

**A UNIFIED APPROACH TO FINITE DEFORMATION
ELASTOPLASTIC ANALYSIS BASED ON THE USE OF
HYPERELASTIC CONSTITUTIVE EQUATIONS**

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By assuming from the outset hyperelastic constitutive behavior, an alternative approach to finite deformation plasticity and viscoplasticity is proposed whereby the need for integration of spatial rate constitutive equations is entirely bypassed. To enhance the applicability of the method, reference is made to a general formulation of plasticity and viscoplasticity which embodies both the multiplicative and additive theories. A new return mapping algorithm capable of accommodating general yield conditions, arbitrary flow and hardening rules and non-constant tangent elasticities is proposed. Finally, a numerical example is presented which illustrates the excellent performance of the method for very large time steps.

1. Introduction

In the finite deformation literature it is often found that the elastic response of the material is spatially formulated in rate form, i.e., as an incremental relation between objective rates of stress and spatial deformation. If special care is not exercised, such incremental relations may not be integrable and thus inconsistent with the notion of hyperelasticity, in the sense that a stored energy potential does not exist. This situation may result in aberrant behavior such as hysteretic dissipation inappropriate for an elastic model [1, 2]. A familiar example is furnished by the assumption frequently made for computational purposes that the spatial tangent elasticity tensor is constant and isotropic. It has been shown in [3] that this widely employed constitutive model is not only incompatible with the notion of hyperelasticity but even fails to define an elastic (non-dissipative) material in the nonlinear range.

As noted in [4], from an algorithmic standpoint special care must be exercised in the integration of spatial rate constitutive equations if the fundamental principle of objectivity is to be preserved. This leads to the notion of incrementally objective integration algorithms [4, 5], and often results in schemes which may add significantly to the computational cost of the analyses [6]. One of the aims of the present paper is to show that this added expense is entirely superfluous. The key fact to be realized is that, even for an inelastic material, the elastic response can be spatially formulated in primitive or non-rate form as a functional

relation between stresses and suitable strain measures. This point appears to have passed largely unnoticed in the computational literature. A notable exception is found in Argyris and Doltsinis [44, 45]. There, the notion of *incremental elastic potential* employed in [42, 43] is replaced by a *hyperelastic* characterization of the elastic response relative to the intermediate configuration, formulated in the context of the so-called *natural approach*. For this purpose, a second-order expansion of the free energy in terms of natural strains is considered. These authors, however, seem to favour an algorithm treatment based on rate formulations as outlined in [46]. Here, in a different context, a general hyperelastic characterization is considered formulated relative to either the intermediate, spatial or material configurations. As a result of this formulation, *the need for integration of spatial rate constitutive equations is entirely bypassed, even in the inelastic case*. Furthermore, truly hyperelastic behavior is obtained and the principle of objectivity is trivially satisfied.

To emphasize the applicability of the method to problems involving inelastic behavior, finite deformation elastoplasticity is considered in detail. In this case, stresses are updated in two steps: the spatial elastic stress-strain relations are first evaluated to produce an elastic stress predictor, which is subsequently mapped onto a suitably updated yield surface. Following the pioneering work of Wilkins [7], similar return-mapping notions have been extensively used in the past [8–12, 41] although the scope of such formulations has been by and large restricted to simple plasticity models such as linearly hardening von Mises and to constant elastic moduli. The problem of extending these schemes to the case of nonlinear elasticity with non-constant tangent elasticity tensors is not a trivial one and has not been heretofore considered in the literature. It is shown in Section 3 how the operator splitting methodology can be used to define computationally efficient return mapping algorithms which are applicable to very general materials exhibiting non-associated plasticity, arbitrary yield criteria and hardening laws and variable elastic moduli. This latter aspect is of particular relevance since, as mentioned above, a formulation of the elastic response consistent with the principles of hyperelasticity cannot possibly result in a constant tangent elasticity tensor.

Finally, in Section 4 a numerical example is presented which demonstrates the excellent performance of the method for very large values of the time step. Further numerical examples are given in [38].

2. Summary of constitutive theory for finite deformation plasticity

The proper formulation of elastoplastic constitutive laws in the finite deformation range has been the subject of considerable conjecture. Differences of opinion have been voiced concerning elastoplastic kinematics and the formulation of flow rules. Lee [13, 14] and others, for instance, have proposed a theory based on a multiplicative decomposition of the deformation gradient,

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \quad (2.1)$$

where the elastic part of the deformation \mathbf{F}^e is obtained by unloading all infinitesimal neighborhoods of the body. This has the effect of introducing a new configuration into the formulation, commonly termed the intermediate configuration, defined as the collection of all unloaded local neighborhoods. This situation is graphically shown in Fig. 1. An account of the geometric concepts underlying the multiplicative decomposition can be found in [15, 47]. For

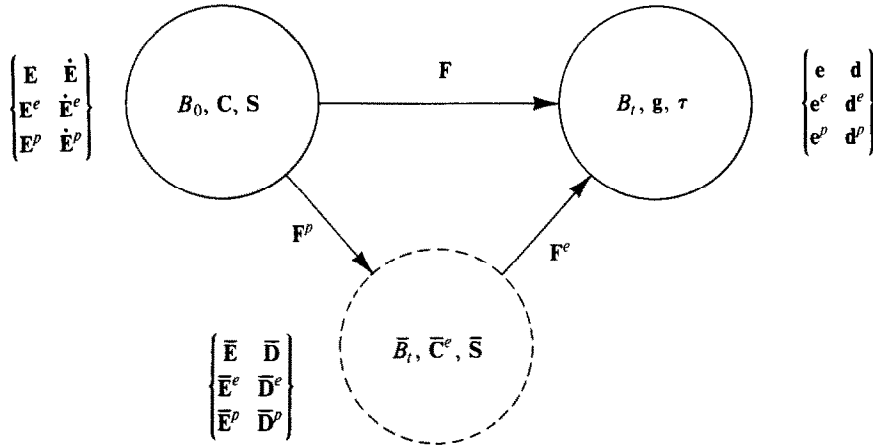


Fig. 1. Schematic representation of material, intermediate and spatial configurations.

polycrystalline solids, such as metals, multiplicative theories are amenable to an elegant physical interpretation based on dislocation mechanics [16].

The multiplicative relation (2.1) does not exhaust, however, all the possibilities concerning an elastic-plastic decomposition of deformation. Green and Naghdi [17] have advocated an additive decomposition of Lagrangian strain,

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p, \quad (2.2)$$

in terms of elastic and plastic components \mathbf{E}^e and \mathbf{E}^p , respectively. Such an additive decomposition rule has been conclusively shown to enjoy a solid thermodynamic foundation [18, 19]. On the other hand, in the computational literature an additive decomposition of the spatial rate of deformation tensor,

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p, \quad (2.3)$$

has been frequently postulated (e.g. [20–24, 41]).

In view of this lack of a standard and generally accepted theoretical framework, this section is devoted to a brief account of a constitutive theory for finite deformation elastoplasticity which will be taken as a basis for subsequent discussions. The multiplicative decomposition (2.1) is adopted as the basic kinematic assumption. However, purely geometric arguments together with the concept of covariance¹ show that within the multiplicative framework material and spatial formulations can be derived in which strains decompose additively into elastic and plastic parts. The main geometric relations involved are summarized in Table 4. The correspondence between additive and multiplicative theories has also been investigated by Green and Naghdi [27] and Nemat-Nasser [28], among others, in a different context.

For simplicity, attention is confined throughout to the isothermal case. Tables 1, 2 and 3 summarize the relevant relations pertaining to the constitutive framework adopted herein. It should be emphasized, however, that the numerical techniques proposed in this paper are not dependent upon this particular set of constitutive assumptions and can be extended to other models without conceptual changes.

¹The notion of covariance expresses the idea of form invariance of the basic field equations with respect to arbitrary superposed diffeomorphisms (see [29] and [31, Section 2.4]). This notion, central to other branches of mechanics such as general relativity, also plays an important role in continuum mechanics (see [25, 26]).

Table 1
Material formulation of elastoplastic constitutive relations

Elastic-plastic decomposition of Lagrangian strain

$$\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p$$

Stress-strain relations

$$\mathbf{S} = \rho_0 \frac{\partial \Psi}{\partial \mathbf{E}^e}(\mathbf{E}^e, \mathbf{E}^p, \mathbf{Q})$$

Flow rule

$$\dot{\mathbf{E}}^p = \dot{\gamma} \mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$$

Hardening laws

$$\dot{\mathbf{Q}} = \dot{\gamma} \mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$$

Yield criterion

$$\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q}) = 0$$

2.1. Material formulation

In the material description, Table 1, the local plastic state of the material is assumed to be characterized by the Lagrangian strain tensor \mathbf{E} , the plastic Lagrangian strain $\mathbf{E}^p \equiv \frac{1}{2}(\mathbf{C}^p - \mathbf{I})$ and the plastic internal variables \mathbf{Q} . Here, \mathbf{C}^p is the plastic right Cauchy–Green tensor defined as $\mathbf{C}^p = \mathbf{F}^p \bar{\mathbf{G}} \mathbf{F}^p$, and $\bar{\mathbf{G}}$ is the metric tensor in the intermediate configuration. Typically, one chooses $\bar{\mathbf{G}} \equiv \mathbf{I}$; i.e., the standard metric in \mathbb{R}^3 . In the present context, the elastic Lagrangian strain \mathbf{E}^e is formally defined as the difference $\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p$. The nature of the plastic variables depends on the particular plastic model under consideration. For instance, for isotropic hardening, von Mises yield criterion and associative flow rule; the vector of internal plastic variables \mathbf{Q} reduces to the yield stress. For isotropic-kinematic hardening, \mathbf{Q} includes both the yield stress and the back-stress tensor defining the location of the elastic domain.

