\mathbb{N})^{\alpha}}<\infty\}.$$

Theorem 4.2.10 ([29]) *The operator $\mathcal{R}$ is a linear and continuous mapping from $Dom(\mathcal{R}) \subset X \otimes (S)_{-1}$ into the space $X \otimes (S)_{-1}$, and in this case the domains of $\mathbb{D}$ and $\mathcal{R}$ coincide, i.e. $Dom(\mathcal{R}) = Dom(\mathbb{D})$.*

Proof. Clearly, if $u \in Dom(\mathbb{D}) \subset X \otimes (S)_{-1,-p}$ then $\mathcal{R}u \in X \otimes (S)_{-1,-p}$. This follows from (4.16) and

$$\|\mathcal{R}u\|^2_{X\otimes(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{J}} |\alpha|^2 \, \|u_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} = \|u\|^2_{Dom(\mathbb{D})} < \infty.$$

□

For $u \in X \otimes \exp(S)_{-1,-p}$ it follows that

$$Dom_{exp}(\mathbb{D}) \subseteq Dom_{exp}(\mathcal{R}).$$

Now we consider the Kondratiev type of q-weighted spaces and characterize the Ornstein-Uhlenbeck operator in the sense of the previous statement. Let $\rho \in [0,1]$. We define the domain $Dom(\mathcal{R})_{-\rho}$ to be the set of all processes $u \in X \otimes (S)_{-\rho}$ represented in the form (4.2) such that the condition

$$\sum_{\alpha\in\mathcal{J}} |\alpha|^2 (\alpha!)^{1-\rho} \|u_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty$$

is satisfied for some $p \in \mathbb{N}_0$. Furthermore we define the domain $Dom_{exp}(\mathcal{R})_{-\rho}$ to be the set of all processes $u \in X \otimes \exp(S)_{-\rho}$, having the chaos expansion of the form (4.2) and satisfying the condition

$$\sum_{\alpha\in\mathcal{J}} |\alpha|^2 (\alpha!)^{1-\rho} \|u_\alpha\|^2_X e^{-p(2\mathbb{N})^\alpha} < \infty$$

for some $p \in \mathbb{N}_0$.

Theorem 4.2.11 *Let $\rho \in [0,1]$. Then we have the following inclusions*

$$Dom(\mathcal{R})_{-\rho} \subseteq Dom(\mathbb{D})_{-\rho} \subseteq Dom_{exp}(\mathbb{D})_{-\rho} \subseteq Dom_{exp}(\mathcal{R})_{-\rho}.$$

We have already seen that for $\rho = 1$ spaces $Dom(\mathcal{R})_{-\rho}$ and $Dom(\mathbb{D})_{-\rho}$ coincide.

4.3 Operators of Fractional Malliavin Calculus

Consider the extension of the operator M from $S'(\mathbb{R}) \to S'(\mathbb{R})$ onto generalized stochastic processes.

Definition 4.3.1 *Let* $\mathbf{M} = M \otimes Id : S'(\mathbb{R}) \otimes (Q)^P_{-1} \to S'(\mathbb{R}) \otimes (Q)^P_{-1}$ *be given by*

$$\mathbf{M}\left(\sum_{\alpha\in\mathcal{I}} a_\alpha(t) \otimes K_\alpha(\omega)\right) = \sum_{\alpha\in\mathcal{I}} M a_\alpha(t) \otimes K_\alpha(\omega). \tag{4.17}$$

Its restriction to $L^2_H(\mathbb{R}) \otimes L^2(P)$ is an isometric mapping

$$L^2_H(\mathbb{R}) \otimes L^2(P) \to L^2(\mathbb{R}) \otimes L^2(P).$$

Example 4.3.1 *In Example 2.8.3 and Example 2.8.4 we have seen that*

$$B^{(H)}_t = \mathbf{M}B_t \;\; in \;\; L^2(\mu) \quad and \quad W^{(H)}_t = \mathbf{M}W_t \;\; in \;\; (S)^\mu_{-1}.$$

In [51], the fractional Malliavin derivative in $L^2(\mu)$ was defined as

$$\mathbb{D}^{(H)} = \mathbf{M}^{-1} \circ \mathbb{D}.$$

Thus, in [29] we extended this notion to a class of singular generalized stochastic processes. For example, on the Kondratiev white noise spaces with Gaussian measure

$$\mathbb{D}^{(H)} : X \otimes (S)^\mu_{-1} \to X \otimes S'(\mathbb{R}) \otimes (S)^\mu_{-1}$$

is given by

$$\begin{aligned} \mathbb{D}^{(H)} F &= \mathbf{M}^{-1} \circ \mathbb{D}\, F \\ &= \mathbf{M}^{-1}\left(\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon^{(k)}}\right) \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes e_k \otimes H_{\alpha-\varepsilon^{(k)}}, \end{aligned} \tag{4.18}$$

for $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha$, $f_\alpha \in X$, $\alpha \in \mathcal{I}$. Note that the domain of the fractional Malliavin derivative coincides with the domain of the classical Malliavin derivative. The following definition holds on a general white noise space (Gaussian, Poissonian, fractional Gaussian or fractional Poissonian).

Definition 4.3.2 *Let* $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha \in X \otimes (S)_{-1}$*, respectively* $X \otimes \exp(S)_{-1}$*. If* $F \in Dom(\mathbb{D})$*, respectively* $F \in Dom_{exp}(\mathbb{D})$*, then the* fractional Malliavin derivative *of* F *is defined by*

$$\mathbb{D}^{(H)} F = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes M^{-1}\mathfrak{e}_k \otimes K_{\alpha-\varepsilon^{(k)}}. \tag{4.19}$$

In the following theorem, P will denote either the Gaussian or Poissonian measure, and P_H will denote their corresponding fractional measures. The notation $(Q)_{-1}$ will refer to either $(S)_{-1}$ or $\exp(S)_{-1}$ with the appropriate measure.

Theorem 4.3.1 ([29]) *Let* $\mathbb{D}$ *and* $\mathbb{D}^{(H)}$ *denote the Malliavin derivative, respectively the fractional Malliavin derivative on* $X \otimes (Q)_{-1}^{P}$*. Let* $\widetilde{\mathbb{D}}$ *denote the Malliavin derivative on* $X \otimes (Q)_{-1}^{P_H}$*. Then,*

$$\mathbb{D}^{(H)} F = \mathbf{M}^{-1} \circ \mathbb{D}\, F = \mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1}\, F, \tag{4.20}$$

for all $F \in Dom(\mathbb{D})$*.*

Proof. We will conduct the proof for the Gaussian case. Since $D^{(H)}F = \mathbf{M}^{-1} \circ \mathbb{D}F$ follows directly from (4.17) and (4.19), we need to prove that (4.18) is equal to $\mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1}F$, where $\widetilde{\mathbb{D}}$ stands for the Malliavin derivative in $L^2(\mu_H)$. Clearly,

$$\begin{aligned}
\mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1}\left(\sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha\right) &= \mathcal{M} \circ \widetilde{\mathbb{D}}\left(\sum_{\alpha\in\mathcal{I}} f_\alpha \otimes \widetilde{\mathcal{H}}_\alpha\right)\\
&= \mathcal{M}\left(\sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes e_k \otimes \widetilde{\mathcal{H}}_{\alpha-\varepsilon^{(k)}}\right)\\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes e_k \otimes H_{\alpha-\varepsilon^{(k)}}.
\end{aligned}$$

□

Example 4.3.2 *It is well known that in* $L^2(\mu)$*, the Malliavin derivative of Brownian motion is* $\mathbb{D}B_t(\omega) = \chi[0,t] = \sum_{k=1}^{\infty} c_k \xi_k$*. Thus,*

$$\mathbb{D}^{(H)} B_t(\omega) = M^{-1}\chi[0,t] = M^{(1-H)}(0,t) = \sum_{k=1}^{\infty} c_k e_k,$$

where

$$\begin{aligned}
c_k &= (\xi_k, \chi[0,t])_{L^2(\mathbb{R})} = (M^{-1}\xi_k, M^{-1}\chi[0,t])_{L^2_{1-H}(\mathbb{R})}\\
&= (e_k, M^{(1-H)}(0,t))_{L^2_{1-H}(\mathbb{R})}.
\end{aligned}$$

Definition 4.3.3 *Let* $\delta : X \otimes S'(\mathbb{R}) \otimes (Q)^P_{-1} \to X \otimes (Q)^P_{-1}$ *denote the Skorokhod integral in sense of Definition 4.2.5 and Theorem 4.2.6. The* fractional Skorokhod integral $\delta^{(H)} : X \otimes S'(\mathbb{R}) \otimes (Q)^P_{-1} \to X \otimes (Q)^P_{-1}$ *is defined for every* $F \in Dom(\delta)$ *by*

$$\delta^{(H)} F = \delta \circ \mathbf{M}\, F. \tag{4.21}$$

Finally, for the Ornstein-Uhlenbeck operator we note that its fractional version coincides with the regular one, i.e. from (4.20) and (4.21) it follows that

$$\mathcal{R}^{(H)} = \delta^{(H)} \circ \mathbb{D}^{(H)} = \delta \circ \mathbf{M} \circ \mathbf{M}^{-1} \circ \mathbb{D} = \delta \circ \mathbb{D} = \mathcal{R}.$$

Chapter 5

Applications of the Chaos Expansion Method to Some Classes of Equations

In this chapter we present some applications of the chaos expansion method to obtain explicit forms of solutions of some classes of stochastic differential equations involving the Malliavin derivative and the Ornstein-Uhlenbeck operator. We provide a general method of solving stochastic differential equations, also known as the propagator method, first introduced by Boris Rozovsky. The Wiener-Itô chaos decomposition of general random processes which appear in equations is used to set all coefficients in the chaos expansion on the left-hand side of the equation equal to the corresponding coefficients on the right-hand side of the equation. With this method we reduce a problem to an infinite system of deterministic equations. Summing up all coefficients of the expansion and proving convergence in an appropriate weighted space of stochastic distributions, one obtains the solution of the initial equation. The equations presented and solved in this chapter are original results of this thesis and are published in [27], [28], [29] and [30]. Other types of equations investigated by the same method can be found in [26], [31], [35], [56], [67].

All stochastic equations solved in this section can be interpreted, by the use of the isometric transformations $\mathcal{U}$ and $\mathcal{M}$ defined in (4.5) and (4.6), in all four white noise spaces, Gaussian, Poissonian, fractional Gaussian and fractional Poissonian white noise spaces, we have considered so far. Also, due to Theorem 4.3.1 the Malliavin derivative and the Skorokhod integral can be interpreted as their fractional counterparts in the corresponding fractional white noise space. With this argumentation we state the equations and solve them in a white noise space of general type.

5.1 Equations With the Malliavin Derivative

At the beginning we apply the chaos expansion transform in order to solve two equations involving the Malliavin derivative, the first order equation involving the Malliavin derivative, and then a generalized eigenvalue problem for the Malliavin derivative. In both cases, solutions obtained by this approach have a simple form and belong to a certain space of weighted singular generalized stochastic processes.

Denote by $r = r(\alpha) = \min\{k \in \mathbb{N} : \alpha_k \neq 0\}$, for nonzero multi-index $\alpha \in \mathcal{I}$. Then the first nonzero component of α is the rth component α_r, i.e. $\alpha = (0, 0, ..., 0, \alpha_r, ..., \alpha_m, 0, 0, ...)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of α, except the rth, which is $\alpha_r - 1$. We call $\alpha_{\varepsilon^{(r)}}$ the *representative* of α and write

$$\alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)}, \quad \alpha \in \mathcal{I}, \ |\alpha| > 0. \tag{5.1}$$

For example, the first nonzero component of $\alpha = (0, 0, 2, 1, 0, 5, 0, 0, ...)$ is its third component. It follows that $r = 3$, $\alpha_r = 2$ and the representative of α is $\alpha_{\varepsilon^{(r)}} = \alpha - \varepsilon^{(3)} = (0, 0, 1, 1, 0, 5, 0, 0, ...)$.

The set $\mathcal{K}_\alpha = \{\beta \in \mathcal{I} : \alpha = \beta + \varepsilon^{(j)}, \text{ for some } j \in \mathbb{N}\}$, $\alpha \in \mathcal{I}$, $|\alpha| > 0$ is a nonempty set, because $\alpha_{\varepsilon^{(r)}} \in \mathcal{K}_\alpha$. Moreover, if $\alpha = n\varepsilon^{(r)}$, $n \in \mathbb{N}$ then $Card(\mathcal{K}_\alpha) = 1$ and in all other cases $Card(\mathcal{K}_\alpha) > 1$. For example if $\alpha = (0, 1, 3, 0, 0, 5, 0, ...)$, then the set $\mathcal{K}_\alpha$ has three elements $\mathcal{K}_\alpha = \{\alpha_{\varepsilon^{(2)}} = (0, 0, 3, 0, 0, 5, 0, ...), (0, 1, 2, 0, 0, 5, 0, ...), (0, 1, 3, 0, 0, 4, 0, ...)\}$.

5.1.1 A first order equation

Let us consider a first order equation involving the Malliavin derivative i.e. an equation of the form

$$\begin{cases} \mathbb{D}u = h, & h \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1} \\ Eu = \widetilde{u}_0, & \widetilde{u}_0 \in X \end{cases}. \tag{5.2}$$

The next result characterizes the family of stochastic processes that can be written as the Malliavin derivative of some singular stochastic process. A necessary and sufficient condition for existence of a solution is stated and the solution is expressed in its explicit form, the chaos expansion form.

