MODEL THEORY FOR INTUITIONISTIC LOGIC

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INTRODUCTION

Three major schools in the philosophy of mathematics developed around the turn of the century--logicism, formalism and intuitionism, as they are usually called. The surge of activity was initiated by Frege who attempted to provide a philosophical basis for mathematics by deriving mathematical principles from logic and exhibiting mathematical objects as logical constructs. His position was essentially a kind of reductionism, and even if he had succeeded, similar kinds of ontological and epistomological questions could have arisen in logic, but at least at that time, the status of logic was considered less controversial than the status of mathematics. Ironically, attention was drawn to the problems in philosophy of mathematics to which Frege addressed himself by Russell's discovery of a paradox in Frege's system and, consequently, in set theory in general--as it was conceived at the time.

The existence of paradoxes in set theory demonstrated conclusively the lack of a sound philosophical basis for mathematics as well as the importance of such a basis. The questions of ontological status of mathematical objects, the meaning of mathematical expressions and the nature of the process through which we arrive at our knowledge of mathematics became not only important philosophical problems, but also issues practically revevant for the working mathematician (to use standard euphemism), confronted with the possibility of inconsistency of his field. It is not surprising, therefore, that both formalism and intuitionism were developed by mathematicians--Hilbert, who probably was the greatest mathematician of the period, and Brouwer, also very respected for his work in topology.

Hilbert's idea was that mathematical practices and results could be justified by encoding them into formal system, and then proving the consistency of those systems--of course by some weaker means, that is, using only some non-controversial practices. It should be stressed that this was only one part of Hilbert's program. The second part, so called finitism, was supposed to provide those secure, non-controversial means for proving consistency of formal systems. This part is essential for Hilbert's philosophical position, because he divides all mathematical statements into real (meaningful) and ideal (meaningless). The real statements are about real objects whose ontology is based on concrete, spatio-temporal objects (e.g., graphic signs), while the ideal statements involve also the notions and procedures which refer to ideal objects (e.g., actual infinity) and cannot be assigned any meaning. The whole point of ideal statements is just to enable us to prove true real statements. Thus, no special ontological assumptions are needed for mathematics because one part can be accommodated inside any general metaphysical conception of the physical world, while for the rest, no ontological claims are made. Since ideal statements possess no meaning on their own, any proof involving ideal statements has to be formalised, that is, derivations must be based on syntactic considerations only (thus, the idea of a formal system as a game of symbols). If a proof of that kind, that is, a proof in a formal system, is a proof of a real statement, it can be justified by a real proof of consistency of the formal system. Of course, any formula of a formal system representing a real statement does indeed have an interpretation, and the whole value and meaning of a consistent formal system consists in its contribution to proving such real statements.

It was a common feature of logicism and formalism that they relied upon the realization of specific mathematical projects, and that they lost their appeal exactly because of the apparent failure of those projects, or more precisely, when the realizability of the projects became highly questionable. The logicist program was not, in fact, destroyed by the Russell's discovery of paradoxes--only Frege's approach. The real arguments against the logicist thesis came as a result of Russell's attempts to save it. Namely, if the theory of types is to be regarded as a successful realization of the logicist mathematical project (i.e., derivation of mathematics from logic), the price has to be paid of acceptance, as a part of logic, of principles which the original philosophical position is not able to account for. But on such a view, the philosophical benefit of the reduction becomes dubious, if not outright nonexistent, because the logic which includes the principles required for the theory of types is burdened by the same kind of philosophical problems the logicist program tries to solve in mathematics. On the other hand, the work on the theory of types, made it fairly clear that no significantly weaker principles could suffice for the realization of the logicist mathematical project.

Similarly, Gödel's discovery that for a consistency proof of formalized arithmetic (or any formal system in which a significant portion of it can be reproduced) even the full power of the formal system in question would not suffice, does not have to be interpreted as destroying the formalist position. Two arguments are possible against such interpretation, incidentally both offered by Gödel. First, it could be that the fault lies with the specific character of the particular type of formalisation used, so the creation of a new kind of formal systems could make the finitist consistency proofs possible. This argument is, however, completely unsubstantiated, and in the absence of any hints about the possible nature of such new formalisms, cannot be taken too seriously. Second, it is possible that a consistency proof for arithmetic could be provided (without circularity), by addition of new principles to finitism. In light of Gentzen's consistency proof, based on ε_0 induction, it seems almost certain that such principles would not be accountable for on the basis of the original philosophical position of finitism. The situation, thus, would arise (similar to that with the theory of types in logicism) where the success in the realization of the mathematical project would require depriving the philosophical position of most of its appeal.

While logicism and formalism tried to solve the philosophical problems of mathematics in a way that would provide justification for the existing practices, intuitionism took those problems more seriously, that is, not just as questions that should be answered in a satisfactory way, but as demonstrating the incoherence of the traditional position as well as the unacceptability of the actual practices allowed by this position. Not only are the ontological views of the traditional position unsatisfactory, but the whole conception of understanding of mathematical expressions is fundamentally mistaken. For intuitionism, therefore, the solution is a radical reconstruction of traditional mathematics, and primarily reinterpretation of its language on the basis of a philosophically sound conception of its nature. To use a metaphor, not the construction of a new, safe foundation for the old building--because the whole structure is beyond repair and the defects in the design cannot be compensated by any foundation--but the abandonment, or even better--demolition of the old house and the construction of a new one, starting of course from the foundation.

The philosophical position on which the intuitionist reconstruction of mathematics is to be based could be briefly outlined as follows. The objects of mathematical research are mental constructions performed by a working mathematician. The mathematical statements are to be understood as being about those mental objects and their properties. Epistemological questions are addressed in a sort of tautological way by requiring that the constructions must be amenable to easy mental inspection and self-evidently correct, and that a construction can be assumed to possess only those properties which it evidently does, or which it can be proved to possess. This proof again has to be a mental construction and has to satisfy the same kinds of requirements.

On this conception, logic is secondary to mathematics, but only in a quite specific sense that logical principles are just mathematical principles of a high degree of generality. So, logic is an integral part of mathematics, not some external doctrine which is supposed to precede it.

Let us examine now in somewhat greater detail the meanings the logical connectives are supposed to have on the intuitionistic interpretation of mathematical language. Those meanings can be specified by determining the contribution logical connectives make to the meaning of sentences in which they occur. Since understanding of the meaning of a sentence amounts to knowing when the sentence can be considered as true, which for the intuitionist means knowing what can be accepted as its proof, the meaning of a logical connective is specified by understanding how it contributes to "proof-criteria" of sentences in which it occurs. So let us see what constitutes a proof of a sentence formed by means of a logical connective from other sentences of which we already are able to recognize whether a construction is a proof. If A and B are sentences, a proof of "A & B" is a construction that consists of a proof of A and a proof of B. A proof of "A v B" is either a proof of A or a proof of B. A proof of "A → B" consists of two parts: first, a construction α, which applied to any proof of A produces a proof of B, and second, a proof that the construction α has this property. In explaining the meaning of the intuitionistic implication, this second part is sometimes neglected. However, if a construction α is given, it need not be self-evident that, when applied to any proof of A (and "any" here has a very strong meaning because it is not prescribed by any formal system, what form a proof might take) α will necessarily give, as a result, a proof of B. Since negation "$\sim A$" is a derived connective defined in terms of implication and absurdity (i.e., "$A \rightarrow \Box$" where $\Box$ denotes $0 = 1$ if we are speaking about arithmetic, or can be taken as a primitive notion otherwise), its meaning is derived from the meaning of implication. Namely, a proof of "$\sim A$" consists of a construction which transforms any proof of A into a proof of a contradiction and a proof of that fact.

A proof of "$\exists x P(x)$" where P is some property, consists of a construction of some object (i.e., a construction which <u>is</u> an object of the desired kind, e.g., a natural number - in arithmetic) and a proof that this object has the property P. Therefore, a proof of an existential statement must provide a specific object which has the claimed property.

A proof of "$\forall x P(x)$" consists of a construction α, which when applied to any construction of an object a (in the range of the variable x) yields a proof of P(a), and a proof that α does indeed have this property. This second part of "$\forall x P(x)$" was also sometimes neglected as in the case of implication but it is necessary for the same reason. For example, if P is a decidable property of natural numbers and if an intuitionistically unacceptable classical proof of $\forall x P(x)$ existed, it would be clear classically that a method α (intuitionistically acceptable) exists which when applied to any natural number n, yields a proof (again intuitionistically acceptable) of P (n). At that moment, however, there would be no proof (intuitionistically) that α will indeed do that whenever applied to a natural number.

It is clear from the explanations given above that the meanings of logical connectives on intuitionistic interpretation are very different from their meanings on classical interpretation and that, therefore, the principles of logic cannot be the same. From a mathematical point of view, however, those explanations must be regarded as vague and unclear. As we saw before, logicism and formalism suffered serious setbacks and as a consequence it would be hard (if not impossible) to argue for the original position of either of them. Intuitionism is still a viable philosophical position but it is not free from problems. One line of argument against it is that the intuitionist interpretation of mathematical language is so vague that it is not amenable to a rigorous analysis by mathematical means and that, unlike logicism and formalism, it could not possibly have been rejected by mathematical arguments. Another type of problem is epistemological. Unless a solipsist position is taken, it is hard to see how intuitionism could escape epistemological questions similar to those which pose some of the hardest problems for Platonist position. Namely, how do we arrive at our knowledge about mental constructions, in particular, other people's constructions? It can be argued that the problematic notion of the intuitive grasp of properties of an abstract object (existing in some outside realm) is replaced with a not significantly less problematic notion of the intuitive grasp of properties of mental constructions, while paying the price of replacing a simple and clear theory of meaning with an undoubtedly complicated and vague one. The seriousness of this kind of objection depends on the future development of intuitionism. Intuitionism could suffer serious damage from such objections if it got into a position (not unlike that position into which logicism arrived with the theory of types) where the further development of intuitionistic mathematics would require the acceptance of some new principles whose nature would be such that their acceptance would render the notion of self-evidence obscure.

Our aim here is neither to attack intuitionism nor, even less, to defend it so we shall leave this argument here. In fact, the type of research to which the present work belongs makes most sense if neither position is adopted as the only true approach. This type of research can be described as a classically conceived and executed semantical analysis of intuitionistic formal systems. Technically, the only connection with intuitionism is that the formal systems which are investigated are usually regarded as depicting the formal properties of parts of intuitionistic mathematics. Otherwise, the formal systems are defined, the models constructed and their properties and interrelations investigated by classical methods without much (if any) regard for intuitionistic acceptability. Such an enterprise could seem somewhat paradoxical in light of some, not uncommon (mis) conceptions about the intuitionist program and its relations to other schools, so some clarifications might be in place.

There are in this approach, which we could tentatively dub "classical model theory for formalized intuitionism", two instances of what might be considered a "contradictio in adjecto":

(1) "formalized intuitionism"

(2) "classical models for intuitionistic theories"

This latter apparent contradiction has two aspects:

(a) from an intuitionistic point of view, classical methods are unacceptable and meaningless and so is, therefore, this kind of model theory.

(b) from a classical point of view, intuitionism is irrelevant, and so are, consequently, its formal systems and their model theory. We shall address now each of these questions.

(1) It is known that Brouwer was strongly opposed to formalism and formalization. But this does not have to be construed as a stand against any formal methods, but rather as against the uncritical belief that mathematics is identical to the set of theorems of some formal system. Therefore, it is not that formal systems are unacceptable as such, but only that they do not convey the whole content of (intuitionistic) mathematics. It is debatable if such view can be ascribed to Brouwer, but it was certainly shared by many of his followers, notably Heyting. But even if formal systems are unsatisfactory, they are necessary for the purposes of communication.

Further justification for formal systems inside intuitionism was drawn from their success. Namely, even if it is granted that formal systems are an impoverished version of mathematical reasoning, it is indisputable that Heyting's predicate calculus is a clear and simple account of all logical principles which are at present actually used in intuitionistic mathematics. It also turned out that the immensely complicated work of Brouwer can be formally derived in this calculus from just a few of his basic principles. Thus, we can say that if formalism is still an evil for the intuitionist, it has become a necessary one and it would be hard to find intuitionists who would not recognize its role and its indispensability for settling some issues, especially technically complicated ones.

As for (2), let us first note that the situation has changed considerably since the time when the three schools waged the fierce battle over which account is the only true and correct one. The relations among the adherents of different philosophical positions are today (with the exception of a few doctrinaire ultraintuitionist and constructivist groups) more that of peaceful coexistence. It is generally considered that none of the proposed solutions is completely satisfactory and the attitude of the majority is that of interest in what other approaches might contribute to the common cause.

(a) As we mentioned before, the explanations of intuitionistic meanings of logical connectives cannot be taken as satisfactory from a mathematical point of view. Their vagueness becomes even more of a problem in light of the obvious impredicativity of the implication and universal quantifier. In order to recognize a proof of $A \to B$, we must refer to the totality of all proofs of A and similarly, to recognize a proof of $\forall x P(x)$ we must refer to the totality of all (constructions of) objects in the range of the variable x. If we have in mind that the given explanations of meanings of connectives do not in any way prescribe what types of constructions could be allowed and that thus, the totality of all proofs of A includes not only those that are not, as yet, constructed but also those proofs which might be based on principles we now know nothing of, it becomes clear that the problem of impredicativity cannot be easily dismissed.

These problems could be resolved by constructing a rigorous semantical theory, and considerable efforts have been made to provide one by intuitionistic methods, but as yet, without a completely satisfactory result. Therefore, it seems quite reasonable, that in the absence of intuitionistic semantics, a classical semantics might be of some help. As it turned out, significant results about intuitionistic formal systems were proved using a classical approach. Such proofs are not in general acceptable for an intuitionist, but even for him they may have some value in the sense that they can indicate what cannot be proved and what possibly could. Furthermore, some proofs obtained, for example, using Kripke models, can be directly transformed into intuitionistically meaningful proofs (cf. Smorynski [35] p. 337). An additional objection against classical semantics is that it does not provide the intended interpretation of logical connectives. But even if (or when) the precise mathematical expression of the intended interpretation is obtained, it still might be useful to study unintended ones, in much the same way as we find it useful to study nonstandard models (of Peano arithmetic, for example) in the classical case.

(b) As we mentioned above, the intuitionist position is the only one which is still viable and, therefore, in the absence of any sound alternative, it cannot be lightly discarded as irrelevant. Further, there is in traditional mathematics a significant and meaningful distinction between constructive and nonconstructive proofs. Although we do not need to accept intuitionistic philosophy to make this distinction, the intuitionistic formal systems provide the only precise and systematic (even if maybe too restrictive) account of the notion of constructive proof. If such an account is taken as important or useful, then the classical model theory of formalized intuitionism makes full sense.

Another point is that even if intuitionistic philosophy is rejected, the existence of intuitionistic mathematics cannot be denied. In order to assess its value (importance) a classically trained mathematician must try to understand it, a task which is made much easier by the existence of classical model theory for intuitionistic systems.