The stress-strain relations may be expressed in terms of a free-energy potential $\Psi(\mathbf{E}^e, \mathbf{E}^p, \mathbf{Q})$. It should be noted that this form of free-energy potential coincides with the one first proposed by Green and Naghdi [17].

General non-associated flow rules and hardening laws governing the evolution of \mathbf{E}^p and \mathbf{Q} can be formulated in terms of a plastic flow direction $\mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$ and plastic moduli $\mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$. Yielding of the material is expressed in terms of a yield function $\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q})$. For instance, in the particular case of an associated flow rule one has $\mathbf{R} = \partial \Phi / \partial \mathbf{S}$. Note that the dependence on \mathbf{C} of \mathbf{R} , \mathbf{H} and Φ need to be included to account for effects such as pressure dependence of the plastic response.

Table 2

Multiplicative formulation of elastoplastic constitutive relations

Elastic-plastic decomposition of deformation gradient

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$$

Stress-strain relations

$$\bar{\mathbf{S}} = \rho_0 \frac{\partial \bar{\Psi}}{\partial \bar{\mathbf{E}}^e} (\bar{\mathbf{E}}^e, \bar{\mathbf{E}}^p, \bar{\mathbf{Q}}, \mathbf{F}^p)$$

Flow rule

$$L_v^p \bar{\mathbf{E}}^p = \bar{\mathbf{D}}^p = \dot{\gamma} \bar{\mathbf{R}}(\bar{\mathbf{S}}, \bar{\mathbf{C}}^e, \bar{\mathbf{Q}})$$

Hardening laws

$$L_v^p \bar{\mathbf{Q}} = \dot{\gamma} \bar{\mathbf{H}}(\bar{\mathbf{S}}, \bar{\mathbf{C}}^e, \bar{\mathbf{Q}})$$

Yield criterion

$$\bar{\Phi}(\bar{\mathbf{S}}, \bar{\mathbf{C}}^e, \bar{\mathbf{Q}}) = 0$$

Table 3

Spatial formulation of elastoplastic constitutive relations

Elastic-plastic decomposition of Almansi strain

$$\mathbf{e}^p = \mathbf{e} - \mathbf{e}^e$$

Stress-strain relations

$$\boldsymbol{\tau} = \rho_0 \frac{\partial \psi}{\partial \mathbf{e}^e} (\mathbf{e}^e, \mathbf{e}^p, \mathbf{q}, \mathbf{F})$$

Flow rule

$$L_v \mathbf{e}^p = \mathbf{d}^p = \dot{\gamma} \mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q})$$

Hardening laws

$$L_v \mathbf{q} = \dot{\gamma} \mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q})$$

Yield criterion

$$\varphi(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) = 0$$

2.2. Multiplicative theory

Relative to the intermediate configuration, Table 2, the local plastic state is characterized by the Lagrangian strain tensors $\bar{\mathbf{E}}^e \equiv \frac{1}{2}(\bar{\mathbf{C}}^e - \bar{\mathbf{G}})$ and $\bar{\mathbf{E}}^p \equiv \frac{1}{2}(\bar{\mathbf{G}} - \bar{\mathbf{b}}^{p-1})$, and some suitable set of plastic internal variables $\bar{\mathbf{Q}}$. Here, $\bar{\mathbf{C}}^e = \mathbf{F}^{et} \mathbf{g} \mathbf{F}^e$ is the elastic right Cauchy–Green relative to the intermediate configuration, $\bar{\mathbf{b}}^{p-1} \equiv \mathbf{F}^{p-t} \mathbf{G}^{-1} \mathbf{F}^{p-1}$ is the plastic Finger deformation tensor, and \mathbf{G} stands for the metric tensor in the reference configuration. Finally, \mathbf{g} denotes the spatial metric tensor. Although in the context of \mathbb{R}^3 one typically makes the identification $\mathbf{G} \equiv \bar{\mathbf{G}} \equiv \mathbf{g}$ with the standard euclidean metric in \mathbb{R}^3 , for conceptual clarity it proves convenient here to maintain the notational distinction between these three metric tensors. The relationship between the *material* tensors \mathbf{C} , \mathbf{C}^p , \mathbf{E}^p and \mathbf{E}^e , and the tensors $\bar{\mathbf{C}}^e$, $\bar{\mathbf{G}}$, $\bar{\mathbf{E}}^p$ and $\bar{\mathbf{E}}^e$ defined on the intermediate configuration, has been summarized for convenience in Table 4, columns 1 and 2. These tensors play an identical role in different configurations. The relationship between these tensors is one of tensorial nature which is defined in the only possible way that geometrically makes sense. In the context of analysis in manifolds (e.g., see [29] or [31, Chapter 1]) the terminology *pull-back/push-forward* is often employed. We shall use this ‘descriptive’ terminology in the sequel.

The relation between $\bar{\mathbf{E}}^e$ and the second Piola–Kirchhoff stress tensor relative to the intermediate configuration, $\bar{\mathbf{S}}$, can be expressed with the aid of an objective free energy potential $\bar{\Psi}(\bar{\mathbf{E}}^e, \bar{\mathbf{E}}^p, \bar{\mathbf{Q}}, \mathbf{F}^p)$. Note that this choice of arguments is entirely consistent with the one made in the material formulation, and simply follows from the latter by push-forward with \mathbf{F}^p . This accounts for including \mathbf{F}^p in the list of arguments of $\bar{\Psi}$.

The plastic response in the intermediate configuration may be characterized by means of a plastic flow direction $\bar{\mathbf{R}}(\bar{\mathbf{S}}, \bar{\mathbf{C}}^e, \bar{\mathbf{Q}})$, plastic moduli $\bar{\mathbf{H}}(\bar{\mathbf{S}}, \bar{\mathbf{C}}^e, \bar{\mathbf{Q}})$ and a yield function $\bar{\Phi}(\bar{\mathbf{S}}, \bar{\mathbf{C}}^e, \bar{\mathbf{Q}})$. Consistent with the dependence in the material description of the plastic response functions on \mathbf{C} , the dependence of $\bar{\mathbf{R}}$, $\bar{\mathbf{H}}$ and $\bar{\Phi}$ on the elastic right Cauchy–Green tensor $\bar{\mathbf{C}}^e$ needs to be included to obtain a fully covariant formulation.

It may be noted from Table 2 that rates of tensors defined in the intermediate configuration are taken relative to the plastic flow, which immediately leads to the notion of *plastic Lie derivative*. This concept, proposed in [26] in the context of elasticity, is the natural extension of the notion of Lie derivative (e.g., [31]), to the present description relative to the intermediate configuration. In a more classical terminology, it may be defined simply as the *convected* derivative relative to the intermediate configuration. As an example, the plastic Lie derivative of $\bar{\mathbf{C}}^e$ is computed by first pulling $\bar{\mathbf{C}}^e$ back to the material configuration to obtain \mathbf{C} , taking the time derivative $\dot{\mathbf{C}}$ and finally pushing it forward into the intermediate configuration according to the expression $L_p^p \bar{\mathbf{C}}^e \equiv \mathbf{F}^{p-t} \dot{\mathbf{C}} \mathbf{F}^{p-1}$. Similar definitions apply to any other tensorial object, in particular to the internal variables $\bar{\mathbf{Q}}$ as recorded in Table 2.

2.3. Spatial formulation

By way of background, it may be recalled that in the context of elasticity the stored energy potential in the spatial description is an objective function of the form $\tilde{\psi}(\mathbf{g}, \mathbf{F})$ [25, 29, 31], where \mathbf{g} is the spatial metric tensor. The tensorial dependence of $\tilde{\psi}$ on \mathbf{g} was first pointed out by Doyle and Ericksen [30] who derived the formula $\boldsymbol{\tau} = 2\rho_0 \partial \tilde{\psi} / \partial \mathbf{g}$, where $\boldsymbol{\tau}$ is the Kirchhoff stress tensor. Since the Almansi strain tensor \mathbf{e} is given by $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{b}^{-1})$, with $\mathbf{b}^{-1} = \mathbf{F}^{-t} \mathbf{F}^{-1}$, we have the equivalent form $\psi(\mathbf{e}, \mathbf{F})$ of the stored energy potential, which in turn leads to the

alternative expression $\tau = \rho_0 \partial \psi / \partial \mathbf{e}$ for the Doyle–Ericksen formula. This formula follows from the fact that $L_v \mathbf{b} \equiv \mathbf{0}$, [25].

