5 Some Aspects of Objectivity

If laws of nature discovered at different places and times were not the same, scientific work would have to be redone at every new place and at each time. We know that the laws of nature we discover have to take the same form, however we are oriented or we set our clock; there is no difference whether we measure distances relative to, for example, east or west, or we date events from, for example, the birth of Christ or the death of Newton. Qualitative and quantitative descriptions of physical phenomena have to remain unchanged even if we make any changes in the point of view from which we observe them. Thus, physical processes do not depend on the change of observer. To ancient natural philosophers this was not so obvious. The mathematical representation of physical phenomena must reflect this invariance.

The following chapter has the task to express this fundamental finding with the concept of objectivity, or frame-indifference, which constitutes an essential part in non-linear continuum mechanics. We introduce the terminology of an observer, consider changes of observers and apply the concept of objectivity to tensor fields. Transformation rules for various kinematical, stress and stress rate quantities under changes of observers are also derived.

It is obvious to claim that material properties must be invariant under changes of observers. This fundamental requirement is expressed through the principle of material frame-indifference. To show how this objectivity requirement restricts elastic material response is the aim of the last section.

Additional information is found in the same monographs which have already been suggested for Chapter 2 (see the reference list on p. 56). Of course, the short list does not contain a comprehensive review of the large number of papers and books available on this subject.

5.1 Change of Observer, and Objective Tensor Fields

Before examining specific constitutive equations for some elastic materials it is first necessary to present a mathematical foundation for the change of an observer and to introduce the concept of objectivity for tensor fields. We study further how velocity and acceleration fields behave under changes of observers.

Observer and Euclidean transformation. The description of a physical process is related directly to the choice of an **observer**, which we denote subsequently by O. An arbitrarily chosen observer in the three-dimensional Euclidean space and in time is equipped to measure

(i) *relative positions* of points in space (with a ruler), and

(ii) *instants of time* (with a clock).

An **event** is noticed by an observer in terms of *position (place* $\mathbf{x}$*)* and *time* t.

Consider two arbitrary events in the Euclidean space characterized by the pairs $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}, t)$ (compare with OGDEN [1997]). We assume that $(\mathbf{x}_0, t_0)$ is 'frozen' as long as the event $(\mathbf{x}, t)$ occurs. An observer records that the pair of points in space is separated by the *distance* $|\mathbf{x} - \mathbf{x}_0|$, and that the *time interval (lapse)* between the events under observation is $t - t_0$. In the following we let the pairs $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}, t)$ map to $(\mathbf{x}_0^+, t_0^+)$ and $(\mathbf{x}^+, t^+)$ so that both the distance $|\mathbf{x} - \mathbf{x}_0|$ and the time interval $t - t_0$ are preserved (see Figure 5.1).

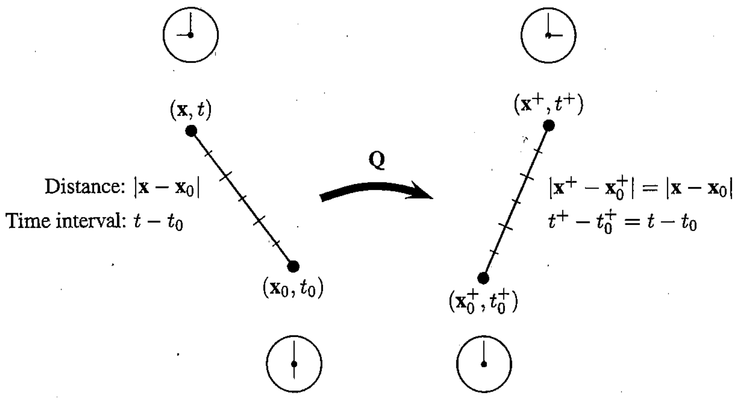

Figure 5.1 Map of two points preserving distance and time interval.

A spatial mapping which satisfies the requirements above may be represented by the time-dependent transformation

$$\mathbf{x}^+ - \mathbf{x}_0^+ = \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0) \quad . \tag{5.1}$$

The point differences $\mathbf{x}^+ - \mathbf{x}_0^+$ and $\mathbf{x} - \mathbf{x}_0$ can be interpreted as vectors which are related through the orthogonal tensor $\mathbf{Q}(t)$, with the well-known property $\mathbf{Q}^{-1}(t) = \mathbf{Q}(t)^{\mathrm{T}}$ (compare with Section 1.2, p. 16). In order to maintain orientations we admit only rotation, consequently, $\mathbf{Q}$ is assumed to be proper orthogonal ($\det\mathbf{Q} = +1$). Hence, with (5.1), we may write the following mathematical expression

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x} \quad , \qquad t^+ = t + \alpha \quad , \tag{5.2}$$

where in regard to $\mathbf{x}^+$ and $\mathbf{x}$ we think of position vectors characterizing two points. For eq. (5.2) we have introduced a vector $\mathbf{c}(t)$, and a real number α denoting the *time-shift*, which are defined to be

$$\mathbf{c}(t) = \mathbf{x}_0^+ - \mathbf{Q}(t)\mathbf{x}_0 \quad , \qquad \alpha = t_0^+ - t_0 \quad . \tag{5.3}$$

Note that both $\mathbf{c}$ and $\mathbf{Q}$ are continuous functions of time which, for convenience, are assumed to be continuously differentiable. The one-to-one mapping of the form (5.2) connecting the pair $(\mathbf{x}, t)$ with its corresponding pair $(\mathbf{x}^+, t^+)$ is frequently referred to as a **Euclidean transformation.**

Change of observer. We assert, for example, that macroscopic properties of materials are not affected by the choice of an observer, a fundamental principle of physics. However, the general aim is to ensure that the stress state in a body and any physical quantity with an intrinsic feature must be **invariant** relative to a particular **change of observer**. Therefore, we expect from a change of observer that *distances* between arbitrary pairs of points in space and *time intervals* between events are preserved. In other words, we require that a different observer O^+ monitors the same relative distances of points and the same time intervals between events under observation.

We can show that the spatial transformation (5.1) or $(5.2)_1$, combined with the time-shift $(5.2)_2$, denote the most general time-dependent change of observer from O to O^+. The event at place $\mathbf{x}$ and at time t recorded by observer O is the same event as that recorded by a different observer O^+ at $\mathbf{x}^+$ and t^+. Note that we are now considering **one** event recorded by **two** (different) observers O and O^+ who are moving relative to each other.

In order to describe a physical process in the (three-dimensional) Euclidean space and on the real time axis we assign to each of the observers a rectangular Cartesian coordinate system, which we characterize by a set of *fixed* basis vectors, i.e. $\{\mathbf{e}_a\}$ and $\{\mathbf{e}_a^+\}$ relative to O and O^+, respectively. We call them **reference frames of the observers**. Hence, any points $\mathbf{x}$ and $\mathbf{x}^+$ may be represented by the position vectors $\mathbf{x} = x_a\mathbf{e}_a$ and $\mathbf{x}^+ = x_a^+\mathbf{e}_a^+$, with x_a and x_a^+ denoting rectangular Cartesian coordinates, as usual. The shift in the time scale between the observer O and O^+ is $t^+ = t + \alpha$.

In order to formulate the change of observer in index notation, we identify $\mathbf{Q}(t)$ as the relative rotation of the reference frames of the observers. Hence, $\mathbf{e}_a^+ = \mathbf{Q}(t)\mathbf{e}_a$

(compare with eq. $(1.182)_1$). Multiplying eqs. $(5.2)_1$ and $(5.3)_1$ by $\mathbf{e}_a^+$ and using identities (1.81), $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ and the relation according to $(1.23)_3$, we find that

$$x_a^+ = \mathbf{x}^+ \cdot \mathbf{e}_a^+ = c_a^+(t) + x_a \;, \qquad c_a^+(t) = \mathbf{c}(t) \cdot \mathbf{e}_a^+ = x_{0a}^+ - x_{0a} \;. \tag{5.4}$$

The mathematical expression $(5.4)_1$ states that the observers O and O^+ assign, with the exception of the shift c_a^+, the same coordinates to the corresponding points $\mathbf{x}$ and $\mathbf{x}^+$.

The reader should be aware that the introduced mapping, i.e. a change of observer, affects the points in space-time and *not* the coordinates of points. However, a change of reference frame changes the coordinates of points (and not the points themselves) and is simply governed by a coordinate transformation, as introduced in Section 1.5.

EXAMPLE 5.1 Consider two arbitrary points of a continuum body identified by their position vectors $\mathbf{x}$ and $\mathbf{y}$ at time t. The events $(\mathbf{x}, t)$ and $(\mathbf{y}, t)$ are recorded by an observer O with the reference frame $\mathbf{e}_a$. A second observer O^+ with the reference frame $\mathbf{e}_a^+$ records the same events at the associated points $\mathbf{x}^+$ and $\mathbf{y}^+$ at time t^+.

Compute the transformation of the spatial vector field $\mathbf{u} = \mathbf{y} - \mathbf{x} = u_a \mathbf{e}_a$ into its counterpart $\mathbf{u}^+ = \mathbf{y}^+ - \mathbf{x}^+ = u_a^+ \mathbf{e}_a^+$ and determine the components u_a (of $\mathbf{u}$) and u_a^+ (of $\mathbf{u}^+$), recorded by the two (different) observers O and O^+, respectively.

Solution. Using (5.2), $\mathbf{u}$ transforms according to

$$\mathbf{u}^+ = \mathbf{y}^+ - \mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{y} - \mathbf{x}) - \mathbf{c}(t) = \mathbf{Q}(t)\mathbf{u} \;. \tag{5.5}$$

With reference to eqs. $(1.23)_3$ and (1.21), the components of the spatial vector fields $\mathbf{u}^+$ and $\mathbf{u}$ relative to one and to the other reference frames of the observers read

$$u_a^+ = \mathbf{u}^+ \cdot \mathbf{e}_a^+ = (y_a^+ - x_a^+)\mathbf{e}_a^+ \cdot \mathbf{e}_a^+ = y_a^+ - x_a^+ \;, \tag{5.6}$$

$$u_a = \mathbf{u} \cdot \mathbf{e}_a = (y_a - x_a)\mathbf{e}_a \cdot \mathbf{e}_a = y_a - x_a \;. \tag{5.7}$$

By means of $(5.4)_1$ we deduce from $(5.6)_3$ that

$$u_a^+ = y_a^+ - x_a^+ = c_a^+ + y_a - c_a^+ - x_a = y_a - x_a \;. \tag{5.8}$$

With $(5.7)_3$ this shows that $u_a^+ = u_a$, signifying that both observers measure the *same* distance. ■

Any spatial vector field $\mathbf{u}$ that transforms according to eq. $(5.5)_3$, i.e.

$$\mathbf{u}^+ = \mathbf{Q}(t)\mathbf{u} \;, \tag{5.9}$$

is said to be **objective** or equivalently **frame-indifferent**. According to Example 5.1, transformation (5.9) implies that two observers O^+ and O, moving relative to each other, record the same coordinates, $u_a^+ = u_a$, which is the meaning of an objective spatial vector field. In general, if a physical quantity is objective then it is independent of an observer.

Consider now a motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ of a continuum body as seen by an arbitrary observer O in space. The motion specifies the place $\mathbf{x}$ at current time t of a certain material point initially at $\mathbf{X}$. A second observer O^+ monitors the *same* motion at the place $\mathbf{x}^+$ and at current time t^+, we write $\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X}, t^+)$. Note that the reference configuration (and any referential position $\mathbf{X}$) is fixed and therefore independent of the change of observer.

Hence, the motion $\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X}, t^+)$ is related to $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ by the Euclidean transformation (5.2), i.e.

$$\boldsymbol{\chi}^+(\mathbf{X}, t^+) = \mathbf{c}(t) + \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X}, t) \;, \qquad t^+ = t + \alpha \;, \tag{5.10}$$

for each point $\mathbf{X}$ and time t.

Velocity and acceleration fields under changes of observers. In general, observers are located at different places in space and move relative to each other, as implied by the time-dependence of $\mathbf{c}(t)$ and $\mathbf{Q}(t)$. Therefore, the descriptions of motions depend on the observers and, consequently, the velocity and acceleration of motion are, in general, *not* objective (frame-indifferent), as shown in the following.

To begin with, we introduce

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \dot{\boldsymbol{\chi}}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \;, \\ \mathbf{v}^+(\mathbf{x}^+, t^+) &= \dot{\boldsymbol{\chi}}^+(\mathbf{X}, t^+) = \frac{\partial \boldsymbol{\chi}^+(\mathbf{X}, t^+)}{\partial t^+} \;, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathbf{a}(\mathbf{x}, t) &= \dot{\mathbf{v}}(\mathbf{x}, t) = \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} \;, \\ \mathbf{a}^+(\mathbf{x}^+, t^+) &= \dot{\mathbf{v}}^+(\mathbf{x}^+, t^+) = \frac{\partial \mathbf{v}^+(\mathbf{x}^+, t^+)}{\partial t^+} \;, \end{aligned} \tag{5.12}$$

which are the spatial velocities and accelerations of a certain point as observed by O and O^+, respectively. Before examining the transformation rules for the velocity and acceleration fields under changes of observers it is first necessary to formulate the inverse relation of $(5.2)_1$ and its material time derivative. Hence, with $\mathbf{Q}(t)^{\mathrm{T}}\mathbf{Q}(t) = \mathbf{I}$, we deduce from $(5.2)_1$

$$\mathbf{x} = \mathbf{Q}(t)^{\mathrm{T}}[\mathbf{x}^+ - \mathbf{c}(t)] \;, \tag{5.13}$$

and using $(2.28)_1$, the product rule of differentiation and $(5.11)_2$, we find that

$$\mathbf{v}(\mathbf{x}, t) = \dot{\overline{\mathbf{Q}(t)^{\mathrm{T}}}}[\mathbf{x}^+ - \mathbf{c}(t)] + \mathbf{Q}(t)^{\mathrm{T}}[\mathbf{v}^+ - \dot{\mathbf{c}}(t)] \;. \tag{5.14}$$

The overbar covers the quantity to which the time differentiation is applied. Further, we define the skew tensor

$$\boldsymbol{\Omega}(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^{\mathrm{T}} = -\boldsymbol{\Omega}(t)^{\mathrm{T}} \tag{5.15}$$

with the property

$$\boldsymbol{\Omega}^2(t) = -\dot{\mathbf{Q}}(t)\mathbf{Q}(t)^{\mathrm{T}}[\dot{\mathbf{Q}}(t)\mathbf{Q}(t)^{\mathrm{T}}]^{\mathrm{T}} = -\dot{\mathbf{Q}}(t)\dot{\overline{\mathbf{Q}(t)^{\mathrm{T}}}} \quad . \tag{5.16}$$

The tensor $\boldsymbol{\Omega}$ represents the spin of the reference frame of observer O relative to the reference frame of observer O^+.

Hence, material time differentiation of the spatial part of (5.10) gives, using (5.11) and the product rule,

$$\mathbf{v}^+(\mathbf{x}^+, t^+) = \dot{\mathbf{c}}(t) + \dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v}(\mathbf{x}, t) \quad . \tag{5.17}$$

From (5.17) we find, with the aid of (5.13) and $(5.15)_1$, the transformation law for the spatial velocity field, namely

$$\mathbf{v}^+ = \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}} + \boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}) \tag{5.18}$$

(suppressing the arguments of functions).

We deduce from relation (5.18) that the spatial velocity field $\mathbf{v}$ is *not* objective under changes of observers following the arbitrary transformation $\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}$. Since the extra terms $\dot{\mathbf{c}}$ and $\boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c})$ are present, the requirement for objectivity, i.e. eq. (5.9), is not satisfied. Hence, the velocity field $\mathbf{v}$ is only objective if

$$\dot{\mathbf{c}} + \boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}) = \mathbf{o} \quad , \tag{5.19}$$

implying a change of observer according to the following time-independent transformation

$$\mathbf{x}^+ = \mathbf{c}_0 + \mathbf{Q}_0\mathbf{x} \qquad \text{with} \qquad \dot{\mathbf{c}}_0 = \mathbf{o} \quad , \qquad \dot{\mathbf{Q}}_0 = \mathbf{O} \quad , \tag{5.20}$$

referred to as a **time-independent rigid transformation**. Therein, the vector $\mathbf{c}_0$ and the orthogonal tensor $\mathbf{Q}_0$ are assumed to be time-independent (constant) quantities. For a time-independent rigid transformation the magnitudes of $\mathbf{v}$ and $\mathbf{v}^+$ are equal, so that $\mathbf{v}^+ = \mathbf{Q}\mathbf{v}$, as would be required for an objective vector field.

The material time differentiation of (5.18) gives, by means of the product rule and eqs. (5.12) and $(5.11)_2$,

$$\mathbf{a}^+ = \mathbf{Q}\mathbf{a} + \ddot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{v} + \dot{\boldsymbol{\Omega}}(\mathbf{x}^+ - \mathbf{c}) + \boldsymbol{\Omega}(\mathbf{v}^+ - \dot{\mathbf{c}}) \tag{5.21}$$

(suppressing the arguments of functions).

Finally, by means of (5.14) and eqs. $(5.15)_1$, $(5.16)_2$, we obtain from (5.21) the transformation

$$\mathbf{a}^+ = \mathbf{Q}\mathbf{a} + \ddot{\mathbf{c}} + (\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega}^2)(\mathbf{x}^+ - \mathbf{c}) + 2\boldsymbol{\Omega}(\mathbf{v}^+ - \dot{\mathbf{c}}) \tag{5.22}$$

for the spatial acceleration field $\mathbf{a}$. Like for the spatial velocity field, the acceleration field is *not* objective for a general change of observer. The terms $\dot{\boldsymbol{\Omega}}(\mathbf{x}^+ - \mathbf{c})$ and $-\boldsymbol{\Omega}^2(\mathbf{x}^+ - \mathbf{c})$ are called the **Euler acceleration** and **centrifugal acceleration**, while the last term in eq. (5.22), i.e. $2\boldsymbol{\Omega}(\mathbf{v}^+ - \dot{\mathbf{c}})$, represents the **Coriolis acceleration**.

An acceleration field is objective under all changes of observer *if and only if* the lengths of $\mathbf{a}^+$ and $\mathbf{a}$ are equal, i.e. $\mathbf{a}^+(\mathbf{x}^+, t^+) = \mathbf{Q}(t)\mathbf{a}(\mathbf{x}, t)$, requiring that

$$\ddot{\mathbf{c}} + (\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega}^2)(\mathbf{x}^+ - \mathbf{c}) + 2\boldsymbol{\Omega}(\mathbf{v}^+ - \dot{\mathbf{c}}) = \mathbf{o} \quad . \tag{5.23}$$

Consequently, this implies that $\dot{\mathbf{c}}$ is constant and that the orthogonal tensor $\mathbf{Q}$ is also constant. A change of observer from O to O^+ of this type, for which the spatial acceleration is *objective*, is called a **Galilean transformation**, and is governed by

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}_0\mathbf{x} \qquad \text{with} \qquad \ddot{\mathbf{c}}(t) = \mathbf{o} \quad , \qquad \dot{\mathbf{Q}}_0 = \mathbf{O} \tag{5.24}$$

for all times t. Here, $\mathbf{Q}_0$ denotes the time-independent orthogonal tensor, and $\mathbf{c}(t) = \mathbf{v}_0 t + \mathbf{c}_0$, with the initial (constant) quantities for $\mathbf{c}_0$ and the velocity $\mathbf{v}_0$.

Objective higher-order tensor fields. A spatial tensor field of order n, $n = 1, 2, \ldots$, i.e. $\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n$, is called **objective** or equivalently **frame-indifferent**, if, during any change of observer, $\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n$ transforms according to

$$(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n)^+ = \mathbf{Q}\mathbf{u}_1 \otimes \cdots \otimes \mathbf{Q}\mathbf{u}_n \quad , \tag{5.25}$$

which holds for every tensor $\mathbf{Q}$ and every vector $\mathbf{u}_n$.

By introducing a spatial second-order tensor field $\mathbf{A}(\mathbf{x}, t)$ to be $\mathbf{u}_1(\mathbf{x}, t) \otimes \mathbf{u}_2(\mathbf{x}, t)$ $(n = 2)$, we find, using (5.9), the important relation

$$\begin{aligned} \mathbf{A}^+(\mathbf{x}^+, t^+) &= [\mathbf{u}_1(\mathbf{x}, t) \otimes \mathbf{u}_2(\mathbf{x}, t)]^+ = \mathbf{Q}(t)\mathbf{u}_1(\mathbf{x}, t) \otimes \mathbf{Q}(t)\mathbf{u}_2(\mathbf{x}, t) \\ &= \mathbf{Q}(t)[\mathbf{u}_1(\mathbf{x}, t) \otimes \mathbf{u}_2(\mathbf{x}, t)]\mathbf{Q}(t)^{\mathrm{T}} = \mathbf{Q}(t)\mathbf{A}(\mathbf{x}, t)\mathbf{Q}(t)^{\mathrm{T}} \quad . \end{aligned} \tag{5.26}$$

For any spatial vector field $\mathbf{u}$ $(n = 1)$, eq. (5.25) reduces to $\mathbf{u}^+(\mathbf{x}^+, t^+) = \mathbf{Q}(t)\mathbf{u}(\mathbf{x}, t)$, which we found through eq. (5.9).

In particular, for $n = 0$ we have a scalar field. It is obvious that any spatial scalar field $\Phi(\mathbf{x}, t)$, recorded by O, is unaffected by a change of observer. Hence, a spatial scalar field Φ is objective if, under all Euclidean transformations (5.2), Φ transforms according to

$$\Phi^+(\mathbf{x}^+, t^+) = \Phi(\mathbf{x}, t) \quad , \tag{5.27}$$

where Φ^+ is the corresponding scalar field recorded by observer O^+.

In summary: the *requirement of objectivity* means that tensor, vector and scalar fields transform under changes of observers according to the laws

$$\left.\begin{aligned} \mathbf{A}^+(\mathbf{x}^+,t^+) &= \mathbf{Q}(t)\mathbf{A}(\mathbf{x},t)\mathbf{Q}(t)^{\mathrm{T}} \quad , \\ \mathbf{u}^+(\mathbf{x}^+,t^+) &= \mathbf{Q}(t)\mathbf{u}(\mathbf{x},t) \quad , \\ \Phi^+(\mathbf{x}^+,t^+) &= \Phi(\mathbf{x},t) \quad , \end{aligned}\right\} \tag{5.28}$$

where $\mathbf{x}^+$ and $\mathbf{x}$ are related by the Euclidean transformation (5.2).

EXAMPLE 5.2 Show that the spatial gradient of an objective vector field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ transforms according to

$$[\mathrm{grad}\mathbf{u}(\mathbf{x},t)]^+ = \mathbf{Q}(t)\mathrm{grad}\mathbf{u}(\mathbf{x},t)\mathbf{Q}(t)^{\mathrm{T}} \quad , \tag{5.29}$$

where $(\mathrm{grad}\mathbf{u})^+ = \partial\mathbf{u}^+/\partial\mathbf{x}^+$ denotes the spatial gradient of the vector $\mathbf{u}^+$ recorded by an observer O^+. Note that in view of $(5.28)_1$ the second-order tensor field $\mathrm{grad}\mathbf{u}$ retains the objectivity property.

Solution. A vector field $\mathbf{u}$ remains unchanged during any change of observer if $\mathbf{u}^+(\mathbf{x}^+,t^+) = \mathbf{Q}(t)\mathbf{u}(\mathbf{x},t)$. Hence, by the chain rule,

$$\frac{\partial\mathbf{u}^+}{\partial\mathbf{x}^+}\frac{\partial\boldsymbol{\chi}^+}{\partial\mathbf{x}} = \mathbf{Q}\frac{\partial\mathbf{u}}{\partial\mathbf{x}} \tag{5.30}$$

(the arguments of the functions have been omitted). With the aid of transformation $(5.10)_1$ we obtain the desired result. Note that, in general, the gradient of an objective tensor field of order n is also objective. ■

EXERCISES

1. Consider objective scalar, vector and tensor fields $\Phi(\mathbf{x},t)$, $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$, respectively.
   (a) Show that $\mathrm{grad}\Phi$, $\mathrm{div}\mathbf{u}$ and $\mathrm{div}\mathbf{A}$ are objective fields during any change of observer.
   (b) Show that the Lie time derivative of a contravariant spatial vector field $\mathbf{u}$, as determined in eq. (2.194), is objective.

2. Using eqs. (5.13)–(5.15) obtain the alternative relation for the spatial acceleration field (5.21) in terms of $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{a}$ in the form
$$\mathbf{a}^+ = \mathbf{Q}\mathbf{a} + \ddot{\mathbf{c}} + \ddot{\mathbf{Q}}\mathbf{x} + 2\dot{\mathbf{Q}}\mathbf{v} \quad ;$$
and show that its gradient is given by
$$(\mathrm{grad}\mathbf{a})^+ = (\ddot{\mathbf{Q}} + 2\dot{\mathbf{Q}}\mathbf{l} + \mathbf{Q}\mathrm{grad}\mathbf{a})\mathbf{Q}^{\mathrm{T}} \quad ; \tag{5.31}$$
where $\mathbf{l} = \mathrm{grad}\mathbf{v}$ denotes the spatial velocity gradient.
3. Assume an objective transformation of the body force (per unit volume), i.e. $\mathbf{b}^+(\mathbf{x}^+,t^+) = \mathbf{Q}(t)\mathbf{b}(\mathbf{x},t)$. Show that the local form of Cauchy's first equation of motion in the spatial description, i.e. (4.53), is only objective under a Galilean transformation.

5.2 Superimposed Rigid-body Motions

In the following section we show that a change of observer may equivalently be viewed as certain rigid-body motions superimposed on the current configuration. We apply this concept to various kinematical quantities and to some stress tensors of importance.

Rigid-body motion. As noted, the fundamental relationship (5.2) describes a change of observer, preserving both the distances between arbitrary pairs of points in space, and time intervals between events under observation.

It is essential to introduce an important equivalent mechanical statement of the specification (5.2): for this purpose we consider a motion $\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X},t^+)$ of a continuum body which differs from another motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$ of the same body by a *superimposed* (possibly time-dependent) *rigid-body motion* and by a *time-shift*, as depicted in Figure 5.2. We emphasize that, in contrast to the considerations of the last section, $\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X},t^+)$ and $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$ are motions of **two** events recorded by a **single** observer O. The rigid-body motion moves the region Ω in space occupied by the body at time t, defined by the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$, to a new region Ω^+ occupied by the same body at t^+, which is given by $\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X},t^+)$. Here and elsewhere we will employ the symbol $(\bullet)^+$ to designate quantities associated with the new region Ω^+.

According to the principle of relativity, the description of a single motion monitored by two (different) observers, as described in the last section, is equivalent to the description of two (different) motions monitored by a single observer. Hence, the pairs $(\mathbf{x},t)$ and $(\mathbf{x}^+,t^+)$, which are defined on regions Ω and Ω^+, are precisely related by the Euclidean transformation (5.2), i.e. $\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}$ and $t^+ = t + \alpha$.

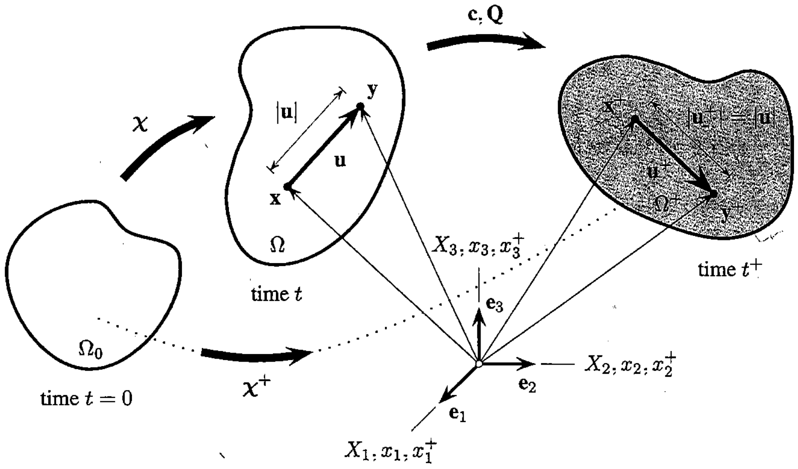

Figure 5.2 Two motions χ^+ and χ of a body (monitored by a single observer) which differ by a superimposed rigid-body motion and by a time-shift. A spatial vector field $\mathbf{u}$ transforms into $\mathbf{u}^+ = \mathbf{Qu}$, with length $|\mathbf{u}^+| = |\mathbf{u}|$.

Within this context the vector $\mathbf{c}$ describes a superimposed (time-dependent, pure) *rigid-body translation* for which any material point moves an identical distance, with the same magnitude and direction at time t. Since $\mathbf{Q}$ is a *proper* orthogonal tensor ($\det\mathbf{Q} = +1$), the orientation is preserved and $\mathbf{Q}$ describes a superimposed (time-dependent, pure) *rigid-body rotation.* For a pure rigid-body rotation, transformation (5.2) reduces to $\mathbf{x}^+ = \mathbf{Q}(t)\mathbf{x}$.

Hence, at each instant of time a rigid-body motion is the composition of a rigid-body translation $\mathbf{c}$ and a rigid-body rotation $\mathbf{Q}$ about an axis of rotation, combined with a time-shift $\alpha = t^+ - t$. The material points occupy the same relative position in each motion (the *angle* between two arbitrary vectors and their *lengths* remain constant).

We now recall Example 5.1 and apply the described concept to the spatial vector field $\mathbf{u} = \mathbf{y} - \mathbf{x}$ located at region Ω (see Figure 5.2). Hence, a rigid-body motion maps the points $\mathbf{x}, \mathbf{y}$ to the associated points $\mathbf{x}^+, \mathbf{y}^+$ located in Ω^+ and the spatial vector $\mathbf{u} = \mathbf{y} - \mathbf{x}$ to $\mathbf{u}^+ = \mathbf{y}^+ - \mathbf{x}^+$.

With (5.2) we may conclude that the distance between the two points $\mathbf{y}^+$ and $\mathbf{x}^+$ remain unchanged. Namely, $\mathbf{y}^+ - \mathbf{x}^+ = \mathbf{Q}(t)(\mathbf{y} - \mathbf{x})$ (compare with eq. (5.1)), which immediately implies, on use of definition (1.15), identity (1.81) and the orthogonality condition $\mathbf{Q}(t)^{\mathrm{T}}\mathbf{Q}(t) = \mathbf{I}$, that $|\mathbf{y}^+ - \mathbf{x}^+| = |\mathbf{y} - \mathbf{x}|$. Consequently the lengths of the vectors $\mathbf{u}^+$ and $\mathbf{u}$ are equal, i.e. $|\mathbf{u}^+| = |\mathbf{u}|$. We say that the spatial vector field $\mathbf{u}$ is objective during the rigid-body motion.

It is trivial but worthy of mention that any material field $\mathcal{F}(\mathbf{X}, t)$ of some physical scalar, vector or tensor quantity, which is characterized as a function of the referential position $\mathbf{X}$ and time t, is unaffected by a rigid-body motion superimposed on Ω. Hence, $\mathcal{F}^+(\mathbf{X}, t^+) = \mathcal{F}(\mathbf{X}, t)$.

Euclidean transformation of various kinematical quantities. The following discussion is concerned with the behavior of various kinematical quantities during a superimposed rigid-body motion.

To begin with, we consider the deformation gradient at the point $\mathbf{x} \in \Omega$ and its associated point $\mathbf{x}^+ \in \Omega^+$, i.e.

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial \mathbf{X}} \ , \qquad \mathbf{F}^+(\mathbf{X}, t^+) = \frac{\partial \mathbf{x}^+(\mathbf{X}, t^+)}{\partial \mathbf{X}} \ . \tag{5.32}$$

Differentiating (5.2) with respect to $\mathbf{X}$ gives the transformation rule

$$\mathbf{F}^+ = \frac{\partial \mathbf{x}^+}{\partial \mathbf{X}} = \mathbf{Q}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{QF} \qquad \text{or} \qquad F_{aA}^+ = \frac{\partial x_a^+}{\partial X_A} = Q_{ab}\frac{\partial x_b}{\partial X_A} = Q_{ab}F_{bA} \tag{5.33}$$

for the deformation gradient (for convenience, we will not indicate subsequently the dependence for the above functions on position and time). Note that the second-order tensor $\mathbf{F}$ is *objective* even though (5.33) does not coincide with the fundamental (objectivity) requirement $(5.28)_1$. However, recall that the deformation gradient $\mathbf{F}$ is a two-point tensor field, in which one index describes material coordinates X_A which are intrinsically independent of the observer. That is why the deformation gradient transforms like a vector according to $(5.28)_2$ and why $\mathbf{F}$ is regarded as objective.

Moreover, let $J = \det\mathbf{F}$ and $J^+ = \det\mathbf{F}^+$. Since the tensor $\mathbf{Q}$ is proper orthogonal ($\det\mathbf{Q} = +1$), eq. $(5.33)_3$ implies, through the property (1.101), that

$$J^+ = J > 0 \ . \tag{5.34}$$

Hence, the scalar field J remains unaltered by a superimposed rigid-body motion. Also the sign of the volume ratio J is preserved, since $\det\mathbf{Q} = +1$.

Next, we recall the unique polar decomposition of the deformation gradient at $\mathbf{x} \in \Omega$ and $\mathbf{x}^+ \in \Omega^+$, i.e.

$$\mathbf{F} = \mathbf{RU} = \mathbf{vR} \ , \qquad \mathbf{F}^+ = \mathbf{R}^+\mathbf{U}^+ = \mathbf{v}^+\mathbf{R}^+ \ . \tag{5.35}$$

Applying $(5.33)_3$ to (5.35), we arrive at the representations

$$\mathbf{R}^+\mathbf{U}^+ = \mathbf{QRU} \ , \qquad \mathbf{v}^+\mathbf{R}^+ = \mathbf{QvR} \ . \tag{5.36}$$

Since the tensor $\mathbf{QR}$ is orthogonal it follows from $(5.36)_1$ that the transformation rules for the rotation tensor $\mathbf{R}$ and the right stretch tensor $\mathbf{U}$ are

$$\mathbf{R}^+ = \mathbf{QR} \ , \qquad \mathbf{U}^+ = \mathbf{U} \ . \tag{5.37}$$

By analogy with the deformation gradient, $\mathbf{R}$ is an objective two-point tensor field. The right stretch tensor is defined with respect to the reference configuration. Hence, $\mathbf{U}$ remains unaltered by a superimposed rigid-body motion and $\mathbf{U}$ is therefore also objective.

From eq. $(5.36)_2$, we obtain, using result $(5.37)_1$ and the orthogonality condition $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, the transformation rule

$$\mathbf{v}^+ = \mathbf{Q}\mathbf{v}\mathbf{Q}^{\mathrm{T}} \quad . \tag{5.38}$$

Clearly, the left stretch tensor $\mathbf{v}$ is objective.

Next, we discuss the spatial velocity gradient according to $(2.141)_4$, i.e. $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. In an analogous manner, the spatial velocity gradient generated by the motion χ^+ reads

$$\mathbf{l}^+ = (\dot{\mathbf{F}}\mathbf{F}^{-1})^+ = \dot{\mathbf{F}}^+(\mathbf{F}^+)^{-1} \quad . \tag{5.39}$$

By deriving eq. $(5.33)_3$ with respect to time, using the product rule, i.e. $\dot{\mathbf{F}}^+ = \dot{\mathbf{Q}}\mathbf{F}+\mathbf{Q}\dot{\mathbf{F}}$, and the inverse relation of $(5.33)_3$, i.e. $(\mathbf{F}^+)^{-1} = \mathbf{F}^{-1}\mathbf{Q}^{\mathrm{T}}$, the spatial velocity gradient follows from $(5.39)_2$ as

$$\mathbf{l}^+ = (\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}})(\mathbf{F}^{-1}\mathbf{Q}^{\mathrm{T}}) = \boldsymbol{\Omega} + \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}} \quad , \tag{5.40}$$

with the skew tensor $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. Since $\boldsymbol{\Omega}$ is present, the spatial velocity gradient $\mathbf{l}$ fails to satisfy the objectivity requirement ($\mathbf{l}^+ \neq \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}}$). Hence, the kinematical quantity $\mathbf{l}$ is not a suitable candidate for formulating constitutive equations, which must be objective.

Euclidean transformation of stress tensors. Let dynamical processes be given by the pairs $(\boldsymbol{\sigma}^+, \chi^+)$ and $(\boldsymbol{\sigma}, \chi)$, where χ^+ and χ are related through (5.10). We now want to show how the Cauchy stress tensors $\boldsymbol{\sigma}^+$ and $\boldsymbol{\sigma}$ are related.

We recall the Cauchy traction vector $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ with the unit vector $\mathbf{n}$ at point $\mathbf{x}$ directed along the outward normal to the boundary surface $\partial\Omega$ of an arbitrary region Ω at time t. A superimposed rigid-body motion transforms region Ω to a new region Ω^+ which is bounded by the associated boundary surface $\partial\Omega^+$ at a later time $t^+ = t + \alpha$. The Cauchy traction vector transforms to $\mathbf{t}^+ = \boldsymbol{\sigma}^+\mathbf{n}^+$ with the unit vector $\mathbf{n}^+$ at point $\mathbf{x}^+$ normal to $\partial\Omega^+$. By taking note that the vectors $\mathbf{t}$ and $\mathbf{n}$ transform according to the objectivity requirement $(5.28)_2$, we obtain that $\mathbf{Q}\mathbf{t} = \boldsymbol{\sigma}^+\mathbf{Q}\mathbf{n}$. A comparison with $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ gives the fundamental transformation rule

$$\boldsymbol{\sigma}^+ = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} \tag{5.41}$$

for the stress tensor. This means that the Cauchy stress tensor is objective.

In order to describe the first Piola-Kirchhoff stress tensor which is generated by the motion $\mathbf{x}^+ = \chi^+(\mathbf{X}, t^+)$, we may write the Piola transformation (3.8) as $\mathbf{P}^+(\mathbf{F}^+)^{\mathrm{T}} = J^+\boldsymbol{\sigma}^+$. Knowing that the scalar J is objective according to eq. (5.34), and using $(5.33)_3$ and (5.41), we find, with the help of (3.8), that

$$\begin{aligned} \mathbf{P}^+(\mathbf{Q}\mathbf{F})^{\mathrm{T}} &= J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} \quad , \\ \mathbf{P}^+(\mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\,J\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{P}(\mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}) \quad , \\ \mathbf{P}^+ &= \mathbf{Q}\mathbf{P} \quad . \end{aligned} \tag{5.42}$$

Since the two-point stress tensor field $\mathbf{P}$ transforms like a vector field according to the objectivity requirement $(5.28)_2$, $\mathbf{P}$ is objective. The second Piola-Kirchhoff stress tensor $\mathbf{S}$ is parameterized by material coordinates only. Therefore, the material tensor field does not depend on any superimposed rigid-body motion, and hence $\mathbf{S} = \mathbf{S}^+$.

Note that all the stress tensors $\boldsymbol{\sigma}$, $\mathbf{P}$ and $\mathbf{S}$ discussed are suitable candidates for the description of material response, which fundamentally is required to be independent of the observer.

EXERCISES

1. Recall the kinematic relations (2.65), (2.69) and (2.79), (2.83). Using the transformation rule $\mathbf{F}^+ = \mathbf{Q}\mathbf{F}$, show that the material strain tensors $\mathbf{C}$ and $\mathbf{E}$ are unaffected by any possible rigid-body motion, i.e.

   $$\mathbf{C}^+ = \mathbf{C} \quad , \qquad \mathbf{E}^+ = \mathbf{E} \quad , \tag{5.43}$$

   and that the spatial strain tensors $\mathbf{b}$ and $\mathbf{e}$ transform according to the rules

   $$\mathbf{b}^+ = \mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}} \quad , \qquad \mathbf{e}^+ = \mathbf{Q}\mathbf{e}\mathbf{Q}^{\mathrm{T}} \quad . \tag{5.44}$$

   Note that all these kinematical quantities are objective, since $\mathbf{C}$ and $\mathbf{E}$ are defined with respect to the reference configuration and the second-order tensor fields $\mathbf{b}$ and $\mathbf{e}$ conform with the requirement of objectivity given in eq. $(5.28)_1$.

2. By eqs. (2.148) and (2.149), the spatial velocity gradient $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is recalled to be the sum of the rate of deformation tensor $\mathbf{d} = \mathbf{d}^{\mathrm{T}}$ and the spin tensor $\mathbf{w} = -\mathbf{w}^{\mathrm{T}}$.

   Show that rigid-body motions involve the transformations

   $$\mathbf{d}^+ = \mathbf{Q}\mathbf{d}\mathbf{Q}^{\mathrm{T}} \quad , \qquad \mathbf{w}^+ = \boldsymbol{\Omega} + \mathbf{Q}\mathbf{w}\mathbf{Q}^{\mathrm{T}} \quad , \tag{5.45}$$

   where $\mathbf{d}$ is objective. Note that $\mathbf{w}$, which is expressed through the skew tensor $\boldsymbol{\Omega}$, is affected by rigid-body motions, and hence $\mathbf{w}$ is not objective.

5.3 Objective Rates

One aim of this section is to perform objective time derivatives, which are essential in order to formulate constitutive equations in the rate form. We focus attention on some important objective stress rates associated with the names *Oldroyd, Green, Naghdi, Jaumann, Zaremba* or *Truesdell.*

Objective rates. The material time derivatives of the objective vector field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and the objective second-order tensor field $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$, which transform according to eqs. $(5.28)_1$ and $(5.28)_2$, are given by means of the product rule of differentiation as

$$\dot{\mathbf{u}}^+ = \mathbf{Q}\dot{\mathbf{u}} + \dot{\mathbf{Q}}\mathbf{u} \quad , \qquad \dot{\mathbf{A}}^+ = \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}\dot{\overline{\mathbf{Q}^{\mathrm{T}}}} \quad . \tag{5.46}$$

Clearly, neither $\dot{\mathbf{u}}$ nor $\dot{\mathbf{A}}$ retains the objectivity requirements (5.28) ($\dot{\mathbf{u}}^+ \neq \mathbf{Q}\dot{\mathbf{u}}$ and $\dot{\mathbf{A}}^+ \neq \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}}$). Note that material time derivatives of objective spatial tensor fields will not, in general, be objective and they are not, therefore, suitable quantities for formulating constitutive equations in the rate form.

This motivates the introduction of objective time derivatives called **objective rates**, which are basically modified material time derivatives. Before proceeding to examine objective rate forms it is first necessary to express the material time derivatives of $\mathbf{Q}$ and $\mathbf{Q}^{\mathrm{T}}$ from relation $(5.45)_2$. With definition $(5.15)_1$ and property $\mathbf{w} = -\mathbf{w}^{\mathrm{T}}$ for the spin tensor we find that

$$\dot{\mathbf{Q}} = \mathbf{w}^+\mathbf{Q} - \mathbf{Q}\mathbf{w} \quad , \qquad \dot{\overline{\mathbf{Q}^{\mathrm{T}}}} = -\mathbf{Q}^{\mathrm{T}}\mathbf{w}^+ + \mathbf{w}\mathbf{Q}^{\mathrm{T}} \quad . \tag{5.47}$$

Hence, substituting $(5.47)_1$ into $(5.46)_1$, we find immediately, by analogy with the transformation rule $(5.28)_2$, that

$$(\dot{\mathbf{u}} - \mathbf{w}\mathbf{u})^+ = \mathbf{Q}(\dot{\mathbf{u}} - \mathbf{w}\mathbf{u}) \quad , \tag{5.48}$$

$$(\mathring{\mathbf{u}})^+ = \mathbf{Q}\mathring{\mathbf{u}} \quad , \tag{5.49}$$

where we have introduced the definition for the **co-rotational rate** of the objective vector field $\mathbf{u}$, i.e.

$$\mathring{\mathbf{u}} = \dot{\mathbf{u}} - \mathbf{w}\mathbf{u} \quad . \tag{5.50}$$

In general, we denote co-rotational rates with the accent $(\mathring{\bullet})$.

By analogy with the above we introduce the co-rotational rate of the objective second-order tensor field $\mathbf{A}$. With the help of eqs. (5.47) and $(5.28)_1$ we find from $(5.46)_2$ after some straightforward recasting that

$$(\dot{\mathbf{A}} - \mathbf{w}\mathbf{A} + \mathbf{A}\mathbf{w})^+ = \mathbf{Q}(\dot{\mathbf{A}} - \mathbf{w}\mathbf{A} + \mathbf{A}\mathbf{w})\mathbf{Q}^{\mathrm{T}} \quad , \tag{5.51}$$

$$(\mathring{\mathbf{A}})^+ = \mathbf{Q}\mathring{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} \quad , \tag{5.52}$$

where we have introduced the definition

$$\mathring{\mathbf{A}} = \dot{\mathbf{A}} - \mathbf{w}\mathbf{A} + \mathbf{A}\mathbf{w} \quad , \tag{5.53}$$

known as the **Jaumann-Zaremba rate**, which is often used in plasticity theory. Obviously, in regard to eqs. (5.49) and (5.52), the co-rotational rates of $\mathbf{u}$ and $\mathbf{A}$ are indeed objective.

If $\mathbf{A}$ is a symmetric tensor we can easily show an interesting property connecting the Jaumann-Zaremba rate and the material time derivative of $\mathbf{A}$. Using (5.53) and the property of double contraction according to (1.95), i.e. $\mathbf{A} : \mathbf{w}\mathbf{A} = \mathbf{A} : \mathbf{A}\mathbf{w}$, we obtain

$$2\mathbf{A} : \mathring{\mathbf{A}} = 2\mathbf{A} : \dot{\mathbf{A}} - 2\mathbf{A} : \mathbf{w}\mathbf{A} + 2\mathbf{A} : \mathbf{A}\mathbf{w} = \dot{\overline{\mathbf{A} : \mathbf{A}}} \quad . \tag{5.54}$$

Finally we define the **convected rates** of $\mathbf{u}$ and $\mathbf{A}$. These are the objective fields

$$\overset{\triangle}{\mathbf{u}} = \dot{\mathbf{u}} + \mathbf{l}^{\mathrm{T}}\mathbf{u} \quad , \qquad \overset{\triangle}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{l}^{\mathrm{T}}\mathbf{A} + \mathbf{A}\mathbf{l} \quad , \tag{5.55}$$

where the accent $(\overset{\triangle}{\bullet})$ indicates convected rates. The rate $\overset{\triangle}{\mathbf{A}}$ is also called the **Cotter-Rivlin rate.**

Objective stress rates. We now focus attention on some of the infinitely many possible objective stress rates that may be defined. The choice of suitable, i.e. objective, stress rates is essential in the formulation of constitutive rate equations, which must be objective.

The **Oldroyd stress rate** of a spatial stress field is defined to be the Lie time derivative of that field. We shall indicate Oldroyd stress rates by the abbreviation `Oldr`. By recalling the concept of Lie time derivatives from Section 2.8, in particular, rule $(2.188)_1$, the Lie time derivative of the contravariant Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\begin{aligned} \pounds_{\mathbf{v}}(\boldsymbol{\sigma}^{\sharp}) &= \mathbf{F}\left[\frac{\mathrm{D}(\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}})}{\mathrm{D}t}\right]\mathbf{F}^{\mathrm{T}} \\ &= \mathbf{F}(\dot{\overline{\mathbf{F}^{-1}}}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} + \mathbf{F}^{-1}\dot{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} + \mathbf{F}^{-1}\boldsymbol{\sigma}\dot{\overline{\mathbf{F}^{-\mathrm{T}}}})\mathbf{F}^{\mathrm{T}} \quad , \end{aligned} \tag{5.56}$$

where the transformations (2.87) and the product rule of differentiation are to be used. Hence, using the identities $(2.145)_2$ and $(2.146)_2$, we conclude that the Oldroyd stress rate of the Cauchy stress $\boldsymbol{\sigma}$ is

$$\mathtt{Oldr}(\boldsymbol{\sigma}) = \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} \quad , \tag{5.57}$$

where $\dot{\boldsymbol{\sigma}}$ denotes the material time derivative of the Cauchy stress tensor.

We now show how the Oldroyd stress rate, $\mathtt{Oldr}(\boldsymbol{\sigma})$, generated by the motion $\boldsymbol{\chi}$, is related to its counterpart $\mathtt{Oldr}(\boldsymbol{\sigma})^+$, generated by $\boldsymbol{\chi}^+$. Considering $\mathtt{Oldr}(\boldsymbol{\sigma})^+ =$

$\dot{\boldsymbol{\sigma}}^+ - \mathbf{l}^+\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^+\mathbf{l}^{+\mathrm{T}}$ and using transformations (5.41) and $(5.40)_2$ in combination with eqs. (5.15) and (5.57), we obtain

$$\begin{aligned}\mathtt{Oldr}(\boldsymbol{\sigma})^+ &= \overline{\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}}^{\boldsymbol{\cdot}} - (\boldsymbol{\Omega} + \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}})\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} \\ &= \mathbf{Q}(\dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}})\mathbf{Q}^{\mathrm{T}} \\ &= \mathbf{Q}\,\mathtt{Oldr}(\boldsymbol{\sigma})\mathbf{Q}^{\mathrm{T}} \quad . \end{aligned} \tag{5.58}$$

Hence, the Oldroyd stress rate is objective. In general, we can prove that Lie time derivatives of objective spatial tensor fields yield objective spatial tensor fields.

By analogy with (5.57), the Oldroyd stress rate of the Kirchhoff stress $\boldsymbol{\tau}$ is

$$\mathtt{Oldr}(\boldsymbol{\tau}) = \dot{\boldsymbol{\tau}} - \mathbf{l}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}} \quad . \tag{5.59}$$

Adopting rule (2.188) for the contravariant Kirchhoff stress tensor $\boldsymbol{\tau}$ and using relation $(3.63)_1$, we obtain the important equation

$$\pounds_{\mathbf{v}}(\boldsymbol{\tau}^\sharp) = \chi_* \left(\frac{\mathrm{D}}{\mathrm{D}t} \chi_*^{-1}(\boldsymbol{\tau}^\sharp) \right) = \mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} \quad , \tag{5.60}$$

relating the Lie time derivative of $\boldsymbol{\tau}$, i.e. $\mathtt{Oldr}(\boldsymbol{\tau})$, and the material time derivative $\dot{\mathbf{S}}$ of the second Piola-Kirchhoff stress $\mathbf{S}$ according to the push-forward operation $(2.87)_1$.

EXAMPLE 5.3 Consider the definitions

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \quad , \qquad \overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} \tag{5.61}$$

of objective stress rates, where $\overset{\triangledown}{\boldsymbol{\sigma}}$ is called the **Green-Naghdi stress rate** and $\overset{\circ}{\boldsymbol{\sigma}}$ the **Jaumann-Zaremba stress rate** (compare with the Jaumann-Zaremba rate (5.53) for any objective second-order tensor field). The spin tensor $\mathbf{w}$ is given by definition (2.149).

Show that both $\overset{\triangledown}{\boldsymbol{\sigma}}$ and $\overset{\circ}{\boldsymbol{\sigma}}$ are special cases of the Oldroyd stress rate of $\boldsymbol{\sigma}$ in the sense that $\overset{\triangledown}{\boldsymbol{\sigma}}$ corresponds to the Lie time derivative (5.56), with $\mathbf{F}$ replaced by the rotation tensor $\mathbf{R}$, and $\overset{\circ}{\boldsymbol{\sigma}}$ is the Lie time derivative (5.56), with the rate of deformation tensor $\mathbf{d}$ set to zero. Discuss the case in which the Green-Naghdi stress rate and the Jaumann-Zaremba stress rate coincide.

Solution. Setting $\mathbf{F} = \mathbf{R}$ in relation (5.56) and employing identities $(2.145)_1$ and $(2.146)_1$ (change $\mathbf{F}$ to $\mathbf{R}$) with the orthogonality condition $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, we obtain

$$\pounds_{\mathbf{v}}(\boldsymbol{\sigma}^\sharp)|_{\mathbf{F}=\mathbf{R}} = \dot{\boldsymbol{\sigma}} - \dot{\mathbf{R}}\mathbf{R}^{-1}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{R}^{-\mathrm{T}}\overline{\mathbf{R}^{\mathrm{T}}}^{\boldsymbol{\cdot}} = \dot{\boldsymbol{\sigma}} - \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \quad , \tag{5.62}$$

where $\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ is a skew tensor according to (2.160).

However, setting $\mathbf{d} = \mathbf{O}$ in the Lie time derivative (5.56), which equivalently means that $\mathbf{l} = \mathbf{w}$, we deduce from (5.57) that

$$\pounds_{\mathbf{v}}(\boldsymbol{\sigma}^\sharp)|_{\mathbf{d}=\mathbf{O}} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} \quad , \tag{5.63}$$

with the skew tensor $\mathbf{w} = -\mathbf{w}^{\mathrm{T}}$.