Theorem 5.1.1 ([30]) *Let* $h = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k \otimes K_\alpha \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$, *with coefficients* $h_{\alpha,k} \in X$ *such that*

$$\frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}}, r} = \frac{1}{\alpha_j} h_{\beta, j}\,, \tag{5.3}$$

for the representative $\alpha_{\varepsilon^{(r)}}$ *of* $\alpha \in \mathcal{I}$, $|\alpha| > 0$ *and all* $\beta \in \mathcal{K}_\alpha$, *such that* $\alpha = \beta + \varepsilon^{(j)}$, *for* $j \geq r$, $r \in \mathbb{N}$. *Then, equation* (5.2) *has a unique solution in* $X \otimes (S)_{-1}$. *The chaos expansion of the generalized stochastic process, which represents the unique solution of equation* (5.2) *is given by*

$$u = \widetilde{u}_0 + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}\in\mathcal{I}} \frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}},r} \otimes K_\alpha. \tag{5.4}$$

Proof. We seek the solution in the form $u = \widetilde{u}_0 + \sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} u_\alpha \otimes K_\alpha$. Thus,

$$\mathbb{D}\left(\widetilde{u}_0 + \sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} u_\alpha \otimes K_\alpha\right) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k \otimes K_\alpha$$

$$\sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} \left(\sum_{k\in\mathbb{N}} \alpha_k\, u_\alpha \otimes \mathfrak{e}_k\right) \otimes K_{\alpha-\varepsilon^{(k)}} = \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k\right) \otimes K_\alpha$$

$$\sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} (\alpha_k+1)\, u_{\alpha+\varepsilon^{(k)}} \otimes \mathfrak{e}_k\right) \otimes K_\alpha = \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k\right) \otimes K_\alpha$$

Due to uniqueness of the Wiener-Itô chaos expansion it follows that, for all $\alpha \in \mathcal{I}$

$$\sum_{k\in\mathbb{N}} (\alpha_k+1) u_{\alpha+\varepsilon^{(k)}} \otimes \mathfrak{e}_k = \sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k.$$

Due to uniqueness of the series expansion in $S'(\mathbb{R})$ we obtain a family of deterministic equations

$$u_{\alpha+\varepsilon^{(k)}} = \frac{1}{\alpha_k+1} h_{\alpha,k}, \quad \text{for all} \quad \alpha \in \mathcal{I},\ k \in \mathbb{N}, \tag{5.5}$$

from which we can calculate u_α, by induction on the length of α.

For $\alpha = (0,0,0,...)$, the equations in (5.5) reduce to $u_{\varepsilon^{(k)}} = h_{\alpha,k}$, $\alpha \in \mathcal{I}$, $k \in \mathbb{N}$, i.e.

$$\begin{cases} u_{(1,0,0,...)} = h_{(0,0,0,...),1} \\ u_{(0,1,0,...)} = h_{(0,0,0,...),2} \\ u_{(0,0,1,0...)} = h_{(0,0,0,...),3} \\ \quad\vdots \end{cases},$$

and we obtain the coefficients u_α for α of length one. Note, u_α are obtained in terms of $h_{\alpha_{\varepsilon(r)},r} = h_{(0,0,0,\dots),r}$, $r \in \mathbb{N}$.

For $|\alpha| = 1$ multi-indices are of the form $\alpha = \varepsilon^{(j)}$, $j \in \mathbb{N}$, so several cases occur. For $j = 1$, $\alpha = \varepsilon^{(1)} = (1, 0, 0, \dots)$, we have

$$\begin{cases} u_{(2,0,0,\dots)} = \frac{1}{2} h_{(1,0,0,\dots),1} \\ u_{(1,1,0,\dots)} = h_{(1,0,0,\dots),2} \\ u_{(1,0,1,0\dots)} = h_{(1,0,0,\dots),3} \\ u_{(1,0,0,1,0\dots)} = h_{(1,0,0,\dots),4} \\ \vdots \end{cases}. \tag{5.6}$$

Continuing, for $j = 2$, $\alpha = \varepsilon^{(2)} = (0, 1, 0, \dots)$ the equations in (5.5) reduce to

$$\begin{cases} u_{(1,1,0,0,\dots)} = h_{(0,1,0,0,\dots),1} \\ u_{(0,2,0,\dots)} = \frac{1}{2} h_{(0,1,0,0,\dots),2} \\ u_{(0,1,1,0\dots)} = h_{(0,1,0,0,\dots),3} \\ u_{(0,1,0,1,0\dots)} = h_{(0,1,0,0,\dots),4} \\ \vdots \end{cases}, \tag{5.7}$$

and then, for $\alpha = \varepsilon^{(3)} = (0, 0, 1, 0, \dots)$ we obtain

$$\begin{cases} u_{(1,0,1,0,\dots)} = h_{(0,0,1,0,\dots),1} \\ u_{(0,1,1,0,\dots)} = h_{(0,0,1,0,\dots),2} \\ u_{(0,0,2,0\dots)} = \frac{1}{2} h_{(0,0,1,0,\dots),3} \\ u_{(0,0,1,1,0\dots)} = h_{(0,0,1,0,\dots),4} \\ \vdots \end{cases}. \tag{5.8}$$

The coefficient $u_{(1,1,0,0,\dots)}$ appears in systems (5.6) and (5.7) and thus the additional condition $h_{(1,0,0,\dots),2} = h_{(0,1,0,0,\dots),1}$ has to hold in order to have a solvable system. Also, from expressions for $u_{(0,1,1,0,\dots)}$ and $u_{(0,1,0,1,\dots.)}$ in (5.7) and (5.8) we obtain conditions $h_{(0,1,0,\dots),3} = h_{(0,0,1,0,\dots),2}$ and $h_{(0,0,0,1,0,\dots),2} = h_{(0,1,0,0,\dots),4}$ respectively, which need to be satisfied, in order to have a unique u_α. In the same manner we obtain all coefficients u_α, for α of the length two, expressed as a function of $h_{\alpha_{\varepsilon(r)},r}$.

Let now $|\alpha| = 2$. Then different combinations for the multi-indices occur: if we choose $\alpha = (1, 1, 0, 0, \dots)$ then (5.5) transforms into the system

$$\begin{cases} u_{(2,1,0,0,\dots)} = \frac{1}{2} h_{(1,1,0,0,\dots),1} \\ u_{(1,2,0,\dots)} = \frac{1}{2} h_{(1,1,0,0,\dots),2} \\ u_{(1,1,1,0\dots)} = h_{(1,1,0,0,\dots),3} \\ u_{(1,1,0,1,0\dots)} = h_{(1,1,0,0,\dots),4} \\ \vdots \end{cases}, \tag{5.9}$$

and if we choose $\alpha = (1,0,1,0,0,...)$, then the equations in (5.5) transform into

$$\begin{cases} u_{(2,0,1,0,...)} = \frac{1}{2}h_{(1,0,1,0,0,...),1} \\ u_{(1,1,1,0,...)} = h_{(1,0,1,0,0,...),2} \\ u_{(1,0,2,0...)} = \frac{1}{2}h_{(1,0,1,0,0,...),3} \\ u_{(1,0,1,1,0...)} = h_{(1,0,1,0,0,...),4} \\ \vdots \end{cases} . \tag{5.10}$$

We continue with $\alpha = (0,1,1,0,0,...)$ and $\alpha = (2,0,0,...)$ and obtain the systems

$$\begin{cases} u_{(1,1,1,0,...)} = h_{(0,1,1,0,0,...),1} \\ u_{(0,2,1,0,...)} = \frac{1}{2}h_{(0,1,1,0,0,...),2} \\ u_{(0,1,2,0...)} = \frac{1}{2}h_{(0,1,1,0,0,...),3} \\ u_{(0,1,1,1,0...)} = h_{(0,1,1,0,0,...),4} \\ \vdots \end{cases} , \quad \text{and} \tag{5.11}$$

$$\begin{cases} u_{(3,0,0,...)} = \frac{1}{3}h_{(2,0,0,...),1} \\ u_{(2,1,0,...)} = h_{(2,0,0,...),2} \\ u_{(2,0,1,0...)} = h_{(2,0,0,...),3} \\ u_{(2,0,0,1,0...)} = h_{(2,0,0,...),4} \\ \vdots \end{cases} \tag{5.12}$$

respectively. For $\alpha = (0,2,0,0,...)$ the system (5.5) transforms into

$$\begin{cases} u_{(1,2,0,0,...)} = h_{(0,2,0,0,...),1} \\ u_{(0,3,0,...)} = \frac{1}{3}h_{(0,2,0,0,...),2} \\ u_{(0,2,1,0...)} = h_{(0,2,0,0,...),3} \\ u_{(0,2,0,1,0...)} = h_{(0,2,0,0,...),4} \\ \vdots \end{cases} . \tag{5.13}$$

Combining with the previous results, we obtain u_α for $|\alpha| = 3$. Two different representations of $u_{(2,1,0,0,...)}$ are given in systems (5.9) and (5.12), so the additional condition $\frac{1}{2}h_{(1,1,0,0,...),1} = h_{(2,0,0,0,...),2}$ follows. We express $u_{(2,1,0,0,...)} = \frac{1}{2}h_{(1,1,0,0,...),1}$ in form of the representative of the multi-index $\alpha = (2,1,0,0,...)$. Since the coefficient $u_{(1,2,0,...)}$ appears both in (5.9) and (5.13), we receive another condition $\frac{1}{2}h_{(1,1,0,0,...),2} = h_{(0,2,0,0,...),1}$, and express $u_{(1,2,0,...)} = h_{(0,2,0,0,...),1}$ by its representative. From (5.9), (5.10) and (5.11) we obtain $u_{(1,1,1,0,...)} = h_{(0,1,1,0,0,...),1}$ and the condition $h_{(1,1,0,0,...),3} = h_{(1,0,1,0,...),2} = h_{(0,1,1,0,0,...),1}$. Then, $\frac{1}{2}h_{(0,1,1,0,...),2} = h_{(0,2,0,...),3}$ follows from (5.11) and (5.13), and $u_{(0,2,1,0,...)} = \frac{1}{2}h_{(0,1,1,0,...),2}$.

We proceed by the same procedure for all multi-index lengths to obtain u_α.

If the set $\mathcal{K}_\alpha$, $\alpha \in \mathcal{I}$, has at least one more element besides the representative $\alpha_{\varepsilon^{(r)}}$ of α, then the condition for the process h is given in the form (5.3). We obtain the coefficients u_α of the solution as functions of the representative $\alpha_{\varepsilon^{(r)}}$

$$u_\alpha = \frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}}, r}, \quad \text{for } |\alpha| \neq 0, \ \alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)},$$

and the form of the solution (5.4).

It remains to prove convergence of the solution (5.4) in $X \otimes (S)_{-1}$. Let $h \in X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-p}$. Then, there exists $p > 0$ such that

$$\|h\|^2_{X\otimes S_{-p}(\mathbb{R})\otimes (S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|h_{\alpha,k}\|^2_X \, (2k)^{-p} \, (2\mathbb{N})^{-p\alpha} < \infty.$$

Note that for $\widetilde{u}_0 \in X$ we have $\|\widetilde{u}_0\|_X = \|\widetilde{u}_0\|_{X\otimes(S)_{-1,-q}}$ for all $q > 0$. Then, the convergence follows from

$$\begin{aligned}
\|u\|^2_{X\otimes(S)_{-1,-2p}} &= \|\widetilde{u}_0\|^2_{X\otimes(S)_{-1,-2p}} + \sum_{\substack{\alpha\in\mathcal{I}, |\alpha|>0, \\ \alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}}} \frac{1}{\alpha_r^2} \|h_{\alpha_{\varepsilon^{(r)}},r}\|^2_X (2\mathbb{N})^{-2p(\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)})} \\
&\leq \|\widetilde{u}_0\|^2_{X\otimes(S)_{-1,-2p}} + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}} \|h_{\alpha_{\varepsilon^{(r)}},r}\|^2_X \, (2r)^{-p} \, (2\mathbb{N})^{-p\alpha} \\
&\leq \|\widetilde{u}_0\|^2_{X\otimes(S)_{-1,-2p}} + \sum_{\alpha\in\mathcal{I}} \sum_{r\in\mathbb{N}} \|h_{\alpha,r}\|^2_X \, (2r)^{-p} (2\mathbb{N})^{-p\alpha} < \infty,
\end{aligned}$$

where we have used the fact that $(2\mathbb{N})^{p\varepsilon^{(r)}}(2\mathbb{N})^{-p\alpha} \leq 1$ for all $\alpha \in \mathcal{I}$, $r \in \mathbb{N}$ $\square$

Special cases

- Assume that the process h is expressed as a product $h = c \otimes g$, $c \in S'(\mathbb{R})$ and $g \in X \otimes (S)_{-1}$.