It is also interesting that the various equivalent formulations of classical semantics for formalized intuitionism involve structures which arise naturally in classical mathematics and are interesting on their own: topological spaces, and especially--Baire space, Heyting algebra (which occur also naturally in the algebraic theory of varieties), model theoretic forcing, sheaves, topoi. For example, the interest in intuitionistic logic was recently revived among some mathematicians by the discovery that the so-called "internal-logic" in a topos is intuitionistic.

The classical model theory for intuitionistic formal systems started developing very early, pratically immediately after Heyting's construction of intuitionistic predicate calculus. Two, in a sense conflicting tendencies were present in this development. One is to construct a semantics that could be accepted by intuitionists. The best result in this direction are the Beth trees, but they still did not achieve that goal. Namely, it turned out that for the proof of the completeness theorem, it is necessary to assume so-called Markov's principle, which is generally regarded by intuitionists as unacceptable. The other tendency is to provide a semantics as simple as possible that could make the intuitionistic meanings of logical connectives clearer to non-intuitionists. In this direction, the best achievement are Kripke models, especially for the case of predicate calculus and first order theories. On the one hand, they are easy to work with and in them most of the technical matters are more transparent than in the other types of models. On the other hand, the intuitive interpretation of Kripke structures as the growing (in time) body of mathematical knowledge provides to a non-intuitionist the simplest (if not completely accurate) insight into the motivations (principles) which lie behind the intuitionistic interpretation of mathematical language.

It can be said that most of the research in this field was motivated by a desire to solve one or another particular problem of different intuitionistic formal systems. Consequently, the results are often isolated with gaps in knowledge in between. While on one subject, fairly sophisticated results might have been obtained, on another similar topic, the most basic facts might not be known. About the only general model theoretic results known thus far are completeness, compactness, Craig's interpolation and Skolem-Löwenheim theorems and a few results about ultraproducts. It was our aim to try to contribute to more systematic development of general model theory for intuitionistic first order logic. In light of the above remarks it seemed natural to choose Kripke models as the principal object of our study. We shall however use other structures, notably saturated theories, whenever that can be advantageous.

A few words at the end about technical conventions. The work is divided into chapters and chapters into sections. Definitions, theorems and other formal statements are consecutively numbered within each section. By "Theorem II 2.1." we shall refer to Theorem 1 of the Section 2. of Chapter II. When referring to Theorems (and other formal statements) in the same chapter or section, we omit the number for the chapter and section repsectively.

The first chapter is a collection of known results and definitions and our contribution, aside from the manner of presentation and arrangement, is minimal. The results in Chapter II and III are practically all new (except Definitions III.1.1. and III.2.1. and Theorem III.2.1.). In our view, the principal contributions are Theorems II.1.1., II.1.2., II 2.1. II.2.2., III.1.1., III.2.2. and III.2.4.

CHAPTER I

1. Syntax

In this section we state the definitions and notational conventions concerning the syntax of the first order intuitionistic logic. For definiteness, we describe also a formal system of intuitionistic predicate logic (IPC). Intuitionistic logic was first formalized by Heyting in 1930 [15]. The system given here is due to Spector [36]. It was chosen because of its intuitive appeal and convenience for model theoretic considerations.

The language of intuitionistic predicate calculus contains the following six types of symbols:

(i) a countable set of individual variables (denoted by x, y, z, ...),

(ii) a countable set of individual constants (denoted by a, b, c, ...),

(iii) for every natural number $n \geq 1$, a countable set of n-ary function symbols (denoted by f^n, g^n, ..., or by f, g, ..., if arity is evident),

(iv) for every natural number $n \geq 1$, a countable set of n-ary relation symbols (denoted by P^n, R^n, ..., or by P, R, ..., if arity is evident),

(v) logical connectives: $\Box$ (the propositional constant for absurdity), $\vee$, &, $\rightarrow$, $\exists$, $\forall$,

vi) auxiliary symbols: parentheses (and) and the comma , .

Terms, atomic formulas and formulas are defined as usual. It should be noted that the definition of atomic formulas includes $\square$. Formulas will be denoted by lower case Greek letters ϕ, ψ, ..., terms by s, t, ..., and sets of formulas by capital Greek letters Γ, Δ, ... A sentence is a formula without free variables.

The axioms are given by the following schemata:

1) $\phi \to \phi$

2) $\phi \,\&\, \psi \to \phi$

3) $\phi \,\&\, \psi \to \psi$

4) $\phi \to \phi \vee \psi$

5) $\psi \to \phi \vee \psi$

6) $\square \to \phi$

7) $\forall x \phi(x) \to \phi(t)$

8) $\phi(t) \to \exists x \phi(x)$

Derivation rules are given by the following schemata:

MP: $\dfrac{\phi \quad \phi \to \psi}{\psi}$ $\quad P_1$: $\dfrac{\phi \to \psi \quad \psi \to \chi}{\phi \to \chi}$

P_2: $\dfrac{\phi \to \chi \quad \psi \to \chi}{\phi \vee \psi \to \chi}$ $\quad P_3$: $\dfrac{\phi \to \psi \quad \phi \to \chi}{\phi \to \psi \,\&\, \chi}$

$$P_4: \frac{\phi \,\&\, \psi \to \chi}{\phi \to (\psi \to \chi)} \qquad P_5: \frac{\phi \to (\psi \to \chi)}{\phi \,\&\, \psi \to \chi}$$

$$Q_1: \frac{\psi \to \phi\ (x)}{\psi \to \forall x\, \phi\ (x)} \qquad Q_2: \frac{\phi\ (x) \to \psi}{\exists x\, \phi\ (x) \to \psi}$$

It is assumed in axioms 7) and 8) that t is a term free for x in ϕ and in rules Q_1 and Q_2 that x is not a free variable of ψ.

$\Gamma \vdash \phi$ means that the formula ϕ is a consequence of the set of hypotheses Γ, in IPC. In that case the application of the rules Q_1 and Q_2 has to be restricted to formulas which do not depend on the hypotheses containing x free, i.e., Q_1 and Q_2 can be applied only to formulas which have been derived from Γ without the use of hypotheses in which x is free. Since we are interested in sentences, this kind of restriction has little effect, because any proof of a sentence from hypotheses can be transformed into a proof satisfying that restriction (cf. Prawitz [30], I.§ 3.). $\vdash \phi$ means of course that ϕ is a theorem of IPC. Γ is consistent if $\Gamma \nvdash \Box$. A theory is a consistent set of sentences. Besides usual conventions about abbreviations, we introduce also $\sim \phi$ as an abbreviation for $\phi \to \Box$, and $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \to \psi) \,\&\, (\psi \to \phi)$.

We list now some characteristic properties of IPC.

1) $\vdash \phi \vee \psi$ implies $\vdash \phi$ or $\vdash \psi$

2) If $\vdash \exists x\, \phi(x)$ and all individual constants occurring in ϕ are among $a_1, \ldots, a_n$ then $\vdash \phi\ (a)$ for some $a \in \{a_1, \ldots, a_n\}$.

If ϕ has no individual constants, then $\vdash \phi(a)$ for arbitrary a, and also $\vdash \forall x\, \phi(x)$.

3) The following classical theorems do not hold in IPC:

$$\phi \vee \sim \phi$$

$$\sim \sim \phi \rightarrow \phi$$

$$(\phi \rightarrow \psi) \rightarrow (\sim \phi \vee \psi)$$

$$\sim (\phi \,\&\, \sim \psi) \rightarrow (\phi \rightarrow \psi)$$

$$(\sim \phi \rightarrow \sim \psi) \rightarrow (\psi \rightarrow \phi)$$

$$\sim (\phi \rightarrow \psi) \rightarrow (\phi \,\&\, \sim \psi)$$

$$(\phi \rightarrow (\psi \vee \chi)) \rightarrow (\phi \rightarrow \psi) \vee (\phi \rightarrow \chi)$$

$$\sim \forall x\, \phi(x) \rightarrow \exists x \sim \phi(x)$$

$$\forall x\, (\phi(x) \vee \psi) \rightarrow \forall x \phi(x) \vee \psi \quad (x \text{ not free in } \psi)$$

$$(\phi \rightarrow \exists x\, \psi(x)) \rightarrow \exists x\, (\phi \rightarrow \psi(x)) \quad (x \text{ not free in } \phi)$$

etc.

We give later simple counter-examples for some of those formulas using Kripke models. It should be noted that if we reverse the major implication sign, each of them (except, of course, the first one) becomes a theorem of IPC.

A countable (intuitionistic) first order language L is defined exactly as in the classical case, except that all logical connectives are primitive symbols. Therefore, it makes sense to speak about a classical relational structure as a structure for L. All it means is that the structure in question is equipped with exactly the required distinguished elements, functions and relations. Consequently, we shall say that a sentence of L is satisfied in a classical structure for L if it is satisfied under classical interpretation of logical connectives (e.g., $P \to Q$, for P and Q atomic, is satisfied iff either P is not satisfied or Q is satisfied).

2. Kripke structures

Kripke structures were, of course, defined by Saul Kripke in 1965 [23]. Structures of a similar kind were designed by Beth [2], [3]. Since they are both dismissed by intuitionists as not capturing the intended meanings of logical connectives, we chose to consider only Kripke structures as more transparent and easier to work with. The present formulation differs inessentially from the original. The differences are mostly the result of an effort to treat Kripke structures, as much as possible, as a generalization of classical model theory, and to use, consequently, standard terminology and notation whenever it is possible, highlighting thus the important differences. We note in passing that there is a modification of Kripke models which is regarded as acceptable by some intuitionists (though it is regarded as controversial by others).

Definition 1:

Let $\mathcal{T} = \langle T, 0, \leq \rangle$ be a partially ordered set with the minimal element 0, and for each $t \in T$ let $\mathfrak{A}_t$ be a classical structure for L. The structure $\langle \mathcal{T} ; \mathfrak{A}_t : t \in T \rangle$ is called a Kripke structure if the following holds for every $s \leq t$ in $\mathcal{T}$:

(i) $A_s \subseteq A_t$ (A_s is the universe of the structure $\mathfrak{A}_s$)

(ii) if c is an individual constant from L then $c^s = c^t$ (by c^s we mean the interpretation of c in $\mathfrak{A}_s$)

(iii) if f is an n-ary function symbol from L and $a_1, \ldots, a_n, b \in A_s$ then $f^s(a_1, \ldots, a_n) = b$ implies $f^t(a_1, \ldots, a_n) = b$

(iv) if R is an n-ary relation symbol from L then $R^s \subseteq (A_s)^n \cap R^t$

Classical relational structures shall be denoted by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, ... possibly with subscripts, and Kripke structures by $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \ldots$. If $\mathfrak{B}_s$ and $\mathfrak{B}_t$ are such that (i)-(iv) hold we say that $\mathfrak{B}_s$ is a positive substructure of $\mathfrak{B}_t$ and write $\mathfrak{B}_s \subseteq^+ \mathfrak{B}_t$. Elements of T are called nodes and o is called the base node. $\models$ is the classical satisfaction relation.

Remark: We could have defined Kripke structures more generally by requiring, for $s \leq t$, only that $\mathfrak{B}_s$ is homomorphically embedded in $\mathfrak{B}_t$. When discussing theories with undecidable equality (for example, in analysis, the equality of real numbers is undecidable, i.e., it is possible that $t \nVdash x = y$ and $t \nVdash \sim x = y$) the definition with homomorphic embeddings would enable us to still use normal models, i.e., models in which equality is interpreted as identity. Since we do not consider the theories with undecidable equality, this generalization would not provide any advantage in the present setting, while the notation would be considerably more complicated.

We define now inductively a relation, called forcing, between elements of T and sentences of $L \cup \bigcup_{t \in T} A_t$. If $t \in T$ and ϕ is a sentence of $L \cup A_t$ we say that t forces ϕ, in symbols $t \Vdash \phi$, if and only if one of the following holds:

(0) ϕ is atomic and $\mathfrak{A}_t \models \phi$ ($\models$ is the classical satisfaction relation, and it is assumed that if $a \in A_t$ appears in ϕ, it is interpreted as itself)

(1) ϕ is $\psi \mathbin{\&} \chi$ and $t \Vdash \psi$ and $t \Vdash \chi$

(2) ϕ is $\psi \vee \chi$ and $(t \Vdash \psi$ or $t \Vdash \chi)$

(3) ϕ is $\exists x\, \psi(x)$ and for some $a \in A_t$, $t \Vdash \psi(a)$

(4) ϕ is $\psi \to \chi$ and for every $s \geq t$ in T, either $s \nVdash \psi$ or $s \Vdash \chi$

(5) ϕ is $\forall x\, \psi(x)$ and for every $s \geq t$ and every $a \in A_s$, $s \Vdash \psi(a)$

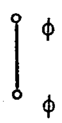
Remark: By definition, $\Box$ is an atomic formula, so it is treated in (0). Since $\sim \phi$ is defined as $\phi \to \Box$, as a consequence of (0) and (4) we have:

(6) $t \Vdash \sim \phi$ iff for every $s \geq t$, $s \nVdash \phi$.

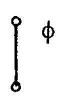
If ϕ is a sentence of L (i.e., a formula without free variables), we say that ϕ holds in $\mathfrak{K} = < T; \mathfrak{A}_t : t \in T >$ iff $0 \Vdash \phi$, or equivalently if for every $t \in T$, $t \Vdash \phi$. $\mathfrak{K}$ is a model of Γ iff each sentence from Γ holds in $\mathfrak{K}$.

If ϕ is a formula of L, when we write $\phi(x_1, \ldots, x_n)$, it will be assumed that all the free variables of ϕ are among $x_1, \ldots, x_n$. If $a_1, \ldots, a_n \in A_t$ then $t \Vdash \phi[a_1, \ldots, a_n]$ means that the sentence $\phi(a_1, \ldots, a_n)$ of $L \cup A_t$, obtained from ϕ by substituting $a_1, \ldots, a_n$ for $x_1, \ldots, x_n$ respectively, is forced by t. If $0 \Vdash \phi[a_1, \ldots, a_n]$ we say that the elements $a_1 \ldots, a_n$ satisfy (or realize) the formula ϕ.

As an illustration of how effective and convenient a tool Kripke structures are, we describe now counter-example models for some of the classical theorems, quoted in Section 1, which do not hold in IPC. To make things more transparent, models will be indicated by diagrams. Partial order of a model is represented by a graph, the lower most vertex corresponding to the base node. To the right of each vertex we write some of the formulas which are assumed to be forced by the corresponding node of the model, and on the left, where necessary, the universe of the structure at that node. In order to simplify the diagrams we sometimes do not exhibit a fact which can be readily derived from what is shown in the picture (for example, we write 

instead of 

).