The Green-Naghdi stress rate and the Jaumann-Zaremba stress rate clearly coincide for $\mathbf{w} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$. This is the case for a rigid-body rotation (recall Example 2.12, eq. (2.162)). ■

The **Truesdell stress rate** of the Cauchy stress, denoted by $\mathtt{Trues}(\boldsymbol{\sigma})$, is defined as the Piola transformation of $\dot{\mathbf{S}}$. Thus,

$$\mathtt{Trues}(\boldsymbol{\sigma}) = J^{-1}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} \quad , \tag{5.64}$$

that is the push-forward of $\dot{\mathbf{S}}$ scaled by the inverse of the volume ratio, $J^{-1} = (\det\mathbf{F})^{-1}$. Hence, using the Piola transformation $(3.65)_1$ and the product rule of differentiation we find from (5.64) that

$$\begin{aligned}\mathtt{Trues}(\boldsymbol{\sigma}) = J^{-1}\mathbf{F}\left[\frac{\mathrm{D}(J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}})}{\mathrm{D}t}\right]\mathbf{F}^{\mathrm{T}} &= J^{-1}\mathbf{F}(\dot{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} \\ &+ J\overline{\mathbf{F}^{-1}}^{\boldsymbol{\cdot}}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\dot{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\boldsymbol{\sigma}\overline{\mathbf{F}^{-\mathrm{T}}}^{\boldsymbol{\cdot}})\mathbf{F}^{\mathrm{T}} \quad . \end{aligned} \tag{5.65}$$

Using relations $(2.178)_2$ and $(2.145)_2$, $(2.146)_2$ we deduce from $(5.65)_2$ that

$$\mathtt{Trues}(\boldsymbol{\sigma}) = \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} + \boldsymbol{\sigma}\mathrm{tr}\mathbf{d} \quad . \tag{5.66}$$

By comparing eqs. (5.57) and (5.59) with (5.66) we may easily deduce the relationships

$$\mathtt{Oldr}(\boldsymbol{\sigma}) = \mathtt{Trues}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}\mathrm{tr}\mathbf{d} \quad , \tag{5.67}$$

$$\mathtt{Oldr}(\boldsymbol{\tau}) = J\,\mathtt{Trues}(\boldsymbol{\sigma}) \tag{5.68}$$

between the Oldroyd stress rate and the Truesdell stress rate.

EXAMPLE 5.4 Suppose that the transformation (5.68) is given. Express $\boldsymbol{\sigma}$ as $J^{-1}\boldsymbol{\tau}$ and derive the relation (5.67) by simply applying the product rule of differentiation to the Oldroyd stress rate.

Solution. With $\boldsymbol{\sigma} = J^{-1}\boldsymbol{\tau}$ and the fact that the directional derivative (in our case the Lie time derivative) satisfies the common rules of differentiation, for example, the product rule, we may write

$$\mathtt{Oldr}(\boldsymbol{\sigma}) = \pounds_{\mathrm{v}}(J^{-1}\boldsymbol{\tau}) = J^{-1}\pounds_{\mathrm{v}}(\boldsymbol{\tau}) + \pounds_{\mathrm{v}}(J^{-1})\boldsymbol{\tau} \quad . \tag{5.69}$$

According to considerations of Section 2.8, the Lie time derivative of a scalar field is equal to the material time derivative of that scalar field, and hence $\pounds_{\mathrm{v}}(J^{-1}) = \dot{\overline{J^{-1}}}$. Therefore, with the chain rule and eqs. $(2.178)_2$ and (5.68), relation $(5.69)_2$ leads to the desired result,

$$\begin{aligned}\mathtt{Oldr}(\boldsymbol{\sigma}) &= J^{-1}\,\mathtt{Oldr}(\boldsymbol{\tau}) + \dot{\overline{J^{-1}}}\boldsymbol{\tau} = J^{-1}\mathtt{Oldr}(\boldsymbol{\tau}) - J^{-1}\boldsymbol{\tau}\mathrm{tr}\mathbf{d} \\ &= \mathtt{Trues}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}\mathrm{tr}\mathbf{d} \quad ; \end{aligned} \tag{5.70}$$

where the relation $\boldsymbol{\sigma} = J^{-1}\boldsymbol{\tau}$ is to be used again. ■

EXERCISES

1. Consider the Cotter-Rivlin rate $\overset{\triangle}{\mathbf{A}}$ defined by $(5.55)_2$.
   (a) Show that $\overset{\triangle}{\mathbf{A}}$ is the Lie time derivative of a covariant tensor field $\mathbf{A}$.
   (b) With (5.53) and $\mathbf{l} = \mathbf{d} + \mathbf{w}$ show that the connection between the Cotter-Rivlin and Jaumann-Zaremba rates is

$$\overset{\triangle}{\mathbf{A}} = \overset{\circ}{\mathbf{A}} + \mathbf{d}\mathbf{A} + \mathbf{A}\mathbf{d} \quad .$$

2. Recall the convected rates $\hat{\mathbf{u}}$ and $\overset{\triangle}{\mathbf{A}}$, as defined in eq. (5.55). Using eqs. $(5.40)_2$, (5.15) and (5.46), (5.28), show that they are objective according to

$$(\hat{\mathbf{u}})^{+} = \mathbf{Q}\hat{\mathbf{u}} \quad , \qquad (\overset{\triangle}{\mathbf{A}})^{+} = \mathbf{Q}\overset{\triangle}{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} \quad .$$

5.4 Invariance of Elastic Material Response

In this section we introduce the principle of material frame-indifference which states basically that material properties do not depend on the change of observer. In particular, we show how it restricts the response of elastic materials and derive objective constitutive equations which are defined to be invariant for all changes of observer. This principle is crucial when constitutive theories such as the theory of elasticity or plasticity are considered.

In the following we consider only the isothermal case for which the absolute temperature Θ remains constant during the process.

Cauchy-elastic materials. A material is called **Cauchy-elastic** or **elastic** if the stress field at time t depends only on the state of deformation (and the state of temperature) at this time t and not on the deformation history (and temperature history). Hence, the stress field of a Cauchy-elastic material is independent of the deformation path (independent of the time). However, note that the actual work done by the stress field on a Cauchy-elastic material does, in general, depend on the deformation path.

A **constitutive equation** (or **equation of state**) represents the intrinsic physical properties of a continuum body. It determines generally the state of stress at any point of that body to any arbitrary motion at time t. A constitutive equation is either regarded as mathematically generalized (axiomatic) or is based upon experimental data (empirical).

The constitutive equation of an isothermal elastic body relates the Cauchy stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ at each place $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ with the deformation gradient $\mathbf{F} = \mathbf{F}(\mathbf{X}, t)$. We may express the constitutive equation in the general form

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{g}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}) \quad , \tag{5.71}$$

where $\mathbf{g}$ is referred to as the **response function** associated with the Cauchy stress tensor $\boldsymbol{\sigma}$.

In equation (5.71), $\boldsymbol{\sigma}$ was allowed to depend upon the referential position $\mathbf{X} \in \Omega_0$ in addition to $\mathbf{F}$. Hence, the stress response varies from one particle to the other. However, for subsequent introductory treatments, it is convenient to restrict our attention on continuum bodies, in which both the Cauchy stress tensor $\boldsymbol{\sigma}$ and the reference mass density ρ_0 are independent of the position $\mathbf{X}$; such bodies are called **homogeneous**.

Hence, instead of (5.71), we write the constitutive equation in the form

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) \quad , \tag{5.72}$$

which determines the stresses $\boldsymbol{\sigma}$ from the given deformation gradient $\mathbf{F}$. From the *mechanical* point of view $\mathbf{g}$ characterizes the material properties of a (isothermal) Cauchy-elastic material, while from the *mathematical* point of view $\mathbf{g}$ is a tensor-valued function of one tensor variable $\mathbf{F}$. The concept of tensor functions, as we will use it here, is explained in Section 1.7. A constitutive equation of the type of (5.72) is often referred to as a **stress relation**.

Note that for homogeneous deformations the corresponding stresses are constant (since $\mathbf{F}$ has the same value at every point of the body) and, interestingly enough, Cauchy's equation of equilibrium (4.54) trivially reduces to $\mathrm{div}\boldsymbol{\sigma} = \mathbf{o}$. For this case the body force $\mathbf{b}$ is zero, which means that a homogeneous deformation of a continuum body occurs without body force.

Principle of material frame-indifference. As already mentioned in previous sections, constitutive equations must be objective (frame-indifferent) with respect to the Euclidean transformation (5.2). In other words, if a constitutive equation is satisfied for a dynamical process $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ then it must also be satisfied for any associated (equivalent) dynamical process $(\boldsymbol{\sigma}^+, \boldsymbol{\chi}^+)$ which is generated by the transformations (5.41) and (5.10). This is a fundamental axiom of mechanics which is known as the **principle of material frame-indifference** or the **principle of material objectivity** or simply as **objectivity** (see TRUESDELL and NOLL [1992, Sections 19, 19A]). If this principle is violated, the constitutive equations are affected by rigid-body motions and meaningless results are obtained.

To begin with, the material frame-indifference of the stress relation (5.72) imposes certain restrictions on the response function $\mathfrak{g}$. We consider a motion $\boldsymbol{\chi}^+$ which differs from $\boldsymbol{\chi}$ by a rigid-body motion superimposed on the current configuration (compare with Figure 5.2). The rigid-body motion maps the region Ω to a new region Ω^+ and the stress relation (5.72) to $\boldsymbol{\sigma}^+ = \mathfrak{g}(\mathbf{F}^+)$. We demand that both regions, namely Ω and Ω^+, are associated with the same function $\mathfrak{g}$ because it is for the *same* elastic material. Hence, using $(5.33)_3$ on the one hand and (5.41), (5.72) on the other hand, we arrive at

$$\boldsymbol{\sigma}^+ = \mathfrak{g}(\mathbf{F}^+) = \mathfrak{g}(\mathbf{QF}) \quad , \qquad \boldsymbol{\sigma}^+ = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathfrak{g}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \quad . \tag{5.73}$$

Combining $(5.73)_1$ and $(5.73)_2$, we find the restriction

$$\mathbf{Q}\mathfrak{g}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} = \mathfrak{g}(\mathbf{QF}) \tag{5.74}$$

on $\mathfrak{g}$ for every nonsingular $\mathbf{F}$ and orthogonal $\mathbf{Q}$. In other words, constitutive equation (5.72) is independent of the observer if the response function $\mathfrak{g}$ satisfies the invariance relation (5.74).

Employing the right polar decomposition $\mathbf{F} = \mathbf{RU}$ on the right-hand side of (5.74), we may write $\mathbf{Q}\mathfrak{g}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} = \mathfrak{g}(\mathbf{QRU})$, where $\mathbf{R}$ is the orthogonal rotation tensor and $\mathbf{U}$ the right stretch tensor. Since the latter relation holds for all proper orthogonal tensors $\mathbf{Q}$, it also holds for the special choice $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$. Hence, using the orthogonality condition $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, we obtain a corresponding *reduced* form of eq. (5.74), i.e.

$$\mathfrak{g}(\mathbf{F}) = \mathbf{R}\mathfrak{g}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \quad , \tag{5.75}$$

for the function $\mathfrak{g}$ and for every $\mathbf{F}$ and $\mathbf{R}$. Therefore, the associated stress relation reads

$$\boldsymbol{\sigma} = \mathbf{R}\mathfrak{g}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \quad , \tag{5.76}$$

which shows that the properties of an elastic material are independent of the rotational part of $\mathbf{F} = \mathbf{RU}$, characterized by $\mathbf{R}$. Note that the reduced constitutive equation (5.76) is compatible with the principle of material frame-indifference which can be shown as follows. By analogy with (5.76), let $\boldsymbol{\sigma}^+ = \mathbf{R}^+\mathfrak{g}(\mathbf{U}^+)(\mathbf{R}^+)^{\mathrm{T}}$ and use eq. (5.37) in order to obtain $\boldsymbol{\sigma}^+ = \mathbf{QR}\mathfrak{g}(\mathbf{U})\mathbf{R}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}$. Hence, by use of (5.76), we obtain once more $\boldsymbol{\sigma}^+ = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$ (compare with (5.41)).

An alternative form of constitutive equation (5.72) follows from Piola transformation (3.8). With (5.72) and the volume ratio $J = \det\mathbf{F}$, we obtain

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \det\mathbf{F}\mathfrak{g}(\mathbf{F})\mathbf{F}^{-\mathrm{T}} = \mathfrak{G}(\mathbf{F}) \quad , \tag{5.77}$$

where we have defined the tensor-valued tensor function $\mathfrak{G}$ associated with the first Piola-Kirchhoff stress tensor $\mathbf{P}$.

By analogy with the above we may now show the material frame-indifference of the stress relation $(5.77)_3$. Considering $\mathbf{P}^+ = \mathfrak{G}(\mathbf{F}^+)$ and, using $(5.33)_3$ on the one hand and (5.42) and $(5.77)_3$ on the other hand, we find that

$$\mathbf{P}^+ = \mathfrak{G}(\mathbf{F}^+) = \mathfrak{G}(\mathbf{QF}) \quad , \qquad \mathbf{P}^+ = \mathbf{QP} = \mathbf{Q}\mathfrak{G}(\mathbf{F}) \quad . \tag{5.78}$$

Equating $(5.78)_1$ and $(5.78)_2$, we find the invariant relation

$$\mathbf{Q}\mathfrak{G}(\mathbf{F}) = \mathfrak{G}(\mathbf{QF}) \tag{5.79}$$

for the function $\mathfrak{G}$ and for every $\mathbf{F}$ and $\mathbf{Q}$. Relations (5.74) and (5.79) are necessary and sufficient conditions for the constitutive equations (5.72) and $(5.77)_3$ to satisfy the principle of material frame-indifference.

A reduced form of constitutive equation $(5.77)_3$ is obtained from restriction (5.79). Setting $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ and replacing $\mathbf{F}$ by its right polar decomposition $\mathbf{RU}$ on the right-hand side of (5.79), we obtain, using $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$,

$$\mathfrak{G}(\mathbf{F}) = \mathbf{R}\mathfrak{G}(\mathbf{U}) \quad . \tag{5.80}$$

The restriction on $\mathfrak{G}$ expressed through eqs. (5.79) and (5.80) are equivalent to the restriction on $\mathfrak{g}$, as given in eqs. (5.74) and (5.75).

Another alternative form of the constitutive equation which turns out to be very useful in the theory of elasticity follows from relation $(3.65)_1$. With the volume ratio $J = \det\mathbf{U}$, i.e. $(2.96)_2$, the polar decomposition $\mathbf{F} = \mathbf{RU}$, the stress relation (5.76), and the fact that the right stretch tensor $\mathbf{U}$ is symmetric, we obtain

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \det\mathbf{U}\mathbf{U}^{-1}\mathfrak{g}(\mathbf{U})\mathbf{U}^{-1} \quad . \tag{5.81}$$

By recalling that $\mathbf{U}$ is the unique square root of the right Cauchy-Green tensor $\mathbf{C}$, we may write $\mathbf{C}^{1/2}$ in the place of $\mathbf{U}$. Defining a tensor-valued tensor function $\mathfrak{H}$ we may introduce finally the second Piola-Kirchhoff stress tensor $\mathbf{S}$ in the form

$$\mathbf{S} = \mathfrak{H}(\mathbf{C}) \quad . \tag{5.82}$$

Since the reference configuration is unaffected by superimposed rigid-body motions, we know that the second Piola-Kirchhoff stress tensor and the right Cauchy-Green tensor simply transform according to $\mathbf{S}^+ = \mathbf{S}$ and $\mathbf{C}^+ = \mathbf{C}$. We conclude that the stress relation (5.82) is independent of the observer.

EXAMPLE 5.5 Investigate if the elastic material given by

$$\boldsymbol{\sigma} = \mathfrak{I}(\mathbf{E}) \quad , \tag{5.83}$$

associated with motion $\boldsymbol{\chi}$, satisfies the principle of material frame-indifference.

Solution. For another motion $\boldsymbol{\chi}^+$ (recorded by a single observer O), assumption (5.83) implies $\boldsymbol{\sigma}^+ = \mathfrak{I}(\mathbf{E}^+)$. By recalling the transformations for the Cauchy stress tensor $\boldsymbol{\sigma}$ and the Green-Lagrange strain tensor $\mathbf{E}$, which satisfy eqs. (5.41) and $(5.43)_2$, and knowing that $\mathfrak{I}$ is the same function for the two (different) motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ (it is for the *same* elastic material), we conclude that

$$\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} = \mathfrak{I}(\mathbf{E}) \quad . \tag{5.84}$$

Note that this relation is only true for $\mathbf{Q} = \mathbf{I}$; thus, constitutive equation (5.83) does *not* satisfy the principle of material frame-indifference. ■

Isotropic Cauchy-elastic materials. We assume that the Cauchy stress tensor $\boldsymbol{\sigma}$ depends on the left Cauchy-Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$. The constitutive equation (5.72) may then be written in the alternative form

$$\boldsymbol{\sigma} = \mathfrak{h}(\mathbf{b}) \quad , \tag{5.85}$$

where $\mathfrak{h}$ is a tensor-valued function of the symmetric second-order tensor $\mathbf{b}$ associated with the Cauchy stress tensor $\boldsymbol{\sigma}$.

In order to find the restriction imposed on the response function $\mathfrak{h}$ by the assumption of material frame-indifference, let $\boldsymbol{\sigma}^+ = \mathfrak{h}(\mathbf{b}^+)$, where the response function $\mathfrak{h}$ is the same for the two motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$. We now use $(5.44)_1$ on the one hand and (5.41), (5.85) on the other hand in order to obtain

$$\boldsymbol{\sigma}^+ = \mathfrak{h}(\mathbf{b}^+) = \mathfrak{h}(\mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}}) \quad , \qquad \boldsymbol{\sigma}^+ = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathfrak{h}(\mathbf{b})\mathbf{Q}^{\mathrm{T}} \quad . \tag{5.86}$$

Combining eqs. $(5.86)_1$ and $(5.86)_2$ we find the fundamental invariance relation

$$\mathbf{Q}\mathfrak{h}(\mathbf{b})\mathbf{Q}^{\mathrm{T}} = \mathfrak{h}(\mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}}) \tag{5.87}$$

for the function $\mathfrak{h}$ and for every tensor $\mathbf{b}$ and orthogonal tensor $\mathbf{Q}$. Hence, constitutive equation (5.85) is independent of the observer if $\mathfrak{h}$ satisfies restriction (5.87).

A specific elastic material which may be described by the constitutive equation in the form (5.85), with property (5.87), is said to be **isotropic**. A tensor-valued function such as $\mathfrak{h}(\mathbf{b})$ is said to be **isotropic** if it satisfies relations of type (5.87). Hence, we refer to $\mathfrak{h}(\mathbf{b})$ as a **tensor-valued isotropic tensor function** of one variable $\mathbf{b}$.

From the physical point of view the condition of isotropy is expressed by the property that the material exhibits no preferred directions. In fact, the stress response of an isotropic elastic material is not affected by the choice of the reference configuration. For a piece of wood, for example, which is of cellular structure, the properties in the direction of the grain differ from those in other directions, so the material certainly is *not* isotropic.

The isotropic tensor function $\mathfrak{h}(\mathbf{b})$, which satisfies (5.87), may be represented in the explicit form

$$\boldsymbol{\sigma} = \mathfrak{h}(\mathbf{b}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{b} + \alpha_2\mathbf{b}^2 \quad , \qquad \alpha_a = \alpha_a[I_1(\mathbf{b}), I_2(\mathbf{b}), I_3(\mathbf{b})] \quad , \tag{5.88}$$

for each $\mathbf{b}$, which is known as the **Rivlin-Ericksen representation theorem**. For a proof of this crucial relation see RIVLIN and ERICKSEN [1955], TRUESDELL and NOLL [1992, Section 12], SPENCER [1980, Appendix] and GURTIN [1981a, pp. 233-235]. Note that the representation theorem (5.88) represents a fundamental requirement for the mathematical form of the stress relation.

Here, α_a, $a = 0, 1, 2$, are three scalar functions called **response coefficients** or **material functions**. Hence, in general, for an *isotropic* material only three parameters are needed in order to describe the stress state. The scalar functions α_a depend on the three invariants of tensor $\mathbf{b}$ and therefore on the current deformation state. The invariants are defined with respect to eqs. (1.170)–(1.172) as

$$I_1(\mathbf{b}) = \mathrm{tr}\mathbf{b} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad , \tag{5.89}$$

$$I_2(\mathbf{b}) = \frac{1}{2}\left[(\mathrm{tr}\mathbf{b})^2 - \mathrm{tr}(\mathbf{b}^2)\right] = \mathrm{tr}\mathbf{b}^{-1}\det\mathbf{b} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 \quad , \tag{5.90}$$

$$I_3(\mathbf{b}) = \det\mathbf{b} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2 \quad , \tag{5.91}$$

where λ_a^2 are the three eigenvalues of the symmetric spatial tensor $\mathbf{b}$, see eq. (2.119). In eq. (5.91), relation $(2.81)_2$ was used.

Representation (5.88) is the *most general form* of a constitutive equation for isotropic elastic materials also known as the **first representation theorem for isotropic tensor functions**. From the constitutive equation (5.88) we deduce that the principal directions of the Cauchy stress tensor $\boldsymbol{\sigma}$ and the left Cauchy-Green tensor $\mathbf{b}$ coincide. Hence, for isotropic elastic materials the two symmetric tensors $\boldsymbol{\sigma}$ and $\mathbf{b}$ are said to be **coaxial** in every configuration.

By analogy with (5.88), the constitutive equation

$$\boldsymbol{\sigma} = \overline{\alpha}_0 \mathbf{I} + \overline{\alpha}_1 \mathbf{d} + \overline{\alpha}_2 \mathbf{d}^2 \ , \qquad \overline{\alpha}_a = \overline{\alpha}_a[\rho, I_1(\mathbf{d}), I_2(\mathbf{d}), I_3(\mathbf{d})] \ , \tag{5.92}$$

characterizes the behavior of a viscous fluid, in particular, of a so-called **Reiner-Rivlin fluid**. Therein, ρ and $\mathbf{d}$ are the spatial mass density and the rate of deformation tensor, while $\overline{\alpha}_a$, $a = 0, 1, 2$, are scalar functions of the invariants I_a, $a = 1, 2, 3$, given in (5.89)–(5.91) with $\mathbf{d}$ replacing $\mathbf{b}$.

In order to find an alternative explicit representation for (5.88), we recall the *Cayley-Hamilton* equation (1.174). Since any tensor satisfies its own characteristic equation, we may write (1.174) as $\mathbf{b}^3 - I_1\mathbf{b}^2 + I_2\mathbf{b} - I_3\mathbf{I} = \mathbf{O}$ and find, by multiplying this equation with $\mathbf{b}^{-1}$, that

$$\mathbf{b}^2 = I_1\mathbf{b} - I_2\mathbf{I} + I_3\mathbf{b}^{-1} \ . \tag{5.93}$$

Eliminating $\mathbf{b}^2$ from (5.88) in favor of $\mathbf{b}^{-1}$ we obtain an alternative representation of a constitutive equation for isotropic elastic materials, i.e.

$$\boldsymbol{\sigma} = \mathfrak{h}(\mathbf{b}) = \beta_0\mathbf{I} + \beta_1\mathbf{b} + \beta_{-1}\mathbf{b}^{-1} \ , \qquad \beta_a = \beta_a[I_1(\mathbf{b}), I_2(\mathbf{b}), I_3(\mathbf{b})] \ , \tag{5.94}$$

where β_a, $a = 0, 1, -1$, are three scalar functions (response coefficients) which, in terms of the three invariants of $\mathbf{b}$, are expressed as

$$\beta_0 = \alpha_0 - I_2\alpha_2 \ , \qquad \beta_1 = \alpha_1 + I_1\alpha_2 \ , \qquad \beta_{-1} = I_3\alpha_2 \ . \tag{5.95}$$

Representation (5.94) is also known as the **second representation theorem for isotropic tensor functions**, see, for example, GURTIN [1981a, p. 235].

Incompressible Cauchy-elastic materials. If the Cauchy-elastic material is **incompressible**, then the stress relation is determined only up to an arbitrary scalar p which can be identified as a pressure-like quantity. The constitutive equations (5.72), (5.76) and (5.82) are then replaced by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{g}(\mathbf{F}) = -p\mathbf{I} + \mathbf{R}\mathbf{g}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \ , \tag{5.96}$$

$$\mathbf{S} = -p\mathbf{C}^{-1} + \mathfrak{H}(\mathbf{C}) \ , \tag{5.97}$$

where the tensor-valued tensor functions $\mathbf{g}(\mathbf{F}), \mathbf{g}(\mathbf{U})$ and $\mathfrak{H}(\mathbf{C})$ need only be defined for the kinematic constraints $\det\mathbf{F} = 1, \det\mathbf{U} = 1$ and $\det\mathbf{C} = 1$, respectively. The indeterminate terms $-p\mathbf{I}$ and $-p\mathbf{C}^{-1}$ are known as **reaction stresses**, which do no work in any motion compatible with above constraints. For incompressible materials the (indeterminate) scalar p required to maintain incompressibility may only be found by means of the equilibrium conditions and the boundary conditions and is *not* specified by a constitutive equation. Note that the scalar p must always be included in a stress relation of an incompressible material.

In an incompressible and isotropic Cauchy-elastic material we replace constitutive equations $(5.88)_1$ and $(5.94)_1$ by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \alpha_1\mathbf{b} + \alpha_2\mathbf{b}^2 \ , \qquad \boldsymbol{\sigma} = -p\mathbf{I} + \beta_1\mathbf{b} + \beta_{-1}\mathbf{b}^{-1} \ , \tag{5.98}$$

respectively, where the response coefficients α_1, α_2 and β_1, β_{-1} depend now only on the two scalar invariants I_1 and I_2 (since $I_3 = \det\mathbf{b} = 1$). Note that the scalars p in eqs. (5.98) differ by the term $\alpha_2 I_2$. For an incompressible and isotropic Cauchy-elastic material the stresses given in (5.98) are determined only up to p. The scalar functions α_0 and β_0 multiplying $\mathbf{I}$ in $(5.88)_1$ and $(5.94)_1$ are absorbed into the reaction stresses. The response coefficients in (5.98) are related by

$$\alpha_1 = \beta_1 - I_1\beta_{-1} \ , \qquad \alpha_2 = \beta_{-1} \ . \tag{5.99}$$

Two special cases result directly from $(5.98)_2$, i.e. the so-called **Mooney-Rivlin model** for incompressible materials, for which β_1 and β_{-1} are constants, and the **neo-Hookean model** for incompressible materials, for which β_1 is constant and $\beta_{-1} = 0$. For a study of these types of material see more in Section 6.5.

EXERCISES

1. Consider the classical **Newtonian fluid**, for which the viscous stress depends linearly on the rate of deformation tensor $\mathbf{d}$. It is the simplest model for a viscous fluid and is given by the constitutive equation

$$\boldsymbol{\sigma} = [-p(\rho) + \lambda(\rho)\mathrm{tr}\mathbf{d}]\mathbf{I} + 2\eta(\rho)\mathbf{d} \ ,$$

characterizing (low molecular weight) liquids and gases such as water, oil or air. Therein, the function $p(\rho)$ depends on the spatial mass density ρ, and λ and η are two parameters characterizing the **viscosity** of the Newtonian fluid.

   (a) Show that the response of this type of fluid conforms to the principle of material frame-indifference.

   (b) Apply the Newtonian constitutive equation to a motion which causes simple shear deformation (see eq. (2.3)). Take a constant viscosity η and show that the only non-vanishing shear stress σ_{12} is

$$\sigma_{12} = \sigma_{21} = \eta\dot{c} \ , \tag{5.100}$$

   with the shear rate $\dot{c}$ (compare with Exercise 4(a),(b) on p. 105).

Note that the Newtonian viscous fluid is simply a special case of the Reiner-Rivlin fluid (5.92) obtained by choosing the response coefficients $\overline{\alpha}_a$, $a = 0, 1, 2$, as $\overline{\alpha}_0 = -p(\rho) + \lambda(\rho)\mathrm{tr}\mathbf{d}$, $\overline{\alpha}_1 = 2\eta(\rho)$ and $\overline{\alpha}_2 = 0$, respectively. However, by setting $\overline{\alpha}_0 = -p(\rho)$ and $\overline{\alpha}_1 = \overline{\alpha}_2 = 0$ we obtain the constitutive equation $\boldsymbol{\sigma} = -p(\rho)\mathbf{I}$ characterizing the material properties of an **elastic fluid**.

2. By recalling the transformation rules (5.41), (5.31) and $(5.40)_2$, $(5.45)_1$ (from Sections 5.1 and 5.2), show that the constitutive equations

$$\boldsymbol{\sigma} = \mathrm{grad}\mathbf{a} + \mathrm{grad}^{\mathrm{T}}\mathbf{a} + 2\mathbf{l}^{\mathrm{T}}\mathbf{l} \ , \qquad \dot{\boldsymbol{\sigma}} = \mathbf{l}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} + \alpha\mathbf{d}$$

   are acceptable forms which conform to the principle of material frame-indifference, where α is a material parameter and $\mathbf{a}, \mathbf{l}$ and $\mathbf{d}$ denote the spatial acceleration field, the spatial velocity gradient and the rate of deformation tensor, respectively.

3. Consider a material which is isotropic and Cauchy-elastic. Let a uniform extension (or compression) in all three directions according to relation (2.131), which corresponds to a triaxial stress state, be given.

   By determining the deformation gradient $\mathbf{F}$ and the left Cauchy-Green tensor $\mathbf{b}$ (with respect to a set of some orthonormal basis vectors $\mathbf{e}_a$) find the most general representation for the Cauchy stress tensor $\boldsymbol{\sigma}$.

4. By means of a pull-back of (5.88) to the reference configuration and a scaling with the inverse volume ratio J^{-1}, show that for isotropic elastic materials the second Piola-Kirchhoff stress tensor $\mathbf{S}$ is coaxial with the right Cauchy-stress tensor $\mathbf{C}$ *if and only if* $\boldsymbol{\sigma}$ is coaxial with $\mathbf{b}$.

6 Hyperelastic Materials

The fundamental equations introduced in Chapters 2-4 are essential to characterize kinematics, stresses and balance principles, and hold for any continuum body for all times. However, they do not distinguish one material from another and remain valid in all branches of continuum mechanics.

For the case of deformable bodies the equations mentioned are certainly not sufficient on their own to determine the material response. Hence, we must establish additional equations in the form of appropriate *constitutive laws* which are furnished to specify the *ideal material* in question. A constitutive law should approximate the observed physical behavior of a *real material* under specific conditions of interest.

Generally we use a functional relationship as a *constitutive equation* and this enables us to specify the stress components in terms of other field functions such as strain and temperature. A constitutive equation determines the state of stress at any point $\mathbf{x}$ of a continuum body at time t and is necessarily different for different types of continuous bodies.

Each field of continuum mechanics deals with certain continuous media including **fluids**, which are liquids or gases (such as water, oil, air etc.) and **solids** (such as rubber, metal, ceramics, wood, living tissue etc.). If the constitutive equations are valid for physical objects such as fluids we call the field of continuum mechanics **fluid mechanics**. Another important field in which constitutive equations are valid for solids is known as **solid mechanics**. Note that fluid and solid mechanics differ only with respect to constitutive equations, but they share the same set of field equations.

The main goal of the next two chapters is to study various constitutive equations within the field of solid mechanics appropriate for approximation techniques such as the finite element method. For the most part we follow the so-called **phenomenological approach**, describing the macroscopic nature of materials as continua. The phenomenological approach is mainly concerned with fitting mathematical equations to experimental data and is particularly successful in solid mechanics (such as classical elastoplasticity). However, phenomenological modeling is not capable of relating the mechanism of deformation to the underlying physical (microscopic) structure of the material.

In Chapter 6 we discuss phenomenological constitutive equations which interrelate the stress components and the strain components within a nonlinear regime. Since we are studying so-called purely *mechanical* theories, thermodynamic variables such as the entropy and the temperature are ignored. However, they are taken into account and elaborated on within Chapter 7. No attempt is made to present a comprehensive list of the various important contributions on constitutive modeling to date. This text discusses some selected material models essential in science, industrial engineering practice and in the field of biomechanics, where structures exhibit large strain behavior, very often within the coupled thermodynamic regime.

Sections 6.1-6.8 consider hyperelastic materials in general, and cover a wide range of important types of material such as isotropic and transversely isotropic materials, incompressible and compressible hyperelastic materials and composite materials, in particular. Some important specifications for rubber-like (or other) materials are also presented. The remaining Sections 6.9-6.11 focus attention on inelastic materials. Based on the concept of internal variables, viscoelastic materials and isotropic hyperelastic materials with damage at finite strains are introduced. Plastic and viscoplastic materials which have the ability to undergo irreversible or permanent deformations are not considerd in this text.

6.1 General Remarks on Constitutive Equations

It is the aim of **constitutive theories** to develop mathematical models for representing the real behavior of matter. Constitutive theories of materials are very important but they are a difficult subject in modern nonlinear continuum mechanics. We make no attempt to conduct a comprehensive review of the large number of constitutive theories. For more on formulating nonlinear constitutive theories see, for example, TRUESDELL and NOLL [1992] and the excellent contributions by *Rivlin* in the 1940s and 1950s collected by BARENBLATT and JOSEPH [1997].

In particular, we present a nonlinear constitutive theory suitable to describe a wide variety of physical phenomena in which the strains may be large, i.e. finite. For the case of a (hyper)elastic material the resulting theory is called **finite (hyper)elasticity theory** or just **finite (hyper)elasticity** for which nonlinear continuum mechanics is the fundamental basis (see GREEN and ADKINS [1970] for an analytical treatment and inter alia LE TALLEC [1994] for numerical solution techniques).

Constitutive equations for hyperelastic materials. A so-called **hyperelastic material** (or in the literature often called a **Green-elastic material**) postulates the existence of a **Helmholtz free-energy function** Ψ, which is defined per unit reference *volume* rather than per unit *mass*.

For the case in which $\Psi = \Psi(\mathbf{F})$ is solely a function of $\mathbf{F}$ or some strain tensor, as introduced in Section 2.4, the Helmholtz free-energy function is referred to as the **strain-energy function** or **stored-energy function** (see Section 4.4, p. 159). Subsequently, we often use the common terminology **strain energy** or **stored energy**. The strain-energy function $\Psi = \Psi(\mathbf{F})$ is a typical example of a scalar-valued function of one tensor variable $\mathbf{F}$, which we assume to be continuous.

A concept of importance in elasticity is **polyconvexity** of strain-energy functions. The global existence theory of solutions, for example, is based on the condition of polyconvexity of strain-energy functions. For an extensive discussion on the underlying issue, see BALL [1977], CIARLET [1988, Chapters 4, 7], MARSDEN and HUGHES [1994, Section 6.4] and ŠILHAVÝ [1997, Sections 17.5, 18.5].

We now restrict attention to **homogeneous materials** in which the distributions of the internal constituents are assumed to be uniform on the continuum scale. For this type of ideal material the strain-energy function Ψ depends only upon the deformation gradient $\mathbf{F}$. Of course, for so-called **heterogeneous materials** (a material that is not homogeneous) Ψ will depend additionally upon the position of a point in the medium.

A hyperelastic material is defined as a subclass of an elastic material, as given in eqs. $(5.77)_3$ and (5.72), whose response functions $\mathfrak{G}$ and $\mathfrak{g}$ have physical expressions of the form

$$\mathbf{P} = \mathfrak{G}(\mathbf{F}) = \frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}} \qquad \text{or} \qquad P_{aA} = \frac{\partial\Psi}{\partial F_{aA}} \ , \tag{6.1}$$

and by use of relation (3.9) for the symmetric Cauchy stress tensor, i.e. $\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}} = \boldsymbol{\sigma}^{\mathrm{T}}$,

$$\begin{aligned} \boldsymbol{\sigma} = \mathfrak{g}(\mathbf{F}) &= J^{-1}\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\mathbf{F}^{\mathrm{T}} = J^{-1}\mathbf{F}\left(\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\right)^{\mathrm{T}} \\ \text{or} \qquad \sigma_{ab} &= J^{-1}F_{Aa}\frac{\partial\Psi}{\partial F_{Ab}} = J^{-1}F_{aA}\frac{\partial\Psi}{\partial F_{bA}} \ . \end{aligned} \tag{6.2}$$

These types of equation we already know as (purely mechanical) constitutive equations (or equations of state). They establish an axiomatic or empirical model as the basis for approximating the behavior of a real material. Such a model we call a **material model** or a **constitutive model**. As is clear from the constitutive equations (6.1) and (6.2) the stress response of hyperelastic materials is derived from a given scalar-valued energy function, which implies that hyperelasticity has a conservative structure.

The derivative of the scalar-valued function Ψ with respect to the tensor variable $\mathbf{F}$ determines the gradient of Ψ and is understood according to the definition introduced in (1.239). It is a second-order tensor which we know as the first Piola-Kirchhoff stress tensor $\mathbf{P}$. The derivation requires that the component function $\Psi(F_{aA})$ is differentiable with respect to all components F_{aA}.

A so-called **perfectly elastic material** is by definition a material which produces locally no entropy (see TRUESDELL and NOLL [1992, Section 80]). In other words, we use subsequently the term 'perfectly' for a certain class of materials which has the special merits that for every admissible process the internal dissipation $\mathcal{D}_{\text{int}}$ is zero (naturally, damage, viscous mechanisms and plastic deformations are excluded). We will consider perfectly elastic materials up to Section 6.9.

We may derive the constitutive equation (6.1) directly from the Clausius-Planck form of the second law of thermodynamics (4.154) which degenerates to an equality for the class of perfectly elastic materials. With the expression (4.154) for $\mathcal{D}_{\text{int}}$ the time differentiation of the strain-energy function, i.e. $\dot{\Psi}(\mathbf{F}) = \partial\Psi(\mathbf{F})/\partial\mathbf{F} : \dot{\mathbf{F}}$, gives

$$\mathcal{D}_{\text{int}} = \mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} = \left(\mathbf{P} - \frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\right) : \dot{\mathbf{F}} = 0 \quad , \tag{6.3}$$

at every point of the continuum body and for all times during the process.

As $\mathbf{F}$ and hence $\dot{\mathbf{F}}$ can be chosen arbitrarily, the expressions in parentheses must be zero. Therefore, as a consequence of the second law of thermodynamics, the physical expression (6.1) holds. We often say that $\mathbf{P}$ is the **thermodynamic force** *work conjugate* to $\mathbf{F}$. This procedure goes back to COLEMAN and NOLL [1963] and COLEMAN and GURTIN [1967], and in the literature is sometimes referred to as the **Coleman-Noll procedure**.

For convenience, throughout this text we require that the strain-energy function vanishes in the reference configuration, i.e. where $\mathbf{F} = \mathbf{I}$. We express this assumption by the **normalization condition**

$$\Psi = \Psi(\mathbf{I}) = 0 \quad . \tag{6.4}$$

From the physical observation we know that the strain-energy function Ψ increases with deformation. In addition to (6.4), we therefore require that

$$\Psi = \Psi(\mathbf{F}) \geq 0 \quad , \tag{6.5}$$

which restricts the ranges of admissible functions occurring in expressions for the strain energy.

The strain-energy function Ψ attains its *global* minimum for $\mathbf{F} = \mathbf{I}$ at thermodynamic equilibrium (in fact, from (6.4), $\Psi(\mathbf{I})$ is zero). We assume that Ψ has no other stationary points in the strain space. Relations (6.4) and (6.5) ensure that the stress in the reference configuration, which we call the **residual stress**, is zero. We say that the reference configuration is **stress-free**.

For the behavior at finite strains we require additionally that the scalar-valued function Ψ must satisfy so-called **growth conditions**. This implies that Ψ tends to $+\infty$ if either $J = \det\mathbf{F}$ approaches $+\infty$ or 0^+, i.e.

$$\left.\begin{aligned} \Psi(\mathbf{F}) &\to +\infty \qquad \text{as} \qquad \det\mathbf{F} \to +\infty \quad , \\ \Psi(\mathbf{F}) &\to +\infty \qquad \text{as} \qquad \det\mathbf{F} \to 0^+ \quad . \end{aligned}\right\} \tag{6.6}$$

Physically, that means that we would require an infinite amount of strain energy in order to expand a continuum body to the infinite range or to compress it to a point with vanishing volume.

For further discussions see the books by, for example, CIARLET [1988] and OGDEN [1997].

Equivalent forms of the strain-energy function. In order to illustrate Ψ we imagine a stretched (rubber) band with a certain amount of energy stored. The strain energy $\Psi(\mathbf{F})$ generated by the motion $\mathbf{x} = \chi(\mathbf{X}, t)$ is assumed to be *objective*. This means, after a (possibly time-dependent) *translation* and *rotation* of the stretched (rubber) band in space, that the amount of energy stored is unchanged.

Hence, the strain energy $\Psi(\mathbf{F})$ must be equal to the strain energy $\Psi(\mathbf{F}^+)$ generated by a second motion $\mathbf{x}^+ = \chi^+(\mathbf{X}, t^+)$ which differs from χ by a superimposed rigid-body motion (recall Section 5.2). Employing the transformation rule for the deformation gradient $(5.33)_3$, we see that Ψ cannot be an arbitrary function of $\mathbf{F}$. In particular, it must obey the restriction

$$\Psi(\mathbf{F}) = \Psi(\mathbf{F}^+) = \Psi(\mathbf{QF}) \tag{6.7}$$

for all tensors $\mathbf{F}$, with $\det\mathbf{F} > 0$, and for all orthogonal tensors $\mathbf{Q}$, since $\mathbf{F}$ transforms into $\mathbf{QF}$, i.e. $F^+_{aA} = Q_{ab}F_{bA}$.

In order to obtain equivalent formulations of (6.7) we take a special choice for $\mathbf{Q}$, namely the transpose of the proper orthogonal rotation tensor, $\mathbf{R}^{\mathrm{T}}$, and use the right polar decomposition $(2.93)_1$. Then, from (6.7), we find that $\Psi(\mathbf{F}) = \Psi(\mathbf{R}^{\mathrm{T}}\mathbf{F}) = \Psi(\mathbf{R}^{\mathrm{T}}\mathbf{RU})$, and finally,

$$\Psi(\mathbf{F}) = \Psi(\mathbf{U}) \quad , \tag{6.8}$$

which holds for arbitrary $\mathbf{F}$.

From (6.8) we learn that Ψ is independent of the rotational part of $\mathbf{F} = \mathbf{RU}$. We conclude that a hyperelastic material depends only on the stretching part of $\mathbf{F}$, i.e. the symmetric right stretch tensor $\mathbf{U}$. It is important to note that the relation $\Psi(\mathbf{F}) = \Psi(\mathbf{U})$ specifies the necessary and sufficient condition for the strain energy to be objective during superimposed rigid-body motions.

Since the right Cauchy-Green tensor and the Green-Lagrange strain tensor are given by $\mathbf{C} = \mathbf{U}^2$ and $\mathbf{E} = (\mathbf{U}^2 - \mathbf{I})/2$, we may express Ψ as a function of the six components C_{AB}, E_{AB} of the symmetric material tensors $\mathbf{C}, \mathbf{E}$, respectively. Hence, we may write

$$\Psi(\mathbf{F}) = \Psi(\mathbf{C}) = \Psi(\mathbf{E}) \quad . \tag{6.9}$$

For notational simplicity, here and elsewhere we will use the same Greek letter Ψ for different strain-energy functions.

Reduced forms of constitutive equations. In the following we present some reduced forms of constitutive equations for hyperelastic materials at finite strains.

Consider the derivative of the strain-energy function $\Psi(\mathbf{F}) = \Psi(\mathbf{C})$ with respect to time t. By means of the chain rule of differentiation, property $(1.93)_1$ and the combination of $(2.168)_1$ and $(2.165)_1$, we obtain the expressions

$$\begin{aligned}\dot{\Psi} &= \mathrm{tr}\left[\left(\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\right)^{\mathrm{T}}\dot{\mathbf{F}}\right] = \mathrm{tr}\left[\left(\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}}\right)\dot{\mathbf{C}}\right] \\ &= \mathrm{tr}\left[\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}}\left(\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}}\right)\right] = 2\,\mathrm{tr}\left(\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}}\right) \quad , \end{aligned} \tag{6.10}$$

which must be valid for arbitrary tensors $\dot{\mathbf{F}}$. Since $\mathbf{C}$ is a symmetric second-order tensor, the gradient of the scalar-valued tensor function $\Psi(\mathbf{C})$, used in (6.10), is also symmetric. From (6.10) we deduce immediately that

$$\left(\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\right)^{\mathrm{T}} = 2\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}} \quad , \tag{6.11}$$

which, when substituted back into $(6.2)_3$, gives an important reduced form of the constitutive equation for hyperelastic materials, namely

$$\begin{aligned}\boldsymbol{\sigma} &= J^{-1}\mathbf{F}\left(\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\right)^{\mathrm{T}} = 2J^{-1}\mathbf{F}\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}} \\ \text{or} \quad \sigma_{ab} &= J^{-1}F_{aA}\frac{\partial\Psi}{\partial F_{bA}} = 2J^{-1}F_{aA}F_{bB}\frac{\partial\Psi}{\partial C_{AB}} \quad . \end{aligned} \tag{6.12}$$

Alternative expressions may be obtained for the Piola-Kirchhoff stress tensors $\mathbf{P}$ (which is non-symmetric) and $\mathbf{S}$ (which is symmetric). From (3.8) and $(3.65)_1$ we find, by means of the stress relation $(6.12)_2$, the chain rule and $2\mathbf{E} = \mathbf{C} - \mathbf{I}$, that

$$\begin{aligned} &\mathbf{P} = 2\mathbf{F}\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}} \quad , \qquad \mathbf{S} = 2\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}} = \frac{\partial\Psi(\mathbf{E})}{\partial\mathbf{E}} \\ \text{or} \quad &P_{aA} = 2F_{aB}\frac{\partial\Psi}{\partial C_{AB}} \quad , \qquad S_{AB} = 2\frac{\partial\Psi}{\partial C_{AB}} = \frac{\partial\Psi}{\partial E_{AB}} \quad . \end{aligned} \tag{6.13}$$

Note that the response function occurring in the general constitutive equation (5.82) is with reference to $(6.13)_2$ determined by $\mathfrak{H}(\mathbf{C}) = 2\partial\Psi(\mathbf{C})/\partial\mathbf{C}$.

The Eshelby tensor and the tensor of chemical potential. The **Eshelby tensor** (or the **(elastic) energy-momentum tensor**) is a crucial quantity in *fracture mechanics* and the *continuum theory of dislocations* (which are not discussed in this text). However, for completeness we present the **(isothermal) referential Eshelby tensor G**, which is, in general, non-symmetric and is defined as

$$\mathbf{G} = -J\mathbf{F}^{\mathrm{T}}\frac{\partial}{\partial\mathbf{F}}\left(\frac{\Psi(\mathbf{F})}{J}\right) \qquad \text{or} \qquad G_{AB} = -JF_{aA}\frac{\partial}{\partial F_{aB}}\left(\frac{\Psi}{J}\right) \tag{6.14}$$

(see ESHELBY [1975] and CHADWICK [1975]), where $J = \det\mathbf{F}$ denotes the volume ratio. The physical dimension of the Eshelby tensor is the same as that of the strain energy.

The symmetric **tensor of chemical potential k** is related to **G** according to

$$\mathbf{k} = \mathbf{F}^{-\mathrm{T}}\mathbf{G}\mathbf{F}^{\mathrm{T}} \qquad \text{or} \qquad k_{ab} = F_{Aa}^{-1}G_{AB}F_{bB} \tag{6.15}$$

see, for example, BOWEN [1976a].

The spatial tensor of chemical potential is required in the theory of *diffusing mixtures* (see TRUESDELL [1984] which contains more details).

Work done on hyperelastic materials. We consider a dynamical process within some closed **time interval** denoted by $[t_1, t_2]$ in which the two arbitrary instants t_1 and t_2 are elements of the interval. The dynamical process is given by a motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ and the stress $\boldsymbol{\sigma}$ with the corresponding Cauchy traction vector $\mathbf{t}$ and the body force $\mathbf{b}$. During the process the body deforms according to the deformation gradient $\mathbf{F} = \mathbf{F}(t)$, with $t \in [t_1, t_2]$.

We say that a dynamical process is **closed** if $\mathbf{F}_1 = \mathbf{F}_2$. We introduce the definitions $\mathbf{F}_1 = \mathbf{F}(t_1)$ and $\mathbf{F}_2 = \mathbf{F}(t_2)$ of the deformation gradients in the **initial configuration** and the **final configuration** of the dynamical process, respectively.

Our next step is to determine the work done by the stress field on a continuum body of unit volume during a certain time interval $[t_1, t_2]$. Consider a body whose material properties are hyperelastic according to the general constitutive equation given in (6.1). Hence, from $(4.101)_1$ and the above definitions we find by means of the chain rule that

$$\int_{t_1}^{t_2}\mathbf{P} : \dot{\mathbf{F}}\,\mathrm{d}t = \int_{t_1}^{t_2}\frac{\partial\Psi}{\partial\mathbf{F}} : \dot{\mathbf{F}}\,\mathrm{d}t = \int_{t_1}^{t_2}\frac{\mathrm{D}\Psi(\mathbf{F})}{\mathrm{D}t}\,\mathrm{d}t = \Psi(\mathbf{F}_2) - \Psi(\mathbf{F}_1) \quad , \tag{6.16}$$

which, for a closed dynamical process with $\mathbf{F}_1 = \mathbf{F}_2$, reduces to

$$\int_{t_1}^{t_2}\mathbf{P} : \dot{\mathbf{F}}\,\mathrm{d}t = \Psi(\mathbf{F}_2) - \Psi(\mathbf{F}_1) = 0 \quad . \tag{6.17}$$

Thus, as distinct from Cauchy-elastic materials (see Section 5.4), the actual work done by the stress field on a hyperelastic material during a certain (closed) time interval depends only on the *initial* and *final* configurations (path independent). In fact, the work is *zero* in closed dynamical processes. This important result also holds for continuum bodies which may undergo inhomogeneous deformations, in which $\Psi = \Psi(\mathbf{F}, \mathbf{X})$ and $\rho_0 = \rho_0(\mathbf{X})$, with $\mathbf{F} = \mathbf{F}(\mathbf{X}, t)$.

EXERCISES

1. Expand the strain-energy function $\Psi(\mathbf{F})$ in the form of tensorial polynomials in the vicinity of the reference configuration, i.e. for $\mathbf{F} = \mathbf{I}$,

$$\Psi(\mathbf{F}) = \Psi(\mathbf{I}) + (\mathbf{F} - \mathbf{I}) : \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} + \cdots \geq 0$$

for all $\mathbf{F} - \mathbf{I}$. Using relations (6.4) and (6.5) show that the stress in the reference configuration is zero.

2. Recall the definitions of the referential Eshelby tensor $\mathbf{G}$ and the tensor of chemical potential $\mathbf{k}$.

   (a) Show that the forms

$$\mathbf{G} = \Psi(\mathbf{F})\mathbf{I} - J\mathbf{F}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} \ , \qquad \mathbf{k} = \Psi(\mathbf{F})\mathbf{I} - \boldsymbol{\tau} \tag{6.18}$$

   are equivalent to those given in (6.14) and (6.15), where $\boldsymbol{\tau}$ is the Kirchhoff stress tensor, as defined in eq. (3.62).

   (b) By applying requirement (3.10), show with $(6.18)_1$ that

$$\mathbf{GC} = \mathbf{CG}^{\mathrm{T}} \ ,$$

   with the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, as defined in eq. (2.66).

6.2 Isotropic Hyperelastic Materials

We now restrict the strain-energy function by a particular property that the material may possess, namely *isotropy*. This property is based on the physical idea that the response of the material, when studied in a stress-strain experiment, is the same in all directions. One example of an (approximately) isotropic material with a wide range of applications is rubber.

In this section we are concerned with the mathematical formulation of isotropy within the context of hyperelasticity.

Scalar-valued isotropic tensor function. We consider an arbitrary point $\mathbf{X}$ of an elastic continuum body occupying the region Ω_0 (reference configuration) at time $t = 0$. A motion χ may carry this point $\mathbf{X} \in \Omega_0$ to a place $\mathbf{x} = \chi(\mathbf{X}, t)$ specifying a location in the region Ω (current configuration) at time t.

