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A THERMODYNAMIC FORMULATION OF FINITE-DEFORMATION ELASTOPLASTICITY WITH HARDENING BASED ON THE CONCEPT OF MATERIAL ISOMORPHISM*

Bob Svendsen†

Federal Institute for Materials Research and Testing, D-12200 Berlin, Germany

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Abstract—This paper presents a thermodynamic formulation of a model for finite deformation of materials exhibiting elastoplastic material behaviour with non-linear isotropic and kinematic hardening. Central to this formulation is the notion that the form of the elastic constitutive relation be unaffected by the plastic deformation or transformation in the material, as commonly assumed in particular in the context of crystal plasticity. When generalized to the phenomenological context, this implies that the internal variable representing plastic deformation is an elastic material isomorphism. Among other things, this requirement on the plastic deformation leads directly to the standard elasto-plastic multiplicative decomposition of the deformation gradient. In addition, a dependence of the plastic part of the free energy on the plastic deformation itself yields a thermodynamic form for the centre of the elastic range of the material, i.e. the back stress. Finally, we show how this approach can be applied to formulate thermodynamic forms for linear, and non-linear Armstrong–Frederick, kinematic hardening models. © 1998 Elsevier Science Ltd. All rights reserved

I. INTRODUCTION AND NOTATION

Over the years, many thermodynamically-based formulations of inelastic material behaviour have appeared (e.g. Coleman, 1964; Green and Naghdi, 1965, 1971; Valanis, 1967; Coleman and Gurtin, 1967; Kratochvíl and Dillon, 1969; Rice, 1971; Mandel, 1972, 1974, 1982; Halphen and Nguyen Quoc Son, 1975; Lubliner, 1973, 1984, 1986, 1987; Chaboche, 1993). Most of these involve invariably, in one way or another (as does the approach taken in this work), the concept of internal variables to represent the history dependence of such material behaviour. In the context of large deformations, the most notable of these is perhaps the representation of inelastic deformation as an internal variable. The most common approach here, of course, is the elastoplastic multiplicative decomposition of the deformation gradient (e.g. Lee, 1969; Clifton, 1972; Haupt, 1985), which forms the basis of most recent formulations of elastoplastic material behaviour, and in particular those aimed at numerical simulation of such behaviour (e.g. Simo, 1992; Miehe, 1994). Among other things, we show in this work that such a decomposition arises naturally in the context of the assumption that plastic deformation does not affect the

*Dedicated to Professor Peter Haupt on the occasion of his 60th birthday.
†E-mail: bob.svendsen@bam.de

form of the elastic constitutive relation, as is in particular the case in classical crystal plasticity. In Noll's terminology (e.g. Truesdell and Noll, 1965 (Sec. 27); Noll, 1967, 1972), such quantities represent *material isomorphisms*; in particular, since the plastic deformation preserves the form of the elastic constitutive relation, we refer to it as an *elastic material isomorphism* here. Early models for inelastic material behaviour utilizing Noll's concept of material isomorphism include Noll (1972) and Wang and Bloom (1974). The concepts of elastic range (e.g. Pipkin and Rivlin, 1965; Owen, 1968, 1970; Del Piero, 1975; Šilhavý and Kratochvíl, 1977; Kratochvíl and Šilhavý, 1977; Lucchesi and Podio-Guidugli, 1988, 1990) and material isomorphism have been combined explicitly* by Šilhavý and Kratochvíl (1977), Kratochvíl and Šilhavý (1977) and, more recently, by Bertram (1993) and Bertram and Kraska (1995), to formulate models for inelastic material behaviour; in particular the latter authors applied their formulation to crystal plasticity. In all of these previous works, the plastic deformation is assumed to be an elastic material isomorphism from the start. A framework for the more general case, i.e. when plastic deformation is not necessarily an elastic material isomorphism, has been formulated recently in a thermodynamic setting by Svendsen (1998).

In the current work, we apply this last approach to examine the aspects and consequences of material isomorphism for the thermodynamic formulation of large-deformation elastoplasticity with non-linear isotropic and kinematic hardening in the context of the internal dissipation inequality. After introducing the basic constitutive considerations and internal dissipation inequality (Section 2), we examine the consequences of the plastic deformation being an elastic material isomorphism (Section 3). Next, particular forms for the plastic part of the free energy are constructed which yield the well-known special cases of Prager (i.e. linear) and non-linear Armstrong–Frederick kinematic hardening (Section 4). Finally, we compare the results of the current formulation with those of previous works (Section 5). In particular, previous thermodynamic formulations of large-deformation kinematic hardening include Dogui and Sidoroff (1985) for the linear case and, more recently, Tsakmakis (1996) in the elastoplastic and Haupt (1995) in the viscoplastic context, for the Armstrong and Frederick (1966) case, and generalizations thereof. Among these previous works, one finds in essence two approaches, the difference being in whether or not the plastic part of the free energy depends explicitly on the plastic deformation (e.g. Dogui and Sidoroff, 1985; Haupt, 1995; this work) or not (e.g. Tsakmakis, 1996; Sievert, 1997). In particular, the latter approach is motivated by that in the small deformation context (e.g. Chaboche, 1993).

Finally, a word on notation. Let bold face, upper case italic letters such as $\mathbf{D}$, $\mathbf{F}$ and $\mathbf{T}$ represent second-order Euclidean tensors, or time-dependent fields of such tensors. In addition, let $\mathbf{A}\cdot\mathbf{B} := \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$ represent the inner product of such tensors, where $\mathbf{A}^{\mathrm{T}}$ represents the transpose of $\mathbf{A}$. Further, let $\mathrm{sym}(\mathbf{A}) := 1/2(\mathbf{A}+\mathbf{A}^{\mathrm{T}})$ and $\mathrm{skw}(\mathbf{A}) := 1/2(\mathbf{A}-\mathbf{A}^{\mathrm{T}})$ represent the symmetric and skew-symmetric parts of any Euclidean tensor $\mathbf{A}$. A scalar-, vector- or tensor-valued function $\psi(\mathbf{A})$ of a Euclidean tensor $\mathbf{A}$ is called positive homogeneous of degree 1 if $\psi(a\mathbf{A}) = a\psi(\mathbf{A})$ holds for all positive real numbers a. For notational simplicity, it proves advantageous to abuse notation in this work and denote mappings and their values by the same symbol. Other notations and mathematical concepts will be introduced as they arise in what follows.

*Without identifying it as such, earlier works such as Fox (1968) and Owen (1968, 1970) in effect represented plastic deformation as an elastic material isomorphism.