In the present context, similar arguments lead to a spatial elastic potential of the form $\psi(\mathbf{e}^e, \mathbf{e}^p, \mathbf{q}, \mathbf{F})$, where $\mathbf{e}^e = \frac{1}{2}(\mathbf{g} - \mathbf{b}^{e-1})$ is the elastic Almansi strain tensor, and $\mathbf{b}^{e-1} = \mathbf{F}^{e-1} \bar{\mathbf{G}} \mathbf{F}^{e-1}$ is the elastic Finger deformation tensor. Note that plastic strains may be defined by means of the difference $\mathbf{e}^p \equiv \mathbf{e} - \mathbf{e}^e$. Alternatively, the relation $\mathbf{e}^e = \mathbf{e} - \mathbf{e}^p$ follows from its material counterpart $\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p$ simply by push-forward operation. We refer again to Table 4 for an explicit expression of the relations between *spatial* and *material* tensors, and tensors defined relative to the intermediate configuration. Note that these are simply pull-back/push-forward type of (tensorial) relations. It should be emphasized that, contrary to common practice in the computational literature, the elastic response of the material is thus formulated as a non-rate functional relation between spatial stresses and elastic strains.

By covariance, the flow rule, hardening laws and yield criterion take the form expressed in Table 3, in terms of τ , the spatial plastic variables, \mathbf{q} , and \mathbf{g} . This latter quantity is the spatial counterpart of \mathbf{C} and is needed in general to obtain scalars from τ and \mathbf{q} and formulate tensorially meaningful plastic relations.

Finally, the Lie derivative $L_v \mathbf{e}^p$ is defined as the push-forward into the current configuration of $\dot{\mathbf{E}}^p$, which takes the form $L_v \mathbf{e}^p = \mathbf{F}^{-1} \dot{\mathbf{E}}^p \mathbf{F}^{-1}$. Similarly, $L_v \mathbf{q}$ is defined as the push-forward of $\dot{\mathbf{Q}}$ into the current configuration. The specific component form of $L_v \mathbf{q}$ depends on the tensorial nature of the plastic variables \mathbf{q} under consideration. The reader unfamiliar with geometric notions such as the push-forward and pull-back operations and the Lie derivative may wish to consult reference [31, Chapter 1] for an excellent review.

REMARK 2.1. In the context of elasticity, one also have (at least) three possible alternative descriptions, namely, the material and spatial descriptions and the *rotated* description [25, 26]. The latter is obtained from the spatial description by pull-back with the rotation tensor arising from the polar decomposition of deformation gradient. Equivalently, the rotated and material

Table 4
Basic definition of kinematical and stress variables

	$B \equiv \text{material}$	Configurations $\bar{B}_i \equiv \text{intermediate}$	$B_i \equiv \text{spatial}$
Basic kinematic tensors	$\mathbf{C} = \mathbf{F}^t \mathbf{g} \mathbf{F}$; $\mathbf{C}^p = \mathbf{F}^{pt} \bar{\mathbf{G}} \mathbf{F}^p$ $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$ $\mathbf{E}^p = \frac{1}{2}(\mathbf{C}^p - \mathbf{G})$ $\mathbf{E}^e = \frac{1}{2}(\mathbf{C} - \mathbf{C}^p)$ $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$	$\bar{\mathbf{C}}^e = \mathbf{F}^{et} \mathbf{g} \mathbf{F}^e$; $\bar{\mathbf{G}}$ $\bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{C}}^e - \mathbf{F}^{p-1} \mathbf{G} \mathbf{F}^{p-1})$ $\bar{\mathbf{E}}^p = \frac{1}{2}(\bar{\mathbf{G}} - \mathbf{F}^{p-1} \mathbf{G} \mathbf{F}^{p-1})$ $\bar{\mathbf{E}}^e = \frac{1}{2}(\bar{\mathbf{C}}^e - \bar{\mathbf{G}})$ $\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \bar{\mathbf{E}}^p$	\mathbf{g} , $\mathbf{b}^{e-1} = \mathbf{F}^{-1} \bar{\mathbf{G}} \mathbf{F}^{-1}$ $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{F}^{-1} \mathbf{G} \mathbf{F}^{-1})$ $\mathbf{e}^p = \frac{1}{2}(\mathbf{b}^{e-1} - \mathbf{F}^{-1} \mathbf{G} \mathbf{F}^{-1})$ $\mathbf{e}^e = \frac{1}{2}(\mathbf{g} - \mathbf{b}^{e-1})$ $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$
Rate of deformation tensors	$\dot{\mathbf{E}}$ $\dot{\mathbf{E}}^p$ $\dot{\mathbf{E}}^e$ $\dot{\mathbf{E}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p$	$\bar{\mathbf{D}} \equiv L_v^p \bar{\mathbf{E}}$ $\bar{\mathbf{D}}^p \equiv L_v^p \bar{\mathbf{E}}^p$ $\bar{\mathbf{D}}^e \equiv L_v^p \bar{\mathbf{E}}^e$ $\bar{\mathbf{D}} = \bar{\mathbf{D}}^e + \bar{\mathbf{D}}^p$	$\mathbf{d} \equiv L_v \mathbf{e}$ $\mathbf{d}^p \equiv L_v \mathbf{e}^p$ $\mathbf{d}^e \equiv L_v \mathbf{e}^e$ $\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p$
Basic stress tensors	\mathbf{S}	$\bar{\mathbf{S}} = \mathbf{F}^p \mathbf{S} \mathbf{F}^{pt}$	$\tau = \mathbf{F} \mathbf{S} \mathbf{F}^t$

descriptions are related by pull-back/push-forward with the stretching tensor. In plasticity, the description relative to the intermediate configuration is the counterpart of the rotated description.

REMARK 2.2. The multiplicative decomposition (2.1), as it stands, is only defined modulo rigid body motions superimposed on \mathbf{F}^p . Attempts to uniquely orient the intermediate configuration have been made by Mandel (e.g. [32, 33]) and others. In view of the indeterminacy underlying (2.1) some authors have advocated flow rules for the plastic spin in the intermediate configuration [34] or for $\dot{\mathbf{F}}^p$ [16].

REMARK 2.3. A general loading/unloading criterion for the elastoplastic problem can be expressed in standard Kuhn–Tucker form as

$$\varphi \leq 0, \quad \dot{\lambda} \geq 0, \quad \varphi \dot{\lambda} = 0. \quad (2.4)$$

These conditions must hold simultaneously along any possible process. For $\varphi < 0$, (2.4)₃ yields $\dot{\lambda} = 0$, i.e. elastic behavior, while plastic flow is characterized by $\dot{\lambda} > 0$ which, as a result of (2.4)₃, implies satisfaction of the yield criterion $\varphi = 0$. $\dot{\lambda}$ is determined during loading by enforcing the so-called consistency condition $\dot{\varphi} = 0$. A convenient characterization of elastic/plastic processes, clearly inspired in the algorithmic concept of elastic predictor, has been recently discussed at length in [41]. This characterization is based on the use of the so-called ‘rate of elastic trial stress’ to sense loading/unloading, and arises naturally in the algorithmic framework discussed below. We refer to [41] for further details.

2.4. Model problem

Specific examples of simple constitutive models of the form summarized in Table 3 are the following.

(1) *Hyperelastic response.* This is an example describing a class of hyperelastic constitutive models proposed in [3] which proves convenient in numerical applications. In a spatial description, the model is defined by

$$\begin{aligned} \psi &= \lambda U(J^e) + \frac{1}{2}\mu I^e - \mu \log J^e, \\ \tau &= \lambda J^e \frac{dU(J^e)}{dJ^e} \mathbf{g}^{-1} + \mu (\mathbf{b}^e - \mathbf{g}^{-1}), \\ J\mathbf{c}^e &= \lambda J^e \frac{d}{dJ^e} \left[J^e \frac{dU(J^e)}{dJ^e} \right] \mathbf{g}^{-1} \otimes \mathbf{g}^{-1} + 2 \left[\mu - \lambda J^e \frac{dU(J^e)}{dJ^e} \right] \mathbf{I}_k, \end{aligned} \quad (2.5)$$

where $J^e = \det(\mathbf{F}^e)$, $J = \det(\mathbf{F})$, I^e is the first invariant of \mathbf{b}^e , and \mathbf{I}_k is the fourth-order symmetric

unit tensor with components $\frac{1}{2}[(g^{-1})^{ik}(g^{-1})^{jl} + (g^{-1})^{il}(g^{-1})^{jk}]$. The fourth-order tensor \mathbf{c}^e is the *spatial elasticity tensor* of the material which is often defined indirectly by push-forward of the *material* elasticity tensor $\rho_0 \partial \mathbf{S} / \partial \mathbf{E}^e$ (e.g., see [2, p. 131, equation (45.2)]). The parameters λ and μ are Lamé type material constants.

The form of the stored energy function (2.5)₁ corresponds to a neo-Hookean material which is extended into the compressible range by adding the extra function $U(J^e)$. This is in fact a particular case of a Hadamard material. A simple choice of $U(J^e)$ is given by

$$U(J^e) = \frac{1}{2}(\log J^e)^2, \quad (2.6)$$

which has proven effective in the context of the penalty method applied to incompressible materials such as the Mooney–Rivlin model [35]. Model (2.5) defines the simplest possible hyperelastic material whose tangent elasticity tensor is isotropic. Further motivation is provided by the fact that the Mooney–Rivlin model is a first approximation to the constitutive behavior of any nonlinear incompressible elastic isotropic material.