We now study the effect of a rigid-body motion superimposed on the *reference configuration*. We postulate that the body occupying the region Ω_0 is translated by the vector $\mathbf{c}$ and rotated by the orthogonal tensor $\mathbf{Q}$ according to

$$\mathbf{X}^{\star} = \mathbf{c} + \mathbf{QX} \ , \tag{6.19}$$

which moves Ω_0 to a new region $\Omega_0^{\star}$ (new reference configuration), and the arbitrary point with position vector $\mathbf{X}$ to a new location identified by the position vector $\mathbf{X}^{\star} \in \Omega_0^{\star}$ (see Figure 6.1).

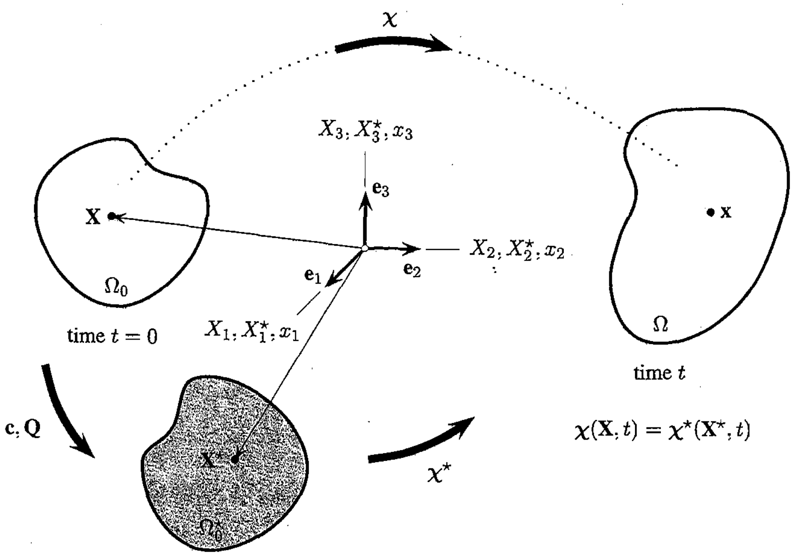

Figure 6.1 Rigid-body motion superimposed on the reference configuration.

We now demand that a different motion $\mathbf{x} = \chi^{\star}(\mathbf{X}^{\star}, t)$ moves $\Omega_0^{\star}$ to the current configuration Ω so that

$$\mathbf{x} = \chi(\mathbf{X}, t) = \chi^{\star}(\mathbf{X}^{\star}, t) \ , \tag{6.20}$$

mapping $\mathbf{X}^{\star}$ to place $\mathbf{x}$. By the chain rule and relation (6.19) the deformation gradient

$\mathbf{F}$ may be expressed as

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}^\star}\mathbf{Q} = \mathbf{F}^\star\mathbf{Q} \\ \text{or} \qquad F_{aA} &= \frac{\partial x_a}{\partial X_A} = \frac{\partial x_a}{\partial X^\star_B} Q_{BA} = F^\star_{aB} Q_{BA} \quad , \end{aligned} \tag{6.21}$$

where $\mathbf{F}^\star = \partial\mathbf{x}/\partial\mathbf{X}^\star$ is defined to be the deformation gradient relative to the region $\Omega_0^\star$. From $(6.21)_3$ we find the important transformation, namely

$$\mathbf{F}^\star = \mathbf{F}\mathbf{Q}^{\mathrm{T}} \qquad \text{or} \qquad F^\star_{aA} = F_{aB} Q_{AB} \quad . \tag{6.22}$$

Hence, we say that a hyperelastic material is **isotropic** relative to the reference configuration Ω_0 if the values of the strain energy $\Psi(\mathbf{F})$ and $\Psi(\mathbf{F}^\star)$ are the same for all orthogonal tensors $\mathbf{Q}$. With (6.22) we may write

$$\Psi(\mathbf{F}) = \Psi(\mathbf{F}^\star) = \Psi(\mathbf{F}\mathbf{Q}^{\mathrm{T}}) \quad . \tag{6.23}$$

In other words, if we can show that a motion of an elastic body superimposed on any particularly translated and/or rotated reference configuration leads to the same strain-energy function at time t, then the material is said to be isotropic. However, if a superimposed rigid-body motion changes the strain-energy function in the sense that (6.23) is not satisfied ($\Psi(\mathbf{F}) \neq \Psi(\mathbf{F}^\star)$) the hyperelastic material is said to be **anisotropic** (see, for example, TRUESDELL and NOLL [1992, Section 33] and OGDEN [1997, Section 4.2.5]).

It is important to mention that relation (6.23) is *fundamentally distinct* from requirement (6.7), which says that the strain energy must be objective during rigid-body motions, i.e. independent of an observer. The later condition holds for **all** materials (it is a fundamental physical requirement and must be satisfied) while the condition of isotropic response (6.23) holds only for **some** materials (it is a material-dependent requirement and may or may not be satisfied), namely for *isotropic materials*, which makes a crucial difference.

In addition, it is important to note that for the material-dependent requirements (6.22) and (6.23) it is the *reference* configuration that has been translated and rotated. For that case the deformation gradient $\mathbf{F}$ is multiplied on the *right* by $\mathbf{Q}^{\mathrm{T}}$, whereby $\mathbf{Q}$ acts with material coordinates, i.e. Q_{AB}. However, for the objectivity requirements $(5.33)_3$ and (6.7), it is the *current* configuration that has been translated and rotated, and $\mathbf{F}$ is multiplied on the *left* by $\mathbf{Q}$, acting with spatial coordinates, i.e. Q_{ab}.

We now suppose that during motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ the strain-energy function may adopt the form $\Psi(\mathbf{F}) = \Psi(\mathbf{C})$; recall eq. $(6.9)_1$. If we restrict the hyperelastic material to isotropic hyperelastic response we require that $\Psi(\mathbf{C}) = \Psi(\mathbf{C}^\star)$, with $\mathbf{C}^\star = \mathbf{F}^{\star\mathrm{T}}\mathbf{F}^\star$. With reference to (6.22) we conclude that

$$\Psi(\mathbf{C}) = \Psi(\mathbf{F}^{\star\mathrm{T}}\mathbf{F}^\star) = \Psi(\mathbf{Q}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{Q}^{\mathrm{T}}) \quad , \tag{6.24}$$

implying with the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, that

$$\Psi(\mathbf{C}) = \Psi(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}) \quad . \tag{6.25}$$

If the requirement for isotropy (6.25) holds for all *symmetric* tensors $\mathbf{C}$ and orthogonal tensors $\mathbf{Q}$, we say that the strain-energy function $\Psi(\mathbf{C})$ is a **scalar-valued isotropic tensor function** of one variable $\mathbf{C}$ or simply an **invariant** of the symmetric tensor $\mathbf{C}$. We can show, if the strain-energy function is an invariant, then its gradient is a tensor-valued isotropic tensor function.

EXAMPLE 6.1 Assume that the hyperelastic material is restricted to isotropic response. Show that the strain energy may be expressed by the identity

$$\Psi(\mathbf{C}) = \Psi(\mathbf{b}) \quad , \tag{6.26}$$

where $\Psi(\mathbf{b})$ characterizes an isotropic function of the (spatial) left Cauchy-Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$.

Solution. We substitute for $\mathbf{Q}$ in condition (6.25) the proper orthogonal rotation tensor $\mathbf{R}$. Then, kinematic relation $(2.109)_3$ implies (6.26), which holds for any isotropic deformation. ■

Constitutive equations in terms of invariants. If a scalar-valued tensor function is an invariant under a rotation, according to (6.25), it may be expressed in terms of the principal invariants of its argument (for example, $\mathbf{C}$ or $\mathbf{b}$), which is a fundamental result for isotropic scalar functions, known as the **representation theorem for invariants** (for a proof see, for example, GURTIN [1981a, p. 231] or TRUESDELL and NOLL [1992, Section 10]).

Having this in mind, the strain energies, as established in eq. (6.26), may be expressed as a set of independent strain invariants of the symmetric Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{b}$, namely, through $I_a = I_a(\mathbf{C})$ and $I_a = I_a(\mathbf{b})$, $a = 1, 2, 3$, respectively. With reference to (6.26), we may write equivalently

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})] = \Psi[I_1(\mathbf{b}), I_2(\mathbf{b}), I_3(\mathbf{b})] \quad . \tag{6.27}$$

Again, (6.27) is exclusively valid for *isotropic* hyperelastic materials satisfying condition (6.25) for all orthogonal tensors $\mathbf{Q}$. Since $\mathbf{C}$ and $\mathbf{b}$ have the same eigenvalues, which are the squares of the principal stretches λ_a^2, $a = 1, 2, 3$, we conclude that

$$I_1(\mathbf{C}) = I_1(\mathbf{b}) \quad , \qquad I_2(\mathbf{C}) = I_2(\mathbf{b}) \quad , \qquad I_3(\mathbf{C}) = I_3(\mathbf{b}) \quad , \tag{6.28}$$

where the three principal invariants are explicitly given in accordance with eqs. (5.89)–

(5.91). Note that for the stress-free reference configuration, the strain-energy functions (6.27), with (5.89)–(5.91), must satisfy the normalization condition (6.4), i.e. $\Psi = 0$, for $I_1 = I_2 = 3$ and $I_3 = 1$. The representation in the form of invariants was established in the classical work of RIVLIN [1948].

In order to determine constitutive equations for isotropic hyperelastic materials in terms of strain invariants, consider a differentiation of $\Psi(\mathbf{C}) = \Psi(I_1, I_2, I_3)$ with respect to tensor $\mathbf{C}$. We assume that $\Psi(\mathbf{C})$ has continuous derivatives with respect to the principal invariants I_a, $a = 1, 2, 3$. By means of the chain rule of differentiation we find

$$\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \Psi}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \Psi}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \Psi}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}} = \sum_{a=1}^{3} \frac{\partial \Psi}{\partial I_a}\frac{\partial I_a}{\partial \mathbf{C}} \ . \tag{6.29}$$

The derivative of the first invariant I_1 with respect to $\mathbf{C}$, as needed for (6.29), gives with $(6.28)_1$, $(5.89)_1$ and the property (1.94) of double contraction

$$\frac{\partial I_1}{\partial \mathbf{C}} = \frac{\partial \mathrm{tr}\mathbf{C}}{\partial \mathbf{C}} = \frac{\partial (\mathbf{I} : \mathbf{C})}{\partial \mathbf{C}} = \mathbf{I} \qquad \text{or} \qquad \frac{\partial I_1}{\partial C_{AB}} = \delta_{AB} \ . \tag{6.30}$$

The derivatives of the remaining two invariants with respect to $\mathbf{C}$ follow from eqs. $(5.90)_1$ and $(5.91)_1$, by means of (6.28), $(6.30)_3$, the chain rule and relations $(1.252)_2$, (1.241) (use the *symmetric* tensor $\mathbf{C}$ in the place of $\mathbf{A}$), and have the forms

$$\begin{aligned} \frac{\partial I_2}{\partial \mathbf{C}} &= \frac{1}{2}\left(2\mathrm{tr}\mathbf{C}\,\mathbf{I} - \frac{\partial \mathrm{tr}(\mathbf{C}^2)}{\partial \mathbf{C}}\right) = I_1\mathbf{I} - \mathbf{C} \ , \qquad && \frac{\partial I_3}{\partial \mathbf{C}} = I_3\mathbf{C}^{-1} \\ \text{or} \qquad \frac{\partial I_2}{\partial C_{AB}} &= I_1\delta_{AB} - C_{AB} \ , && \frac{\partial I_3}{\partial C_{AB}} = I_3 C_{AB}^{-1} \ . \end{aligned} \tag{6.31}$$

Substituting (6.29)–(6.31) into constitutive equation $(6.13)_2$ gives the *most general form* of a stress relation in terms of the three strain invariants, which characterizes isotropic hyperelastic materials at finite strains, i.e.

$$\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = 2\left[\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{I} - \frac{\partial \Psi}{\partial I_2}\mathbf{C} + I_3\frac{\partial \Psi}{\partial I_3}\mathbf{C}^{-1}\right] \ . \tag{6.32}$$

The gradient of the invariant $\Psi(\mathbf{C}) = \Psi(I_1, I_2, I_3)$ has the simple representation (6.32), which is a fundamental relationship in the theory of finite hyperelasticity. Note that (6.32) is a general representation for three dimensions, in which Ψ may adopt any scalar-valued isotropic function of one symmetric second-order tensor variable.

Multiplication of $(6.32)_2$ by tensor $\mathbf{C}$ from the right-hand side or from the left-hand side leads to the same result. We say that $\partial\Psi(\mathbf{C})/\partial\mathbf{C}$ **commutes** (or, is **coaxial**) with $\mathbf{C}$ in the sense that

$$\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}}\mathbf{C} = \mathbf{C}\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \ , \tag{6.33}$$

which is an essential consequence of isotropy.

Next, we present the spatial counterpart of constitutive equation (6.32). According to relation (3.66), the Cauchy stress $\boldsymbol{\sigma}$ follows from the second Piola-Kirchhoff stress $\mathbf{S}$ by the Piola transformation $\boldsymbol{\sigma} = J^{-1}\mathbf{FSF}^{\mathrm{T}}$. By multiplying the tensor variables $\mathbf{I}, \mathbf{C}, \mathbf{C}^{-1}$ with $\mathbf{F}$ from the left-hand side and with $\mathbf{F}^{\mathrm{T}}$ from the right-hand side, we may write by means of the left Cauchy-Green tensor $\mathbf{b} = \mathbf{FF}^{\mathrm{T}}$, that $\mathbf{FIF}^{\mathrm{T}} = \mathbf{FF}^{\mathrm{T}} = \mathbf{b}$, $\mathbf{FCF}^{\mathrm{T}} = (\mathbf{FF}^{\mathrm{T}})^2 = \mathbf{b}^2$, $\mathbf{FC}^{-1}\mathbf{F}^{\mathrm{T}} = (\mathbf{FF}^{-1})(\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}}) = \mathbf{I}$. With (6.32) we deduce from $\boldsymbol{\sigma} = J^{-1}\mathbf{FSF}^{\mathrm{T}}$ that

$$\boldsymbol{\sigma} = 2J^{-1}\left[I_3\frac{\partial \Psi}{\partial I_3}\mathbf{I} + \left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{b} - \frac{\partial \Psi}{\partial I_2}\mathbf{b}^2\right] \ . \tag{6.34}$$

Following arguments analogous to those which led from (5.88) to (5.94), we find an alternative form to (6.34), namely

$$\boldsymbol{\sigma} = 2J^{-1}\left[\left(I_2\frac{\partial \Psi}{\partial I_2} + I_3\frac{\partial \Psi}{\partial I_3}\right)\mathbf{I} + \frac{\partial \Psi}{\partial I_1}\mathbf{b} - I_3\frac{\partial \Psi}{\partial I_2}\mathbf{b}^{-1}\right] \ . \tag{6.35}$$

By comparing (6.34) and (6.35) with (5.88) and (5.94) we may derive the response coefficients α_a, $a = 0, 1, 2$, and β_a, $a = 0, 1, -1$, in terms of the strain-energy function (6.27). We obtain the forms

$$\begin{gathered} \alpha_0 = 2I_3^{1/2}\frac{\partial \Psi}{\partial I_3} \ , \qquad \alpha_1 = 2I_3^{-1/2}\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right) \ , \\ \alpha_2 = -2I_3^{-1/2}\frac{\partial \Psi}{\partial I_2} \ , \end{gathered} \tag{6.36}$$

$$\begin{gathered} \beta_0 = 2I_3^{-1/2}\left(I_2\frac{\partial \Psi}{\partial I_2} + I_3\frac{\partial \Psi}{\partial I_3}\right) \ , \qquad \beta_1 = 2I_3^{-1/2}\frac{\partial \Psi}{\partial I_1} \ , \\ \beta_{-1} = -2I_3^{1/2}\frac{\partial \Psi}{\partial I_2} \ , \end{gathered} \tag{6.37}$$

which specify the first and the second representation theorem for isotropic tensor functions, i.e. (5.88) and (5.94), respectively.

Note that in order to formulate constitutive equations which are not restricted to isotropic response and which satisfy the objectivity requirement, it is appropriate to use quantities which are referred to the reference configuration. Within the context of hyperelasticity it is obvious to use the right Cauchy-stress tensor $\mathbf{C}$ and its work conjugate stress field, i.e. the second Piola-Kirchhoff stress tensor $\mathbf{S}$.

Constitutive equations derived from $\Psi(\mathbf{b})$ and $\Psi(\mathbf{v})$. If the strain-energy function depends on the (symmetric) left Cauchy-Green tensor $\mathbf{b}$, then the isotropic hyperelastic response is

$$\boldsymbol{\sigma} = 2J^{-1}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}}\mathbf{b} = 2J^{-1}\mathbf{b}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} \qquad \text{or} \qquad \sigma_{ab} = 2J^{-1}b_{ac}\frac{\partial \Psi}{\partial b_{cb}} \tag{6.38}$$

(see also TRUESDELL and NOLL [1992, Section 85, p. 313]), where $\Psi(\mathbf{b})$ is a scalar-valued isotropic function of the tensor variable $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$. This constitutive equation plays an essential part in isotropic finite hyperelasticity. On comparison with eq. (5.85) we deduce that the right-hand side of eq. (6.38) corresponds to the response function $\mathfrak{h}(\mathbf{b})$.

Proof. In order to obtain the constitutive equation (6.38), we start by differentiating the postulated strain-energy function $\Psi(\mathbf{b})$ with respect to time t. Considering symmetries, a straightforward algebraic manipulation gives, by means of the chain rule, relation (2.171) and the property (1.95) of double contraction,

$$\begin{aligned}\dot{\Psi} &= \frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} : \dot{\mathbf{b}} = \frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} : (\mathbf{l}\mathbf{b} + \mathbf{b}\mathbf{l}^{\mathrm{T}}) \\ &= 2\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} : \mathbf{l}\mathbf{b} = 2\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}}\mathbf{b} : \mathbf{l} \ , \end{aligned} \tag{6.39}$$

where $\mathbf{l}$ is the spatial velocity gradient, in general a non-symmetric tensor.

With respect to eqs. (6.32) and (6.33), taking $\mathbf{b}$ in place of $\mathbf{C}$, we deduce that $\partial\Psi(\mathbf{b})/\partial\mathbf{b}$ commutes with the symmetric second-order tensor $\mathbf{b}$, in the sense that

$$\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}}\mathbf{b} = \mathbf{b}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} \ , \tag{6.40}$$

implying the symmetry of tensor $(\partial\Psi(\mathbf{b})/\partial\mathbf{b})\mathbf{b}$.

However, from (4.154), we know that for perfectly elastic materials (for which $\mathcal{D}_{\mathrm{int}} = 0$) the stress power $\mathrm{w}_{\mathrm{int}}$ per unit reference volume equals $\dot{\Psi}$. The combination of identity $(4.113)_1$, i.e. $\mathrm{w}_{\mathrm{int}} = J\boldsymbol{\sigma} : \mathbf{d}$, and eq. $(6.39)_4$ with the requirement (6.40) for isotropy implies

$$J\boldsymbol{\sigma} : \mathbf{d} = 2\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}}\mathbf{b} : \mathbf{d} = 2\mathbf{b}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} : \mathbf{d} \ , \tag{6.41}$$

where the rate of deformation tensor $\mathbf{d}$ is the symmetric part of $\mathbf{l}$. For this relation we used the fact that the double contraction of a symmetric tensor and a skew tensor is zero.

Simple arguments reduce relation (6.41) to the desired fundamental constitutive equation (6.38) for isotropic response. ■

Since the left stretch tensor $\mathbf{v} = \mathbf{b}^{1/2}$ is the unique square root of $\mathbf{b}$, the strain energy may also be expressed as an isotropic function of $\mathbf{v}$; thus, relation (6.26) may be extended to $\Psi(\mathbf{C}) = \Psi(\mathbf{b}) = \Psi(\mathbf{v})$. Note that for notational convenience, we do not distinguish between different strain-energy functions Ψ.

With respect to constitutive equation (6.38) we derive an equivalent form in terms of the left stretch tensor $\mathbf{v}$. Analogous to the procedure which was used to establish eq. (6.11) we find that

$$2\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} = \frac{\partial \Psi(\mathbf{v})}{\partial \mathbf{v}}\mathbf{v}^{-1} \ , \tag{6.42}$$

so that (6.38) reads

$$\boldsymbol{\sigma} = J^{-1}\frac{\partial \Psi(\mathbf{v})}{\partial \mathbf{v}}\mathbf{v} = J^{-1}\mathbf{v}\frac{\partial \Psi(\mathbf{v})}{\partial \mathbf{v}} \ , \tag{6.43}$$

which is another important stress relation characterizing the behavior of isotropic hyperelastic materials at finite strains. Note that, since $\mathbf{v}$ is the unique square root of $\mathbf{b}$, $\partial\Psi/\partial\mathbf{b}$ also commutes with $\mathbf{v}$.

Constitutive equations in terms of principal stretches. If the strain-energy function Ψ is an invariant, we may regard Ψ as a function of the principal stretches λ_a, $a = 1, 2, 3$. In the place of (6.27), we may represent Ψ in the form

$$\Psi = \Psi(\mathbf{C}) = \Psi(\lambda_1, \lambda_2, \lambda_3) \ . \tag{6.44}$$

For the stress-free reference configuration the normalization condition (6.4) takes on the form $\Psi(1, 1, 1) = 0$.

Consider the left stretch tensor $\mathbf{v} = \mathbf{b}^{1/2}$ describing the deformed state of an isotropic hyperelastic material. From the eigenvalue problem $(2.118)_3$ we know that λ_a denote the three principal stretches (the real eigenvalues) of $\mathbf{v}$. Since the principal directions of $\mathbf{v}$ coincide with those of $\mathbf{b}$ (compare with eqs. $(2.118)_3$ and $(2.119)_2$) they also coincide with the principal directions of the Cauchy stress tensor $\boldsymbol{\sigma}$ (recall representation (5.88)).

Consequently, with respect to (6.43) the principal Cauchy stresses σ_a, $a = 1, 2, 3$, simply result in

$$\sigma_a = J^{-1}\lambda_a\frac{\partial \Psi}{\partial \lambda_a} \ , \qquad a = 1, 2, 3 \ , \tag{6.45}$$

with the volume ratio

$$J = \lambda_1\lambda_2\lambda_3 \ , \tag{6.46}$$

according to $(5.91)_3$.

In addition to (6.45), we introduce equivalent relations for the three principal Piola-Kirchhoff stresses P_a and S_a, namely

$$P_a = \frac{\partial \Psi}{\partial \lambda_a} \ , \qquad S_a = \frac{1}{\lambda_a}\frac{\partial \Psi}{\partial \lambda_a} \ , \qquad a = 1, 2, 3 \tag{6.47}$$

(compare with the following Example 6.2), which may be expressed in terms of the Cauchy stresses (6.45) as

$$P_a = J\lambda_a^{-1}\sigma_a \ , \qquad S_a = J\lambda_a^{-2}\sigma_a \ , \qquad a = 1, 2, 3 \ . \tag{6.48}$$

Constitutive relations (6.45) and (6.47) show that principal stresses in an isotropic hyperelastic material depend only upon the principal stretches. They are simply obtained by differentiating the strain-energy function with respect to the corresponding principal stretches.

EXAMPLE 6.2 Consider the strain energy $\Psi(\mathbf{C}) = \Psi(\lambda_1, \lambda_2, \lambda_3)$. Obtain the constitutive equations in the spectral forms

$$\boldsymbol{\sigma} = \sum_{a=1}^{3} \sigma_a \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a \ , \tag{6.49}$$

$$\mathbf{P} = \sum_{a=1}^{3} P_a \hat{\mathbf{n}}_a \otimes \hat{\mathbf{N}}_a \ , \qquad \mathbf{S} = \sum_{a=1}^{3} S_a \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a \ , \tag{6.50}$$

where σ_a and P_a, S_a, $a = 1, 2, 3$, are the principal values of the Cauchy stress tensor $\boldsymbol{\sigma}$ and the two Piola-Kirchhoff stress tensors $\mathbf{P}$, $\mathbf{S}$ according to the expressions (6.45) and (6.47), respectively. The orthonormal vectors $\hat{\mathbf{N}}_a$ and $\hat{\mathbf{n}}_a = \mathbf{R}\hat{\mathbf{N}}_a$, $a = 1, 2, 3$, denote the principal referential and spatial directions (axes of stretch), respectively.

These constitutive equations describe isotropic response of hyperelastic materials and hold *if and only if* $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

Solution. We start with constitutive equation $(6.50)_2$ which is expressed in terms of second Piola-Kirchhoff stresses S_a, $a = 1, 2, 3$. We compute the derivative of the isotropic function $\Psi(\mathbf{C})$ with respect to the symmetric tensor $\mathbf{C}$. By means of the chain rule and kinematic relation (2.125), we obtain for the general case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = \sum_{a=1}^{3} \frac{\partial \Psi}{\partial \lambda_a^2} \frac{\partial \lambda_a^2}{\partial \mathbf{C}} = \sum_{a=1}^{3} \frac{\partial \Psi}{\partial \lambda_a^2} \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a \ . \tag{6.51}$$

In $(6.51)_2$, λ_a^2 are the eigenvalues (the squares of the principal stretches) and $\hat{\mathbf{N}}_a$ the corresponding eigenvectors (principal referential directions) of $\mathbf{C}$ (compare with the eigenvalue problem (2.117)). With $(6.51)_2$ and the chain rule we find from $(6.13)_2$ that

$$\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = \sum_{a=1}^{3} \underbrace{\frac{1}{\lambda_a} \frac{\partial \Psi}{\partial \lambda_a}}_{S_a} \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a \ , \tag{6.52}$$

which gives the desired results $(6.47)_2$ and $(6.50)_2$.

By the use of $(6.52)_2$, the relation according to (1.58) and eq. $(2.134)_1$, i.e. $\mathbf{F}\hat{\mathbf{N}}_a = \lambda_a \hat{\mathbf{n}}_a$, $a = 1, 2, 3$, the spectral form of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ may be found from transformation (3.67) as

$$\begin{aligned}\mathbf{P} = \mathbf{F}\mathbf{S} &= \mathbf{F}\left(\sum_{a=1}^{3} \frac{1}{\lambda_a} \frac{\partial \Psi}{\partial \lambda_a} \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a\right) \\ &= \sum_{a=1}^{3} \frac{1}{\lambda_a} \frac{\partial \Psi}{\partial \lambda_a} (\mathbf{F}\hat{\mathbf{N}}_a) \otimes \hat{\mathbf{N}}_a = \sum_{a=1}^{3} \underbrace{\frac{\partial \Psi}{\partial \lambda_a}}_{P_a} \hat{\mathbf{n}}_a \otimes \hat{\mathbf{N}}_a \ .\end{aligned} \tag{6.53}$$

Similarly, having in mind the results $(6.53)_4$ and $(2.134)_1$, transformation (3.9) gives, using (3.10), the spectral form

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{P}^{\mathrm{T}} = J^{-1}\mathbf{F}\left(\sum_{a=1}^{3} \frac{\partial \Psi}{\partial \lambda_a} (\hat{\mathbf{n}}_a \otimes \hat{\mathbf{N}}_a)^{\mathrm{T}}\right) = \sum_{a=1}^{3} \underbrace{J^{-1}\lambda_a \frac{\partial \Psi}{\partial \lambda_a}}_{\sigma_a} \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a \tag{6.54}$$

of the Cauchy stress tensor, where the property $\mathbf{F}(\hat{\mathbf{n}}_a \otimes \hat{\mathbf{N}}_a)^{\mathrm{T}} = (\mathbf{F}\hat{\mathbf{N}}_a) \otimes \hat{\mathbf{n}}_a$, $a = 1, 2, 3$, was used (compare with relations which are analogous to (1.85) and (1.58)). ■

EXERCISES

1. By analogy with the procedure which led to (6.11), obtain the eq. (6.42) and relation

$$\left(\frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}\right)^{\mathrm{T}} = 2\mathbf{F}^{\mathrm{T}} \frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} \ . \tag{6.55}$$

2. Rewrite the spectral representations of constitutive equations (6.49) and (6.50) for a given strain-energy function of the particular form $\Psi = \Psi(\ln\lambda_1, \ln\lambda_2, \ln\lambda_3)$. Consider the general case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

3. Take the strain energy Ψ as a function of the principal stretches characterizing the behavior of isotropic hyperelastic materials. Let at least one principal stretch be equal to the other.

   (a) For the case in which we have two equal principal stretches, namely $\lambda_1 = \lambda_2 \neq \lambda_3$, obtain the constitutive equations

$$\boldsymbol{\sigma} = J^{-1}\lambda_1 \frac{\partial \Psi}{\partial \lambda_1} (\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2) + J^{-1}\lambda_3 \frac{\partial \Psi}{\partial \lambda_3} \hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3 \ ,$$

$$\mathbf{P} = \frac{\partial \Psi}{\partial \lambda_1} (\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{N}}_1 + \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{N}}_2) + \frac{\partial \Psi}{\partial \lambda_3} \hat{\mathbf{n}}_3 \otimes \hat{\mathbf{N}}_3 \ ,$$

$$\mathbf{S} = \frac{1}{\lambda_1} \frac{\partial \Psi}{\partial \lambda_1} (\hat{\mathbf{N}}_1 \otimes \hat{\mathbf{N}}_1 + \hat{\mathbf{N}}_2 \otimes \hat{\mathbf{N}}_2) + \frac{1}{\lambda_3} \frac{\partial \Psi}{\partial \lambda_3} \hat{\mathbf{N}}_3 \otimes \hat{\mathbf{N}}_3 \ .$$

(b) Using the property $(1.65)_2$ for the second-order unit tensor $\mathbf{I}$ and relation $(2.124)_3$ for the rotation tensor $\mathbf{R}$, show that for $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

$$\boldsymbol{\sigma} = \sigma \sum_{a=1}^{3} \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a = \sigma \mathbf{I} \quad ,$$

$$\mathbf{P} = P \sum_{a=1}^{3} \hat{\mathbf{n}}_a \otimes \hat{\mathbf{N}}_a = P\mathbf{R} \quad , \qquad \mathbf{S} = S \sum_{a=1}^{3} \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a = S\mathbf{I} \quad ,$$

with the scalar-valued scalar functions $\sigma = J^{-1}\lambda \partial\Psi/\partial\lambda$ and with $P = \partial\Psi/\partial\lambda$, $S = \lambda^{-1}\partial\Psi/\partial\lambda$.

6.3 Incompressible Hyperelastic Materials

Numerous polymeric materials can sustain finite strains without noticeable volume changes. Such types of material may be regarded as incompressible so that only isochoric motions are possible. For many cases, this is a common idealization and accepted assumption often invoked in continuum and computational mechanics. In this section we present the constitutive foundation of incompressible hyperelastic materials.

Incompressible hyperelasticity. Materials which keep the volume constant throughout a motion are characterized by the incompressibility constraint

$$J = 1 \quad , \tag{6.56}$$

or by some other equivalent expressions according to (2.179) (recall the expression (2.52) for the volume ratio J). In general, a material which is subjected to an internal constraint, of which incompressibility is the most common, is referred to as a **constrained material.**

In order to derive general constitutive equations for incompressible hyperelastic materials, we may postulate the strain-energy function

$$\Psi = \Psi(\mathbf{F}) - p(J-1) \quad , \tag{6.57}$$

where the strain energy Ψ is defined for $J = \det\mathbf{F} = 1$. The scalar p introduced in (6.57) serves as an indeterminate *Lagrange multiplier*, which can be identified as a **hydrostatic pressure**. Note that the scalar p may only be determined from the equilibrium equations and the boundary conditions. It represents a workless reaction to the kinematic constraint on the deformation field.

Differentiating eq. (6.57) with respect to the deformation gradient $\mathbf{F}$ and using identity (2.176), we arrive at a general constitutive equation for the first Piola-Kirchhoff stress tensor $\mathbf{P}$. Hence, eq. (6.1) may be adopted in the form

$$\mathbf{P} = -p\mathbf{F}^{-\mathrm{T}} + \frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}} \quad . \tag{6.58}$$

An alternative derivation of (6.58) is obtained by reference to the expression $(6.3)_2$. For incompressible hyperelasticity, $\dot{\mathbf{F}}$ is not arbitrary anymore and the expressions in parentheses of $(6.3)_2$ need not be zero. However, $(6.3)_2$ must be satisfied for every $\dot{\mathbf{F}}$ which is governed by the incompressibility constraint in the form of $\dot{J} = \mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}} = 0$ (recall $(2.179)_4$). Consequently, adding the zero term to $(6.3)_2$, we find that

$$\left(\mathbf{P} - \frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}} + p\mathbf{F}^{-\mathrm{T}}\right) : \dot{\mathbf{F}} = 0 \quad . \tag{6.59}$$

With standard arguments, the Coleman-Noll procedure implies physical expression (6.58).

Multiplying eq. (6.58) by $\mathbf{F}^{-1}$ from the left-hand side, we conclude from $(3.65)_2$ that the second Piola-Kirchhoff stress tensor $\mathbf{S}$ takes on the form

$$\mathbf{S} = -p\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} + \mathbf{F}^{-1}\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}} = -p\mathbf{C}^{-1} + 2\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}} \quad , \tag{6.60}$$

where the inverse of relation (2.65), i.e. $\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$, and identity (6.11) are to be used.

However, multiplying eq. (6.58) by $\mathbf{F}^{\mathrm{T}}$ from the right-hand side, we conclude from (3.9) that the symmetric Cauchy stress tensor $\boldsymbol{\sigma}$ may be expressed as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\mathbf{F}^{\mathrm{T}} = -p\mathbf{I} + \mathbf{F}\left(\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}\right)^{\mathrm{T}} \quad . \tag{6.61}$$

The fundamental constitutive equations (6.58), (6.60) and (6.61) are the *most general forms* used to define incompressible hyperelastic materials at finite strains. Equations $(6.60)_2$ and (6.61) are associated with (5.97) and (5.96). Note that the response functions $\mathfrak{H}(\mathbf{C})$ and $\mathbf{g}(\mathbf{F})$ occurring in (5.97) and $(5.96)_1$ are identified by $\mathfrak{H}(\mathbf{C}) = 2\partial\Psi(\mathbf{C})/\partial\mathbf{C}$ and $\mathbf{g}(\mathbf{F}) = (\partial\Psi(\mathbf{F})/\partial\mathbf{F})\mathbf{F}^{\mathrm{T}} = \mathbf{F}(\partial\Psi(\mathbf{F})/\partial\mathbf{F})^{\mathrm{T}}$, i.e. constitutive equation $(6.2)_2$ for $J = 1$.

Incompressible isotropic hyperelasticity. For the case of isotropy we have already pointed out that the dependence of Ψ on the Cauchy-Green tensors $\mathbf{C}$ or $\mathbf{b}$ may be expressed by their three strain invariants (see eq. (6.27)). However, for the incompressible case we consider the kinematic constraint, namely $I_3 = \det\mathbf{C} = \det\mathbf{b} = 1$. Therefore, the two principal invariants I_1 and I_2 are the only independent deformation variables.

For a review on the theory of incompressible isotropic hyperelasticity see, for example, OGDEN [1982, 1986].

A suitable strain-energy function for incompressible isotropic hyperelastic materials is, in view of (6.27), given by

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C})] - \frac{1}{2}p(I_3 - 1) = \Psi[I_1(\mathbf{b}), I_2(\mathbf{b})] - \frac{1}{2}p(I_3 - 1) \quad , \tag{6.62}$$

where $p/2$ serves as an indeterminate *Lagrange multiplier*.

In order to examine the associated constitutive equation in terms of the two principal strain invariants I_1, I_2, we proceed by deriving $(6.62)_1$ with respect to tensor $\mathbf{C}$. Analogous to the procedure which led to (6.32) we find, using the chain rule, eqs. $(6.30)_3$, (6.31) and the constraint $I_3 = 1$, that

$$\mathbf{S} = 2\frac{\partial \Psi(I_1, I_2)}{\partial \mathbf{C}} - \frac{\partial [p(I_3 - 1)]}{\partial \mathbf{C}} = -p\mathbf{C}^{-1} + 2\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{I} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{C} \quad , \tag{6.63}$$

which is basically constitutive equation (6.32), in which the term $I_3(\partial\Psi/\partial I_3)$ is substituted by $-p/2$.

A push-forward operation of $(6.63)_2$ and an elimination of $\mathbf{b}^2$ in favor of $\mathbf{b}^{-1}$ (see relation (5.93)) yields two alternative forms of $\boldsymbol{\sigma}$, corresponding to eqs. (6.34) and (6.35), namely

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right)\mathbf{b} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{b}^2 \quad , \tag{6.64}$$

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\frac{\partial \Psi}{\partial I_1}\mathbf{b} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{b}^{-1} \quad . \tag{6.65}$$

Note that the scalars p in eqs. (6.64) and (6.65) differ by the term $2I_2(\partial\Psi/\partial I_2)$. By comparing (6.64) and (6.65) with (5.98) we obtain explicitly the response coefficients α_1, α_2 and β_1, β_{-1} as

$$\alpha_1 = 2\left(\frac{\partial \Psi}{\partial I_1} + I_1\frac{\partial \Psi}{\partial I_2}\right) \quad , \qquad \alpha_2 = -2\frac{\partial \Psi}{\partial I_2} \quad , \tag{6.66}$$

$$\beta_1 = 2\frac{\partial \Psi}{\partial I_1} \quad , \qquad \beta_{-1} = -2\frac{\partial \Psi}{\partial I_2} \quad . \tag{6.67}$$

In order to find a constitutive equation for incompressible materials which is associated with (6.38), we recall the transformation (6.55). Then, (6.61) gives the constitutive equation

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{I} + 2\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}}\mathbf{b} = -p\mathbf{I} + 2\mathbf{b}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} \\ \text{or} \qquad \sigma_{ab} &= -p\delta_{ab} + 2b_{ac}\frac{\partial \Psi}{\partial b_{cb}} \end{aligned} \tag{6.68}$$

in terms of the spatial strain variable $\mathbf{b}$. This is only valid for incompressible isotropic hyperelastic materials.

If we express Ψ as a function of the three principal stretches λ_a we write $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) - p(J - 1)$ in the place of (6.57), with the indeterminate *Lagrange multiplier* p. Using $\partial J/\partial\lambda_a = J\lambda_a^{-1}$, $a = 1, 2, 3$, which is relation (2.176) expressed in principal stretches, eqs. (6.45) and (6.47) are then replaced by

$$\sigma_a = -p + \lambda_a\frac{\partial \Psi}{\partial \lambda_a} \quad , \qquad a = 1, 2, 3 \quad , \tag{6.69}$$

$$P_a = -\frac{1}{\lambda_a}p + \frac{\partial \Psi}{\partial \lambda_a} \quad , \qquad S_a = -\frac{1}{\lambda_a^2}p + \frac{1}{\lambda_a}\frac{\partial \Psi}{\partial \lambda_a} \quad , \qquad a = 1, 2, 3 \quad , \tag{6.70}$$

with the three principal Cauchy stresses σ_a and the Piola-Kirchhoff stresses P_a, S_a. These stress relations incorporate the unknown scalar p, which must be determined from the equilibrium equations and the boundary conditions. The incompressibility constraint $J = 1$ takes on the form

$$\lambda_1\lambda_2\lambda_3 = 1 \quad , \tag{6.71}$$

leaving two independent stretches as the deformation measures. Expressing the first and second Piola-Kirchhoff stresses in terms of the Cauchy stresses (6.69), we obtain, by analogy with (6.48), $P_a = \lambda_a^{-1}\sigma_a$ and $S_a = \lambda_a^{-2}\sigma_a$, $a = 1, 2, 3$.

EXAMPLE 6.3 Consider a thin sheet of incompressible hyperelastic material which is embedded in a reference frame of (right-handed) coordinate axes with a fixed set of orthonormal basis vectors $\mathbf{e}_a$, $a = 1, 2, 3$. Suppose that the axes are aligned with the major faces of the sheet.

A deformation created by the stretch ratios λ_1, λ_2 along the directions $\mathbf{e}_1, \mathbf{e}_2$ results in a *(homogeneous) biaxial deformation* with the kinematic relation (2.132). The associated stress state is assumed to be *plane* throughout the sheet so that the Cauchy stress components $\sigma_{13}, \sigma_{23}, \sigma_{33}$ are equal to zero which is in accordance with (3.59).

Show that the biaxial stress state of the homogeneous problem is of the form

$$\sigma_1 = 2(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2})\left(\frac{\partial \Psi}{\partial I_1} + \lambda_2^2\frac{\partial \Psi}{\partial I_2}\right) \quad , \tag{6.72}$$

$$\sigma_2 = 2(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2})\left(\frac{\partial \Psi}{\partial I_1} + \lambda_1^2\frac{\partial \Psi}{\partial I_2}\right) \tag{6.73}$$

(see RIVLIN [1948, eq. (6.5)]), with the principal invariants $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}$ and $I_3 = 1$.

Solution. Since the tensors $\boldsymbol{\sigma}$ and $\mathbf{b}$ are coaxial for isotropic elastic materials (recall p. 201), the principal stresses follow from (6.65),

$$\sigma_a = -p + 2\left(\lambda_a^2 \frac{\partial \Psi}{\partial I_1} - \frac{1}{\lambda_a^2}\frac{\partial \Psi}{\partial I_2}\right) \;, \qquad a = 1, 2, 3 \;, \tag{6.74}$$

where λ_a^2 are the three eigenvalues of the left Cauchy-Green tensor $\mathbf{b}$ (see the eigenvalue problem $(2.119)_2$). This relation was first presented by RIVLIN [1948].

With the condition of incompressibility (6.71) in the form of $\lambda_3 = (\lambda_1\lambda_2)^{-1}$ and the boundary condition $\sigma_3 = 0$ we may determine p explicitly. For $a = 3$, we deduce from (6.74) that

$$\sigma_3 = 0 \qquad \longrightarrow \qquad p = 2\left(\frac{1}{\lambda_1^2\lambda_2^2}\frac{\partial \Psi}{\partial I_1} - \lambda_1^2\lambda_2^2\frac{\partial \Psi}{\partial I_2}\right) \;. \tag{6.75}$$

This result substituted back into eq. (6.74) leads to the nonzero stress components σ_1 and σ_2. ■

EXERCISES

1. Consider a thin sheet of incompressible hyperelastic material ($I_3 = 1$) with the same setting as formulated in Example 6.3.

   (a) Consider a **simple tension** for which $\lambda_1 = \lambda$. Then, obeying incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$, the equal stretch ratios in the transverse directions are, by symmetry, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$. Show that for this mode of deformation the homogeneous stress state reduces to $\sigma_1 = \sigma, \sigma_2 = \sigma_3 = 0$, with

   $$\sigma = 2\left(\lambda^2 - \frac{1}{\lambda}\right)\left(\frac{\partial \Psi}{\partial I_1} + \frac{1}{\lambda}\frac{\partial \Psi}{\partial I_2}\right) \;,$$

   where the invariants are $I_1 = 2\lambda^{-1} + \lambda^2, I_2 = \lambda^{-2} + 2\lambda$.

   As a special case of the biaxial deformation, as discussed in Example 6.3, consider an *equibiaxial deformation* for which $\lambda_1 = \lambda_2 = \lambda, \lambda_3 = \lambda^{-2}$ and $\sigma_1 = \sigma_2 = \sigma, \sigma_3 = 0$. Show that

   $$\sigma = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(\frac{\partial \Psi}{\partial I_1} + \lambda^2\frac{\partial \Psi}{\partial I_2}\right) \;,$$

   with $I_1 = 2\lambda^2 + \lambda^{-4}, I_2 = \lambda^4 + 2\lambda^{-2}$.

   (b) Consider a homogeneous *pure shear deformation* with the kinematic relation $\lambda_1 = \lambda, \lambda_2 = 1, \lambda_3 = 1/\lambda$ (compare with eq. (2.133)). Show that the nonzero Cauchy stress components are

   $$\sigma_1 = 2\left(\lambda^2 - \frac{1}{\lambda^2}\right)\left(\frac{\partial \Psi}{\partial I_1} + \frac{\partial \Psi}{\partial I_2}\right) \;, \tag{6.76}$$

   $$\sigma_2 = 2\left(1 - \frac{1}{\lambda^2}\right)\left(\frac{\partial \Psi}{\partial I_1} + \lambda^2\frac{\partial \Psi}{\partial I_2}\right) \;, \tag{6.77}$$

   evaluated for $I_1 = I_2 = \lambda^2 + \lambda^{-2} + 1$.

2. Consider a thin sheet of incompressible hyperelastic material with the same setting as formulated in Example 6.3 but subjected to a homogeneous *simple shear deformation* which is caused by a motion in the form of (2.3) (compare also with Exercise 2 on p. 93).

   (a) Show that the associated stress state is completely defined by

   $$\begin{aligned} \sigma_{11} &= -p + 2(1 + c^2)\frac{\partial \Psi}{\partial I_1} - 2\frac{\partial \Psi}{\partial I_2} \;, \\ \sigma_{22} &= -p + 2\frac{\partial \Psi}{\partial I_1} - 2(1 + c^2)\frac{\partial \Psi}{\partial I_2} \;, \\ \sigma_{33} &= -p + 2\frac{\partial \Psi}{\partial I_1} - 2\frac{\partial \Psi}{\partial I_2} \;, \\ \sigma_{12} &= \mu c \;, \qquad \mu = 2\left(\frac{\partial \Psi}{\partial I_1} + \frac{\partial \Psi}{\partial I_2}\right) \end{aligned} \tag{6.78}$$

   (with $I_1 = I_2 = 3 + c^2$), where $\mu > 0$, called the **shear modulus**, is a measure of resistance to distortion and p is a scalar to be determined from the boundary conditions.

   In addition, show that the angle θ of the two principal directions acting in the plane normal to $\hat{\mathbf{n}}_3$ is given by $\tan 2\theta = 2/c$.

   (b) Consider a *plane stress state* throughout the sheet in the sense that the face of the body normal to the direction $\mathbf{e}_3$ is free of surface tractions, i.e. $\sigma_{13} = \sigma_{23} = \sigma_{33} \equiv 0$. Show that the nonzero Cauchy stress components are

   $$\sigma_{11} = 2c^2\frac{\partial \Psi}{\partial I_1} \;, \qquad \sigma_{22} = -2c^2\frac{\partial \Psi}{\partial I_2} \;, \qquad \sigma_{12} = \mu c \;.$$

6.4 Compressible Hyperelastic Materials

A material which can undergo changes of volume is said to be **compressible**. Foamed elastomers, for example, are able to sustain finite strains with volume changes. The only restriction on this class of materials is that the volume ratio J must be positive.

In this section we introduce suitable constitutive equations in order to characterize compressible hyperelastic materials, and we discuss isotropy as a special case.

Compressible hyperelasticity. Since some materials behave quite differently in bulk and shear it is most beneficial to split the deformation locally into a so-called *volumetric part* and an *isochoric part*, originally proposed by FLORY [1961] and successfully applied within the context of isothermal finite strain elasticity by, for example, LUBLINER [1985], SIMO and TAYLOR [1991a], OGDEN [1997] and within the context of finite strain elastoplasticity by, for example, SIMO et al. [1985] among many others.

In particular, we consider the deformation gradient **F** and the corresponding strain measure $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. Rather than dealing directly with **F** and **C** we perform a multiplicative decomposition of **F** into *volume-changing (dilational)* and *volume-preserving (distortional)* parts, often used in elastoplasticity (see, for example, LEE [1969]). We write

$$\mathbf{F} = (J^{1/3}\mathbf{I})\overline{\mathbf{F}} = J^{1/3}\overline{\mathbf{F}} \quad , \qquad \mathbf{C} = (J^{2/3}\mathbf{I})\overline{\mathbf{C}} = J^{2/3}\overline{\mathbf{C}} \quad . \tag{6.79}$$

The terms $J^{1/3}\mathbf{I}$ and $J^{2/3}\mathbf{I}$ are associated with volume-changing deformations, while $\overline{\mathbf{F}}$ and $\overline{\mathbf{C}} = \overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}}$ are associated with volume-preserving deformations of the material, with

$$\det\overline{\mathbf{F}} = \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3 = 1 \qquad \text{and} \qquad \det\overline{\mathbf{C}} = (\det\overline{\mathbf{F}})^2 = 1 \quad , \tag{6.80}$$

where

$$\overline{\lambda}_a = J^{-1/3}\lambda_a \quad , \qquad a = 1, 2, 3 \tag{6.81}$$

characterize the so-called **modified principal stretches**. We call $\overline{\mathbf{F}}$ and $\overline{\mathbf{C}}$ the **modified deformation gradient** and the **modified right Cauchy-Green tensor**, respectively. A material for which dilational changes require a much higher exterior work than volume-preserving changes is called a **nearly incompressible** (or **slightly compressible**) material, for which the compressibility effects are small.

The concept of the multiplicative decomposition of **F** is supported additionally by the field of computational mechanics. For example, to avoid numerical complications in the finite element analysis of slightly compressible materials it is often advantageous to separate numerical treatments of the volumetric and isochoric parts of the deformation gradient **F**; this will be discussed in Sections 8.5 and 8.6.

Before proceeding to examine constitutive equations for compressible hyperelastic materials it is first necessary to stick to kinematics and to compute the derivative of the modified right Cauchy-Green tensor $\overline{\mathbf{C}}$ relative to the symmetric tensor **C**. By means of $(5.91)_2$, we obtain from $(6.31)_2$, $\partial J^2/\partial\mathbf{C} = J^2\mathbf{C}^{-1}$. Using the chain rule we arrive at

$$\frac{\partial J}{\partial \mathbf{C}} = \frac{J}{2}\mathbf{C}^{-1} \qquad \text{and} \qquad \frac{\partial J^{-2/3}}{\partial \mathbf{C}} = -\frac{1}{3}J^{-2/3}\mathbf{C}^{-1} \quad . \tag{6.82}$$

Finally, according to the inverse of relation $(6.79)_2$, property (1.256) and relation $(6.82)_2$, we obtain the fourth-order tensor

$$\begin{aligned}\frac{\partial\overline{\mathbf{C}}}{\partial\mathbf{C}} &= \frac{\partial(J^{-2/3}\mathbf{C})}{\partial\mathbf{C}} = J^{-2/3}\left(\mathbb{I} + J^{2/3}\mathbf{C}\otimes\frac{\partial J^{-2/3}}{\partial\mathbf{C}}\right) \\ &= J^{-2/3}\underbrace{(\mathbb{I} - \frac{1}{3}\mathbf{C}\otimes\mathbf{C}^{-1})}_{\mathbb{P}^{\mathrm{T}}} = J^{-2/3}\mathbb{P}^{\mathrm{T}} \quad ,\end{aligned} \tag{6.83}$$

in which $\mathbb{P}^{\mathrm{T}}$ defines the transpose of the fourth-order tensor $\mathbb{P}$ governed by the identity (1.157). We call $\mathbb{P}$ the **projection tensor** with respect to the reference configuration, therefore expressed through **C**. With the associated property (1.159), the relation for the projection tensor $\mathbb{P}$ reads, with reference to $(6.83)_3$, as

$$\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C} \quad , \tag{6.84}$$

where $\mathbb{I}$ denotes the fourth-order unit tensor, as defined in eq. $(1.160)_1$, with representation (1.161).

Earlier we agreed to study the purely mechanical theory. To characterize processes within an isothermal situation at constant temperature, we postulate a unique *decoupled* representation of the strain-energy function $\Psi = \Psi(\mathbf{C})$ (per unit reference volume). It is based on kinematic assumption $(6.79)_2$ and of the specific form

$$\Psi(\mathbf{C}) = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}(\overline{\mathbf{C}}) \quad , \tag{6.85}$$

where $\Psi_{\mathrm{vol}}(J)$ and $\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})$ are given scalar-valued functions of J and $\overline{\mathbf{C}}$ which are assumed to be objective. They describe the so-called **volumetric** (or **dilational**) **elastic response** and the **isochoric** (or **distortional**) **elastic response** of the material, respectively.

Additionally, we require that Ψ_{vol} is a *strictly convex* function taking on its unique minimum at $J = 1$ (for formal definitions of strictly convex functions see, for example, OGDEN [1997, Appendix 1]). With reference to normalization condition (6.4) we claim that $\Psi_{\mathrm{vol}}(J) = 0$ and $\Psi_{\mathrm{iso}}(\overline{\mathbf{C}}) = 0$ hold *if and only if* $J = 1$ and $\overline{\mathbf{C}} = \mathbf{I}$, respectively.