II. BASIC CONSTITUTIVE ASSUMPTIONS AND INTERNAL DISSIPATION

We assume in this work that the material in question behaves elastoplastically in the sense that it exhibits rate-independent hysteresis (e.g. Haupt, 1993). In the context of the internal variable approach being pursued here, each inelastic process is represented by the evolution of a corresponding internal variable; for the moment, we collectively designate these by $\boldsymbol{\xi}$. Assuming further for simplicity in this work that the temperature is constant, the basic constitutive relations of the model then take the forms:

$$\begin{aligned} \psi &= \psi(\mathbf{F}, \boldsymbol{\xi}, \dot{\mathbf{F}}), \\ \mathbf{S} &= \mathbf{S}(\mathbf{F}, \boldsymbol{\xi}, \dot{\mathbf{F}}), \\ \dot{\boldsymbol{\xi}} &= \dot{\boldsymbol{\xi}}(\mathbf{F}, \boldsymbol{\xi}, \dot{\mathbf{F}}), \end{aligned} \tag{1}$$

where ψ represents the referential free energy density, and $\mathbf{S} := \mathbf{F}^{-1}\mathbf{K}\mathbf{F}^{-\mathrm{T}}$ the second Piola–Kirchhoff, $\mathbf{K} := \det(\mathbf{F})\mathbf{T}$ the Kirchhoff, and $\mathbf{T}$ the Cauchy, stress tensors, respectively. As usual, the internal variables $\boldsymbol{\xi}$ represent quantities whose evolution is not accounted for by the usual balance relations, requiring the constitutive relations (1) to be evolutionary in form (e.g. Valanis, 1967; Coleman and Gurtin, 1967; Lubliner, 1973; Haupt, 1993). As usual, the deformation gradient $\mathbf{F}$ and its rate $\dot{\mathbf{F}}$ are determined via the balance relations or controlled externally.

The basic thermodynamic development in this work is based, as usual, on the referential internal dissipation rate density δ, i.e. the difference between work preformed on the material and work stored in the material per unit time, which in the isothermal case can be written in the form

$$\delta = \mathbf{S}\cdot\mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}} - \dot{\psi} \tag{2}$$

relative to $\mathbf{S}$. Next, substituting the constitutive relation $(1)_1$ into eqn (2), we obtain

$$\delta = \delta(\mathbf{F}, \boldsymbol{\xi}, \dot{\mathbf{F}}) = [\mathbf{S} - \mathbf{F}^{-1}\psi_{,\mathbf{F}}]\cdot\mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}} - \psi_{,\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}} - \psi_{,\dot{\mathbf{F}}}\cdot\ddot{\mathbf{F}}. \tag{3}$$

Note that δ is linear in the independent variable $\ddot{\mathbf{F}}$. For δ to be greater than or equal to zero for any admissible thermodynamic process $(\mathbf{F},\boldsymbol{\xi},\dot{\mathbf{F}})$ in which $\ddot{\mathbf{F}}$ can vary independently, then, we must have

$$\psi_{,\dot{\mathbf{F}}} = \mathbf{0}, \tag{4}$$

yielding the reduced form

$$\psi = \psi(\mathbf{F}, \boldsymbol{\xi}) \tag{5}$$

for the referential free energy density ψ, as well as that

$$\delta = \delta(\mathbf{F}, \boldsymbol{\xi}, \dot{\mathbf{F}}) = [\mathbf{S} - \mathbf{F}^{-1}\psi_{,\mathbf{F}}]\cdot\mathbf{F}^{T}\dot{\mathbf{F}} - \psi_{,\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}} \tag{6}$$

for the referential dissipation rate density in the current constitutive context.

Now, since internal variables (or at least those of interest here) can always be defined or introduced in an observer-invariant* fashion, the requirement of material frame-indifference reduces $(1)_1$ to

$$\psi = \psi(\mathbf{C}, \xi), \tag{7}$$

where $\mathbf{C} := \mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green deformation tensor. This requirement likewise reduces $(1)_{2,3}$ to

$$\begin{aligned} \mathbf{S} &= \mathbf{S}(\mathbf{C}, \xi, \dot{\mathbf{C}}), \\ \dot{\xi} &= \dot{\xi}(\mathbf{C}, \xi, \dot{\mathbf{C}}). \end{aligned} \tag{8}$$

An immediate consequence of material frame-indifference is that

$$\psi_{,\mathbf{F}} = 2\mathbf{F}\psi_{,\mathbf{C}} \tag{9}$$

via eqn (7). Assuming now that the material is non-polar, i.e.

$$\mathbf{S}^T = \mathbf{S}, \tag{10}$$

and introducing the velocity gradient $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$, deformation rate $\mathbf{D} := \mathrm{sym}(\mathbf{L})$, continuum spin $\mathbf{W} := \mathrm{skw}(\mathbf{L})$ and using the results $\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}$ and $\mathbf{F}^T\dot{\mathbf{F}} = {}^1/_2\dot{\mathbf{C}} + \mathbf{F}^T\mathbf{W}\mathbf{F}$, eqn (6) reduces to

$$\delta = \delta(\mathbf{C}, \xi, \dot{\mathbf{C}}) = {}^1/_2[\mathbf{S} - 2\psi_{,\mathbf{C}}]\cdot\dot{\mathbf{C}} + \delta_P \tag{11}$$

with $\mathrm{skw}(\psi_{,\mathbf{C}}) = \mathbf{0}$. In this last result,

$$\delta_P = \delta_P(\mathbf{C}, \xi, \dot{\mathbf{C}}) := -\psi_{,\xi}\cdot\dot{\xi} \tag{12}$$

represents the "plastic" dissipation rate density, which Lubliner (1984, 1986) has refered to as the plastic dissipation function, with which we work in what follows.

For the material behaviour to be rate-independent, the reduced constitutive forms (eqn (8)) for $\mathbf{S}$ and $\dot{\xi}$ must be positive homogeneous of order 0 and 1, respectively, in $\dot{\mathbf{C}}$, as usual. On this basis, the internal dissipation rate density δ from eqn (11) is then positive homogeneous of degree 1 in $\dot{\mathbf{C}}$, i.e. not linear in $\dot{\mathbf{C}}$. As such, the usual "elastic" form

$$\mathbf{S} = 2\psi_{,\mathbf{C}} \tag{13}$$

for $\mathbf{S}$ is, in general, not necessary to satisfy $\delta \geq 0$, but rather only sufficient. Lubliner (1973) has shown that, in the special case of a loading function-based formulation of elastoplasticity, certain additional assumptions lead to the necessity of eqn (13) as well to insure

*In particular, one finds such an observer-independent representation for (tensorial) plastic deformation or transformation in, e.g. Fox (1968), Wang and Bloom (1974), Šilhavý (1977), Šilhavý and Kratochvíl (1977), Kratochvíl and Šilhavý (1977) or Bertram (1993).