1) $\underline{\phi \vee \sim \phi}$. A simple counter-example model is defined by $T = \{0, 1\}$, $0 \leq 1$, $0 \nVdash \phi$, $1 \Vdash \phi$. This is represented by the following diagram: 

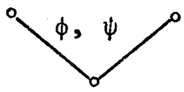
The same model can serve also as a counter-example for $\sim \sim \phi \rightarrow \phi$ (because $0 \Vdash \sim \sim \phi$)

2) $(\phi \rightarrow \psi) \rightarrow (\sim \phi \vee \psi)$. The model is given by  ϕ, ψ

Obviously, $0 \nVdash \psi$ and $0 \nVdash \sim \phi$ but $0 \Vdash \phi \rightarrow \psi$.

3) $\sim(\phi \,\&\, \sim \psi) \rightarrow (\phi \rightarrow \psi)$. The model is given by  ψ / ϕ

Clearly $0 \nVdash \sim \psi$ and $1 \nVdash \sim \psi$, so $0 \Vdash \sim (\phi \,\&\, \sim \psi)$ but $0 \Vdash \phi$ and $0 \nVdash \psi$ so $0 \nVdash \phi \rightarrow \psi$.

4) $(\phi \rightarrow (\psi \vee \chi)) \rightarrow (\phi \rightarrow \psi) \vee (\phi \rightarrow \psi)$. In this case a counter-example model cannot be linear, so the simplest model is represented by:  ϕ, ψ ϕ, χ

Here, $0 \nVdash \phi \rightarrow \psi$ because there is a node above it which forces ϕ and does not force ψ, and similarly $0 \nVdash \phi \rightarrow \chi$.

5) $(\forall x \, \phi(x) \rightarrow \psi) \rightarrow \exists x \, (\phi(x) \rightarrow \psi)$ (x not free in ψ).

The model is described by: $\{a, b\}$  $\phi(a)$ / $\{a\}$

Here $0 \nVdash \forall x \, \phi(x)$ and $1 \nVdash \forall x \, \phi(x)$ so trivially $0 \Vdash \forall x \phi(x) \rightarrow \psi$, but $1 \Vdash \phi(a)$ and $1 \nVdash \psi$ so $0 \nVdash \exists x \, (\phi(x) \rightarrow \psi)$.

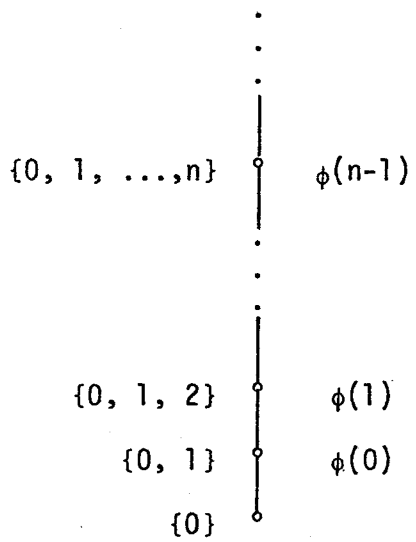
6) $(\phi \rightarrow \exists x \, \psi(x)) \rightarrow \exists x \, (\phi \rightarrow \psi(x))$ (x not free in ϕ).

$\{a, b\}$  $\phi, \psi(b)$ / $\{a\}$

Obviously, $0 \Vdash \phi \rightarrow \exists x \, \psi(x)$, but since $A_0 = \{a\}$

$0 \nVdash \exists x \, (\phi \rightarrow \psi(x))$

7) $\sim \forall x\, \phi(x) \rightarrow \exists x \sim \phi(x)$. Let T = N, with the standard ordering and let $A_n = \{0,1, \ldots, n\}$ and for $m < n$ let $n \Vdash \phi(m)$. This can be represented by:



Clearly for every n, $n \nVdash \phi(n)$, so $n \nVdash \forall x\, \phi(x)$ and $0 \Vdash \sim \forall x\, \phi(x)$. On the other hand, obviously $0 \nVdash \exists x \sim \phi(x)$, because $1 \Vdash \phi(0)$. In fact, for every n, $n \nVdash \exists x \sim \phi(x)$ so this model is also a model for $\sim \forall x\, \phi(x)\ \&\ \sim \exists x \sim \phi(x)$, which is classically false. Furthermore, for every n, $n \nVdash \phi(n) \vee \sim \phi(n)$, so $n \nVdash \forall x\, (\phi(x) \vee \sim \phi(x))$ and thus $0 \Vdash \sim \forall x\, (\phi(x) \vee \sim\phi(x))$. Therefore, the same model provides a counter-example for $\sim\sim \forall x\, (\phi(x) \vee \sim \phi(x))$, which demonstrates that the double negation of a classical theorem does not have to be an intuitionistic theorem, if it involves a universal quantifier. (Note that the first two nodes of the given model suffice as a counter-example for $\sim \forall x\, \phi(x) \rightarrow \exists x \sim \phi(x)$. The infinite model is necessary, however, for the other two formulas.)

We list now a few major facts about Kripke models. $\Gamma \models_K \phi$ means that every Kripke model of Γ is a model of ϕ ("every model" here, of course, means every one which is a structure for the language of Γ and ϕ).

<u>Theorem 1</u> (Soundness)

If $\vdash \phi$ then $\models_K \phi$

Proof: Straightforward verification of axioms and rules of inference.

<u>Theorem 2</u>

If ϕ is forced by a node of a Kripke structure, it is also forced by every node above (i.e., $s \Vdash \phi(a_1, \ldots, a_n)$ for $a_1, \ldots a_n \in A_s$ and $s \leq t$ implies $t \Vdash \phi(a_1, \ldots, a_n)$.

Proof: For atomic formulas, theorem holds by definition. For other formulas, it is proved by easy induction on complexity of ϕ.

<u>Definition 2</u> (Truncation)

If $\mathfrak{K} = \langle \mathcal{T}; \mathfrak{A}_t : t \in T \rangle$ is a Kripke structure and $s \in T$, we define a new Kripke structure $\mathfrak{K}_s = \langle \mathcal{T}_s; \mathfrak{A}_t : t \in T_s \rangle$, called <u>truncation of $\mathfrak{K}$ at s</u>, by defining $T_s = \{t \in T : s \leq t\}$ and $\mathcal{T}_s = \langle T_s, s, \leq \rangle$. The structures $\mathfrak{A}_t$, for $t \in T_s$, are the same. Forcing relation is defined as usual, and denoted by $\Vdash_s$.

Theorem 3

If $\mathfrak{K}_s$ is a truncation of $\mathfrak{K}$ at s, $t \in T_s$ and $a_1, \ldots, a_n \in A_t$, then $t \Vdash \phi(a_1, \ldots, a_n)$ iff $t \Vdash_s \phi(a_1, \ldots, a_n)$.

Proof: Inspection of the definition of forcing reveals that $t \Vdash \phi$ depends in each case only on the classical structure at that node and/or nodes above it. Therefore, the theorem follows by easy induction on the complexity of ϕ.

Theorem 4 (Strong completeness)

$$\Gamma \vdash \phi \quad \text{iff} \quad \Gamma \models_K \phi$$

Proof: One direction is an easy consequence of soundness Theorem and Deduction Theorem for IPC. The proof of the difficult direction we postpone until the end of Section 4.

3. Heyting algebras

The first characteristic model of intuitionistic propositional calculus was constructed by Jaskowski in 1936 in the form of an infinite matrix [17]. A more standard model, in the form of the lattice of open sets of a topological space with appropriately defined operation for $\rightarrow$, was developed by Stone [37] and Tarski [38] as a generalization of the power-set (i.e., Boolean algebra) semantics of the classical propositional calculus. The same type of model was later formulated in algebraic terms (i.e., in terms of closure algebras) by McKinsey and Tarski ([26], [27] and [28]).Heyting algebras (or pseudo-Boolean algebras as they are also called) arise in the dual of closure algebras (i.e., in algebras with interior instead of closure operation). They are usually preferred to the dual formulation (sometimes called Brouwerian lattices) because they better exhibit some properties of intuitionistic logic.

Definition 1.

An algebraic structure $\langle H, v, \&, \to, 0, 1, \rangle$ is called a Heyting algebra if $\langle H, v, \&, 0, 1\rangle$ is a distributive lattice with the least element 0 and the greatest element 1, and the binary operation $\to$ satisfies $a \leq b \to c$ iff $a \,\&\, b \leq c$.

In fact, the theory of Heyting algebras is an equational theory. One set of axioms consists in the axioms for lattices expressed in terms of v, &, 0 and 1, e.g., $x \,\&\, 1 = x$, $x \,\&\, x = x$, $x \,\&\, y = y \,\&\, x$, $x \,\&\, (y \,\&\, z) = (x \,\&\, y) \,\&\, z$, plus the same axioms for v, plus $x \,\&\, (y \vee x) = x$ and $x \vee (y \,\&\, x) = x$ together with the following four equations for $\to$:

$$x \to x = 1$$

$$x \,\&\, (x \to y) = x \,\&\, y$$

$$y \,\&\, (x \to y) = y$$

$$x \to (y \,\&\, z) = (x \to y) \,\&\, (x \to z)$$

While in the lattice axioms v and & are treated equally, the axioms for $\to$ exhibit the lack of connection between $\to$ and v in Heyting algebras.

This feature reflects the difficulties in dealing with formulas which involve implication and disjunction in intuitionistic formal systems.

The Heyting algebra approach is very convenient for propositional logic and is analogous to the Boolean algebra approach in the classical case. Mostowski's extension of the approach to predicate logic [29] is related to Boolean-valued models. Scott developed a similar approach (only based on topology instead of on algebra) for intuitionistic analysis. Today algebraic and topological considerations of intuitionistic formal systems are often presented in the more general setting of sheaves.

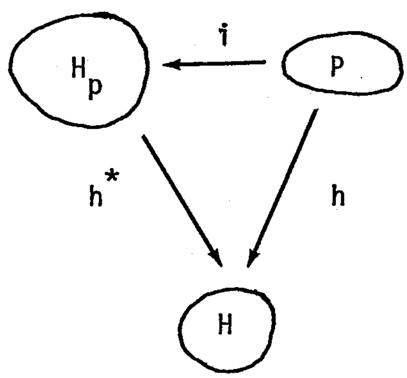
Interesting and for our purpose the most important, examples of Heyting algebras are Lindenbaum algebras of intuitionistic formal systems. We consider first the propositional case. Let P be a countable set of propositional letters and let S be the set of all formulas formed, in the usual way, from P and connectives v, & and $\rightarrow$ (we assume $\square \in P$). We denote elements of S by ϕ, ψ, ... For $\phi \in S$ let $|\phi| = \{\psi \in S: \vdash \phi \leftrightarrow \psi\}$ (here, of course, $\vdash$ denotes derivability in intuitionistic propositional calculus which is obtained from IPC by deleting axioms 7) and 8) and rules Q1 and Q2). Let $H_p = \{ |\phi| : \phi \in S\}$. Then it is easy to verify that H_p with operations:

$$|\phi| \vee |\psi| = |\phi \vee \psi|$$

$$|\phi| \,\&\, |\psi| = |\phi \,\&\, \psi|$$

$$|\phi| \rightarrow |\psi| = |\phi \rightarrow \psi|$$

and distinguished elements $0 = |\square|$ and $1 = |\square \rightarrow \square|$, constitutes a Heyting algebra (we use intentionally the same symbols for operations of a Heyting algebra and logical connectives because this both stresses the similarity and improves readability while the possibility of misunderstanding is minimal). In fact, H_p is a free Heyting algebra on the set of generators $\{|p| : p \in P\}$, as can be seen from the following argument. Let H be an arbitrary Heyting algebra and $h:P \rightarrow H$ a valuation of propositional letters in H. Then there exists a unique homomorphism $h^* : H_p \rightarrow H$, defined by $h^*(|p|) = h(p)$ for $p \in P$, and by usual induction on complexity for $\phi \in S-P$ (e.g., $h^*(|\phi \;\&\; \psi|) = h^*(|\phi|) \;\&\; h^*(|\psi|)$, where clearly the first & denotes the logical connective and the second &, the operation in H). Obviosuly, the following diagram commutes (where $i : P \rightarrow H_p$ ($i(p) = |p|$) is the natural inclusion)



Due to this fact, it is enough to consider only H_p. Namely, given a Heyting algebra H and a valuation $h:P \rightarrow H$, what interests us is only the subalgebra of H generated by the set $\{h(p) : p \in P\}$. This algebra is isomorphic to a quotient algebra of H_p obtained in the following way. Let $h^*: H_p \to H$ be defined as before and let $\bar{F} = \{|\phi| \in H_p : h^*(|\phi|) = 1_H\}$. We can define now the quotient algebra $H_{p/F}$ whose elements are $|\phi|_F = \{|\psi| \in H_p$: for some $a \in F$, $|\phi|$ & $a = |\psi|$ & $a\} = \{|\psi| : |\phi \leftrightarrow \psi| \in F\} = \{|\psi| : h^*(|\phi \leftrightarrow \psi|) = 1\}$, and operations are those induced by H_p (e.g., $|\phi|_F$ & $|\psi|_F$ = $=|\,|\phi|$ & $|\psi|\,|_F = |\phi$ & $\psi|_F$). If $H' \subseteq H$ is the smallest subalgebra of H containing the set $\{h(p) : p \in P\}$, it is easy to show that the function f, defined by $f(|\phi|_F) = h^*(|\phi|)$, is an isomorphism of $H_{p/F}$ onto H'. Here F represents the set of formulas which are valid in H under valuation h, in effect a propositional theory, and $H_{p/F}$ describes, in a sense, the consequences of that theory. This can be made clearer in the following way.

Let $\Gamma = \{A_i : i \in I\}$ be the set of axioms of a propositional theory. That theory is consistent if and only if the set $\Gamma° = \{|A_i| : i \in I\} \subseteq H_p$ has the finite intersection property. If this is the case, then the set $\Gamma°$ can be extended to a proper filter F (i.e., $F \neq H_p$). The filter F corresponds to the deductive closure of Γ because $\Gamma \vdash \phi$ iff $|\phi| \in F$. Conversely, if a filter F in H_p is given, then it determines a theory $\Gamma = \{\phi : |\phi| \in F\}$. A theory is said to have <u>disjunction property</u> if $\Gamma \vdash \phi \vee \psi$ implies $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$. A filter F is <u>prime</u> if $a \vee b \in F$ implies $a \in F$ or $b \in F$. It follows immediately that a filter F in H_p is prime if and only if the theory $\Gamma = \{\phi : |\phi| \in F\}$ has disjunction property.