  Theorem 5.1.2 ([30]) *Let* $c = \sum_{k\in\mathbb{N}} c_k \, \mathfrak{e}_k \in S'(\mathbb{R})$ *and* $g = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes K_\alpha \in X \otimes (S)_{-1}$ *with coefficients* $g_\alpha \in X$ *such that*

  $$\frac{1}{\alpha_r} g_{\alpha_{\varepsilon^{(r)}}} c_r = \frac{1}{\alpha_j} g_\beta \, c_j, \tag{5.14}$$

  holds for all $\beta \in \mathcal{K}_\alpha$, $j \geq r$, $r \in \mathbb{N}$, *and their representative* $\alpha_{\varepsilon^{(r)}}$. *Then*

  $$\mathbb{D}u = c \otimes g, \quad Eu = \widetilde{u}_0, \quad \widetilde{u}_0 \in X, \tag{5.15}$$

  has a unique solution in $X \otimes (S)_{-1}$ *given by*

  $$u = \widetilde{u}_0 + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}\in\mathcal{I}} \frac{1}{\alpha_r} g_{\alpha_{\varepsilon^{(r)}}} c_r \otimes K_\alpha. \tag{5.16}$$

Proof. Providing an analogous procedure as in the previous theorem, we reduce equation (5.15) to a family of deterministic equations

$$u_{\alpha+\varepsilon^{(k)}} = \frac{1}{\alpha_k + 1} g_\alpha \, c_k, \quad \text{for all} \quad \alpha \in \mathcal{I},\, k \in \mathbb{N}, \tag{5.17}$$

from which, by induction on $|\alpha|$, we obtain the coefficients u_α of the solution u, as functions of the representative $\alpha_{\varepsilon^{(r)}}$. Let $\alpha \in \mathcal{I}$, $|\alpha| > 0$ be given by (5.1). Condition (5.14) implies $u_\alpha = \frac{1}{\alpha_r} g_{\alpha_{\varepsilon^{(r)}}} c_r$. The proof of convergence of the solution (5.16) in $X \otimes (S)_{-1}$ follows in the same way as in the previous theorem. □

- Especially, if we choose $c = \mathfrak{e}_i$, for fixed $i \in \mathbb{N}$, then equation (5.15) transforms into

$$\begin{cases} \mathbb{D}u = \mathfrak{e}_i \otimes g, & g \in X \otimes (S)_{-1} \\ Eu = \widetilde{u}_0, & \widetilde{u}_0 \in X \end{cases}. \tag{5.18}$$

Theorem 5.1.3 ([30]) *Let $g \in X \otimes (S)_{-1}$. Then* (5.18) *has a unique solution in $X \otimes (S)_{-1}$ of the form*

$$u = \widetilde{u}_0 + \sum_{n \in \mathbb{N}} \frac{1}{n} g_{(n-1)\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}}, \tag{5.19}$$

if and only if g is of the form

$$g = \sum_{n=0}^{\infty} g_{n\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}} = \sum_{n=0}^{\infty} g_{n\varepsilon^{(i)}} \left(I(\mathfrak{e}_i)\right)^{\lozenge n}, \tag{5.20}$$

where $I(\cdot)$ represents the Itô integral.

Proof. Let $u \in X \otimes (S)_{-1}$ be a process of the form (5.19). Then, $u \in Dom(\mathbb{D})$ and from

$$\mathbb{D}u = \sum_{n=1}^{\infty} \frac{1}{n} g_{(n-1)\varepsilon^{(i)}} \otimes n\, K_{(n-1)\varepsilon^{(i)}} \otimes \mathfrak{e}_n = \sum_{n=1}^{\infty} g_{n\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}} \otimes \mathfrak{e}_n$$

follows that it is a solution to (5.18).

Conversely, let a process $g \in X \otimes (S)_{-1}$ be of the form (5.20). Then, following the notation of Theorem 5.1.2, $c = \mathfrak{e}_i$ has the expansion $c = \sum_{k=1}^{\infty} c_k \, \mathfrak{e}_k$, where $c_k = 1$ for $k = i$ and $c_k = 0$ for $k \neq i$, $k \in \mathbb{N}$.

The family of equations (5.17) transforms to the family of deterministic equations

$$\left\{\begin{array}{rl} (\alpha_i+1)\, u_{\alpha+\varepsilon^{(i)}} = g_\alpha, & g_\alpha \in X \\ u_{\alpha+\varepsilon^{(k)}} = 0, & k=1,2,3...,\, k\neq i \end{array}\right., \quad \alpha\in\mathcal{I}. \tag{5.21}$$

If (5.20) holds, then for fixed $i\in\mathbb{N}$, $g_\alpha = 0$, for all $\alpha\neq n\varepsilon^{(i)}$, and from (5.21) similarly as in Theorem 5.1.2 the coefficients are obtained by induction on $|\alpha|$,

$$u_\alpha = \left\{\begin{array}{rl} \frac{1}{n} g_{(n-1)\varepsilon^{(i)}}, & \alpha = n\varepsilon^{(i)} \\ 0, & \alpha\neq n\varepsilon^{(i)} \end{array}\right., \; n\in\mathbb{N}.$$

The chaos expansion of the solution is

$$u = \widetilde{u}_0 + \sum_{n\in\mathbb{N}} \frac{1}{n}\, g_{(n-1)\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}} = \widetilde{u}_0 + \sum_{n\in\mathbb{N}} \frac{1}{n}\, g_{(n-1)\varepsilon^{(i)}} \otimes (I(\mathfrak{e}_k))^{\lozenge n}.$$

Convergence in $X\otimes(S)_{-1}$ can be proven by the same method as in Theorem 5.1.2. Clearly, there exists $p\in\mathbb{N}$, such that

$$\begin{aligned} \|u\|^2_{X\otimes(S)_{-1,-p}} &= \|\widetilde{u}_0\|^2_X + \sum_{n=1}^{\infty} \frac{1}{n^2}\, \|g_{(n-1)\varepsilon^{(i)}}\|^2_X (2\mathbb{N})^{-pn\varepsilon^{(i)}} \\ &\leq \|\widetilde{u}_0\|^2_X + \sum_{n=1}^{\infty} \|g_{(n-1)\varepsilon^{(i)}}\|^2_X (2\mathbb{N})^{-p(n-1)\varepsilon^{(i)}} \\ &= \|\widetilde{u}_0\|^2_X + \sum_{n=0}^{\infty} \|g_{n\varepsilon^{(i)}}\|^2_X (2\mathbb{N})^{-pn\varepsilon^{(i)}} < \infty. \end{aligned}$$

□

5.1.2 A generalized eigenvalue problem

Consider the equation

$$\left\{\begin{array}{rl} \mathbb{D}u = C\otimes u, & C\in S'(\mathbb{R}) \\ Eu = \widetilde{u}_0, & \widetilde{u}_0\in X. \end{array}\right. \tag{5.22}$$

Motivation for studying this equation can be found in optimal control problems. In particular, in [43] a special type of this equation appeared when stochastic maximum principle was applied to an optimal control problem. The solution is an $\mathcal{F}$-measurable Malliavin differentiable random variable and is obtained by applying the Clark-Ocone formula. General solution methods of this type of Malliavin differential equations were not discussed. In [53] partial information stochastic control problem of a system of forward-backward stochastic differential equations driven by Lévy process was studied. Linear homogeneous partial information Malliavin-differential type equation appeared in risk minimization of the terminal wealth in financial markets by using representation of convex risk measure, i.e. in terms of g-expectations.

The eigenvalue problem is studied also in [33] and [36].

Theorem 5.1.4 ([27]) *Let $C=\sum_{k=1}^{\infty} c_k\,\xi_k \in S'(\mathbb{R})$. If $c_k \geq \frac{1}{2k}$, for all $k\in\mathbb{N}$ then equation (5.22) has a unique solution in $X\otimes(Sc)_{-1}$, given by*

$$u=\widetilde{u}_0\otimes \sum_{\alpha=(\alpha_1,\alpha_2,\ldots)\in\mathcal{I}} \Big(\prod_{k=1}^{\infty}\frac{c_k^{\alpha_k}}{\alpha_k!}\Big)\,K_\alpha=\widetilde{u}_0\otimes\sum_{\alpha\in\mathcal{I}}\frac{c^\alpha}{\alpha!}\,K_\alpha. \tag{5.23}$$

Proof. Using the chaos expansion method, we transform equation (5.22) into the system of deterministic equations

$$(\alpha_k+1)\,u_{\alpha+\varepsilon^{(k)}}=u_\alpha\,c_k,\qquad \alpha\in\mathcal{I},\quad k\in\mathbb{N}. \tag{5.24}$$

The solution is obtained by induction with respect to the length of multi-indices α. From $Eu=\widetilde{u}_0$ it follows that $u_{(0,0,0,\ldots)}=\widetilde{u}_0$.

Starting with $|\alpha|=0$ i.e. $\alpha=(0,0,0,\ldots)$ equations in (5.24) reduce to

$$\left\{\begin{array}{r} u_{(1,0,0,0,\ldots)}=\widetilde{u}_0\,c_1\\ u_{(0,1,0,0,\ldots)}=\widetilde{u}_0\,c_2\\ u_{(0,0,1,0,0,\ldots)}=\widetilde{u}_0\,c_3\\ \vdots\\ u_{\varepsilon^{(k)}}=\widetilde{u}_0\,c_k\end{array}\right.,\quad k\in\mathbb{N}. \tag{5.25}$$

and we receive the coefficients u_α for α of length one.

Next, for $|\alpha|=1$ we have $\alpha=\varepsilon^{(i)}$, $i=1,2,\ldots$
If $\alpha=(1,0,0,0,\ldots)$ then from (5.24) and (5.25) we obtain

$$\left\{\begin{array}{r} u_{(2,0,0,0,\ldots)}=\frac{1}{2}u_{(1,0,0,0,\ldots)}c_1=\frac{1}{2!}c_1^2\,\widetilde{u}_0\\ u_{(1,1,0,0,\ldots)}=u_{(1,0,0,0,\ldots)}c_2=c_1c_2\,\widetilde{u}_0\\ u_{(1,0,1,0,0,\ldots)}=u_{(1,0,0,0,\ldots)}\,c_3=c_1c_3\,\widetilde{u}_0\\ \vdots\end{array}\right.. \tag{5.26}$$

If $\alpha=(0,1,0,0,\ldots)$ then from (5.24) and (5.25) we have

$$\left\{\begin{array}{r} u_{(1,1,0,0,\ldots)}=u_{(0,1,0,0,\ldots)}\,c_1=c_1c_2\,\widetilde{u}_0\\ u_{(0,2,0,0,\ldots)}=\frac{1}{2}u_{(0,1,0,0,\ldots)}\,c_2=c_2^2\,\widetilde{u}_0\\ u_{(0,1,1,0,\ldots)}=u_{(0,1,0,0,\ldots)}\,c_3=c_2c_3\,\widetilde{u}_0\\ \vdots\end{array}\right.. \tag{5.27}$$

Continuing with $\alpha = \varepsilon^{(k)}$, $k \geq 3$ we obtain all u_α of length two.

For $|\alpha| = 2$ from system of equations (5.24) and results obtained in previous step (5.26), (5.27),... we obtain u_α, for $|\alpha| = 3$.

We start with $\alpha = (1, 1, 0, 0, ...)$ and obtain the family

$$\begin{cases} u_{(2,1,0,0,...)} = \frac{1}{2} u_{(1,1,0,0,...)}\, c_1 = \frac{1}{2} c_1^2 c_2\, \widetilde{u}_0 \\ u_{(1,2,0,0,...)} = \frac{1}{2} u_{(1,1,0,0,...)}\, c_2 = \frac{1}{2} c_1 c_2^2\, \widetilde{u}_0 \\ u_{(1,1,1,0,...)} = u_{(1,1,0,0,...)}\, c_3 = c_1 c_2 c_3\, \widetilde{u}_0 \\ \vdots \end{cases},$$

then continue with $\alpha = (2, 0, 0, ...)$ and receive

$$\begin{cases} u_{(3,0,0,0,...)} = \frac{1}{3} u_{(2,0,0,0,...)}\, c_1 = \frac{1}{3!} c_1^3\, \widetilde{u}_0 \\ u_{(2,1,0,0,...)} = u_{(2,0,0,0,...)}\, c_2 = \frac{1}{2} c_1^2 c_2\, \widetilde{u}_0 \\ u_{(2,0,1,0,...)} = u_{(2,0,0,0,...)}\, c_3 = c_1^2 c_3\, \widetilde{u}_0 \\ \vdots \end{cases},$$

and so on. We proceed by the same procedure for all multi-index lengths to obtain u_α in the form

$$u_\alpha = \widetilde{u}_0 \otimes \frac{c_1^{\alpha_1}}{\alpha_1!} \cdot \frac{c_2^{\alpha_2}}{\alpha_2!} \cdot \frac{c_3^{\alpha_3}}{\alpha_3!} \cdots, \text{ for all } \alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots) \in \mathcal{I},$$

and the form of the solution (5.23).

It remains to prove the convergence of the solution (5.23) in the space $X \otimes (Sc)_{-1}$, i.e. to prove that, for some $p > 0$

$$\|u\|^2_{X\otimes(Sc)_{-1,-p,c}} = \sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_X\, (2\mathbb{N}\, c)^{-p\alpha} < \infty.$$

From assumption $c_k \geq \frac{1}{2k}$, for all $k \in \mathbb{N}$, it follows that $\sum_{\alpha\in\mathcal{I}} (2\mathbb{N}c)^{-p\alpha} < \infty$ if $p > 0$. Then, for $p > 3$, we have

$$\begin{aligned} \|u\|^2_{X\otimes(Sc)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|\widetilde{u}_0\|^2_X\, \frac{c^{2\alpha}}{(\alpha!)^2} (2\mathbb{N}c)^{-p\alpha} \\ &\leq \|\widetilde{u}_0\|^2_X \sum_{\alpha\in\mathcal{I}} c^{2\alpha} (2\mathbb{N})^{-p\alpha} c^{-p\alpha} \\ &\leq \|\widetilde{u}_0\|^2_X \sum_{\alpha\in\mathcal{I}} c^{-(p-2)\alpha} \sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty. \end{aligned}$$

With this statement we complete the proof. □

Special cases

- Especially, for $C = \xi_i$, for fixed $i \in \mathbb{N}$, the equation (5.22) transforms into

$$\begin{cases} \mathbb{D}u = \xi_i \otimes u \\ Eu = \widetilde{u}_0, \quad \widetilde{u}_0 \in X. \end{cases} \tag{5.28}$$

The chaos expansion of the generalized stochastic process $u \in X \otimes (S)_{-1}$ which represents the solution of (5.28) is given by

$$u = \widetilde{u}_0 \otimes \sum_{n=0}^{\infty} \frac{c_i^n}{n!} K_{n\epsilon^{(i)}} = \widetilde{u}_0 \otimes \sum_{n=0}^{\infty} \frac{1}{n!} K_{n\epsilon^{(i)}} = \widetilde{u}_0 \otimes \exp^{\lozenge} I(\xi_i),$$

where $I(\xi_i)$ represents the Itô integral of the Hermite function ξ_i, $i \in \mathbb{N}$. Using the generating property of Hermite polynomials (1.1) we obtain another form of the solution

$$u = \widetilde{u}_0 \otimes \exp\left(I(\xi_i) - \frac{1}{2}\right) = \widetilde{u}_0 \otimes \varepsilon_{\xi_i},$$

where ε_{ξ_i} is the normalized stochastic exponential of ξ_i, defined in (2.7).