$\delta \geq 0$, the most important of these being the assumption that elastic processes result in no dissipation. If eqn (13) was, in general, both necessary and sufficient to satisfy $\delta \geq 0$, then the condition

$$\delta_P \geq 0 \tag{14}$$

on the plastic dissipation δ_P would follow automatically from $\delta \geq 0$; since this, however, is not the case in general, eqn (14) does not follow from these, and is itself only sufficient to satisfy $\delta \geq 0$, again in general. Since it is not the purpose of the current work to pursue such additional conditions needed to obtain the necessity of eqns (13) and (14) as well, they are simply adopted as additional and independent constitutive assumptions for simplicity in what follows.

As can be motivated by simple rheological models (e.g. Dogui and Sidoroff, 1985; Maugin, 1992; Chaboche, 1993; Haupt, 1995), we assume in the elastoplastic case that ψ as given in eqn (7) can be split into a sum of elastic ψ_E and plastic ψ_P, parts. As such, ψ_E will depend on the current total deformation $\mathbf{C}$, while ψ_P, does not. In particular, we assume that energy can become stored in the material as a result of plastic deformation, which we represent in this work via a deformation-like internal variable* $\mathbf{P}$ accounting for the effect of (tensorial) plastic deformation (i.e. the "collective" or "effective" deformation of all active glide systems, grain boundaries and so on, in the material) on the material behaviour. Since such deformation may in general influence the elastic response of the material, we assume that ψ_E also depends in general on it. Restricting the remaining inelastic processes to non-linear kinematic and isotropic hardening, we assume that these do not influence the elastic response of the material. On the other hand, they may result in additional storage of energy in the material during inelastic processes. To account for this, we assume further that, beyond $\mathbf{P}$, ψ_P, depends on two deformation-like internal variables $\mathbf{Y}$ and ε. As will be seen in Section 4 in the context of the Armstrong–Frederick model for non-linear kinematic hardening, the evolution of $\mathbf{Y}$ governs, e.g. the saturation of the back stress. Analogously, ε is thermodynamically conjugate to the yield stress, as will also be seen in what follows.

On this basis, then, ψ takes the additive form†

$$\psi(\mathbf{C}, \xi) = \psi_E(\mathbf{C}, \mathbf{P}) + \psi_P(\mathbf{P}, \mathbf{Y}, \varepsilon) \tag{15}$$

with $\xi = (\mathbf{P}, \mathbf{Y}, \varepsilon)$. As will be seen in the next section, the form $\psi_E(\mathbf{C},\mathbf{P})$ for ψ_E in eqn (15) depending on the total deformation and its plastic part represents a conceptual generalization to the current large-deformation context of the usual small-deformation assumption that ψ_E depends on the "elastic part" of the deformation (when $\mathbf{P}$ is assumed to be an

*Here, we are using the notation from Mandel (1972) for this variable (see also Mandel, 1974, 1982). From the phenomenological point of view, this tensor could, in general, represent, in an effective sense, more complex or general inelastic processes than the conceptually analogous plastic "deformation gradient" $\mathbf{F}_P$, arising in the usual multiplicative elastoplastic decomposition of $\mathbf{F}$ from crystal plasticity (e.g. Lee, 1969). As will be shown in the next section, however, under the further constitutive assumption that $\mathbf{P}$ is an elastic material isomorphism, $\mathbf{P}$ can in fact be identified constitutively with $\mathbf{F}_P$.

†As already mentioned in the introduction, a basic difference arises here between the current approach and that of, e.g. Kratochvíl and Dillon (1969), Mandel (1972, 1974, 1982), Lubliner (1984, 1986) and Tsakmakis (1996) in that we allow ψ_P to depend on $\mathbf{P}$. On the other hand, this is conceptually consistent with one approach taken in Dogui and Sidoroff (1985) and Haupt (1995). This assumption has consequences for the thermodynamic formulation of back stress, as shown in Sections 4 and 5.

elastic material isomorphism; see Section 3). Note that inelastic processes such as damage, which may influence the form of the elastic response of the material, are excluded for simplicity here. Substituting finally eqn (15) into (14), it reduces to

$$\delta_{\mathrm{P}} = \delta_{\mathrm{P}}(\mathbf{C}, \mathbf{P}, \mathbf{Y}, \varepsilon, \dot{\mathbf{C}}) = -\psi_{,\mathbf{P}} \cdot \dot{\mathbf{P}} - \psi_{,\mathbf{Y}} \cdot \dot{\mathbf{Y}} - \psi_{,\varepsilon} \dot{\varepsilon}. \tag{16}$$

The corresponding evolution relations for these variables then take the forms

$$\begin{aligned} \dot{\mathbf{P}} &= \mathbf{L}_{\mathrm{P}}(\mathbf{C}, \mathbf{P}, \mathbf{Y}, \varepsilon, \dot{\mathbf{C}})\mathbf{P}, \\ \dot{\mathbf{Y}} &= \dot{\mathbf{Y}}(\mathbf{C}, \mathbf{P}, \mathbf{Y}, \varepsilon, \dot{\mathbf{C}}), \\ \dot{\varepsilon} &= \dot{\varepsilon}(\mathbf{C}, \mathbf{P}, \mathbf{Y}, \varepsilon, \dot{\mathbf{C}}), \end{aligned} \tag{17}$$

via eqn (8). In the current context of simple materials, $\mathbf{P}$ is interpreted as a tensor-valued inelastic process transforming the reference "state" of an infinitesimal neighborhood of each material point into a corresponding inelastically-transformed one, whose local evolution is given by eqn $(17)_1$. On the other hand, $\mathbf{Y}$ is interpreted as an inelastic process with respect to reference configuration (e.g. like $\mathbf{C}$ or $\mathbf{S}$) in this work; such an interpretation for $\mathbf{Y}$ arises naturally in the context of particular cases, as will be shown for example in the Armstrong–Frederick case in Section 4. Such more general aspects of the current formulation have been discussed in more detail elsewhere (Svendsen, 1998); here, we focus primarily on the thermodynamic formulation of kinematic hardening afforded by the above framework, subject to the further assumption that $\mathbf{P}$ represents an elastic material isomorphism, to which we now turn.