It is generally considered that an intuitionistic theory should have disjunction property. Therefore, we work only with prime filters. It is easy now to demonstrate the connection between Kripke models and Heyting algebras. We note, first, that the definition of Kripke structure for propositional language is obtained from Definition 2.1. by replacing classical structure $\mathfrak{A}_t$ with a set P_t of propositional letters and replacing clauses (i) - (iv) by

(*) $P_s \subseteq P_t$ (for $s \leq t$)

and clause (o) by

(**) $t \Vdash p$ iff $p \in P_t$

Let $\underline{F}$ be the set of all prime filters in H_p, including the trivial filter {1}. Note that ϕ is a theorem of intuitionistic propositional calculus iff $|\phi| = 1$. If we take the partially ordered set $< \underline{F}, \{1\}, \subseteq >$ and define $P_F = \{p \in P : |p| \in F\}$ for $F \in \underline{F}$ and $p \in P$, we obtain a Kripke model $\mathfrak{K}_p = \langle < \underline{F}, \{1\}, \subseteq > ; P_F : F \in \underline{F} \rangle$. We call this model canonical because it can be proved that $\{1\} \Vdash \phi$ iff ϕ is a theorem of intuitionistic propositional calculus. The proof is routine and is based on several standard facts about filters (e.g., if $|\phi \rightarrow \psi| \notin F$ then there exist $F' \in \underline{F}$ such that $F \subseteq F'$ and $|\phi| \in F'$ and $|\psi| \notin F'$). In addition, any propositional Kripke model (for the same set of propositional letters P) can be embedded into $\mathfrak{K}_p$ in the following way.

Lemma 1.

For any propositional Kripke model $\mathfrak{K} = \langle T; P_t : t \in T \rangle$ there exists an embedding $e : T \to \underline{F}$ such that for every $t \in T$ and formula ϕ, $t \Vdash \phi$ iff $e(t) \Vdash_e \phi$ (where $\Vdash_e$ is the forcing relation for the model $\langle\langle\{e(t) : t \in T\}, e(o), \subseteq\rangle; P_t : t \in T\rangle$).

Proof: Observe that for any t, $\{\phi : t \Vdash \phi\}$ is a deductively closed theory with disjunction property. Therefore, the corresponding filter in H_p, $F_t = \{|\phi| : t \Vdash \phi\}$, is prime. If we define now $e(t) = F_t$, we see that $|p| \in F_t$ iff $p \in P_t$ iff $F_t \Vdash_e p$. Standard induction on complexity of ϕ shows $t \Vdash \phi$ iff $|\Phi| \in F_t$ iff $F_t \Vdash_e \phi$.

By the same method, starting from an arbitrary Heyting algebra H and a valuation $h : P \to H$, we can construct a Kripke model on the p.o. set of prime filters of H, such that $h(\phi) = 1_H$ iff $0 \Vdash \phi$.

Conversely, given a Kripke model $\mathfrak{K} = \langle T; P_t : t \in T\rangle$, we can obtain a Heyting algebra H and a valuation h which makes exactly those formulas valid which are forced by the base node. Let $H = \{U \subseteq T : s \in U \text{ and } s \leq t \text{ implies } t \in U\}$ and for $p \in P$, let $h(p) = \{t \in T : t \Vdash p\}$. By Theorem 2.2. h (p) is an element of H for every $p \in P$. Operations v and & are defined as set-theoretic union and intersection, constants are $1 = T$ and $0 = \emptyset$ and $U \to V = \bigcup \{W \in H : U \cap W \subseteq V\}$. It is easy to show by induction on the complexity of ϕ, that the value of ϕ in H under valuation h is $\{t : t \Vdash \phi\}$ and therefore ϕ is valid iff in $\mathfrak{K}$, $0 \Vdash \phi$.

Let L be a first order intuitionistic language. Assume, for simplicity, that does not contain function symbols and let Rel be the set of all relation symbols and Ind the set of all individual constants of L. We can define now the Lindenbaum algebra of sentences of L. Let S_t be the set of all sentences of L, $|\phi| = \{\psi \in St : \vdash \phi \leftrightarrow \psi\}$ (now $\vdash$ means "theorem in IPC") and $H_L = \{|\phi| : \phi \in St\}$. The operations are defined the same way as in propositional case. The only difference is that in addition to axioms of Heyting algebra, two special infinitary conditions hold:

$$|\exists x \phi (x)| = \sup \{|\phi (c)| : c \in \text{Ind}\} \text{ and}$$
$$|\forall x \phi (x)| = \inf \{|\phi (c)| : c \in \text{Ind}\}.$$

The infinitary operations sup and inf are defined in the usual way from the ordering $\leq$, which inturn is defined by $x \leq y$ iff $x \& y = x$ iff $x \vee y = y$ or in this case also by $|\phi| \leq |\psi|$ iff $\vdash \phi \rightarrow \psi$.

All the results mentioned above about the connection between filters and theories also hold in this context. In addition, we shall consider filters which have the property

$$|\exists x \phi (x)| \in F \text{ implies } |\phi (c)| \in F.$$

Such filters are called <u>existential</u>. We shall show in the next section how proper, prime, existential filters (ppef) in H_L correspond to L-saturated theories.

4. Saturated theories

Saturated (intuitionistic) theories were proposed as an alternative semantics for intuitionistic first order logic by Aczel [1]. They were also used by Fitting [5], Gabbay [9], Thomason [39] and Smorynski [35] for proving various model-theoretic results.

Let L be a countable first order intuitionistic language. Since it is much easier to work with theories in which function symbols are replaced by relations, we assume that L does not contain any function symbols So let Rel be the set of all relation symbols and Ind the countable set of all individual constants. It will be convenient to consider Ind as partitioned into disjoint countable sets C_n $(n \in \omega)$. We shall speak in this section only about sentences of L.

We introduce now some new notation : Ind (ϕ) is the set of individual constants occurring in ϕ and Ind $(\Gamma) = \bigcup \{\text{Ind}(\phi) : \phi \in \Gamma\}$. Also, let Cn $(\Gamma) = \{\phi : \Gamma \vdash \phi$ and Ind $(\phi) \subseteq$ Ind $(\Gamma)\}$.

Definition 1.

Γ is a saturated theory if and only if

(i) Γ is deductively closed (i.e., Cn $(\Gamma) = \Gamma$)

(ii) Γ is consistent (i.e., $\Box \notin \Gamma$)

(iii) Γ has the disjunction property
(i.e., $\phi \vee \psi \in$ Cn (Γ) implies $\phi \in$ Cn (Γ) or $\psi \in$ Cn (Γ))

(iv) Γ has the existential instantiation property (i.e., if $\exists x \ \phi(x) \in Cn(\Gamma)$ then $\phi(c) \in Cn(\Gamma)$ for some individual constant c).

If Ind $(\Gamma) = C$, we also say that Γ is C-saturated.

It immediately follows from the definition of forcing and Theorem 2.1. that the set of all sentences forced at a node of a Kripke structure is saturated. The converse also holds, that is, every saturated theory is the set of all sentences forced at a node of a Kripke model. Let us first cite, without proofs, a few lemmas from [1]. We can assume, without loss of generality, that Ind $(\Gamma) = C_0$, so that all extensions of Γ we talk about can be assumed to be in the same language L (otherwise we could add to L a countable set of new individual constants, paritioned as above, and speak only about theories in the new language).

Lemma 1.

If Γ is a consistent set of sentences, Ind $(\phi) \subseteq$ Ind (Γ) and $\Gamma \not\Vdash \phi$ then there is a saturated extension $\Delta \supseteq \Gamma$ such that $\phi \notin \Delta$.

Lemma 2.

(a) If Γ is consistent, Ind $(\phi \rightarrow \psi) \subseteq$ Ind (Γ) and $\Gamma \not\vdash \phi \rightarrow \psi$ then there is a saturated $\Delta \supseteq \Gamma$ such that

$$\phi \in \Delta \text{ and } \psi \notin \Delta$$

(b) If Γ is consistent, Ind $(\forall x \ \phi(x)) \subseteq$ Ind (Γ) and $\Gamma \not\vdash \forall x \ \phi(x)$ then there is a saturated $\Delta \supseteq \Gamma$ and $c \in$ Ind (Δ) such that

$$\phi(c) \notin \Delta.$$

As we said, all extension Δ mentioned in Lemma 1 and 2 can be taken to be in L. To make things more transparent we can also assume Ind $(\Delta) = \bigcup_{n=0}^{k} C_n$ for some k. If Γ is consistent, let

$$S_\Gamma = \{\Delta : \Gamma \subseteq \Delta, \Delta \text{ is saturated and Ind } (\Delta) = \bigcup_{n=0}^{k} C_n \text{ for some } k\}.$$

It follows from Lemma 1.

Lemma 3.

$\Gamma \vdash \phi$ iff $\phi\varepsilon\Delta$ for every $\Delta \varepsilon S_\Gamma$ such that Ind $(\phi) \subseteq$ Ind (Δ)

We can construct now a canonical Kripke model for a saturated theory Γ. Note that, if Γ is saturated, then $\Gamma \varepsilon S_\Gamma$, so $< S_\Gamma, \Gamma, \subseteq >$ is a paritally ordered set with the least element Γ. For every $\Delta \varepsilon S_\Gamma$ define A_Δ = Ind (Δ) and for each n-ary relation symbol $R\varepsilon Rel$, $R^\Delta = \{< c_1, \ldots, c_n > : R(c_1, \ldots c_n) \varepsilon \Delta\}$. Thus we get a classical structure $\mathfrak{U}_\Delta = \langle A_\Delta, \{R^\Delta : R \varepsilon Rel\} \rangle$. It is easy to see that

$$\mathfrak{M}_\Gamma = \langle < S_\Gamma, \Gamma, \subseteq >; \mathfrak{U}_\Delta : \Delta \varepsilon S_\Gamma \rangle$$ is a Kripke structure with a

countable universe at each node. It follows from Lemmas 1-3 that for every $\Delta\varepsilon S_\Gamma$, $\Delta \Vdash \phi$ iff $\phi \varepsilon \Delta$. Therefore:

Lemma 4.

Γ is saturated if and only of there is a Kripke structure such that Γ is the set of all sentences forced at its base node.

The proof of the Strong Completeness Theorem (2.4.) follows from

Lemmas 3. and 4. Namely, if $\Gamma \vdash \phi$, for every Kripke model of Γ the set of sentences forced at the base node will be some $\Delta \varepsilon S_\Gamma$ and if Ind $(\phi) \subseteq$ Ind (Δ), by Lemma 3. $\phi \varepsilon \Delta$. Conversely, if $\Gamma \nvdash \phi$, again by Lemma 3, there exists $\Delta \varepsilon S_\Gamma$ such that Ind $(\phi) \subseteq$ Ind (Δ) and $\phi \notin \Delta$, so $\mathfrak{M}_\Delta$ will be a model of Γ in which ϕ does not hold, so

$\Gamma \not\models_k \phi$.

<u>Remark</u>: Strong Completeness Theorem and results equivalent to Lemmas 1-4 were proved independently by Aczel [1], Fitting [5] and Thomason [39]. Aczel's proof was more general in the sense that he worked with arbitrary sets of individual constants and Kripke models in which classical structures can be indexed by a class instead of set, but as he states it, this generalization does not have any major significance.

At the end, let us say a few words about the connection between Kripke models, saturated theories and Heyting algebra. Heyting algebra semantics for IPC is defined by assigning to each n-ary relation symbol a function mapping n-tuples of individuals (elements of the fixed domain) into elements of a Heyting algebras. Since we are interested only in sentences, we can restrict our attention to the Lindenbaum algebra H_L (defined at the end of Section 3).

It is straightforward that if Γ is a saturated theory (in L) then $F(\Gamma) = \{|\phi| \varepsilon H_L : \phi \varepsilon \Gamma\}$ is a proper, prime existential filter (ppef) in H_L, and conversely, if F is ppef in H_L then $\Gamma(F) = \{\phi: |\phi| \varepsilon F\}$ is a saturated theory. Note that by this process we do not return to the same Γ because Ind $(\Gamma\ (F(\Gamma)))$ = Ind (i.e., the set of all individual constants of L). However, if $St(C) = \{\phi \in St : Ind(\phi) \subseteq C\}$, then:

<u>Lemma 5.</u>

$$\Gamma(F(\Gamma)) \cap St\ (Ind(\Gamma)) = \Gamma$$

Proof: By definition,

$$\Gamma(F(\Gamma)) = \{\phi : |\phi| \in F\ (\Gamma)\ \} = \{\phi : \text{for some } \psi \in \Gamma,\ \vdash \phi \leftrightarrow \psi\}.$$

Therefore

$$\Gamma(F(\Gamma)) \cap St\ (Ind(\Gamma)) = \{\phi : Ind\ (\phi) \subseteq Ind\ (\Gamma) \text{ and for some } \psi \in \Gamma,\ \vdash \phi \leftrightarrow \psi\}.$$

But, Ind $(\phi) \subseteq$ Ind (Γ), $\psi \in \Gamma$ and $\vdash \phi \leftrightarrow \psi$ implies $\phi \in Cn(\Gamma) = \Gamma$, which proves the Lemma.

The converse, $F(\Gamma\ (F)) = F$, for F a ppef in H_L, is obvious. We shall need also the following Lemma.

<u>Lemma 6.</u>

Let F be a ppef in H_L and let $H_{L/F}$ be the quotient algebra.

If filter $D \supseteq F$ is ppef in H_L then $D' = \{|\phi|_F \in H_{L/F} : |\phi| \in D\}$ is ppef in $H_{L/F}$. Conversely, if E' is a ppef in $H_{L/F}$ then

$E = \{|\phi| : |\phi|_F \in E'\}$ is a ppef in H_L, containing F.

Proof: The proof consists in a routine checking of definitions. We show a few cases for illustration. If $|\phi|_F$, $|\psi|_F \varepsilon D'$, then $|\phi|$, $|\psi| \varepsilon D'$, so $|\phi \ \& \ \psi| \varepsilon D$ and $|\phi \ \& \ \psi|_F \varepsilon D'$. By definition $|\phi|_F \ \& \ |\psi|_F = ||\phi| \ \& \ |\psi||_F = |\phi \ \& \ \psi|_F$ so it follows that $|\phi|_F \ \& \ |\psi|_F \varepsilon D'$.

Suppose $|\phi|_F \varepsilon D'$ and $|\phi|_F \leq_F |\psi|_F$. By definition of $H_{L/F}$, this is equivalent to $|\phi \rightarrow \psi| \varepsilon F$. Since $F \subsetneq D$ it follows that $|\phi \rightarrow \psi| \varepsilon D$. But $|\phi|_F \varepsilon D'$ implies $|\phi| \varepsilon D$, so $|\psi| \varepsilon D$ and $|\psi|_F \varepsilon D'$, etc.

It is clear now that if Γ is a saturated theory and $F = F(\Gamma)$ is the associated filter in H_L, the set of all proper, prime and existential filters in $H_{L/F}$ corresponds to the set S_Γ of all saturated extensions of Γ, which in turn determines the canonical Kripke model of Γ. Therefore, we could say that saturated theories are, in a sense, a link between Kripke models and Heyting algebras.