Remark 5.1.1 *In [31] it is proved that $\mathbb{D}\varepsilon_h = h\,\varepsilon_h$, for deterministic h, i.e. the family of normalized stochastic exponential represents the family of eigenfunctions of the operator $\mathbb{D}$, thus Theorem 5.1.4 gives a more general result.*

- If we choose $C = 0$ then equation (5.22) transforms to

$$\mathbb{D}u = 0,\ Eu = \widetilde{u}_0, \quad \widetilde{u}_0 \in X \tag{5.29}$$

and has a unique trivial solution $u = \widetilde{u}_0$ in the space X.

5.2 An Equation Involving the Ornstein-Uhlenbeck Operator

In this section we solve a stochastic equation involving generalized stochastic processes and the Ornstein-Uhlenbeck operator $\mathcal{R}$.

Let $P(t) = p_m t^m + p_{m-1}t^{m-1} + ... + p_1 t + p_0$, $t \in \mathbb{R}$ be a polynomial of degree m with real coefficients. Then,

$$P(\mathcal{R}) = p_m \mathcal{R}^m + p_{m-1}\mathcal{R}^{m-1} + ... + p_1\mathcal{R} + p_0 Id,$$

where Id is the identity operator.

Recall that a family of orthogonal polynomials K_α, i.e. a family of the Fourier-Hermite and the Charlier polynomials, is the family of eigenfunctions of the Ornstein-Uhlenbeck operator, and the corresponding eigenvalues are $|\alpha|$, i.e.

$$\mathcal{R}\, K_\alpha = |\alpha|\, K_\alpha, \quad \alpha \in \mathcal{I}.$$

If we apply the operator $\mathcal{R}$ onto the K_α successively k times, we obtain

$$\mathcal{R}^k\,(K_\alpha) = |\alpha|^k\, K_\alpha, \quad k \in \mathbb{N}, \text{ for } \alpha \in \mathcal{I}.$$

Action of the operator $\mathcal{R}$ onto a singular generalized stochastic process u is given by (4.16).

Theorem 5.2.1 ([27])

(i) Let P be a polynomial such that $P(k) \neq 0$, $k \in \mathbb{N}_0$. Then equation

$$P(\mathcal{R})u = g, \text{ where } g \in X \otimes (S)_{-1,-p} \text{ for some } p > 0, \tag{5.30}$$

has a unique solution in $X \otimes (S)_{-1}$ given by

$$u = \sum_{\alpha\in\mathcal{I}} \frac{g_\alpha}{P(|\alpha|)} \otimes K_\alpha. \tag{5.31}$$

(ii) Let P be a polynomial such that $P(k) = 0$ for $k \in M$, where M is a finite subset of $\mathbb{N}_0$ and let $c_i \in X$, $i \in M$. Equation

$$P(\mathcal{R})u = g,\ g \in X \otimes (S)_{-1},\ u_\alpha = c_i,\ |\alpha| = i,\ i \in M,$$

has a unique solution in $X \otimes (S)_{-1}$, given by

$$u = \sum_{\alpha\in\mathcal{I},\, |\alpha|\notin M} \frac{g_\alpha}{P(|\alpha|)} \otimes K_\alpha + \sum_{|\alpha|=i\in M} c_i \otimes K_\alpha. \tag{5.32}$$

Proof. Note that the Fourier-Hermite respectively the Charlier polynomials K_α are eigenfunctions also for the operator $P(\mathcal{R})$:

$$P(\mathcal{R})\, K_\alpha = P(|\alpha|)\, K_\alpha, \quad \alpha \in \mathcal{I}.$$

Assume that $u \in X \otimes (S)_{-1}$ is a generalized stochastic process of the form (4.2). Then

$$P(\mathcal{R})u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes P(\mathcal{R})\, K_\alpha = \sum_{\alpha\in\mathcal{I}} P(|\alpha|)u_\alpha \otimes K_\alpha. \tag{5.33}$$

Thus, $P(\mathcal{R})$ maps $Dom(\mathbb{D}) \subset X\otimes(S)_{-1,-p} \to X\otimes(S)_{-1,-p-r}$ for $r > 1+2m$, where r depends on the growth of $P(|\alpha|)$.
Note that for all $\alpha \in \mathcal{I}$, $|\alpha| \neq 0$, $|P(|\alpha|)| \leq (2\mathbb{N})^{m\alpha}|\alpha|$. Then

$$\begin{aligned}
\|P(\mathcal{R})u\|^2_{X\otimes(S)_{-1,-p-r}} &= \sum_{\alpha\in\mathcal{I}} \|P(|\alpha|)\,u_\alpha\|^2_X (2\mathbb{N})^{-(p+r)\alpha} \\
&= |P(0)|^2\,\|u_{(0,0,0,\cdots)}\|^2_X + \sum_{\alpha\in\mathcal{I},|\alpha|>0} |P(|\alpha|)|^2\|u_\alpha\|^2_X(2\mathbb{N})^{-(p+r)\alpha} \\
&\leq |P(0)|^2\|u_{(0,0,0,\cdots)}\|^2_X + \sum_{\alpha\in\mathcal{I},|\alpha|>0} |\alpha|^2(2\mathbb{N})^{2m\alpha}\|u_\alpha\|^2_X(2\mathbb{N})^{-(p+r)\alpha} \\
&\leq D \sum_{\alpha\in\mathcal{I},|\alpha|>0} |\alpha|^2\|u_\alpha\|^2_X(2\mathbb{N})^{-p\alpha} < \infty,
\end{aligned}$$

where $D = |P(0)|^2 + \sum_{\alpha\in\mathcal{I},|\alpha|>0}(2\mathbb{N})^{-(r-2m)\alpha}$, for $r > 2m+1$. We can also conclude that $P(\mathcal{R})$ is a continuous and bounded operator.

Let $g = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes K_\alpha$, where $g_\alpha \in X$, $\alpha \in \mathcal{I}$. Then by (5.33):

$$\sum_{\alpha\in\mathcal{I}} P(|\alpha|)u_\alpha \otimes K_\alpha = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes K_\alpha.$$

Due to the uniqueness of the Wiener-Itô chaos expansion, the last equation transforms to the system of deterministic equations

$$P(|\alpha|)u_\alpha = g_\alpha, \quad \text{for all } \alpha \in \mathcal{I}.$$

Now we prove (i). Since $P(|\alpha|) \neq 0$ for all $\alpha \in \mathcal{I}$, it follows that $u_\alpha = \frac{g_\alpha}{P(|\alpha|)}$ and equation (5.30) has a unique formal solution of the form (5.31).

It remains to prove convergence of the solution in $X\otimes(S)_{-1,-p}$, for some $p > 0$. Note that there exists $C > 0$ such that $|P(|\alpha|)| \geq C$ for all $\alpha \in \mathcal{I}$. Thus,

$$\|u\|^2_{X\otimes(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} \|\frac{g_\alpha}{P(|\alpha|)}\|^2_X (2\mathbb{N})^{-p\alpha} \leq \frac{1}{C^2}\sum_{\alpha\in\mathcal{I}} \|g_\alpha\|^2_X(2\mathbb{N})^{-p\alpha} < \infty,$$

because $g \in X\otimes(S)_{-1,-p}$. Thus equation (5.30) has a unique solution $u \in X\otimes(S)_{-1,-p}$.

The proof of assertion (ii) simply follows by the the previous analysis. The coefficients of the solution u are given by

$$u_\alpha = \begin{cases} \frac{g_\alpha}{P(|\alpha|)}, & |\alpha| \notin M \\ c_i, & |\alpha| = i \in M, \end{cases}$$

and the solution has the form (5.32) if and only if $g_\alpha = 0$, for $|\alpha| \in M$.

Note, if there exists at least one $\widetilde{\alpha} \in M$ such that $g_{\widetilde{\alpha}} \neq 0$, then equation (5.30) has no solution. □

5.3 An Equation Involving the Exponential of the Ornstein-Uhlenbeck Operator

Consider now a stochastic differential equation of the form

$$e^{c\mathcal{R}}u = h, \tag{5.34}$$

where $e^{c\mathcal{R}} = \sum_{k=0}^{\infty} \frac{c^k \mathcal{R}^k}{k!}$, $c \in \mathbb{R}$ and $h \in X \otimes \exp(S)_{-1,-p}$ is a singular generalized stochastic process.

Theorem 5.3.1 ([30]) *Let $h \in X \otimes \exp(S)_{-1,-p}$, for some $p > 0$. Then, there exists $q > 0$ such that equation* (5.34) *has a unique generalized solution in $X \otimes \exp(S)_{-1,-q}$ given by the form*

$$u = \sum_{\alpha \in \mathcal{I}} e^{-c|\alpha|} h_\alpha \otimes K_\alpha. \tag{5.35}$$

Proof. Assume $u \in X \otimes \exp(S)_{-1,-p}$ is a generalized stochastic process of the form (4.2), satisfying condition (4.3) with $q_\alpha^{-p} = e^{-p(2\mathbb{N})^\alpha}$.

Note that the differential operator $e^{c\mathcal{R}}$ satisfies the identity

$$e^{c\mathcal{R}} K_\alpha = \sum_{k=0}^{\infty} \frac{c^k \mathcal{R}^k}{k!} K_\alpha = \sum_{k=0}^{\infty} \frac{c^k |\alpha|^k}{k!} K_\alpha = e^{c|\alpha|} K_\alpha, \quad \alpha \in \mathcal{I}.$$

Then

$$e^{c\mathcal{R}} u = \sum_{\alpha \in \mathcal{I}} e^{c|\alpha|} u_\alpha \otimes K_\alpha, \qquad u_\alpha \in X. \tag{5.36}$$

For $c > 0$ the operator $e^{c\mathcal{R}}$ is a continuous and bounded mapping from $X \otimes \exp(S)_{-1,-p}$ into $X \otimes \exp(S)_{-1,-q}$, for some $q > p+2c$. From $e^{|\alpha|} \leq e^{(2\mathbb{N})^\alpha}$, $\alpha \in \mathcal{I}$ it it follows that

$$\begin{aligned}
\|e^{c\mathcal{R}}u\|^2_{X\otimes\exp(S)_{-1,-q}} &= \sum_{\alpha\in\mathcal{I}} e^{2c|\alpha|} \|u_\alpha\|^2_X e^{-q(2\mathbb{N})^\alpha} \\
&\leq \sum_{\alpha\in\mathcal{I}} e^{2c|\alpha|}\, e^{-p(2\mathbb{N})^\alpha}\, \|u_\alpha\|^2_X\, e^{-(q-p)(2\mathbb{N})^\alpha} \\
&\leq \left(\sum_{\alpha\in\mathcal{I}} e^{2c|\alpha|} e^{-(q-p)(2\mathbb{N})^\alpha}\right) \left(\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_X e^{-p(2\mathbb{N})^\alpha}\right) \\
&\leq \left(\sum_{\alpha\in\mathcal{I}} e^{-(q-p-2c)(2\mathbb{N})^\alpha}\right) \|u\|^2_{X\otimes\exp(S)_{-1,-p}} < \infty,
\end{aligned}$$

for $q > p + 2c$.

If $c \leq 0$ then the operator $e^{c\mathcal{R}}$ is a continuous and bounded mapping from $X \otimes \exp(S)_{-1,-p}$ into $X \otimes \exp(S)_{-1,-q}$, for $q > p$:

$$\begin{aligned} \|e^{c\mathcal{R}}u\|^2_{X\otimes\exp(S)_{-1,-q}} &= \sum_{\alpha\in\mathcal{I}} e^{2c|\alpha|}\|u_\alpha\|^2_X e^{-q(2\mathbb{N})^\alpha} \\ &\leq \left(\sum_{\alpha\in\mathcal{I}} e^{-(q-p)(2\mathbb{N})^\alpha}\right) \|u\|^2_{X\otimes\exp(S)_{-1,-p}} < \infty. \end{aligned}$$

Let $h \in X \otimes \exp(S)_{-1,-p}$ be of the form $h = \sum_{\alpha\in\mathcal{I}} h_\alpha \otimes K_\alpha$ such that $h_\alpha \in X$ and

$$\sum_{\alpha\in\mathcal{I}} \|h_\alpha\|^2_X \, e^{-p(2\mathbb{N})^\alpha} < \infty. \tag{5.37}$$

We are looking for the solution u of (5.34) in the form (4.2) where $u_\alpha \in X$ are the coefficients to be determined.

We apply (5.36) to transform equation (5.34) into the system of deterministic equations

$$e^{c|\alpha|}u_\alpha = h_\alpha, \qquad \alpha \in \mathcal{I}.$$

Thus, $u_\alpha = e^{-c|\alpha|}h_\alpha$ and we obtain a unique solution of equation (5.34) in the form (5.35).