III. ELASTIC MATERIAL ISOMORPHISM

From the classical case of crystal plasticity, one has the notion that the (inelastic) deformation resulting from the motion of dislocations in active glide systems, and so on, does not affect the structure of the crystal lattice. In other words, such deformation does not change its symmetry. On the phenomenological level, where we are dealing in general with material symmetry rather than crystal symmetry, one could generalize this idea into the assumption that plastic deformation does not affect the form of the elastic constitutive relation. In the context of eqn (15), such an assumption implies a special dependence of ψ_{E} on $\mathbf{P}$. Indeed, $\mathbf{P}$ will not change the form of eqn (13) if and only if the dependence of ψ_{E} on $\mathbf{P}$ takes the special form*

$$\text{EMI} \quad \psi_{\mathrm{E}}(\mathbf{C}, \mathbf{P}) = \varphi_{\mathrm{E}}(\mathbf{P}^{-\mathrm{T}}\mathbf{C}\mathbf{P}^{-1}),$$

*The stronger assumption

$$\psi(\mathbf{C}, \mathbf{P}, \alpha) = \varphi(\mathbf{P}^{-\mathrm{T}}\mathbf{C}\mathbf{P}^{-1}, \alpha),$$

on the dependence of the entire referential free energy density ψ on $\mathbf{P}$ (in the guise of $\mathbf{F}_{\mathrm{P}}$, α representing all other internal variables) has been commented on by Lubliner (1984, 1986), who, however, seems to have been unaware of its connection with the concept of material isomorphism. This stronger assumption underlies almost all of the standard thermodynamic formulations of inelastic material behaviour, e.g. Kratochvil and Dillon (1969), Mandel (1972, 1974, 1982), Lubliner (1984, 1986), Chaboche (1993), Tsakmakis (1996) and Sievert (1997).

where EMI stands for "elastic material isomorphism". Originally, this notion stems from Noll (see, Noll, 1972, Sec. 9), who refers to time-independent transformations between (the tangent spaces of) two distinct material points which preserve some aspect of their material response as material isomorphisms. Here, we are using it in a slightly different fashion, i.e. it refers to a time-dependent process between two states of the same material point leaving some aspect of its material behaviour unchanged. Since the material behaviour in question here is elastic, we refer to $\mathbf{P}$ in this context as an elastic material isomorphism. Note that EMI represents in essence both (1) a restriction on the dependence of ψ_E on $\mathbf{P}$, and (2) the definition of $\mathbf{P}$ as an elastic material isomorphism. In particular, the elastic part of ψ as restricted in EMI depends on $\mathbf{P}$ only through its ("push-forward") tensorial action* $\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}$ on $\mathbf{C}$. Substituting EMI into eqn (13), we see that, as claimed, such a dependence of ψ_E on $\mathbf{P}$ does indeed preserve the form

$$\mathbf{S} = 2\psi_{,\mathbf{C}} = 2\mathbf{P}^{-1}\varphi_{E,\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}}\mathbf{P}^{-T} \tag{18}$$

of the elastic constitutive relation, as well as that

$$4(\psi_{,\mathbf{C}})_{,\mathbf{C}}[\mathbf{A}] = 4\mathbf{P}^{-1}(\varphi_{E,\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}})_{,\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}}[\mathbf{P}^{-T}\mathbf{A}\mathbf{P}^{-1}]\mathbf{P}^{-T} \tag{19}$$

of the elasticity tensor with respect to any symmetric second-order tensor $\mathbf{A}$. Transforming eqn (18) forward with $\mathbf{F}$, and using the definition of $\mathbf{C}$, yields the form

$$2\mathbf{F}\psi_{,\mathbf{F}^T\mathbf{F}}\mathbf{F}^T = 2(\mathbf{F}\mathbf{P}^{-1})\varphi_{E,(\mathbf{F}\mathbf{P}^{-1})^T(\mathbf{F}\mathbf{P}^{-1})}(\mathbf{F}\mathbf{P}^{-1})^T \tag{20}$$

for the elastic Kirchhoff stress; the same of course can be done for the elasticity tensor in eqn (19).

Since $2\mathbf{F}\psi_{,\mathbf{F}^T\mathbf{F}}\mathbf{F}^T$ in eqn (20) represents the elastic Kirchhoff stress, the result of eqn (20) implies that, from the material behaviour point of view, the combination

$$\mathbf{E} := \mathbf{F}\mathbf{P}^{-1} \tag{21}$$

represents the "elastic" part of $\mathbf{F}$. Indeed, introducing the corresponding right Cauchy–Green deformation tensor

$$\mathbf{C}_E := \mathbf{E}^T\mathbf{E} \tag{22}$$

eqn (18) becomes

$$2\psi_{,\mathbf{C}} = 2\mathbf{P}^{-1}\varphi_{E,\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}}\mathbf{P}^{-T} = 2\mathbf{P}^{-1}\varphi_{E,\mathbf{C}_E}\mathbf{P}^{-T}, \tag{23}$$

while eqn (20) takes the form

$$2\mathbf{F}\psi_{,\mathbf{C}}\mathbf{F}^T = 2\mathbf{E}\varphi_{E,\mathbf{C}_E}\mathbf{E}^T \tag{24}$$

both via eqn (21). From the material behaviour point of view, then, $\mathbf{E}$ can be identified with the standard elastic "deformation gradient" $\mathbf{F}_E$, and $\mathbf{P}$ with $\mathbf{F}_P$, when EMI holds. Note that this constitutive derivation of the elastoplastic decomposition of $\mathbf{F}$ in the

*The essential aspect of this action is its tensorial nature; the push-forward form of this action is chosen merely for convenience, i.e. so as to obtain correspondence of $\mathbf{P}$ with $\mathbf{F}_P$ (and not $\mathbf{F}_P^{-1}$) under this assumption.

current formulation depends only on (1) the (local) representation of plastic deformation in the material via $\mathbf{P}$, and (2) the assumption EMI. Indeed, without this last assumption, there is really no reason in the current formulation to relate $\mathbf{F}$ and $\mathbf{P}$ in the manner of eqn (21) at all. Furthermore, the "derivation" of $\mathbf{F}=\mathbf{F}_{\mathrm{E}}\mathbf{F}_{\mathrm{P}}$ in the context of EMI clearly implies that it is independent of the interpretation of the intermediate configuration as being stress-free (e.g. Lee and Liu, 1967; Lee, 1969). This is also implied from a formal point of view by the recent work of Bertram (1993) (see also Bertram and Kraska, 1995), and from a (generalized) kinematic point of view by that of Del Piero and Owen (1993) (see Owen, 1992) on structured deformations applied to elastoplasticity.