CHAPTER II

1. Some Properties of Forcing

In this section we shall prove a few results about the connection between forcing, $\Vdash$ and (classical) satisfaction relation, $\models$.

A formula is called <u>positive</u> if it is built up from atomic formulas using only v, & and $\exists$. Let $P = \{\phi : \phi \text{ is positive}\}$. A trivial inspection of clauses (0)-(3) of the definition of forcing yields

<u>Lemma</u> 1.

Let $\mathfrak{K} = \langle T; \mathfrak{B}_t : t\varepsilon T\rangle$ be a Kripke structure, $t\varepsilon T$ and $\phi\varepsilon P$. Then:

(i) $t \Vdash \phi$ if $\mathfrak{B}_t \models \phi$.

(ii) if $\mathfrak{B}_t \models \phi$ then for every $s \geq t$, $\mathfrak{B}_s \models \phi$.

(iii) if $\mathfrak{B}_t \not\models \phi$ then for every $s \leq t$, $\mathfrak{B}_s \not\models \phi$.

<u>Definition 1</u>.

Let $S_o = P$ and if S_n is already defined, let S_{n+1} be the smallest set of formulas satisfying:

(1) $S_n \subseteq S_{n+1}$

(2) if ϕ, $\psi\varepsilon S_n$ then $(\sim\phi \rightarrow \psi)\varepsilon S_{n+1}$

(3) if ϕ, $\psi\varepsilon S_n$ then $\sim\sim\phi$, $\sim \forall x \sim \phi$ and $\sim(\phi \rightarrow \sim \psi)$ are in S_{n+1}

(4) if ϕ, $\psi\varepsilon S_n$ then $(\phi \;\&\; \psi)$, $(\phi \vee \psi)$ and $\exists x\phi$ are in S_{n+1}

Finally, let $S_\omega = \bigcup_{n\varepsilon\omega} S_n$

<u>Theorem 1</u>.

If $\phi(x_1, \ldots, x_n)$ is in S_ω then for any Kripke structure $\mathfrak{K} = \langle T; \mathfrak{A}_t : t\varepsilon T\rangle$, any $t\varepsilon T$ and any $a_1, \ldots, a_n\varepsilon A_t$

$\mathfrak{A}_t \models \phi[a_1, \ldots, a_n]$ implies $t \Vdash [a_1, \ldots, a_n]$.

Proof: ϕ is in S_ω just in case it is in S_n, for some n. So we prove, by induction, that for every n, the theorem holds for all formulas from S_n. Since $S_o = P$, the theorem holds for S_o, by Lemma 1. Suppose that it holds for S_n and let $\phi \varepsilon S_{n+1} - S_n$ (without loss of generality, we shall suppress the free variables in ϕ). There are seven cases.

(i) ϕ is $(\sim \psi \rightarrow \chi)$, where ψ and χ are in S_n. Then $\mathfrak{A}_t \models \phi$ means $\mathfrak{A}_t \models \psi \vee \chi$, so either $\mathfrak{A}_t \models \psi$ or $\mathfrak{A}_t \models \chi$. By induction hypothesis then $t \Vdash \psi$ or $t \Vdash \chi$. If $t \Vdash \psi$, for every $s \geq t$, $s \nVdash \sim\psi$. If $t \Vdash \chi$, for every $s \geq t$, $s \Vdash \chi$. In either case it follows that $t \Vdash \sim \psi \rightarrow \chi$.

(ii) ϕ is $\sim \sim\psi$, for some $\psi \varepsilon S_n$. $\mathfrak{A}_t \models \phi$ implies then $\mathfrak{A}_t \models \psi$, so by induction hypothesis $t \Vdash \psi$ and consequently $t \Vdash \sim \sim\psi$.

(iii) ϕ is $\sim \forall x \sim\psi(x)$, for some $\psi \varepsilon S_n$. $\mathfrak{A}_t \models \phi$ implies $\mathfrak{A}_t = \exists x\ \psi(x)$ so there is an $a \varepsilon A_t$ such that $\mathfrak{A}_t \models \psi[a]$. By induction hypothesis then $t \Vdash \psi[a]$. It follows that $t \Vdash \sim \sim\psi[a]$ and $t \Vdash \exists x \sim \sim \psi(x)$ so $t \Vdash \sim \forall x \sim \psi(x)$.

(iv) ϕ is $\sim(\psi \rightarrow \sim \chi)$, where $\psi, \chi \varepsilon S_n$. $\mathfrak{A}_t \models \phi$ implies $\mathfrak{A}_t \models \psi \mathbin{\&} \chi$, so by induction hypothesis $t \Vdash \psi \mathbin{\&} \chi$. Since $(\psi \mathbin{\&} \chi) \rightarrow \sim(\psi \rightarrow \sim\chi)$ is a theorem, it follows that $t \Vdash \phi$.

The other three cases, when ϕ is $\psi \mathbin{\&} \chi$, $\psi \vee \chi$ or $\exists x\ \psi$, are easily proved, in a similar fashion.

<u>Definition 2</u>.

Let S be the set of all formulas (from L) which are classically equivalent to, and intuitionistically implied by, a formula from S_ω, i.e., $S = \{\phi$: for some $\psi \varepsilon S_\omega$, $\vdash_c \psi \leftrightarrow \phi$ and $\vdash \psi \rightarrow \phi\}$.

Corollary 1.

If $\phi(x_1, \ldots, x_n)$ is in S then

$\mathfrak{A}_t \models [a_1, \ldots, a_n]$ implies $t \Vdash \phi[a_1, \ldots, a_n]$

Proof: Suppose that $\mathfrak{A}_t \models \phi$ and let $\psi \varepsilon S_\omega$ be such that $\vdash_c \psi \leftrightarrow \phi$ and $\vdash \psi \rightarrow \phi$. Then $\mathfrak{A}_t \models \psi$ and by Theorem 1, $t \Vdash \psi$ so $t \models \phi$.

Definition 3.

Let $R_o = P \cup \{\sim \phi : \phi \varepsilon S\}$ and if R_n is already defined let R_{n+1} be the smallest set of formulas satisfying:

(1) $R_n \subseteq R_{n+1}$

(2) if $\phi \varepsilon S$ and $\psi \varepsilon R_n$ then $(\phi \rightarrow \psi) \varepsilon R_{n+1}$

(3) if ϕ, $\psi \varepsilon R_n$ then $\forall x\, \phi$, $\exists x\, \phi$, $(\phi \vee \psi)$ and $(\phi \,\&\, \psi)$ are in R_{n+1}

Let $R_\omega = \bigcup_{n \varepsilon \omega} R_n$

Theorem 2.

If $\phi(x_1, \ldots, x_n)$ is in R_ω then for any $\mathfrak{K} = \langle T; \mathfrak{A}_t : t \varepsilon T\rangle$, any $t \varepsilon T$ and any $a_1, \ldots, a_n \varepsilon A_t$

$t \Vdash \phi[a_1, \ldots, a_n]$ implies $\mathfrak{A}_t \models \phi[a_1, \ldots, a_n]$

Proof: by induction. Suppose ϕ is in R_o and $t \Vdash \phi$. If $\phi \varepsilon P$, $\mathfrak{A}_t \models \phi$ by Lemma 1. If not, there is a formula $\psi \varepsilon S$ such that ϕ is $\sim \psi$. Then $t \nVdash \psi$ and by Corollary 1. $\mathfrak{A}_t \nvDash \psi$, so $\mathfrak{A}_t \models \sim \psi$. Suppose now that the theorem holds for $\phi \varepsilon R_n$ and let $\phi \varepsilon R_{n+1} - R_n$ and $t \Vdash \phi$. There are five cases:

(i) ϕ is $\psi \rightarrow \chi$, where $\psi \varepsilon S$ and $\chi \varepsilon R_n$. Now $t \Vdash \psi \rightarrow \chi$ implies that $t \nVdash \psi$ or $t \Vdash \chi$. If $t \nVdash \psi$, by Corollary 1. $\mathfrak{A}_t \nvDash \psi$ and if $t \Vdash \chi$, by induction hypothesis $\mathfrak{A}_t \models \chi$. In either case $\mathfrak{A}_t \models \psi \to \chi$.

Other cases, when ϕ is $\forall x\, \psi$, $\exists x\, \psi$, $(\psi \vee \chi)$ or $(\psi \,\&\, \chi)$ are trivial.

Definition 5.

Let R = $\{\phi$: there is a formula $\psi \varepsilon R_\omega$ such that $\vdash_c \phi \leftrightarrow \psi$ and $\vdash \phi \to \psi\}$.

Corollary 2.

If $\phi \varepsilon R$ then $t \Vdash \phi$ implies $\mathfrak{A}_t \models \phi$.

2. Omitting Types Theorem

Some explanation is due here of what is meant by "omitting types theorem". Since classically equivalent formulas are not necessarily intuitionistically equivalent, different (equivalent) formulations of the classical theorem may appear here as candidates for different theorems. In the case of omitting types theorem we start with a consistent theory Γ and a set of formulas. $\Sigma = \{\sigma(x):\sigma$ has at most x free $\}$, in the same language. The theorem states that if Γ locally omits Σ then it has a model omitting Σ. Now, there are at least four, classically equivalent, formulations of the "locally omitting" condition:

for every sentence $\exists x\, \phi(x)$ in the same language as Γ and consistent with Γ, there is a formula $\sigma(x) \varepsilon \Sigma$ such that

(i) Γ is consistent with $\exists x\, (\phi \,\&\, \sim\sigma)$

(ii) Γ is consistent with $\exists x \sim(\phi \to \sigma)$

(iii) Γ is consistent with $\sim\forall x (\phi \to \sigma)$

(iv) $\Gamma \nvDash \forall x (\phi \to \sigma)$

Intuitionistically, (i) is equivalent to (ii), (ii) implies (iii) and (iii) implies (iv), but in general case none of the reverse implications holds, as the simple counter-examples will show. Let us see first, what each of (i) through (iv) means in terms of Kripke structures.

(i) There is a model $\mathfrak{K}$ of Γ with an element $a \varepsilon A_0$ (the universe at the base node) such that $0 \Vdash \phi[a]$. Furthermore, for every $t \varepsilon T$, $t \nVdash \sigma[a]$.

(ii) There is a model $\mathfrak{K}$ of Γ with an element $a \varepsilon A_0$ such that every node has a node above it which forces $\phi[a]$ and does not force $\sigma[a]$.

(iii) There is a model $\mathfrak{K}$ of Γ such that for every node t, there is a node $s \geq t$ and an element $a \varepsilon A_s$ such that $s \Vdash \phi[a]$ and $s \nVdash \sigma[a]$.

(iv) There is a model of Γ and a node t in it and an element $a \varepsilon A_t$ such that $t \Vdash \phi[a]$ and $t \nVdash \sigma[a]$.

$\exists x (\phi \& \sim\sigma) \to \exists x \sim(\phi \to \sigma)$ and $\exists x \sim(\phi \to \sigma) \to \sim \forall x (\phi \to \sigma)$ are theorems and obviously $\forall x (\phi \to \sigma)$ cannot hold in a model in which $\sim\forall x (\phi \to \sigma)$ holds. Although $\exists x \sim(\phi \to \sigma) \to \exists x(\phi \& \sim\sigma)$ is not a theorem, (ii) implies (i) because from the model whose existence was assumed in (ii), we can obtain, by truncation, a model for $\Gamma \cup \{\exists x (\phi \ \&\ \sim\sigma)\}$. Namely, if t_0 is such that $t_0 \Vdash \phi[a]$ and $t_0 \nVdash \sigma[a]$ then clearly $t_0 \Vdash \sim\sigma[a]$, so let $T' = \{s \in T : t_0 \leq s\}$ and let $\mathcal{T}' = \langle T', t_0, \leq \rangle$. The model $\mathfrak{K}' = \langle \mathcal{T}'; \mathfrak{A}_s : s \in T' \rangle$ is the model required in (i). To see that (iii) does not imply (ii), suppose that $\Gamma \vdash \sim\exists x \sim\sigma(x)$. A model from (ii) must have, as we saw, $t_0 \Vdash \sim\sigma[a]$ for some t_0 and a. But this means that $t_0 \Vdash \exists x \sim\sigma(x)$, which cannot happen in a model for Γ. The following example shows that $\sim\exists x \sim\sigma(x)$ is consistent with $\sim \forall x(\phi \rightarrow \sigma)$. Let $\mathcal{T} = \langle N, 0, \leq \rangle$. $A_n = \{0.1, \ldots, n\}$ and suppose $n \Vdash \sigma[m]$ for $m < n$ and $n \Vdash \phi[m]$ for every n, $m \in \omega$. Then for every $n \in \omega$ $n \Vdash \phi[n]$ and $n \nVdash \sigma[n]$, so $n \nVdash \forall x (\phi \rightarrow \sigma)$ and $n \nVdash \exists x \sim\sigma(x)$. Finally, (iv) does not imply (iii), because a model in (iv) might have nodes which force $\forall x (\phi \rightarrow \sigma)$ (it is enough that $\Gamma \vdash \sim\sim\forall x (\phi \rightarrow \sigma)$).

Hence, we have three alternatives for the antecedent of omitting types theorem. As for the consequent, we should have at least the following: there is a model of Γ in which no element of the universe at the base node realizes Σ, i.e., for every $a \in A_0$ there is $\sigma(x) \in \Sigma$ such that $0 \nVdash \sigma[a]$. This is in accord with the basic intuition about types of elements and about models omitting types just in case they do not contain elements of a certain kind, since the universe at the base node is in a sense the principal domain of a Kripke model.

We shall prove now the following.

<u>Theorem 1</u>

Let L be a countable first order intuitionistic language with equality and let Γ be a consistent set of sentences in L and Σ a set of formulas in L with at most one variable, x, free. Suppose that for every sentence $\exists x\, \phi(x)$ in L, consistent with Γ, there is a formula $\sigma(x)\varepsilon\Sigma$ such that $\exists x(\phi(x)\ \&\ {\sim}\sigma(x))$ is consistent with Γ. Then there exists a model of Γ in which no element of the universe at the base node realizes Σ. Moreover, this model can be taken to have a countable universe at each node and for each element $\underline{a}$ of the universe at the base node, there is $\sigma(x)\varepsilon\Sigma$ such that ${\sim}\sigma[a]$ holds in the model.

Proof: Let $C = \{c_0, c_1, \ldots\}$ be a countable set of individual constants not occurring in L and let $L' = L \cup C$. We shall prove the theorem by extending Γ to an L'-saturated theory Γ_ω (i.e., Γ_ω is saturated and $Ind(\Gamma_\omega) \subseteq L'$) with the property that for every individual constant c from L' there is a formula $\sigma(x)\varepsilon\Sigma$ such that ${\sim}\sigma(c)\varepsilon\Gamma\omega$. Then canonical Kripke model obtained from $S_\Gamma = \{\Delta : \Gamma\omega \subseteq \Delta$ and Δ is $(L' \cup C')$-saturated for some countable set C' of individual constants$\}$ is the desired model.