We now determine constitutive equations for compressible hyperelastic materials. In order to particularize the second law of thermodynamics through the Clausius-Planck inequality (4.154) to the specific strain energy (6.85) at hand we determine the derivative of Ψ with respect to time t first. By means of the chain rule we obtain from (6.85)

$$\dot{\Psi} = \frac{\mathrm{d}\Psi_{\mathrm{vol}}(J)}{\mathrm{d}J}\dot{J} + \frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}} : \dot{\overline{\mathbf{C}}} \quad . \tag{6.86}$$

Hence, we need to compute $\dot{J}$ and $\dot{\overline{\mathbf{C}}}$, which, with eqs.$(6.82)_1$ and $(6.83)_4$, simply results in $\dot{J} = \partial J/\partial \mathbf{C} : \dot{\mathbf{C}} = J\mathbf{C}^{-1} : \dot{\mathbf{C}}/2$ and $\dot{\overline{\mathbf{C}}} = 2(\partial\overline{\mathbf{C}}/\partial\mathbf{C}) : \dot{\mathbf{C}}/2 = 2J^{-2/3}\mathbb{P}^{\mathrm{T}} : \dot{\mathbf{C}}/2$. Having this in mind, with the stress power $w_{\mathrm{int}} = \mathbf{S} : \dot{\mathbf{C}}/2$ per unit reference volume and relation (6.86), we may deduce from (4.154) that

$$\mathcal{D}_{\mathrm{int}} = \left(\mathbf{S} - J\frac{\mathrm{d}\Psi_{\mathrm{vol}}(J)}{\mathrm{d}J}\mathbf{C}^{-1} - J^{-2/3}\mathbb{P} : 2\frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\right) : \frac{\dot{\mathbf{C}}}{2} = 0 \ , \tag{6.87}$$

where the identity (1.157) is to be used. Since we consider perfectly elastic materials the internal dissipation $\mathcal{D}_{\mathrm{int}}$ must vanish.

The standard Coleman-Noll procedure leads to constitutive equations for compressible hyperelastic materials, in which the stress response constitutes an additive split of $(6.13)_2$, i.e. $\mathbf{S} = 2\partial\Psi(\mathbf{C})/\partial\mathbf{C}$. In particular, the second Piola-Kirchhoff stress $\mathbf{S}$ consists of a purely **volumetric contribution** and a purely **isochoric contribution**, i.e. $\mathbf{S}_{\mathrm{vol}}$ and $\mathbf{S}_{\mathrm{iso}}$, respectively. We write

$$\mathbf{S} = 2\frac{\partial\Psi(\mathbf{C})}{\partial\mathbf{C}} = \mathbf{S}_{\mathrm{vol}} + \mathbf{S}_{\mathrm{iso}} \ . \tag{6.88}$$

This split is based on the definitions

$$\mathbf{S}_{\mathrm{vol}} = 2\frac{\partial\Psi_{\mathrm{vol}}(J)}{\partial\mathbf{C}} = Jp\mathbf{C}^{-1} \ , \tag{6.89}$$

$$\begin{aligned}\mathbf{S}_{\mathrm{iso}} &= 2\frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})}{\partial\mathbf{C}} = J^{-2/3}(\mathbb{I} - \frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}) : \overline{\mathbf{S}} \\ &= J^{-2/3}\mathrm{Dev}\overline{\mathbf{S}} = J^{-2/3}\mathbb{P} : \overline{\mathbf{S}} \ ,\end{aligned} \tag{6.90}$$

with the constitutive equations for the **hydrostatic pressure** p and the **fictitious second Piola-Kirchhoff stress** $\overline{\mathbf{S}}$ defined by

$$p = \frac{\mathrm{d}\Psi_{\mathrm{vol}}(J)}{\mathrm{d}J} \quad \text{and} \quad \overline{\mathbf{S}} = 2\frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}} \ . \tag{6.91}$$

It is important to note that in contrast to incompressible materials, the scalar function p is specified by a constitutive equation. The projection tensor $\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}$ in (6.90) furnishes the physically correct deviatoric operator in the *Lagrangian description*, i.e. $\mathrm{Dev}(\bullet) = (\bullet) - (1/3)[(\bullet) : \mathbf{C}]\mathbf{C}^{-1}$, so that

$$\mathrm{Dev}\overline{\mathbf{S}} : \mathbf{C} = 0 \ . \tag{6.92}$$

The characterization of the stress response in the material description in terms of the projection tensor $\mathbb{P}$ leads to a convenient short-hand notation (see also, for example, HOLZAPFEL [1996a]).

EXAMPLE 6.4 Consider the decoupled strain-energy function (6.85) with the associated stress relation (6.88). Perform a Piola transformation according to (3.66) and obtain the additive decomposition

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathrm{vol}} + \boldsymbol{\sigma}_{\mathrm{iso}} \tag{6.93}$$

of the Cauchy stress tensor $\boldsymbol{\sigma}$, where the purely volumetric and isochoric stress contributions are defined by

$$\boldsymbol{\sigma}_{\mathrm{vol}} = p\mathbf{I} \ , \qquad \boldsymbol{\sigma}_{\mathrm{iso}} = J^{-1}\overline{\mathbf{F}}(\mathbb{P} : \overline{\mathbf{S}})\overline{\mathbf{F}}^{\mathrm{T}} \ , \tag{6.94}$$

$\overline{\mathbf{F}}$ being the modified deformation gradient. The constitutive equations for the hydrostatic pressure p and the fictitious second Piola-Kirchhoff stress $\overline{\mathbf{S}}$ are given by (6.91).

Solution. A push-forward operation on the second Piola-Kirchhoff stress tensor $\mathbf{S}$ to the current configuration and a scaling with the inverse of the volume ratio transforms $(6.88)_1$ to

$$\boldsymbol{\sigma} = J^{-1}\chi_*(\mathbf{S}^\sharp) = 2J^{-1}\mathbf{F}\left(\frac{\partial\Psi_{\mathrm{vol}}(J)}{\partial\mathbf{C}} + \frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})}{\partial\mathbf{C}}\right)\mathbf{F}^{\mathrm{T}} \ , \tag{6.95}$$

where the decoupled form (6.85) is to be used. Hence, considering the first term on the right-hand side, we obtain, using $(6.89)_2$ and $\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$,

$$2J^{-1}\mathbf{F}\frac{\partial\Psi_{\mathrm{vol}}(J)}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}} = J^{-1}\mathbf{F}\left(Jp\mathbf{C}^{-1}\right)\mathbf{F}^{\mathrm{T}} = p\mathbf{I} \ , \tag{6.96}$$

which is the volumetric Cauchy stress contribution $\boldsymbol{\sigma}_{\mathrm{vol}}$ defining a hydrostatic stress state, as discussed on p. 125.

Considering the second term on the right-hand side of eq. $(6.95)_2$, we obtain, using $(6.90)_4$ and the kinematic assumption $(6.79)_1$,

$$2J^{-1}\mathbf{F}\frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}} = J^{-5/3}\mathbf{F}(\mathbb{P} : \overline{\mathbf{S}})\mathbf{F}^{\mathrm{T}} = J^{-1}\overline{\mathbf{F}}(\mathbb{P} : \overline{\mathbf{S}})\overline{\mathbf{F}}^{\mathrm{T}} \ , \tag{6.97}$$

which is the isochoric Cauchy stress contribution $\boldsymbol{\sigma}_{\mathrm{iso}}$. ■

Compressible isotropic hyperelasticity. A suitable *decoupled* representation of the strain-energy function for compressible *isotropic* hyperelastic materials is, by analogy with assumption (6.85), given by

$$\Psi(\mathbf{b}) = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}(\overline{\mathbf{b}}) \ , \tag{6.98}$$

with the multiplicative split of the left Cauchy-Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ in the form

$$\mathbf{b} = (J^{2/3}\mathbf{I})\overline{\mathbf{b}} = J^{2/3}\overline{\mathbf{b}} \tag{6.99}$$

(compare with eq. $(6.79)_2$). The terms $J^{2/3}\mathbf{I}$ and $\overline{\mathbf{b}} = \overline{\mathbf{F}}\overline{\mathbf{F}}^{\mathrm{T}}$ represent the volume-changing (dilational) and volume-preserving contributions to the deformation. We call $\overline{\mathbf{b}}$ the **modified left Cauchy-Green tensor**, with $\det\overline{\mathbf{b}} = 1$.

The derivative of the volume ratio $J = (\det\mathbf{b})^{1/2}$ and the modified left Cauchy-Green tensor $\overline{\mathbf{b}}$ relative to $\mathbf{b}$ is given by $(6.82)_1$ and $(6.83)_3$ (with $\mathbf{C}$ replaced by $\mathbf{b}$). We obtain

$$\frac{\partial J}{\partial \mathbf{b}} = \frac{J}{2}\mathbf{b}^{-1} \quad , \qquad \frac{\partial \overline{\mathbf{b}}}{\partial \mathbf{b}} = J^{-2/3}(\mathbb{I} - \frac{1}{3}\mathbf{b} \otimes \mathbf{b}^{-1}) \quad . \tag{6.100}$$

Following arguments analogous to those which led to eqs. (6.88)–(6.91), we obtain from (6.98) the spatial version of constitutive equations which are expressed in terms of J and $\overline{\mathbf{b}}$. Given entirely in the spatial description and characterizing the isotropic behavior of compressible hyperelastic materials, we have

$$\boldsymbol{\sigma} = 2J^{-1}\mathbf{b}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}} = 2J^{-1}\frac{\partial \Psi(\mathbf{b})}{\partial \mathbf{b}}\mathbf{b} = \boldsymbol{\sigma}_{\mathrm{vol}} + \boldsymbol{\sigma}_{\mathrm{iso}} \quad , \tag{6.101}$$

where the stress contributions are defined by

$$\boldsymbol{\sigma}_{\mathrm{vol}} = 2J^{-1}\mathbf{b}\frac{\partial \Psi_{\mathrm{vol}}(J)}{\partial \mathbf{b}} = p\mathbf{I} \quad , \tag{6.102}$$

$$\begin{aligned}
\boldsymbol{\sigma}_{\mathrm{iso}} &= 2J^{-1}\mathbf{b}\frac{\partial \Psi_{\mathrm{iso}}(\overline{\mathbf{b}})}{\partial \mathbf{b}} = 2J^{-1}\mathbf{b}J^{-2/3}(\mathbb{I} - \frac{1}{3}\mathbf{b}^{-1} \otimes \mathbf{b}) : \frac{\partial \Psi_{\mathrm{iso}}(\overline{\mathbf{b}})}{\partial \overline{\mathbf{b}}} \\
&= \mathbf{b}(J^{-2/3}\mathbb{I} - \frac{1}{3}\mathbf{b}^{-1} \otimes \overline{\mathbf{b}})\overline{\mathbf{b}}^{-1} : 2J^{-1}\frac{\partial \Psi_{\mathrm{iso}}(\overline{\mathbf{b}})}{\partial \overline{\mathbf{b}}}\overline{\mathbf{b}} \\
&= \underbrace{(\mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I})}_{\mathbb{P}} : \overline{\boldsymbol{\sigma}} = \mathrm{dev}\overline{\boldsymbol{\sigma}} \\
&= \mathbb{P} : \overline{\boldsymbol{\sigma}} \quad .
\end{aligned} \tag{6.103}$$

Use has been made of eqs. (6.99) and (6.100) and properties (1.95) and (1.155).

The constitutive equation for the hydrostatic pressure p is given in $(6.91)_1$ and the **fictitious Cauchy stress tensor** $\overline{\boldsymbol{\sigma}}$ is defined to be

$$\overline{\boldsymbol{\sigma}} = 2J^{-1}\overline{\mathbf{b}}\frac{\partial \Psi_{\mathrm{iso}}(\overline{\mathbf{b}})}{\partial \overline{\mathbf{b}}} = 2J^{-1}\frac{\partial \Psi_{\mathrm{iso}}(\overline{\mathbf{b}})}{\partial \overline{\mathbf{b}}}\overline{\mathbf{b}} \quad . \tag{6.104}$$

In $(6.103)_4$ we have introduced, additionally, the projection tensor

$$\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I} \tag{6.105}$$

which furnishes the physically correct deviatoric operator in the *Eulerian description*, i.e. $\mathrm{dev}(\bullet) = (\bullet) - (1/3)[(\bullet) : \mathbf{I}]\mathbf{I}$, so that

$$\mathrm{dev}\overline{\boldsymbol{\sigma}} : \mathbf{I} = 0 \quad . \tag{6.106}$$

For the characterization of the stress response in the spatial description in terms of the projection tensor $\mathbb{P}$ see also, for example, the work of MIEHE [1994].

Note the similar structure of the stress relations in the Lagrangian description (6.88)–(6.91) to those presented in (6.101)–(6.104).

Compressible isotropic hyperelasticity in terms of invariants. We now introduce a strain-energy function for compressible *isotropic* hyperelastic materials in terms of strain invariants. By analogy with the decoupled representation (6.85), or (6.98), we write

$$\Psi = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}[\bar{I}_1(\overline{\mathbf{C}}), \bar{I}_2(\overline{\mathbf{C}})] = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}[\bar{I}_1(\overline{\mathbf{b}}), \bar{I}_2(\overline{\mathbf{b}})] \quad , \tag{6.107}$$

with the first two strain invariants $\bar{I}_1$ and $\bar{I}_2$ of the symmetric modified Cauchy-Green tensors. Since $\overline{\mathbf{C}}$ and $\overline{\mathbf{b}}$ have the same eigenvalues, we deduce that

$$\bar{I}_1 = \bar{I}_1(\overline{\mathbf{C}}) = \bar{I}_1(\overline{\mathbf{b}}) \quad , \qquad \bar{I}_2 = \bar{I}_2(\overline{\mathbf{C}}) = \bar{I}_2(\overline{\mathbf{b}}) \quad . \tag{6.108}$$

The strain invariants $\bar{I}_a$, $a = 1, 2, 3$, are referred to as the **modified invariants** and are defined by

$$\bar{I}_1 = \mathrm{tr}\overline{\mathbf{C}} = \mathrm{tr}\overline{\mathbf{b}} \quad , \tag{6.109}$$

$$\bar{I}_2 = \frac{1}{2}\left[(\mathrm{tr}\overline{\mathbf{C}})^2 - \mathrm{tr}(\overline{\mathbf{C}}^2)\right] = \frac{1}{2}\left[(\mathrm{tr}\overline{\mathbf{b}})^2 - \mathrm{tr}(\overline{\mathbf{b}}^2)\right] \quad , \tag{6.110}$$

$$\bar{I}_3 = \det\overline{\mathbf{C}} = \det\overline{\mathbf{b}} \quad . \tag{6.111}$$

With the kinematic assumption $(6.79)_2$, or (6.99), and properties (1.92) and (1.101), we conclude from (6.109)–(6.111) with reference to (5.89)–(5.91) that

$$\bar{I}_1 = J^{-2/3}I_1 \quad , \qquad \bar{I}_2 = J^{-4/3}I_2 \quad , \qquad \bar{I}_3 = 1 \quad . \tag{6.112}$$

Finally, we formulate the associated constitutive equation in terms of the volume ratio J and the modified invariants $\bar{I}_1, \bar{I}_2$, which reads in the material description as

$$\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{S}_{\mathrm{vol}} + \mathbf{S}_{\mathrm{iso}} \quad , \tag{6.113}$$

with the volumetric contribution $\mathbf{S}_{\mathrm{vol}}$ to the second Piola-Kirchhoff stress, i.e. (6.89), and the isochoric contribution $\mathbf{S}_{\text{iso}}$, defined by

$$\mathbf{S}_{\text{iso}} = 2\frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \mathbf{C}} = J^{-2/3}\mathbb{P} : \overline{\mathbf{S}} \quad . \tag{6.114}$$

The isochoric second Piola-Kirchhoff stress tensor $\mathbf{S}_{\text{iso}}$ is $J^{-2/3}$ multiplied by the double contraction of the fourth-order projection tensor $\mathbb{P}$, see eq.(6.84), with the fictitious second Piola-Kirchhoff stress tensor $\overline{\mathbf{S}}$, which is here defined as

$$\overline{\mathbf{S}} = 2\frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \overline{\mathbf{C}}} = \overline{\gamma}_1 \mathbf{I} + \overline{\gamma}_2 \overline{\mathbf{C}} \quad , \tag{6.115}$$

with the two response coefficients given by

$$\overline{\gamma}_1 = 2\left(\frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \bar{I}_2}\right) \quad , \qquad \overline{\gamma}_2 = -2\frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \bar{I}_2} \quad . \tag{6.116}$$

The details are left to be supplied as an exercise by the reader.

For the stress response of compressible isotropic hyperelastic materials in terms of the volume ratio J and two of the modified principal stretches $\overline{\lambda}_a$ as independent variables, see the study by OGDEN [1997, Section 7.2.3].

EXERCISES

1. Consider the modified right Cauchy-Green tensor $\overline{\mathbf{C}}$ according to $(6.79)_2$ and properties (1.159), (1.134). For the fourth-order projection tensor $\mathbb{P}$ obtain the identities

$$\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C} = \mathbb{I} - \frac{1}{3}\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}} = \mathbb{I} - \frac{1}{3}(\overline{\mathbf{C}} \otimes \overline{\mathbf{C}}^{-1})^{\mathrm{T}} \quad , \qquad \mathbb{P}^n = \mathbb{P} \quad ,$$

   where n is a positive integer.

2. Show that the properties (6.92) and (6.106) hold.

3. Consider the strain-energy function $\Psi_{\text{iso}}[\bar{I}_1(\overline{\mathbf{C}}), \bar{I}_2(\overline{\mathbf{C}})]$ in terms of the modified invariants $\bar{I}_1 = J^{-2/3} I_1$ and $\bar{I}_2 = J^{-4/3} I_2$.

   (a) Show that the derivatives of $\bar{I}_1$ and $\bar{I}_2$ with respect to tensor $\overline{\mathbf{C}}$ are

$$\frac{\partial \bar{I}_1}{\partial \overline{\mathbf{C}}} = \mathbf{I} \quad , \qquad \frac{\partial \bar{I}_2}{\partial \overline{\mathbf{C}}} = \bar{I}_1 \mathbf{I} - \overline{\mathbf{C}} \quad . \tag{6.117}$$

   (b) Use the chain rule and the results (6.117) and $(6.83)_4$ in order to obtain the constitutive equation $(6.114)_2$ (with eqs. (6.115) and (6.116)) in the material description.

4. Consider the strain-energy function $\Psi = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}[\bar{I}_1(\overline{\mathbf{b}}), \bar{I}_2(\overline{\mathbf{b}})]$ with the associated constitutive equation for compressible isotropic hyperelastic materials in the spatial description, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{vol}} + \boldsymbol{\sigma}_{\text{iso}}$.

   (a) By analogy with the above Exercise 3(b), obtain the constitutive equation for the isochoric (Cauchy) stress contribution $\boldsymbol{\sigma}_{\text{iso}} = \mathbb{p} : \overline{\boldsymbol{\sigma}}$, where $\mathbb{p}$ is the fourth-order projection tensor, i.e. eq.(6.105), and $\overline{\boldsymbol{\sigma}}$ the fictitious Cauchy stress tensor, defined as

$$\overline{\boldsymbol{\sigma}} = J^{-1}(\overline{\gamma}_1 \overline{\mathbf{b}} + \overline{\gamma}_2 \overline{\mathbf{b}}^2) \quad . \tag{6.118}$$

   The response coefficients $\overline{\gamma}_1$, $\overline{\gamma}_2$ are equivalent to those given in (6.116).

   (b) Eliminate $\overline{\mathbf{b}}^2$ from (6.118) in favor of $\overline{\mathbf{b}}^{-1}$ and derive the equivalent form

$$\overline{\boldsymbol{\sigma}} = J^{-1}(\overline{\gamma}_0 \overline{\mathbf{b}} + \overline{\gamma}_2)\overline{\mathbf{b}}^{-1} \qquad \text{with} \qquad \overline{\gamma}_0 = 2\frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \bar{I}_1}$$

   for the fictitious Cauchy stress tensor, which is responsible for volume-preserving deformations. The response coefficient $\overline{\gamma}_2$ is given by $(6.116)_2$.

   Hint: Recall identity (5.93).

6.5 Some Forms of Strain-energy Functions

From previous sections we have learnt that the stress response of hyperelastic materials is derived from the given strain-energy function Ψ. Numerous *specific* forms of strain-energy functions to describe the elastic properties of incompressible as well compressible materials have been proposed in the literature and more or less efficient new specific forms are published on a daily basis.

The aim of this section is to specify some forms of strain-energy functions which are well tried within the constitutive theory of finite elasticity and frequently employed in the literature. In particular, we present a selection of representative examples of Ψ known from rubber elasticity describing *isotropic* hyperelastic materials within the isothermal regime (for a collection of constitutive models for rubber see the book edited by DORFMANN and MUHR [1999]). We start by presenting suitable strain-energy functions for *incompressible* materials and continue with some particular forms which are able to describe *compressibility*.

Ogden model for incompressible (rubber-like) materials. The only materials undergoing finite strains relative to an equilibrium state are biomaterials such as biological soft tissues and solid polymers such as **rubber-like materials**. On the latter we will focus subsequently. If we subject vulcanized rubber to very high hydrostatic pressures, we observe that it undergoes very small volume changes. To change the shape of a piece of rubber is very much easier than to change its volume. For the purpose of computational analyses, rubber is often regarded as incompressible with the constraint condition $J = \lambda_1\lambda_2\lambda_3 = 1$.

A very sophisticated development for simulating incompressible (rubber-like) materials in the phenomenological context is due to OGDEN [1972a, 1982] and [1997, Chapter 7]. The postulated strain energy is a function of the principal stretches λ_a, $a = 1, 2, 3$, is computationally simple, and plays a crucial role in the theory of finite elasticity. It describes the changes of the principal stretches from the reference to the current configuration and has the form

$$\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3) \ . \tag{6.119}$$

On comparison with the linear theory we obtain the (consistency) condition

$$2\mu = \sum_{p=1}^{N} \mu_p\alpha_p \qquad \text{with} \qquad \mu_p\alpha_p > 0 \ , \qquad p = 1, \ldots, N \ , \tag{6.120}$$

where the parameter μ denotes the classical **shear modulus** in the reference configuration, known from the linear theory.

In equation (6.119), N is a positive integer which determines the number of terms in the strain-energy function, μ_p are (constant) shear moduli and α_p are dimensionless constants, $p = 1, \ldots, N$. It emerges that only three pairs of constants ($N = 3$) are required to give an excellent correlation with experimental stress-deformation data (see TRELOAR [1944] and TRELOAR [1975]) for simple tension, equibiaxial tension and pure shear of vulcanized rubber over a very large strain range. Many scientists consider the experimental data of TRELOAR [1944] to be the essential rubber data. For a more detailed discussion of the correlation with the experimental data and for additional sources, see the works by, for example, OGDEN [1972a, 1986, 1987, 1992a, 1997], TRELOAR [1975, Section 11.2], TWIZELL and OGDEN [1983] and BEATTY [1987, Sections 8-11].

Typical values of the constants $\alpha_p, \mu_p, p = 1, 2, 3$, are

$$\left.\begin{aligned} \alpha_1 &= 1.3 & \mu_1 &= 6.3 \cdot 10^5 \mathrm{N/m^2} \\ \alpha_2 &= 5.0 & \mu_2 &= 0.012 \cdot 10^5 \mathrm{N/m^2} \\ \alpha_3 &= -2.0 & \mu_3 &= -0.1 \cdot 10^5 \mathrm{N/m^2} \end{aligned}\right\} \tag{6.121}$$

which determine the shear modulus $\mu = 4.225 \cdot 10^5 \mathrm{N/m^2}$ according to $(6.120)_1$.

VALANIS and LANDEL [1967] have postulated the hypothesis that the strain energy $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3)$ may be written as the sum of *three separate* functions $\omega(\lambda_a)$, $a = 1, 2, 3$, which depend on the principal stretches, we write $\Psi = \omega(\lambda_1) + \omega(\lambda_2) + \omega(\lambda_3)$. This additive decomposition of the strain energy is known as the **Valanis-Landel hypothesis**.

Hence, in view of the Valanis-Landel hypothesis the strain-energy function due to Ogden may be written in the equivalent form

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{a=1}^{3} \omega(\lambda_a) \qquad \text{with} \qquad \omega(\lambda_a) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_a^{\alpha_p} - 1) \ . \tag{6.122}$$

According to OGDEN [1986, 1997], separation (6.122) may also be motivated by data obtained from biaxial experiments of JONES and TRELOAR [1975].

EXAMPLE 6.5 Consider an incompressible hyperelastic membrane under *biaxial* deformation with kinematic assumption (2.132). In particular, the two principal stretches λ_1 and λ_2 are given. According to the membrane theory assume a plane stress state and specify the Cauchy stresses in the plane of the membrane by applying Ogden's strain-energy function.

Solution. The three principal values σ_a of the Cauchy stresses are given according to relation (6.69). Using (6.119) we find, after differentiation, that

$$\sigma_a = -p + \sum_{p=1}^{N} \mu_p\lambda_a^{\alpha_p} \ , \qquad a = 1, 2, 3 \ , \tag{6.123}$$

where p is a scalar not specified by a constitutive equation. It is determined from a boundary condition, namely by the requirement that $\sigma_3 = 0$ which allows p to be expressed explicitly from (6.123), setting $a = 3$, as

$$p = \sum_{p=1}^{N} \mu_p\lambda_3^{\alpha_p} \ . \tag{6.124}$$

Combining (6.124) and (6.123) we obtain the two nonzero stress components

$$\sigma_1 = \sum_{p=1}^{N} \mu_p[\lambda_1^{\alpha_p} - (\lambda_1\lambda_2)^{-\alpha_p}] \ , \tag{6.125}$$

$$\sigma_2 = \sum_{p=1}^{N} \mu_p[\lambda_2^{\alpha_p} - (\lambda_1\lambda_2)^{-\alpha_p}] \ , \tag{6.126}$$

where the incompressibility constraint $\lambda_3 = (\lambda_1\lambda_2)^{-1}$ has been used. ■

Mooney-Rivlin, neo-Hookean, Varga model for incompressible (rubber-like) materials. As a special case we obtain from eq. (6.119) the **Mooney-Rivlin model**, the **neo-Hookean model** and the **Varga model** (see Mooney [1940], Rivlin [1948, 1949a, b], Treloar [1943a, b] and Varga [1966], respectively).

For example, the very useful *Mooney-Rivlin model* results from (6.119) by setting $N = 2, \alpha_1 = 2, \alpha_2 = -2$. Using the strain invariants I_1, I_2 as presented by $(5.89)_2$ and $(5.90)_3$, with the constraint condition $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1$, we find from (6.119) that

$$\begin{aligned}\Psi &= c_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + c_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3) \\ &= c_1(I_1 - 3) + c_2(I_2 - 3) \quad , \end{aligned} \tag{6.127}$$

with the constants $c_1 = \mu_1/2$ and $c_2 = -\mu_2/2$. Adopting $(6.120)_1$ the shear modulus μ has the value $\mu_1 - \mu_2$.

The classical strain energy $\Psi = \Psi(I_1, I_2)$ of the Mooney-Rivlin form is often employed in the description of the behavior of isotropic rubber-like materials. *Mooney* derived it on the basis of mathematical arguments employing considerations of symmetry.

The *neo-Hookean model* results from (6.119) by setting $N = 1, \alpha_1 = 2$. Using the first principal invariant I_1, see eq. $(5.89)_2$, we find from (6.119) that

$$\Psi = c_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) = c_1(I_1 - 3) \quad , \tag{6.128}$$

with the constant $c_1 = \mu_1/2$ and the shear modulus $\mu = \mu_1$ according to $(6.120)_1$.

This strain-energy function involves a single parameter only and provides a mathematically simple and reliable constitutive model for the nonlinear deformation behavior of isotropic rubber-like materials. It relies on phenomenological considerations and includes typical effects known from nonlinear elasticity within the small strain domain. However, the important strain-energy function (6.128) may also be motivated from the **statistical theory** in which vulcanized rubber is regarded as a three-dimensional network of long-chain molecules that are connected at a few points. A brief discussion is given in Section 7.2 on p. 318.

Constitutive relations for the Mooney-Rivlin and the neo-Hookean model follow from (6.65) by means of $(6.127)_2$ and $(6.128)_2$. Derivatives of Ψ with respect to the strain invariants I_1 and I_2 give the simple associated stress relations $\boldsymbol{\sigma} = -p\mathbf{I} + 2c_1\mathbf{b} - 2c_2\mathbf{b}^{-1}$ and $\boldsymbol{\sigma} = -p\mathbf{I} + 2c_1\mathbf{b}$, respectively. Compare also the considerations on p. 203.

As the last special case of Ogden's model we introduce the model by *Varga*. It results from (6.119) by setting $N = 1, \alpha_1 = 1$, i.e.

$$\Psi = c_1(\lambda_1 + \lambda_2 + \lambda_3 - 3) \quad , \tag{6.129}$$

with the constant $c_1 = \mu_1$ and the shear modulus $\mu = \mu_1/2$ according to $(6.120)_1$.

Note that of all constitutive approaches given, the Ogden model with $N = 3$ excellently replicates the finite strain behavior of rubber-like materials; see, for example, Ogden [1972a] for an analytical treatment or Duffett and Reddy [1983], Sussman and Bathe [1987], Simo and Taylor [1991a] and Miehe [1994] for a numerical simulation, among many others. The assumptions made in the Varga model, the neo-Hookean model (obeying *(Gaussian) statistical theory*) or in the Mooney-Rivlin model are rather simple. Consequently, these types of constitutive model are not able to capture the finite extensibility domain of polymer chains (see Treloar [1976]).

EXAMPLE 6.6 This example has the aim of investigating the inflation of a spherical *(incompressible rubber) balloon* with different material models. Analyses of balloon inflations have some applications in producing, for example, meteorological balloons for high-altitude measurements or balloon-tipped catheters for clinical treatments. Inflation experiments of spherical neoprene balloons were carried out by Alexander [1971].

In particular, compute the inflation pressure p_i, i.e. the internal pressure in the balloon, and the circumferential Cauchy stress σ as a function of the circumferential stretch λ of the balloon. Let the initial (zero-pressure) radius of the rubber balloon be $R = 10.0$ and the initial thickness of the wall be $H = 0.1$. For the geometrical situation of the spherical balloon in the reference and current configuration see Figure 6.2.

On the basis of the described four prototypes of constitutive models, that are the *Ogden, Mooney-Rivlin, neo-Hookean* and *Varga models*, study the different mechanical behavior and compare the solutions, drawn in a diagram. Do not consider aspherical modes which clearly develop during the inflation process.

The material properties for the Ogden model are given according to (6.121), with the shear modulus $\mu = 4.225 \cdot 10^5 \mathrm{N/m}^2$ in the reference configuration. For the Mooney-Rivlin model take $c_1 = 0.4375\mu, c_2 = 0.0625\mu$ $(c_1/c_2 = 7)$, as suggested by Anand [1986], for the neo-Hookean model $c_1 = \mu/2$, and for the Varga model take $c_1 = 2\mu$.

Solution. We know from a perfect sphere under inflation pressure p_i that every direction in the plane of the sphere is a principal direction. Hence, the stretch ratio is $\lambda = \lambda_1 = \lambda_2$ which characterizes an equibiaxial deformation. The associated circumferential Cauchy stress is $\sigma = \sigma_1 = \sigma_2$ (while $\sigma_3 = 0$ by the assumption of plane stress). Hence, constitutive equations (6.125) and (6.126) reduce to a single relation, namely

$$\sigma = \sum_{p=1}^{N} \mu_p(\lambda^{\alpha_p} - \lambda^{-2\alpha_p}) \quad , \tag{6.130}$$

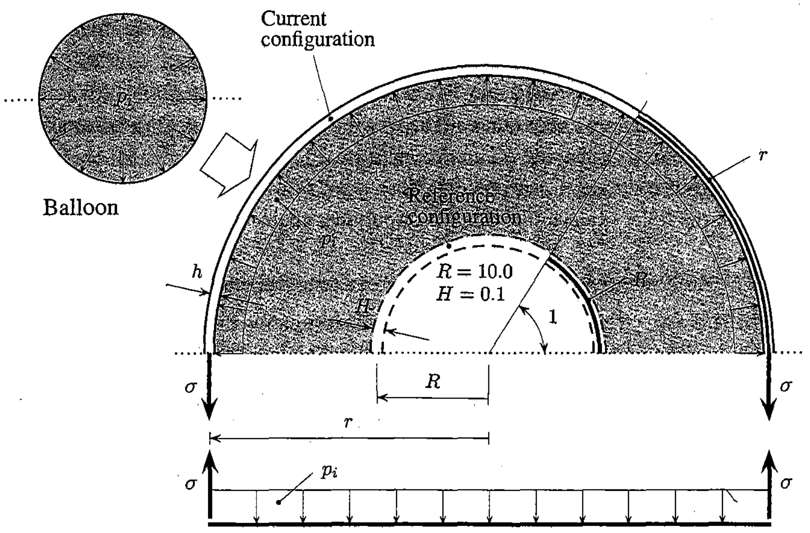

Figure 6.2 Geometry of a spherical balloon in the reference and the current configuration, showing only one hemisphere.

which is plotted in Figure 6.3 for different material parameters. The figure illustrates the relationships between the Cauchy stress σ and the circumferential stretch λ of any point of the rubber balloon for various constitutive models.

By equilibrium we find from Figure 6.2 (free-body diagram) that $r^2\pi p_i = 2r\pi h\sigma$, where r and h denote the radius and the wall thickness of the rubber balloon in the current configuration. According to this condition we find that

$$p_i = 2\frac{h}{r}\sigma \ . \tag{6.131}$$

In view of the kinematical situation of the inflated balloon (see Figure 6.2) the stretch λ at a certain point of the balloon is r/R. Incompressibility requires that the wall volume is conserved, which means that $4\pi r^2 h = 4\pi R^2 H$. With this condition we find that $\lambda_3 = h/H = 1/\lambda^2$ which denotes the stretch in the direction perpendicular to the surface of the sphere, indeed $\lambda_1\lambda_2\lambda_3 = \lambda^2\lambda_3 = 1$.

Using these relations and constitutive equation (6.130) we may find from (6.131) the analytical expression

$$p_i = 2\frac{H}{R}\sum_{p=1}^{N}\mu_p(\lambda^{\alpha_p-3} - \lambda^{-2\alpha_p-3}) \tag{6.132}$$

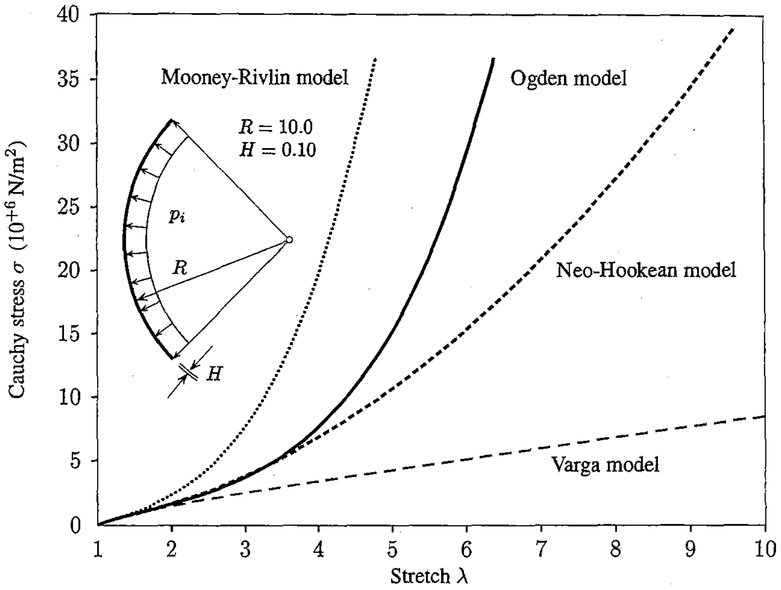

Figure 6.3 Geometry and Cauchy stress σ versus stretch λ of any point of the balloon.

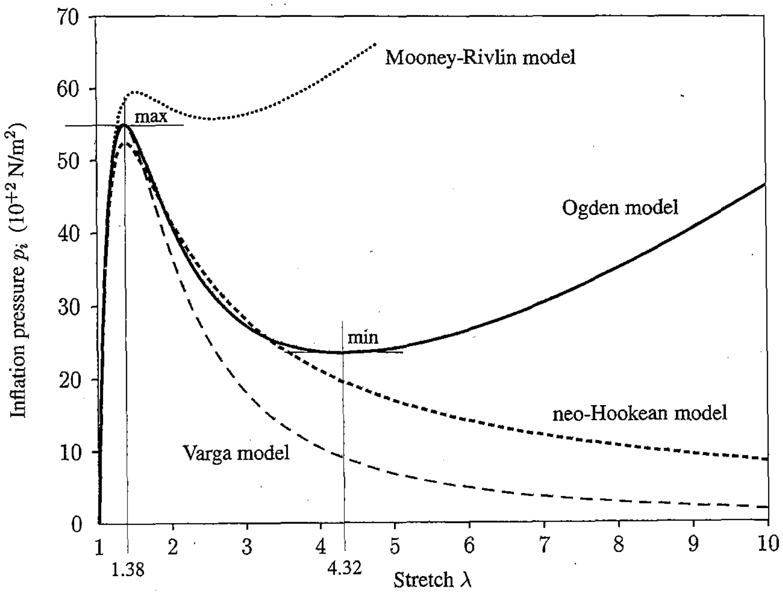

Figure 6.4 Inflation pressure p_i versus stretch λ of any point of the balloon.

for the inflation pressure p_i, which is plotted in Figure 6.4, for $N = 1, 2, 3$, and with material parameters as given above.

The analytical solutions of the six parameter material model proposed by Ogden are in excellent agreement with experimental data by TRELOAR [1944] (see also NEEDLEMAN [1977]) who solved the problem on the basis of the Ritz-Galerkin method. Experimental data show a very stiff initial stage (as seen with any party balloon) in which the inflation pressure p_i rises steeply with the circumferential stretch. After the pressure has reached a maximum the rubber balloon will suddenly *'snap through'*, and a release of the pressure will allow it to *'snap back'* (see Figure 6.4, see also the studies by OGDEN [1972a] and BEATTY [1987] among others). This effect is caused by the deformation dependent pressure load, is dynamic in character and known as **snap buckling**. The pressure-stretch path clearly shows the existence of a local maximum and minimum, the maximum and minimum pressures are at $\lambda = 7^{1/6} = 1.38$ and $\lambda = 4.32$, respectively.

The curve for the Mooney-Rivlin model, with $c_1/c_2 = 7$, shows the characteristic behavior of a spherical rubber balloon, but, however, the results based on Ogden's model (and Treloar's experimental data on vulcanized rubber) are significantly different.

The neo-Hookean and Varga form of the strain energy reproduces more or less the real behavior of the balloon for small strain ranges. Reasonable correlations for all material models are obtained at the low strain level. However, for finite strains the typical characteristic of the load-deflection curve cannot be reproduced with the simplified neo-Hookean and Varga model.

Experimental investigations show that the balloon develops a bulge on one side and becomes aspherical (compare with NEEDLEMAN [1977]). The bifurcations of pressurized elastic spherical shells from an analytical point of view are studied by HAUGHTON and OGDEN [1978], HAUGHTON [1980] and ERICKSEN [1998, Chapter 5]. Compressibility effects are considered by HAUGHTON [1987]. ■

Yeoh, Arruda and Boyce model for incompressible (rubber-like) materials. Nearly all practical engineering elastomers contain reinforcing fillers such as *carbon black* (in natural rubber vulcanizate) or *silica* (in silicone rubber). These finely distributed fillers, which have typical dimensions of the order of $1.0 - 2.0 \cdot 10^{-12}$m, form physical and chemical bonds with the polymer chains. The fine filler particles are added to the elastomers in order to improve their physical properties which are mainly tensile and tear strength, or abrasion resistance. The associated stress-strain behavior is observed to be highly nonlinear (see, for example, GENT [1962]). Carbon-black filled rubbers have important applications in the manufacture of automotive tyres and many other engineering components.

It turns out that for carbon-black filled rubbers the strain-energy functions described hitherto in this section are not adequate to approximate the observed physical behavior. For example, consider a simple shear deformation of a filler-loaded rubber. Physical observations show that the shear modulus μ of the material varies with deformation in a significant way. To be more specific, μ decreases with increasing deformation initially and then rises again at large deformations (see YEOH [1990, Figure 2]). The associated relation for the shear stress is clearly nonlinear. Now, taking, for example, the Mooney-Rivlin model according to strain-energy function $(6.127)_2$, then, from the explicit expression $(6.78)_2$ we may specify a shear modulus

$$\mu = 2\left(\frac{\partial \Psi}{\partial I_1} + \frac{\partial \Psi}{\partial I_2}\right) = 2(c_1 + c_2) > 0 \quad . \tag{6.133}$$

The relation for the shear stress is, however, linear with the constant slope $2(c_1 + c_2)$, i.e. the shear modulus. Apparently the Mooney-Rivlin (and its neo-Hookean specialization) model is too simple for the characterization of the elastic properties of carbon-black filled rubber vulcanizates.

The phenomenological material model by YEOH [1990] is motivated in order to simulate the mechanical behavior of carbon-black filled rubber vulcanizates with the typical stiffening effect in the large strain domain. Published data for filled rubbers (see KAWABATA and KAWAI [1977] and SEKI et al. [1987]) suggest that $\partial\Psi/\partial I_2$ is numerically close to zero. *Yeoh* made a simplifying assumption that $\partial\Psi/\partial I_2$ is equal to zero and proposed a three-term strain-energy function where the second strain invariant does not appear. It has the specific form

$$\Psi = c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3 \quad , \tag{6.134}$$

where c_1, c_2, c_3 are material constants which must satisfy certain restrictions.

Since by (6.5) the strain-energy function Ψ is either *zero* (in which case Ψ has only one real root, corresponding to $I_1 = 3$) or *positive*, we must have $I_1 > 3$ (note that for an incompressible material $I_1 \geq 3$ with the equality only in the reference configuration). Hence, the (convex) strain-energy function increases monotonically with I_1 and $\partial\Psi/\partial I_1 = 0$ has no real roots. From the discriminants of the respective cubic and quadratic equations in $(I_1 - 3)$ the appropriate restrictions on the values for c_1, c_2, c_3 may be determined.

Recall the simple shear deformation example of a filled rubber from above once more. With the strain energy (6.134) we now conclude from eq. $(6.133)_1$ that

$$\mu = 2c_1 + 4c_2(I_1 - 3) + 6c_3(I_1 - 3)^2 > 0 \quad . \tag{6.135}$$

The shear modulus μ involves first-order and second-order terms in $(I_1 - 3)$ and approximates the observed nonlinear physical behavior with satisfying accuracy (provided $c_2 < 0$ and $c_1 > 0, c_3 > 0$).

Another material model for the response of rubber which has a similar structure to (6.134) is due to ARRUDA and BOYCE [1993]. It is, however, a statistical model where the parameters are physically linked to the chain orientations involved in the deformation of the three-dimensional network structure of the rubber. The molecular network structure is represented by an eight-chain model which replaces classical three and four-chain models. The individual polymer chains in the network are described by the non-Gaussian statistical theory and are able to capture the finite extensibility domain. The physically based constitutive model possesses symmetry with respect to the principal stretch space.

The strain-energy function is derived from the *inverse Langevin function* (see, for example, TRELOAR [1975, Chapter 6]) by means of Taylor's expansion (compare with TRELOAR [1954]). Here we present the first three terms for the strain energy, i.e.

$$\Psi = \mu\left[\frac{1}{2}(I_1-3)+\frac{1}{20n}(I_1^2-9)+\frac{11}{1050n^2}(I_1^3-27)+\ldots\right] , \qquad (6.136)$$

where μ denotes the shear modulus and n is the number of segments (each of the same length) in a chain, freely jointed together at chemical cross-links. For a more detailed explanation of the underlying concept of statistical thermodynamics the reader is referred to Section 7.2 of this text.

In this two parameter model the first strain invariant I_1 may be linked to the stretch in a chain, λ_{chain}, by the expression $\sqrt{I_1}=\sqrt{3}\lambda_{\mathrm{chain}}$. The chain stretch λ_{chain} is defined to be the current chain length divided by the initial chain length. An advantageous feature of the eight-chain model (6.136) is that all chains stretch equally under uniform extension and biaxial extension.

For further network models which consider chain interactions see, for example, the book by TRELOAR [1975, Chapter 6] and the articles by FLORY and ERMAN [1982] and ANAND [1996].

Ogden model for compressible (rubber-like) materials. Rubber-like materials in the 'rubbery' state used in engineering are often slightly compressible and associated with minor dilatational deformations. Compressibility is accounted for by the addition of a strain energy Ψ_{vol}, describing the purely *volumetric elastic response* (see the framework of compressible hyperelasticity, Section 6.4). For our considerations, in particular, we use the decoupled representation of the strain-energy function $\Psi(\lambda_1,\lambda_2,\lambda_3)=\Psi_{\mathrm{vol}}(J)+\Psi_{\mathrm{iso}}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3)$ expressed in terms of principal stretches.

For rubber-like materials, OGDEN [1972b] proposed a volumetric response function in terms of the volume ratio J of the following form

$$\Psi_{\mathrm{vol}}(J)=\kappa\mathcal{G}(J) \qquad \text{with} \qquad \mathcal{G}=\beta^{-2}(\beta\ln J+J^{-\beta}-1) , \qquad (6.137)$$

for $\beta>0$. The scalar-valued scalar function $\mathcal{G}$ characterizes a strictly convex function, and κ and β denote the constant **bulk modulus** in the reference configuration and an **(empirical) coefficient**, respectively. The strain energy (6.137) satisfies the normalization condition, $\Psi_{\mathrm{vol}}(1)=0$. Note that this empirical function meets experimental results with excellent accuracy (see OGDEN [1972b, Figure 1]), indicating that rubber-like materials are (slightly) compressible. In particular, for $\beta=9$, the distribution of the hydrostatic pressure is in good agreement with experimental data of ADAMS and GIBSON [1930] and BRIDGMAN [1945].

An alternative version of $(6.137)_2$, due to SIMO and MIEHE [1992], is obtained by setting $\beta=-2$ to give

$$\mathcal{G}=\frac{1}{4}(J^2-1-2\ln J) . \qquad (6.138)$$

The second part of the decoupled strain energy, i.e. $\Psi_{\mathrm{iso}}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3)$, describes the purely *isochoric elastic response* in terms of modified principal stretches $\bar\lambda_a=J^{-1/3}\lambda_a$, $a=1,2,3$. We have

$$\Psi_{\mathrm{iso}}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3)=\sum_{a=1}^{3}\varpi(\bar\lambda_a) \qquad \text{with} \qquad \varpi(\bar\lambda_a)=\sum_{p=1}^{N}\frac{\mu_p}{\alpha_p}(\bar\lambda_a^{\alpha_p}-1) ; \qquad (6.139)$$

and with the condition (6.120).

EXAMPLE 6.7 Consider the decomposed structure (6.137) and (6.139) of the strain energy and the additive split of the second Piola-Kirchhoff stress tensor $(6.88)_2$. With specifications $(6.137)_2$ and (6.138) find the purely *volumetric contribution* $\mathbf{S}_{\mathrm{vol}}$ of the stress response in the explicit form. In addition, with $(6.139)_2$, find the spectral decomposition of $\mathbf{S}_{\mathrm{iso}}$, i.e. the purely *isochoric contribution.*

Solution. In order to particularize the volumetric stress $\mathbf{S}_{\mathrm{vol}}$ it is only necessary to derive the term $\mathrm{d}\Psi_{\mathrm{vol}}(J)/\mathrm{d}J$ (see eq. $(6.91)_1$). From eq. (6.89) we find, using $(6.91)_1$ and the relation for the purely volumetric elastic response in the form of $(6.137)_1$, that

$$\mathbf{S}_{\mathrm{vol}}=2\frac{\partial\Psi_{\mathrm{vol}}(J)}{\partial\mathbf{C}}=Jp\mathbf{C}^{-1} ; \qquad p=\frac{\mathrm{d}\Psi_{\mathrm{vol}}(J)}{\mathrm{d}J}=\kappa\frac{\mathrm{d}\mathcal{G}(J)}{\mathrm{d}J} , \qquad (6.140)$$

in which, with the strain-energy functions $(6.137)_2$ and (6.138), we obtain the specification

$$\frac{\mathrm{d}\mathcal{G}(J)}{\mathrm{d}J}=\frac{1}{\beta J}\left(1-\frac{1}{J^\beta}\right) ; \qquad \frac{\mathrm{d}\mathcal{G}(J)}{\mathrm{d}J}=\frac{1}{2J}(J^2-1) . \qquad (6.141)$$

As a second step we particularize the isochoric stress $\mathbf{S}_{\mathrm{iso}}$ in respect of the strain energy (6.139). Before proceeding it is first necessary to provide the relation $\partial\bar\lambda_a/\partial\lambda_b$.

Recall (6.81), i.e. $\overline{\lambda}_a = J^{-1/3}\lambda_a$, $a = 1,2,3$, and relation $(6.82)_1$ which, when formulated in principal stretches, reads $\partial J/\partial\lambda_a = J\lambda_a^{-1}$. Thus, we have

$$\begin{aligned}\frac{\partial\overline{\lambda}_a}{\partial\lambda_b} &= \frac{\partial(J^{-1/3}\lambda_a)}{\partial\lambda_b} = J^{-1/3}(\delta_{ab} - \frac{1}{3}J^{-1}\frac{\partial J}{\partial\lambda_b}\lambda_a) \\ &= J^{-1/3}(\delta_{ab} - \frac{1}{3}\overline{\lambda}_a\overline{\lambda}_b^{-1}) \quad ; \qquad a,b = 1,2,3 \quad ; \qquad (6.142)\end{aligned}$$

which is relation (6.83) expressed through the modified principal stretches $\overline{\lambda}_a$.

Hence, by analogy with (6.52), we obtain the isochoric stress response in terms of principal values for the general case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, namely

$$\mathbf{S}_{\text{iso}} = 2\frac{\partial\Psi_{\text{iso}}(\overline{\lambda}_1,\overline{\lambda}_2,\overline{\lambda}_3)}{\partial\mathbf{C}} = \sum_{a=1}^{3}\underbrace{\frac{1}{\lambda_a}\frac{\partial\Psi_{\text{iso}}}{\partial\lambda_a}}_{S_{\text{iso}\,a}}\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_a \quad , \qquad (6.143)$$

where $S_{\text{iso}\,a}$, $a = 1,2,3$, are the principal values of the second Piola-Kirchhoff stress tensor $\mathbf{S}_{\text{iso}}$ and $\hat{\mathbf{N}}_a$, $a = 1,2,3$, denote the principal referential directions.

By use of the chain rule and relation $(6.142)_3$, a straightforward computation from $(6.143)_2$ gives the explicit expressions

$$S_{\text{iso}\,a} = \frac{1}{\lambda_a}\frac{\partial\Psi_{\text{iso}}}{\partial\lambda_a} = \frac{1}{\lambda_a^2}\left(\overline{\lambda}_a\frac{\partial\Psi_{\text{iso}}}{\partial\overline{\lambda}_a} - \frac{1}{3}\sum_{b=1}^{3}\overline{\lambda}_b\frac{\partial\Psi_{\text{iso}}}{\partial\overline{\lambda}_b}\right) \quad ; \qquad a = 1,2,3 \quad (6.144)$$

for the principal isochoric stress values (compare with OGDEN [1997, Section 7.2.3]). The summation symbol (which could be omitted) emphasizes that the index b is repeated, meaning summation over $1,2,3$. However, there is no summation over the index a. Using the relation for the purely isochoric elastic response in the form of (6.139), we achieve finally the term $\partial\Psi_{\text{iso}}/\partial\overline{\lambda}_a$, $a = 1,2,3$, in the specific form

$$\frac{\partial\Psi_{\text{iso}}}{\partial\overline{\lambda}_a} = \frac{\partial\overline{w}(\overline{\lambda}_a)}{\partial\overline{\lambda}_a} = \sum_{p=1}^{N}\mu_p\overline{\lambda}_a^{\alpha_p-1} \quad ; \qquad a = 1,2,3 \qquad (6.145)$$

(see also the derivation by SIMO and TAYLOR [1991a]).

~~The~~ complete stress response, as given through (6.140), (6.141) and the spectral decomposition (6.143)–(6.145), serves as a meaningful basis for finite element analyses of constitutive models for isotropic hyperelastic materials at finite strains. ■

Similarly to the compressible version of Ogden's model we can reformulate the *Mooney-Rivlin*, *neo-Hookean*, *Varga*, *Yeoh*, *Arruda and Boyce* models, i.e. (6.127)–(6.129), (6.134), (6.136), as *decoupled* representations. We have just to replace λ_a, I_a by the *modified* quantities $\overline{\lambda}_a, \overline{I}_a$, as defined in (6.81), (6.109)–(6.111) and to add a suitable volumetric response function Ψ_{vol}, for example, $(6.137)_2$ or (6.138).