The notion that plastic deformation does not affect the form of the elastic response of the material is, of course, much older than that of a (elastic) material isomorphism. Indeed, one finds this notion (at least tacitly) in the context of the continuum theory of dislocations in works such as Eckart (1948), whose local geometric natural reference state, i.e. a forerunner of the modern "intermediate configuration", is tacitly based on this (see also Besseling, 1968). In the realm of continuum mechanics, one finds in the work of Fox (1968) from the continuum dislocation point of view, and in that of Owen (1968, 1970) from the contitutive point of view, the plastic deformation represented (at least tacitly) as an elastic material isomorphism. In particular, Fox (1968) introduces the plastic deformation as a (time-dependent) material uniformity for the elastic behaviour of the material. Such a representation of the plastic deformation was formulated in detail by Wang and Bloom (1974) in their work on material uniformity and inhomogeneity in anelastic bodies. In the context of Noll's (1972) new theory of simple materials, Šilhavý and Kratochvíl (1977) and Kratochvíl and Šilhavý (1977) defined their plastic "distortion" as an elastic material isomorphism from the start. More recently, this concept finds tacit use in the work of, e.g. Lucchesi and Podio-Guidugli (1988, 1990) on materials with elastic range, and explicit use in that of Bertram (1993) or Bertram and Kraska (1995) in the case of crystal plasticity. Note that, in all of these works, the plastic deformation or transformation is introduced either tacitly or explicitly as such an isomorphism from the start. In some cases of texture development, in particular when this leads to a corresponding change in the (material) symmetry of the elastic material behaviour, however, the plastic deformation no longer represents such an isomorphism. As such, a more general formulation is necessary in which the plastic deformation is not necessarily introduced as such an isomorphism from the start.

Beyond eqn (23), EMI also yields the particular form

$$-\psi_{,\mathbf{P}} = 2\mathbf{C}_{\mathrm{E}}\varphi_{\mathrm{E},\mathbf{C}_{\mathrm{E}}}\mathbf{P}^{-\mathrm{T}} - \psi_{\mathrm{P},\mathbf{P}} = \mathbf{M}\mathbf{P}^{-\mathrm{T}} - \psi_{\mathrm{P},\mathbf{P}} \tag{25}$$

for the quantity $-\psi_{,\mathbf{P}}$ thermodynamically conjugate to $\dot{\mathbf{P}}$ in eqn (16) via (15) and the chain rule, where

$$M := 2\mathbf{C}_{\mathrm{E}}\varphi_{\mathrm{E},\mathbf{C}_{\mathrm{E}}} \tag{26}$$

represents Mandel's (1972) stress tensor. Substituting then eqn (25) into (16) yields the form

$$\delta_{\mathrm{P}} = \delta_{\mathrm{P}}(\mathbf{C},\mathbf{P},\mathbf{Y},\varepsilon,\dot{\mathbf{C}}) = [M - \psi_{\mathrm{P},\mathbf{P}}\mathbf{P}^{\mathrm{T}}]\cdot\mathbf{L}_{\mathrm{P}} - \psi_{\mathrm{P},\mathbf{Y}}\cdot\dot{\mathbf{Y}} - \psi_{\mathrm{P},\varepsilon}\dot{\varepsilon} \tag{27}$$

for the plastic dissipation function via eqn $(17)_1$. This last form of δ_{P} implies that $\mathbf{M}-\psi_{\mathrm{P},\mathbf{P}}\mathbf{P}^{\mathrm{T}}$ is thermodynamically conjugate to $\mathbf{L}_{\mathrm{P}}$ (note that $\dot{\mathbf{Y}}$ and $\dot{\varepsilon}$ are not linearly

dependent on $\mathbf{L}_P$; see Section 4). In the context of an associated formulation of elastoplasticity as based on the notion of elastic range, or more specifically, as based on a yield function, the form of eqn (27) is consistent with the interpretation of the tensor

$$\mathbf{X} := \psi_{P,\mathbf{P}}\mathbf{P}^T \tag{28}$$

as the centre of the elastic range of the material. As such eqn (28) represents a general thermodynamic form for the back stress, a basic result of the current approach. Besides depending upon EMI, note that this thermodynamic form for $\mathbf{X}$ is also dependent on the additive form of eqn (15) assumed for ψ in this work. In an analogous fashion, the quantity

$$\sigma := -\psi_{P,\varepsilon} \tag{29}$$

thermodynamically conjugate to the deformation-like internal variable ε in eqn (27) whose evolution is associated with isotropic hardening, can be interpreted as the yield stress of the material (e.g. Chaboche, 1993).

The results of eqns (26) and (28) imply that neither $\mathbf{M}$ nor $\mathbf{X}$ are in general symmetric (i.e. with respect to the Euclidean metric); since φ_{E,C_E} is symmetric, however, $\mathbf{M}$ is not an arbitrary linear transformation, but rather satisfies the auxiliary condition

$$\mathbf{M} = \mathbf{C}_E\mathbf{M}^T\mathbf{C}_E^{-1}, \tag{30}$$

implying that only six of its nine components are independent, i.e. that it is 6-dimensional, as noted also by Lubliner (1984, 1986). Although not symmetric with respect to the Euclidean metric, the condition in eqn (30) implies that $\mathbf{M}$ is, in general, symmetric with respect to the metric on the "intermediate" configuration induced by $\mathbf{E}^{-1}$ and the Euclidean metric on this configuration via "pull-back". The consequences of this symmetry for the formulation of associated plasticity, as well as the general material symmetry considerations for $\mathbf{X}$ in the form given in eqn (28), have been investigated elsewhere (Svendsen, 1998). As already stated above, here we focus instead on the application of the above formulation to the case of linear and non-linear Armstrong–Frederick kinematic hardening, our next task.

IV. APPLICATION: ARMSTRONG–FREDERICK KINEMATIC HARDENING

In the large deformation context, the classical Armstrong and Frederick (1966) evolution relation for $\mathbf{X}$ (see also, e.g. Chaboche and Rousselier (1983), for the geometric linear case) may be expressed in the form

$$\overset{*}{\mathbf{X}} = c\mathbf{D}_P - \dot{\kappa}\mathbf{X} \tag{31}$$

relative to the intermediate configuration, where $\overset{*}{\mathbf{X}}$ is some objective derivative of $\mathbf{X}$, c is a non-negative material constant mediating the increase of $\mathbf{X}$ with increasing plastic deformation, $\mathbf{D}_P := \mathrm{sym}(\mathbf{L}_P)$ is the symmetric part of $\mathbf{L}_P$, and κ is a further internal variable which increases monotonically with increasing plastic deformation, resulting in the damping of $\mathbf{X}$ as usual. In the classical Armstrong and Frederick (1966) case, one assumes, for example, $\kappa = bs$, where b is the original Armstrong–Frederick saturation parameter, and s is a plastic arc length.