(2) *Plastic response model.* An example of a plasticity model widely used in computation is furnished by the von Mises yield criterion with isotropic hardening. To illustrate the role played by the spatial metric tensor \mathbf{g} and the right Cauchy–Green tensor \mathbf{C} in a covariant formulation of plasticity, material and spatial versions of the von Mises model are given next. In a spatial setting, the von Mises yield function reads

$$\varphi(\boldsymbol{\tau}, \mathbf{g}, \kappa) \equiv \frac{1}{2} \boldsymbol{\tau}' : [\mathbf{I}_{\mathbf{g}} : \boldsymbol{\tau}'] - \kappa^2 \equiv \frac{1}{2} \tau'^{ij} \tau'^{kl} g_{ik} g_{jl} - \kappa^2, \quad (2.7a)$$

where the components of the deviatoric Kirchhoff stress tensor $\boldsymbol{\tau}'$ are given by

$$\tau'^{ij} = \tau^{ij} - \frac{1}{3}(\tau^{kl} g_{kl})(g^{-1})^{ij}, \quad (2.7b)$$

and κ is the yield stress in shear. An alternative material formulation involves the yield function

$$\Phi(\mathbf{S}, \mathbf{C}, \kappa) = \frac{1}{2} S'^{IJ} S'^{KL} C_{IK} C_{JL} - \kappa^2, \quad (2.8a)$$

where the stress deviator consistent with (2.7b) is defined as

$$S'^{IJ} = S^{IJ} - \frac{1}{3}(S^{KL} C_{KL})(C^{-1})^{IJ}. \quad (2.8b)$$

Note that the hydrostatic pressure p is given by the equivalent expressions $3Jp = \tau^{ij} g_{ij} \equiv S'^{IJ} C_{IJ}$ which are utilized in spatial and material definitions, (2.7b) and (2.8b), of the stress deviator. Thus, it is apparent that \mathbf{g} and \mathbf{C} need to be included in the formulation of the plastic response. Note further that an identical role is played by $\bar{\mathbf{C}}^e$ in the intermediate configuration.

3. Numerical formulation

In this section, a number of numerical techniques is proposed that allow a systematic treatment of constitutive models of the general type discussed above within the context of

finite element analysis. The stress update algorithm herein proposed falls within the category of elastic predictor-return mapping algorithms widely used in computational plasticity. However, our formulation departs from currently employed procedures in that:

(i) The need for integration algorithms for elastic rate constitutive relations is entirely bypassed by formulating the elastic relations in non-rate form.

(ii) General hyperelastic models are considered and the definition of the return mapping is capable of accommodating non-constant elastic moduli.

(iii) The algorithm is applicable to completely general plasticity models including non-associated flow rules and arbitrary yield criteria.

(iv) Although the spatial formulation is herein emphasized, the proposed algorithm equally applies in a material setting.

In particular, our formulation of the elastic response is truly hyperelastic and does not involve objective stress rates. Thus, the elastic predictor is reduced to a mere function evaluation. Algorithmic requirements pertaining to integration of spatial rate constitutive equations such as incremental objectivity [4, 5] are not an issue here owing to the inherently objective nature of the hyperelastic constitutive relations. In addition, the elastic predictor is infinitely accurate since no algorithmic approximations are involved.

3.1. Elastic-plastic operator split

In any numerical scheme employed for the analysis of elastoplastic problems it eventually becomes necessary to update state variables such as stresses, strains and plastic parameters. In the context of finite element analysis using isoparametric elements, stress updates take place at the Gauss points and are performed for a given incremental deformation. The problem to be addressed, therefore, is that of updating the *known* state variables \mathbf{F}_n , \mathbf{F}_n^p , \mathbf{q}_n and τ_n associated with a *converged* configuration B_n into their corresponding *updated* values \mathbf{F}_{n+1} , \mathbf{F}_{n+1}^p , \mathbf{q}_{n+1} and τ_{n+1} on the *updated* configuration B_{n+1} in a manner consistent with the constitutive assumptions expressed in Table 3. In this process, the incremental displacements, \mathbf{u} , defining the geometry update $B_n \rightarrow B_{n+1}$ are assumed given.

A number of authors has advocated the use of so-called return mapping algorithms for integration of elastoplastic constitutive relations [7–12]. The applicability of such algorithms has been for the most part restricted to simple plasticity models such as linearly hardening von Mises model with constant elastic moduli. However, many materials of engineering interest such as concrete and soils exhibit nonlinear elastic response, non-associated plasticity and complex yield criteria, flow rules and hardening laws. Furthermore, it has been shown in [3] that an elastic material cannot possibly have constant and isotropic tangent elasticities in the finite deformation range. Therefore, an integration scheme for elastoplastic constitutive relations should be able to accommodate general *non-constant* tangent stiffness compliances.

We show next how an operator splitting methodology proposed in [36, Chapter 3] in the context of linearized kinematics can be conveniently extended to define efficient return mapping procedures which are capable of dealing with fully nonlinear elastic response and complex plastic models.

3.1.1. Spatial formulation

For the purpose of this discussion, the deformation gradients are taken to be prescribed functions of time

$$\mathbf{F} = \hat{\mathbf{F}}(t). \quad (3.1)$$

For simplicity, in what follows we shall ignore the possibility of a dependence of the potential ψ on the plastic variables \mathbf{q} . Let us start by rephrasing the spatial constitutive relations in Table 3 as a set of *equations of evolution* of the following form:

$$\begin{aligned} \mathbf{d} &\equiv \mathbf{d}^e + \mathbf{d}^p = \hat{\mathbf{d}}(t), \\ L_v \boldsymbol{\tau} &= J \mathbf{c}^e : \mathbf{d}^e + J \mathbf{c}^p : \mathbf{d}^p, \\ L_v \mathbf{e}^p &\equiv \mathbf{d}^p = \dot{\gamma} \mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}), \quad L_v \mathbf{q} = \dot{\gamma} \mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}), \end{aligned} \quad (3.2)$$

where $\hat{\mathbf{d}}(t) \equiv (\hat{\mathbf{F}}(t) \hat{\mathbf{F}}^{-1}(t))^S$ is given. The elastic constitutive equations have been formulated in rate form simply by taking the Lie derivative of stress-strain relations $\boldsymbol{\tau} = \rho_0 \partial \psi / \partial \mathbf{e}^e$. To this end, one defines

$$\mathbf{c}^e \equiv \rho \frac{\partial^2 \psi}{\partial \mathbf{e}^e \partial \mathbf{e}^e}, \quad \mathbf{c}^p \equiv \rho \frac{\partial^2 \psi}{\partial \mathbf{e}^e \partial \mathbf{e}^p}, \quad (3.3)$$

where \mathbf{c}^e is the spatial tangent elasticity tensor. In addition the fact has been used that

$$L_v \mathbf{e}^e = \mathbf{d}^e, \quad L_v \mathbf{e}^p = \mathbf{d}^p. \quad (3.4)$$

The plastic rate parameter $\dot{\gamma}$ is determined from the requirement that the yield condition,

$$\varphi(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) = 0, \quad (3.5)$$

be identically satisfied during plastic loading. Note that any other objective stress rate can be used in (3.2)₂ by suitably adjusting the right-hand side. In particular, if an ‘elastic’ Lie derivative relative to the intermediate configuration is employed, the term in (3.2)₂ involving \mathbf{d}^p no longer appears explicitly.

3.1.2. Elastic-plastic operator split

As first noted in [36], return mapping algorithms are a natural consequence of the fact that constitutive relations (3.2) can be ‘split’ into elastic and plastic parts. The former is *deformation driven* and is given by

$$\begin{aligned} \mathbf{d} &\equiv \mathbf{d}^e + \mathbf{d}^p = \hat{\mathbf{d}}(t), \\ L_v \boldsymbol{\tau} &= J \mathbf{c}^e : \mathbf{d}, \quad L_v \mathbf{e}^p \equiv \mathbf{d}^p = \mathbf{0}, \quad L_v \mathbf{q} = \mathbf{0}. \end{aligned} \quad (3.6)$$

On the other hand, the plastic part of the constitutive relations reduces to

$$\begin{aligned} \mathbf{d} &\equiv \mathbf{d}^e + \mathbf{d}^p = \mathbf{0}, \\ \frac{\partial}{\partial t} \boldsymbol{\tau} &= -J(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{d}^p, \end{aligned}$$

$$\frac{\partial}{\partial t} \mathbf{e}^p = \dot{\gamma} \mathbf{r}(\tau, \mathbf{g}, \mathbf{q}), \quad \frac{\partial}{\partial t} \mathbf{q} = \dot{\gamma} \mathbf{h}(\tau, \mathbf{g}, \mathbf{q}). \quad (3.7)$$

It should be noted that (3.6) and (3.7) do indeed add up to the total rate constitutive relations (3.2), consistent with the notion of operator split.

The decoupled equations (3.6) and (3.7) are amenable to the following interpretation.