For example, the decoupled strain-energy function for the Mooney-Rivlin model has the form

$$\Psi(J,\overline{I}_1,\overline{I}_2) = \Psi_{\text{vol}}(J) + c_1(\overline{I}_1 - 3) + c_2(\overline{I}_2 - 3) \quad . \qquad (6.146)$$

However, material models are often presented in a *coupled* form. The compressible Mooney-Rivlin model, for example, may be given as

$$\Psi(J,I_1,I_2) = c(J-1)^2 - d\ln J + c_1(I_1 - 3) + c_2(I_2 - 3) \quad , \qquad (6.147)$$

where c is a material constant and d defines a (dependent) parameter with certain restrictions. By recalling the assumption that the reference configuration is stress-free we may deduce from (6.147) that $d = 2(c_1 + 2c_2)$. The first two terms in (6.147) were proposed by CIARLET and GEYMONAT [1982] in a slightly different form (see also CIARLET [1988, Section 4.10]).

Another example is the *coupled* form of the compressible neo-Hookean model given by the strain-energy function

$$\Psi(I_1,J) = \frac{c_1}{\beta}(J^{-2\beta} - 1) + c_1(I_1 - 3) \quad , \qquad \beta = \frac{\nu}{1-2\nu} \qquad (6.148)$$

(see, for example, BLATZ [1971]), with the constants $c_1 = \mu/2$ and β. The material parameters μ and ν denote the **shear modulus** and **Poisson's ratio**, respectively.

Blatz and Ko model. For foamed elastomers which cannot be regarded as incompressible BLATZ and KO [1962] and OGDEN [1972b] proposed a strain-energy function which combines theoretical arguments and experimental data (performed on certain solid polyurethane rubbers and foamed polyurethane elastomers). It is based on a *coupled* function of volumetric and isochoric parts according to

$$\begin{aligned}\Psi(I_1,I_2,I_3) = {}& f\frac{\mu}{2}\left[(I_1 - 3) + \frac{1}{\beta}(I_3^{-\beta} - 1)\right] \\ &+(1-f)\frac{\mu}{2}\left[\left(\frac{I_2}{I_3} - 3\right) + \frac{1}{\beta}(I_3^{\beta} - 1)\right] \quad , \qquad (6.149)\end{aligned}$$

in which μ and ν denote the shear modulus and Poisson's ratio, and $f \in [0,1]$ is an interpolation parameter. By means of the incompressibility constraint $I_3 = 1$, eq. (6.149) reduces to the Mooney-Rivlin form introduced in eq. (6.127) (with the constants $c_1 = f\mu/2$ and $c_2 = (1-f)\mu/2$).

Another special case of the strain energy (6.149) may be found by taking $f = 1$, leading to the compressible neo-Hookean model introduced in eq. (6.148) (with $I_3 = J^2$ and the constant $\mu/2 = c_1$). An interesting description of the Blatz and Ko model was presented by BEATTY and STALNAKER [1986] and BEATTY [1987].

EXERCISES

1. For the description of isotropic hyperelastic materials at finite strains we recall the important class of strain-energy functions Ψ in terms of principal invariants. Study some models suitable to describe compressible materials and particularize the associated stress relations.

   (a) Firstly, we consider the *coupled* form of the compressible *Mooney-Rivlin, neo-Hookean, Blatz and Ko* models according to the given strain energies (6.147)–(6.149), respectively. By means of (6.32) deduce the stress relation

$$\mathbf{S} = 2\frac{\partial \Psi(I_1, I_2, I_3)}{\partial \mathbf{C}} = \gamma_1 \mathbf{I} + \gamma_2 \mathbf{C} + \gamma_3 \mathbf{C}^{-1} \quad ,$$

   with the three response coefficients $\gamma_1 = 2(\partial\Psi/\partial I_1 + I_1 \partial\Psi/\partial I_2)$, $\gamma_2 = -2\partial\Psi/\partial I_2$, $\gamma_3 = 2I_3\partial\Psi/\partial I_3$ for the second Piola-Kirchhoff stress tensor $\mathbf{S}$ as specified in Table 6.1.

	Mooney-Rivlin model (6.147)	*neo-Hookean* model (6.148)	*Blatz and Ko* model (6.149)
γ_1	$2(c_1 + c_2 I_1)$	$2c_1$	$\mu f + \xi I_1/2$
γ_2	$-2c_2$	0	$-\xi/2$
γ_3	$2cJ(J-1) - d$	$-2c_1 I_3^{-\beta}$	$-\mu f I_3^{-\beta}$ $-\xi(I_2 - I_3^{\beta+1})/2$

Table 6.1 Specified coefficients for the constitutive equations of some materials in the coupled form.

   For notational simplicity we have introduced the non-negative parameter $\xi = 2\mu(1-f)/I_3$. Note that the response coefficients $\gamma_1, \gamma_2, \gamma_3$ for the *neo-Hookean* model may be found as a special case of the *Blatz and Ko* model by taking $\xi = 0$ (with the constant $\mu f = 2c_1$).

   (b) Secondly, we consider the *decoupled* form of the compressible *Mooney-Rivlin* model (6.146) and the compressible *neo-Hookean* model (obtained by setting $c_2 = 0$ in the Mooney-Rivlin model). In addition, we consider the decoupled versions of the *Yeoh* model and the *Arruda and Boyce* model, i.e.

$$\Psi = \Psi_{\text{vol}} + c_1(\bar{I}_1 - 3) + c_2(\bar{I}_1 - 3)^2 + c_3(\bar{I}_1 - 3)^3 \quad ;$$

$$\Psi = \Psi_{\text{vol}} + \mu\left[\frac{1}{2}(\bar{I}_1 - 3) + \frac{1}{20n}(\bar{I}_1^2 - 9) + \frac{11}{1050n^2}(\bar{I}_1^3 - 27) + \ldots\right]$$

   (just replace I_1 in eqs. (6.134), (6.136) by the modified first invariant $\bar{I}_1$). Derive the associated stress relations

$$\overline{\mathbf{S}} = 2\frac{\partial \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2)}{\partial \overline{\mathbf{C}}} = \overline{\gamma}_1 \mathbf{I} + \overline{\gamma}_2 \overline{\mathbf{C}}$$

   for the fictitious second Piola-Kirchhoff stress tensor, with the specified response coefficients $\overline{\gamma}_1, \overline{\gamma}_2$ (see eq. (6.116)) according to Table 6.2.

	Mooney-Rivlin model (6.146)	*neo-Hookean* model	*Yeoh* model (6.134)	*Arruda and Boyce* model (6.136)
$\overline{\gamma}_1$	$2(c_1 + c_2 \bar{I}_1)$	$2c_1$	$2c_1 + 4c_2(\bar{I}_1 - 3)$ $+6c_3(\bar{I}_1 - 3)^2$	$\mu[1 + (1/5n)\bar{I}_1$ $+(11/175n^2)\bar{I}_1^2 + \ldots]$
$\overline{\gamma}_2$	$-2c_2$	0	0	0

Table 6.2 Specified coefficients for the constitutive equations of some materials in the decoupled form.

2. Consider a spherical balloon of incompressible hyperelastic material ($\lambda_1\lambda_2\lambda_3 = 1$). The material is characterized by a strain-energy function in terms of principal stretches according to

$$\Psi = \frac{\mu}{\alpha}(\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3) \quad , \qquad (6.150)$$

   with material constants μ and α. For $\alpha = 2$ we obtain the classical neo-Hookean model.

   (a) Determine the inflation pressure p_i as a function of the circumferential stretch λ in the form

$$p_i = 2\mu\frac{H}{R}(\lambda^{\alpha-3} - \lambda^{-2\alpha-3}) \quad ,$$

   where H and R are the initial (zero-pressure) thickness and radius of the spherical balloon, respectively.

   (b) Show that the function $p_i = p_i(\lambda)$ has a relative maximum if $0 < \alpha < 3$ and a relative minimum if $-3/2 < \alpha < 0$ (see OGDEN [1972a]).

   Consequently, balloons made of materials which are described by the strain-energy function (6.150) will not *'snap through'* for $\alpha \gg 3$. Typical examples for this type of (stable) material behavior are biomaterials such as an *artery* (for the mechanics of the arterial wall see the excellent survey text by HUMPHREY [1995], and the papers by HOLZAPFEL et al. [2000] and HOLZAPFEL [2001]) or a *ventricle* (see NEEDLEMAN et al. [1983]). Specific results on membrane biomechanics including illustrative examples from the literature are reviewed in the article by HUMPHREY [1998].

3. Consider a thin sheet of incompressible hyperelastic material with the same setting as formulated in Example 6.3. The homogeneous stress response of the material is assumed to be isotropic and based on Ogden's model. Discuss the stress states (which are plane throughout the sheet) for the following two modes of deformations (for the associated kinematic relations, compare with Exercise 1 on p. 226):

   (a) Consider a *simple tension* for which $\lambda_1 = \lambda$ $(\lambda_2 = \lambda_3)$. Show that the only nonzero Cauchy stress σ, in the direction of the applied stretch λ, is

   $$\sigma = \sum_{p=1}^{N} \mu_p(\lambda^{\alpha_p} - \lambda^{-\alpha_p/2}) \ .$$

   In addition, find the stress required to produce a final extended length of $\lambda = 2$ for each of the Mooney-Rivlin and the neo-Hookean models.

   (b) Consider a homogeneous *pure shear deformation* and show that the biaxial stress state $(\sigma_3 = 0)$ of the problem is of the form

   $$\sigma_1 = \sum_{p=1}^{N} \mu_p(\lambda^{\alpha_p} - \lambda^{-\alpha_p}) \ , \qquad \sigma_2 = \sum_{p=1}^{N} \mu_p(1 - \lambda^{-\alpha_p}) \ .$$

   (c) Compute the associated constitutive equations given in (a) and (b) for the Mooney-Rivlin, neo-Hookean and Varga models and compare the results with Ogden's model (plot the relation between the Cauchy stress and the associated stretch ratio for each material model).

4. The so-called **Saint-Venant Kirchhoff model** is characterized by the strain-energy function

   $$\Psi(\mathbf{E}) = \frac{\gamma}{2}(\mathrm{tr}\mathbf{E})^2 + \mu \mathrm{tr}\mathbf{E}^2 \qquad (6.151)$$

   (see, for example, CIARLET [1988, p. 155]), in which $\gamma > 0$ and $\mu > 0$ are the two constants of **Lamé**. The Lamé constant γ is usually denoted in the literature by the symbol λ. However, in order to avoid confusion with the stretch ratio λ we use a different symbol for it. The Saint-Venant Kirchhoff model is a classical nonlinear model for compressible hyperelastic materials often used for metals. Note that the volume ratio J does not appear explicitly in this material model.

   (a) From the given strain energy $\Psi(\mathbf{E})$ derive the second Piola-Kirchhoff stress $\mathbf{S}$, which linearly depends on the Green-Lagrange strain $\mathbf{E}$.

   (b) Consider the one-dimensional case of the constitutive equation derived in (a). For a *uniform deformation* of a rod (with uniform cross-section), i.e. $x = \lambda X$, derive the relation between the nominal stress P and the associated stretch ratio λ (which is a cubic equation in λ) and plot the function $P = P(\lambda)$.

   Show that $P(\lambda)$ is not monotonic in compression and derive the critical stretch value $\lambda_{\mathrm{crit}} = (1/3)^{1/2}$ at which the Saint-Venant Kirchhoff model fails (zero stiffness, the tangent of $P(\lambda)$ at λ_{crit} is horizontal). This failure is not influenced by the material constants γ and μ.

   In addition show that the material model does not satisfy the growth condition $(6.6)_2$ (in fact for $\lambda \to 0^+$ the stress tends to zero which is physically unrealistic). CIARLET [1988] showed, with the proof by RAOULT [1986], that the Saint-Venant Kirchhoff model does not satisfy the requirement of polyconvexity either.

   Note that this material model is suitable for large displacements but it is *not* recommended to use it for large compressive strains.

5. In the first term in eq. (6.151) replace $\mathrm{tr}\mathbf{E}$ by $\ln J$ and γ by $\kappa > 0$ in order to obtain a **modified Saint-Venant Kirchhoff model** of the form

   $$\Psi(\mathbf{E}) = \frac{\kappa}{2}(\ln J)^2 + \mu \mathrm{tr}\mathbf{E}^2 \ , \qquad (6.152)$$

   where $J = \det\mathbf{F}$ denotes the volume ratio. The proposed material model (6.152) circumvents the serious drawbacks of the classical Saint-Venant Kirchhoff model (see Exercise 4) when used for large compressive strains.

   (a) From the strain energy (6.152) derive the second Piola-Kirchhoff stress $\mathbf{S} = \mathbf{S}(\mathbf{C})$ as a function of the right Cauchy-Green tensor $\mathbf{C}$. The result is similar to a stress relation proposed by CURNIER [1994, eq. (6.113)] which also has the aim of avoiding the defects of the classical Saint-Venant Kirchhoff model occurring at large compressive strains.

   (b) Consider a one-dimensional problem as described in Exercise 4(b) and obtain the nominal stress P as a function of λ. Discuss the function $P(\lambda)$ for the two regions $\lambda > 1$, $\lambda < 1$ and show that the modified Saint-Venant Kirchhoff model satisfies the growth condition in the sense that the stress tends to (minus) infinity for $\lambda \to 0^+$.

6.6 Elasticity Tensors

In order to obtain solutions of nonlinear (initial boundary-value) problems in computational finite elasticity and inelasticity so-called **incremental/iterative solution techniques** of *Newton's type* are frequently applied to solve a sequence of linearized problems.

This strategy requires knowledge of the linearized constitutive equation, here presented in both the material and spatial descriptions. The underlying technique was first introduced in the mechanics of solids and structures by HUGHES and PISTER [1978]. The process of linearizing constitutive equations is a very important task in computational mechanics and the main objective of this section. For more on the concept of linearization, which is basically differentiation, see Section 8.4.

Material and spatial representations of the elasticity tensor. Consider the nonlinear second Piola-Kirchhoff stress tensor $\mathbf{S}$ of a point at a certain time t. We look at $\mathbf{S}$ as a nonlinear tensor-valued tensor function of one variable. We assume this variable to be the right Cauchy-Green tensor $\mathbf{C}$.

First of all we do not assume that the stress tensor is derived from a strain-energy function Ψ. According to considerations (1.247) and (1.248) we are now in a position to determine the total differential

$$d\mathbf{S} = \mathbb{C} : \frac{1}{2} d\mathbf{C} \quad , \tag{6.153}$$

in which we have introduced the definition

$$\mathbb{C} = 2\frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} \qquad \text{or} \qquad C_{ABCD} = 2\frac{\partial S_{AB}}{\partial C_{CD}} \quad , \tag{6.154}$$

which, by means of the chain rule, reads, in terms of the Green-Lagrange strain tensor $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$,

$$\mathbb{C} = \frac{\partial \mathbf{S}(\mathbf{E})}{\partial \mathbf{E}} \qquad \text{or} \qquad C_{ABCD} = \frac{\partial S_{AB}}{\partial E_{CD}} \quad . \tag{6.155}$$

The quantity $\mathbb{C}$ characterizes the gradient of function $\mathbf{S}$ and relates the work conjugate pairs of stress and strain tensors. It measures the change in stress which results from a change in strain and is referred to as the **elasticity tensor** in the *material description* or the **referential tensor of elasticities**. It is a tensor of rank four with the four indices A, B, C, D.

The elasticity tensor $\mathbb{C}$ is always symmetric in its first and second slots, i.e. AB, and in its third and fourth slots, i.e. CD,

$$C_{ABCD} = C_{BACD} = C_{ABDC} \tag{6.156}$$

(we have, in general, 36 independent components at each strain state).

We say $\mathbb{C}$ possesses the **minor symmetries**. The symmetry condition (6.156) is independent of the existence of a strain-energy function Ψ and holds for *all* elastic materials. Note that the minor symmetry of $\mathbb{C}$ follows from the symmetries of the right Cauchy-Green tensor $\mathbf{C}$ (or equivalently from the Green-Lagrange strain tensor $\mathbf{E}$) and the second Piola-Kirchhoff stress tensor $\mathbf{S}$.

If we assume the existence of a scalar-valued energy function Ψ (hyperelasticity), then $\mathbf{S}$ may be derived from Ψ according to $\mathbf{S} = 2\partial\Psi(\mathbf{C})/\partial\mathbf{C}$ (see $(6.13)_2$). Hence, using (6.154), we arrive at the crucial relation

$$\mathbb{C} = 4\frac{\partial^2 \Psi(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}} \qquad \text{or} \qquad C_{ABCD} = 4\frac{\partial^2 \Psi}{\partial C_{AB} \partial C_{CD}} \tag{6.157}$$

for the elasticities in the material description, with the symmetries

$$\mathbb{C} = \mathbb{C}^{\mathrm{T}} \qquad \text{or} \qquad C_{ABCD} = C_{CDAB} \quad . \tag{6.158}$$

We say $\mathbb{C}$ possesses the **major symmetries**. Thus, tensor $\mathbb{C}$ has only 21 independent components at each strain state. The condition (6.158) is a necessary and sufficient condition for a material to be *hyperelastic*. The symmetry condition $C_{ABCD} = C_{CDAB}$ is often referred to as the definition of hyperelasticity. Hence, the major symmetry of $\mathbb{C}$ is basically *equivalent* to the existence of a strain-energy function. Note that the major symmetry of the elasticity tensor is associated with the symmetry of the **(tangent) stiffness matrix** arising in a finite element discretization procedure.

The **elasticity tensor** in the *spatial description* or the **spatial tensor of elasticities**, denoted by $\mathbb{c}$, is defined as the push-forward operation of $\mathbb{C}$ times a factor of J^{-1} (see MARSDEN and HUGHES [1994, Section 3.4]), in other texts the definition of $\mathbb{c}$ frequently excludes the factor J^{-1}. It is the *Piola transformation* of $\mathbb{C}$ on each large index so that

$$\mathbb{c} = J^{-1}\chi_*(\mathbb{C}) \quad , \qquad \mathrm{c}_{abcd} = J^{-1} F_{aA} F_{bB} F_{cC} F_{dD}\, C_{ABCD} \quad , \tag{6.159}$$

with the minor symmetries

$$\mathrm{c}_{abcd} = \mathrm{c}_{bacd} = \mathrm{c}_{abdc} \quad , \tag{6.160}$$

and additionally for hyperelasticity we have the major symmetries $\mathbb{c} = \mathbb{c}^{\mathrm{T}}$ or $\mathrm{c}_{abcd} = \mathrm{c}_{cdab}$. The fourth-order tensors $\mathbb{C}$ and $\mathbb{c}$ are crucial within the *concept of linearization*, as will become apparent in Chapter 8, particularly in Section 8.4.

The spatial representation of eq. (6.153) can be shown to be

$$\pounds_{\mathbf{v}}(\boldsymbol{\tau}^{\sharp}) = J\mathbb{c} : \mathbf{d} \tag{6.161}$$

(for a proof see Section 8.4, p. 398), in which $\pounds_{\mathbf{v}}(\boldsymbol{\tau}^{\sharp})$, $\mathbf{d}$ and $\mathbb{c}$ denote the objective Oldroyd stress rate (5.59) of the contravariant Kirchhoff stress tensor $\boldsymbol{\tau}$, the rate of deformation tensor (2.148), and the spatial elasticity tensor, as defined in eq. (6.159), respectively. A material is said to be **hypoelastic** if the associated rate equations of the form (6.161) are not obtained from a (scalar-valued) energy function. For more on hypoelastic materials see the classical and detailed account by TRUESDELL and NOLL [1992, Sections 99-103].

Systematic treatments of the elasticity tensors have been given by, for example, TRUESDELL and TOUPIN [1960, Sections 246-249], CHADWICK and OGDEN [1971a, b], HILL [1981], TRUESDELL and NOLL [1992, Sections 45, 82] and OGDEN [1997, Chapter 6].

Decoupled representation of the elasticity tensor. Based on the kinematic assumption $(6.79)_2$ and the decoupled structure of the strain-energy function (6.85) we derive the associated elasticity tensor. We focus attention solely on the material description of the elasticity tensor.

The elasticity tensor (6.154) may be written in the *decoupled* form

$$\mathbb{C} = 2\frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \mathbb{C}_{\text{vol}} + \mathbb{C}_{\text{iso}} \quad , \tag{6.162}$$

which represents the completion of the additive split of the stress response (6.88).

In relation (6.162) we introduced the definitions

$$\mathbb{C}_{\text{vol}} = 2\frac{\partial \mathbf{S}_{\text{vol}}}{\partial \mathbf{C}} \quad , \qquad \mathbb{C}_{\text{iso}} = 2\frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}} \tag{6.163}$$

of the purely *volumetric contribution* $\mathbb{C}_{\text{vol}}$ and the purely *isochoric contribution* $\mathbb{C}_{\text{iso}}$.

By analogy with eq. (6.157) we express the two contributions $\mathbb{C}_{\text{vol}}$ and $\mathbb{C}_{\text{iso}}$ in terms of the strain-energy function Ψ. Before this exploitation we introduce the definition

$$\frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} = -\mathbf{C}^{-1} \odot \mathbf{C}^{-1} \tag{6.164}$$

of the fourth-order tensor $\partial \mathbf{C}^{-1}/\partial \mathbf{C}$, for convenience (recall Example 1.11, p. 43, and take $\mathbf{A} = \mathbf{C}$ in relation (1.249)), where the symbol $\odot$ has been introduced to denote the tensor product according to the rule

$$-(\mathbf{C}^{-1} \odot \mathbf{C}^{-1})_{ABCD} = -\frac{1}{2}(C_{AC}^{-1} C_{BD}^{-1} + C_{AD}^{-1} C_{BC}^{-1}) = \frac{\partial C_{AB}^{-1}}{\partial C_{CD}} \quad . \tag{6.165}$$

Starting with $(6.163)_1$, a straightforward computation yields, with definition $(6.89)_2$, property (1.256), the derivative of J and $\mathbf{C}^{-1}$ with respect to $\mathbf{C}$, i.e. eqs. $(6.82)_1$ and (6.164), and the product rule of differentiation,

$$\begin{aligned} \mathbb{C}_{\text{vol}} &= 2\frac{\partial \mathbf{S}_{\text{vol}}}{\partial \mathbf{C}} = 2\frac{\partial (Jp\mathbf{C}^{-1})}{\partial \mathbf{C}} \\ &= 2\mathbf{C}^{-1} \otimes \left(p\frac{\partial J}{\partial \mathbf{C}} + J\frac{\partial p}{\partial \mathbf{C}} \right) + 2Jp\frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\ &= J\tilde{p}\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2Jp\mathbf{C}^{-1} \odot \mathbf{C}^{-1} \quad . \end{aligned} \tag{6.166}$$

For convenience, we have introduced the scalar function $\tilde{p}$, defined by

$$\tilde{p} = p + J\frac{\mathrm{d}p}{\mathrm{d}J} \quad , \tag{6.167}$$

with the constitutive equation for p given in $(6.91)_1$. Note that the only values which must be specified for a given material are p and $\tilde{p}$.

The following example shows a lengthy but *representative* derivation of an elasticity tensor.

EXAMPLE 6.8 Show the following explicit expression for the second contribution to the elasticity tensor, i.e. the isochoric part $\mathbb{C}_{\text{iso}}$, as defined in eq. $(6.163)_2$,

$$\begin{aligned} \mathbb{C}_{\text{iso}} = {} & \mathbb{P} : \overline{\mathbb{C}} : \mathbb{P}^{\mathrm{T}} + \frac{2}{3}\mathrm{Tr}(J^{-2/3}\overline{\mathbf{S}})\tilde{\mathbb{P}} \\ & - \frac{2}{3}(\mathbf{C}^{-1} \otimes \mathbf{S}_{\text{iso}} + \mathbf{S}_{\text{iso}} \otimes \mathbf{C}^{-1}) \end{aligned} \tag{6.168}$$

(compare also with HOLZAPFEL [1996a]), which is based on the definitions

$$\overline{\mathbb{C}} = 2J^{-4/3}\frac{\partial \overline{\mathbf{S}}}{\partial \overline{\mathbf{C}}} = 4J^{-4/3}\frac{\partial^2 \Psi_{\text{iso}}(\overline{\mathbf{C}})}{\partial \overline{\mathbf{C}} \partial \overline{\mathbf{C}}} \quad , \qquad \mathrm{Tr}(\bullet) = (\bullet) : \mathbf{C} \quad , \tag{6.169}$$

$$\tilde{\mathbb{P}} = \mathbf{C}^{-1} \odot \mathbf{C}^{-1} - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \tag{6.170}$$

of the fourth-order **fictitious elasticity tensor** $\overline{\mathbb{C}}$ in the *material description*, the trace $\mathrm{Tr}(\bullet)$ and the **modified projection tensor** $\tilde{\mathbb{P}}$ of fourth-order.

Solution. Starting from the definition of $\mathbb{C}_{\text{iso}}$, i.e. $(6.163)_2$, we find, using $(6.90)_4$ and property (1.256), that

$$\begin{aligned} \mathbb{C}_{\text{iso}} &= 2\frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}} = 2\frac{\partial (J^{-2/3}\mathbb{P} : \overline{\mathbf{S}})}{\partial \mathbf{C}} \\ &= 2(\mathbb{P} : \overline{\mathbf{S}}) \otimes \frac{\partial J^{-2/3}}{\partial \mathbf{C}} + 2J^{-2/3}\frac{\partial (\mathbb{P} : \overline{\mathbf{S}})}{\partial \mathbf{C}} \quad . \end{aligned} \tag{6.171}$$

The first term in this equation yields, through property $(6.82)_2$ and definition $(6.90)_4$,

$$2(\mathbb{P} : \overline{\mathbf{S}}) \otimes \frac{\partial J^{-2/3}}{\partial \mathbf{C}} = -\frac{2}{3}(J^{-2/3}\mathbb{P} : \overline{\mathbf{S}}) \otimes \mathbf{C}^{-1} = -\frac{2}{3}\mathbf{S}_{\text{iso}} \otimes \mathbf{C}^{-1} \quad , \tag{6.172}$$

which gives the last expression of (6.168) we show.

Hence, in the following we analyze exclusively the second term in (6.171). With the definition of the projection tensor (6.84), identities $(1.160)_1$, (1.152) and the chain rule, we have

$$2J^{-2/3}\frac{\partial(\mathbb{P}:\overline{\mathbf{S}})}{\partial\mathbf{C}} = 2J^{-2/3}\frac{\partial}{\partial\mathbf{C}}\left(\overline{\mathbf{S}}-\frac{1}{3}(\mathbf{C}^{-1}\otimes\mathbf{C}):\overline{\mathbf{S}}\right)$$
$$= 2J^{-2/3}\left(\frac{\partial\overline{\mathbf{S}}}{\partial\overline{\mathbf{C}}}-\frac{1}{3}\frac{\partial(\overline{\mathbf{S}}:\mathbf{C})\mathbf{C}^{-1}}{\partial\overline{\mathbf{C}}}\right):\frac{\partial\overline{\mathbf{C}}}{\partial\mathbf{C}} \quad , \qquad (6.173)$$

and finally, using the definition of the fourth-order tensor $\partial\overline{\mathbf{C}}/\partial\mathbf{C}$, i.e. $(6.83)_4$, definition $(6.169)_1$ and property (1.256),

$$2J^{-2/3}\frac{\partial(\mathbb{P}:\overline{\mathbf{S}})}{\partial\mathbf{C}} = [\overline{\mathbb{C}}-\frac{1}{3}(\mathbb{A}_1+\mathbb{A}_2)]:\mathbb{P}^{\mathrm{T}} \quad , \qquad (6.174)$$

in which the definitions

$$\mathbb{A}_1 = 2J^{-4/3}\mathbf{C}^{-1}\otimes\frac{\partial(\overline{\mathbf{S}}:\mathbf{C})}{\partial\overline{\mathbf{C}}} \quad , \qquad \mathbb{A}_2 = 2J^{-4/3}(\overline{\mathbf{S}}:\mathbf{C})\frac{\partial\mathbf{C}^{-1}}{\partial\overline{\mathbf{C}}} \qquad (6.175)$$

are to be used.

In order to study (6.175) in more detail we apply property (1.255), the chain rule, identity $(1.160)_1$, definition $(6.169)_1$ of $(6.175)_1$ and definition (6.164) of $(6.175)_2$ to give

$$\mathbb{A}_1 = \mathbf{C}^{-1}\otimes\left(\mathbf{C}:\overline{\mathbb{C}}+2J^{-2/3}\overline{\mathbf{S}}\right) = \mathbf{C}^{-1}\otimes\mathbf{C}:\overline{\mathbb{C}}+2\mathbf{C}^{-1}\otimes J^{-2/3}\overline{\mathbf{S}} \quad , \qquad (6.176)$$

$$\mathbb{A}_2 = -2J^{-2/3}(\overline{\mathbf{S}}:\mathbf{C})\mathbf{C}^{-1}\odot\mathbf{C}^{-1} \quad . \qquad (6.177)$$

Eqs. (6.176) and (6.177) substituted back into (6.174) yield, using identity $(1.160)_1$ and definition $(6.169)_2$,

$$2J^{-2/3}\frac{\partial(\mathbb{P}:\overline{\mathbf{S}})}{\partial\mathbf{C}} = (\mathbb{I}-\frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}):\overline{\mathbb{C}}:\mathbb{P}^{\mathrm{T}}-\frac{2}{3}\mathbf{C}^{-1}\otimes J^{-2/3}\overline{\mathbf{S}}:\mathbb{P}^{\mathrm{T}}$$
$$+\frac{2}{3}\mathrm{Tr}(J^{-2/3}\overline{\mathbf{S}})\mathbf{C}^{-1}\odot\mathbf{C}^{-1}:\mathbb{P}^{\mathrm{T}} \quad . \qquad (6.178)$$

With the definition of the projection tensor (6.84), then with definition $(6.90)_4$, according to identity (1.157), and by means of (6.164) with rule (1.254) and (6.170) we find finally

$$2J^{-2/3}\frac{\partial(\mathbb{P}:\overline{\mathbf{S}})}{\partial\mathbf{C}} = \mathbb{P}:\overline{\mathbb{C}}:\mathbb{P}^{\mathrm{T}}-\frac{2}{3}\mathbf{C}^{-1}\otimes\mathbf{S}_{\mathrm{iso}}$$
$$+\frac{2}{3}\mathrm{Tr}(J^{-2/3}\overline{\mathbf{S}})\underbrace{(\mathbf{C}^{-1}\odot\mathbf{C}^{-1}-\frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}^{-1})}_{\tilde{\mathbb{P}}} \quad , \qquad (6.179)$$

which is identical to the remaining terms in (6.168). ■

The two tensor expressions $(6.166)_4$ and (6.168) in the material description represent explicit forms which are generally applicable to any compressible hyperelastic material of interest. Since we have already computed the stress relation $\mathbf{S} = \mathbf{S}_{\mathrm{vol}}+\mathbf{S}_{\mathrm{iso}}$, with the terms $\mathbf{S}_{\mathrm{vol}} = Jp\mathbf{C}^{-1}$ and $\mathbf{S}_{\mathrm{iso}} = J^{-2/3}\mathbb{P}:\overline{\mathbf{S}}$, it is a straightforward task to set up the associated elasticity tensor $\mathbb{C} = \mathbb{C}_{\mathrm{vol}}+\mathbb{C}_{\mathrm{iso}}$. All that remains is to determine $\tilde{p}$, $\overline{\mathbb{C}}$ and $\tilde{\mathbb{P}}$ from relations (6.167), $(6.169)_1$ and (6.170), respectively. Since the nonlinear functions p and $\overline{\mathbf{S}}$, $\mathbf{S}_{\mathrm{iso}}$, occurring in $(6.166)_4$ and (6.168), are already known from the stress relation, a different material model only affects the elasticity tensor $\mathbb{C}$ through the scalar function $\tilde{p} = \tilde{p}(\Psi_{\mathrm{vol}})$ and the fourth-order fictitious elasticity tensor $\overline{\mathbb{C}} = \overline{\mathbb{C}}(\Psi_{\mathrm{iso}})$.

Expressions $(6.166)_4$ and (6.168) exhibit a clear structure and are fundamental within the finite element method in preserving quadratic rate of convergence near the solution point, when **Newton's method** is used as the associated solution technique.

The two explicit relations $(6.166)_4$ and (6.168) were specified within a so-called **consistent linearization process** of the associated stress tensor. The notion 'consistent linearization' means a linearization of *all* quantities which are related to the nonlinear problem. In the community of computational mechanics the elasticity tensors $(6.166)_4$ and (6.168) are frequently referred to as **algorithmic** or **consistent linearized tangent moduli** in the material and spatial descriptions.

For some complex problems the set up of the analytical tangent moduli is a difficult and time-consuming task. This is why the tangent moduli are also computed on a numerical basis, which turns out to be a straightforward and convenient technique in order to linearize sophisticated stress relations such as, for example, the stress response of materials associated with anisotropic damage at finite strains. For a numerical computation of consistent tangent moduli in large-strain inelasticity see, for example, MIEHE [1996], which contains more details and references on the underlying concept of approximation. However, manipulations which have led to $(6.166)_4$ and (6.168) may also be carried out with some of the commercially available mathematical software-packages having the feature of symbolic computation.

Elasticity tensor in terms of principal stretches. Consider an isotropic hyperelastic material characterized by the strain-energy function $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3)$, with the principal stretches $\lambda_1, \lambda_2, \lambda_3$.

The aim is now to derive the spectral form of the elasticity tensor $\mathbb{C}$ in the *material description*, namely

$$\mathbb{C} = \sum_{a,b=1}^{3}\frac{1}{\lambda_b}\frac{\partial S_a}{\partial\lambda_b}\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_b$$
$$+\sum_{\substack{a,b=1\\a\neq b}}^{3}\frac{S_b-S_a}{\lambda_b^2-\lambda_a^2}(\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b+\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_a) \quad ; \qquad (6.180)$$

with the principal second Piola-Kirchhoff stresses

$$S_a = 2\frac{\partial \Psi}{\partial \lambda_a^2} = \frac{1}{\lambda_a}\frac{\partial \Psi}{\partial \lambda_a} \quad , \qquad a = 1, 2, 3 \quad , \tag{6.181}$$

and the set $\{\hat{\mathbf{N}}_a\}$, $a = 1, 2, 3$, of orthonormal eigenvectors of the right Cauchy-Green tensor $\mathbf{C}$. They define principal referential directions at a point $\mathbf{X}$, with the conditions $|\hat{\mathbf{N}}_a| = 1$ and $\hat{\mathbf{N}}_a \cdot \hat{\mathbf{N}}_b = \delta_{ab}$.

The important fourth-order tensor $\mathbb{C}$ in a more general setting was given by OGDEN [1997, Section 6.1.4]. Compare also the work of CHADWICK and OGDEN [1971a, b] with some differences in notation.

The proof of representation (6.180) is as follows:

Proof. In order to prove relation (6.180) we follow an approach which takes advantage of isotropy. We know from Chapter 5 that for isotropic elastic materials the second Piola-Kirchhoff stress tensor $\mathbf{S}$ is coaxial with the right Cauchy-Green tensor $\mathbf{C}$, so that $\mathbf{S}$ has the same principal directions as $\mathbf{C}$.

For notational convenience we use henceforth a rate formulation rather than an infinitesimal formulation. In particular, we now compute the material time derivatives of the stress and strain tensors $\mathbf{S}$ and $\mathbf{C}$, and compare them with the rate form of relation (6.153), i.e. $\dot{\mathbf{S}} = \mathbb{C} : \dot{\mathbf{C}}/2$, in order to obtain the elasticity tensor $\mathbb{C}$, as defined in (6.154) and specified in (6.180).

To begin with, consider a set of orthonormal basis vectors $\mathbf{e}_a$, $a = 1, 2, 3$, fixed in space. Consequently, the set $\{\hat{\mathbf{N}}_a\}$, $a = 1, 2, 3$, of orthonormal eigenvectors may be governed by the transformation law

$$\hat{\mathbf{N}}_a = \mathbf{Q}\mathbf{e}_a \quad , \qquad a = 1, 2, 3 \tag{6.182}$$

(compare with eq. (1.182), Section 1.5), where $\mathbf{Q}$ denotes an orthogonal tensor with components $Q_{ab} = \mathbf{e}_a \cdot \hat{\mathbf{N}}_b = \cos\theta(\mathbf{e}_a, \hat{\mathbf{N}}_b)$, representing the cosine of the angle between the fixed basis vectors $\mathbf{e}_a$ and the orthonormal eigenvectors $\hat{\mathbf{N}}_b$ (principal referential directions). Tensor $\mathbf{Q}$ is characterized by the *orthogonality condition*, i.e. $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$.

Next, we compute the material time derivative of the principal referential directions $\hat{\mathbf{N}}_a$. Since the basis vectors are assumed to be fixed in space ($\dot{\mathbf{e}}_a = \mathbf{o}$), we may write $\dot{\hat{\mathbf{N}}}_a = \dot{\mathbf{Q}}\mathbf{e}_a$, $a = 1, 2, 3$. By expanding this equation with the orthogonality condition and by means of the skew tensor $\boldsymbol{\Omega}$, as introduced in (5.15), and transformation (6.182), we may eliminate the basis $\{\mathbf{e}_a\}$ and find that

$$\dot{\hat{\mathbf{N}}}_a = (\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}})\mathbf{Q}\mathbf{e}_a = \boldsymbol{\Omega}\hat{\mathbf{N}}_a \quad , \qquad a = 1, 2, 3 \quad . \tag{6.183}$$

Note that the components of the skew tensor $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^{\mathrm{T}}$ with respect to the basis $\{\mathbf{e}_a\}$ are obtained from $(6.183)_2$ in the form

$$\Omega_{ab} = \hat{\mathbf{N}}_a \cdot \boldsymbol{\Omega}\hat{\mathbf{N}}_b = \hat{\mathbf{N}}_a \cdot \dot{\hat{\mathbf{N}}}_b = -\Omega_{ba} \quad , \tag{6.184}$$

with $\Omega_{aa} = 0$. By means of identity $(1.65)_2$, we may deduce from $(6.183)_2$ the representation $\boldsymbol{\Omega} = \sum_{a=1}^{3} \dot{\hat{\mathbf{N}}}_a \otimes \hat{\mathbf{N}}_a$.

Knowing $\dot{\hat{\mathbf{N}}}_a$, we now can determine the material time derivative of the spectral representation of $\mathbf{C}$. Converting eq. (2.128) to the rate form, we obtain, by means of eq. $(6.183)_2$ and the spectral decomposition of the right Cauchy-Green tensor, i.e. $\mathbf{C} = \sum_{a=1}^{3} \lambda_a^2 \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a$,

$$\begin{aligned} \dot{\mathbf{C}} - \sum_{a=1}^{3} 2\lambda_a\dot{\lambda}_a\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a &= \sum_{a=1}^{3} \lambda_a^2(\dot{\hat{\mathbf{N}}}_a \otimes \hat{\mathbf{N}}_a + \hat{\mathbf{N}}_a \otimes \dot{\hat{\mathbf{N}}}_a) \\ &= \sum_{a=1}^{3} \lambda_a^2(\boldsymbol{\Omega}\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a + \hat{\mathbf{N}}_a \otimes \boldsymbol{\Omega}\hat{\mathbf{N}}_a) \\ &= \boldsymbol{\Omega}\mathbf{C} - \mathbf{C}\boldsymbol{\Omega} \quad . \end{aligned} \tag{6.185}$$

From $(6.185)_2$ using (6.184) we deduce that

$$\dot{\mathbf{C}} = \sum_{a=1}^{3} 2\lambda_a\dot{\lambda}_a\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a + \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \Omega_{ab}(\lambda_b^2 - \lambda_a^2)\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_b \quad , \tag{6.186}$$

where $2\lambda_a\dot{\lambda}_a = \hat{\mathbf{N}}_a \cdot \dot{\mathbf{C}}\hat{\mathbf{N}}_a = \dot{C}_{aa}$, $a = 1, 2, 3$ (compare with eq. (2.129)), denote *normal* components (diagonal elements) and $\Omega_{ab}(\lambda_b^2 - \lambda_a^2) = \hat{\mathbf{N}}_a \cdot \dot{\mathbf{C}}\hat{\mathbf{N}}_b = \dot{C}_{ab}$, $a \neq b$, denote *shear* components of $\dot{\mathbf{C}}$ (off-diagonal elements) with respect to the basis $\{\hat{\mathbf{N}}_a\}$.

By isotropy, $\mathbf{S}$ has the same principal directions as $\mathbf{C}$. Hence, recall $(6.52)_2$, i.e. $\mathbf{S} = \sum_{a=1}^{3} S_a\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a$, with $S_a = 1/\lambda_a(\partial\Psi/\partial\lambda_a)$, $a = 1, 2, 3$, we obtain, by analogy with (6.186)

$$\dot{\mathbf{S}} = \sum_{a=1}^{3} \dot{S}_a\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a + \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \Omega_{ab}(S_b - S_a)\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_b \quad , \tag{6.187}$$

in which the material time derivative of the principal second Piola-Kirchhoff stresses is defined to be

$$\dot{S}_a = \sum_{b=1}^{3} \frac{\partial S_a}{\partial \lambda_b^2}\dot{\lambda}_b^2 = \sum_{b=1}^{3} \frac{\partial S_a}{\partial \lambda_b}\dot{\lambda}_b \quad . \tag{6.188}$$

By expanding the numerator and denominator of the second term in (6.187) with $\lambda_b^2 - \lambda_a^2$ and by means of $(6.188)_2$, eq. (6.187) can be rephrased as

$$\dot{\mathbf{S}} = \sum_{a,b=1}^{3} \frac{\partial S_a}{\partial \lambda_b}\dot{\lambda}_b\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a + \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \Omega_{ab}(\lambda_b^2 - \lambda_a^2)\frac{S_b - S_a}{\lambda_b^2 - \lambda_a^2}\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_b \quad . \tag{6.189}$$

On comparing the derived eqs. (6.189) and (6.186) with (6.153) in the rate form, i.e. $\dot{\mathbf{S}} = \mathbb{C} : \dot{\mathbf{C}}/2$, we find by inspection that the elasticity tensor $\mathbb{C}$ emerges, as given by eq. (6.180). ■

For the case in which two or even all three eigenvalues λ_a^2 of $\mathbf{C}$ (and also of $\mathbf{b}$) are equal, the associated two or three stresses S_a are also equal, by isotropy. Hence, the divided difference $(S_b - S_a)/(\lambda_b^2 - \lambda_a^2)$ in expression (6.180) represents an indeterminate form of type $\frac{0}{0}$. However, it can be shown that the divided difference is well-defined as λ_b approaches λ_a. Namely, applying *l'Hôpital's rule*, we see simply that

$$\lim_{\lambda_b \to \lambda_a} \frac{S_b - S_a}{\lambda_b^2 - \lambda_a^2} = \frac{\partial S_b}{\partial \lambda_b^2} - \frac{\partial S_a}{\partial \lambda_b^2} \tag{6.190}$$

(compare also with the work of CHADWICK and OGDEN [1971b]). Consequently, the elasticity tensor, as defined in eq. (6.180), is valid for the three cases: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, $\lambda_1 = \lambda_2 \neq \lambda_3$ and $\lambda_1 = \lambda_2 = \lambda_3$.

Finally, in order to set up the spectral form of the elasticity tensor $\mathbb{c}$ in the *spatial description* we use the Piola transformation of $\mathbb{C}$ for principal values. According to (6.159), this gives $c_{abcd} = J^{-1}\lambda_a\lambda_b\lambda_c\lambda_d C_{abcd}$, with the principal stretches $\lambda_1, \lambda_2, \lambda_3$ and the volume ratio $J = \lambda_1\lambda_2\lambda_3$. A straightforward computation leads to

$$\begin{aligned} \mathbb{c} = {} & \sum_{a,b=1}^{3} J^{-1}\lambda_a^2\lambda_b \frac{\partial S_a}{\partial \lambda_b} \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_b \otimes \hat{\mathbf{n}}_b \\ & + \sum_{\substack{a,b=1 \\ a \neq b}}^{3} \frac{\sigma_b\lambda_a^2 - \sigma_a\lambda_b^2}{\lambda_b^2 - \lambda_a^2} (\hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_b \otimes \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_b + \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_b \otimes \hat{\mathbf{n}}_b \otimes \hat{\mathbf{n}}_a) \ , \end{aligned} \tag{6.191}$$

with the principal Cauchy stresses $\sigma_a = J^{-1}\lambda_a^2 S_a$, $a = 1, 2, 3$ (see the inverse of eq. $(6.48)_2$), and the principal spatial directions $\hat{\mathbf{n}}_a$, $a = 1, 2, 3$, which are the orthonormal eigenvectors of $\mathbf{v}$ (and also of $\mathbf{b}$), with $|\hat{\mathbf{n}}_a| = 1$ and $\hat{\mathbf{n}}_a \cdot \hat{\mathbf{n}}_b = \delta_{ab}$. From the property (2.120) we know that the two-point tensor $\mathbf{R}$ rotates the principal referential directions $\hat{\mathbf{N}}_a$ into the principal spatial directions $\hat{\mathbf{n}}_a$.

If $\lambda_a = \lambda_b$ we may conclude that $\sigma_a = \sigma_b$, by isotropy. Hence, the divided difference $(\sigma_b\lambda_a^2 - \sigma_a\lambda_b^2)/(\lambda_b^2 - \lambda_a^2)$ in expression (6.191) gives us $\frac{0}{0}$ and must therefore be determined applying *l'Hôpital's rule*. Differentiating the numerator and denominator by λ_b and taking the limits $\lambda_b \to \lambda_a$, the divided difference becomes

$$\begin{aligned} \lim_{\lambda_b \to \lambda_a} \frac{\sigma_b\lambda_a^2 - \sigma_a\lambda_b^2}{\lambda_b^2 - \lambda_a^2} &= \left(\frac{\partial \sigma_b}{\partial \lambda_b}\lambda_a^2 - \frac{\partial \sigma_a}{\partial \lambda_b}\lambda_b^2 - 2\lambda_b\sigma_a \right) \frac{1}{2\lambda_b} \\ &= \frac{1}{2}\lambda_a \left(\frac{\partial \sigma_b}{\partial \lambda_b} - \frac{\partial \sigma_a}{\partial \lambda_b} \right) - \sigma_a \ . \end{aligned} \tag{6.192}$$

An alternative version of solution $(6.192)_2$ in terms of the principal second Piola-Kirchhoff stresses, frequently used in other texts, is left as an exercise.

EXERCISES

1. For the description of isotropic hyperelastic materials at finite strains consider the strain-energy function $\Psi = \Psi(I_1, I_2, I_3)$ in the *coupled* form, with the principal invariants I_a, $a = 1, 2, 3$.

   (a) Use the stress relation $(6.32)_2$, the chain rule and the derivatives of the invariants with respect to $\mathbf{C}$, i.e. eqs. $(6.30)_3$ and (6.31), in order to obtain the *most general form* of the elasticity tensor $\mathbb{C}$ in terms of the three principal invariants

   $$\begin{aligned} \mathbb{C} &= 2\frac{\partial \mathbf{S}}{\partial \mathbf{C}} = 4\frac{\partial^2 \Psi(I_1, I_2, I_3)}{\partial \mathbf{C} \partial \mathbf{C}} \\ &= \delta_1 \mathbf{I} \otimes \mathbf{I} + \delta_2(\mathbf{I} \otimes \mathbf{C} + \mathbf{C} \otimes \mathbf{I}) + \delta_3(\mathbf{I} \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{I}) \\ &\quad + \delta_4 \mathbf{C} \otimes \mathbf{C} + \delta_5(\mathbf{C} \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{C}) \\ &\quad + \delta_6 \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + \delta_7 \mathbf{C}^{-1} \odot \mathbf{C}^{-1} + \delta_8 \mathbb{I} \ , \end{aligned} \tag{6.193}$$

   with the coefficients $\delta_1, \ldots, \delta_8$ defined by

   $$\left.\begin{aligned} &\delta_1 = 4\left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_1} + 2I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + \frac{\partial \Psi}{\partial I_2} + I_1^2 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} \right) \ , \\ &\delta_2 = -4\left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} \right) \ , \\ &\delta_3 = 4\left(I_3 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} + I_1 I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \right) \ , \qquad \delta_4 = 4\frac{\partial^2 \Psi}{\partial I_2 \partial I_2} \ , \\ &\delta_5 = -4I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \ , \qquad \delta_6 = 4\left(I_3 \frac{\partial \Psi}{\partial I_3} + I_3^2 \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} \right) \ , \\ &\delta_7 = -4I_3 \frac{\partial \Psi}{\partial I_3} \ , \qquad \delta_8 = -4\frac{\partial \Psi}{\partial I_2} \ . \end{aligned}\right\} \tag{6.194}$$

   The fourth-order tensors $\mathbf{C}^{-1} \odot \mathbf{C}^{-1}$ and $\mathbb{I}$ in eq. $(6.193)_3$ are defined according to (6.164) and (1.160), respectively.

   (b) Particularize the coefficients δ_a, $a = 1, \ldots, 8$, (6.194), for the compressible *Mooney-Rivlin, neo-Hookean, Blatz and Ko* models, i.e. eqs. (6.147)–(6.149). For convenience, summarize the (nonzero) coefficients for the three material models to form the entries of Table 6.3.

   For notational simplicity we have introduced the abbreviation $\xi = 2\mu(1 - f)/I_3$ with $\xi \geq 0$. Note that the coefficients δ_a, $a = 1, \ldots, 8$, for the *neo-Hookean* model are simple those of the *Blatz and Ko* model obtained by setting $\xi = 0$ and $\mu f = 2c_1$.

   Compare the corresponding constitutive equations of the three material models of Exercise 1(a) on p. 248, with the specified coefficients summarized in Table 6.1.

	Mooney-Rivlin model (6.147)	*neo-Hookean* model (6.148)	*Blatz and Ko* model (6.149)
δ_1	$4c_2$		ξ
δ_3			$-\xi I_1$
δ_5			ξ
δ_6	$2cJ(2J-1)$	$4\beta c_1 I_3^{-\beta}$	$2\mu f\beta I_3^{-\beta}+\xi(I_2+\beta I_3^{\beta+1})$
δ_7	$-2[2cJ(J-1)-d]$	$4c_1 I_3^{-\beta}$	$2\mu f I_3^{-\beta}+\xi(I_2-I_3^{\beta+1})$
δ_8	$-4c_2$		$-\xi$

Table 6.3 Specified coefficients for the elasticity tensors of some materials in the coupled form.

2. The strain-energy function $\Psi(J,\bar{I}_1,\bar{I}_2)=\Psi_{\text{vol}}(J)+\Psi_{\text{iso}}(\bar{I}_1,\bar{I}_2)$ is given in terms of the volume ratio J and the two modified principal stretches $\bar{I}_1,\bar{I}_2$. This type of strain energy in *decoupled* representation is suitable for the characterization of compressible isotropic materials at finite strains.

   (a) The associated decoupled elasticity tensor $\mathbb{C}$ is given by (6.162), with the volumetric and isochoric parts $(6.166)_4$ and (6.168), respectively. Particularize the fictitious elasticity tensor $\overline{\mathbb{C}}$ in the *material description*, i.e. eq. $(6.169)_1$, to the specific strain energy at hand.