Since $\mathbf{X}$ is, in the Armstrong–Frederick case, symmetric, the corresponding thermodynamic form of eqn (28) for $\mathbf{X}$ must also be symmetric. As shown by Svendsen (1998), this will be the case when the symmetry group of ψ with respect to the intermediate configuration is given by the entire orthogonal group (in which case $\mathbf{M}$ from eqn (26) is also symmetric, i.e. the material is elastically isotropic). Indeed, in that case, we obtain the reduction

$$\psi_{\mathrm{P}}(\mathbf{P}, \mathbf{Y}, \varepsilon) = \psi_{\mathrm{P}}(\mathbf{U}_{\mathrm{P}}, \mathbf{Y}, \varepsilon) \tag{32}$$

of ψ_{P} via the polar decomposition $\mathbf{P} = \mathbf{R}_{\mathrm{P}}\mathbf{U}_{\mathrm{P}}$ of $\mathbf{P}$. Since ψ_{P} is otherwise arbitrary at this point, any function of $\mathbf{U}_{\mathrm{P}}$ can also be used in ψ_{P} without loss of generality. By analogy with the elastic case we can then choose in particular the function $\mathbf{C}_{\mathrm{P}} = \mathbf{U}_{\mathrm{P}}^{\mathrm{T}}\mathbf{U}_{\mathrm{P}}$ of $\mathbf{U}_{\mathrm{P}}$, such that

$$\psi_{\mathrm{P},\mathbf{P}} = 2\mathbf{P}\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}}, \tag{33}$$

and so

$$\mathbf{X} = \psi_{\mathrm{P},\mathbf{P}}\mathbf{P}^{\mathrm{T}} = 2\mathbf{P}\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}}\mathbf{P}^{\mathrm{T}} \tag{34}$$

from eqn (28) is now symmetric. On this basis,

$$\mathbf{X}_C := 2\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}} \tag{35}$$

represents the constitutive part of $\mathbf{X}$, i.e.

$$\mathbf{X} = \mathbf{P}\mathbf{X}_C\mathbf{P}^{\mathrm{T}} \tag{36}$$

and so

$$\overset{*}{\mathbf{X}} = \mathbf{P}\dot{\mathbf{X}}_C\mathbf{P}^{\mathrm{T}} = \dot{\mathbf{X}} - \mathbf{L}_{\mathrm{P}}\mathbf{X} - \mathbf{X}\mathbf{L}_{\mathrm{P}}^{\mathrm{T}} \tag{37}$$

On this basis, the thermodynamic form of eqn (34) is consistent with $\mathbf{X}$ belonging to the first family of dual stress–deformation tensors of Haupt and Tsakmakis (1989) (see Haupt and Tsakmakis, 1986; Svendsen and Tsakmakis, 1994).

Now, assuming that $\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}}$ is independent of ε, the time derivative of eqn (35), yields

$$\dot{\mathbf{X}}_C = 2\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}\mathbf{C}_{\mathrm{P}}}[\dot{\mathbf{C}}_{\mathrm{P}}] + 2\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}\mathbf{Y}}[\dot{\mathbf{Y}}]. \tag{38}$$

Compatibility of this last result with eqns (31) and (37) then requires the constraint

$$4\mathbf{P}\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}\mathbf{C}_{\mathrm{P}}}[\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}]\mathbf{P}^{\mathrm{T}} = c\mathbf{A} \tag{39}$$

on the form of $\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}\mathbf{C}_{\mathrm{P}}}$ for all symmetric $\mathbf{A}$, with the form

$$\dot{\mathbf{Y}} = -(\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}\mathbf{Y}})^{-1}[\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}}]\dot{\kappa} \tag{40}$$

for the evolution of $\mathbf{Y}$, which is then directly "proportional" to that of κ, such that $\mathbf{Y}$ evolves with κ. To obtain these last two results, we have used the identity

$$\dot{\mathbf{C}}_{\mathrm{P}} = 2\mathbf{P}^{\mathrm{T}}\mathbf{D}_{\mathrm{P}}\mathbf{P}, \tag{41}$$

and assumed that $\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}\mathbf{Y}}$ is invertible.

A particular form of ψ_P satisfying these latter requirements is given by

$$\psi_P(\mathbf{C}_P, \mathbf{Y}, \varepsilon) = {}^1/_2\mathbf{C}_P\cdot\mathcal{H}(\mathbf{Y}) - {}^1/_4 c\ln(\det(\mathbf{C}_P)) - h(\varepsilon) \tag{42}$$

where $\mathcal{H}(\mathbf{Y})$ represents a differentiable, symmetric-tensor-valued function of $\mathbf{Y}$ possessing additional properties which are apparent from the context of what follows, and h is a scalar-valued isotropic hardening function. Perhaps the simplest form that $\mathcal{H}$ could take would be $\mathcal{H}(\mathbf{Y}) = c\mathbf{Y}$, with $\mathbf{Y}$ symmetric. In this latter case, ψ is isotropic with respect to both the intermediate and reference configurations (Svendsen, 1998); indeed, for this particular choice of $\mathcal{H}$, note that ψ_P as given in eqn (42), is an isotropic function of its arguments.

Now, from eqn (42), we obtain

$$\mathbf{X}_C = 2\psi_{P,\mathbf{C}_P} = \mathcal{H}(\mathbf{Y}) - {}^1/_2 c\mathbf{C}_P^{-1}, \tag{43}$$

as well as

$$4\psi_{P,\mathbf{C}_P\mathbf{C}_P}[\mathbf{A}] = c\mathbf{C}_P^{-1}\mathbf{A}\mathbf{C}_P^{-1} \tag{44}$$

and

$$\psi_{P,\mathbf{C}_P\mathbf{Y}} = {}^1/_2\mathcal{H}_{,\mathbf{Y}}. \tag{45}$$

Further, we then have the evolution relation

$$\dot{\mathbf{Y}} = (\mathcal{H}_{,\mathbf{Y}})^{-1}[{}^1/_2 c\mathbf{C}_P^{-1} - \mathcal{H}(\mathbf{Y})]\dot{\kappa} \tag{46}$$

for $\mathbf{Y}$, as well as that

$$\dot{\mathbf{X}}_C = c\mathbf{P}^{-1}\mathbf{D}_P\mathbf{P}^{-T} - 2\psi_{P,\mathbf{C}_P}\dot{\kappa} \tag{47}$$

for $\mathbf{X}_C$ via

$$\dot{\overline{{}^1/_2\mathbf{C}_P^{-1}}} = -{}^1/_2\mathbf{C}_P^{-1}\dot{\mathbf{C}}_P\mathbf{C}_P^{-1} = \mathbf{P}^{-1}\mathbf{D}_P\mathbf{P}^{-T} \tag{48}$$

eqns (46) and (43). Finally, note that the evolution relation (40) for $\mathbf{Y}$, and the particular form in eqn (42) for ψ_P, in turn yield the form