Finally, the convergence of the solution in $X \otimes \exp(S)_{-1,-p}$, in case of $c > 0$, follows directly from (5.37). But, in case of $c \leq 0$ the solution converges in the space $X \otimes \exp(S)_{-1,-q}$, for some $q > p - 2c$, i.e.

$$\begin{aligned} \|u\|^2_{X\otimes\exp(S)_{-1,-q}} &= \sum_{\alpha\in\mathcal{I}} e^{-2c|\alpha|}\|h_\alpha\|^2_X e^{-q(2\mathbb{N})^\alpha} \\ &\leq \Big(\sum_{\alpha\in\mathcal{I}} e^{-2c|\alpha|}\, e^{-(q-p)(2\mathbb{N})^\alpha}\Big)\Big(\sum_{\alpha\in\mathcal{I}} \|h_\alpha\|^2_X \, e^{-p(2\mathbb{N})^\alpha}\Big) \\ &\leq M\, \|h\|^2_{X\otimes\exp(S)_{-1,-p}} < \infty, \end{aligned}$$

where $M = \sum_{\alpha\in\mathcal{I}} e^{-(q-p+2c)(2\mathbb{N})^\alpha} < \infty$ for $q > p - 2c$. □

5.4 The Stochastic Dirichlet Problem Driven by The Ornstein-Uhlenbeck Operator: Approach by The Fredholm Alternative for Chaos Expansions

This section is devoted to the stochastic version of the Fredholm alternative in the framework of chaos expansion methods on white noise probability space. We apply the results to solve the Dirichlet problem generated by an elliptic second order differential operator with stochastic coefficients, stochastic input data and boundary conditions, and with the Ornstein-Uhlenbeck operator as a perturbation term. The stochastic Dirichlet problem was studied in [57], [58], [67] and as a conclusion to this series of papers in [28] we introduced the Malliavin derivative and its related operator, the Ornstein-Uhlenbeck operator, into this setting. The following results represent the main contribution of this dissertation to the Malliavin differential theory in white noise framework.

In [28] we studied a stochastic Dirichlet problem with a perturbation term driven by the Ornstein-Uhlenbeck operator

$$\begin{aligned} L\lozenge u(x,\omega)+cP(\mathcal{R})u(x,\omega)&=h(x,\omega)+\sum_{i=1}^{n}D_i f^i(x,\omega), \qquad x\in I,\omega\in\Omega,\\ u(x,\omega)\restriction_{\partial I}&=g(x,\omega), \end{aligned} \tag{5.38}$$

where I is an open bounded subset of $\mathbb{R}^n$, $c\in\mathbb{R}$, and L is a stochastic differential operator of the form

$$\begin{aligned} L\lozenge u(x,\omega)=\sum_{i=1}^{n}D_i\Big(\sum_{j=1}^{n}a^{ij}(x,\omega)\lozenge D_j u(x,\omega)+b^i(x,\omega)\lozenge u(x,\omega)\Big)\\ +\sum_{i=1}^{n}c^i(x,\omega)\lozenge D_i u(x,\omega)+d(x,\omega)\lozenge u(x,\omega), \end{aligned} \tag{5.39}$$

where $a^{ij}(x,\omega),b^i(x,\omega),c^i(x,\omega),d(x,\omega)\in L^\infty(I)\otimes(S)_{-1,-(p-4)}$, $i,j=1,2,\dots,n$, P is a polynomial with coefficients in $\mathbb{R}$, $h,f^i\in L^2(I)\otimes(S)_{-1,-p}$ and $g\in W^{1,2}(I)\otimes(S)_{-1,-p}$.

In [57] and [58] the stochastic Dirichlet problem of the form

$$\begin{aligned} L\lozenge u(x,\omega)&=h(x,\omega)+\sum_{i=1}^{n}D_i f^i(x,\omega), \qquad x\in I,\omega\in\Omega,\\ u(x,\omega)\restriction_{\partial I}&=g(x,\omega), \end{aligned} \tag{5.40}$$

is considered and we showed the existence of a unique solution assuming that L is an elliptic operator with essentially bounded coefficients satisfying standard conditions. The maximum principle and the approach developed in [5] for the deterministic Dirichlet problem are used.

In the framework considered, the coefficients of L are stochastic processes, thus in physical interpretation corresponding equation with constant coefficients to the equation (5.40) can be understood as a diffusion process in a stochastic anisotropic medium, with transport and creation also dependent on some random factors, and with a stochastic boundary value. Example of a stochastic anisotropic medium is a medium consisting of two randomly mixed immiscible fluids.

In [57] and [58] is proved that in order to obtain a solution of (5.40), the operator should generate a bilinear form that is both coercive and continuous. Assuming that all input data f, g and the coefficients of L are in $(S)_{-1,-p}$ (for fixed x) and that L is elliptic, one obtains ellipticity (and thus also coercivity of the associated bilinear form) in $(S)_{-1,-q}$ for $q \geq p$. On the other hand, the associated bilinear form is continuous on $(S)_{-1,-q}$ for $q \leq p$. Since both conditions must hold, it is necessary to hold p fixed and work only in $(S)_{-1,-p}$. Thus, it is of great interest to develop Fredholm alternative type theorems holding in $(S)_{-1,-p}$, which will be the first topic of this section. We will find conditions for the operator A acting on Kondratiev spaces, under which equations of the form $f - A(f) = g$ have a unique solution.

In [28] we proved solvability and uniqueness of the solution to (5.38) under assumptions made only on the expectation of L and certain conditions on the positivity of the perturbation term. In particular, when $c = 0$, (5.38) reduces to the equation considered in [57] and [58], but with much less restrictive conditions on L. This is one of the important contributions of this section and is included in this thesis as its original and the most important part. We will prove that there is a solution in $(S)_{-1,-p}$ for p large enough.

The method used in all equations is the *chaos expansion method*, i.e the *propagator method*. With this method we reduce the stochastic partial differential equations to an infinite triangular system of partial differential equations, which can be solved by induction. Summing up all coefficients of the expansion and proving convergence in an appropriate weighted space, one obtains the solution of the initial stochastic partial differential equation.

5.4.1 The Fredholm alternative for chaos expansions

Now we prove a general form of the Fredholm alternative theorem for mappings given by chaos expansions.

Definition 5.4.1 *Let X be a Banach space and X' its dual space. The operator $A : X \to X'$ is an* operator of FA-type *if it satisfies the Fredholm alternative:*

- *either the equation $f - A(f) = 0$ has a nontrivial solution $f \in X$*
- *or the equation $f - A(f) = g$ has a unique solution $f \in X$ for each $g \in X'$.*

It is well known that if A is a compact operator $A : V \to V$ where V is a Hilbert space, then it is of FA-type. Note that the embedding $Id : W_0^{1,2}(I) \to W^{1-\varepsilon,2}(I)$ is compact for $\varepsilon > 0$, but not as a mapping $Id : W_0^{1,2}(I) \to W_0^{1,2}(I)$.

We will use the following fact in Hilbert spaces V. If $A : V \to V$ is a compact operator, then for every $c \geq 0$ the equation $f(1-c) - Af = 0$ has only the trivial solution if and only if equation $f(1-c)-Af = g$ has a unique solution for each $g \in V$. For $c = 0$ this statement reduces to the classical Fredholm alternative theorem.

In the following theorems we provide some sufficient conditions (other then compactness) under which an operator is of FA-type.

Theorem 5.4.1 ([28]) *Let X be a Banach space and $T_\alpha : X \to X'$, $\alpha \in \mathcal{I}$, be a family of FA-type operators that are uniformly bounded by a constant $K > 0$. Let $p \in \mathbb{N}$. Consider the mapping $\hat{T} : X \otimes (S)_{-1,-p} \to X' \otimes (S)_{-1,-p}$ defined by*

$$\hat{T}(\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes K_\alpha) = \sum_{\alpha\in\mathcal{I}} T_\alpha(u_\alpha) \otimes K_\alpha. \tag{5.41}$$

Then,

- *either the equation $f - \hat{T}(f) = 0$ has a nontrivial solution $f \in X \otimes (S)_{-1,-p}$*
- *or the equation $f - \hat{T}(f) = g$ has a unique solution $f \in X \otimes (S)_{-1,-p}$ for each $g \in X' \otimes (S)_{-1,-p}$.*

In the second case, the operator $(Id - \hat{T})^{-1}$ whose existence is asserted there, is also a bounded operator.

Especially, if $K < 1$, then $\hat{T}$ is a contraction mapping, thus $f - \hat{T}(f) = g$ has a unique solution for all $g \in X' \otimes (S)_{-1,-p}$.

Proof. For $u \in X \otimes (S)_{-1,-p}$ we have

$$\sum_{\alpha\in\mathcal{I}} \|T_\alpha(u_\alpha)\|_{X'}^2 (2\mathbb{N})^{-p\alpha} \leq K \sum_{\alpha\in\mathcal{I}} \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty,$$

thus the operator T in (5.41) is continuous.

Assume now that $f - \hat{T}(f) = 0$ has only the trivial solution $f = 0$. This means that in the expansion $f = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha$, $f_\alpha = 0$ for all $\alpha \in \mathcal{I}$, i.e. each equation $f_\alpha - T_\alpha(f_\alpha) = 0$ has only the trivial solution $f_\alpha = 0$.

Consider now the equation $f - \hat{T}(f) = g$ i.e.

$$\sum_{\alpha\in\mathcal{I}} (f_\alpha - T_\alpha(f_\alpha)) \otimes K_\alpha = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes K_\alpha.$$

Due to uniqueness of the Wiener-Itô chaos expansion this is equivalent to

$$f_\alpha - T_\alpha(f_\alpha) = g_\alpha, \quad g_\alpha \in X', \text{ for all } \alpha \in \mathcal{I}.$$

But since T_α is FA-type, and $f_\alpha - T_\alpha(f_\alpha) = 0$ has only the trivial solution, from the Fredholm alternative it follows that there exists a unique solution $f_\alpha \in X$ solving $f_\alpha - T_\alpha(f_\alpha) = g_\alpha$. Every generalized stochastic process is uniquely determined by its coefficients in the chaos expansion, thus $f = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha$ is the unique solution of $f - \hat{T}(f) = g$.

It remains to prove that $\sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha$ converges in $X \otimes (S)_{-1,-p}$ i.e. that

$$\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty.$$

Since T_α are uniformly bounded, it follows that $(Id - T_\alpha)^{-1}$ are also uniformly bounded, and from $f_\alpha = g_\alpha + T_\alpha(f_\alpha)$ it follows that

$$\|f_\alpha\|_X \leq C \|g_\alpha\|_{X'},$$

where $C = \max_{\alpha\in\mathcal{I}} \|(Id - T_\alpha)^{-1}\|_{op}$.

Thus,

$$\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} \leq C^2 \sum_{\alpha\in\mathcal{I}} \|g_\alpha\|_{X'}^2 (2\mathbb{N})^{-p\alpha} < \infty.$$

Especially, if $K < 1$, then

$$C = \max_{\alpha\in\mathcal{I}} \|(Id - T_\alpha)^{-1}\|_{op} \leq \max_{\alpha\in\mathcal{I}} (1 - \|T_\alpha\|_{op})^{-1} \leq (1 - K)^{-1}. \qquad \square$$

Corollary 5.4.1 ([28]) *Let X be a Banach space and $T_\alpha : X \to X'$, be a family of compact operators that are uniformly bounded by a constant $K > 0$. Let $p \in \mathbb{N}$ and let $\hat{T} : X \otimes (S)_{-1,-p} \to X' \otimes (S)_{-1,-p}$ be an operator of the form (5.41). Let $Q_\alpha : X' \to X'$, $\alpha \in \mathcal{I}$, be a family of uniformly bounded operators and consider the mapping $\hat{Q} : X' \otimes (S)_{-1,-p} \to X' \otimes (S)_{-1,-p}$ defined by*

$$\hat{Q}(\sum_{\alpha\in\mathcal{I}} u_\alpha \otimes K_\alpha) = \sum_{\alpha\in\mathcal{I}} Q_\alpha(u_\alpha) \otimes K_\alpha. \tag{5.42}$$

Then,

- *either the equation $f - \hat{Q}(\hat{T}(f)) = 0$ has a nontrivial solution $f \in X \otimes (S)_{-1,-p}$*
- *or the equation $f - \hat{Q}(\hat{T}(f)) = g$ has a unique solution $f \in X \otimes (S)_{-1,-p}$ for each $g \in X' \otimes (S)_{-1,-p}$.*

Proof. Since T_α is compact and Q_α is continuous, $Q_\alpha \circ T_\alpha$ is a compact operator for each $\alpha \in \mathcal{I}$ and the assertion follows from the classical Fredholm alternative similarly as in Theorem 5.4.1. □

For operators which can not be represented in the form (5.41) or (5.42) the following theorem will provide sufficient conditions under which these operators are of FA-type. Before we state the theorem we will explain the framework.

Consider an operator $A : X \otimes (S)_{-1,-p} \to X' \otimes (S)_{-1,-p}$. Let the equation $f - A(f) = 0$ have only the trivial solution $f = 0$ in $X \otimes (S)_{-1,-p}$. Let $g = \sum_{\gamma\in\mathcal{I}} g_\gamma \otimes K_\gamma \in X' \otimes (S)_{-1,-p}$ and assume that the equation

$$f - A(f) = g$$

can be reduced (using the chaos expansion on both the left and the right-hand side) to a lower triangular system of the form

$$f_\gamma - a_\gamma(f_\gamma) = g_\gamma + F_\gamma(\{f_\alpha, \alpha < \gamma\}), \quad \gamma \in \mathcal{I},$$

for some family of FA-type operators $a_\gamma : X \to X'$, $\gamma \in \mathcal{I}$, and some family of functions F_γ, $\gamma \in \mathcal{I}$, so that F_γ depends on f_α, $\alpha < \gamma$, but not on $\alpha \geq \gamma$ (this means that the system is lower triangular and thus can be solved by induction on γ). Since by assumption $f_\gamma - a_\gamma(f_\gamma) = 0$ has only the trivial solution for every $\gamma \in \mathcal{I}$ and a_γ is of FA-type, it follows that there exists a unique f_γ solving $f_\gamma - a_\gamma(f_\gamma) = g_\gamma + F_\gamma(\{f_\alpha, \alpha < \gamma\})$ given by

$$f_\gamma = (Id - a_\gamma)^{-1}(g_\gamma + F_\gamma(\{f_\alpha, \alpha < \gamma\})).$$

Now, if $\sum_{\gamma\in\mathcal{I}} f_\gamma \otimes K_\gamma$ converges in $X'\otimes(S)_{-1,-p}$, then by uniqueness of the Wiener-Itô chaos expansion it follows that $f=\sum_{\gamma\in\mathcal{I}} f_\gamma\otimes K_\gamma$ is the unique solution to $f-A(f)=g$.