   Start with the constitutive equation for the fictitious second Piola-Kirchhoff stress $\overline{\mathbf{S}}$, as defined in eq. (6.115), and use the derivatives of $\bar{I}_1,\bar{I}_2$ with respect to tensor $\overline{\mathbf{C}}$, i.e. (6.117), in order to obtain the *most general form* of $\overline{\mathbb{C}}$ in terms of $\bar{I}_1$ and $\bar{I}_2$ in the form

$$\begin{aligned} J^{4/3}\overline{\mathbb{C}} &= 2\frac{\partial\overline{\mathbf{S}}}{\partial\overline{\mathbf{C}}} = 4\frac{\partial^2\Psi_{\text{iso}}(\bar{I}_1,\bar{I}_2)}{\partial\overline{\mathbf{C}}\partial\overline{\mathbf{C}}} \\ &= \bar{\delta}_1\mathbf{I}\otimes\mathbf{I}+\bar{\delta}_2(\mathbf{I}\otimes\overline{\mathbf{C}}+\overline{\mathbf{C}}\otimes\mathbf{I})+\bar{\delta}_3\overline{\mathbf{C}}\otimes\overline{\mathbf{C}}+\bar{\delta}_4\mathbb{I} \quad , \end{aligned}$$

   with the fourth-order unit tensor $\mathbb{I}$ defined by (1.160) and the coefficients $\bar{\delta}_a$, $a=1,\ldots,4$, by

$$\left.\begin{aligned} \bar{\delta}_1 &= 4\left(\frac{\partial^2\Psi_{\text{iso}}}{\partial\bar{I}_1\partial\bar{I}_1}+2\bar{I}_1\frac{\partial^2\Psi_{\text{iso}}}{\partial\bar{I}_1\partial\bar{I}_2}+\frac{\partial\Psi_{\text{iso}}}{\partial\bar{I}_2}+\bar{I}_1^2\frac{\partial^2\Psi_{\text{iso}}}{\partial\bar{I}_2\partial\bar{I}_2}\right) \quad , \\ \bar{\delta}_2 &= -4\left(\frac{\partial^2\Psi_{\text{iso}}}{\partial\bar{I}_1\partial\bar{I}_2}+\bar{I}_1\frac{\partial^2\Psi_{\text{iso}}}{\partial\bar{I}_2\partial\bar{I}_2}\right) \quad , \\ \bar{\delta}_3 &= 4\frac{\partial^2\Psi_{\text{iso}}}{\partial\bar{I}_2\partial\bar{I}_2} \quad , \qquad \bar{\delta}_4 = -4\frac{\partial\Psi_{\text{iso}}}{\partial\bar{I}_2} \quad . \end{aligned}\right\} \tag{6.195}$$

   (b) To be specific, take the decoupled form of the compressible *Mooney-Rivlin* model (6.146) and the compressible *neo-Hookean* model (set $c_2=0$). Additionally, take the *Yeoh* model and the *Arruda and Boyce* model of the forms

$$\Psi=\Psi_{\text{vol}}+c_1(\bar{I}_1-3)+c_2(\bar{I}_1-3)^2+c_3(\bar{I}_1-3)^3 \quad ,$$

$$\Psi=\Psi_{\text{vol}}+\mu\left[\frac{1}{2}(\bar{I}_1-3)+\frac{1}{20n}(\bar{I}_1^2-9)+\frac{11}{1050n^2}(\bar{I}_1^3-27)+\ldots\right] \quad ,$$

   corresponding to eqs. (6.134) and (6.136), respectively.

   Specify the coefficients $\bar{\delta}_a$, $a=1,\ldots,4$, (6.195), for the four material models in question and summarize the result in the form of a table.

	Mooney-Rivlin model (6.146)	*neo-Hookean* model	*Yeoh* model (6.134)	*Arruda and Boyce* model (6.136)
$\bar{\delta}_1$	$4c_2$	0	$8[c_2+3c_3(\bar{I}_1-3)]$	$\mu[2/5n$ $+(44/175n^2)\bar{I}_1+\ldots]$
$\bar{\delta}_4$	$-4c_2$	0	0	0

Table 6.4 Specified coefficients for the elasticity tensors of some materials in the decoupled form.

   Note that the associated constitutive equations, with the specified coefficients as summarized in Table 6.2, are presented in Exercise 1(b) on p. 248.

3. Consider a compressible isotropic material characterized by the strain-energy function in the *decoupled* form of $\Psi(\lambda_1,\lambda_2,\lambda_3)=\Psi_{\text{vol}}(J)+\Psi_{\text{iso}}(\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3)$, with the volume ratio $J=\lambda_1\lambda_2\lambda_3$ and the modified principal stretches $\bar{\lambda}_a=J^{-1/3}\lambda_a$, $a=1,2,3$. The associated decoupled structure of the elasticity tensor $\mathbb{C}$ in the material description is given as $\mathbb{C}(\lambda_1,\lambda_2,\lambda_3)=\mathbb{C}_{\text{vol}}+\mathbb{C}_{\text{iso}}$, with the *volumetric* contribution $\mathbb{C}_{\text{vol}}$ specified by the expression $(6.166)_4$.

   (a) Show that the spectral form of the *isochoric* contribution $\mathbb{C}_{\text{iso}}$ may be given by

$$\begin{aligned} \mathbb{C}_{\text{iso}} = &\sum_{a,b=1}^{3}\frac{1}{\lambda_b}\frac{\partial S_{\text{iso}\,a}}{\partial\lambda_b}\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_b \\ &+\sum_{\substack{a,b=1\\a\neq b}}^{3}\frac{S_{\text{iso}\,b}-S_{\text{iso}\,a}}{\lambda_b^2-\lambda_a^2}(\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b \\ &\qquad\qquad+\hat{\mathbf{N}}_a\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_b\otimes\hat{\mathbf{N}}_a) \quad , \end{aligned} \tag{6.196}$$

   where $S_{\text{iso}\,a}=(\partial\Psi_{\text{iso}}/\partial\lambda_a)/\lambda_a$, $a=1,2,3$, denote the principal values of the second Piola-Kirchhoff stress tensor $\mathbf{S}_{\text{iso}}$.

(b) In order to specify the elasticity tensor (6.196), take Ogden's model (6.139) and recall the isochoric hyperelastic stress response in terms of the three principal stresses $S_{\text{iso}\,a}$, $a = 1, 2, 3$, determined by the closed form expression $(6.144)_2$ using (6.145) (compare with Example 6.7).

For this important class of material models particularize the coefficients of the first part of the elasticity tensor (6.196) by means of the given strain-energy function (6.139), i.e. $\Psi_{\text{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3) = \sum_{a=1}^{3} \sum_{p=1}^{N} (\mu_p/\alpha_p)(\overline{\lambda}_a^{\alpha_p} - 1)$. Show that

$$\frac{1}{\lambda_b} \frac{\partial S_{\text{iso}\,a}}{\partial \lambda_b} = \frac{J^{-1/3}}{\lambda_b} \sum_{c=1}^{3} \frac{\partial S_{\text{iso}\,a}}{\partial \overline{\lambda}_c} \left(\delta_{cb} - \frac{1}{3} \overline{\lambda}_c \overline{\lambda}_b^{-1} \right) \tag{6.197}$$

$$= \begin{cases} \lambda_a^{-2} \lambda_b^{-2} \displaystyle\sum_{p=1}^{N} \mu_p \alpha_p \left(\frac{1}{3} \overline{\lambda}_a^{\alpha_p} + \frac{1}{9} \sum_{c=1}^{3} \overline{\lambda}_c^{\alpha_p} \right) & \text{for } a = b \quad , \\ \lambda_a^{-2} \lambda_b^{-2} \displaystyle\sum_{p=1}^{N} \mu_p \alpha_p \left(-\frac{1}{3} \overline{\lambda}_a^{\alpha_p} - \frac{1}{3} \overline{\lambda}_b^{\alpha_p} + \frac{1}{9} \sum_{c=1}^{3} \overline{\lambda}_c^{\alpha_p} \right) & \text{for } a \neq b \quad . \end{cases}$$

This representation was given by SIMO and TAYLOR [1991a, Example 2.2], with some differences in notation.

4. Consider a strain-energy function in terms of principal stretches λ_a, $a = 1, 2, 3$, and principal invariants I_a, $a = 1, 2, 3$, according to $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) = \Psi(I_1, I_2, I_3)$.

   Write down the associated constitutive equation for the principal second Piola-Kirchhoff stresses S_a, $a = 1, 2, 3$, in terms of the three principal invariants (compare with eq. $(6.32)_2$). By use of this result show that the divided difference $(S_b - S_a)/(\lambda_b^2 - \lambda_a^2)$ is well-defined as λ_b approaches λ_a and yields

$$\lim_{\lambda_b \to \lambda_a} \frac{S_b - S_a}{\lambda_b^2 - \lambda_a^2} = -2 \frac{\partial \Psi}{\partial I_2} - 2 I_3 \frac{\partial \Psi}{\partial I_3} \lambda_a^{-2} \lambda_b^{-2} \quad .$$

5. By means of *l'Hôpital's rule* derive the alternative version of (6.192) in terms of the principal second Piola-Kirchhoff stresses S_a, i.e.

$$\lim_{\lambda_b \to \lambda_a} \frac{\sigma_b \lambda_a^2 - \sigma_a \lambda_b^2}{\lambda_b^2 - \lambda_a^2} = J^{-1} \lambda_a^2 \lambda_b^2 \left(\frac{\partial S_b}{\partial \lambda_b^2} - \frac{\partial S_a}{\partial \lambda_b^2} \right) \quad .$$

6. Consider the constitutive equation (6.38) in terms of the (spatial) left Cauchy-Green tensor $\mathbf{b}$ representing isotropic hyperelastic response. Deduce from the constitutive rate equation (6.161) that the associated elasticity tensor $\mathbb{c}$ in the spatial description has the explicit form

$$J \mathbb{c} = 4 \mathbf{b} \frac{\partial^2 \Psi(\mathbf{b})}{\partial \mathbf{b} \partial \mathbf{b}} \mathbf{b} \quad . \tag{6.198}$$

   Hint: Recall relation (3.62), the Oldroyd stress rate (5.59) of the Kirchhoff stress, kinematic relation (2.171) and use the chain rule.

   Representation (6.198) was given by MIEHE and STEIN [1992]. In their work the definition of the elasticity tensor $\mathbb{c} = \chi_*(\mathbb{C})$ in the spatial description excludes the factor J^{-1}.

7. Consider the additive split of the Cauchy stress (6.101)–(6.104) in terms of J and $\overline{\mathbf{b}}$, based on the strain-energy function of the form (6.98). Show that the associated elasticity tensor $\mathbb{c}$ in the spatial description may be written in the *decoupled* form

$$\mathbb{c} = \mathbb{c}_{\text{vol}} + \mathbb{c}_{\text{iso}} \quad ,$$

   with the definitions

$$J \mathbb{c}_{\text{vol}} = 4 \mathbf{b} \frac{\partial^2 \Psi_{\text{vol}}(J)}{\partial \mathbf{b} \partial \mathbf{b}} \mathbf{b} = J(\tilde{p} \mathbf{I} \otimes \mathbf{I} - 2p \mathbb{I}) \qquad \text{with} \qquad \tilde{p} = p + J \frac{\mathrm{d}p}{\mathrm{d}J} \quad ,$$

$$\begin{aligned} J \mathbb{c}_{\text{iso}} &= 4 \mathbf{b} \frac{\partial^2 \Psi_{\text{iso}}(\overline{\mathbf{b}})}{\partial \mathbf{b} \partial \mathbf{b}} \mathbf{b} \\ &= \mathbb{p} : \overline{\mathbb{c}} : \mathbb{p} + \frac{2}{3} \operatorname{tr}(\overline{\boldsymbol{\tau}}) \mathbb{p} - \frac{2}{3} (\mathbf{I} \otimes \boldsymbol{\tau}_{\text{iso}} + \boldsymbol{\tau}_{\text{iso}} \otimes \mathbf{I}) \end{aligned}$$

   of the purely volumetric contribution $\mathbb{c}_{\text{vol}}$ and the purely isochoric contribution $\mathbb{c}_{\text{iso}}$, the latter being based on the spatial projection tensor $\mathbb{p} = \mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I}$ and on the definitions $\overline{\boldsymbol{\tau}} = J \overline{\boldsymbol{\sigma}}$, $\boldsymbol{\tau}_{\text{iso}} = J \boldsymbol{\sigma}_{\text{iso}}$, with $\boldsymbol{\sigma}_{\text{iso}} = \mathbb{p} : \overline{\boldsymbol{\sigma}}$, as given in (6.103) and (6.104).

   In addition, we introduced the definitions of the fourth-order **fictitious elasticity tensor** $\overline{\mathbb{c}}$ in the spatial description and the trace $\operatorname{tr}(\bullet)$ according to

$$\overline{\mathbb{c}} = 4 \overline{\mathbf{b}} \frac{\partial^2 \Psi_{\text{iso}}(\overline{\mathbf{b}})}{\partial \overline{\mathbf{b}} \partial \overline{\mathbf{b}}} \overline{\mathbf{b}} \quad , \qquad \operatorname{tr}(\bullet) = (\bullet) : \mathbf{I} \quad .$$

   For an explicit derivation, see MIEHE [1994, Appendix A].

6.7 Transversely Isotropic Materials

Numerous materials are composed of a **matrix material** (or in the literature often called **ground substance**) and one or more **families of fibers**. This type of material, which we call a **composite material** or **fiber-reinforced composite**, is heterogeneous in the sense that it has different compositions throughout the body. We consider only composite materials in which the fibers are continuously arranged in the matrix material. These types of composites have strong directional properties and their mechanical responses are regarded as **anisotropic**.

The challenge in the design of fiber-reinforced composites is to combine the matrix material and the fibers in such a way that the resulting material is most efficient for the desired application. For engineering applications composite materials provide many advantages over monolithic materials such as high stiffness and strength, low weight and thermal expansion and corrosion resistance. However, the drawbacks in using composite materials seem to be the high costs when compared with those of monolithic (more classical) materials and, from the practical point of view, limited knowledge of how to combine these types of material.

A material which is reinforced by only *one* family of fibers has a *single* preferred direction. The stiffness of this type of composite material in the fiber direction is typically much greater than in the directions orthogonal to the fibers. It is the simplest representation of material anisotropy, which we call **transversely isotropic** with respect to this preferred direction. The material response along directions orthogonal to this preferred direction is isotropic. These composite materials are employed in a variety of applications in industrial engineering and medicine. For manufacturing and fabrication processes and for typical features and properties of transversely isotropic materials the reader should consult, for example, the textbook by HERAKOVICH [1998].

The aim of the following section is to investigate transversely isotropic materials capable of supporting *finite elastic* strains. As mentioned above, all fibers have a single preferred direction. However, the fibers are assumed to be continuously distributed throughout the material. We derive appropriate constitutive equations which are based solely on a continuum approach (excluding micromechanical considerations). Constitutive equations which model transversely isotropic materials in the *small elastic* strain regime are well established and may be found, for example, in the textbooks by TSAI and HAHN [1980], DANIEL and ISHAI [1994], HERAKOVICH [1998] and JONES [1999].

Kinematic relation and structure of the free energy. We consider a continuum body $\mathcal{B}$ which initially occupies a typical region Ω_0 at a fixed *reference time* $t = 0$. The region is known as a fixed *reference configuration* of that body $\mathcal{B}$. A point in Ω_0 may be characterized by the position vector $\mathbf{X}$ (with material coordinates X_A, $A = 1, 2, 3$) related to a fixed set of axes. At a subsequent time $t > 0$ the continuum body is in a *deformed configuration* occupying a region Ω. The associated point in Ω is characterized by the position vector $\mathbf{x}$ (with spatial coordinates x_a, $a = 1, 2, 3$) related to the same fixed set of axes. For more details about the relevant notation recall Section 2.1.

We suppose that the only anisotropic property of the solid comes from the presence of the fibers. To start with, for a material which is reinforced by only one family of fibers, the stress at a material point depends not only on the deformation gradient $\mathbf{F}$ but also on that single preferred direction, which we call the **fiber direction**. The direction of a fiber at point $\mathbf{X} \in \Omega_0$ is defined by a unit vector field $\mathbf{a}_0(\mathbf{X})$, $|\mathbf{a}_0| = 1$, with material coordinates a_{0A}. The fiber under a deformation moves with the material points of the continuum body and arrives at the deformed configuration Ω. Hence, the new fiber direction at the associated point $\mathbf{x} \in \Omega$ is defined by a unit vector field $\mathbf{a}(\mathbf{x}, t)$, $|\mathbf{a}| = 1$, with spatial coordinates a_a. For subsequent use it is beneficial to review the section on material and spatial strain tensors introduced on pp. 76-81.

Allowing length changes of the fibers, we must determine the *stretch* λ of the fiber along its direction $\mathbf{a}_0$. It is defined as the ratio between the length of a fiber element in the deformed and reference configuration. By combining eq. (2.62) with (2.73) we find that

$$\lambda \mathbf{a}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\mathbf{a}_0(\mathbf{X}) \quad , \tag{6.199}$$

which relates the fiber directions in the reference and the deformed configurations.

Consequently, since $|\mathbf{a}| = 1$, we find the square of stretch λ following the symmetry

$$\lambda^2 = \mathbf{a}_0 \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{a}_0 = \mathbf{a}_0 \cdot \mathbf{C}\mathbf{a}_0 \quad , \tag{6.200}$$

which we already have introduced in relation (2.64). This means, that the fiber stretch depends on the fiber direction of the undeformed configuration, i.e. the unit vector field $\mathbf{a}_0$, and the strain measure, i.e. the right Cauchy-Green tensor $\mathbf{C}$.

We now assume the transversely isotropic material to be hyperelastic, characterized by a Helmholtz free-energy function Ψ per unit reference volume. Because of the directional dependence on the deformation, expressed by the unit vector field $\mathbf{a}_0$, we require that the free energy depends explicitly on both the right Cauchy-Green tensor $\mathbf{C}$ and the fiber direction $\mathbf{a}_0$ in the reference configuration.

Since the sense of $\mathbf{a}_0$ is immaterial, Ψ is taken as an *even* function of $\mathbf{a}_0$. Hence, by introducing the tensor product $\mathbf{a}_0 \otimes \mathbf{a}_0$, Ψ may be expressed as a function of the two arguments $\mathbf{C}$ and $\mathbf{a}_0 \otimes \mathbf{a}_0$. The tensor $\mathbf{a}_0 \otimes \mathbf{a}_0$ (with Cartesian components $a_{0A}a_{0B}$) is of order two. For the Helmholtz free-energy function we may therefore write

$$\Psi = \Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) \quad . \tag{6.201}$$

From previous sections we know that the free energy must be independent of the coordinate system; hence $\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0)$ must be objective. Since $\mathbf{C}$ and $\mathbf{a}_0 \otimes \mathbf{a}_0$ are defined with respect to the reference configuration (which is fixed), they are unaffected by a rigid-body motion superimposed on the current configuration. Consequently, the principle of material frame-indifference of the postulated free energy $\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0)$ is satisfied trivially.

EXAMPLE 6.9 The free energy $\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0)$ must be unchanged if both the *matrix material* and the *fibers* in the reference configuration undergo a rotation around a certain axis described by the proper orthogonal tensor $\mathbf{Q}$.

Show that the requirement for transversely isotropic hyperelastic materials formally reads

$$\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = \Psi(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{a}_0 \otimes \mathbf{a}_0\mathbf{Q}^{\mathrm{T}}) \quad , \tag{6.202}$$

which holds for all proper orthogonal tensors $\mathbf{Q}$.

Solution. For the solution it is beneficial to review eqs. (6.19)–(6.25) of Section 6.2.

A rotation of the reference configuration by tensor $\mathbf{Q}$ transforms a typical point $\mathbf{X}$ into position $\mathbf{X}^\star = \mathbf{Q}\mathbf{X}$. Consequently, fiber direction $\mathbf{a}_0$ transforms into the new fiber direction $\mathbf{a}_0^\star = \mathbf{Q}\mathbf{a}_0$ so that $\mathbf{a}_0^\star \otimes \mathbf{a}_0^\star$ becomes $\mathbf{Q}\mathbf{a}_0 \otimes \mathbf{a}_0\mathbf{Q}^{\mathrm{T}}$. Now, after a subsequent motion of the rotated reference configuration, $\mathbf{X}^\star$ maps into position $\mathbf{x}$. Thus, the deformation gradient $\mathbf{F}^\star$ and the strain measure $\mathbf{C}^\star = \mathbf{F}^{\star\mathrm{T}}\mathbf{F}^\star$ relative to the rotated reference configuration are $\mathbf{F}^\star = \mathbf{F}\mathbf{Q}^{\mathrm{T}}$ (compare with (6.22)) and $\mathbf{C}^\star = \mathbf{F}^{\star\mathrm{T}}\mathbf{F}^\star = \mathbf{Q}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}$, respectively.

We say that a hyperelastic material is transversely isotropic relative to a reference configuration if the identity $\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = \Psi(\mathbf{C}^\star, \mathbf{a}_0^\star \otimes \mathbf{a}_0^\star)$ is satisfied for all proper orthogonal tensors $\mathbf{Q}$. Hence, restriction (6.202) follows directly. Note that in view of (6.202), Ψ may be seen as a scalar-valued *isotropic* tensor function of the two tensor variables $\mathbf{C}$ and $\mathbf{a}_0 \otimes \mathbf{a}_0$. ■

According to (6.27), an isotropic hyperelastic material may be represented by the first three invariants I_1, I_2, I_3 of either $\mathbf{C}$ or $\mathbf{b}$, characterized in (5.89)–(5.91). These invariants can be used to fulfil requirement (6.25), i.e. $\Psi(\mathbf{C}) = \Psi(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}})$ for all $(\mathbf{Q}, \mathbf{C})$. Following SPENCER [1971, 1984], two additional (new) scalars, I_4 and I_5, are necessary to form the *integrity bases* of the tensors $\mathbf{C}$ and $\mathbf{a}_0 \otimes \mathbf{a}_0$ and to satisfy relation (6.202). They are the so-called **pseudo-invariants** of $\mathbf{C}$ *and* $\mathbf{a}_0 \otimes \mathbf{a}_0$, which are given by

$$I_4(\mathbf{C}, \mathbf{a}_0) = \mathbf{a}_0 \cdot \mathbf{C}\mathbf{a}_0 = \lambda^2 \quad , \qquad I_5(\mathbf{C}, \mathbf{a}_0) = \mathbf{a}_0 \cdot \mathbf{C}^2\mathbf{a}_0 \quad . \tag{6.203}$$

The two pseudo-invariants I_4, I_5 arise directly from the anisotropy and contribute to the free energy. They describe the properties of the fiber family and its interaction with the other material constituents. Note that invariant I_4 is equal to the square of the stretch λ in the fiber direction $\mathbf{a}_0$ (compare with eq. (6.200)).

For the definition of the *integrity bases* and the related *theory of invariants* see the lecture notes by SCHUR [1968], the articles by SPENCER [1971] and ZHENG [1994, and references therein]. For applications in continuum mechanics the reader is referred to the works by RIVLIN [1970], BETTEN [1987a, Chapter D and 1987b], TRUESDELL and NOLL [1992] among others. A brief review of the theory of invariants may also be found in SCHRÖDER [1996].

For a transversely isotropic material, the free energy can finally be written in terms of the five independent scalar invariants, and eq. (6.27)$_1$, valid for isotropic material response, and may consequently be expanded according to

$$\Psi = \Psi\,[I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)] \quad . \tag{6.204}$$

The free energy (6.204) provides a fundamental basis for deriving the associated constitutive equations.

Constitutive equations in terms of invariants. In order to derive the constitutive equations we apply (6.13)$_2$. Then, by use of the chain rule, the second Piola-Kirchhoff stress tensor $\mathbf{S}$ is given as a function of the five scalar invariants, i.e.

$$\mathbf{S} = 2\frac{\partial\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0)}{\partial\mathbf{C}} = 2\sum_{a=1}^{5} \frac{\partial\Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0)}{\partial I_a}\frac{\partial I_a}{\partial\mathbf{C}} \quad , \tag{6.205}$$

in which $\partial I_1/\partial\mathbf{C}$ and $\partial I_2/\partial\mathbf{C}$, $\partial I_3/\partial\mathbf{C}$ are given by (6.30) and (6.31), respectively.

The remaining derivatives follow from (6.203) and have the forms

$$\frac{\partial I_4}{\partial\mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{a}_0 \qquad \text{or} \qquad \frac{\partial I_4}{\partial C_{AB}} = a_{0\,A}a_{0\,B} \quad , \tag{6.206}$$

$$\begin{aligned} \frac{\partial I_5}{\partial\mathbf{C}} &= \mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0 \\ \text{or} \qquad \frac{\partial I_5}{\partial C_{AB}} &= a_{0\,A}C_{BC}a_{0\,C} + a_{0\,B}C_{AC}a_{0\,C} \quad . \end{aligned} \tag{6.207}$$

Finally, (6.205) reads, with eqs. (6.30), (6.31), (6.206) and (6.207),

$$\begin{aligned} \mathbf{S} = 2\Bigg[&\left(\frac{\partial\Psi}{\partial I_1} + I_1\frac{\partial\Psi}{\partial I_2}\right)\mathbf{I} - \frac{\partial\Psi}{\partial I_2}\mathbf{C} + I_3\frac{\partial\Psi}{\partial I_3}\mathbf{C}^{-1} \\ &+ \frac{\partial\Psi}{\partial I_4}\mathbf{a}_0 \otimes \mathbf{a}_0 + \frac{\partial\Psi}{\partial I_5}(\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0)\Bigg] \quad , \end{aligned} \tag{6.208}$$

which extends the constitutive equation (6.32) by the addition of the last two terms.

Using arguments similar to those used for the derivation of the spatial version of the stress relation (6.34), namely a push-forward operation on the material stress tensor $\mathbf{S}$ by the motion χ, we arrive, using (6.199) and (6.203)$_1$, at

$$\begin{aligned} \boldsymbol{\sigma} = 2J^{-1}\Bigg[&I_3\frac{\partial\Psi}{\partial I_3}\mathbf{I} + \left(\frac{\partial\Psi}{\partial I_1} + I_1\frac{\partial\Psi}{\partial I_2}\right)\mathbf{b} - \frac{\partial\Psi}{\partial I_2}\mathbf{b}^2 \\ &+ I_4\frac{\partial\Psi}{\partial I_4}\mathbf{a} \otimes \mathbf{a} + I_4\frac{\partial\Psi}{\partial I_5}(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} \otimes \mathbf{a})\Bigg] \quad . \end{aligned} \tag{6.209}$$

Recall that the unit vector $\mathbf{a}(\mathbf{x}, t)$ denotes the fiber direction in the deformed configuration while $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the (second-order) left Cauchy-Green tensor. Observe the similar structure of the last two terms in the stress relation (6.209) to that presented in (6.208).

The associated elasticity tensors in the material and spatial descriptions follow by means of expressions (6.154) and (6.159), respectively (for an explicit derivation compare with the work of WEISS et al. [1996]). For implementations of large strain transversely isotropic models in a finite element program see WEISS et al. [1996], SCHRÖDER [1996], BONET and BURTON [1998] among others.

Incompressible transversely isotropic materials. We now consider transversely isotropic materials with an *incompressible isotropic* matrix material.

Firstly, we study the case in which the embedded fibers are *extensible*. Since we assume incompressibility of the isotropic matrix material, i.e. $I_3 = 1$, we are able to postulate a free energy in terms of the remaining four independent invariants. Because of the incompressibility constraint $I_3 = 1$ the free energy Ψ is enhanced by an indeterminate *Lagrange multiplier* $p/2$ which is identified as a reaction pressure. In view of (6.62) we have the assumption

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)] - \frac{1}{2}p(I_3 - 1) \quad . \tag{6.210}$$

The associated stress relations given in the reference and current configurations are basically those presented by (6.63) and (6.64) supplemented by the fourth and fifth term in eqs. (6.208) and (6.209), respectively.

Secondly, we study an incompressible isotropic matrix material which is continuously reinforced throughout by *inextensible* fibers. This means that $\lambda = 1$ and, in view of $(6.203)_1$, the fourth invariant is equal to one. With this additional internal constraint, the free energy Ψ is a function only of I_1, I_2, which are responsible for the hyperelastic isotropic matrix material, and I_5, which is responsible for the fibers. By adding the term $q(I_4 - 1)/2$ to the free energy Ψ we obtain the function

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_5(\mathbf{C}, \mathbf{a}_0)] - \frac{1}{2}p(I_3 - 1) - \frac{1}{2}q(I_4 - 1) \quad , \tag{6.211}$$

where $q/2$ is an additional indeterminate *Lagrange multiplier*.

The associated stress relations in the Lagrangian and Eulerian descriptions for the transversely isotropic materials with an *incompressible isotropic* matrix material and *inextensible* fibers (with direction $\mathbf{a}_0$) are the extended constitutive equations (6.63) and (6.65), i.e.

$$\begin{aligned} \mathbf{S} = {} & -p\mathbf{C}^{-1} - q\mathbf{a}_0 \otimes \mathbf{a}_0 + 2\left(\frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2}\right)\mathbf{I} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{C} \\ & + 2\frac{\partial \Psi}{\partial I_5}(\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0) \quad , \end{aligned} \tag{6.212}$$

$$\boldsymbol{\sigma} = -p\mathbf{I} - q\mathbf{a} \otimes \mathbf{a} + 2\frac{\partial \Psi}{\partial I_1}\mathbf{b} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{b}^{-1} + 2\frac{\partial \Psi}{\partial I_5}(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} \otimes \mathbf{a}) \quad . \tag{6.213}$$

Note that the indeterminate terms $-q\mathbf{a}_0 \otimes \mathbf{a}_0$ and $-q\mathbf{a} \otimes \mathbf{a}$ are identified as fiber reaction stresses which respond to the inextensibility constraint $I_4 = 1$.

EXERCISES

1. Starting from the pseudo-invariants I_4 and I_5, i.e. eqs. (6.203), show their derivative with respect to $\mathbf{C}$, eqs. (6.206) and (6.207).

2. We characterize a transversely isotropic material by the *decoupled* free energy in the form

$$\Psi = \Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}(\bar{I}_1, \bar{I}_2, \bar{I}_4, \bar{I}_5) \quad , \tag{6.214}$$

where Ψ_{vol} and Ψ_{iso} are the volumetric and isochoric contributions to the hyperelastic response (recall Section 6.4). The modified invariants $\bar{I}_1, \bar{I}_2$ are given according to eqs. (6.109) and (6.110), while $\bar{I}_3 = \det\overline{\mathbf{C}} = 1$ (note that $\bar{I}_1, \bar{I}_2, \bar{I}_3$ are the modified principal invariants of the modified tensor $\overline{\mathbf{C}} = J^{-2/3}\mathbf{C}$). The remaining modified pseudo-invariants are expressed by $\bar{I}_4 = J^{-2/3}I_4$ and $\bar{I}_5 = J^{-4/3}I_5$.

   (a) Having in mind the free energy (6.214) and the derivatives (6.206) and (6.207), show that the constitutive equation $\mathbf{S} = Jp\mathbf{C}^{-1} + J^{-2/3}\mathbb{P} : \overline{\mathbf{S}}$ specializes to

$$\begin{aligned} \overline{\mathbf{S}} &= 2\frac{\partial \Psi_{\mathrm{iso}}(\bar{I}_1, \bar{I}_2, \bar{I}_4, \bar{I}_5)}{\partial \overline{\mathbf{C}}} = 2\sum_{\substack{a=1 \\ a\neq 3}}^{5} \frac{\partial \Psi_{\mathrm{iso}}}{\partial \bar{I}_a}\frac{\partial \bar{I}_a}{\partial \overline{\mathbf{C}}} \\ &= \bar{\gamma}_1\mathbf{I} + \bar{\gamma}_2\overline{\mathbf{C}} + \bar{\gamma}_4\mathbf{a}_0 \otimes \mathbf{a}_0 + \bar{\gamma}_5(\mathbf{a}_0 \otimes \overline{\mathbf{C}}\mathbf{a}_0 + \mathbf{a}_0\overline{\mathbf{C}} \otimes \mathbf{a}_0) \quad , \end{aligned} \tag{6.215}$$

   with the response coefficients

$$\bar{\gamma}_4 = 2\frac{\partial \Psi_{\mathrm{iso}}}{\partial \bar{I}_4} \quad , \qquad \bar{\gamma}_5 = 2\frac{\partial \Psi_{\mathrm{iso}}}{\partial \bar{I}_5}$$

   for the fictitious second Piola-Kirchhoff stress $\overline{\mathbf{S}}$. Note that the coefficients $\bar{\gamma}_1$ and $\bar{\gamma}_2$ reflect the isotropic stress response, as given in eqs. (6.116).

   (b) By recalling Section 6.6, a closed form expression for the elasticity tensor $\mathbb{C}$ in the material description is given by relation (6.162), with contributions $\mathbb{C}_{\mathrm{vol}}$, i.e. $(6.166)_4$, and $\mathbb{C}_{\mathrm{iso}}$, i.e. (6.168).

By use of the important property (1.256) and the constitutive equation for the fictitious second Piola-Kirchhoff stress (6.215)$_3$, show that the fictitious elasticity tensor $\overline{\mathbb{C}}$ in the *material description* takes on the form

$$\begin{aligned} J^{4/3}\overline{\mathbb{C}} &= 2\frac{\partial \overline{\mathbf{S}}}{\partial \overline{\mathbf{C}}} = 4\frac{\partial^2 \Psi_{\mathrm{iso}}(\bar{I}_1, \bar{I}_2, \bar{I}_4, \bar{I}_5)}{\partial \overline{\mathbf{C}} \partial \overline{\mathbf{C}}} \\ &= \bar{\delta}_1 \mathbf{I} \otimes \mathbf{I} + \bar{\delta}_2 (\mathbf{I} \otimes \overline{\mathbf{C}} + \overline{\mathbf{C}} \otimes \mathbf{I}) + \bar{\delta}_3 \overline{\mathbf{C}} \otimes \overline{\mathbf{C}} + \bar{\delta}_4 \mathbb{I} \\ &\quad + \bar{\delta}_5 (\mathbf{I} \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{I}) \\ &\quad + \bar{\delta}_6 \left(\overline{\mathbf{C}} \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \overline{\mathbf{C}}\right) \\ &\quad + \bar{\delta}_7 \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 + \bar{\delta}_8 \left(\mathbf{I} \otimes \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} + \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} \otimes \mathbf{I}\right) \\ &\quad + \bar{\delta}_9 \left(\overline{\mathbf{C}} \otimes \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} + \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} \otimes \overline{\mathbf{C}}\right) + \bar{\delta}_{10} \left(\frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} \otimes \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}}\right) \\ &\quad + \bar{\delta}_{11} \left(\mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} + \frac{\partial \bar{I}_5}{\partial \overline{\mathbf{C}}} \otimes \mathbf{a}_0 \otimes \mathbf{a}_0\right) + \bar{\delta}_{12} \frac{\partial^2 \bar{I}_5}{\partial \overline{\mathbf{C}} \partial \overline{\mathbf{C}}} \quad , \end{aligned}$$

with the fourth-order unit tensor $\mathbb{I}$ defined by (1.160) and the coefficients $\bar{\delta}_a$, $a = 5, \ldots, 12$, by

$$\left. \begin{aligned} \bar{\delta}_5 &= 4\left(\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_1 \partial \bar{I}_4} + \bar{I}_1 \frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_2 \partial \bar{I}_4}\right) \quad , \qquad \bar{\delta}_6 = -4\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_2 \partial \bar{I}_4} \quad , \\ \bar{\delta}_7 &= 4\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_4 \partial \bar{I}_4} \quad , \qquad \bar{\delta}_8 = 4\left(\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_1 \partial \bar{I}_5} + \bar{I}_1 \frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_2 \partial \bar{I}_5}\right) \quad , \\ \bar{\delta}_9 &= -4\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_2 \partial \bar{I}_5} \quad , \qquad \bar{\delta}_{10} = 4\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_5 \partial \bar{I}_5} \quad , \\ \bar{\delta}_{11} &= 4\frac{\partial^2 \Psi_{\mathrm{iso}}}{\partial \bar{I}_4 \partial \bar{I}_5} \quad , \qquad \bar{\delta}_{12} = 4\frac{\partial \Psi_{\mathrm{iso}}}{\partial \bar{I}_5} \quad . \end{aligned} \right\}$$

Note that the coefficients $\bar{\delta}_a$, $a = 1, \ldots, 4$, were given previously in relations (6.195) and reflect the isotropic contributions.

6.8 Composite Materials with Two Families of Fibers

In the following we discuss appropriate constitutive equations for the *finite elastic* response of fiber-reinforced composites in which the matrix material is reinforced by *two* families of fibers. We assume that the fibers are continuously distributed throughout the material so that the continuum theory of fiber-reinforced composites is the constitutive theory of choice.

There are many different fibers and matrix materials now in use for composite materials. Examples of specific fibers for structural applications are *boron* and *glass*. The latter is an important engineering fiber with high strength and low cost. Further examples are *carbon* and *graphite* (the difference is in the carbon content), the organic fiber *aramid* and the ceramic fibers *silicon carbide* and *alumina* among others. Many specific matrix materials are available for the use in composites; for example, *thermoplastic polymers*, *thermoset polymers*, *metals* (such as aluminum, titanium and copper) and *ceramics*.

Vast numbers of applications in industry are concerned with composite materials, such as the finite elastic response of belts and high pressure tubes, steel reinforced rubber used in tyres, and integrated circuits used in electronic computing devices. Typical medical applications are lightweight wheelchairs and implant devices such as hip joints (see also the textbook by HERAKOVICH [1998, Chapter 1]). The five-volume encyclopedia of composites edited by LEE [1990, 1991] includes a detailed account of special types of fiber, matrix materials and composites as engineering materials. Typical engineering properties, manufacturing and fabrication processes and details on how to use composite materials for different applications are also provided.

However, it is important to note that numerous organisms such as the human body, animals and plants are heterogeneous systems of various composite biomaterials. The textbooks by FUNG [1990, 1993, 1997] are concerned with the biomechanics of various biomaterials, soft tissues and organs of the human body. One important example of a fibre-reinforced biomaterial is the artery. The layers of the arterial wall are composed mainly of an isotropic matrix material (associated with the *elastin*) and two families of fibers (associated with the *collagen*) which are arranged in symmetrical spirals (for arterial histology see RHODIN [1980]). For mechanical properties and constitutive equations of arterial walls, see the reviews by, for example, HAYASHI [1993], HUMPHREY [1995] and the data book edited by ABÉ et al. [1996, Chapter 2]. A simple finite element simulation of the orthotropic biomechanical behavior of the arterial wall is provided by HOLZAPFEL et al. [1996d, 1996e] and HOLZAPFEL and WEIZSÄCKER [1998]. For a review of finite element models for arterial wall mechanics, see the article by SIMON et al. [1993].

Free energy and constitutive equations. We may now consider a body built up of a matrix material with two families of fibers each of which is unidirectional with preferred direction. The matrix material is assumed to be hyperelastic. The preferential fiber directions in the reference and the current configuration are denoted by the unit vector fields $\mathbf{a}_0, \mathbf{g}_0$ and $\mathbf{a}, \mathbf{g}$, respectively. By analogy with relation (6.201) we may postulate the free energy

$$\Psi = \Psi(\mathbf{C}, \mathbf{A}_0, \mathbf{G}_0) \tag{6.216}$$

per unit reference volume. For notational simplicity we have introduced the abbreviations $\mathbf{A}_0 = \mathbf{a}_0 \otimes \mathbf{a}_0$ and $\mathbf{G}_0 = \mathbf{g}_0 \otimes \mathbf{g}_0$, frequently referred to as **structural tensors.**

The free energy must be unchanged if the fiber-reinforced composite (i.e. a hyperelastic (matrix) material with two families of fibers) in the reference configuration undergoes a rotation described by the proper orthogonal tensor $\mathbf{Q}$. Using arguments similar to those used for a single fiber family (see the previous Section 6.7), the requirement for this type of composite is, in view of (6.202), given by

$$\Psi(\mathbf{C}, \mathbf{A}_0, \mathbf{G}_0) = \Psi(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_0\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{G}_0\mathbf{Q}^{\mathrm{T}}) \quad , \tag{6.217}$$

which holds for all tensors $\mathbf{Q}$ (recall Example 6.9). Here, Ψ is a scalar-valued *isotropic* tensor function of the three tensor variables $\mathbf{C}$, $\mathbf{A}_0$ and $\mathbf{G}_0$.

According to SPENCER [1971, 1984], requirement (6.217) is satisfied if Ψ is a function of the set of invariants

$$\left.\begin{array}{ccc} I_1(\mathbf{C}) \ , & I_2(\mathbf{C}) \ , & I_3(\mathbf{C}) \ , \\ & I_4(\mathbf{C}, \mathbf{a}_0) \ , \quad I_5(\mathbf{C}, \mathbf{a}_0) \ , & \\ I_6(\mathbf{C}, \mathbf{g}_0) = \mathbf{g}_0 \cdot \mathbf{C}\mathbf{g}_0 \ , & & I_7(\mathbf{C}, \mathbf{g}_0) = \mathbf{g}_0 \cdot \mathbf{C}^2\mathbf{g}_0 \ , \\ I_8(\mathbf{C}, \mathbf{a}_0, \mathbf{g}_0) = (\mathbf{a}_0 \cdot \mathbf{g}_0)\mathbf{a}_0 \cdot \mathbf{C}\mathbf{g}_0 \ , & & I_9(\mathbf{a}_0, \mathbf{g}_0) = (\mathbf{a}_0 \cdot \mathbf{g}_0)^2 \ . \end{array}\right\} \tag{6.218}$$

The three invariants I_1, I_2, I_3 are identical to those from the isotropic theory presented in eqs. (5.89)–(5.91). The pseudo-invariants I_4, I_5 are given by eq. (6.203) and characterize one family of fibers with direction $\mathbf{a}_0$. The pseudo-invariants $I_4, \ldots, I_9$ are associated with the anisotropy generated by the two families of fibers. The dot product $\mathbf{a}_0 \cdot \mathbf{g}_0$ is a geometrical constant determining the cosine of the angle between the two fiber directions in the reference configuration. Therefore, the invariant I_9 does not depend on the deformation and is subsequently no longer considered. Note that I_4 and I_6 are equal to the squares of the stretch in the fiber directions $\mathbf{a}_0$ and $\mathbf{g}_0$, respectively.

The constitutive equation for the second Piola-Kirchhoff stress $\mathbf{S}$ follows from the postulated free energy (6.216) by differentiation with respect to $\mathbf{C}$. By means of the chain rule, $\mathbf{S}$ is given as a function of the remaining eight scalar invariants in the form

$$\mathbf{S} = 2\frac{\partial\Psi(\mathbf{C}, \mathbf{A}_0, \mathbf{G}_0)}{\partial\mathbf{C}} = 2\sum_{a=1}^{8} \frac{\partial\Psi(I_1, \ldots, I_8)}{\partial I_a}\frac{\partial I_a}{\partial\mathbf{C}} \quad , \tag{6.219}$$

in which $\partial I_1/\partial\mathbf{C}, \ldots, \partial I_3/\partial\mathbf{C}$ and $\partial I_4/\partial\mathbf{C}$, $\partial I_5/\partial\mathbf{C}$ are given by eqs. (6.30), (6.31) and (6.206), (6.207), respectively.

The remaining derivatives of the invariants follow from $(6.218)_6$–$(6.218)_8$ and have the forms

$$\frac{\partial I_6}{\partial\mathbf{C}} = \mathbf{G}_0 \qquad \text{or} \qquad \frac{\partial I_6}{\partial C_{AB}} = g_{0A}g_{0B} = G_{AB} \quad , \tag{6.220}$$

$$\begin{aligned} \frac{\partial I_7}{\partial\mathbf{C}} &= \mathbf{g}_0 \otimes \mathbf{C}\mathbf{g}_0 + \mathbf{g}_0\mathbf{C} \otimes \mathbf{g}_0 \\ \text{or} \qquad \frac{\partial I_7}{\partial C_{AB}} &= g_{0A}C_{BC}g_{0C} + g_{0B}C_{AC}g_{0C} \quad , \end{aligned} \tag{6.221}$$

$$\begin{aligned} \frac{\partial I_8}{\partial\mathbf{C}} &= \frac{1}{2}(\mathbf{a}_0 \cdot \mathbf{g}_0)(\mathbf{a}_0 \otimes \mathbf{g}_0 + \mathbf{g}_0 \otimes \mathbf{a}_0) \\ \text{or} \qquad \frac{\partial I_8}{\partial C_{AB}} &= \frac{1}{2}(a_{0C}g_{0C})(a_{0A}g_{0B} + g_{0A}a_{0B}) \quad . \end{aligned} \tag{6.222}$$

Using arguments similar to those used for deriving the stress relation (6.208), we obtain the explicit expressions

$$\begin{aligned} \mathbf{S} = 2\Bigg[&\left(\frac{\partial\Psi}{\partial I_1} + I_1\frac{\partial\Psi}{\partial I_2}\right)\mathbf{I} - \frac{\partial\Psi}{\partial I_2}\mathbf{C} + I_3\frac{\partial\Psi}{\partial I_3}\mathbf{C}^{-1} \\ &+ \frac{\partial\Psi}{\partial I_4}\mathbf{A}_0 + \frac{\partial\Psi}{\partial I_5}(\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0) + \frac{\partial\Psi}{\partial I_6}\mathbf{G}_0 \\ &+ \frac{\partial\Psi}{\partial I_7}(\mathbf{g}_0 \otimes \mathbf{C}\mathbf{g}_0 + \mathbf{g}_0\mathbf{C} \otimes \mathbf{g}_0) + \frac{1}{2}\frac{\partial\Psi}{\partial I_8}(\mathbf{a}_0 \cdot \mathbf{g}_0)(\mathbf{a}_0 \otimes \mathbf{g}_0 + \mathbf{g}_0 \otimes \mathbf{a}_0)\Bigg] \quad . \end{aligned} \tag{6.223}$$

The anisotropic stress response (6.223) extends relation (6.208) by the addition of the last three terms. Note that $(\partial\Psi/\partial I_4)\mathbf{A}_0$ and $(\partial\Psi/\partial I_6)\mathbf{G}_0$ characterize the (decoupled) stress contributions arising only from the fibers.

Orthotropic hyperelastic materials. If $\mathbf{a}_0 \cdot \mathbf{g}_0 = 0$, the two families of fibers have orthogonal directions. Then, the material is said to be **orthotropic** in the reference configuration with respect to the planes normal to the fibers and the surface in which the fibers lie. Since now two directions in space $(\mathbf{a}_0, \mathbf{g}_0)$ are preferred with respect to the mechanical response of the composite, the remaining third direction orthogonal to the fiber plane is also a preferred direction. The mechanical response in the third direction is governed by the matrix material. The list of invariants (6.218) reduces to the first seven and the free energy has the form $\Psi = \Psi(I_1, \ldots, I_7)$.

A further special case may be found under the assumption that the isotropic matrix material is *incompressible*, i.e. $I_3 = 1$. Additionally, the families of fibers may also be *inextensible* in the two fiber directions $\mathbf{a}_0$ and $\mathbf{g}_0$, consequently $I_4 = 1$ and $I_6 = 1$. For this case a suitable Helmholtz free-energy function is given by

$$\begin{aligned} \Psi = \;& \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_5(\mathbf{C}, \mathbf{a}_0), I_7(\mathbf{C}, \mathbf{g}_0)] \\ & - \frac{1}{2}p(I_3 - 1) - \frac{1}{2}q(I_4 - 1) - \frac{1}{2}r(I_6 - 1) \quad , \end{aligned} \tag{6.224}$$

with the indeterminate *Lagrange multipliers* $p/2$, $q/2$, $r/2$.

The constitutive equations for an orthotropic material composed of an *incompressible isotropic* matrix material and *inextensible* fibers (with directions $\mathbf{a}_0$ and $\mathbf{g}_0$) depend only on the invariants I_1, I_2 and I_5, I_7. The constitutive equations in the Lagrangian and Eulerian descriptions, extending eqs. (6.212) and (6.213), follow from (6.224) and are

$$\begin{aligned}
\mathbf{S} &= 2\sum_{\substack{a=1\\ a\neq 3,4,6}}^{7} \frac{\partial\Psi(I_1,I_2,I_5,I_7)}{\partial I_a}\frac{\partial I_a}{\partial \mathbf{C}} - \frac{\partial}{\partial \mathbf{C}}\left[p(I_3-1)+q(I_4-1)+r(I_6-1)\right] \\
&= -p\mathbf{C}^{-1} - q\mathbf{A}_0 - r\mathbf{G}_0 + 2\left(\frac{\partial\Psi}{\partial I_1} + I_1\frac{\partial\Psi}{\partial I_2}\right)\mathbf{I} - 2\frac{\partial\Psi}{\partial I_2}\mathbf{C} \\
&\quad +2\frac{\partial\Psi}{\partial I_5}(\mathbf{a}_0\otimes\mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C}\otimes\mathbf{a}_0) + 2\frac{\partial\Psi}{\partial I_7}(\mathbf{g}_0\otimes\mathbf{C}\mathbf{g}_0 + \mathbf{g}_0\mathbf{C}\otimes\mathbf{g}_0) \quad , \qquad (6.225)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\sigma} &= 2\mathbf{F}\frac{\partial\Psi}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}} = -p\mathbf{I} - q\mathbf{A} - r\mathbf{G} + 2\frac{\partial\Psi}{\partial I_1}\mathbf{b} - 2\frac{\partial\Psi}{\partial I_2}\mathbf{b}^{-1} \\
&\quad +2\frac{\partial\Psi}{\partial I_5}(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b}\otimes\mathbf{a}) + 2\frac{\partial\Psi}{\partial I_7}(\mathbf{g}\otimes\mathbf{b}\mathbf{g} + \mathbf{g}\mathbf{b}\otimes\mathbf{g}) \quad , \qquad (6.226)
\end{aligned}$$

respectively. Herein, the indeterminate terms $-p\mathbf{C}^{-1}$, $-q\mathbf{A}_0$, $-r\mathbf{G}_0$ (and $-p\mathbf{I}$, $-q\mathbf{A}$, $-r\mathbf{G}$) are identified as reaction stresses associated with the constraints $I_3 = 1$, $I_4 = 1$, $I_6 = 1$, with the pressure-like quantity p and the fiber tensions q, r, respectively. In eq. $(6.226)_2$ we have introduced the definitions $\mathbf{A} = \mathbf{a}\otimes\mathbf{a}$ and $\mathbf{G} = \mathbf{g}\otimes\mathbf{g}$ of the structural tensors $\mathbf{A}$ and $\mathbf{G}$. Recall that $\mathbf{a}(\mathbf{x},t)$, $\mathbf{g}(\mathbf{x},t)$ denote the fiber directions in the deformed configuration while $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the spatial strain tensor of second-order.

However, if the two families of fibers are mechanically equivalent – and not necessarily orthogonal – then the material is said to be **locally orthotropic** in the reference configuration with respect to the mutually orthogonal planes which bisect the two fiber families (with directions $\mathbf{a}_0$ and $\mathbf{g}_0$) and the surface in which the fibers lie. Then, Ψ is a function of the first eight invariants listed in (6.218) and is symmetric with respect to interchanges of $\mathbf{a}_0$ and $\mathbf{g}_0$.

It can finally be shown that for a *locally orthotropic* material Ψ can be expressed as a function of the seven invariants

$$\left.\begin{array}{llll} I_1(\mathbf{C}) \;, & I_2(\mathbf{C}) \;, & I_3(\mathbf{C}) \;, & I_8(\mathbf{C},\mathbf{a}_0,\mathbf{g}_0) \;, \\ I_9(\mathbf{C},\mathbf{a}_0,\mathbf{g}_0) \;, & I_{10}(\mathbf{C},\mathbf{a}_0,\mathbf{g}_0) \;, & I_{11}(\mathbf{C},\mathbf{a}_0,\mathbf{g}_0) & \end{array}\right\} \qquad (6.227)$$

(see SPENCER [1984]), with I_1, I_2, I_3 and I_8 given by eqs. (5.89)–(5.91) and $(6.218)_8$, and the definitions

$$I_9 = I_4 + I_6 \;, \qquad I_{10} = I_4 I_6 \;, \qquad I_{11} = I_5 + I_7 \qquad (6.228)$$

for the remaining three pseudo-invariants.

EXERCISES

1. Consider a *locally orthotropic* material with the free energy Ψ expressed as a function of the invariants presented by (6.227) using (6.228). Assume an *incompressible isotropic* matrix material and two families of *inextensible* fibers.

   Show that the constitutive equation for the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\begin{aligned}
\boldsymbol{\sigma} &= -p\mathbf{I} - q\mathbf{A} - r\mathbf{G} + 2\frac{\partial\Psi}{\partial I_1}\mathbf{b} - 2\frac{\partial\Psi}{\partial I_2}\mathbf{b}^{-1} + (\mathbf{a}_0\cdot\mathbf{g}_0)\frac{\partial\Psi}{\partial I_8}(\mathbf{a}\otimes\mathbf{g} + \mathbf{g}\otimes\mathbf{a}) \\
&\quad +2\frac{\partial\Psi}{\partial I_{11}}(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b}\otimes\mathbf{a} + \mathbf{g}\otimes\mathbf{b}\mathbf{g} + \mathbf{g}\mathbf{b}\otimes\mathbf{g}) \quad ,
\end{aligned}$$

   where the first three terms characterize reaction stresses.

2. We characterize a compressible composite with two families of fibers by the *decoupled* representation of the free energy

$$\Psi = \Psi(\mathbf{C},\mathbf{A}_0,\mathbf{G}_0) = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}(\bar{I}_1,\bar{I}_2,\bar{I}_4,\ldots,\bar{I}_8) \qquad (6.229)$$

   (compare also with Exercise 2 on p. 271), with the volumetric and isochoric parts Ψ_{vol} and Ψ_{iso}, and the modified invariants $\bar{I}_1, \bar{I}_2$ given by eqs. (6.109) and (6.110) ($\bar{I}_3 = \det\overline{\mathbf{C}} = 1$). The remaining modified pseudo-invariants are $\bar{I}_a = J^{-2/3}I_a$, $a = 4, 6, 8$, and $\bar{I}_a = J^{-4/3}I_a$, $a = 5, 7$.