$$\begin{aligned}\delta_P &= \delta_P(\mathbf{C}, \mathbf{P}, \mathbf{Y}, \varepsilon, \dot{\mathbf{C}})\\ &= [\mathbf{M} - \mathbf{X}]\cdot\mathbf{L}_P + \psi_{P,\mathbf{Y}}\cdot(\psi_{P,\mathbf{C}_P\mathbf{Y}})^{-1}[\psi_{P,\mathbf{C}_P}]\dot{\kappa} - \psi_{P,\varepsilon}\dot{\varepsilon}\\ &= [\mathbf{M} - \mathbf{X}]\cdot\mathbf{L}_P + [{}^1/_2\mathbf{C}_P\cdot\mathcal{H}(Y) - {}^3/_4 c]\dot{\kappa} + \sigma\dot{\varepsilon}\end{aligned} \tag{49}$$

for the plastic dissipation function of eqn (27), with $\sigma = h'$ from eqns (29) and (42). As usual, this last form is satisfied sufficiently when each term is individually greater than or equal to zero. Noting that $\dot{\kappa}\geq 0$, this will in particular be the case for the second term when

$$\mathbf{C}_P\cdot\mathcal{H}(\mathbf{Y}) \geq {}^3/_2 c \tag{50}$$

holds, representing a restriction on the form of the function $\mathcal{H}(\mathbf{Y})$ in relation to $\mathbf{C}_P$. In particular, assuming $\mathbf{P}(0) = \mathbf{I}$, $\mathbf{Y}(0) = \mathbf{I}$, $\mathbf{X}(0) = 0$, we have $\mathbf{X}_C(0) = 0$ via eqn (36), and so $\mathcal{H}(\mathbf{Y}(0)) = {}^1/_2 c\mathbf{I}$ from eqn (43), in which case $\mathbf{C}_P(0)\cdot\mathcal{H}(\mathbf{Y}(0)) = {}^1/_2 c\mathbf{I}\cdot\mathbf{I} = {}^3/_2 c$. Initially, then, this condition is satisfied identically.

The above formulation can also be carried out with respect to the second family of dual stress–deformation tensors of Haupt and Tsakmakis (1989) as well; in this case, the Armstrong–Frederick relation takes the form

$$\overset{*}{\mathbf{X}} = -c\mathbf{D}_{\mathrm{P}} - \dot{\kappa}\mathbf{X} \tag{51}$$

where now

$$\overset{*}{\mathbf{X}} = \dot{\mathbf{X}} + \mathbf{L}_{\mathrm{P}}^{\mathrm{T}}\mathbf{X} + \mathbf{X}\mathbf{L}_{\mathrm{P}} \tag{52}$$

A thermodynamic form for this relation as well as ψ_{P} can be obtained in the context of the current formulation when we, so to speak, simply replace $\mathbf{C}_{\mathrm{E}}$ and $\mathbf{C}_{\mathrm{P}}$ by their inverses in the formulation. Indeed, we then obtain the form

$$\mathbf{M} = 2\mathbf{C}_{\mathrm{E}}\varphi_{\mathrm{E},\mathbf{C}_{\mathrm{E}}} = -2\varphi_{\mathrm{E},\mathbf{C}_{\mathrm{E}}^{-1}}\mathbf{C}_{\mathrm{E}}^{-1} \tag{53}$$

for $\mathbf{M}$ from eqn (26), as well as

$$\mathbf{X} = 2\mathbf{P}\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}}\mathbf{P}^{\mathrm{T}} = -2\mathbf{P}^{-\mathrm{T}}\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}^{-1}}\mathbf{P}^{-1} \tag{54}$$

for $\mathbf{X}$ from eqn (54). Likewise,

$$\mathbf{X}_C = -2\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}^{-1}} \tag{55}$$

then holds for $\mathbf{X}_{\mathrm{C}}$. Following the same procedure as above with respect to the first family, one can then establish in the same fashion that the form

$$\psi_{\mathrm{P}}(\mathbf{C}_{\mathrm{P}}^{-1}, \mathbf{Y}, \varepsilon) = {}^1/_2\mathbf{C}_{\mathrm{P}}^{-1}\cdot\mathcal{H}(\mathbf{Y}) + {}^1/_4 c\ln(\det(\mathbf{C}_{\mathrm{P}}^{-1})) - h(\varepsilon) \tag{56}$$

for ψ_{P}, as well as that

$$\dot{\mathbf{Y}} = -(\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}^{-1}\mathbf{Y}})^{-1}[\psi_{\mathrm{P},\mathbf{C}_{\mathrm{P}}^{-1}}]\dot{\kappa} \tag{57}$$

for $\mathbf{Y}$, analogous to eqns (42) and (40), respectively, yield eqns (51) and (52) via (54) and time-differentiation.

V. DISCUSSION

As already alluded to in the previous sections, one finds in the literature essentially two approaches to the thermodynamic formulation of the back stress via internal variables. Conceptually speaking, both of these approaches for the case of linear kinematic hardening are considered in Dogui and Sidoroff (1985). Following these authors, we will refer to the approach in which ψ_{P} depends explicitly on $\mathbf{P}$ as Approach 1, and the other as Approach 2. The variables associated with each approach will be labeled correspondingly. For the sake of simplicity, we also neglect isotropic hardening here, focusing solely on kinematic hardening.

As already mentioned above, formulations representing versions of Approach 1 can be found in Dogui and Sidoroff (1985), whose approach was limited to linear kinematic hardening, and has been generalized in the current work to the non-linear Armstrong–

Frederick case, as well as in Haupt (1995), who also dealt with this latter case. In particular, Dogui and Sidoroff (1985) discussed the form*

$$\mathbf{X}_1 = {}^1/_2 c[\mathbf{B}_P - \mathbf{I}] \tag{58}$$

for $\mathbf{X}_1$, whose time derivative yields a Prager-type model for linear kinematic hardening with respect to the intermediate configuration, as well as the form

$$\overset{*}{\mathbf{X}}_1 = \dot{\mathbf{X}}_1 - \mathbf{L}_P\mathbf{X}_1 - \mathbf{X}_1\mathbf{L}_P^T \tag{59}$$

for $\overset{*}{\mathbf{X}}_1$ analogous to the current work, with $\mathbf{B}_P = \mathbf{F}_P\mathbf{F}_P^T$ the plastic left Cauchy–Green deformation. Although they didn't obtain the corresponding form for ψ_P, this can be done, yielding

$$\psi_P = \psi_P(\mathbf{F}_P) = \psi_P(\mathbf{B}_P) = {}^1/_4 c[\mathrm{tr}(\mathbf{B}_P) - \ln \det(\mathbf{B}_P)] \tag{60}$$

in the current context. As shown in the current work, because ψ_P depends solely on $\mathbf{F}_P$ in their formulation, it is limited to linear kinematic hardening. On the other hand, Haupt (1995) worked with the form