Elastic part of the constitutive equations. In the elastic equations (3.6), the plastic response of the material is ‘frozen’, so that plastic strains and plastic variables are merely pushed forward into subsequent configurations. Furthermore, since $\mathbf{d}^p = \mathbf{0} = \mathbf{F}^{p-1} \dot{\mathbf{C}}^p \mathbf{F}^{p-1}$, it follows that $\dot{\mathbf{C}}^p = \mathbf{0}$ and hence $\dot{\mathbf{F}}^p = \mathbf{0}$. Consequently, the evolution of the state variables introduced by the elastic equations takes place for a *fixed intermediate configuration*, and all of the prescribed deformation $\hat{\mathbf{d}}(t)$ goes into elastically straining the material. Thus, the elastic deformation gradients $\mathbf{F}^e = \hat{\mathbf{F}}(t) \mathbf{F}^{p-1} \equiv \hat{\mathbf{F}}^e(t)$ become themselves given functions of time. Finally, since our formulation is *hyperelastic*, the elastic equations (3.6) are directly *integrable* and stresses are simply given by the elastic relations

$$\boldsymbol{\tau} = \rho_0 \left. \frac{\partial \psi}{\partial \mathbf{e}^e} \right|_{\hat{\mathbf{F}}^e(t)}. \quad (3.8)$$

Plastic part of the constitutive equations. In the plastic equations, on the other hand, one has $\mathbf{d} = \mathbf{0}$ and hence the *spatial configuration remains fixed*. Under these conditions, the Lie derivative simply reduces to partial differentiation with respect to time and, in particular, $\partial \mathbf{e}^p / \partial t = \mathbf{d}^p$. Thus, the plastic equations (3.7) may be recast as

$$\begin{aligned} \frac{\partial}{\partial t} \boldsymbol{\tau} &= -\dot{\gamma} J(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r}(\tau, \mathbf{g}, \mathbf{q}), \\ \frac{\partial}{\partial t} \mathbf{q} &= \dot{\gamma} \mathbf{h}(\tau, \mathbf{g}, \mathbf{q}), \end{aligned} \quad (3.9)$$

or, dividing through by $\dot{\gamma}$,

$$\begin{aligned} \frac{d\boldsymbol{\tau}}{d\gamma} &= -J(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r}(\tau, \mathbf{g}, \mathbf{q}), \\ \frac{d\mathbf{q}}{d\gamma} &= \mathbf{h}(\tau, \mathbf{g}, \mathbf{q}). \end{aligned} \quad (3.10)$$

These equations define a *relaxation* of the stresses $\boldsymbol{\tau}$ towards a suitably updated elastic domain. Such a plastic relaxation process is completed as soon as the yield condition (3.5) is satisfied.

3.1.3. Material formulation

The elastic-plastic splitting methodology defined above can be alternatively formulated in a material setting. In this case, entirely analogous arguments point to the following choice of

decoupled equations (elastic and plastic, respectively):

$$\begin{aligned}
 \dot{\mathbf{E}} &= \dot{\hat{\mathbf{E}}}(t), & \dot{\mathbf{E}} &= \mathbf{0}, \\
 \dot{\mathbf{S}} &= \mathbf{A}^e(\mathbf{E}^e, \mathbf{E}^p) : \dot{\mathbf{E}}, & \dot{\mathbf{S}} &= -[\mathbf{A}^e(\mathbf{E}^e, \mathbf{E}^p) - \mathbf{A}^p(\mathbf{E}^e, \mathbf{E}^p)] : \dot{\mathbf{E}}^p, \\
 \dot{\mathbf{E}}^p &= \mathbf{0}, & \dot{\mathbf{E}}^p &= \dot{\gamma} \mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q}), \\
 \dot{\mathbf{Q}} &= \mathbf{0}, & \dot{\mathbf{Q}} &= \dot{\gamma} \mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q}),
 \end{aligned} \tag{3.11}$$

where $\hat{\mathbf{E}}(t)$ is again a given function of time, $\mathbf{A}^e \equiv \rho_0 \partial^2 \Psi / \partial \mathbf{E}^e \partial \mathbf{E}^e$ is the material tangent elasticity tensor, and $\mathbf{A}^p \equiv \rho_0 \partial^2 \Psi / \partial \mathbf{E}^e \partial \mathbf{E}^p$. As before, the elastic part of the constitutive relations defines a process of elastic straining in which the plastic deformations \mathbf{E}^p remain unchanged while the stresses \mathbf{S} are evaluated through the elastic relations from the known elastic strains $\hat{\mathbf{E}}^e(t) = \hat{\mathbf{E}}(t) - \mathbf{E}^p$. On the other hand, the plastic equations define a relaxation process for stresses and plastic variables which continues until the yield criterion

$$\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q}) = 0 \tag{3.12}$$

is satisfied.

REMARK 3.1. The material tensors \mathbf{A}^e and \mathbf{A}^p defined above are related to the spatial tensors \mathbf{Jc}^e and \mathbf{Jc}^p given by (3.3), through push-forward with \mathbf{F} .

3.2. Stress update: General return mapping algorithm

Based on the elastic-plastic split (3.6), (3.7), a return algorithm can be conveniently defined by first solving the elastic equations (3.6) to obtain an elastic predictor, which is then taken as an initial condition for the plastic relaxation equations (3.7). The resulting procedure is graphically shown in Fig. 4. For associated perfect plasticity, for instance, it is seen that the plastic equations define a return path for the stresses which runs along the steepest descents of the yield function, Fig. 2. It should be noted, however, that the steepest descent direction in stress space is determined based on the tensor $(\mathbf{c}^e - \mathbf{c}^p)$.

For the perfectly plastic von Mises model with infinitesimal isotropic elasticity, the return path for stresses is clearly radial. In general, however, the return path defined by (3.7) is not known in advance nor can it be determined analytically. It becomes therefore necessary to compute the return path for the stresses numerically. An efficient algorithm for this purpose is listed in Table 5, together with the remaining steps in the update procedure. As may be seen, the return mapping is defined iteratively. At every iteration the yield function φ is linearized about the current values $\tau_{n+1}^{(i)} \mathbf{q}_{n+1}^{(i)}$ and $\varphi_{n+1}^{(i)}$. Such a linearized yield function defines a straight intersection or 'cut' on the plane $\varphi = 0$ onto which $\tau_{n+1}^{(i)}$ and $\mathbf{q}_{n+1}^{(i)}$ are projected to obtain the next iteration $\tau_{n+1}^{(i+1)}$ and $\mathbf{q}_{n+1}^{(i+1)}$. It should be noted that such projection involves the current elastic moduli $\mathbf{c}_{n+1}^{e(i)}$. The initial conditions for the return procedure $\tau_{n+1}^{(0)}$ and $\mathbf{q}_{n+1}^{(0)}$ are taken to coincide with the elastic predictor.

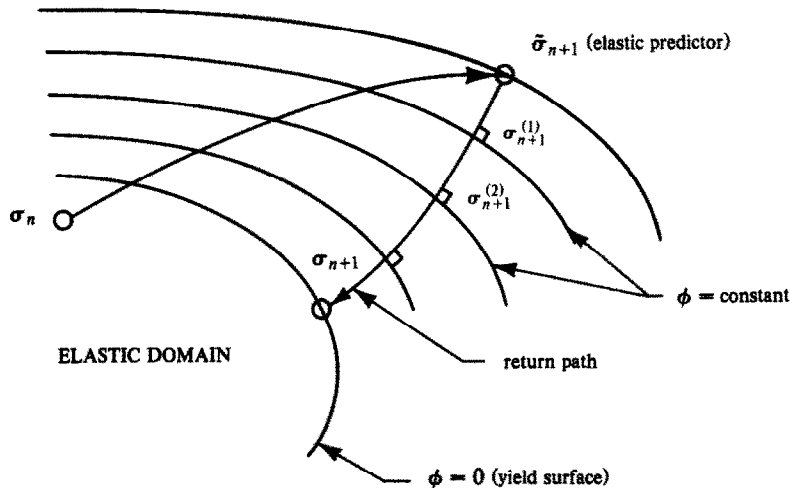


Fig. 2. Geometric interpretation for the case of perfect plasticity of a general return mapping algorithm based on an elastic-plastic split of the constitutive equations. The elastic relations define an elastic predictor $\tilde{\sigma}_{n+1}$ which is subsequently returned to the yield surface along the steepest descent path of the yield function ϕ . The steepest direction is determined in terms of the local metric defined by the elastic tangent moduli.

3.2.1. Geometric interpretation

A geometric interpretation of the proposed algorithm is given in Fig. 3. As we may see, the elastic predictor is returned to the yield surface in successive steps. Each one of these steps involves a projection of the stresses onto a straight (linear) approximation to the yield surface or 'cut'. In the limit, such cuts become tangent to the yield surface and plastic consistency is restored at a *quadratic* convergence rate. For an associated flow rule, the computed return path is indeed an approximation to the steepest descent path as defined by the tensor $(c^e - c^p)$.

3.2.2. Stability and consistency

From general results concerning the operator splitting methodology (e.g., see [36, Chapter 3] for a review) it immediately follows that the proposed algorithm is consistent with the constitutive relations in Table 3. Furthermore, unconditional stability follows automatically provided that both the elastic predictor and the plastic corrector are separately unconditionally stable. As for the former, unconditional stability is trivially achieved due to the exact nature of the algorithm. On the other hand, the equations (3.10) defining the return path are clearly dissipative and the corresponding trajectories contractive provided the yield function is convex and the plastic flow direction r derives from a convex potential or loading function [36, Chapter 3]. Under these conditions, the return algorithm is unconditionally stable and so is the overall update procedure.