   Use the free energy (6.229) to particularize the fictitious second Piola-Kirchhoff stress $\overline{\mathbf{S}}$ which appears in the constitutive equation for $\mathbf{S} = Jp\mathbf{C}^{-1} + J^{-2/3}\mathbb{P} : \overline{\mathbf{S}}$. Show that

$$\overline{\mathbf{S}} = 2\frac{\partial\Psi_{\mathrm{iso}}(\bar{I}_1,\bar{I}_2,\bar{I}_4,\ldots,\bar{I}_8)}{\partial\overline{\mathbf{C}}} = 2\sum_{\substack{a=1\\ a\neq 3}}^{8}\frac{\partial\Psi_{\mathrm{iso}}}{\partial\bar{I}_a}\frac{\partial\bar{I}_a}{\partial\overline{\mathbf{C}}} \quad ,$$

   with the explicit expressions

$$\frac{\partial\bar{I}_1}{\partial\overline{\mathbf{C}}} = \mathbf{I} \;, \qquad \frac{\partial\bar{I}_2}{\partial\overline{\mathbf{C}}} = \bar{I}_1\mathbf{I} - \overline{\mathbf{C}} \;, \qquad \frac{\partial\bar{I}_4}{\partial\overline{\mathbf{C}}} = \mathbf{A}_0 \;,$$

$$\frac{\partial\bar{I}_5}{\partial\overline{\mathbf{C}}} = \mathbf{a}_0\otimes\overline{\mathbf{C}}\mathbf{a}_0 + \mathbf{a}_0\overline{\mathbf{C}}\otimes\mathbf{a}_0 \;, \qquad \frac{\partial\bar{I}_6}{\partial\overline{\mathbf{C}}} = \mathbf{G}_0 \;,$$

$$\frac{\partial\bar{I}_7}{\partial\overline{\mathbf{C}}} = \mathbf{g}_0\otimes\overline{\mathbf{C}}\mathbf{g}_0 + \mathbf{g}_0\overline{\mathbf{C}}\otimes\mathbf{g}_0 \;, \qquad \frac{\partial\bar{I}_8}{\partial\overline{\mathbf{C}}} = \frac{1}{2}(\mathbf{a}_0\cdot\mathbf{g}_0)(\mathbf{a}_0\otimes\mathbf{g}_0 + \mathbf{g}_0\otimes\mathbf{a}_0) \;.$$

6.9 Constitutive Models with Internal Variables

Many materials used in the fields of engineering and physics are inelastic. It turns out that the constitutive models introduced hitherto are not adequate to describe this class of materials, for which every admissible process is dissipative. Within the remaining sections of this chapter we study inelastic materials and, based on the concept of internal variables, we derive constitutive models for viscoelastic materials and hyperelastic materials with isotropic damage.

Concept of internal variables. The current thermodynamic state of thermoelastic materials can be determined solely by the current values of the deformation gradient $\mathbf{F}$ and the temperature Θ. Variables such as $\mathbf{F}$ or Θ are *measurable* and *controllable* quantities and are *accessible* to direct observation. In practice these type of variables are usually called **external variables**.

The current thermodynamic state of materials that involve dissipation can be determined by a finite number of so-called **internal variables**, or in the literature sometimes called **hidden variables** (hidden to the eyes of external observers). These additional thermodynamic state variables, which we denote collectively by $\boldsymbol{\xi}$, are supposed to describe aspects of the internal structure of materials associated with irreversible (dissipative) effects. Note that strain (stress) and temperature (entropy) depend on these internal variables. The evolution of internal variables replicates indirectly the history of the deformation, and hence they are often also termed **history variables**. Materials that involve dissipative effects we refer to as **dissipative materials** or **materials with dissipation**.

Hence, the **concept of internal variables** postulates that the current thermodynamic state at a point of a dissipative material is specified by the triple $(\mathbf{F}, \Theta, \boldsymbol{\xi})$ (the current thermodynamic state may be imagined as a *fictitious* state of thermodynamic equilibrium). Then, the current thermodynamic state is represented in a finite-dimensional state space and described by the current values (and *not* by their past history) of the deformation gradient, the temperature and the finite number of internal variables.

The nature of internal variables may be physical, describing the physical structure of materials. In the course of phenomenological experiments one may be able to *identify* internal variables; however, they are certainly *not controllable* or *observable*.

We use the internal variables as phenomenological variables which are constructed mathematically. They are *mechanical* (or thermal, or even chemical or electrical ...) state variables describing structural properties within a macroscopic framework, such as the 'dashpot displacements' in viscoelastic models, damage, inelastic strains, dislocation densities, point-defects and so on. Hence, here we introduce both external and internal variables as macroscopic quantities without referring to the internal microstructure of the material.

The concept of internal variables serves as a profound basis for the development of constitutive equations for dissipative materials studied in the following section.

Constitutive equations and internal dissipation. The existence of non-equilibrium states that do evolve with time is an essential feature of inelastic materials.

Two typical examples of irreversible processes known from classical mechanics which govern non-equilibrium states are **relaxation** and **creep**. Relaxation (and creep) is the time-dependent return to the (new) equilibrium state after a disturbance. In general, stress will decrease with time at a *fixed (constant) strain*, which is referred to as *relaxation*, while during a *creeping* process strain will increase with time at a *fixed (constant) stress*. For an illustration of these two simple processes, see Figure 6.5. The (strain or stress) response of removing a strain or stress is called **recovery**. A viscoelastic behavior of a material is characterized by **hysteresis**. The term 'hysteresis' means that the loading and unloading curves do not coincide. It represents the non-recoverable energy when a material is loaded to a point and then unloaded.

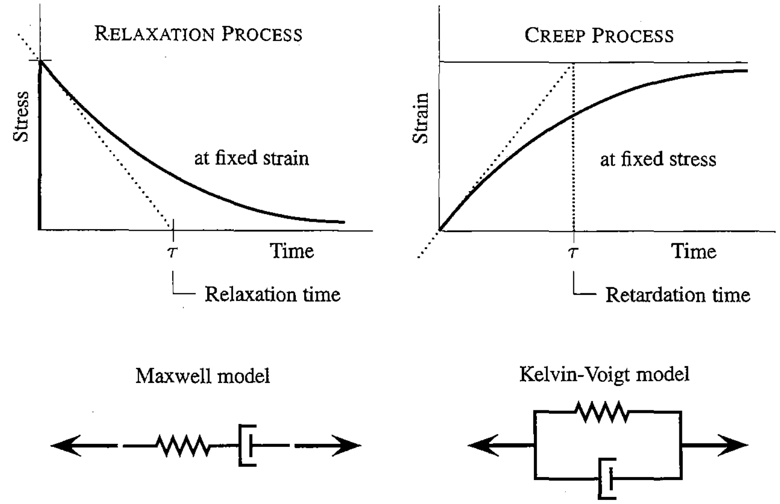

Figure 6.5 Maxwell and Kelvin-Voigt models associated with relaxation and creep behavior.

The **Maxwell model** (a dashpot is arranged in series with a spring) and the **Kelvin-Voigt model** (a dashpot is arranged in parallel with a spring), two mechanical models known from linear viscoelasticity, are frequently used to discuss relaxation and creep behavior. These models combine 'viscous' (or fluid-like) with 'elastic' (or solid-like) behavior. Under the action of a constant deformation (strain), the Maxwell model is supposed to produce instantaneously a stress response by the **spring** which is followed by an exponential stress relaxation due to the **dashpot**. On the other hand, the Kelvin-Voigt model is supposed to produce no immediate deformation for a constant load (stress). However, in a Kelvin-Voigt model a deformation (strain) will be created with time according to an exponential function. Within the realm of non-equilibrium thermodynamics the viscoelastic deformation mechanisms of these material models are not reversible.

The rate of decay of the stress and strain in a viscoelastic process is characterized by the so-called **relaxation time** $\tau \in (0, \infty)$, with dimension of time, known from linear viscoelasticity. The parameter τ associated with a creeping process is often referred to as the **retardation time**.

The constitutive equations introduced hitherto are no longer sufficient to describe dissipative materials. The vast majority of constitutive models that are used to approximate the physical behavior of real nonlinear inelastic materials are developed on the basis of internal variables.

In this chapter we remain within an isothermal framework, in which the temperature is assumed to be constant ($\Theta = \Theta_0$). Hence, we postulate a Helmholtz free-energy function Ψ which defines the thermodynamic state by the observable variable $\mathbf{F}$ and a set of additional internal history variables $\boldsymbol{\xi}_\alpha$, $\alpha = 1, \dots, m$, to be specified for the particular problem. We write

$$\Psi = \Psi(\mathbf{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m) \quad , \qquad (6.230)$$

where the second-order tensors $\boldsymbol{\xi}_\alpha$, $\alpha = 1, \dots, m$, represent the dissipation mechanism of the material. They are linked to the irreversible relative movement of the material inside the system and describe the deviations from equilibrium (see, for example, VALANIS [1972]). An assumption of the form (6.230) can easily be adjusted to describe a rich variety of *porous, viscous* or *plastic materials*. The actual number of the phenomenological internal variables needs to be chosen for each different material and may vary from one theory and (boundary) condition to another; for example, the size of the specimen under observation. However, the definition of internal variables should be chosen so that they somehow replicate the underlying internal microstructure of the material (even though they are introduced as macroscopic quantities).

In general, the internal variables may take on *scalar, vector* or *tensor* values. Here the internal variables are all denoted by second-order tensors.

In order to particularize the Clausius-Planck inequality of the form (4.154) to the free energy Ψ at hand, we must differentiate (6.230) with respect to time. By means of the chain rule we obtain $\dot{\Psi}(\mathbf{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m) = \partial\Psi/\partial\mathbf{F} : \dot{\mathbf{F}} + \sum_{\alpha=1}^{m} \partial\Psi/\partial\boldsymbol{\xi}_\alpha : \dot{\boldsymbol{\xi}}_\alpha$, and finally, with the expression for the stress power $\mathrm{w}_{\mathrm{int}} = \mathbf{P} : \dot{\mathbf{F}}$ per unit reference volume, we find from (4.154) that

$$\mathcal{D}_{\mathrm{int}} = \left(\mathbf{P} - \frac{\partial\Psi(\mathbf{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m)}{\partial\mathbf{F}}\right) : \dot{\mathbf{F}} - \sum_{\alpha=1}^{m} \frac{\partial\Psi(\mathbf{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m)}{\partial\boldsymbol{\xi}_\alpha} : \dot{\boldsymbol{\xi}}_\alpha \geq 0 \quad . \qquad (6.231)$$

In order to satisfy $\mathcal{D}_{\mathrm{int}} \geq 0$ we apply the Coleman-Noll procedure. For arbitrary choices of the tensor variable $\dot{\mathbf{F}}$, we deduce a physical expression for the first Piola-Kirchhoff stress $\mathbf{P}$ and a remainder inequality governing the non-negativeness of the internal dissipation $\mathcal{D}_{\mathrm{int}}$ (required by the second law of thermodynamics). We have

$$\mathbf{P} = \frac{\partial\Psi(\mathbf{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m)}{\partial\mathbf{F}} \quad , \qquad \mathcal{D}_{\mathrm{int}} = \sum_{\alpha=1}^{m} \boldsymbol{\Xi}_\alpha : \dot{\boldsymbol{\xi}}_\alpha \geq 0 \quad , \qquad (6.232)$$

which must hold at every point of the continuum body and for all times during a thermodynamic process. In $(6.232)_2$ we have defined the internal (second-order) tensor variables $\boldsymbol{\Xi}_\alpha$, $\alpha = 1, \dots, m$, which are related (conjugate) to $\boldsymbol{\xi}_\alpha$ through the **internal constitutive equations**

$$\boldsymbol{\Xi}_\alpha = -\frac{\partial\Psi(\mathbf{F}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m)}{\partial\boldsymbol{\xi}_\alpha} \quad , \qquad \alpha = 1, \dots, m \quad . \qquad (6.233)$$

The additional constitutive equations (6.233) restrict the free energy Ψ and relate the gradient of the free energy Ψ with respect to the internal variables $\boldsymbol{\xi}_\alpha$ to the associated internal variables $\boldsymbol{\Xi}_\alpha$, $\alpha = 1, \dots, m$. Note that the presence of additional variables in the free energy (6.230) justifies additional constitutive equations. A physical motivation of restriction (6.233) may be given by several examples, one of which, stemming from linear viscoelasticity, is presented on p. 286; in particular, see eq. (6.251).

In constitutive equations $(6.232)_1$ and (6.233) the tensor variables $\mathbf{F}$ and $\boldsymbol{\xi}_\alpha$ are associated with the thermodynamic forces $\mathbf{P}$ and $\boldsymbol{\Xi}_\alpha$, respectively. A constitutive model which is characterized by the set of equations (6.231)–(6.233) is called an **internal variable model**.

For the case in which the internal variables $\boldsymbol{\xi}_\alpha$ are not needed to characterize the thermodynamic state of a system, then, the internal dissipation $\mathcal{D}_{\mathrm{int}}$ in $(6.232)_2$ is zero (the material is considered to be perfectly elastic) and all relations from previous sections of this chapter may be applied. In order to describe materials without dissipative character, the set of equations (6.231)–(6.233) simply reduces to (6.3) and $(6.1)_1$.

Evolution equations and thermodynamic equilibrium. The derived set of equations (6.232) and (6.233) must be complemented by a kinetic relation, which describes the evolution of the involved internal variable $\boldsymbol{\xi}_\alpha$ and the associated dissipation mechanism. Consequently, suitable **equations of evolution** (rate equations) are required in order to describe the way an irreversible process evolves.

The only restriction on these equations is thermodynamic admissibility, i.e. the satisfaction of the fundamental inequality $(6.232)_2$ characterizing local entropy production. The missing equations for the evolution of the internal variables $\boldsymbol{\xi}_\alpha$ may be written, for example, as

$$\dot{\boldsymbol{\xi}}_\alpha(t) = \boldsymbol{\mathcal{A}}_\alpha(\mathbf{F}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m) \ , \qquad \alpha = 1, \ldots, m \ . \tag{6.234}$$

The evolution of the system is described by $\boldsymbol{\mathcal{A}}_\alpha$, $\alpha = 1, \ldots, m$, which are tensor-valued functions of $1 + m$ tensor variables.

Every system will tend towards a state of **thermodynamic equilibrium**, which implies that the observable and internal variables reach equilibrium under a prescribed stress or strain; they remain constant at any particle of the system with time. Hence, the behavior at the equilibrium state may be considered as a limiting case and does not depend upon time.

In view of eq. (6.234), the definition of an equilibrium state now requires the additional conditions

$$\boldsymbol{\mathcal{A}}_\alpha(\mathbf{F}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m) = \mathbf{O} \ , \qquad \alpha = 1, \ldots, m \ . \tag{6.235}$$

Hence, $\dot{\boldsymbol{\xi}}_\alpha$ may be seen as the rate of change with which $\boldsymbol{\xi}_\alpha(t)$ tends toward its equilibrium.

In an elastic continuum, every state is an equilibrium state. The internal dissipation $\mathcal{D}_{\text{int}}$ at equilibrium is zero, which characterizes, for instance, a perfectly elastic material, as pointed out in Section 6.1.

6.10 Viscoelastic Materials at Large Strains

Many materials of practical interest appear to behave in a markedly viscoelastic manner over a certain range of stresses and times. The mechanical behavior of, for example, *thermoplastic elastomers* (actually rubber-like materials) or some other types of *natural* and *synthetic polymers* are associated with relaxation and/or creep phenomena, which are important design factors (see, for example, MCCRUM et al. [1997], SPERLING [1992] and WARD and HADLEY [1993]). Problems that involve relaxation and/or creep effects determine irreversible processes and belong to the realm of *non-equilibrium thermodynamics*. For a detailed introduction of the linear and nonlinear theory of viscoelasticity the reader is referred to the book by CHRISTENSEN [1982]. Experimental investigations are documented by, for example, SULLIVAN [1986], LION [1996] and MIEHE and KECK [2000].

In the following we characterize the thermodynamic state of such problems explicitly by means of an internal variable model as introduced in the previous section. A description solely via external variables is also possible, but it emerges that such types of formulation are not preferred for numerical realizations using the finite element method.

Numerous viscoelastic materials can often not be modeled adequately within limits by means of a linear theory. Here we postulate a three-dimensional **viscoelastic model** suitable for finite strains and small perturbations away from the equilibrium state. In contrast to several theories of viscoelasticity (see, for example, the pioneering paper by GREEN and TOBOLSKY [1946]) the present phenomenological approach is *not* restricted to isotropy. For theories that account for finite perturbations away from the equilibrium state, the reader is referred to, for example, KOH and ERINGEN [1963], HAUPT [1993a, b] and REESE and GOVINDJEE [1998a].

Additionally we follow a phenomenological approach that does not consider the underlying molecular structure of the physical object.

Structure of the free energy with internal variables. In particular, we choose an approach which applies the concept of internal variables motivated by SIMO [1987] and followed by, for example, GOVINDJEE and SIMO [1992b, 1993], HOLZAPFEL [1996a], KALISKE and ROTHERT [1997], SIMO and HUGHES [1998, Chapter 10] and HOLZAPFEL and GASSER [2001].

Our study is based on the theory of compressible hyperelasticity within the isothermal regime, as discussed in Section 6.4. We postulate a *decoupled* representation of the Helmholtz free-energy function Ψ. The free energy uses the multiplicative decomposition of the deformation gradient into dilational and volume-preserving parts. Our present approach is in contrast to that which uses the multiplicative decomposition of the deformation gradient into elastic (rate-independent) and permanent (viscous) parts (see SIDOROFF [1974] and LUBLINER [1985] among others).

The change of Ψ within an isothermal viscoelastic process from the reference to the current configuration is given as

$$\Psi(\mathbf{C}, \boldsymbol{\Gamma}_1, \ldots, \boldsymbol{\Gamma}_m) = \Psi^\infty_{\text{vol}}(J) + \Psi^\infty_{\text{iso}}(\overline{\mathbf{C}}) + \sum_{\alpha=1}^{m} \Upsilon_\alpha(\overline{\mathbf{C}}, \boldsymbol{\Gamma}_\alpha) \ , \tag{6.236}$$

valid for some closed time interval $t \in [0, T]$ of interest. We assume that each contribution to the free energy Ψ must satisfy the normalization condition (6.4), i.e.

$$\Psi^\infty_{\text{vol}}(1) = 0 \ , \qquad \Psi^\infty_{\text{iso}}(\mathbf{I}) = 0 \ , \qquad \sum_{\alpha=1}^{m} \Upsilon_\alpha(\mathbf{I}, \mathbf{I}) = 0 \ . \tag{6.237}$$

A material which is characterized by the free energy (6.236) for any point and time we call a **viscoelastic material**.

The first two terms in (6.236), i.e. $\Psi^\infty_{\text{vol}}(J)$ and $\Psi^\infty_{\text{iso}}(\overline{\mathbf{C}})$, are strain-energy functions per unit reference volume and characterize the *equilibrium state* of the solid. They can be identified as the terms presented by eq. (6.85) describing the *volumetric elastic response* and the *isochoric elastic response* as $t \to \infty$, respectively. In fact, the superscript $(\bullet)^\infty$ characterizes functions which represent the hyperelastic behavior of sufficiently slow processes.

The additional third term in (6.236), i.e. the 'dissipative' potential $\sum_{\alpha=1}^{m} \Upsilon_\alpha$, is responsible for the viscoelastic contribution and extends the decoupled strain-energy function (6.85) to the viscoelastic regime. The scalar-valued functions Υ_α, $\alpha = 1,\ldots,m$, represent the so-called **configurational free energy** of the viscoelastic solid and characterize the *non-equilibrium state*, i.e. the behavior of relaxation and creep.

Motivated by experimental data we assume a time-dependent change of the system caused purely by *isochoric* deformations. Hence, the volumetric response remains *fully elastic* and the configurational free energy is a function of the modified right Cauchy-Green tensor $\overline{\mathbf{C}}$ and a set of strain-like internal variables (history variables) not accessible to direct observation, here denoted by $\boldsymbol{\Gamma}_\alpha$, $\alpha = 1,\ldots,m$. Each hidden tensor variable $\boldsymbol{\Gamma}_\alpha$ characterizes the relaxation and/or creep behavior of the material. They are considered to be (inelastic) strains *akin* to the strain measure $\overline{\mathbf{C}}$, with $\boldsymbol{\Gamma}_\alpha = \mathbf{I}$, $\alpha = 1,\ldots,m$, at the (stress-free) reference configuration. The viscoelastic behavior is, in particular, modeled by $\alpha = 1,\ldots,m$ viscoelastic processes with corresponding *relaxation times* (or *retardation times*) $\tau_\alpha \in (0,\infty)$, $\alpha = 1,\ldots,m$.

Note that the set of $1+m$ tensor variables $(\mathbf{C},\boldsymbol{\Gamma}_1,\ldots,\boldsymbol{\Gamma}_m)$ completely characterizes the isothermal viscoelastic state.

Decoupled volumetric-isochoric stress response. In order to obtain the associated constitutive equations describing viscoelastic behavior at finite strains we specify postulate (6.236).

Following arguments analogous to those which led from (6.230) to eqs. (6.232) and (6.233), we obtain physical expressions for the (symmetric) second Piola-Kirchhoff stress $\mathbf{S}$ and the non-negative internal dissipation (local entropy production) $\mathcal{D}_{\text{int}}$ in the forms

$$\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C},\boldsymbol{\Gamma}_1,\ldots,\boldsymbol{\Gamma}_m)}{\partial \mathbf{C}} \ , \qquad \mathcal{D}_{\text{int}} = -\sum_{\alpha=1}^{m} 2\frac{\partial \Upsilon_\alpha(\overline{\mathbf{C}},\boldsymbol{\Gamma}_\alpha)}{\partial \boldsymbol{\Gamma}_\alpha} : \frac{1}{2}\dot{\boldsymbol{\Gamma}}_\alpha \geq 0 \ . \tag{6.238}$$

Starting from the decoupled free energy (6.236), a straightforward computation leads to an additive split of $\mathbf{S}$, as already derived for purely elastic compressible hyperelastic materials (see Section 6.4). We have

$$\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C},\boldsymbol{\Gamma}_1,\ldots,\boldsymbol{\Gamma}_m)}{\partial \mathbf{C}} = \mathbf{S}_{\text{vol}}^{\infty} + \mathbf{S}_{\text{iso}} \ , \tag{6.239}$$

with the definition

$$\mathbf{S}_{\text{iso}} = \mathbf{S}_{\text{iso}}^{\infty} + \sum_{\alpha=1}^{m} \mathbf{Q}_\alpha \tag{6.240}$$

of the isochoric contributions. In eqs. $(6.239)_2$ and (6.240) the quantities

$$\mathbf{S}_{\text{vol}}^{\infty} = J\frac{\mathrm{d}\Psi_{\text{vol}}^{\infty}(J)}{\mathrm{d}J}\mathbf{C}^{-1} \ , \qquad \mathbf{S}_{\text{iso}}^{\infty} = J^{-2/3}\mathbb{P} : 2\frac{\partial \Psi_{\text{iso}}^{\infty}(\overline{\mathbf{C}})}{\partial \overline{\mathbf{C}}} \tag{6.241}$$

determine volumetric and isochoric contributions, which we take to be fully elastic. In relation $(6.241)_2$ the (fourth-order) projection tensor $\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}$ furnishes the deviatoric operator in the *Lagrangian description*. Note that for these elastic contributions we may apply the framework of compressible hyperelasticity and adopt relations (6.88)–(6.91) by using $\Psi_{\text{vol}}^{\infty}$ and $\Psi_{\text{iso}}^{\infty}$ instead of Ψ_{vol} and Ψ_{iso}.

In (6.240) we have introduced additional internal tensor variables $\mathbf{Q}_\alpha$, $\alpha = 1,\ldots,m$, which may be interpreted as **non-equilibrium stresses** in the sense of non-equilibrium thermodynamics. Note that the symbol $\mathbf{Q}$ has already been used and must not be confused with the orthogonal tensor. As can be seen from (6.240) the isochoric second Piola-Kirchhoff stress is decomposed into an **equilibrium part** and a **non-equilibrium part** characterized by the elastic response of the system $\mathbf{S}_{\text{iso}}^{\infty}$ and the viscoelastic response $\sum_{\alpha=1}^{m} \mathbf{Q}_\alpha$, respectively.

By analogy with (6.90) we have defined the relationship

$$\begin{aligned} \mathbf{Q}_\alpha &= 2\frac{\partial \Upsilon_\alpha(\overline{\mathbf{C}},\boldsymbol{\Gamma}_\alpha)}{\partial \mathbf{C}} = J^{-2/3}(\mathbb{I} - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C}) : \overline{\mathbf{Q}}_\alpha \\ &= J^{-2/3}\mathrm{Dev}\overline{\mathbf{Q}}_\alpha = J^{-2/3}\mathbb{P} : \overline{\mathbf{Q}}_\alpha \ , \qquad \alpha = 1,\ldots,m \end{aligned} \tag{6.242}$$

for the second-order tensors $\mathbf{Q}_\alpha$, with the definition

$$\overline{\mathbf{Q}}_\alpha = 2\frac{\partial \Upsilon_\alpha(\overline{\mathbf{C}},\boldsymbol{\Gamma}_\alpha)}{\partial \overline{\mathbf{C}}} \ , \qquad \alpha = 1,\ldots,m \tag{6.243}$$

of the so-called **fictitious non-equilibrium stresses** $\overline{\mathbf{Q}}_\alpha$. As can be seen from (6.242), $\mathbf{Q}_\alpha$ is the deviatoric projection of $\overline{\mathbf{Q}}_\alpha$ times $J^{-2/3}$, with projection tensor $\mathbb{P}$.

Motivated by the (mechanical) equilibrium equations for the linear viscoelastic solid (see the following Example 6.10, in particular, eq. (6.251)), we conclude further that $\mathbf{Q}_\alpha$ are variables related (conjugate) to $\boldsymbol{\Gamma}_\alpha$, $\alpha = 1,\ldots,m$, with the internal constitutive equations

$$\mathbf{Q}_\alpha = -2\frac{\partial \Upsilon_\alpha(\overline{\mathbf{C}},\boldsymbol{\Gamma}_\alpha)}{\partial \boldsymbol{\Gamma}_\alpha} \ , \qquad \alpha = 1,\ldots,m \ . \tag{6.244}$$

These conditions restrict the configurational free energy $\sum_{\alpha=1}^{m} \Upsilon_\alpha$ in view of $(6.242)_1$. Hence, the internal dissipation $\mathcal{D}_{\text{int}}$ in eq. $(6.238)_2$ equivalently reads $\mathcal{D}_{\text{int}} = \sum_{\alpha=1}^{m} \mathbf{Q}_\alpha : \dot{\boldsymbol{\Gamma}}_\alpha/2 \geq 0$.

The condition for *thermodynamic equilibrium* (compare with eq. (6.235)) implies that for $t \to \infty$ the stresses in eq. (6.240) reach equilibrium, which means that $\mathbf{Q}_\alpha = -2\partial\Upsilon_\alpha/\partial\boldsymbol{\Gamma}_\alpha|_{t\to\infty} \equiv \mathbf{O}$, $\alpha = 1,\ldots,m$, and hence, $\mathbf{Q}_\alpha$ characterize the current *'distance from equilibrium'*. Consequently, the dissipation at equilibrium is zero as seen from $(6.238)_2$ and (6.244). In other words, at thermodynamic equilibrium the material responds as perfectly elastic; general finite elasticity is recovered.

EXAMPLE 6.10 By using a simple *spring-and-dashpot model* find a meaningful rheological interpretation for the phenomenological viscoelastic constitutive model presented. Start with a one-dimensional and linear approach. Derive physically motivated evolution equations for the internal variables.

Solution. To begin with linear geometry consider a *rheological model*, as illustrated in Figure 6.6. It is a one-dimensional **generalized Maxwell model** with a free **spring** on one end and an arbitrary number m of **Maxwell elements** arranged in parallel.

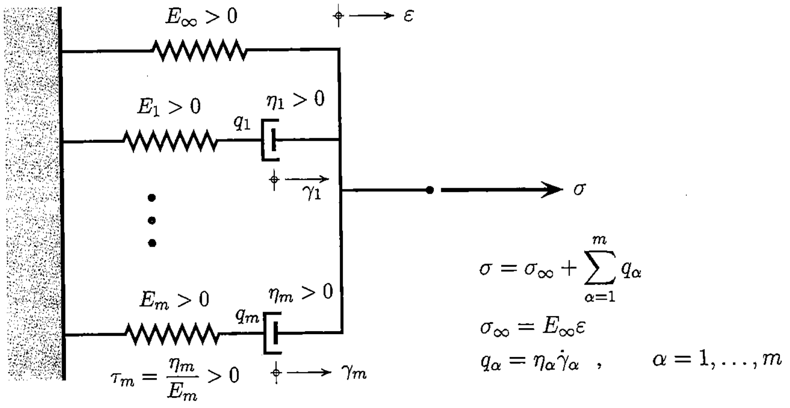

Figure 6.6 Rheological model.

The viscoelastic model in Figure 6.6, which we call temporarily a **mechanical device**, displays both *relaxation* and *creep behavior*. It is a suitable simple model to represent quantitatively the mechanical behavior of real viscoelastic materials. The mechanical device is assumed to have unit area and unit length so that stresses and strains are to be interpreted as forces and extensions (contractions), respectively.

We assume that the solid behavior is modeled by a set of springs responding linear elastically according to **Hooke's law**. The stiffnesses of the free spring on one end and the spring for the so-called α-Maxwell element are determined by **Young's moduli** $E_\infty > 0$ and $E_\alpha > 0$, $\alpha = 1, \ldots, m$, respectively. The flow behavior is modeled by a **Newtonian viscous fluid** responding like a **dashpot**. The viscous fluid of the α-Maxwell element is specified adequately by the material constant $\eta_\alpha > 0$, called the **viscosity**. Based on physics all these parameters are positive.

Let σ be the total stress applied to the *generalized Maxwell model* and ε be an *external variable* which measures the total linear strain due to the stress. By equilibrium, the total stress applied to the device is found to be

$$\sigma = \sigma_\infty + \sum_{\alpha=1}^{m} q_\alpha \tag{6.245}$$

(see Figure 6.6), where the definition of the stress at equilibrium, i.e. $\sigma_\infty = E_\infty \varepsilon$, is to be used. The *internal variables* q_α, $\alpha = 1, \ldots, m$, are the non-equilibrium stresses in the dashpot of the α-Maxwell element characterizing the dissipation mechanism of the viscoelastic model.

The stresses q_α, $\alpha = 1, \ldots, m$, acting on each dashpot are related to the associated internal variables γ_α, which we interpret as (inelastic) strains on each dashpot. In particular, for a Newtonian viscous fluid, q_α are set to be proportional to the current *'distance from equilibrium'*, i.e. the *strain rates* $\dot{\gamma}_\alpha$. We adopt the linear constitutive equation by *Newton*, i.e. $q_\alpha = \eta_\alpha \dot{\gamma}_\alpha$, $\alpha = 1, \ldots, m$. On the other hand, the stress in the spring of the α-Maxwell element is determined by $q_\alpha = E_\alpha(\varepsilon - \gamma_\alpha)$ (see Figure 6.6). Consequently, the stresses (not necessarily at equilibrium) acting on each dashpot is

$$q_\alpha = \eta_\alpha \dot{\gamma}_\alpha = E_\alpha(\varepsilon - \gamma_\alpha) \ , \qquad \alpha = 1, \ldots, m \ . \tag{6.246}$$

Hence, time differentiation of $(6.246)_2$, i.e. $\dot{q}_\alpha = E_\alpha(\dot{\varepsilon} - \dot{\gamma}_\alpha)$, implies by means of $(6.246)_1$ the important evolution equations

$$\dot{q}_\alpha + \frac{q_\alpha}{\tau_\alpha} = \dot{\overline{E_\alpha \varepsilon}} \ , \qquad \alpha = 1, \ldots, m \tag{6.247}$$

for the internal variables within the one-dimensional and linear regime, where the definition of the *relaxation time* (or *retardation time*) $\tau_\alpha = \eta_\alpha / E_\alpha > 0$, $\alpha = 1, \ldots, m$, is to be used.

Since q_α and $\dot{\gamma}_\alpha$ are the stresses and the strain rates acting on each dashpot, we are in the position to define the *rate of work dissipated* within the considered device. By means of $(6.246)_1$, the internal dissipation takes on the form

$$\mathcal{D}_{\text{int}} = \sum_{\alpha=1}^{m} q_\alpha \dot{\gamma}_\alpha = \sum_{\alpha=1}^{m} \eta_\alpha (\dot{\gamma}_\alpha)^2 \geq 0 \ , \tag{6.248}$$

which is always non-negative, since $\eta_\alpha > 0$. It disappears at equilibrium.

We now define the strain energy $\psi(\varepsilon, \gamma_1, \ldots, \gamma_m) = \psi_\infty(\varepsilon) + \sum_{\alpha=1}^{m} \upsilon_\alpha(\varepsilon, \gamma_\alpha)$, with the quadratic forms $\psi_\infty(\varepsilon) = \frac{1}{2} E_\infty \varepsilon^2$ and $\upsilon_\alpha(\varepsilon, \gamma_\alpha) = \frac{1}{2} E_\alpha (\varepsilon - \gamma_\alpha)^2$, and the normalization conditions $\psi_\infty(0) = 0$ and $\upsilon_\alpha(0, 0) = 0$, $\alpha = 1, \ldots, m$. The physically motivated strain energy ψ determines the energy *stored* elastically in the springs of the device, as illustrated in Figure 6.6. The strain energy $\upsilon_\alpha = \upsilon_\alpha(\varepsilon, \gamma_\alpha)$ is responsible for the viscoelastic contribution and is related to the *α-relaxation (retardation) process* with relaxation (retardation) time $\tau_\alpha \in (0,\infty)$.

Differentiation of ψ with respect to the total strain ε gives the total stress σ applied to the device. On comparison with (6.245) we conclude that

$$\frac{\partial \psi(\varepsilon, \gamma_1, \ldots, \gamma_m)}{\partial \varepsilon} = \sigma_\infty(\varepsilon) + \sum_{\alpha=1}^{m} q_\alpha(\varepsilon, \gamma_\alpha) = \sigma \quad , \tag{6.249}$$

where the physical expressions

$$\sigma_\infty = \frac{\mathrm{d}\psi_\infty(\varepsilon)}{\mathrm{d}\varepsilon} = E_\infty \varepsilon \quad , \qquad q_\alpha = \frac{\partial \upsilon_\alpha(\varepsilon, \gamma_\alpha)}{\partial \varepsilon} = E_\alpha(\varepsilon - \gamma_\alpha) \tag{6.250}$$

for the stress at equilibrium $\sigma_\infty(\varepsilon)$ and the non-equilibrium stresses $q_\alpha(\varepsilon, \gamma_\alpha)$, $\alpha = 1, \ldots, m$, are to be used.

Finally, the derivative of ψ with respect to the internal variables γ_α gives with $(6.250)_2$ (or (6.246)) the associated non-equilibrium stresses q_α in the dashpots. The resulting internal constitutive equations read

$$-\frac{\partial \upsilon_\alpha(\varepsilon, \gamma_\alpha)}{\partial \gamma_\alpha} = E_\alpha(\varepsilon - \gamma_\alpha) = q_\alpha \quad , \qquad \alpha = 1, \ldots, m \quad , \tag{6.251}$$

which, when substituted into $(6.248)_1$, gives the internal dissipation $\mathcal{D}_{\text{int}}$ expressed through the strain energy, i.e. $\mathcal{D}_{\text{int}} = -\sum_{\alpha=1}^{m} (\partial \upsilon_\alpha / \partial \gamma_\alpha) \dot{\gamma}_\alpha$.

Note that the general stress relation (6.239) with definition (6.240) may be identified as the three-dimensional and nonlinear version of the linear rheological model (6.249), which, in view of Figure 6.6, decomposes the stresses in *equilibrium* and *non-equilibrium* parts. In addition, the internal constitutive equations (6.244) and definition $(6.242)_1$ may be considered as the three-dimensional generalization of (6.251) and $(6.250)_2$ and also its extension to the finite strain regime. ■

Evolution equations and their solutions. In order to describe the way a viscoelastic process evolves it is necessary to specify complementary *equations of evolution* so that the local entropy production, i.e. the inequality $(6.238)_2$, is satisfied. In particular, we look for a law which governs the internal variables $\mathbf{Q}_\alpha$, $\alpha = 1, \ldots, m$, introduced as *isochoric* non-equilibrium stresses. We require that the evolution equations have a physical basis and provide a good approximation to the observed physical behavior of real materials in the large strain regime. In addition, we require that they are suitable for efficient time integration algorithms that are accessible for use within a finite element procedure.

We motivate the evolution equations for the three-dimensional and nonlinear deformation regime by reference to the relationship (6.247). Having this in mind, an obvious choice of appropriate (linear) evolution equations for each of the internal variables has the form

$$\dot{\mathbf{Q}}_\alpha + \frac{\mathbf{Q}_\alpha}{\tau_\alpha} = \dot{\mathbf{S}}_{\text{iso}\,\alpha} \quad , \qquad \alpha = 1, \ldots, m \quad , \tag{6.252}$$

where (6.252) is valid for some semi-open time interval $t \in (0, T]$, in which the value 0 is not included in the interval. Here we employ a superposed dot to designate the material time derivative as usual. The values $\mathbf{Q}_\alpha|_{t=0} = \mathbf{O}$, $\alpha = 1, \ldots, m$, for the internal variables at initial time $t = 0$ are assumed to be zero, since we agreed to start from a stress-free reference configuration.

In the linear differential equations (6.252), the tensors $\mathbf{S}_{\text{iso}\,\alpha}$ characterize isochoric second Piola-Kirchhoff stresses corresponding to the strain energies $\Psi_{\text{iso}\,\alpha}(\overline{\mathbf{C}})$ of the system (with $\Psi_{\text{iso}\,\alpha}(\mathbf{I}) = 0$ in the reference configuration) responsible for the α-relaxation (retardation) process with relaxation (retardation) time $\tau_\alpha \in (0, \infty)$, $\alpha = 1, \ldots, m$. The definition of the material variables $\mathbf{S}_{\text{iso}\,\alpha}$ is based on structure (6.90) and has the form

$$\mathbf{S}_{\text{iso}\,\alpha} = J^{-2/3}\mathbb{P} : \overline{\mathbf{S}}_\alpha \quad , \qquad \overline{\mathbf{S}}_\alpha = 2\frac{\partial \Psi_{\text{iso}\,\alpha}(\overline{\mathbf{C}})}{\partial \overline{\mathbf{C}}} \quad , \qquad \alpha = 1, \ldots, m \quad , \tag{6.253}$$

where $(6.253)_2$ define the constitutive equations for the fictitious second Piola-Kirchhoff stresses $\overline{\mathbf{S}}_\alpha$. A particular stress $\mathbf{S}_{\text{iso}\,\alpha}$ depends only on the *external variable* $\overline{\mathbf{C}}$, i.e. the modified right Cauchy-Green tensor introduced in $(6.79)_2$.

The linear equations (6.252) are straightforward generalizations of eqs. (6.247), which are physically based. Both tensor quantities $\mathbf{Q}_\alpha$ and $\mathbf{S}_{\text{iso}\,\alpha}$ contribute to the isochoric response of the system. It emerges that the structure (6.252) introduced here is suitable for efficient time integration algorithms as we discuss later in this section.

For the case of non-constant relaxation times τ_α, the convenient concept of **'modified' time** is used in order to obtain linear evolution equations of the type (6.252). Within this concept, τ_α is kept fixed during the process; for more details see, for example, Knauss and Emri [1981] and Govindjee and Simo [1993].

Fairly simple closed form solutions of the linear evolution equations (6.252), which are valid for some semi-open time interval $t \in (0, T]$, are given by the **convolution integrals**

$$\begin{aligned}\mathbf{Q}_\alpha &= \exp(-T/\tau_\alpha)\mathbf{Q}_{\alpha\,0^+} \\ &+ \int_{t=0^+}^{t=T} \exp[-(T-t)/\tau_\alpha]\dot{\mathbf{S}}_{\text{iso}\,\alpha}(t)\mathrm{d}t \quad , \qquad \alpha = 1, \ldots, m\end{aligned} \tag{6.254}$$

(the proof is omitted). The instantaneous response $\mathbf{Q}_{\alpha\,0^+}$ (set of initial conditions) is given by

$$\mathbf{Q}_{\alpha\,0^+} = J_{0^+}^{-2/3}\mathbb{P}_{0^+} : 2\frac{\partial \Upsilon_\alpha(\overline{\mathbf{C}}_{0^+}, \mathbf{\Gamma}_{\alpha\,0^+})}{\partial \overline{\mathbf{C}}_{0^+}} \quad , \qquad \alpha = 1, \ldots, m \tag{6.255}$$

(compare with eqs. (6.242)$_4$, (6.243)), with J_{0+}, $\overline{\mathbf{C}}_{0+}$, $\mathbf{\Gamma}_{\alpha\,0+}$, $\mathbb{P}_{0+} = \mathbb{I} - \frac{1}{3}\mathbf{C}_{0+}^{-1} \otimes \mathbf{C}_{0+}$ defining, respectively, the volume ratio, the modified right Cauchy-Green tensor, the internal variables and the projection tensor at time $t = 0^+$.

The evolution of the internal variables is governed mainly by the strain-energy functions $\Psi_{\text{iso}\,\alpha}$ via relation (6.253). However, if a viscoelastic medium such as a thermoplastic elastomer is composed of identical polymer chains we can motivate the assumption that $\Psi_{\text{iso}\,\alpha}$ is replaceable by the strain-energy function $\Psi_{\text{iso}}^{\infty}$, which is responsible for the isochoric elastic response as $t \rightarrow \infty$. We adopt the expression

$$\Psi_{\text{iso}\,\alpha}(\overline{\mathbf{C}}) = \beta_\alpha^\infty \Psi_{\text{iso}}^\infty(\overline{\mathbf{C}}) \quad , \qquad \alpha = 1, \dots, m \tag{6.256}$$

(which is due to GOVINDJEE and SIMO [1992b]), where $\beta_\alpha^\infty \in [0, \infty)$ are given non-dimensional **strain-energy factors** associated with $\tau_\alpha \in (0, \infty)$, $\alpha = 1, \dots, m$. Consequently, the stresses $\mathbf{S}_{\text{iso}\,\alpha}$, $\alpha = 1, \dots, m$, as introduced in (6.253), may be replaced by means of (6.256) and (6.241)$_2$. We write

$$\mathbf{S}_{\text{iso}\,\alpha} = J^{-2/3}\mathbb{P} : 2\beta_\alpha^\infty \frac{\partial \Psi_{\text{iso}}^\infty(\overline{\mathbf{C}})}{\partial \overline{\mathbf{C}}} = \beta_\alpha^\infty \mathbf{S}_{\text{iso}}^\infty(\overline{\mathbf{C}}) \quad , \qquad \alpha = 1, \dots, m \quad . \tag{6.257}$$

Hence, the material time derivative of the stress tensors, $\dot{\mathbf{S}}_{\text{iso}\,\alpha}$, which govern evolution equations (6.252), are replaced by $\beta_\alpha^\infty \dot{\mathbf{S}}_{\text{iso}}^\infty$.

In summary: the phenomenological viscoelastic model valid over any range of strains is described by constitutive equations (6.239)–(6.243), evolution equations (6.252) with solutions (6.254) and replacements (6.257). Note that with reference to assumption (6.256) the model problem is completely determined by the specification of only two scalar-valued functions, namely $\Psi_{\text{vol}}^\infty(J)$ and $\Psi_{\text{iso}}^\infty(\overline{\mathbf{C}})$, a crucial advantage of the introduced finite strain viscoelastic model.

It is important to emphasize that the described constitutive model fits within the framework of so-called **simple materials** with memory, which are expressed in the general form by $\mathbf{S}(t) = \mathfrak{F}_{s=0^+}^{\infty}[\mathbf{C}(t-s), \mathbf{C}(t)]$ (see, for example, MALVERN [1969, p. 400, eq. (6.7.62)]), with $T = t - s$, where $\mathfrak{F}$ is a (response) functional depending on the history of $\mathbf{C}$ from $T = -\infty$ to $T = t$. For further discussion of this issue the interested reader is referred to MALVERN [1969, Section 6.7, and references therein].

Time integration algorithm. The total second Piola-Kirchhoff stress tensor $\mathbf{S}$ is computed according to relations (6.239)–(6.241) with the *volumetric* and *isochoric* (elastic) response $\mathbf{S}_{\text{vol}}^\infty$ and $\mathbf{S}_{\text{iso}}^\infty$ and the contribution due to the *non-equilibrium stresses* $\sum_{\alpha=1}^{m} \mathbf{Q}_\alpha$ as given by the convolution integrals in the form of (6.254).

For the solutions of the crucial *Cauchy's equations of motion* (see the local forms (4.53) and (4.68)) the stress tensor is required. The main goal of the following is to outline an appropriate **update algorithm** for the total stresses suitable for implementation in a finite element program. The update procedure is realized in the reference configuration, and hence the objectivity requirement based on a Euclidean transformation is trivially satisfied (see Chapter 5). The key of the update algorithm is the *numerical integration* of the convolution integrals (6.254).

In order to obtain the *algorithmic update* of the second Piola-Kirchhoff stress $\mathbf{S}$ we consider a partition (time discretization) $\bigcup_{n=0}^{M}[t_n, t_{n+1}]$ of the closed time interval $t \in [0^+, T]$ of interest, where $0^+ = t_0 < \dots < t_{M+1} = T$. We now concentrate attention on a typical closed **time sub-interval** $[t_n, t_{n+1}]$, with

$$\Delta t = t_{n+1} - t_n \tag{6.258}$$

characterizing the associated **time increment**.

Assume now that up to a certain time t_n the stress $\mathbf{S}_n$ satisfies the equilibrium equation and that the displacement field $\mathbf{u}_n$, the tensor variables

$$\left.\begin{aligned} \mathbf{F}_n &= \mathbf{I} + \operatorname{Grad}\mathbf{u}_n \quad , & J_n &= \det\mathbf{F}_n \quad , \\ \mathbf{C}_n &= \mathbf{F}_n^{\mathrm{T}}\mathbf{F}_n \quad , & \overline{\mathbf{C}}_n &= J_n^{-2/3}\mathbf{C}_n \end{aligned}\right\} \tag{6.259}$$

(see (2.45)$_2$, (2.52), (2.66)$_1$, (6.79)$_2$), and the stress $\mathbf{S}_n$ (determined via the associated constitutive equation) are specified uniquely by the given motion χ_n at time t_n.

Within a *strain-driven* type of numerical procedure, the aim is now to advance the solution to time $t_{n+1} = \Delta t + t_n$ and *update* all relevant quantities. At first we make an *initial guess* for χ_{n+1}, known as a **trial solution**, and update the prescribed loads. Within a classical solution technique, such as *Newton's method*, the new motion χ_{n+1} at time t_{n+1} is corrected iteratively until the balance principles are satisfied within a given tolerance of accuracy. To check equilibrium at time t_{n+1} the tensor variables

$$\left.\begin{aligned} \mathbf{F}_{n+1} &= \mathbf{I} + \operatorname{Grad}\mathbf{u}_{n+1} \quad , & J_{n+1} &= \det\mathbf{F}_{n+1} \quad , \\ \mathbf{C}_{n+1} &= \mathbf{F}_{n+1}^{\mathrm{T}}\mathbf{F}_{n+1} \quad , & \overline{\mathbf{C}}_{n+1} &= J_{n+1}^{-2/3}\mathbf{C}_{n+1} \end{aligned}\right\} \tag{6.260}$$

have to be computed. This process is straightforward since the new motion χ_{n+1} with the updated displacement field $\mathbf{u}_{n+1}$ is considered to be given. The remaining second Piola-Kirchhoff stress at time t_{n+1} is again determined uniquely via the associated constitutive equation. In particular, the so-called **algorithmic stress** at time t_{n+1} reads as

$$\mathbf{S}_{n+1} = (\mathbf{S}_{\text{vol}}^\infty + \mathbf{S}_{\text{iso}}^\infty + \sum_{\alpha=1}^{m} \mathbf{Q}_\alpha)|_{n+1} \quad . \tag{6.261}$$

Since all required strain measures at t_{n+1} are known, the first two stress contributions, i.e. $\mathbf{S}_{\text{vol}\,n+1}^\infty$ and $\mathbf{S}_{\text{iso}\,n+1}^\infty$, are determined via (6.241), which, in the present notation, reads as

$$\mathbf{S}_{\text{vol}\,n+1}^\infty = 2\frac{\partial \Psi_{\text{vol}}^\infty(J_{n+1})}{\partial \mathbf{C}_{n+1}} \quad , \qquad \mathbf{S}_{\text{iso}\,n+1}^\infty = 2\frac{\partial \Psi_{\text{iso}}^\infty(\overline{\mathbf{C}}_{n+1})}{\partial \mathbf{C}_{n+1}} \quad . \tag{6.262}$$

The third term in (6.261), which is the viscoelastic stress contribution $\sum_{\alpha=1}^{m} \mathbf{Q}_{\alpha\, n+1}$ based on (6.254), remains to be evaluated.

The following derivation is related to the approach by SIMO [1987], which bypasses the need for **incremental objectivity** as proposed by HUGHES and WINGET [1980]. Incremental objectivity requires that the algorithmic constitutive equations must be objective (frame-indifferent) during a superimposed (time-dependent) rigid-body motion. Incremental objectivity represents the numerical version of the principle of material frame-indifference, as introduced in Section 5.4.

We now split the convolution integral (6.254) into the form of

$$\int_{0^+}^{t_{n+1}} (\bullet)\mathrm{d}t = \int_{0^+}^{t_n} (\bullet)\mathrm{d}t + \int_{t_n}^{t_{n+1}} (\bullet)\mathrm{d}t \quad . \tag{6.263}$$

Hence, the internal variables $\mathbf{Q}_\alpha$, $\alpha = 1,\ldots,m$, at t_{n+1} are given by

$$\begin{aligned} \mathbf{Q}_{\alpha\, n+1} = {} & \exp(-t_{n+1}/\tau_\alpha)\mathbf{Q}_{\alpha\, 0^+} + \int\limits_{t=0^+}^{t=t_n} \exp[-(t_{n+1}-t)/\tau_\alpha]\dot{\mathbf{S}}_{\mathrm{iso}\,\alpha}(t)\mathrm{d}t \\ & + \int\limits_{t=t_n}^{t=t_{n+1}} \exp[-(t_{n+1}-t)/\tau_\alpha]\dot{\mathbf{S}}_{\mathrm{iso}\,\alpha}(t)\mathrm{d}t \quad , \qquad \alpha = 1,\ldots,m \quad . \end{aligned} \tag{6.264}$$

In order to simplify (6.264) we apply relation (6.258) to all three terms. For the first two terms we use the standard property

$$\exp[-(\Delta t + \beta)/\tau_\alpha] = \exp(-\Delta t/\tau_\alpha)\exp(-\beta/\tau_\alpha) \tag{6.265}$$

for the exponential function, for any constants Δt, τ_α and parameter β which takes on values t_n and $t_n - t$. In addition to (6.265) we use the second-order accurate **mid-point rule** on the third term $\int_{t_n}^{t_{n+1}}(\bullet)\mathrm{d}t$ of eq. (6.264), which means that the time variable t is approximated by $(t_{n+1} + t_n)/2$. We deduce from (6.264) that

$$\begin{aligned} \mathbf{Q}_{\alpha\, n+1} = {} & \exp(-\Delta t/\tau_\alpha)[\exp(-t_n/\tau_\alpha)\mathbf{Q}_{\alpha\, 0^+} + \int\limits_{t=0^+}^{t=t_n} \exp[-(t_n-t)/\tau_\alpha]\dot{\mathbf{S}}_{\mathrm{iso}\,\alpha}(t)\mathrm{d}t] \\ & +\exp(-\Delta t/2\tau_\alpha) \int\limits_{t=t_n}^{t=t_{n+1}} \dot{\mathbf{S}}_{\mathrm{iso}\,\alpha}(t)\mathrm{d}t \quad , \qquad \alpha = 1,\ldots,m \quad . \end{aligned} \tag{6.266}$$

Note that the terms within the brackets are $\mathbf{Q}_\alpha$ at time t_n (compare with eq. (6.254)). By solving the last term in (6.266) and by means of assumption (6.257), we may write

$$\begin{aligned} \mathbf{Q}_{\alpha\, n+1} = {} & \exp(2\xi_\alpha)\mathbf{Q}_{\alpha\, n} \\ & +\exp(\xi_\alpha)\beta_\alpha^\infty(\mathbf{S}_{\mathrm{iso}\, n+1}^\infty - \mathbf{S}_{\mathrm{iso}\, n}^\infty) \quad , \qquad \xi_\alpha = -\frac{\Delta t}{2\tau_\alpha} \quad , \end{aligned} \tag{6.267}$$

for $\alpha = 1,\ldots,m$.