In the following theorem we provide a general concept which is based on the procedure described above.

Theorem 5.4.2 ([28]) *Let X be a Banach space and $Q : X\otimes(S)_{-1}\to X'\otimes(S)_{-1}$ an operator of the form $Q=A+B+C$, where*

1. *$C=c\,P(\mathcal{R})$, for some $c\in\mathbb{R}$ and $\mathcal{R}=\delta\mathbb{D}$ is the Ornstein-Uhlenbeck operator, $P(\mathcal{R})$ the differential operator $P(\mathcal{R})=p_m\mathcal{R}^m+p_{m-1}\mathcal{R}^{m-1}+\ldots+p_1\mathcal{R}+p_0Id$ and P is a polynomial of degree m with real coefficients such that $cP(k)\le 0$, $k\ge k_0$, $k,k_0\in\mathbb{N}_0$.*

2. *$A(\sum_{\gamma\in\mathcal{I}} f_\gamma\otimes K_\gamma)=\sum_{\gamma\in\mathcal{I}} a_\gamma(f_\gamma)\otimes K_\gamma$ and $a_\gamma : X\to X'$ are compact operators such that*
$$\sup_{\gamma\in\mathcal{I}}\left(\frac{1}{1-cP(|\gamma|)-\|a_\gamma\|_{op}}\right)<K, \tag{5.43}$$
for some constant $K>0$.

3. *$B(\sum_{\gamma\in\mathcal{I}} f_\gamma\otimes K_\gamma)=\sum_{\gamma\in\mathcal{I}}\sum_{|\gamma-\beta|>0} b_\beta(f_{\gamma-\beta})K_\gamma$ for some bounded operators $b_\gamma : X\to X'$ and there exists $p>0$ such that*
$$K\sum_{|\beta|>0}\|b_\beta\|_{op}(2\mathbb{N})^{\frac{-p\beta}{2}}<\frac{1}{\sqrt{2}}. \tag{5.44}$$

Let the equation $f-Q(f)=0$ have only the trivial solution $f=0$ in $X\otimes(S)_{-1}$. Then, for every $g\in X'\otimes(S)_{-1}$ there exists a unique $f\in X\otimes(S)_{-1}$ solving
$$f-Qf=g. \tag{5.45}$$

Proof. Equation $f-Qf=0$ is equivalent to
$$f-(A(f)+c\,P(\mathcal{R})(f)+B(f))=0 \qquad \text{and}$$
$$\sum_{\gamma\in\mathcal{I}}\left(f_\gamma-a_\gamma(f_\gamma)-cP(|\gamma|)f_\gamma-\sum_{|\gamma-\beta|>0} b_\beta(f_{\gamma-\beta})\right)\otimes K_\gamma=0.$$
Due to uniqueness of the Wiener-Itô chaos expansion this is equivalent to
$$f_\gamma(1-cP(|\gamma|))-a_\gamma(f_\gamma)-\sum_{|\gamma-\beta|>0} b_\beta(f_{\gamma-\beta})=0,\qquad \gamma\in\mathcal{I}. \tag{5.46}$$

Since $f-Q(f)=0$ has only the trivial solution $f=0$ in $X\otimes(S)_{-1}$, it follows that for each $\gamma\in\mathcal{I}$ equation (5.46) has only the trivial solution $f_\gamma=0$. Since a_γ is compact, the classical Fredholm alternative implies that

$$f_\gamma(1-cP(|\gamma|))-a_\gamma(f_\gamma)=\sum_{|\gamma-\beta|>0} b_\beta(f_{\gamma-\beta})+g_\gamma \tag{5.47}$$

has a unique solution

$$f_\gamma=((1-cP(|\gamma|))\,Id-a_\gamma)^{-1}(g_\gamma+\sum_{|\gamma-\beta|>0} b_\beta(f_{\gamma-\beta})),\qquad \gamma\in\mathcal{I},$$

such that

$$\|f_\gamma\|_X\leq\frac{1}{1-cP(|\gamma|)-\|a_\gamma\|_{op}}\left(\|g_\gamma\|_{X'}+\sum_{|\gamma-\beta|>0}\|b_\beta\|_{op}\|f_{\gamma-\beta}\|_X\right),\quad \gamma\in\mathcal{I}.$$

We will prove that $\sum\limits_{\gamma\in\mathcal{I}} f_\gamma\otimes K_\gamma$ converges in $X\otimes(S)_{-1}$. Indeed,

$$\sum_{\gamma\in\mathcal{I}}\|f_\gamma\|_X^2(2\mathbb{N})^{-p\gamma}\leq K^2\sum_{\gamma\in\mathcal{I}}\left(\|g_\gamma\|_{X'}+\sum_{|\gamma-\beta|>0}\|b_\beta\|_{op}\|f_{\gamma-\beta}\|_X\right)^2(2\mathbb{N})^{-p\gamma}$$

$$\leq 2K^2\left(\sum_{\gamma\in\mathcal{I}}\|g_\gamma\|_{X'}^2(2\mathbb{N})^{-p\gamma}+(\sum_{|\gamma-\beta|>0}\|b_\beta\|_{op}(2\mathbb{N})^{-\frac{p\gamma}{2}})^2\sum_{\gamma\in\mathcal{I}}\|f_\gamma\|_X^2(2\mathbb{N})^{-p\gamma}\right)$$

by the Hölder-Young inequality. Thus,

$$\left(1-2K^2(\sum_{|\gamma-\beta|>0}\|b_\beta\|_{op}(2\mathbb{N})^{-\frac{p\gamma}{2}})^2\right)\sum_{\gamma\in\mathcal{I}}\|f_\gamma\|_X^2(2\mathbb{N})^{-p\gamma}\leq 2K^2\sum_{\gamma\in\mathcal{I}}\|g_\gamma\|_{X'}^2(2\mathbb{N})^{-p\gamma}.$$

By assumption (5.44), there exists $p>0$ large enough so that

$$M=1-2K^2(\sum_{|\gamma-\beta|>0}\|b_\beta\|_{op}(2\mathbb{N})^{-\frac{p\gamma}{2}})^2>0.$$

This implies

$$\sum_{\gamma\in\mathcal{I}}\|f_\gamma\|_X^2(2\mathbb{N})^{-p\gamma}\;\leq\;\frac{2K^2}{M}\sum_{\gamma\in\mathcal{I}}\|g_\gamma\|_{X'}^2(2\mathbb{N})^{-p\gamma}<\infty.$$

□

Remark 5.4.1

(1) *If* a_γ, $\gamma \in \mathcal{I}$ *is a family of uniformly bounded operators and* $1 - cP(|\gamma|) - \|a_\gamma\|_{op} \neq 0$, $\gamma \in \mathcal{I}$, *then condition* (5.43) *holds.*

(2) *If* b_γ, $\gamma \in \mathcal{I}$ *are uniformly bounded operators then there exists* $p \in \mathbb{N}_0$ *large enough such that* (5.44) *holds, thus* (5.45) *always has a solution in* $X \otimes (S)_{-1}$. *Otherwise,* b_γ *might be bounded but not uniformly and then condition* (5.44) *is essential.*

The following example shows that the conditions of the above theorems are sufficient, but not necessary.

Example 5.4.1 *The identity operator* $Id : (S)_{-1,-p} \to (S)_{-1,-p}$ *is of FA-type although it is not compact and it does not satisfy* $\|Id\| < 1$. *Clearly,* $u - u = 0$ *has nontrivial solutions, i.e. every element in* $(S)_{-1,-p}$ *satisfies this equation. Note also that* $Id : (S)_{-1,-p} \to (S)_{-1,-q}$ *is a compact embedding for each* $q > p$.

Example 5.4.2 *Let* $a \in (S)_{-1}$ *be a stochastic process such that* $E(a) = a_{(0,0,\dots)} \in (-1,1)$. *Then the equation*

$$u - a\Diamond u = g,$$

has a unique solution for each $g \in (S)_{-1}$. *Indeed, in [6] it was recently proved that there exist no zero divisors for the Wick product in* $(S)_{-1}$. *Thus, from* $(1-a)\Diamond u = 0$ *it follows that either* $u = 0$ *or* $a = 1$. *Since* $E(a) \neq 1$, *it follows that* $u = 0$. *So,* $u - a\Diamond u = 0$ *has only the trivial solution. Note that* $u - a\Diamond u = g$ *can be reduced to a lower triangular form as in* (5.47)*:*

$$u_\gamma - a_{(0,0,\dots)} u_\gamma = g_\gamma + \sum_{|\gamma-\alpha|>0} a_{\gamma-\alpha} u_\alpha,$$

where $a = \sum_{\gamma\in\mathcal{I}} a_\gamma K_\gamma$, $a_\gamma \in \mathbb{R}$, $\gamma \in \mathcal{I}$. *Applying the same procedure as in Theorem 5.4.2 we obtain by induction on* $|\gamma|$*:*

$$u_\gamma = (Id - a_{(0,0,\dots)})^{-1}(g_\gamma + \sum_{|\gamma-\alpha|>0} a_{\gamma-\alpha} u_\alpha),$$

where u_α, $\alpha < \gamma$, *are known from the previous steps. Choose* $p \in \mathbb{N}$ *large enough such that*

$$\sum_{|\alpha|>0} \frac{|a_\alpha|}{1 - |a_{(0,0,\dots)}|} (2\mathbb{N})^{-\frac{p\alpha}{2}} < \frac{1}{\sqrt{2}}.$$

Then, $\sum_{\gamma\in\mathcal{I}} |u_\gamma|^2 (2\mathbb{N})^{-p\alpha} < \infty$, *and* $u = \sum_{\gamma\in\mathcal{I}} u_\gamma K_\gamma$ *is the unique solution to* $u - a\Diamond u = g$.

In the following example we consider the case of Sobolev spaces $X = W_0^{1,2}(I)$, $X' = W^{-1,2}(I)$, and I is an open subset of $\mathbb{R}$.

Example 5.4.3 *Let $a = \sum_{\alpha\in\mathcal{I}} a_\alpha \otimes K_\alpha$ be a singular generalized process such that $\tilde{a} \in L^\infty(I) \otimes (S)_{-1,-p/2}$. Consider the mapping $A : W^{1,2}(I) \otimes (S)_{-1,-p} \to W^{1,2}(I) \otimes (S)_{-1,-p}$ defined by*

$$A(u) = a\lozenge u, \quad u \in W^{1,2}(I) \otimes (S)_{-1,-p}.$$

In Lemma 4.1.1 we proved that A is a continuous mapping.

In general, A is not a compact mapping, but if $\|E(a)\|_{L^\infty} < 1$ and if

$$\sum_{|\alpha|>0} \frac{\|a_\alpha\|_{L^\infty}}{1 - \|E(a)\|_{L^\infty}} (2\mathbb{N})^{-\frac{p\alpha}{2}} < \frac{1}{\sqrt{2}},$$

then A is of FA-type.

Especially, if $a = W$ is white noise given by $W_x(\omega) = \sum_{k=1}^\infty \xi_k(x) H_{\varepsilon^{(k)}}(\omega)$, $x \in \mathbb{R}$, $\omega \in \Omega$, then $E(W) = 0$, and

$$\sum_{k=1}^\infty \|\xi_k\|_{L^\infty} (2k)^{-\frac{p}{2}} \leq 2(\pi)^{-\frac{1}{4}} \sum_{k=1}^\infty (2k)^{-\frac{p}{2}}$$

by the uniform boundedness of the Hermite functions. Now we can choose p large enough such that $\sum_{k=1}^\infty (2k)^{-\frac{p}{2}} < \frac{\pi^{\frac{1}{4}}}{2\sqrt{2}}$. Thus, the equation

$$u - W\lozenge u = g$$

has a unique solution $u \in W^{1,2}(I) \otimes (S)_{-1,-p}$ for each $g \in W^{1,2}(I) \otimes (S)_{-1,-p}$.

5.4.2 Applications to the Dirichlet problem

In this section we continue with the assumption that $X = W_0^{1,2}(I)$, $X' = W^{-1,2}(I)$ and I is an open bounded subset of $\mathbb{R}^n$.

Consider the stochastic Dirichlet problem with a perturbation term driven by the Ornstein-Uhlenbeck operator, given by (5.38)

$$L\lozenge u(x,\omega) + cP(\mathcal{R})u(x,\omega) = h(x,\omega) + \sum_{i=1}^n D_i f^i(x,\omega), \qquad x \in I, \omega \in \Omega,$$
$$u(x,\omega) \restriction_{\partial I} = g(x,\omega),$$

where L is a stochastic differential operator of the form (5.39), c is a real constant, $\mathcal{R} = \delta\mathbb{D}$ and P is a polynomial of degree m with real coefficients. Denote by $\mathbf{h} = h + \sum_{i=1}^{n} D_i f^i$. Denote by

$$L_\alpha = \sum_{i=1}^{n} D_i(\sum_{j=1}^{n} a_\alpha^{ij}(x)D_j + b_\alpha^i(x)) + \sum_{i=1}^{n} c_\alpha^i(x)D_i + d_\alpha(x).$$

For each $\alpha \in \mathcal{I}$, L_α is a deterministic linear differential operator. Recall that $E(a^{ij}) = a^{ij}_{(0,0,0,\ldots)}$, $E(b^i) = b^i_{(0,0,0,\ldots)}$, $E(c^i) = c^i_{(0,0,0,\ldots)}$, $E(d) = d_{(0,0,0,\ldots)}$.