REMARK 3.2. It should be noted that the proposed return algorithm does not require explicit knowledge of the derivatives of the elasticity tensor c^e , the plastic flow direction r or the plastic hardening moduli h . Nevertheless quadratic convergence of the stress iterates towards the elastic domain is achieved.

Table 5
Stress update algorithm

Step 1. Geometric update

$$\varphi_{n+1} = \varphi_n + \mathbf{u}$$

$$\mathbf{F}_u = \mathbf{I} + \nabla \mathbf{u}$$

$$\mathbf{F}_{n+1} = \mathbf{F}_u \mathbf{F}_n$$

Step 2. Elastic predictor

$$\mathbf{F}_{n+1}^{e(0)} = \mathbf{F}_n^p$$

$$\mathbf{F}_{n+1}^{e(0)} = \mathbf{F}_{n+1} \mathbf{F}_n^{p-1}$$

$$\tau_{n+1}^{(0)} = \rho_0 \left. \frac{\partial \psi^e}{\partial \mathbf{e}^e} \right|_{\mathbf{F}_{n+1}^{e(0)}}$$

$$\mathbf{q}_{n+1}^{(0)} = \text{push-forward of } \mathbf{q}_n \text{ by } \mathbf{F}_u$$

Step 3. Check for yielding

$$\varphi_{n+1}^{(0)} \equiv \varphi(\tau_{n+1}^{(0)}, \mathbf{q}_{n+1}^{(0)}) \leq 0 ?$$

YES $\mathbf{F}_{n+1}^p = \mathbf{F}_{n+1}^{e(0)}$; $\mathbf{q}_{n+1} = \mathbf{q}_{n+1}^{(0)}$; $\tau_{n+1} = \tau_{n+1}^{(0)}$; EXIT

NO $i = 0$

Step 4. Plastic correctors

$$\Delta \gamma = \frac{\varphi_{n+1}^{(i)}}{\left(\frac{\partial \varphi}{\partial \tau} \right)_{n+1}^{(i)} : J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)} - \left(\frac{\partial \varphi}{\partial \mathbf{q}} \right)_{n+1}^{(i)} \cdot \mathbf{h}_{n+1}^{(i)}}$$

$$\tau_{n+1}^{(i+1)} = \tau_{n+1}^{(i)} - \Delta \gamma J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)}$$

$$\mathbf{q}_{n+1}^{(i+1)} = \mathbf{q}_{n+1}^{(i)} + \Delta \gamma \mathbf{h}_{n+1}^{(i)}$$

Step 5. Convergence check

$$\varphi(\tau_{n+1}^{(i+1)}, \mathbf{q}_{n+1}^{(i+1)}) \leq \text{TOL} ?$$

YES Compute \mathbf{F}_{n+1}^e from $\tau_{n+1}^{(i+1)}$ (see Remark 3.3 on isotropy)

$$\mathbf{F}_{n+1}^p = \mathbf{F}_{n+1} \mathbf{F}_{n+1}^{e-1}, \mathbf{q}_{n+1} = \mathbf{q}_{n+1}^{(i+1)}, \tau_{n+1} = \tau_{n+1}^{(i+1)}; \text{EXIT}$$

NO $i \leftarrow i + 1$; GO TO Step 4

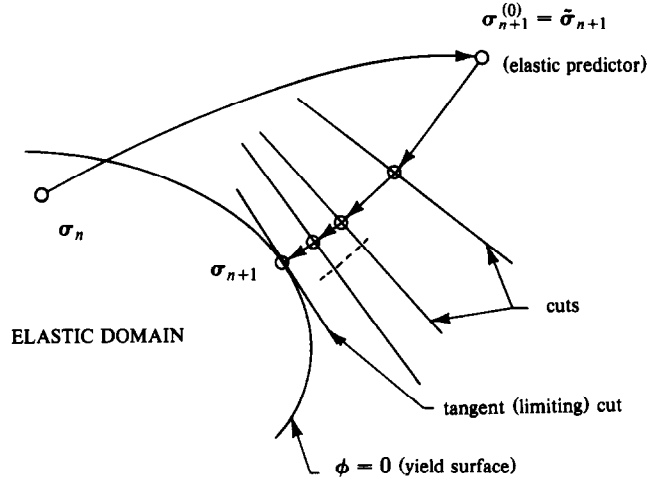


Fig. 3. Numerical implementation of the return mapping algorithm shown in Fig. 2. The elastic predictor $\tilde{\sigma}_{n+1}$ is returned to the yield surface in successive steps. At every step, the updated stresses $\sigma_{n+1}^{(i+1)}$ are computed by projecting the previous iteration $\sigma_{n+1}^{(i)}$ onto the trace on the plane $\phi = 0$ of a linear approximation to the yield function at $\sigma_{n+1}^{(i)}$ or 'cut'. In the limit, such cuts become tangent to the yield surface and plastic consistency is recovered at a quadratic convergence rate.

REMARK 3.3. If the elastic response is *isotropic* a standard argument reveals that the *elastic* potential ψ in the spatial description depends on \mathbf{F}^e through the elastic left Cauchy–Green tensor \mathbf{b}^e . Thus, knowledge of τ at a given configuration only determines uniquely \mathbf{b}^e or, equivalently, the left stretch tensor $\mathbf{V}^e = \mathbf{b}^{e/2}$ is an *unloading* process. In the isotropic case we may set, following Lee [13],

$$\mathbf{F}^e = \mathbf{V}^e, \quad \mathbf{F}^p = \mathbf{F}\mathbf{F}^{e-1}. \quad (3.13)$$

This is the procedure herein adopted in the finite element implementation of the isotropic elastic finite deformation model (2.5). Further computational details may be found in [38].

REMARK 3.4. The stress update algorithm proposed above is amenable to an entirely equivalent formulation in the material setting, based on the elastic-plastic split (3.11). Some features of such material implementation are noteworthy. For instance, the plastic strains \mathbf{E}^p are seen to remain constant during the elastic predictor and thus play the role of fixed parameters in the computation of the elastic tangent moduli $\mathbf{C}^e(\mathbf{E}^e, \mathbf{E}^p)$. On the other hand, the plastic relaxation takes places under constant deformations \mathbf{C} which thus behave as fixed parameters in the definition of the plastic flow direction $\mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$, plastic moduli $\mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$ and yield function $\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q})$.

3.3. Extension to viscoplasticity

The formulation heretofore presented readily extends to the case of viscoplasticity. Assuming for simplicity linear viscosity, the flow rule and hardening laws take the same form

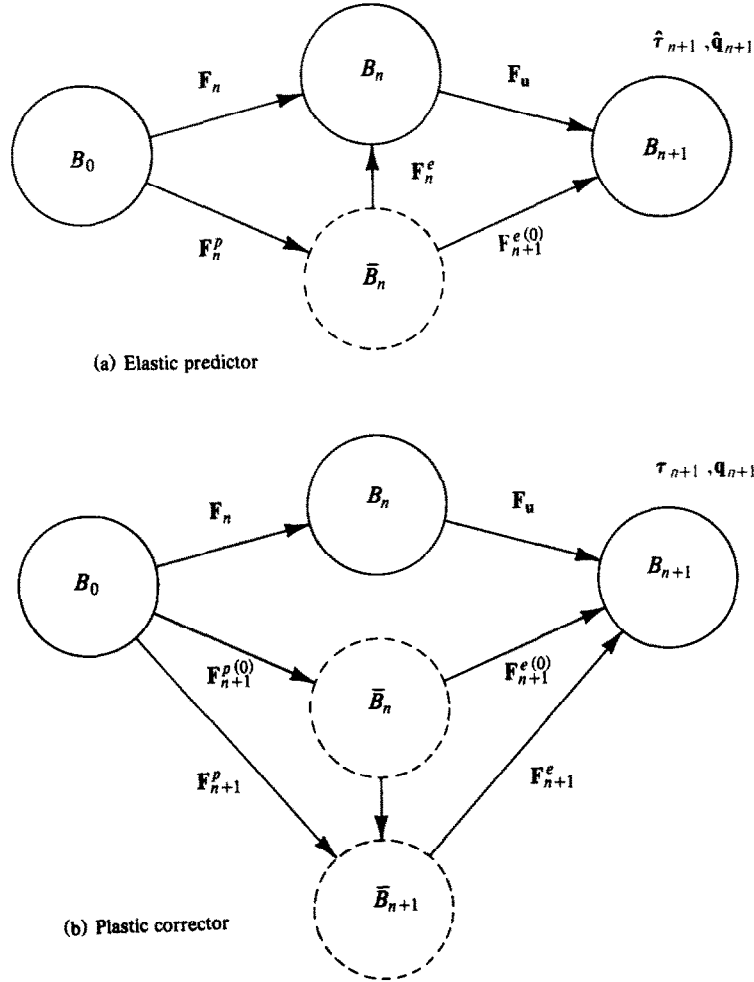


Fig. 4. Geometric aspects of the elastic-plastic splitting methodology. (a) The elastic predictor takes place under constant intermediate configuration and the incremental deformations \mathbf{F}_u strain the body elastically. (b) The plastic corrector leaves the updated spatial configuration invariant while the intermediate configuration relaxes plastically.