$$\psi_P = \psi_P(\mathbf{B}_P, \mathbf{Y}_1) = {}^1/_2 c(\mathbf{A}_P - \mathbf{Y}_1)\cdot(\mathbf{A}_P - \mathbf{Y}_1) \tag{61}$$

with $\mathbf{A}_P := {}^1/_2(\mathbf{I} - \mathbf{B}_P^{-1})$ the plastic Almansi strain tensor, and $\mathbf{Y}_1$ a strain-like internal variable accounting for non-linear hardening. The evolution of this last variable is given by

$$\overset{*}{\mathbf{Y}}_1 = b\dot{s}(\mathbf{A}_P - \mathbf{Y}_1) - 2(\mathbf{D}_P\mathbf{A}_P + \mathbf{A}_P\mathbf{D}_P) \tag{62}$$

with $\overset{*}{\mathbf{Y}}_1$ taking the form of eqn (59). The form of eqn (62) represents a sufficient condition to satisfy $\delta_P \geq 0$. In contrast to the approach taken in the current work, however, he defined the back stress analogous to Tsakmakis (1996) as discussed below, i.e.

$$\mathbf{X}_1 := c[1 + 2(\mathbf{A}_P - \mathbf{Y}_1)](\mathbf{A}_P - \mathbf{Y}_1), \tag{63}$$

rather than directly via a relation of the form given in eqn (28).

As discussed in the text, Approach 2 is based on the stronger form

$$\psi(\mathbf{C}, \mathbf{P}, \alpha) = \varphi(\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}, \alpha) \tag{64}$$

of EMI tacitly assumed in most previous thermodynamic formulations (Kratochvíl and Dillon, 1969; Mandel, 1972, 1974, 1982; Lubliner, 1973, 1984, 1986, 1987; Chaboche, 1993; Tsakmakis, 1996; Sievert, 1997). The fact that a qualitative difference exists between this approach and the one taken in this work becomes immediately clear when one realizes that eqn (64) requires ψ_P to be independent of $\mathbf{P}$, in which case $\mathbf{X}$ as given by eqn (28)

*To be exact, the back stress $\mathbf{X}$ formulated by Dogui and Sidoroff (1985) in this context is actually $\mathbf{X}_1$ as given in eqn (58) rotated forward to the current configuration, i.e. $\mathbf{R}_E\mathbf{X}_1\mathbf{R}_E^T$.

would vanish! Instead of an explicit dependence of ψ_P on $\mathbf{P}$, one works with the form

$$\psi(\mathbf{C}, \mathbf{P}, \mathbf{Y}_2) = \psi_E(\mathbf{C}_E) + \psi_P(\mathbf{Y}_2) \tag{65}$$

of ψ in this case, in which appears the deformation-like variable $\mathbf{Y}_2$. For example, the model of Chaboche (1993), when generalized to finite deformation, takes the form

$$\psi_P(\mathbf{Y}_2) = {}^2/_3 c\mathbf{Y}_2 \cdot \mathbf{Y}_2 \tag{66}$$

(relative to the intermediate configuration; his α is represented here by $\mathbf{Y}_2$), with

$$\overset{*}{\mathbf{Y}}_2 = \mathbf{D}_P - \varphi(s)\dot{s}\mathbf{Y}_2 \text{ and } \overset{*}{\mathbf{Y}}_2 = \dot{\mathbf{Y}}_2 \tag{67}$$

for the evolution of $\mathbf{Y}_2$ analogous to eqn (31) (neglecting static recovery). Further, Chaboche (1993) assumes that the back stress $\mathbf{X}_2$ is thermodynamically conjugate to $\mathbf{Y}_2$, yielding

$$\mathbf{X}_2 = \psi_{P,\mathbf{Y}_2} = {}^1/_3 c\mathbf{Y}_2 \tag{68}$$

from eqn (66). As discussed in more detail in Svendsen *et al.* (1998), this last approach also encompasses the formulation of large-deformation kinematic hardening in polycrystalline metals based on "micromechanical" considerations and a constitutive model for $\mathbf{W}_P := \text{skw}(\mathbf{L}_P)$ as advocated, e.g. by Dafalias (1983, 1985), Loret (1983), Aifantis (1987), Paulun and Pecherski (1987, 1992) or Van der Giessen (1991). Indeed, in this case, we have the form

$$\dot{\mathbf{R}}_E = \mathbf{W}\mathbf{R}_E - \mathbf{R}_E\mathbf{W}_P + \mathbf{O}(|\ln \mathbf{U}_E|) \tag{69}$$

for the evolution of $\mathbf{R}_E$ in the case of small elastic deformations (i.e. $|\ln \mathbf{U}_E 1$), as well as the forms

$$\begin{aligned} \mathbf{K} &= \mathbf{R}_E\mathbf{M}\mathbf{R}_E^T + \mathbf{O}(|\ln\ \mathbf{U}_E|) \\ \mathbf{A}_2 &= \mathbf{R}_E\mathbf{X}_2\mathbf{R}_E^T + \mathbf{O}(|\ln\ \mathbf{U}_E|) \end{aligned} \tag{70}$$

for the Kirchhoff and current back stress tensors, respectively. Consequently, one need only specify a constitutive model for $\mathbf{W}_P$, as done in all the above-mentioned works.

Although of this type, the model by Tsakmakis (1996) is yet more sophisticated. Indeed, on the basis of the Chaboche (1993) form of eqn (66) for ψ_P, he introduces the further stress-like internal variable

$$\mathbf{Z}_2 := \psi_{P,\mathbf{Y}_2} = c\mathbf{Y}_2 \tag{71}$$

which evolves via the form

$$\overset{*}{\mathbf{Z}}_2 = \mathbf{D}_P - b\dot{s}\mathbf{Z}_2 \text{ with } \overset{*}{\mathbf{Z}}_2 = \dot{\mathbf{Z}}_2 - \mathbf{L}_P\mathbf{Z}_2 - \mathbf{Z}_2\mathbf{L}_P^T \tag{72}$$

analogous to eqn (67), and then defines the back stress via the relation

$$\mathbf{X}_2 := (\mathbf{I} + 2\mathbf{Y}_2)\mathbf{Z}_2. \tag{73}$$

This last form for $\mathbf{X}_2$ is motivated by eqn (26) for the Mandel stress $\mathbf{M}$, $(\mathbf{I}+2\mathbf{Y}_2)$ corresponding to $\mathbf{C}_E$ and $\mathbf{Z}_2$ to $2\psi_{E,\mathbf{C}_E}$. Indeed, Tsakmakis (1996) interprets $\mathbf{Y}_2$ as a strain-like internal variable analogous to the elastic Green strain tensor, and $\mathbf{Z}_2$ as a stress-like internal variable analogous to the elastic second Piola–Kirchhoff stress tensor.

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