Let $E_0 = \{\exists x\, \phi_i(x) : i\varepsilon\omega$ and ϕ_i is in the language $L'\}$ and $D_0 = \{\phi_i \vee \psi_i : i\varepsilon\omega$ and $\phi_i \vee \psi_i$ is in the language $L'\}$ be some enumerations of all existential and disjunctive, respectively, sentences in L' and let $\Gamma_0 = \Gamma$. If Γ_n, E_n and D_n $(n\varepsilon\omega)$ are already defined, we proceed as follows depending on which of the three cases applies:

Case 1: $n = 3K$

Let $\exists x\, \phi(x)$ be the first sentence from E_n such that

$\Gamma_n \vdash \exists x\, \phi(x)$ and let c be the first constant from C not occurring in Γ_n or ϕ. Set $\Gamma_{n+1} = \Gamma_n \cup \{\phi(c)\}$. $E_{n+1} = E_n - \{\exists x\, \phi(x)\}$ and $D_{n+1} = D_n$. We claim that Γ_{n+1} is consistent if Γ_n is . Suppose not, i.e., $\Gamma_{n+1} \vdash \Box$. This means that for some $\phi_1, \ldots, \phi_j \varepsilon \Gamma_n$, $\vdash (\phi(c) \;\&\; \phi_1 \;\&\; \ldots \;\&\; \phi_j) \to \Box$. Since c occurs only in ϕ, this implies

$$\vdash (\exists x\, \phi(x) \;\&\; \phi_1 \;\&\; \ldots \;\&\; \phi_j) \to \Box \quad \text{and} \quad \vdash (\phi_1 \;\&\; \ldots \;\&\; \phi_j) \to (\exists x\, \phi(x) \to \Box)$$

so Γ_n is inconsistent (because $\Gamma_n \vdash \exists x\, \phi(x)$).

<u>Case 2: n = 3k+1</u>

Let $\phi \vee \psi$ be the first sentence from D_n such that $\Gamma_n \vdash \phi \vee \psi$. If Γ_n is consistent with ϕ let $\Gamma_{n+1} = \Gamma_n \cup \{\phi\}$. Otherwise let $\Gamma_{n+1} = \Gamma_n \cup \{\psi\}$. In either case let $D_{n+1} = D_n - \{\phi \vee \psi\}$ and $E_{n+1} = E_n$. Again, we claim that Γ_{n+1} is consistent if Γ_n is. For suppose not. Then $\Gamma_n \cup \{\phi\} \vdash \Box$ and $\Gamma_n \cup \{\psi\} \vdash \Box$. By the same argument as above, we get $\Gamma_n \vdash \phi \to \Box$ and $\Gamma_n \vdash \psi \to \Box$ and consequently $\Gamma_n \vdash (\phi \vee \psi) \to \Box$, so Γ_n would be inconsistent.

<u>Case 3: n = 3k+2</u>

Thus far we have constructed $\Gamma_n = \Gamma \cup \{\phi_1, \ldots, \phi_n\}$. Let all individual constants from C occurring in Γ_n be among $c_k, c_{i1}, \ldots, c_{im}$ (we may assume $k \neq i_1, \ldots, k \neq im$). Let $\phi(c_k, c_{i1}, \ldots, c_{im}) = \phi_1 \;\&\; \ldots \;\&\; \phi_n$ and if $x, x_1, \ldots, x_n$ are individual variables not occurring in $\phi(c_k, c_{i1}, \ldots, c_{im})$ let $\phi(x) = \exists x_1 \ldots \exists x_m\, \phi(x, x_1, \ldots x_m)$. Now $\exists x\, \phi(x)$ is a sentence in **L** and it is consistent with Γ so there is a formula $\sigma(x)\varepsilon\Sigma$ such that $\exists x(\phi(x) \& \sim\sigma(x))$ is consistent with Γ. Let $\Gamma_{n+1} = \Gamma_n \cup \{\sim \sigma(c_k)\}$, $E_{n+1} = E_n$ and $D_{n+1} = D_n$. If Γ_n was consistent, Γ_{n+1} is also. Otherwise, we would have $\vdash (\psi_1 \& \ldots \& \psi_j \& \phi(c_k, c_{i1}, \ldots, c_{im}) \& \sim\sigma(c_k)) \rightarrow \Box$ for some $\psi_1, \ldots, \psi_j \varepsilon \Gamma$. Since none of $c_k, c_{i1}, \ldots, c_{im}$ occurs in $\psi_1, \ldots, \psi_j$, this implies

$$\vdash (\psi_1 \& \ldots \& \psi_j \& \exists x (\phi(x) \& \sim\sigma(x))) \rightarrow \Box \text{ and } \Gamma \vdash \exists x (\phi(x) \& \sim\sigma(x)) \rightarrow \Box .$$

As we assumed that Γ_n is consistent and that Γ is consistent with $\exists x (\phi(x) \& \sim\sigma(x))$, this is a contradiction.

Finally, let $\Gamma_\omega = \bigcup_{n\varepsilon\omega} \Gamma_n$. To prove that Γ_ω is $\mathbf{L}'$-saturated we have to show the following four facts:

(1) $C_n(\Gamma_\omega) = \Gamma_\omega$. Suppose $\Gamma_\omega \vdash \phi$ and ϕ is a sentence in $\mathbf{L}'$. $\Gamma_\omega \vdash \phi$ iff $\Gamma_n \vdash \phi$ for some $n\varepsilon\omega$. But $\Gamma_n \vdash \phi$ implies $\Gamma_n \vdash \phi \vee \phi$, so for some $3_{k+1} \geq n$, $\phi \vee \phi$ will be the first consequence of Γ_{3k+1} in the list D_{3k+1} and $\Gamma_{3n+2} = \Gamma_{3n+1} \cup \{\phi\}$, so $\phi\varepsilon\Gamma_\omega$.

(2) Γ_ω is consistent, i.e., $\Gamma_\omega \nvdash \Box$. For suppose $\Gamma_\omega \vdash \Box$. Then $\Gamma_n \vdash \Box$ for some $n\varepsilon\omega$, but this is impossible since the construction was performed in such a way that Γ_n is consistent, provided Γ is consistent and "locally omits" Σ.

(3) If $\phi \vee \psi\varepsilon\Gamma_\omega$ then $\phi\varepsilon\Gamma_\omega$ or $\psi\varepsilon\Gamma_\omega$. $\phi \vee \psi\varepsilon\Gamma_\omega$ means that for some $n\varepsilon\omega$, $\phi \vee \psi \varepsilon\Gamma_n$. Then for some $3k+1$, $\phi \vee \psi$ is the first consequence of Γ_{3k+1} in the list D_{3k+1} and consequently either $\Gamma_{3k+2} = \Gamma_{3k+1} \cup \{\phi\}$ or $\Gamma_{3k+2} = \Gamma_{3k+1} \cup \{\psi\}$.

(4) If $\exists\, x\, \phi(x) \varepsilon \Gamma_\omega$ then $\phi(c) \varepsilon \Gamma_\omega$ for some $c \varepsilon L'$. As before, $\exists\, x\, \phi(x) \varepsilon \Gamma_\omega$ means $\exists\, x\, \phi(x) \varepsilon \Gamma_n$ for some $n \varepsilon \omega$ and so for some 3k, $\exists\, x\, \phi(x)$ is the first consequence of Γ_{3k} in the list E_{3k}. Then for some $c \varepsilon C$, $\Gamma_{3k+1} = \Gamma_{3k} \cup \{\phi\ (c)\}$.

Obviously, no $c_k \varepsilon C$ can realize Σ since $\Gamma_{3k+3} = \Gamma_{3k+2} \cup \{\sim\sigma\ (c_k)\}$, for some $\sigma(x) \varepsilon \Sigma$. If $d \varepsilon Ind(\Gamma)$ then $\Gamma \vdash \exists\, x\ (x=d)$ and $\exists\, x\ (x=d) \varepsilon E_0$, so for some n, $\exists\, x(x=d)$ is the first consequence of Γ_{3n} in the list E_{3n}. For some $c_k \varepsilon C$ then $(c_k=d) \varepsilon \Gamma\omega$ so d cannot realize Σ.

The present theorem can be generalized in two directions: to n-types, that is, for the case when Σ is a set of formulas with n free variables, and to the case of simultaneous omitting of countably many types. We show now the second case.

<u>Theorem 2</u>

Let Γ and each of $\Sigma_q (q \varepsilon \omega)$ be as in the statement of Theorem 1, in particular suppose that, for every sentence $\exists\, x\, \phi(x)$ consistent with Γ and every $q \varepsilon \omega$ there is a formula $\sigma(x) \varepsilon \Sigma_q$ such that $\exists\, x\ (\phi(x)\ \&\ \sim\sigma(x))$ is consistent with Γ. Then, Γ has a model omitting each of Σ_q. As in Theorem 1, there is in fact a model of Γ, with a countable universe at each node, such that for each element <u>a</u> of the universe A_0 at the base node o and each $q \varepsilon \omega$, there is a formula $\sigma(x) \varepsilon \Sigma_q$ with $o \Vdash \sim\sigma[a]$.

Proof: The proof is virtually the same except the Case 3, n = 3k+2. Instead of treating c_k, if $k=2^p\ (2q+1)$ we work with c_p and the type Σ_q. As in the proof of Theorem 1, $\Gamma_n = \Gamma \cup \{\phi_1, \ldots, \phi_n\}$ and if all the constants from C occurring in $\phi_1, \ldots, \phi_n$ are among $c_p, c_{i1}, \ldots, c_{im}$ (with $p \neq i_1, \ldots, p \neq i_m$) let $\phi(c_p, c_{i1}, \ldots, c_{im}) = \phi_1 \& \ldots \& \phi_n$, and $\phi(x) = \exists x_1 \ldots \exists x_m \phi(x, x_1, \ldots, x_m)$. Again, $\exists x \phi(x)$ is consistent with Γ so there is $\sigma(x) \varepsilon \Sigma_q$ such that $\exists x (\phi(x) \& \sim\sigma(x))$ is consistent with Γ. By the same kind of argument $\Gamma_{n+1} = \Gamma_n \cup \{\sim\sigma(c_p)\}$ is consistent, and the proof proceeds the same way as in the case of Theorem 1.

CHAPTER III

1. Prime products of saturated theories

The classical analogue of saturated theories are the so called complete Henkin theories. There is a natural one-to-one correspondence between such theories and classical structures, so the results of classical model theory can be formulated in terms of classical saturated theories.

If classical model theory is thus regarded as the theory of classical saturated theories, intuitionistic model theory formulated as the theory of intuitionistic saturated theories appears to be its smooth generalization. In [1] Aczel suggested that such an approach might be useful for obtaining results in intuitionistic model theory. In particular, he suggested that ultraproduct construction, if formulated in terms of saturated theories instead of in terms of structures, could be carried out for intuitionistic saturated theories. It turns out, as we show now, that it is enough to take the filter over which the product is reduced to be prime.

Let Γ_i $(i\varepsilon I)$ be a collection of saturated theories and let $C_i =$ Ind (Γ_i) (we use the same notational conventions as in Section 4 of Chapter I). Let F be a prime filter over I and let $C = \prod_F C_i$ be the reduced product of the C_i $(i\varepsilon I)$ defined in the usual way. If $a\varepsilon \prod_{i\varepsilon I} C_i$ let $a/_F = \{b\varepsilon \prod_{i\varepsilon I} C_i : \{i\varepsilon I : a(i) = b(i)\}\varepsilon F\}$. Let $S_t(C)$ be the set of all sentences whose all individual constants are in C. If $\phi\varepsilon S_t(C)$ and $\mathrm{Ind}(\phi) = \{a_{1/F}, \ldots, a_{n/F}\}$ let $\phi^i = \phi(a_1(i), \ldots, a_n(i))$. Contrary to the usual procedure where we first define the (algebraic) product of structures and then prove Los's theorem for it, here Los's theorem will hold by definition and it will be proved that the obtained set of sentences is a "structure" of the required kind, i.e., a saturated theory.

Definition 1

Let $\prod_F \Gamma_i = \{\phi\varepsilon S_t(C):\{i:\phi^i\varepsilon\Gamma_i\}\varepsilon F\}$. $\prod_F \Gamma_i$ is called the reduced product of saturated theories Γ_i ($i\varepsilon I$) over the filter F. If the filter F is prime, $\prod_F \Gamma_i$ is called a prime product.

Theorem 1

If the filter F is prime, the set of sentences $\Gamma = \prod_F \Gamma_i$ is a saturated theory.

Proof: (1) Γ is deductively closed, that is, $C_n(\Gamma) = \Gamma$. We prove the nontrivial direction. Let $\phi\varepsilon C_n(\Gamma)$. Then there are $\psi_1, \ldots, \psi_n\varepsilon\Gamma$ such that $\vdash \psi_1 \,\&\, \ldots \,\&\, \psi_n \to \psi$. It is a special case of Lemma I,4.3. that a sentence ψ is a theorem of IPC if and only if $\psi\varepsilon\Delta$ for every saturated Δ such that $Ind(\psi) \subseteq Ind(\Delta)$. Therefore $(\psi_1{}^i \,\&\, \ldots \,\&\, \psi_n{}^i \to \psi^i)\varepsilon\Gamma_i$ for every $i\varepsilon I$. On the other hand $\psi_1\varepsilon\Gamma$ implies by definition of Γ that $\{i:\psi_1{}^i\varepsilon\Gamma_i\}\varepsilon F$. If $p_k = \{i:\psi_k{}^i\varepsilon\Gamma_i\}$ for $k = 1, \ldots, n$, then $p = p_1 \cap \ldots \cap p_n$ is in F. But $\{i:(\psi_1{}^i \,\&\, \ldots \,\&\, \psi_n{}^i)\varepsilon\Gamma_i\} = p_1 \cap \ldots \cap p_n\varepsilon F$ so $\{i:\phi^i\varepsilon\Gamma_i\} \supseteq \{i:(\psi_1{}^i \,\&\, \ldots \,\&\, \psi_n{}^i \to \phi^i)\varepsilon\Gamma_i\} \cap \{i: \psi_1{}^i \,\&\, \ldots \,\&\, \psi_n{}^i)\varepsilon\Gamma_i\} = I \cap p = p\varepsilon F$. Therefore, $\phi\varepsilon\Gamma$.

(2) Γ is consistent, i.e., $\square \notin \Gamma$. Since all Γ_i are consistent, $\square \notin \Gamma_i$ for every $i\varepsilon I$, so by definition $\square$ cannot be in Γ.