as in Table 3 with $\dot{\lambda}$ being replaced by φ/η , where η is the viscosity coefficient. Accordingly, an analogous operator split now applies in which (3.6) remains unaltered and the plastic relaxation equations (3.9) are replaced by

$$\begin{aligned} \frac{\partial}{\partial t} \tau &= -\frac{\varphi}{\eta} J(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r}(\tau, \mathbf{g}, \mathbf{q}), \\ \frac{\partial}{\partial t} \mathbf{q} &= \frac{\varphi}{\eta} \mathbf{h}(\tau, \mathbf{g}, \mathbf{q}), \end{aligned} \quad (3.14)$$

where plastic loading ($\varphi > 0$) is assumed. The rate of change of φ is governed by the following equation:

$$\frac{\partial}{\partial t} \varphi = \frac{\partial \varphi}{\partial \tau} : \frac{\partial \tau}{\partial t} + \frac{\partial \varphi}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial t} = -\frac{\varphi}{\eta} \left[\frac{\partial \varphi}{\partial \tau} : J(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r} - \frac{\partial \varphi}{\partial \mathbf{q}} \cdot \mathbf{h} \right], \quad (3.15)$$

which is the viscous counterpart of the consistency condition of inviscid plasticity. Defining an *instantaneous relaxation time* by the expression

$$\bar{t} \equiv \frac{\eta}{\frac{\partial \varphi}{\partial \tau} : J(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r} - \frac{\partial \varphi}{\partial \mathbf{q}} \cdot \mathbf{h}}; \quad (3.16)$$

(3.15) may be rephrased as

$$\frac{\partial \varphi}{\partial t} = -\frac{\varphi}{\bar{t}}. \quad (3.17)$$

In view of (3.14) and (3.17) the following conclusions may be drawn:

- (i) The return path defined by (3.14) coincides with the return path corresponding to inviscid plasticity.
- (ii) As the plastic relaxation proceeds, the location of the stress point in the return path defined by (3.14) is governed by (3.17).

These results enable one to formulate an algorithm for the numerical integration of the elasto-viscoplastic constitutive relations based upon the following notions. The elastic predictor and return path are computed as in the inviscid case. In particular the return path comprises a sequence of straight segments in stress space directed towards the yield surface. The difference, however, is that only partial relaxation now takes place; thus the stress point does not reach the yield surface. To compute the final location of the stress point in the return path, one may proceed as follows.

(a) Within a generic straight segment (i) in the return path, the relaxation time is taken to be constant and equal to a value $\bar{t}^{(i)}$ computed according to (3.16) from the initial conditions for the segment $\tau_{n+1}^{(i)}$, and $\mathbf{q}_{n+1}^{(i)}$.

(b) Thus, within a typical straight segment the variation of the yield function value is given by the exponential relation

$$\varphi \approx \varphi_{n+1}^{(i)} \exp(-\Delta t / \bar{t}^{(i)}), \quad (3.18)$$

where Δt is the time elapsed since the entrance of the stress point into segment (i).

(c) The total time $\Delta t^{(i)}$ spent by the stress point in segment (i) is therefore given by

$$\Delta t^{(i)} = \bar{t}^{(i)} \log \frac{\varphi_{n+1}^{(i)}}{\varphi_{n+1}^{(i+1)}}. \quad (3.19)$$

(d) The end of the relaxation process is characterized by the condition that $\sum_i \Delta t^{(i)} = h$, where h is the time step size and the sum extends to all traversed segments.

The resulting algorithm is summarized in Table 6. The simplicity and generality of the algorithm should be noted. The same general results mentioned above regarding *consistency*

Table 6

Stress update algorithm: Viscoplasticity

Step 1. Geometric update

Step 2. Elastic predictor

Step 3. Check for yielding

$$\varphi_{n+1}^{(0)} \equiv \varphi(\tau_{n+1}^{(0)}, \mathbf{q}_{n+1}^{(0)}) \leq 0 ?$$

YES $\mathbf{F}_{n+1}^p = \mathbf{F}_{n+1}^{p(0)}$; $\mathbf{q}_{n+1} = \mathbf{q}_{n+1}^{(0)}$; $\tau_{n+1} = \tau_{n+1}^{(0)}$; EXITNO $i = 0$, $t^{(0)} = 0$

Step 4. Plastic correctors

$$\bar{t}^{(i)} = \frac{\eta}{\left(\frac{\partial \varphi}{\partial \tau}\right)_{n+1}^{(i)} : J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)} - \left(\frac{\partial \varphi}{\partial \mathbf{q}}\right)_{n+1}^{(i)} \cdot \mathbf{h}_{n+1}^{(i)}}$$

$$\Delta \gamma = \frac{\varphi_{n+1}^{(i)} \bar{t}^{(i)}}{\eta}$$

$$\tau_{n+1}^{(i+1)} = \tau_{n+1}^{(i)} - \Delta \gamma J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)}$$

$$\mathbf{q}_{n+1}^{(i+1)} = \mathbf{q}_{n+1}^{(i)} + \Delta \gamma \mathbf{h}_{n+1}^{(i)}$$

$$\Delta t^{(i)} = \bar{t}^{(i)} \log(\varphi_{n+1}^{(i)} / \varphi_{n+1}^{(i+1)})$$

$$t^{(i+1)} = t^{(i)} + \Delta t^{(i)}$$

Step 5. Check for end of relaxation process

$$t^{(i+1)} \geq h$$

YES $\Delta t^{(i)} = h - t^{(i)}$

$$\Delta \gamma = \frac{\varphi_{n+1}^{(i)} \bar{t}^{(i)}}{\eta} [1 - \exp(-\Delta t^{(i)} / \bar{t}^{(i)})]$$

$$\tau_{n+1} = \tau_{n+1}^{(i)} - \Delta \gamma J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)}$$

$$\mathbf{q}_{n+1} = \mathbf{q}_{n+1}^{(i)} + \Delta \gamma \mathbf{h}_{n+1}^{(i)}$$

Compute $\mathbf{F}_{n+1}^e \tau_{n+1}^{(i+1)}$ (see Remark 3.3. on isotropy)

$$\mathbf{F}_{n+1}^p = \mathbf{F}_{n+1} \mathbf{F}_{n+1}^{e-1}, \text{ EXIT}$$

NO $i \leftarrow i + 1$; GO TO Step 4

and *stability* of the algorithm apply in the present context. We finally note that as $\eta \rightarrow 0$ the integration scheme for *inviscid* plasticity is recovered. This is consistent with well-known results concerning the inviscid limit of viscoplasticity (e.g., see [36, Chapter 3] for a review).

3.4. Boundary value problem: Consistent tangent operator

In order to formulate a well-defined problem, in addition to the constitutive relations we need to consider the momentum balance equation and suitable boundary and initial conditions. A weak formulation of the resulting boundary value problem can be given as follows. Let $\mathbf{b}(\mathbf{x})$ be the body force, $\mathbf{a}(\mathbf{x})$ the spatial acceleration field and $\bar{\mathbf{t}}(\mathbf{x})$ the traction vector specified on the Neumann boundary $\partial_\sigma B_n$. Furthermore, let the deformation mapping φ

be prescribed on the Dirichlet boundary $\partial_u B_0$ as $\varphi|_{\partial_u B_0} = \bar{\varphi}$. As usual we require that $\partial_u B_n \cap \partial_\sigma B_n = \emptyset$ and $\partial_u B_n \cup \partial_\sigma B_n = \partial B_n$, where one has $\partial_u B_n = \varphi(\partial_u B_0)$. Then, the weak form of the momentum balance equation at time t_n reads

$$G(\varphi, \eta) \equiv \int_{B_n} \rho \mathbf{a} \cdot \eta \, dB + \int_{B_n} \boldsymbol{\sigma} : \nabla \eta \, dB - \int_{B_n} \mathbf{b} \cdot \eta \, dB - \int_{\partial_\sigma B_n} \bar{\mathbf{t}} \cdot \eta \, dS = 0, \quad (3.20)$$

for any admissible variation η such that $\eta|_{\partial_u B_n} = \mathbf{0}$. Since the treatment of the transient dynamic problem plays no role in the present discussion, we shall ignore inertia effects and confine our attention to the static case.

Within the context of finite element analysis, the solution of problem (3.20) is accomplished by an iterative scheme such as Newton's method. Typically, one solves a sequence of linearized problems defined as

$$\begin{aligned} DG(\varphi_{n+1}^{(i)}, \eta) \cdot \mathbf{u}_{n+1}^{(i)} &\equiv \int_{B_n} \left[\frac{1}{J} \text{tr}(\nabla \eta \tau \nabla \mathbf{u}) + \nabla \eta : (\mathbf{c}_{n+1}^{(i)} : \nabla \mathbf{u}_{n+1}^{(i)}) \right] dB \\ &= -G(\varphi_{n+1}^{(i)}, \eta), \end{aligned} \quad (3.21)$$

until the residual $G(\varphi_{n+1}^{(i)}, \eta)$ vanishes to within a prescribed tolerance.