Now we impose the following conditions on the operator $L + cP(\mathcal{R})Id$:

- $E(L) = L_{(0,0,0,\ldots)}$ is elliptic i.e. there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} E(a^{ij})(x)\psi_i\psi_j \geq \lambda|\psi|^2, \quad x \in I, \psi \in \mathbb{R}^n, \tag{5.48}$$

- $$a^{ij}, b^i, c^i, d \in L^\infty(I) \otimes (S)_{-1,-(p-4)} \quad i, j = 1, 2, \ldots, n, \tag{5.49}$$

- $$\langle E(d) + cP(|\alpha|), \varphi\rangle_{L^2(I)} - \sum_{i=1}^{n} \langle E(b^i), D_i\varphi\rangle_{L^2(I)} \leq 0, \tag{5.50}$$

  for all $\varphi \in W_0^{1,2}(I),\ \varphi \geq 0,\ \alpha \in \mathcal{I}$,

- $$cP(k) \leq 0, \quad k \geq k_0,\ k, k_0 \in \mathbb{N}. \tag{5.51}$$

Proposition 5.4.1 ([28]) *Assume that the operator $L + cP(\mathcal{R})Id$ satisfies (5.48), (5.49), (5.50) and (5.51). If $u \in W_0^{1,2}(I) \otimes (S)_{-1,-p}$ satisfies equation*

$$L\Diamond u(x,\omega) + cP(\mathcal{R})u(x,\omega) = 0$$

in $I \times \Omega$, then $u = 0$.

Proof. By (5.48) $L_{(0,0,\ldots)} = E(L)$ is an elliptic (deterministic) linear differential operator and equation $L\Diamond u(x,\omega) + cP(\mathcal{R})\, u(x,\omega) = 0$ can be reduced to a system of equations

$$\big(L_{(0,0,\ldots)} + cP(\mathcal{R})Id\big)\, u_\gamma = \sum_{|\gamma-\beta|>0} L_{\gamma-\beta}\, u_\beta,$$

that can be solved by induction on $|\gamma|$ to obtain that $u_\gamma = 0$ for all $\gamma \in \mathcal{I}$. Clearly, $u = \sum_{\gamma\in\mathcal{I}} u_\gamma \otimes K_\gamma = 0$ converges in $W_0^{1,2}(I)\otimes(S)_{-1,-p}$ for all $p \in \mathbb{N}_0$. Thus, $u = 0$ is the unique solution to $L\lozenge u(x,\omega) + cP(\mathcal{R})u(x,\omega) = 0$. $\square$

Let us stress that the boundary condition in (5.38) is interpreted in the sense that $(u(x,\omega) - g(x,\omega)) \restriction_{\partial I} = 0$ in $W_0^{1,2}(I) \otimes (S)_{-1,-p}$. First we note that it suffices to solve the Dirichlet problem (5.38) for zero boundary values. Namely, for $\hat{u}(x,\omega) = u(x,\omega) - g(x,\omega)$ we have by linearity of the operator $L\lozenge + cP(\mathcal{R})$ that

$$\begin{aligned}
&L\lozenge\hat{u} + cP(\mathcal{R})\hat{u}\\
&\quad = L\lozenge u + cP(\mathcal{R})u - L\lozenge g - cP(\mathcal{R})g\\
&\quad = h + \sum_{i=1}^n D_i f^i - \left(\sum_{i=1}^n D_i(\sum_{j=1}^n a^{ij}\lozenge D_j g + b^i\lozenge g) + \sum_{i=1}^n c^i\lozenge D_i g + d\lozenge g\right) - cP(\mathcal{R})g\\
&\quad = h - \sum_{i=1}^n c^i\lozenge D_i g - d\lozenge g - cP(\mathcal{R})g + \sum_{i=1}^n D_i\left(f^i - \sum_{j=1}^n a^{ij}\lozenge D_j g - b^i\lozenge g\right)\\
&\quad = \hat{h} + \sum_{i=1}^n D_i \hat{f}^i,
\end{aligned}$$

where $\hat{h} = h - \sum_{i=1}^n c^i\lozenge D_i g - d\lozenge g - cP(\mathcal{R})g$ and $\hat{f}^i = f^i - \sum_{j=1}^n a^{ij}\lozenge D_j g - b^i\lozenge g$, $i = 1,2,\ldots,n$. Clearly, $\hat{u}\restriction_{\partial I} = 0$. Thus, any stochastic Dirichlet problem of the form (5.38) can be reduced to the case with zero boundary condition. Moreover, if $h, f^i \in L^2(I)\otimes(S)_{-1,-p}$ and $g \in W^{1,2}(I)\otimes(S)_{-1,-p}$, then $\hat{h}, \hat{f}^i \in L^2(I)\otimes(S)_{-1,-p}$, $\hat{u} \in W_0^{1,2}(I)\otimes(S)_{-1,-p}$ and $u = \hat{u} + g \in W^{1,2}(I)\otimes(S)_{-1,-p}$.

Theorem 5.4.3 ([28]) *Let the operator L and the polynomial P satisfy conditions* (5.48), (5.49), (5.50) *and* (5.51). *Then for $h, f^i \in L^2(I)\otimes(S)_{-1,-p}$, $i = 1,2,\ldots,n$ and for $g \in W^{1,2}(I)\otimes(S)_{-1,-p}$ the stochastic Dirichlet problem* (5.38) *has a unique solution $u \in W^{1,2}(I)\otimes(S)_{-1,-p}$.*

Proof. Without loss of generality we may assume that $g = 0$ and p is large enough so that (in accordance with (5.44)) we have

$$\sum_{|\alpha|>0} \frac{\|L_\gamma\|_{L^\infty}}{\|L_{(0,0,\ldots)}\|_{L^\infty}}(2\mathbb{N})^{-\frac{p\gamma}{2}} < \frac{1}{\sqrt{2}}, \tag{5.52}$$

where $\|L_\gamma\|_{L^\infty} = \max_{1\le i,j\le n}\{\|a_\gamma^{ij}\|_{L^\infty}, \|b_\gamma^i\|_{L^\infty}, \|c_\gamma^i\|_{L^\infty}, \|d_\gamma\|_{L^\infty}\}$.

Let $\|L_\gamma\|_{op}$, $\gamma\in\mathcal{J}$, denote the operator norm of $L_\gamma : W^{1,2}(I)\to W^{-1,2}(I)$, which has the property (see [57], [58]) that

$$\|L_\gamma\|_{op} \le 4 \max_{1\le i,j\le n}\{\|a_\gamma^{ij}\|_{L^\infty}, \|b_\gamma^i\|_{L^\infty}, \|c_\gamma^i\|_{L^\infty}, \|d_\gamma\|_{L^\infty}\}.$$

Clearly, if (5.49) holds, then from $\sum_{\gamma\in\mathcal{J}}\|L_\gamma\|_{L^\infty}(2\mathbb{N})^{-\frac{p\gamma}{2}} < \infty$ it follows that $\sum_{\gamma\in\mathcal{J}}\|L_\gamma\|_{op}(2\mathbb{N})^{-\frac{p\gamma}{2}} < \infty$. Also, (5.52) implies that

$$\sum_{|\gamma|>0}\frac{\|L_\gamma\|_{op}}{\|L_{(0,0,\dots)}\|_{op}}(2\mathbb{N})^{-\frac{p\gamma}{2}} < \frac{1}{\sqrt{2}}.$$

Observe that $L\lozenge u + P(\mathcal{R})u = h + \sum_{i=1}^n D_i f^i$ can be written as

$$\big(L_{(0,0,\dots)} + cP(|\gamma|)Id\big)\,u_\gamma(x) = \mathbf{h}_\gamma(x) - \sum_{\substack{\beta\in\mathcal{J}\\ |\gamma-\beta|>0}} L_{\gamma-\beta}u_\beta(x), \quad u_\gamma(x)\restriction_{\partial I}=0,\ \gamma\in\mathcal{J}.$$

This system is lower triangular and can be solved by induction on $|\gamma|$. In each step the operator that is involved, $L_{(0,0,\dots)} + cP(|\gamma|)Id$ is by the assumptions (5.48), (5.49) and (5.50) a deterministic elliptic linear differential operator with bounded coefficients. From (classical) deterministic theory ([5]) it follows that the last equation has a unique solution and that

$$\|(L_{(0,0,\dots)} + cP(|\gamma|)Id)^{-1}\|_{op} \ne 0,\ \gamma\in\mathcal{J}.$$

Let us show that $\{L_{(0,0,\dots)} + cP(k)Id,\ k\in\mathbb{N}_0\}$ is a bounded family of operators. We have, for $m_0>0$ large enough

$$\|L_{(0,0,\dots)} + cP(k)Id\|_{op} \ge |c||P(k)| - \|L_{(0,0,\dots)}\|_{op} \ge |c|m_0 - \|L_{(0,0,\dots)}\|_{op} \ge \frac{|c|m_0}{2},$$

thus there exists $C>0$ such that

$$\|(L_{(0,0,\dots)} + cP(k)Id)^{-1}\|_{op} \le C, \quad k\in\mathbb{N}_0. \tag{5.53}$$

Let $\gamma=(0,0,\dots)$. Since $h_{(0,0,\dots)}, f^i_{(0,0,\dots)} \in L^2(I)$, $i=1,\dots,n$, from the deterministic theory of elliptic PDEs (see e.g. [5]) it follows that the Dirichlet problem

$$(L_{(0,0,\dots)} + cP(0)Id)\,u_{(0,0,\dots)}(x) = \mathbf{h}_{(0,0,\dots)}(x), \quad u_{(0,0,\dots)}(x)\restriction_{\partial I}=0$$

has a unique weak solution $u_{(0,0,\ldots)} \in W_0^{1,2}(I)$. Moreover, there exists the inverse operator $(L_{(0,0,\ldots)} + cP(0)Id)^{-1}$ as a bounded operator

$$(L_{(0,0,\ldots)} + cP(0)Id)^{-1} : \; L^2(I) \to W_0^{1,2}(I)$$

and the estimate

$$\|u_{(0,0,\ldots)}\|_{W^{1,2}} \le \|(L_{(0,0,\ldots)} + cP(0)Id)^{-1}\|_{op} \cdot \|\mathbf{h}_{(0,0,\ldots)}\|_{L^2} \le C\|\mathbf{h}_{(0,0,\ldots)}\|_{L^2},$$

holds, where C is the constant from (5.53).

Let $\gamma = (1,0,0,\ldots)$. From the previous step we already obtained $u_{(0,0,\ldots)}$, so it remains to solve the problem

$$\left(L_{(0,0,\ldots)} + cP(1)Id\right) u_{(1,0,\ldots)}(x) = \mathbf{h}_{(1,0,\ldots)}(x) - L_{(1,0,\ldots)}u_{(0,0,\ldots)}(x)$$

with a zero boundary condition. Since all coefficients in $L_{(1,0,\ldots)}$ are $L^\infty(I)$ functions, $u_{(0,0,\ldots)} \in W^{1,2}(I)$, after differentiating in the weak sense we will have $L_{(1,0,\ldots)}u_{(0,0,\ldots)} \in L^2(I)$. By assumption, $\mathbf{h}_{(1,0,\ldots)} \in L^2(I)$, $i = 1,\ldots,n$, thus there exists a unique weak solution $u_{(1,0,\ldots)} \in W_0^{1,2}(I)$. Moreover,

$$\begin{aligned}\|u_{(1,0,\ldots)}\|_{W^{1,2}} &\le \|(L_{(0,0,\ldots)} + cP(1)Id)^{-1}\|_{op} \cdot \left(\|\mathbf{h}_{(1,0,\ldots)}\|_{L^2} + \|L_{(1,0,\ldots)}\|_{op} \cdot \|u_{(0,0,\ldots)}\|_{L^2}\right)\\ &\le C\left(\|\mathbf{h}_{(1,0,\ldots)}\|_{L^2} + \|L_{(1,0,\ldots)}\|_{op} \cdot \|u_{(0,0,\ldots)}\|_{L^2}\right), \qquad (5.54)\end{aligned}$$

and the constant C is the same as in the previous step.

In the same manner we obtain u_γ for $\gamma = (0,1,0,\ldots),\ldots$, in general for all $|\gamma| = 1$.

Let now $|\gamma| = 2$. For example, if $\gamma = (2,0,0,\ldots)$ the problem we obtain will have the form

$$\left(L_{(0,0,\ldots)} + cP(2)Id\right) u_{(2,0,\ldots)}(x) = \mathbf{h}_{(2,0,\ldots)}(x) - L_{(2,0,\ldots)}u_{(0,0,\ldots)}(x) - L_{(1,0,\ldots)}u_{(1,0,\ldots)}(x).$$

If for example, $\gamma = (1,1,0,\ldots)$, then we have to solve

$$\left(L_{(0,0,\ldots)} + cP(2)Id\right) u_{(1,1,0,\ldots)}(x) = \mathbf{h}_{(1,1,\ldots)}(x) - L_{(1,0,\ldots)}u_{(0,1,\ldots)}(x) - L_{(0,1,\ldots)}u_{(1,0,\ldots)}(x).$$

In any case, the right-hand of the equation involves known terms determined in the previous steps, while on the left-hand side in each step only the elliptic operator $L_{(0,0,\ldots)}$ and the perturbation term $cP(2)$ are involved. Thus, we obtain the weak solutions u_γ for each γ of length two. For each u_γ an estimate of the form (5.54) holds, e.g.

$$\begin{aligned}&\|u_{(2,0,\ldots)}\|_{W^{1,2}} \le \|(L_{(0,0,\ldots)} + cP(2)Id)^{-1}\|_{op}\\ &\quad \cdot \left(\|\mathbf{h}_{(2,0,\ldots)}\|_{L^2} + \|L_{(2,0,\ldots)}\|_{op} \cdot \|u_{(0,0,\ldots)}\|_{L^2} + \|L_{(1,0,\ldots)}\|_{op} \cdot \|u_{(1,0,\ldots)}\|_{L^2}\right)\\ &\le C\left(\|\mathbf{h}_{(2,0,\ldots)}\|_{L^2} + \|L_{(2,0,\ldots)}\|_{op} \cdot \|u_{(0,0,\ldots)}\|_{L^2} + \|L_{(1,0,\ldots)}\|_{op} \cdot \|u_{(1,0,\ldots)}\|_{L^2}\right).\end{aligned}$$

For each $\gamma \in \mathcal{I}$ we will solve a deterministic Dirichlet probem of the form

$$\left(L_{(0,0,\dots)} + cP(|\gamma|)Id\right) u_\gamma = \mathbf{h}_\gamma - \sum_{|\gamma-\beta|>0} L_{\gamma-\beta} u_\beta,$$

where the right-hand side becomes more complicated in each step, but involves only known terms for which the problem can be solved. Moreover, we obtain norm estimates in each step with the same constant C. This follows from the fact that on the left-hand side of the Dirichlet problem $L_{(0,0,\dots)} + cP(|\gamma|)Id$ plays always the role of the differential operator, and we have proved in (5.53) that

$$\|(L_{(0,0,\dots)} + cP(|\gamma|)Id)^{-1}\|_{op}$$

is uniformly bounded by C.