(3) Γ has the disjunction property. Let $\phi \vee \psi \varepsilon \Gamma$. It means $\{i:\phi^i \vee \psi^i \varepsilon \Gamma_i\} \varepsilon F$. Since each Γ_i has the disjunction property it follows: $\phi^i \vee \psi^i \varepsilon \Gamma_i$ iff $\phi^i \varepsilon \Gamma_i$ or $\psi^i \varepsilon \Gamma_i$. Let $p_1 = \{i:\phi^i \varepsilon \Gamma_i\}$ and $p_2 = \{i:\psi^i \varepsilon \Gamma_i\}$. Then $p_1 \cup p_2 = \{i:\phi^i \vee \psi^i \varepsilon \Gamma_i\} \varepsilon F$ and since F is a prime filter it follows $p_1 \varepsilon F$ or $p_2 \varepsilon F$. Therefore, either $\phi \varepsilon \Gamma$ or $\psi \varepsilon \Gamma$.

(4) Γ has the existential instantiation property. We need here a weak version of the axiom of choice. (This is not, of course, any problem because we use set theory liberally.) Suppose $\exists x \phi(x) \varepsilon \Gamma$. This means that $p = \{i:\exists x \phi^i(x) \varepsilon \Gamma_i\}$ is in F. Since every Γ_i is saturated, $\exists x \phi^i(x) \varepsilon \Gamma_i$ implies $\phi^i(a_i) \varepsilon \Gamma_i$, for some $a_i \varepsilon C_i$. Let $a \varepsilon \prod_{i \varepsilon I} C_i$ be such that $a(i) = a_i$ for $i \varepsilon p$. Then $\{i:\phi^i(a(i)) \varepsilon \Gamma_i\} \supseteq p \varepsilon F$, so $\phi(a/_F) \varepsilon \Gamma$.

There is an interesting connection between this construction and Kripke models. Namely, if $\mathfrak{M}_i (i \varepsilon I)$ is a collection of Kripke models and F a prime filter over I and if Γ_i is the set of all sentences which hold in $\mathfrak{M}_i$, each Γ_i is saturated, so we can obtain the prime product $\Gamma = \prod_F \Gamma_i$. Now, there is a canonical Kripke model $\mathfrak{M}_\Gamma$, defined as in Section 4. of Chapter II, for which Γ is the set of all sentences forced at the base node. Thus, we have a Kripke structure $\mathfrak{M}_\Gamma$, such that for any sentence ϕ with $Ind(\phi) \subseteq Ind(\Gamma)$, ϕ holds in $\mathfrak{M}_\Gamma$ (i.e., $\Gamma \Vdash \phi$) if and only if $\{i:\phi^i$ holds in $\mathfrak{M}_i\} \varepsilon F$. If $\mathfrak{A}_\Gamma$ is the classical structure at the base node Γ of $\mathfrak{M}_\Gamma$, by definition of $\mathfrak{M}_\Gamma$ it is the case that $A_\Gamma = Ind(\Gamma)$. Then, if $\mathfrak{A}_i$ is the classical structure at the base node of $\mathfrak{M}_i$ (for every $i \varepsilon I$), we have

Lemma 1

$$\prod_F \mathfrak{A}_i = \mathfrak{A}_\Gamma$$

Proof: Since $\Gamma = \prod_F \Gamma_i$ and $\mathrm{Ind}(\Gamma_i) = A_i$, by definition we have $A_\Gamma = \prod_F A_i$, so we have to prove only that the structure is the same. Let R be an n-ary relation symbol and $a_{1/F}, \ldots, a_{n/F} \varepsilon A_\Gamma$. $\prod_F \mathfrak{A}_i \models R(a_{1/F}, \ldots, a_{n/F})$ iff $\{i: \mathfrak{A}_i \models R(a_1(i), \ldots, an(i))\} \varepsilon F$ iff $\{i: R(a_1(i), \ldots, a_n(i)) \varepsilon \Gamma_i\} \varepsilon F$ iff $R(a_{1/F}, \ldots, a_{n/F}) \varepsilon \Gamma$ iff $\mathfrak{A}_\Gamma \models R(a_{1/F}, \ldots, a_{n/F})$.

As an indication of the usefulness of products of saturated theories we give now a proof of the compactness theorem.

Theorem 2

Let Σ be a set of sentences without individual constants. Σ has a model if and only if every finite subset of Σ has a model.

Proof: Let I be the set of all finite subsets of Σ and for each $\Delta \varepsilon I$, let $\mathfrak{M}_\Delta$ be a model of Δ. If Γ_Δ is the set of all sentences forced at the base node of $\mathfrak{M}_\Delta$, clearly Γ_Δ is saturated and $\Delta \subseteq \Gamma_\Delta$. For $\Delta \varepsilon I$ let $\Delta^* = \{\Delta' \varepsilon I: \Delta \subseteq \Delta'\}$. The set $\{\Delta^*: \Delta \varepsilon I\}$ has the finite intersection property (because if $\Delta_1 = \{\phi_1, \ldots, \phi_n\}$ and $\Delta_2 = \{\psi_1, \ldots, \psi_m\}$, then $\Delta_1^* \cap \Delta_2^* = \{\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m\}^*$) so it can be enlarged to a prime filter F. Let now $\Gamma = \prod_F \Gamma_\Delta$ be the reduced product of the theories Γ_Δ over the filter F. We claim that the cannonical model $\mathfrak{M}_\Gamma$ of Γ is a model of Σ. It is enough to show that $\Sigma \subseteq \Gamma$. Let $\phi \varepsilon \Sigma$. Since ϕ has no individual constants, $\phi^\Delta = \phi$ for every $\Delta \varepsilon I$. Therefore $\{\Delta: \phi \varepsilon \Gamma_\Delta\} \supseteq \{\phi\}^* \varepsilon F$ so $\phi \varepsilon \Gamma$.

2. <u>Reduced products of Kripke structures</u>

In this section we define reduced products of Kripke structures and prove some preservation results. Ultraproducts of a similar kind were discussed by Cleave [4] and Gabbay [8] and [10].

Let $\mathfrak{M}_i = \langle \mathcal{T}_i; \mathfrak{A}_t : t \varepsilon T_i \rangle$ for $i \varepsilon I$, be a family of Kripke structures for the language L, that is, structures of the same type where $\mathcal{T}_i = \langle T_i, 0_i, \leq_i \rangle$ are partially ordered sets. In order to simplify notation, we may assume that all T_i are mutually disjoint. Let F be a filter over I and let $\mathcal{T} = \langle T, 0, \leq \rangle$ be the reduced product $\prod_F \mathcal{T}_i$. Note that since the theory of partial order with a least element is a Horn theory, $\mathcal{T}$ will be a partially ordered set with the least element 0, for an arbitrary filter F. For $i \varepsilon I$, let $A_i = \bigcup_{t \varepsilon T_i} A_t$ and

let $A = \prod_F A_i$. We denote elements of T by $\alpha_F, \beta_F, \ldots$, where $\alpha_F = \{\beta \varepsilon \prod_{i \varepsilon I} T_i ; \{i \varepsilon I : \alpha(i) = \beta(i)\} \varepsilon F\}$ and elements of A by ξ_F, $\eta_F, \ldots$, where $\xi_F = \{\eta \varepsilon \prod_{i \varepsilon I} A_i : \{i \varepsilon I : \xi(i) = \eta(i)\} \varepsilon F\}$.

Of course, $0 = \{\beta : \{i : \beta(i) = 0_i\} \varepsilon F\}$ and $\alpha_F \leq \beta_F$ iff $\{i : \alpha(i) \leq_i \beta(i)\} \varepsilon F$. The reduced product of structures $\mathfrak{M}_i$ will be a Kripke structure, having $\mathcal{T}$ as its partially ordered set and for each $\alpha_F \varepsilon T$ a classical structure $\mathfrak{A}_{\alpha_F}$ defined in the following way:

(i) The universe of $\mathfrak{A}_{\alpha_F}$ is

$$A_{\alpha_F} = \{\xi_F \varepsilon A : \{i : \xi(i) \varepsilon A_{\alpha(i)}\} \varepsilon F\}.$$

That $A\alpha_F$ is unambiguously defined, i.e., that $\xi_F \varepsilon A\alpha_F$ does not depend on the choice of the representatives ξ and α of equivalence classes ξ_F and α_F, follows from these two, easily verifiable, facts:

(1) $\xi_F \varepsilon A\alpha_F$ and $\xi_F = \eta_F$ implies $\eta_F \varepsilon A\alpha_F$

(2) $\alpha_F = \beta_F$ implies $A\alpha_F = A\beta_F$

(ii) If c is an individual constant from L and if $c^{\alpha(i)}$ is the interpretation of c in $\mathfrak{A}_{\alpha(i)}$ for $i\varepsilon I$, the interpretation of c in $\mathfrak{A}_{\alpha_F}$ is

$$c^{\alpha F} = \{\xi \varepsilon \prod_{i\varepsilon I} A_i : \{i:\xi(i) = c^{\alpha(i)}\}\varepsilon F\}$$

Obviously $c^{\alpha F} \varepsilon A\alpha_F$ because for every $i\varepsilon I$ $c^{\alpha(i)} \varepsilon A_{\alpha(i)}$. Since for any $s, t, \varepsilon T_i, c^s = c^t$ it follows that $c^{\alpha F} = c^{\beta F}$ for any $\alpha_F, \beta_F \varepsilon T$.

(iii) If f is an n-ary function symbol from L and its interpretation in $A_{\alpha(i)}$ is $f^{\alpha(i)}$ then $f^{\alpha F}$ is defined by

$$f^{\alpha F}(\overline{\xi}_F) = \eta_F \text{ iff } \{i:f^{\alpha(i)}(\overline{\xi}(i)) = \eta(i)\}\varepsilon F.$$

(In order to further simplify notation, from now on we write $\overline{f}$ for $f^{\alpha F}$, $\overline{\xi}_F$ for $<\xi_F^1, \ldots, \xi_F^n>$ and $\overline{\xi}(i)$ for $<\xi^1(i), \ldots, \xi^n(i)>$.) We have to show that $\overline{f}$ is indeed a function on $A\alpha_F$, defined unambiguously, that is, we have to show

(1) $\overline{f}$ is always defined. Let $\xi^1_F, \ldots, \xi^n_F \varepsilon A\alpha_F$ and let $X_k = \{i:\xi^k(i)\varepsilon A_{\alpha(i)}\}$ for $k = 1, \ldots, n$. Then $X = \bigcap_{k=1}^{n} X_k \varepsilon F$, and for $i\varepsilon X$ each $\xi^k(i)$ is in $A\alpha_{(i)}$ so $f^{\alpha(i)}(\overline{\xi}(i))$ is defined. Let η be such that $\eta(i) = f^{\alpha(i)}(\overline{\xi}(i))$ for $i\varepsilon X$ and arbitrary otherwise.

Then $\{i:f^{\alpha(i)}(\overline{\xi}(i)) = \eta(i)\} \supseteq X\varepsilon F$ so $\overline{f}(\overline{\xi}_F) = \eta_F$.

(2) $\overline{f}$ is single-valued. Suppose $\overline{f}(\overline{\xi}_F) = \eta_F$ and $\overline{f}(\overline{\xi}_F) = \zeta_F$, that is, $X = \{i:f^{\alpha(i)}(\overline{\xi}(i)) = \eta(i)\}\varepsilon F$ and $Y = \{i:f^{\alpha(i)}(\overline{\xi}(i)) = \zeta(i)\}\varepsilon F$. Then $X\cap Y\varepsilon F$ and since $f^{\alpha(i)}$ is a function, for every $i\varepsilon X\cap Y$, $\eta(i) = \zeta(i)$ so $\eta_F = \zeta_F$.

(3) The definition is unambiguous, i.e., it does not depend on the choice of representatives of equivalence classes. The proof of this claim is similar to the proofs already given, so we leave it out.

(iv) If R is an n-ary relation symbol from L, its interpretation is

$$R^{\alpha_F} = \{<\xi^1_F, \ldots, \xi^n_F> \ \varepsilon\ (A\alpha_F)^n : \{i:<\xi^1(i), \ldots, \xi^n(i)>\varepsilon R^{\alpha(i)}\}\varepsilon F\}$$

As above, it is easy to show that this definition is unambiguous.

REMARK: The more intuitive definition $\mathfrak{A}\alpha_F = \prod_F \mathfrak{A}\alpha(i)$ (as in [10]) is not correct since in that case $A\alpha_F$ can be different from

$A\beta_F$ even if $\alpha_F = \beta_F$.

Let $\mathfrak{M} = \langle T;\ \mathfrak{A}\alpha_F : \alpha_F \varepsilon T \rangle$. The forcing relation is defined as usual.

Definition 1

The structure $\mathfrak{M}$ is called the reduced product of $\mathfrak{M}_i$ over Filter F (and denoted by $\prod_F \mathfrak{M}_i$).

If F is an ultrafilter, the analog of Łos's theorem holds:

Theorem 1

If F is an ultrafilter $\mathfrak{M} = \prod_F \mathfrak{M}_i$, $\alpha_F \varepsilon T$, $\phi(x_1, \ldots, x_n)$ - any formula and $\xi^1{}_F, \ldots, \xi^n{}_F \varepsilon A\alpha_F$ then $\alpha_F \Vdash \phi[\bar{\xi}_F]$ iff $\{i : \alpha(i) \Vdash \phi(\bar{\xi}(i))\} \varepsilon F$.

The analogues of a few other classical results concerning ultraproducts were proved in [4] and [10]. We shall discuss now reduced products in general, that is, products reduced over arbitrary filters.

It is known in classical model theory that Horn sentences are the reduced product sentences (i.e., a sentence is preserved under reduced products if and only if it is equivalent to a Horn sentence), so it would be natural to try to prove some analogue of this result for reduced products of Kripke structures. As in other cases, however, equivalent classical definitions give rise here to different notions. Namely, we could define basic Horn formulas in at least four nonequivalent ways:

1° $\sim P \vee \ldots \vee \sim P_n \vee P$

2° $\sim (P_1 \& \ldots \& P_n) \vee P$

3° $(P_1 \& \ldots \& P_n) \rightarrow P$

4° $\sim (P_1 \& \ldots \& P_n \& \sim P)$

where $P_1, \ldots, P_n, P$ are atomic formulas. In each case the meaning, on classical interpretation, essentially is the same: whenever all of $P_1, \ldots, P_n$ hold, P also holds. The interest in Horn sentences derives partly from the fact that this is the basic form of a mathematical statement. On intuitionistic interpretation, only 3° can be taken to have that kind of meaning. In terms of Kripke models, 1° means that if each of $P_1, \ldots, P_n$ gets to be forced, at some node or other, P has to be forced already at the base node, while 2° requires that, only if at some node all of $P_1, \ldots, P_n$ get to be forced at the same time. 4° says just that, if at some node t all of $P_1, \ldots, P_n$ are forced, P must be forced at some node above t. It seems natural, therefore, to use 3° for definition of basic Horn formulas. It turns out that those formulas are preserved under reduced products.