The convergence rate of iterative scheme (3.4) is essentially governed by the choice of tangent moduli $\mathbf{c}_{n+1}^{(i)}$ which, as recently noted in [37], depends in turn on the iteration scheme adopted. In a typical iteration $i + 1$ within a time step $[t_n, t_{n+1}]$, the variables $(\cdot)_{n+1}^{(i+1)}$ may be obtained by means of the algorithm in Table 5 and from either

- (a) the values $(\cdot)_{n+1}^{(i)}$ corresponding to the previous *non-converged* iteration, or
- (b) the *converged* values $(\cdot)_n$ from the previous time step.

Both procedures define algorithms which are *consistent* with the field equations. However, scheme (a) introduces a 'history dependence' of the converged values on intermediate *non-converged* iterates. This may pose difficulties due to the strong *path dependence* of plasticity models. Spurious unloadings at some Gauss points may also occur as a result of this procedure. By contrast, history dependence on intermediate *non-converged* values is eliminated with the use of scheme (b), and 'fictitious' numerical unloading is therefore prevented. The undesirable features arising from the use of scheme (a) and the convenience of using an approach analogous to scheme (b) advocated here, have been also noted by Argyris, Doltsinis and Kleiber [43].

For a stress update algorithm such as the one proposed in Section 3.1, the updated stresses $\tau_{n+1}^{(i+1)}$ becomes a function of the incremental displacements $\nabla \mathbf{u}$, as well as of suitable initial conditions consistent with the update strategy (a) or (b). The essential point emphasized in [37] is that the tangent moduli appearing in (3.4) should be derived by consistent linearization of the update procedure, in order to achieve the asymptotic rate of quadratic convergence characteristic of Newton's method. If scheme (a) is adopted, the consistent tangent moduli coincide with the classical elastoplastic ones obtained by enforcing the consistency condition. On the other hand, if update scheme (b) is adopted, the consistent tangent moduli are no longer the

classical elastoplastic moduli. In fact, use of these moduli may result in a loss of the quadratic rate of asymptotic convergence, as shown in [37].

Explicit expressions for the tangent moduli consistent with the update strategy (b) and associated with several widely used plastic models are given in [37]. These expressions are developed in the context of return mapping algorithms and for linearized kinematics. For simple yield conditions such as von Mises, and nonlinear isotropic and kinematic hardening rules, the results in [37] may be extended to the present setting. In general, however, the task of evaluating the consistent tangent moduli in closed form may prove exceedingly laborious. It would appear, therefore, that a general purpose implementation of the physically more compelling algorithm based on the updating procedure (b), may require the use of quasi-Newton or secant-Newton methods [39] for the solution of resulting nonlinear algebraic problem.

4. A numerical example

There is a number of important issues involved in the finite element implementation of the formulations and corresponding stress update algorithms discussed in the previous sections. Among them should be mentioned:

- (i) The treatment of constraints in the finite deformation range such as incompressibility of the plastic flow. In the context of the linear theory, the importance of a proper treatment of this constraint was first emphasized by Nagtegaal, Parks and Rice [40].
- (ii) The use of consistent linearization procedures, as discussed in [31, Chapter 4], to obtain elastoplastic tangent moduli consistent with the stress update algorithm for specific models.

These and related computational issues are treated in [38] together with the discussion of several numerical experiments.

Our intention here is simply to illustrate the formulation heretofore discussed by means of a classical example: the thick wall cylinder under internal pressure. For this purpose we consider perfect plastic behavior with a von Mises yield condition, and elastic response governed by the hyperelastic constitutive equations (2.5), (2.6).

The results of the numerical experiment are shown in Fig. 5 and Fig. 6, together with the exact solution for the case of a rigid plastic material. In this figure, α designates the inner radius in the current configuration. The following features pertaining to these results are noteworthy:

- (i) The final configuration of the cylinder, with inner radius $\alpha = 85.10$ and an outer radius of 88.8, is attained in 15 time steps. Note that the initial values of the inner and outer radius are 10 and 20, respectively.
- (ii) Within each time step, convergence with the classical Newton's method, in the energy norm and to a tolerance of $TOL = 10^{-18}$, is achieved in 4–5 iterations.
- (iii) For the purpose of comparison with the rigid plastic solution the material properties are chosen as to obtain *infinitesimal* elastic deformations. In spite of the extremely large loading step employed, excellent agreement between the asymptotically exact analytic solution and the computed results is found.

The reason for the excellent rate of convergence exhibited by the solution process may be

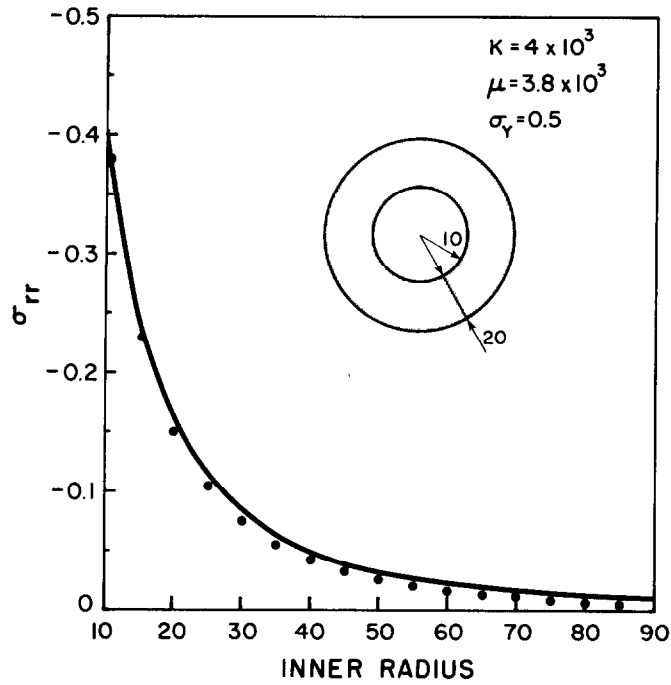


Fig. 5. Thick wall cylinder under internal pressure.

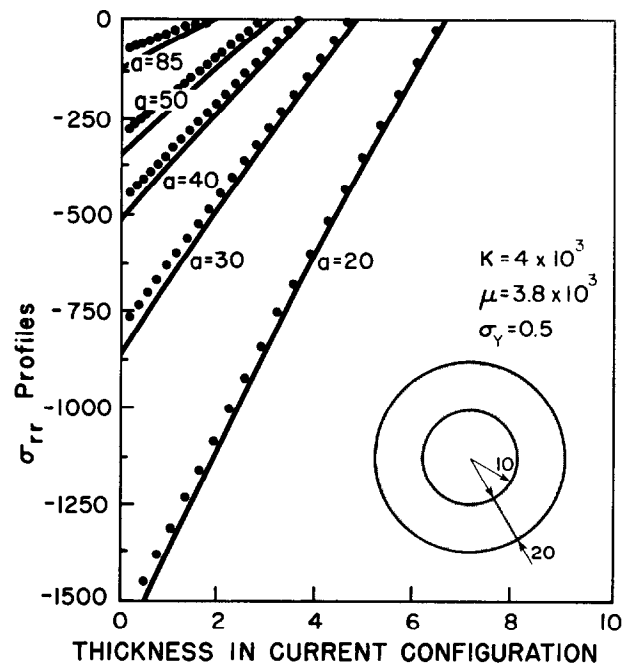


Fig. 6. Thick wall cylinder under internal pressure.

found in the use of tangent moduli consistent with the stress update algorithm. These tangent moduli *are* obtained from the stress update algorithm by a consistent linearization procedure.

5. Summary and conclusions

A unified approach to finite deformation elastoplasticity which embodies both additive and multiplicative theories has been presented. Taking the multiplicative decomposition of the deformation gradient as a point of departure, an additive decomposition of Lagrangian and Almansi strains follows. Such an additive decomposition carries over to the corresponding material and spatial rates of deformation.

From a numerical standpoint, the proposed formulation has several far-reaching consequences. First, the use of hyperelastic constitutive models entirely avoids the need for integration of rate constitutive relations. In particular, so called incrementally objective integration algorithms are no longer needed, even in the context of rate-dependent viscoplastic models. Second, an operator-splitting methodology can be used to exploit the additive decomposition of the deformation rates. On this basis, a general class of return-mapping algorithms capable of accommodating arbitrary yield criteria, flow rules, hardening laws and variable tangent elastic compliances has been derived. It should be emphasized that the elastic predictor in the stress update procedure reduces to a mere function evaluation. The return mapping takes an iterative form whereby stresses converge towards a suitably updated yield surface at a quadratic rate. Accuracy and unconditional stability are guaranteed by general results pertaining to the operator-splitting methodology. The proposed numerical schemes apply to both rate-dependent and rate-independent plastic models.

Owing to the covariant character of the theoretical framework and of the proposed algorithms adopted, selection of the material, intermediate or spatial descriptions as a basis for numerical computations is a simple matter of choice. In fact, the results obtained from any one of the descriptions should be identical.

The accuracy of the method, even for very large time steps has been illustrated by means of an numerical experiment. It is also noted that use of consistent tangent moduli results in excellent rates of convergence.

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