We will prove now that the series $\sum_{\gamma\in\mathcal{I}} u_\gamma(x) \otimes K_\gamma(\omega)$ converges in $W^{1,2}(I) \otimes (S)_{-1,-p}$, and this will define the solution

$$u(x,\omega) = \sum_{\gamma\in\mathcal{I}} \left(L_{(0,0,\dots)} + cP(|\gamma|)Id\right)^{-1} \left(\mathbf{h}_\gamma(x) - \sum_{\substack{\beta\in\mathcal{I} \\ |\gamma-\beta|>0}} L_{\gamma-\beta} u_\beta(x) \right) \otimes K_\gamma(\omega).$$

Indeed, from the estimates (5.53), (5.54) and the generalized Hölder inequality we obtain

$$\sum_{\gamma\in\mathcal{I}} \|u_\gamma\|^2_{W^{1,2}} (2\mathbb{N})^{-p\gamma} \leq$$

$$2C^2 \left(\sum_{\gamma\in\mathcal{I}} \|\mathbf{h}_\gamma\|^2_{L^2} (2\mathbb{N})^{-p\gamma} + \sum_{\gamma\in\mathcal{I}} \left(\sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|>0}} \|L_\alpha\|_{op} \cdot \|u_\beta\|_{L^2} \right)^2 (2\mathbb{N})^{-p\gamma} \right) \leq$$

$$2C^2 \left(\sum_{\gamma\in\mathcal{I}} \|\mathbf{h}_\gamma\|^2_{L^2} (2\mathbb{N})^{-p\gamma} + \left(\sum_{|\alpha|>0} \|L_\alpha\|_{op} (2\mathbb{N})^{-\frac{p\alpha}{2}} \right)^2 \sum_{\beta\in\mathcal{I}} \|u_\beta\|^2_{L^2} (2\mathbb{N})^{-p\beta} \right).$$

Clearly, $K = \sum_{\gamma\in\mathcal{I}} \|\mathbf{h}_\gamma\|^2_{L^2} (2\mathbb{N})^{-p\gamma} < \infty$ since $h, f^i \in L^2(I) \otimes (S)_{-1,-p}$, and $\Lambda = \sum_{|\alpha|>0} \|L_\alpha\|_{op} (2\mathbb{N})^{-\frac{p\alpha}{2}} < \infty$ by assumption (5.49). Thus,

$$\sum_{\gamma\in\mathcal{I}} \|u_\gamma\|^2_{W^{1,2}} (2\mathbb{N})^{-p\gamma} \leq 2C^2 (K + \Lambda^2 \sum_{\gamma\in\mathcal{I}} \|u_\gamma\|^2_{W^{1,2}} (2\mathbb{N})^{-p\gamma}).$$

By (5.53) we have that $1 - 2C^2\Lambda^2 > 0$. Thus,

$$\sum_{\gamma\in\mathcal{I}} \|u_\gamma\|^2_{W^{1,2}}(2\mathbb{N})^{-p\gamma} \leq \frac{2C^2K}{1-2C^2\Lambda^2} < \infty.$$

This means that $u = \sum_{\gamma\in\mathcal{I}} u_\gamma(x) \otimes K_\gamma(\omega) \in W^{1,2}(I) \otimes (S)_{-1,-p}$ is the unique solution (uniqueness follows from uniqueness of the Wiener-Itô chaos expansion representation of stochastic processes) of the Dirichlet problem.□

Example 5.4.4 *Assume the coefficients of the operator L are uniformly bounded i.e. $\|L_\gamma\|_{L^\infty} \leq M$, $\gamma \in \mathcal{I}$. Choose p large enough such that $\sum_{\gamma\in\mathcal{I},|\gamma|>0}(2\mathbb{N})^{-\frac{p\gamma}{2}} < \frac{\|L_{(0,0,\dots)}\|_{L^\infty}}{M\sqrt{2}}$. Then, condition* (5.52) *is satisfied. For example, white noise W and* $\exp^{\Diamond W}$ *have uniformly bounded coefficients in their chaos expansion.*

Epilogue

The chaos expansion approach of the Malliavin calculus, the calculus of variations in infinite dimensional analysis, interpreted in the white noise setting provides a unified approach valid for both continuous and discontinuous measures, and can be carried over to the Lévy processes. This is left as a possibility to consider for further research.

Another possibility for generalization of the concept of the Malliavin calculus of singular generalized stochastic processes is to take advantage of Colombeau generalized function spaces.

Additionally, the theory can be developed for a wider class of operators generated, for example by the Lévy-Laplacian and the symmetrized Lévy-Laplacian, and be applied to linear and nonlinear equations. The operator semigroup technique can also be considered within this framework.

Further applications to stochastic partial differential equations, the modeling of probabilistic properties of their solutions and the studying of numerical approximations of their solutions remain as enticing possibilities for future investigations.

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Curriculum Vitae

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Academic Degrees Awards

- Bachelor of Sciences (Mathematics), October 1997, Faculty of Mathematics, University of Belgrade, GPA: 8,55.

Education

- 1992-1997 : Undergraduate studies at Faculty of Mathematics, University of Belgrade, Major: graduate mathematician in theoretic and applied mathematics
- 2000-2004 : Master studies at Faculty of Mathematics, University of Belgrade, Major: Probability and statistics
- 2007 - Present : Doctoral studies at Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Major: Analysis and probability

Study Abroad Experience

- November 2010 - February 2011, Unit for Engineering Mathematics, Department of Civil Engineering, University of Innsbruck, Austria, supervised by Prof. Michael Oberguggenberger

Scholarship

- Scholarship of the Council of the Scholarship Foundation of the Republic of Austria for Undergraduates, Graduates and Postgraduates, given by the ÖAD for study stay in Austria

Work Experience

- 1997 - 1998 Project Research Assistant, Department of Computer Science, Faculty of Mathematics, University of Belgrade
- 1998 - 2004 Junior Assistant, Faculty of Civil Engineering, University of Belgrade (part time)
- 2006-2007 Teacher of Mathematics, Statistics and Computer Science, Logos International School of Belgrade (part time)
- 1999 Junior Assistant and from 2009 - present Assistant, Department of Mathematics and Informatics, Faculty of Traffic and Transport Engineering, University of Belgrade

Teaching

- 1999 - present Conducting exercises for the courses in Mathematics I, II and III, Probability and Statistics, Computer Science, Visual Basic.

Research

- 2010 - present Engaged at the project *Research and develop of optimal control methods for traffic flow by the use of energy optimization criteria* financed by the Ministry of science of Republic of Serbia

Scientific Impact

Attended several national and international conferences, seminars, summer schools, apart from those listed below.

Presentations at Conferences and Workshops

- *Chaos expansion of general random processes on fractional white noise space*, 12th Serbian Mathematical Congress, Novi Sad, 2008.
- *Equations with Malliavin derivative*, 15^{th} General meeting of European Women in Mathematics, Novi Sad, 2009.
- *Generalized solutions of stochastic differential equations*, Generalized Functions 2009, Vienna, Austria, 2009.
- *Chaos expansion transform: Application to the equations driven by Gaussian and Poissonian white noise*, 6^{th} International Conference on Lévy Processes: Theory and Applications, Dresden, Germany, 2010.

- DAAD intensive course on *Chaos, Expansions and Itô calculus*, several lectures were given, Novi Sad, Serbia, 2010.

- *Chaos expansion methods for stochastic differential equations involving the Malliavin derivative*, 8^{th} International Congress of the International Society for Analysis, its Applications and Computation, Moscow, Russian Republic, 2011.

- *Propagator method for stochastic differential equations*, 16^{th} General meeting of European Women in Mathematics, Barcelona, 2011.

- *Chaos expansion methods for stochastic differential equations* , Workshop Stochastic Analysis, Lévy processes and (B)SDEs, Innsbruck, Austria, 2011.

Publication list

- T. Levajković, D. Seleši, *Chaos expansion of generalized random processes on fractional white noise space*, Proceedings of 12th Serbian Mathematical Congress, Novi Sad Journal of Mathematics 38(3), pp 137–146, 2009.

- T. Levajković, S. Pilipović, D. Seleši, *Chaos expansions: Applications to a generalized eigenvalue problem for the Malliavin derivative*, Integral Transforms and Special Functions 22 (2), pp 97–105, 2011.

- T. Levajković, S. Pilipović, D. Seleši, *The stochastic Dirichlet problem driven by the Ornstein-Uhlenbeck operator: Approach by the Fredholm alternative for chaos expansions*, Stochastic Analysis and Applications, Vol. 29, pp 317–331, 2011.

- T. Levajković, D. Seleši, *Chaos expansion methods for stochastic differential equations involving the Malliavin derivative Part I*, Publications de l'Institut Mathématique Belgrade, Nouvelle série 90(104), pp 65–85, 2011.

- T. Levajković, D. Seleši, *Chaos expansion methods for stochastic differential equations involving the Malliavin derivative Part II*, Publications de l'Institut Mathématique Belgrade, Nouvelle série 90(104), pp 85–98, 2011.

Other Interests

Mountaineering and scuba diving (PADI Advanced Open Water diver).

Биографија

Рођена сам 1.1.1974. у Београду, где сам завршила основну школу и Прву београдску гимназију. Дипломирала сам на Математичком факултету у Београду 1997. године на смеру *Теоријска математика и примене* и исте године уписала последипломске магистарске студије из *Вероватноће и статистике* на Математичком факултету у Београду. Докторске студије на Департману за Математику и информатику Природно-математичког факултета Универзитета у Новом Саду уписала сам 2007. године, из области *Анализа и вероватноћа*. Од маја 1999. године сам запослена на Саобраћајном факултету у Београду.

Као стипендиста ÖAD сам у периоду од новембра 2010. године до фебруара 2011. године боравила у Аустрији, на Институту за базичне науке, Грађевинског факултета Универзитета у Инсбруку.

Имала сам излагања на више међународних конференција, предавања у оквиру семинара и летњих школа. Коаутор сам пет научних радова, једне књиге и две збирке задатака.

Од 2010. године сам ангажована на пројекту *Исраживање и развој оптималних метода управљања протоком саобраћаја*, који финансира Министарсво просвете и науке Републике Србије.

Рекреативно се бавим роњењем и планинарењем.

25. децембар 2011. Тијана Левајковић

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Abstract: In this dissertation we study the main properties of the operators of Malliavin calculus defined on a set of singular generalized stochastic processes, which admit chaos expansion representation form in terms of orthogonal polynomial basis and having values in a certain weighted space of stochastic distributions in white noise framework.

In the first part of the dissertation we focus on white noise spaces and introduce the fractional Poissonian white noise space. All four types of white noise spaces obtained (Gaussian, Poissonian, fractional Gaussian and fractional Poissonian) can be identified through unitary mappings.

As a contribution to the Malliavin differential theory, theorems which characterize the operators of Malliavin calculus, extended from the space of square integrable random variables to the space of generalized stochastic processes were obtained. Moreover the connections with the corresponding fractional versions of these operators are emphasized and proved.

Several examples of stochastic differential equations involving the

operators of the Malliavin calculus, solved by use of the chaos expansion method, have found place in the last part of the dissertation. Particularly, obtained results are applied to solving a generalized eigenvalue problem with the Malliavin derivative and a stochastic Dirichlet problem with a perturbation term driven by the Ornstein-Uhlenbeck operator.
AB

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УНИВЕРЗИТЕТ У НОВОМ САДУ
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Наслов рада: Малиавенов рачун за хаос експанзије уопштених стохастичких процеса са применама у неким класама диференцијалних једначина
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ПО

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Важна напомена:
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Извод: Предмет истраживања ове докторске дисертације је теоријско разматрање главних својстава оператора Малиавеновог рачуна, дефинисаних на класи уопштених стохастичких процеса, који се могу развити у ред по бази, израженој у облику фамилије ортогоналних полинома и са вредностима у неком тежинском простору стохастичких дистрибуција, на простору белог шума.

Први део дисертације је посвећен изучавању разних класа простора белог шума и као резултат, уведен је простор фракционог Поасоновог белог шума. Показано је да су сви добијени простори белог шума међусобно повезани унитарним пресликавањима.

Као допринос Малиавеновој диференцијалној теорији, формулисане су теореме које описују Малиавенове операторе на уопштеним стохастичким процесима. Такође су уочене и истакнуте везе ових оператора са одговарајућим фракционим Малиавеновим операторима.

Метод хаос експанзија, примењен на решавање стохастичких диференцијалних једначина у којима фигуришу Малиавенов извод и Орнштајн-Уленбеков оператор, презентован је у завршном делу дисертације. Конкретно, представљена су решења уопштеног проблема сопствених вредности за оператор Малиавеновог извода као и Дирихлеовог проблема са пертурбацијама генерисаним дејством Орнштајн-Уленбековог оператора.

ИЗ

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