Definition 2

A formula ψ of L is called basic Horn formula iff

$$\psi = (\phi_1 \& \ldots \& \phi_n) \rightarrow \phi$$

where $\phi_1, \ldots, \phi_n, \phi$ are atomic formulas.

ψ is a Horn formula iff it is built up from basic Horn formulas using only the connectives &, $\exists$ and $\forall$.

Consider the following condition for a formula ϕ:

$(L)_\phi$ for every α_F and $\overline{\xi}_F$.

$$\alpha_F \Vdash \phi[\overline{\xi}_F] \text{ iff } \{i:\alpha(i) \Vdash \phi[\overline{\xi}(i)]\}\varepsilon F$$

Lemma 1

$(L)_\phi$ and $(L)_\psi$ implies $(L)_{\phi\&\psi}$.

Proof: Let $X = \{i:\alpha(i) \Vdash \phi[\overline{\xi}(i)]\}$ and

$$Y = \{i:\alpha(i) \Vdash \psi[\overline{\xi}(i)]\}.$$

$\alpha_F \Vdash (\phi \& \psi)[\overline{\xi}_F]$ iff $\alpha_F \Vdash \phi[\overline{\xi}_F]$ and $\alpha_F \Vdash \psi[\overline{\xi}_F]$ iff $X\varepsilon F$ and $Y\varepsilon F$. Then

$$\{i:\alpha_F \Vdash (\phi \& \psi)[\overline{\xi}(i)]\} = X \cap Y\varepsilon F$$

Corollary: Since $(L)_\phi$ holds for atomic ϕ, by definition, it follows from Lemma 1 that $(L)_\phi$ holds when ϕ is a conjunction of atomic formulas.

Lemma 2

Basic Horn formulas are preserved under reduced products, that is, if $\phi_1, \ldots, \phi_n, \phi$ are atomic formulas then for every $\alpha_F\varepsilon T$ and $\overline{\xi}_F\varepsilon^\omega(A\alpha_F)$

$$\{i: \alpha(i) \Vdash (\phi_1 \& \ldots \& \phi_n \rightarrow \phi)[\overline{\xi}(i)]\}\ \varepsilon F \text{ implies}$$

$$\alpha_F \Vdash (\phi_1 \& \ldots \& \phi_n \rightarrow \phi)[\overline{\xi}_F].$$

Proof: By definition of forcing $\alpha_F \Vdash (\phi_1 \ldots \& \phi_n \rightarrow \phi)[\overline{\xi}_F]$

iff for every $\beta_F \geq \alpha_F$ $(\beta_F \varepsilon T)$ either

$\beta_F \nVdash (\phi_1 \& \ldots \& \phi_n)[\overline{\xi}_F]$ or $\beta_F \Vdash \phi[\overline{\xi}_F]$.

Let $X = \{i : \alpha(i) \leq_i \beta(i)\}$,

$Y = \{i : \beta(i) \Vdash (\phi_1 \& \ldots \& \phi_n)[\overline{\xi}(i)]\}$ and

$Z = \{i : \beta(i) \Vdash \phi[\overline{\xi}(i)]\}$, and suppose $X \varepsilon F$.

By Lemma 1. $\beta_F \Vdash (\phi_1 \& \ldots \& \phi_n)[\overline{\xi}_F]$ iff $Y \varepsilon F$, so if $Y \notin F$

we are done. Suppose then $Y \varepsilon F$ and let

$V = \{i : \alpha(i) \Vdash (\phi_1 \& \ldots \& \phi_n \rightarrow \phi)[\overline{\xi}(i)]\} \varepsilon F$.

Now $X \cap Y \cap V \varepsilon F$, but $i \varepsilon X \cap Y \cap V$ implies $i \varepsilon Z$, so $X \cap Y \cap V \subseteq Z$ and $Z \varepsilon F$.

Since ϕ is atomic it follows that $\beta_F \Vdash \phi[\overline{\xi}_F]$.

Theorem 2.

Horn formulas are preserved under reduced products, that is, if ϕ is a Horn formula, $\{i : \alpha(i) \Vdash \phi[\overline{\xi}(i)]\} \varepsilon F$ implies $\alpha_F \Vdash \phi[\overline{\xi}_F]$

Proof: We already know that the theorem holds for basic Horn formulas. For non-basic Horn formulas the proof proceeds by induction on complexity of the construction of ϕ from basic Horn formulas. Since the valuation $\overline{\xi}_F$ plays no role in the proof, it will not be exhibited.

(a) If the theorem holds for formulas ϕ and ψ, it also holds for $\phi \& \psi$. Suppose $X = \{i : \alpha(i) \Vdash \phi \& \psi\} \varepsilon F$ and let $Y = \{i : \alpha(i) \Vdash \phi\}$ and $Z = \{i : \alpha(i) \Vdash \psi\}$. Then $X \subseteq Y$ and $X \subseteq Z$, so $Y \varepsilon F$ and $Z \varepsilon F$. By induction hypothesis then $\alpha_F \Vdash \phi$ and $\alpha_F \Vdash \psi$, so $\alpha_F \Vdash \phi \& \psi$.

(b) If the theorem holds for $\phi(x)$, it also holds for $\exists x\, \phi(x)$. Suppose $X = \{i : \alpha(i) \Vdash \exists x\, \phi(x)\}$. $\alpha(i) \Vdash \exists x\, \phi(x)$ implies $\alpha(i) \Vdash \phi(a_i)$ for some $a_i \varepsilon A\alpha(i)$. Let $\xi(i) = a_i$ for $i \varepsilon X$ and let $\xi(i)$ be an arbitrary element of $A_\alpha(i)$ for $i \notin X$. Then $\xi_F \varepsilon A_{\alpha_F}$ and $X \subseteq \{i : \alpha(i) \Vdash \phi[\xi(i)]\}$. By induction hypothesis then $\alpha_F \Vdash \phi[\xi_F]$ so $\alpha_F \Vdash \exists x\, \phi(x)$

(c) If the theorem holds for $\phi(x)$, it also holds for $\forall x \phi(x)$. Suppose $V = \{i : \alpha(i) \Vdash \forall x\, \phi(x)\} \varepsilon F$. By the definition of forcing, $\alpha_F \Vdash \forall x \phi(x)$ iff for every $\beta_F \varepsilon T$ and $\xi_F \varepsilon A$, $\alpha_F \leq \beta_F$ and $\xi_F \varepsilon A_{\beta_F}$ implies $\beta_F \Vdash \phi[\xi_F]$. Let β_F and ξ_F be given and suppose $X = \{i : \alpha(i) \leq \beta(i)\} \varepsilon F$ and $Y = \{i : \xi(i) \varepsilon A_\beta(i)\} \varepsilon F$.

Consider the set $Z = \{i : \beta(i) \Vdash \phi[\xi(i)]\}$. Now $i \varepsilon V \cap X \cap Y$ implies $i \varepsilon Z$, so we have $Z \supseteq V \cap X \cap Y \varepsilon F$, and by induction hypothesis it follows that $\beta \Vdash \phi[\xi_F]$. This holds for every $\beta_F \geq \alpha_F$ and $\xi_F \varepsilon A_{\beta_F}$ so $\alpha_F \Vdash \forall x \phi(x)$.

As we shall see now, the analogy with the classical case stops here. Namely, the class of formulas preserved under reduced products is much broader than the class of formulas which are intuitionistically equivalent to Horn formulas (it is clear, of course, that if a formula is preserved under all reduced products of Kripke models, it must be classically equivalent to a Horn formula because the reduced products of classical structures are a special case of reduced products of Kripke structures--when all the partially ordered sets are singletons). Consider the following condition:

$(*)_\phi \quad \alpha_F \Vdash \phi[\bar{\xi}_F]$ implies $\{i : \alpha(i) \Vdash \phi[\bar{\xi}(i)]\} \in F$

Lemma 3

If $\{i : \alpha(i) \Vdash (\sim\phi \vee \psi) [\bar{\xi}(i)]\} \in F$ and $(*)_\phi$ holds, then either $\alpha_F \Vdash \sim\phi[\bar{\xi}]$ or

$\{i : \alpha(i) \Vdash \psi[\bar{\xi}(i)]\} \in F$

Proof: Suppose $\alpha_F \nVdash \sim\phi$ (again, we suppress $\bar{\xi}_F$). Then there exist

$\beta_F \geq \alpha_F$ such that $\beta_F \Vdash \phi$.

Let $X = \{i : \alpha(i) \Vdash \sim\phi \vee \psi\} \in F$, $Y = \{i : \alpha(i) \leq \beta(i)\} \in F$ and let $Z = \{i : \beta(i) \Vdash \phi\}$, $U = \{i : \alpha(i) \Vdash \sim \phi\}$ and $V = \{i : \alpha(i) \Vdash \psi\}$. We have to show $V \in F$. Clearly $(*)_\phi$ implies

$Z \in F$ so $Y \cap Z \in F$. Also $U \cup V = X \in F$, so $(Y \cap Z) \cap (U \cup V) \in F$.

However, $Y \cap Z \cap U = \emptyset$ so it follows that $Y \cap Z \cap V \varepsilon F$ and therefore $V \varepsilon F$.

Lemma 4.

If ϕ is a positive formula (i.e., built up from atomic formulas with the connectives $\vee$, &, and $\exists$) then the condition $(*)_\phi$ holds for any reduced product, any node α_F in it and any valuation

$$\bar{\xi}_F \; \varepsilon \; (A_{\alpha_F})^\omega.$$

Proof: By induction. The Lemma holds for atomic formulas by definition.

(a) Suppose $\phi = \psi \,\&\, \chi$. Then $\alpha_F \Vdash \psi \,\&\, \chi$ implies $\alpha_F \Vdash \psi$ and $\alpha_F \Vdash \chi$. By induction hypothesis it follows that

$X = \{i : \alpha(i) \Vdash \psi\} \, \varepsilon F$ and $Y = \{i : \alpha(i) \Vdash \chi\} \, \varepsilon F$.

Therefore $\{i : \alpha(i) \Vdash \psi \,\&\, \chi\} = X \cap Y \, \varepsilon F$.

The other two cases: $\phi = \psi \vee \chi$ and $\phi = \exists x \, \psi(x)$ are treated similarly.

Definition 3

A formula ϕ is said to be a (GBH)-formula iff

$\phi = \sim \psi_1 \vee \ldots \vee \sim \psi_n \vee \psi$ where $\psi_1, \ldots, \psi_n$ are positive and ψ is a Horn formula.

Theorem 3

If ϕ is a (GBH)-formula then $\{i : \alpha(i) \Vdash \phi[\overline{\xi}(i)]\} \varepsilon F$ implies $\alpha_F \Vdash \phi[\overline{\xi}_F]$

Proof: If $\phi = \sim\psi_1 \vee \ldots \vee \sim\psi_n \vee \psi$ and ψ_1 is positive, By Lemma 4. the condition $(*)_{\psi_1}$ holds, so we can apply Lemma 3. to

$\sim\psi_1 \vee (\sim\psi_2 \vee \ldots \vee \sim\psi_n \vee \psi)$. If $\alpha_F \nVdash \sim\psi_1$ we can repeat the same procedure. If, at the end, $\alpha_F \nVdash \sim\psi_k$ for every $k \varepsilon \{1, \ldots, n\}$, then by Theorem 2, $\alpha_F \Vdash \psi$.

Theorem 4

If a formula ϕ is built up from (GBH)-formulas with the connectives &, $\exists$ and $\forall$, it is preserved under reduced products, i.e., $\{i : \alpha(i) \Vdash \phi[\overline{\xi}(i)]\} \varepsilon F$ implies $\alpha_F \Vdash \phi[\overline{\xi}_F]$.

Proof: The proof is practically the same as the proof of Theorem 2 except that we have to start from Theorem 3 instead of from Lemma 2.

CONCLUSION

The general strategy of this work was not to translate the notions and procedures of classical model theory into the context of classical semantics for intuitionistic formal systems but to concentrate on notions and procedures which arise naturally in the setting of Kripke structures, primarily, and other constructions, eventually, and to try to obtain results which might provide us insights about intuitionistic formal theories analogous to insights about classical logic provided by the results of classical model theory.

For example, so called, complete Henkin type theories (i.e., theories which posess a "witness" individual constant for each existential consequence) arise naturally in classical model theory. Completeness is obviously associated with the fact that in a model, a formula either holds or does not hold, and in the latter case, by definition, its negation holds. In the case of intuitionistic theories, however, the idea of completeness not only is absolutely foreign to the spirit and intentions of intuitionism but also does not correspond to any feature of the model theory. Namely, in general case, the set of formulas which hold in a Kripke model (i.e., which are forced by its base node) is not complete. The natural notion which arises in the context of Kripke models is that of a saturated theory. It naturally arises also in the context of Heyting algebras, as a special kind of prime filter. We showed in Section 1 of Chapter III that a simple analogue of ultraproduct construction can be defined in terms of saturated theories. The application of it which was given there, for a proof of compactness theorem, does not, however, fully utilize the potential of this construction. It was shown that the filter over which the reduction is made can be taken to be prime. In some cases it could be an important advantage to consider instead of the Boolean algebra of the power set of indices, some Heyting algebra defined on the power set, and a prime filter in it.

Another example where the straight forward translation from classical model theory is not very successful is that of submodels and homomorphism. Those notions can be defined for Kripke models and they are algebraically meaningful, but they do not appear as a very natural part of Kripke model theory because there are no natural classes of formulas which would be preserved under such operations. This is due, of course, to the fact that formulas of IPC do not possess equivalent prenex normal forms. Consequently, there is no hierarchy for the complexity of formulas. Section 1 of Chapter II is, in a sense, addressed to that problem.

Two operations which are natural and specific for Kripke model theory are truncation and collection. Especially the collection appears to be a very powerful tool, in a sense similar in its strength to model theoretic forcing in classical model theory. It has been extensively studied and by applying it de Jongh and Smorynski [20], Smorynski [35] and Weinstein [41] obtained very important results about intuitionistic arithmetic and analysis. We could say that collection is the perfect example of the kind of notions and procedures one should aim at discovering in the Kripke model theory.

Aside from forcing, two of the most powerful tools of classical model theory are ultraproducts and omitting types theorem. In Chapter III we discussed two kinds of products, prime products of saturated theories in Section 1 and ultra- and reduced products of Kripke structures in Section 2. Prime products of saturated theories are certainly simpler and seem to be more natural than ultra-products of Kripke structures. They are also potentially more powerful because the filter is required only to be prime--a notion which seems to be a natural substitute in this setting for the notion of ultrafilter. However, reduced product in general, cannot be defined in terms of saturated theories and in that respect products of Kripke structures have a certain advantage. We proved several preservation results for products of Kripke structures, but further research in that direction would be useful. One possible outcome could be determination of some complexity-hierarchy of formulas of IPC. In Chapter II we proved an omitting types theorem. Its applications could be numerous. One example would be proving the existence of a Kripke model of Heyting arithmetic in which Church's thesis holds.

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