After rearranging eqs. $(6.267)_1$ we arrive finally at a second-order accurate **recurrence update formula** for the internal stresses in a simple format, namely

$$\mathbf{Q}_{\alpha\, n+1} = \beta_\alpha^\infty \exp(\xi_\alpha)\mathbf{S}_{\mathrm{iso}\, n+1}^\infty + \boldsymbol{\mathcal{H}}_{\alpha\, n} \quad , \qquad \alpha = 1,\ldots,m \quad , \tag{6.268}$$

$$\boldsymbol{\mathcal{H}}_{\alpha\, n} = \exp(\xi_\alpha)[\exp(\xi_\alpha)\mathbf{Q}_{\alpha\, n} - \beta_\alpha^\infty \mathbf{S}_{\mathrm{iso}\, n}^\infty] \quad , \tag{6.269}$$

and with definition $(6.267)_2$ of the dimensionless parameters ξ_α. In recurrence relation (6.268) we have introduced the *(algorithmic) history term* $\boldsymbol{\mathcal{H}}_{\alpha\, n}$, $\alpha = 1,\ldots,m$. This term is determined by the internal history variables $\mathbf{Q}_{\alpha\, n}$ and $\mathbf{S}_{\mathrm{iso}\, n}^\infty$, which are known from the previous step serving as an 'initial' data base.

The recurrence update formula of the type (6.268) was proposed by TAYLOR et al. [1970, and references therein]. Instead of the time integration algorithm outlined above we may use other algorithmic updates for the total stresses. For a slightly different structure see, for example, SIMO [1987] and co-workers, and HOLZAPFEL [1996a]. For an application of the described viscoelastic model to fiber-reinforced composites at finite strains see the recent paper by HOLZAPFEL and GASSER [2001].

Elasticity tensor in material description. The importance of *consistent linearized tangent moduli* in the solution of nonlinear problems by incremental/iterative techniques of Newton's type was emphasized in Section 6.6.

We now determine the consistent linearization of the constitutive model presented above, noting the algorithmic stress (6.261) with relations (6.262), (6.268), (6.269). In view of definition (6.154) and the decoupled stress relation (6.261), the associated *algorithmic elasticity tensor* in the material description at time t_{n+1} may be written in the form

$$\mathbb{C}_{n+1} = (\mathbb{C}_{\mathrm{vol}}^\infty + \mathbb{C}_{\mathrm{iso}}^\infty + \sum_{\alpha=1}^{m} \mathbb{C}_{\mathrm{vis}}^\alpha)|_{n+1} \quad , \tag{6.270}$$

where the first two contributions to $\mathbb{C}_{n+1}$ are given by (6.163), which, in the present notation, reads as

$$\mathbb{C}_{\mathrm{vol}\, n+1}^\infty = 2\frac{\partial \mathbf{S}_{\mathrm{vol}\, n+1}^\infty}{\partial \mathbf{C}_{n+1}} \quad , \qquad \mathbb{C}_{\mathrm{iso}\, n+1}^\infty = 2\frac{\partial \mathbf{S}_{\mathrm{iso}\, n+1}^\infty}{\partial \mathbf{C}_{n+1}} \quad . \tag{6.271}$$

Explicit expressions for (6.271) are found in (6.166) and (6.168).

The third (viscoelastic) contribution to $\mathbb{C}_{n+1}$, i.e. $\sum_{\alpha=1}^{m} \mathbb{C}_{\mathrm{vis}}^\alpha$ at t_{n+1}, is derived using the expression for the internal stresses (6.268). Note that the derivative of the (algorithmic) history term $\boldsymbol{\mathcal{H}}_{\alpha\, n}$, $\alpha = 1,\ldots,m$, (which defines quantities at t_n) with respect to $\mathbf{C}_{n+1}$ is zero. Hence, the third contribution to the elasticity tensor is

$$\begin{aligned} \mathbb{C}_{\mathrm{vis}\, n+1}^\alpha & = 2\frac{\partial \mathbf{Q}_{\alpha\, n+1}}{\partial \mathbf{C}_{n+1}} \\ & = \delta_\alpha \mathbb{C}_{\mathrm{iso}\, n+1}^\infty \quad , \qquad \delta_\alpha = \beta_\alpha^\infty \exp(\xi_\alpha) \quad , \qquad \alpha = 1,\ldots,m \quad . \end{aligned} \tag{6.272}$$

Remarkably, the viscoelastic contribution to the algorithmic elasticity tensor may be expressed as the **viscoelastic factors** δ_α, $\alpha = 1,\ldots,m$, governing the time-dependent part, and by $\mathbb{C}^{\infty}_{\mathrm{iso}\,n+1}$, which is associated with the isochoric elastic response as $t \to \infty$.

Hence, relation (6.270) may be rewritten by means of (6.272) as

$$\mathbb{C}_{n+1} = \mathbb{C}^{\infty}_{\mathrm{vol}\,n+1} + (1+\delta)\mathbb{C}^{\infty}_{\mathrm{iso}\,n+1} \ , \qquad \delta = \sum_{\alpha=1}^{m} \delta_\alpha \ . \tag{6.273}$$

If $\delta = \sum_{\alpha=1}^{m} \beta_\alpha^\infty \exp(\xi_\alpha)$ tends to zero, the algorithmic elasticity tensor reduces to (6.271) and compressible finite hyperelasticity is recovered.

The implementation of the viscoelastic model described above into a finite element program is based on the derived algorithmic stress (6.261) and the associated algorithmic elasticity tensor (6.273). The algorithmic update is carried out at each Gauss point of a finite element. The presented formulation only needs a particularization of the two strain-energy functions $\Psi^{\infty}_{\mathrm{vol}}(J)$ and $\Psi^{\infty}_{\mathrm{iso}}(\overline{\mathbf{C}})$. If we use a strain energy which is expressed in principal stretches, then the algorithmic update is actually performed on the principal values.

For an efficient computational application of the iterative process, see the works by SIMO [1987], GOVINDJEE and SIMO [1992b], HOLZAPFEL [1996a] and HOLZAPFEL and GASSER [2001].

EXERCISES

1. Using the relations

$$\dot{\mathbf{Q}}_\alpha + \frac{\mathbf{Q}_\alpha}{\tau_\alpha} = \exp(-t/\tau_\alpha)\frac{\mathrm{D}}{\mathrm{D}t}[\exp(t/\tau_\alpha)\mathbf{Q}_\alpha] \ , \qquad \alpha = 1,\ldots,m$$

   and integration over time interval $t \in (0,T]$ obtain the convolution integrals (6.254), which are the closed form solutions of the linear differential equations (6.252).

2. *Relaxation test.* Assume a thin sheet of incompressible material in the undeformed configuration which may undergo viscoelastic deformations in the large strain domain. The sheet is stretched to $\lambda_1 = \lambda$ ($\lambda_2 = \lambda_3 = \lambda^{-1/2}$) in one direction (simple tension) and is *fixed* subsequently at this elongation. The deformation is assumed to be homogeneous.

   The viscoelastic behavior of the material is based on a phenomenological Maxwell-type model with a free spring at one end and one Maxwell element arranged in parallel ($m = 1$). The underlying strain energy is due to Ogden, as introduced in Section 6.5. We assume the six-parameter model and use the typical values according to (6.121).

   (a) For a certain closed time interval $t \in [0,T]$ compute an explicit expression for the evolution of the remaining non-vanishing internal stress $Q = Q(\lambda,t)$ and the Cauchy stress $\sigma = \sigma(\lambda,t)$ (the stresses in the transverse directions are zero). Discuss the internal dissipation $\mathcal{D}_{\mathrm{int}}$ along the time interval and determine thermodynamic equilibrium.

   (b) For $t \in [0,T]$ plot the stress decay (Q,σ) at a stretch ratio $\lambda = 5$, with the strain-energy factor $\beta_1^\infty = 1$ and the relaxation time $\tau = 10s$. Give a physical interpretation of the relaxation time τ.

3. *Creep test.* Consider the specimen and the viscoelastic constitutive model as described in the previous exercise, but now let a Cauchy stress σ be applied in one direction up to a certain value σ_0 and then be held constant. The stress causes a homogeneous deformation characterized by the stretch ratios $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$ (simple tension). The underlying strain energy is of neo-Hookean type.

   (a) Derive the stress relation in the form of a nonlinear differential equation of first-order.

   (b) Solve the differential equation by means of Newton's method and use the derived recurrence update formula, or alternatively solve it with the *Runge-Kutta* method. For this purpose write a computer program or simply use some commercially available mathematical software-package. For a certain time domain $t \in [0,T]$ plot the stress evolution (Q,σ), and the stretch evolution λ (for a fixed σ_0), $\beta_1^\infty = 1$ and $\tau = 10s$.

6.11 Hyperelastic Materials with Isotropic Damage

Continuum damage theories are either **micromechanical** or **phenomenological** in nature. Microscopic approaches are certainly the best, but necessitate a strong physical background, and models are mathematically complex and often difficult to identify.

The aim of this section is to formulate a three-dimensional and rate-independent isotropic damage model for the large strain domain describing the Mullins effect. We employ **continuum damage mechanics** (often abbreviated as CDM) and follow a purely phenomenological approach, which leads to damage models describing the *macroscopic* constitutive behavior of materials containing distributed microcracks. The material model introduced is suitable for numerical procedures.

For additional information on the subject of constitutive models within the context of continuum damage mechanics, see the monographs by, for example, KACHANOV

[1986], Lemaitre [1996] and Krajcinovic [1996, Chapter 4] and the review article by de Souza Neto et al. [1998], which also describes techniques for the numerical simulation of (isotropic) internal damage in finitely deformed solids.

Mullins effect. Many rubber-like materials consist of a cross-linked elastomer substance with a distribution of small carbon particles as fillers (see the account of filled elastomers in Mark and Erman [1988, Chapter 20]). A piece of filler-loaded rubber subjected to a series of loadings typically displays pronounced (strain-induced) **stress softening** associated with damage, known as the **Mullins effect** (this effect was pointed out in the early pioneering work of Mullins [1947]; for a detailed description see, for example, Johnson and Beatty [1993a, b]). Nearly all practical engineering rubbers contain carbon particles as fillers and exhibit a certain degree of Mullins effect, which is regarded as essentially being caused by the fillers.

In order to explain the main features of this stress softening phenomenon we consider a strain-controlled cyclic tension test of a piece of filled rubber with two different strain amplitudes and neglect viscoelastic effects (slow strain rates). The cyclic loading and unloading process starts from its unstressed (initial) virgin state 0 and follows a path A, which we call the **primary loading path** (see Figure 6.7). After subsequent unloading initiated from any point 1 on the primary loading path the piece of rubber follows path B and completely returns to the unstressed state 0 (for real rubber this will hardly ever occur). Note that after the test piece has been subjected to a load up to the point 1 the initial properties of the virgin material containing fillers are changed permanently (see Mullins [1947]). The first loading and unloading cycle involves dissipation which is represented by the area between the curves A and B (hysteresis behavior). The area is a measure for non-recoverable energy.

When the material is re-loaded the stress-strain behavior follows the path B again and if a strain beyond the point 1 (at which unloading began) is applied, the path D is activated. It is a continuation of the primary loading path A. For additional unloading which begins at any point 2 of the primary loading path the rubber is retraced back to the unstressed state 0 along the path C. Note that the shape of the second stress-strain cycle differs significantly from the first one. Path C retraces the piece of rubber back to 2 on re-loading and the primary loading path is activated again.

In summary: the stiffness of rubbers containing reinforcing fillers such as carbon black decreases as a result of extensional loading and unloading. The material properties associated with the initial extension of rubber compounds may differ significantly from those associated with successive deformation. With reference to Figure 6.7 we recognize that for a given strain level the stress required on the first unloading and re-loading path B (and also on the second unloading and re-loading path C) is *less* than that on the primary loading path A, and is the essential feature of the Mullins effect.

There are a few theories in the literature aimed at explaining the microscopic damage mechanism. One class of theory is based on the idea that the internal damage is caused by debonding of rubber molecular chains attached between the filler particles (see Bueche [1960, 1961]). The higher the (macroscopic) deformation of the rubber the higher the strain-induced damage. Based on statistical arguments the stress softening effect may be predicted for a fixed level of strain due to previous higher strains in the material.

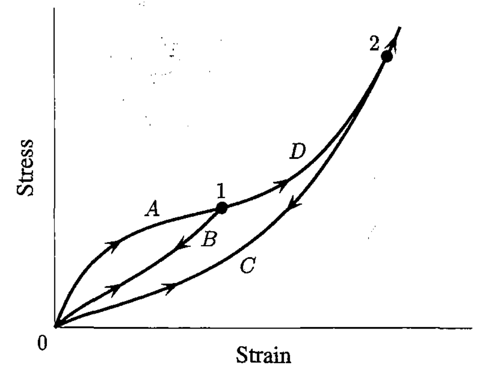

Figure 6.7 Cyclic tension test displaying Mullins effect.

The other class of theory for explaining the microscopic damage mechanism goes back to Mullins and Tobin [1957] and Mullins [1969]. They proposed that initially the filler-loaded rubber exists in a so-called *hard phase* which degrades into a *soft phase* with increasing strain. The transition between the two phases is characterized by a damage parameter which is associated with a strain-amplification function. Johnson and Beatty [1993a, b] have adopted the two-phase approach, which shows good agreement with experimental data in simple tension.

The Mullins effect was observed experimentally in uniaxial cyclic extension tests performed by, for example, Mullins [1947], Mullins and Tobin [1957] (who provided experimental data for loading), Bueche [1961], Mullins and Tobin [1965] (experimental data obtained from the elastic and swelling behavior of filler-loaded rubbers), Harwood et al. [1965], Harwood and Payne [1966a, b], Stern [1967] and Mullins [1969].

It is important to emphasize that in practice several other (inelastic) effects arise under extensional loading and unloading and we address these in brief. From the experimental observation it is known that, besides the Mullins effect (which is an idealized phenomenon), the shape of the stress-strain curves is essentially rate and temperature dependent. In addition, the shape of a piece of carbon-black filled rubber after unloading differs significantly from its virgin shape. This interesting effect caused by reinforcement leads to *residual strains*, which are responsible for the change of shape (in the rubber industry also called *permanent set*). For suitable constitutive models incorporating residual strains see HOLZAPFEL et al. [1999] and OGDEN and ROXBURGH [1999b]. Additional inelastic effects such as relaxation and/or creep depend strongly on the content of solid fillers (see, for example, SO and CHEN [1991, and references therein]).

An experimental investigation of the large strain time-dependent behavior of carbon-black filled Chloroprene rubber subjected to different loading conditions and a constitutive model based on micromechanical considerations is presented by BERGSTRÖM and BOYCE [1998]. A series of uniaxial strain-controlled cyclic experiments on cylindrical specimens of carbon-black filled rubbers between 100% in tension and 30% in compression, including inelastic effects at room temperature, is presented by LION [1996] and MIEHE and KECK [2000]. Phenomenological material models for filled rubbery polymers are also proposed therein. A cyclic tension process with three different strain amplitudes and 12 loading cycles were performed on a virgin specimen. The process shows clearly that the magnitude of the resulting stress softening depends on the number of loading cycles and the strain amplitude. The experimental studies also consider relaxation periods in tension and compression at constant strain. The work of MIEHE and KECK [2000] also provides experimental stress-deformation curves of pre-damaged specimens under monotonous tension and compression and cyclic tension/compression. For an experimental investigation of a filler-loaded tread compound at different temperature levels see LION [1997a].

Damage model in coupled material description. Now we are concerned with the continuum formulation of the (ideal) Mullins effect and neglect rate and temperature dependency as well as residual strains. Additionally, in the phenomenological model we do not consider the presence of carbon-black fillers. We choose a *Lagrangian* description and express the relevant equations in terms of the right Cauchy-Green tensor.

Consider an isothermal elastic process, and postulate a Helmholtz free-energy function Ψ in the *coupled* form

$$\Psi = \Psi(\mathbf{C},\zeta) = (1-\zeta)\Psi_0(\mathbf{C}) \ , \tag{6.274}$$

where Ψ_0 is the **effective strain-energy function** of the hypothetical *undamaged* material, with the normalization condition $\Psi_0(\mathbf{I}) = 0$ and the restriction $\Psi_0(\mathbf{C}) \geq 0$. The factor $(1-\zeta)$ is known as the **reduction factor**, first proposed by KACHANOV [1958] who modeled the creep rupture of metals as a uniaxial problem (*Kachanov* actually introduced $(1-\zeta)$ as the 'integrity' parameter). Here the *internal variable* $\zeta \in [0,1]$ is a *scalar*, referred to as the **damage variable.** The damage variable describes an (ideally) isotropic damage process and is related to the ultimate failure of the material. Note that the strain energy Ψ_0 is assumed to be objective as usual.

Of course this simple type of damage model has limited use in practice, but describes both the *dissipation mechanisms* and the *irreversible rearrangements of the structure*. To refine the model more general (tensorial) forms are needed, especially to describe anisotropic damage.

In order to obtain the stress relation we differentiate first (6.274) with respect to time. Using the chain rule we find that

$$\dot{\Psi} = (1-\zeta)\frac{\partial\Psi_0(\mathbf{C})}{\partial\mathbf{C}} : \dot{\mathbf{C}} - \Psi_0(\mathbf{C})\dot{\zeta} \ , \tag{6.275}$$

with the rate of change of the right Cauchy-Green tensor and the damage variable, i.e. $\dot{\mathbf{C}}$, (2.168), and $\dot{\zeta}$, respectively.

As a particularization of the Clausius-Planck inequality, as given in (4.154), we obtain, by means of (6.275),

$$\mathcal{D}_{\text{int}} = \left(\mathbf{S} - (1-\zeta)2\frac{\partial\Psi_0(\mathbf{C})}{\partial\mathbf{C}}\right) : \frac{\dot{\mathbf{C}}}{2} + \Psi_0(\mathbf{C})\dot{\zeta} \geq 0 \ , \tag{6.276}$$

and therefore the second Piola-Kirchoff stress tensor $\mathbf{S}$ and the *non-negative* internal dissipation $\mathcal{D}_{\text{int}}$ are

$$\mathbf{S} = (1-\zeta)\mathbf{S}_0 \qquad \text{with} \qquad \mathbf{S}_0 = 2\frac{\partial\Psi_0(\mathbf{C})}{\partial\mathbf{C}} \ , \tag{6.277}$$

$$\mathcal{D}_{\text{int}} = f\dot{\zeta} \geq 0 \qquad \text{with} \qquad f = \Psi_0(\mathbf{C}) \geq 0 \ . \tag{6.278}$$

In constitutive equation (6.277) the quantity $\mathbf{S}_0$ denotes the **effective second Piola-Kirchhoff stress tensor**. The dissipation inequality (6.278) clearly shows that damage is a dissipative process, the quantity f therein denotes the **thermodynamic force** which governs the damage evolution. In continuum damage mechanics the thermodynamic quantity f has the meaning of the effective strain energy Ψ_0 released per unit reference volume.

The thermodynamic force f is related (conjugate) to the internal variable ζ according to

$$f = \Psi_0(\mathbf{C}) = -\frac{\partial\Psi}{\partial\zeta} \ , \tag{6.279}$$

see relation (6.274). Therefore, instead of controlling the damage process by the internal variable ζ we can equivalently use its conjugate quantity, i.e. the effective strain energy Ψ_0.

The evolution of f is given by

$$\dot{f} = 2\frac{\partial\Psi_0(\mathbf{C})}{\partial\mathbf{C}} : \frac{\dot{\mathbf{C}}}{2} = \mathbf{S}_0 : \frac{\dot{\mathbf{C}}}{2} \ , \tag{6.280}$$

recognizing that $\dot{f}$ characterizes the **effective stress power** per unit reference volume according to (4.113).

As noted above we assume that the total damage accumulation is based on *Mullins* type (stress) softening of the material. For a strain-controlled cyclic loading process with a fixed strain amplitude this type of phenomenological damage occurs only within the first cycle. We adopt the smooth function $\zeta = \zeta(\alpha)$ as the damage variable, with conditions $\zeta(0) = 0$ and $\zeta(\infty) \in [0,1]$. The phenomenological variable α describes the **discontinuous damage**. A constitutive particularization of the damage variable ζ may, according to MIEHE [1995a], be given by

$$\zeta = \zeta(\alpha) = \zeta_\infty [1 - \exp(-\alpha/\iota)] \quad , \tag{6.281}$$

where ζ_∞ describes the dimensionless **maximum** (possible) **damage** and ι is referred to as the **damage saturation parameter**.

We now aim to determine the *discontinuous damage variable* α over the past history up to the current time, i.e. the history time interval $[0,t]$. To control a discontinuous damage process we use the evolution of the effective strain-energy function Ψ_0. Within the closed time interval $[0,t]$ we take the phenomenological variable α to be related to the *maximum value* of Ψ_0 and write

$$\alpha(t) = \max_{s\in[0,t]} \Psi_0(s) \quad , \tag{6.282}$$

where $s \in [0,t]$ denotes the history variable. Thus, it emerges that α is the **maximum thermodynamic force** with the same dimension as the effective strain energy per unit reference volume. Definition (6.282) was employed by, for example, DE SOUZA NETO et al. [1994] and by MIEHE [1995a], while the rate-independent damage model of SIMO [1987] invoked the *principle of strain-equivalence* (see the chapter on damage mechanics in the book by LEMAITRE and CHABOCHE [1990, Chapter 7]). Other strain-based models for the finite strain domain are due to GOVINDJEE and SIMO [1991, 1992a, b], JOHNSON and BEATTY [1993a, b] among others.

Relation (6.282) generalizes the one-dimensional damage model (for small strains) proposed by GURTIN and FRANCIS [1981b]. It uses the hypothesis that the current state of damage is characterized by the maximum axial strain attained in the history of deformation. Here we do not recall the many important works in the area of small strain damage mechanics.

A straightforward refinement of the isotropic damage model may be obtained by including damage effects governed by *continuous damage accumulation*. In particular, the work of MIEHE [1995a], which is exclusively presented in an *Eulerian* setting, describes continuous damage accumulation. It is based on the arc-length of the effective strain-energy function.

A fairly simple and efficient energy-based damage model to describe the main features of the Mullins effect in filled rubber was proposed by OGDEN and ROXBURGH [1999a]. The material model is composed of a (classical) strain-energy function describing the primary loading path from the unstressed virgin state and an additive damage function responsible for the unloading path which is initiated from any point on the primary loading path. The formulation is based on the **concept of pseudo-elasticity** in which the material is treated as one elastic material in loading and another elastic material in unloading. This idea was used by, for example, FUNG et al. [1979] within the context of modelling arterial walls. It has the significant advantage of convenient and simple description of the stress-strain relationships in cyclic loading and their numerical (finite element) realization.

Damage criterion and damage evolution. We now define a **damage criterion** in the *strain space* at any time of the loading process in the form

$$\phi(\mathbf{C}, \alpha) = f(\mathbf{C}) - \alpha \le 0 \quad , \tag{6.283}$$

with the **damage function** ϕ. For $\phi < 0$ no evolution of damage occurs. The second possible situation is $\phi = 0$ which characterizes the so-called **damage surface** with normal

$$\mathbf{N} = \frac{\partial \phi(\mathbf{C}, \alpha)}{\partial \mathbf{C}} = \frac{\partial f(\mathbf{C})}{\partial \mathbf{C}} \quad . \tag{6.284}$$

For a fixed α, the damage surface delimits the strain space in which the behavior of the material is considered to be fully elastic (no damage accumulation occurs). A representation of the damage surface with the associated normal in the principal strain space is shown in Figure 6.8.

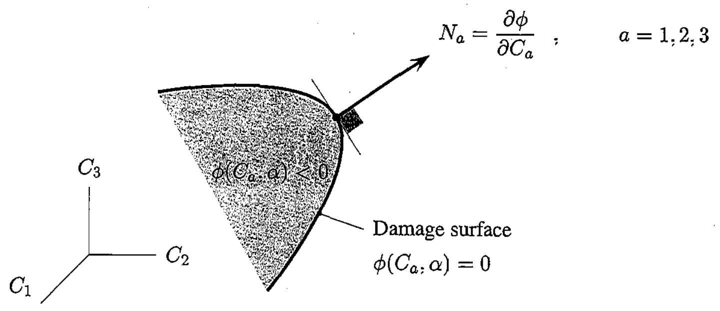

Figure 6.8 Illustration of the damage criterion in the principal strain space.

Hence, a double contraction of the two tensors $\mathbf{N}$ and $\dot{\mathbf{C}}$ gives, using $(6.284)_2$, the scalar $\mathbf{N} : \dot{\mathbf{C}} = \partial f(\mathbf{C})/\partial \mathbf{C} : \dot{\mathbf{C}} = \dot{f}(\mathbf{C})$. Borrowing the terminology from *strain space* *plasticity* (see NAGHDI and TRAPP [1975] and SIMO and HUGHES [1998, p. 84]), at $\phi = 0$ we must distinguish between

$$\phi = 0 \qquad \text{and} \qquad \begin{cases} \dot{f} < 0 \ , \\ \dot{f} = 0 \ , \\ \dot{f} > 0 \ , \end{cases} \tag{6.285}$$

describing *unloading*, *neutral loading* and *loading*, respectively.

Finally, the evolution of the maximum thermodynamic force α, i.e. (6.282), is given, with $(6.280)_2$, by

$$\dot{\alpha} = \begin{cases} \dot{f} = \mathbf{S}_0 : \dfrac{\dot{\mathbf{C}}}{2} & \text{if} \quad \phi = 0 \quad \text{and} \quad \dot{f} > 0 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{6.286}$$

Advancement of damage only occurs for the case of loading, and the initial condition for α is zero. The evolution equation (6.286), which clearly shows the discontinuous property of this damage model, corresponds to those given in GURTIN and FRANCIS [1981b], SIMO [1987] and MIEHE [1995a].

Coupled representation of the elasticity tensor. According to relations (6.153) and (6.154) we derive the symmetric fourth-order elasticity tensor $\mathbb{C}$ in the *material description*. Starting with the constitutive equation (6.277), we have

$$\dot{\mathbf{S}} = (1-\zeta) 2 \frac{\partial \mathbf{S}_0}{\partial \mathbf{C}} : \frac{1}{2}\dot{\mathbf{C}} - \mathbf{S}_0 \dot{\zeta} \ , \tag{6.287}$$

where the rate of damage $\dot{\zeta}$ takes on the form

$$\dot{\zeta} = \zeta'(\alpha)\dot{\alpha} \qquad \text{with} \qquad \zeta'(\alpha) = \frac{\partial \zeta}{\partial \alpha} \ . \tag{6.288}$$

Substituting (6.288) into (6.287), we obtain, using (6.286), the evolution of the stress tensor in the form

$$\dot{\mathbf{S}} = \begin{cases} [(1-\zeta)\mathbb{C}_0 - \zeta'(\alpha)\mathbf{S}_0 \otimes \mathbf{S}_0] : \dfrac{\dot{\mathbf{C}}}{2} & \text{if} \quad \phi = 0 \quad \text{and} \quad \dot{f} > 0 \ , \\ (1-\zeta)\mathbb{C}_0 : \dfrac{\dot{\mathbf{C}}}{2} & \text{otherwise} \ , \end{cases} \tag{6.289}$$

with the **effective elasticity tensor** $\mathbb{C}_0$ in the *material description*. For an *undamaged material*, $\mathbb{C}_0$ is defined to be

$$\mathbb{C}_0 = 2 \frac{\partial \mathbf{S}_0(\mathbf{C})}{\partial \mathbf{C}} \ . \tag{6.290}$$

By comparing $(6.289)_1$ with (6.153) we find that the terms within the bracket characterize the elasticity tensor $\mathbb{C}$ in the material description. The second term in the rate equation $(6.289)_1$ governs the damage that causes stress softening of the material.

Damage model in decoupled material description. To complete our considerations of finite strain elasticity with isotropic damage in the sense of decoupled volumetric-isochoric response (introduced due to a multiplicative split of the deformation gradient (6.79)), we postulate finally a *decoupled* representation of $\Psi = \Psi(\mathbf{C}, \zeta)$ in accordance with (6.85) in the form

$$\Psi(\mathbf{C}, \zeta) = \Psi_{\mathrm{vol}}(J) + (1-\zeta)\Psi_{0\,\mathrm{iso}}(\overline{\mathbf{C}}) \ . \tag{6.291}$$

Here, Ψ_{vol} is a strictly convex function (with the minimum at $J = 1$) which describes the *volumetric* elastic response. The second function $\Psi_{0\,\mathrm{iso}}$ denotes the **isochoric effective strain energy** of the *undamaged* material, which describes the *isochoric* elastic response. Hence, the damage phenomenon is assumed to affect only the isochoric part of the deformation, as proposed by, for example, SIMO [1987]. We require that $\Psi_{\mathrm{vol}}(J) = 0$ and $\Psi_{0\,\mathrm{iso}}(\overline{\mathbf{C}}) = 0$ hold *if and only if* $J = 1$ and $\overline{\mathbf{C}} = \mathbf{I}$, respectively.

Consider the structure (6.291), the purely volumetric contribution to the stress and the elasticity tensor are presented by eqs. (6.89) and (6.166), respectively. The *isochoric* contribution to the stress is, by analogy with (6.277), given by

$$\mathbf{S}_{\mathrm{iso}} = (1-\zeta)\mathbf{S}_{0\,\mathrm{iso}} \qquad \text{with} \qquad \mathbf{S}_{0\,\mathrm{iso}} = 2\frac{\partial \Psi_{0\,\mathrm{iso}}(\overline{\mathbf{C}})}{\partial \mathbf{C}} \ . \tag{6.292}$$

The isochoric contribution to the elasticity tensor includes damage and is, by analogy with (6.289), given by

$$\dot{\mathbf{S}}_{\mathrm{iso}} = \begin{cases} [(1-\zeta)\mathbb{C}_{0\,\mathrm{iso}} - \zeta'(\alpha)\mathbf{S}_{0\,\mathrm{iso}} \otimes \mathbf{S}_{0\,\mathrm{iso}}] : \dfrac{\dot{\mathbf{C}}}{2} & \text{if} \quad \phi = 0 \quad \text{and} \quad \dot{f} > 0 \ , \\ (1-\zeta)\mathbb{C}_{0\,\mathrm{iso}} : \dfrac{\dot{\mathbf{C}}}{2} & \text{otherwise} \ , \end{cases} \tag{6.293}$$

with the isochoric part $\mathbb{C}_{0\,\mathrm{iso}}$ of the *effective elasticity tensor* in the material description. For an *undamaged material*, $\mathbb{C}_{0\,\mathrm{iso}}$ is defined to be

$$\mathbb{C}_{0\,\mathrm{iso}} = 2\frac{\partial \mathbf{S}_{0\,\mathrm{iso}}}{\partial \mathbf{C}} \ . \tag{6.294}$$

Explicit forms of $(6.292)_2$ and (6.294) are given by (6.90) and (6.168), respectively.

By analogy with eq. $(6.278)_2$ the thermodynamic force f has here the meaning of the isochoric effective strain energy $\Psi_{0\,\mathrm{iso}}(\overline{\mathbf{C}})$ of the undamaged material. Within the decoupled framework of volumetric-isochoric response, eqs. $(6.278)_2$ and $(6.280)_2$ take on the forms

$$f = \Psi_{0\,\mathrm{iso}}(\overline{\mathbf{C}}) \geq 0 \qquad \text{and} \qquad \dot{f} = \mathbf{S}_{0\,\mathrm{iso}} : \frac{\dot{\mathbf{C}}}{2} \ . \tag{6.295}$$

The applicability of the constitutive damage model thus described is limited to sufficiently slow processes (viscous effects are not considered). However, the damage model may easily be combined with the viscoelastic model as proposed in the last section. A suitable *decoupled* free energy for characterizing finite-strain viscoelastic damage mechanisms might be given, with reference to (6.236) and (6.291), as $\Psi(\mathbf{C},\zeta,\boldsymbol{\Gamma}_1,\ldots,\boldsymbol{\Gamma}_m) = \Psi_{\text{vol}}^{\infty}(J)+(1-\zeta)[\Psi_{0\,\text{iso}}^{\infty}(\overline{\mathbf{C}})+\sum_{\alpha=1}^{m}\Upsilon_{\alpha\,0}(\overline{\mathbf{C}},\boldsymbol{\Gamma}_\alpha)]$. Here, $\Psi_{0\,\text{iso}}^{\infty}(\overline{\mathbf{C}})$ and $\sum_{\alpha=1}^{m}\Upsilon_{\alpha\,0}(\overline{\mathbf{C}},\boldsymbol{\Gamma}_\alpha)$ denote, respectively, the strain energy and the configurational free energy (per unit reference volume) for the hyperelastic *undamaged* material. Both functions are associated with the isochoric response.

EXERCISES

1. *Pure shear with isotropic damage.* Consider a thin sheet of (incompressible) hyperelastic material which is subjected to a homogeneous pure shear deformation with the kinematic relation $\lambda_1 = \lambda, \lambda_2 = 1, \lambda_3 = 1/\lambda$ (compare with Exercise 1(b) on p. 226). The stress state of this mode of deformation is characterized by σ_1, σ_2 and $\sigma_3 = 0$ (recall eqs. (6.76), (6.77)). The material is supposed to undergo stress softening of Mullins type.

    (a) Based on the isotropic damage model introduced, compute the loading path up to $\lambda = 2$ followed by the unloading path back to $\lambda = 1$. Apply the constitutive particularization of the phenomenological damage variable $\zeta = \zeta(\alpha)$, as given in eq. (6.281), and use the strain-energy function of the Mooney-Rivlin form, i.e. $\Psi = c_1(I_1 - 3) + c_2(I_2 - 3)$, with the ratio of the material constants $c_1/c_2 = 7$.

    (b) Plot the loading and unloading path in the form of the two functions $\sigma_1 = \sigma_1(\lambda)$ and $\sigma_2 = \sigma_2(\lambda)$ with the Mooney-Rivlin parameters $c_1 = 0.4375\mu$, $c_2 = 0.0625\mu$ and the shear modulus in the reference configuartion, i.e. $\mu = 4.225 \cdot 10^5 \mathrm{N/m^2}$. In addition, take $\zeta_\infty = 0.8$ for the maximum damage and $\iota = 0.3 \cdot 10^6 \mathrm{N/m^2}$ for the damage saturation parameter.

2. *Equibiaxial deformation with isotropic damage.* Consider an equibiaxial deformation of an incompressible material, which may be modeled by the strain energy due to Ogden. Recall the stress strain relation derived in Example 6.6 on p. 239, i.e. $\sigma = \sum_{p=1}^{N} \mu_p(\lambda^{\alpha_p} - \lambda^{-2\alpha_p})$. During a load cycle $\lambda = 1 \to 3 \to 1 \to 5 \to 1$ the material accumulates damage of Mullins type.

    Plot the load cycle $\sigma = \sigma(\lambda)$ with the typical values of the constants α_p, μ_p, $p = 1,\ldots,3$, for Ogden's model given by (6.121). Assume the maximum damage $\zeta_\infty = 0.8$ and the damage saturation parameter $\iota = 1.0 \cdot 10^6 \mathrm{N/m^2}$.

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7 Thermodynamics of Materials

Thermodynamics is the science of energy, which studies processes in systems outside the thermodynamic equilibrium state. The term 'thermodynamics' comes from the Greek words θέρμη and δύναμις meaning 'heat' and 'force' (or 'power'), respectively. Thermodynamics has long been a fundamental part of engineering. It constitutes a concept of great generality which is based on a few, simple hypotheses. Today the name 'thermodynamics' is interpreted as including all aspects of *energy*.

There are two principle ways of dealing with thermodynamics: long ago it was recognized that in the real world physical objects are compositions of molecules which are formed by atoms and even smaller subatomic particles. The special field in which the laws of classical mechanics (or quantum mechanics) are applied to large groups of individual particles (molecules, atoms) is called **statistical thermodynamics**. This approach investigates the correlation between the average behavior of particles and the macroscopic properties of a system on a statistical basis. The traditional classical, or phenomenological, approach to the study of thermodynamics, in which the molecular structure of a physical object is disregarded (the object is considered as continuous matter with no microscopic holes) is called **classical thermodynamics** or **(phenomenological) continuum thermodynamics**. It may be viewed as a unified field theory of *mechanics* and *thermodynamics* in which all thermodynamic state variables depend on position and time.

The essential feature of continuum thermodynamics is the derivation of *constitutive equations* (for the *stress tensor*, the *entropy*, the *heat flux vector*) from the basic physical principles of thermodynamics representing the individual (mechanical and thermodynamic) material properties of matter. However, for the formulation of constitutive equations we have several possible choices of independent and dependent variables. In this chapter the list of independent variables is supplemented by non-mechanical variables such as *temperature*, *entropy* and their gradients. We shall combine the (isothermal) constitutive theory of *finite (visco)elasticity*, as introduced in Chapter 6, with the theory of *heat conduction* under *transient* conditions. Solutions of the resulting **coupled thermomechanical problem** are able to describe the interaction between the *mechanical field* and the *thermal field*. In this chapter we study the thermodynamics of continuous media and, in particular, two different classes of constitutive models within the nonlinear constitutive theory of **finite thermoelasticity** and **finite thermoviscoelasticity**.

The only materials undergoing finite strains and temperature changes relative to an equilibrium state are biological soft tissues and rubber-like materials. As known from statistical thermodynamics of rubber elasticity, extended rubber chains tend to return to a less-ordered curled up-state which is characterized by a higher conformation entropy. The thermomechanical behavior of solid polymers is almost entirely based on an entropy concept. Therefore, in this chapter we start out with the aim of reviewing the crucial difference between rubber and metal within a thermodynamic context. In order to describe the three-dimensional network of rubber by means of the Helmholtz free-energy function, some insights in the statistical thermodynamics of rubber and the (molecular) network theory are presented briefly. We restrict attention to the *Gaussian* statistical theory, which enables us to characterize the thermoelastic behavior of a (molecular) network within small strains.

Within this statistical context, the neo-Hookean model, as derived in Section 6.5, is motivated. In the subsequent sections, quite independently of the network theory, we follow an approach to a macroscopic continuum formulation of *thermoelastic* and *thermoviscoelastic materials* by making use of continuum particles. We introduce a constitutive model for the thermoelastic behavior of materials and present a thermodynamic extension of the classical strain-energy function originally proposed by *Ogden*. The material model is set up in order to reproduce the realistic physical stress-strain-temperature response of rubber-like materials.

Moreover, a study of one-dimensional problems of finite thermoelasticity is presented. Distinctive attention is paid to the so-called *thermoelastic inversion phenomena*, a remarkable property of rubber-like materials.

The last section in this chapter is concerned with the study of thermodynamics in terms of internal variables. A constitutive model for highly deformable media that accounts for several thermomechanical coupling effects is examined. The proposed phenomenological model is capable of describing relaxation and/or creep phenomena within the thermomechanical regime.

7.1 Physical Preliminaries

As a basis for our next studies we present an introductory review of some of the interesting physical aspects of the thermoelastic behavior of amorphous solid polymers, that are chemically cross-linked, for example, by sulphur bridges.

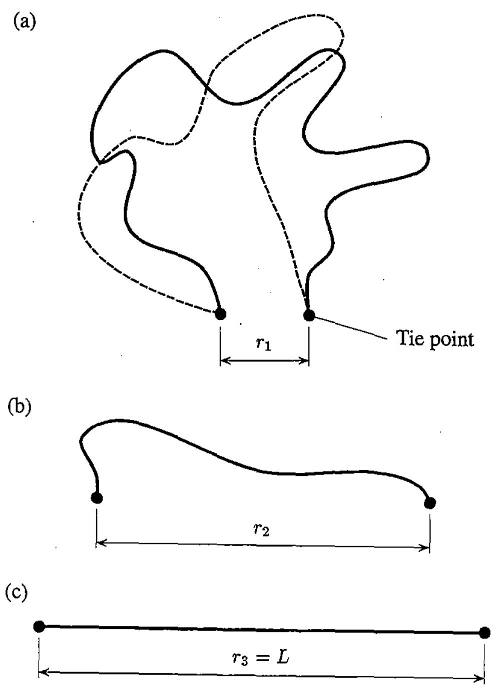

Figure 7.1 Single polymer chain lying between two tie points with various distances r. Two possible conformations are shown in (a). The number of possible conformations decreases with increasing end-to-end distance (b). Fully extended chain showing the limiting case with only one possible conformation (c).

Statistical concept. Based on several physical techniques we know that amorphous polymers are composed of bundles of long-chain molecules (which may be imagined as strings) having a high degree of flexibility. Figure 7.1 shows a model for a single polymer chain lying between successive points of cross-linkage, which we call **tie points**. The single polymer chain forms a typical segment in the coherent and three-dimensional network of rubber. This model is due to GUTH and MARK [1935] and KUHN [1938, 1946]. The distance between the tie points of the chain molecule, denoted by r, we call subsequently **end-to-end distance** (or **separation**). The distance r is a parameter that characterizes a molecular **conformation**. The name conformation comes from chemistry and refers to different shapes (arrangements) of a chain molecule. The most powerful physical technique now available for determining conformations of chain molecules is *small-angle neutron scattering* (for more details see, for example, SPERLING [1992, Section 5.2]).

Figure 7.1(a) shows a polymer chain with the end-to-end distance r_1 of the tie points, which are assumed to be fixed in space. The distance r_1 is much smaller than the **contour length** L which is the length of the fully extended chain. Consequently, the chain may take on an enormous range of possible conformations, two of which are shown in Figure 7.1(a). Obviously, the number of possible conformations decreases for a larger end-to-end distance, and in the limit the number of possible conformations diminishes to only one if the chain is in its most extended state, as illustrated in Figure 7.1(c). Then the value of the end-to-end distance reaches its maximum, i.e. r_3, and equals the contour length L, and the chain is straight. The most crumpled conformation occurs when r tends to zero, and the tie points coincide. Clearly the end-to-end distance r of a chain characterizes the molecular conformation.

The *statistical theory* of rubber elasticity (see, for example, the notable works by TRELOAR [1943a, b], JAMES and GUTH [1943, 1949] and FLORY [1953, Section XI-3, and references therein]), which is basically set up on these concepts makes use of the idea to express the number of conformations that a chain molecule can assume as an **'entropic effect'**. The chains occur in randomly coiled conformations in the unstretched state, as seen in Figure 7.1(a), and as the chains are extended the number of conformations and the entropy decrease.

We proceed now to model this characteristic behavior and to analyze the conformations of a chain molecule. However, just for clarity, consider first the problem in *one dimension* and project the conformation on one coordinate axis, say the x_1-axis (the chain may be imagined as being constrained artificially so that the tie points lie on the x_1-axis). The conformations of the individual chains are distributed in a random manner. The **probability** $p(x_1)\mathrm{d}x_1$ that the end-to-end distance of a chain lies in the interval between x_1 and $x_1 + \mathrm{d}x_1$ is expressed by the **Gaussian distribution function**

$$p(x_1)\mathrm{d}x_1 = \frac{b}{\pi^{1/2}}\exp(-b^2x_1^2)\mathrm{d}x_1 \ , \qquad (7.1)$$

where $p(x_1)$ is the **probability density** (per unit length) and b is a parameter of the model. This parameter is a measure of a representative length, as we see later in Example 7.1, p. 313. For an explicit derivation of the Gaussian distribution function (7.1) for a chain in one dimension the interested reader is referred to the classical book by FLORY [1953, Appendix A of Chapter X].

The Gaussian function is a bell-shaped curve, which provides numerous applications in engineering practice and statistics. The most probable value of x_1, i.e. the maximum of the Gaussian function, may be found by differentiating eq. (7.1) with respect to x_1 and occurs at $x_1 = 0$. The probability decreases monotonically as x_1^2 increases. Note that the entropy may be interpreted as a quantitative measure of probability (microscopic randomness and disorder) by using a fundamental finding which is due to *Boltzmann* and *Planck*. This will be made clear in Section 7.2.

For a more comprehensive survey of the general concepts of statistical mechanics, a terminology which was introduced by *Gibbs*, the reader may be referred to the books by FLORY [1969], TRELOAR [1975], WEINER [1983], CALLEN [1985, Part II] and MARK and ERMAN [1988]; see also the review paper by GUTH [1966].

Rubber versus 'hard' solids. One of the remarkable differences between *rubber* and *'hard' solids*, such as *metals, glasses, ceramics, crystals*, etc. lies in the effect of temperature. The following crucial physical properties, explored and quantified in a set of experiments by *Joule*, exhibit the distinctive behavior of rubber (see JOULE [1859, p. 105]):

(i) a piece of vulcanized rubber subjected to a weight produces a slight *cooling effect* in the very low strain range and changes to a *heating effect* by increasing the weight, and

(ii) rubber will *contract* its length under tension when its temperature is raised (it is not very known that healthy human and animal arteries also *shrink* upon heating, a phenomenon that was pointed out in the early work of ROY [1880–1882] for the first time).

These results are based on a (previous) simple qualitative observation by GOUGH [1805] that a rapidly stretched *rubber band* (adiabatic straining) brought into slight contact with the lips as a sensitive detector feels warm. On the other hand a stretched rubber band in thermodynamic equilibrium feels cold after releasing the tension. This (thermoelastic) coupling phenomenon entered the literature as the so-called **Gough-Joule effect**. Note that the behavior of a metallic spring is in striking contrast to a rubber band. A metallic spring cools continuously on elastic stretching. This is the opposite behavior of a rubber band which warms on stretching, a remarkable experimental observation (see Box 1).

The properties of rubber are well-known above the **glass transition temperature** (see, for example, CYR [1988], WARD and HADLEY [1993]), and are characterized by

(i) extremely long-range extensibility, typically 300–500% extension for vulcanized natural rubber (i.e., without carbon black or other reinforcing fillers) and even more for synthetic rubber (generating low mechanical stresses), accompanied by

(ii) full recovery to the initial dimensions without mechanical and thermal hysteresis within the lower temperature domain of the 'rubbery' region.

Below the glass transition temperature, flexibility and mobility of the chains are so reduced that rubber behaves like a brittle glass (the material consists of rigid crystals). The glass transition temperature, for example, for *natural rubber* and *butyl rubber* is −73°C (see MARK and ERMAN [1988, Chapter 2]).

Based on experimental observations by ANTHONY et al. [1942] (see also TRELOAR [1975, Chapters 2, 13]) the retractive force in a *real rubber* is approximately 90% based on entropy (for additional information the reader is referred to the papers by SHEN and CROUCHER [1975] and CHADWICK and CREASY [1984]).

Polymers	**Metals**
Entropic elasticity	Energetic elasticity
⇒ Total stress is entirely caused by a change in entropy with deformation.	⇒ Entropy does not change with deformation at all.
⇒ Internal energy does not change with deformation at all.	⇒ Total stress is internal energy driven, which changes rapidly with deformation.
⇒ Elasticity arises through entropic straightening of a polymer chain, followed by recoiling into a conformation of maximum entropy, see Figure 7.1 (iso-volumetric phenomena).	⇒ Elasticity arises through energetic increases due to distance changes between atoms against atomic attractive forces, followed by removing the interatomic forces back to its initial dimensions (in general, substantial volume changes accompany deformation).
'Polymers are entropic'	'Metals are energetic'
⇒ A piece of rubber warms on stretching.	⇒ An elastic metallic spring cools on stretching.
⇒ A rubber band under constant tensile force substantially will shrink upon heating and expand upon cooling.	⇒ An elastic metallic spring under constant tensile force will expand upon heating and shrink upon cooling.

From a thermodynamic point of view, the work done in elongating a rubber band is unlike the work produced by stretching a coiled elastic metallic spring.

Box 1 Composition of the concepts in polymers (ideal rubbers) and 'hard' solids (metals, glasses, ceramics, crystals, etc.) in the elastic range.

For an *ideal rubber*, for which by definition one property is incompressibility (the volume remains constant (locally and globally) during a mechanical process), the retractive force is, however, purely determined by changes in entropy and the internal energy does not change with deformation at all, i.e. a significant characteristic of rubber elasticity. We term this type of rubber-like material **'entropic elastic'**.

However, elasticity of metals, glasses, ceramics or crystals arises basically through removing atoms from their equilibrium positions, accompanied by rapid internal energy changes, while the entropy does not change at all (see, for example, HILL [1975] and ERICKSEN [1977]). These materials with a regular atomic structure and usually with high strength are typically called **'energetic elastic'**. They exhibit, in general, substantial volume changes on deformations and stand in sharp contrast to rubber-like solids. For a general overview, the corresponding concepts in *polymers* (ideal rubber) and *'hard' solids* (metals) are summarized in Box 1.

Natural rubber (cis-polyisoprene) does not recover completely. In order to achieve dimensional stability and deformations which are completely reversible in the 'rubbery' state, vulcanization of the rubber (typically done for commercial products) is required. Within a vulcanization process polymer chains are chemically connected to other chains at different locations to produce a cross-linked monolithic three-dimensional network (see SPERLING [1992]). On the other hand crystallization occurring in the highly stretched rubber is influenced by vulcanization. Crystallization is lower for a higher concentration of sulfur used in the vulcanization process (see TRELOAR [1975, pp. 16-23]).

It is important to note that the material properties of highly stretched crystalline rubber become anisotropic and that the heat of crystallization is much larger than produced by the mentioned thermoelastic *Gough-Joule effect*. Moreover, due to frictional losses during the deformation process an additional heat is generated. In addition, it is mentioned that real networks contain defects (see, for example, MULLINS and THOMAS [1960] and SCANLAN [1960]). Crystallinity effects and network imperfections lie outside the scope of this text.

7.2 Thermoelasticity of Macroscopic Networks

A typical vulcanized rubber may be considered as the assembly of long-chain molecules. Each chain is attached at both ends and thus produces one giant molecule, which we call the **(molecular) network**. From the irregular three-dimensional network we draw conclusions regarding its material properties. In the following, motivated by statistical thermodynamics, we describe the material properties of rubber through the Helmholtz free-energy function Ψ.

The freely jointed chain in three dimensions. We assume that the molecular network contains N chains per unit volume which is often referred to as the **network density**.

We consider first a representative single polymer chain in space **detached** from the network, which means that the chain is taken *out* of the network. Our aim is to compute the entropy of this chain and to study its thermodynamic behavior. The chain with contour length L is cross-linked at the tie points O and P of the network (see Figure 7.2). One end of the chain is attached to the fixed origin O of the x_1, x_2, x_3-coordinate system. The other end is given by the **end-to-end vector** $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, pointing to P and characterizing a certain number of different shapes.

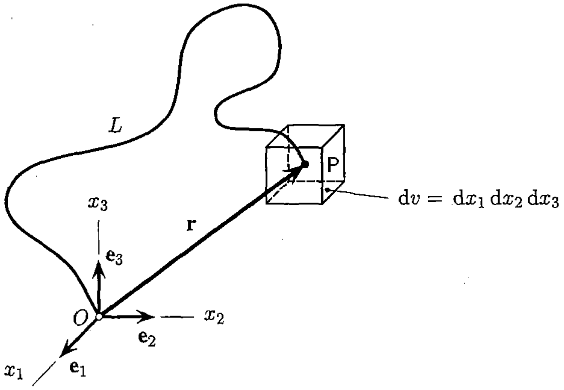

Figure 7.2 A representative single polymer chain OP detached from the network.

Further, we assume a so-called **Gaussian chain** which is defined so that the distance between the tie points (chain ends) O and P, i.e. $r = |\mathbf{r}|$, is considerably less than the contour length L, i.e.

$$r \ll L \quad . \tag{7.2}$$

Hence, we follow the context of the **Gaussian statistical theory** of elasticity which is valid for problems where only small strains are involved.

The contour length L of a single chain is commonly considered to be an assembly of n (statistical) segments joined together, each of length l so that $L = nl$. We suppose that there is no correlation between the directions of the successive segments. Based on this simple mechanical model we may determine the so-called **mean value** $\bar{r}$ of the end-to-end distance r for this **freely jointed chain** observed at one instant of time. This

is given by

$$\overline{r^2} = nl^2 \quad , \tag{7.3}$$

where the mean-square value $\overline{r^2}$ denotes the average over r^2. For a more detailed explanation the interested reader is referred to the textbooks by, for example, McCrum et al. [1997, Section 2.8] or Ward and Hadley [1993, Section 3.3] and left to study Exercise 1 on p. 320.

We wish to calculate the probability that the tie point P lies within the infinitesimal volume element of size $\mathrm{d}v = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ at P (see Figure 7.2). By analogy with eq. (7.1) we may introduce the probability densities $p(x_2)$ and $p(x_3)$ which are associated with the x_2-axis and the x_3-axis, respectively. In addition, it is possible to show that $p(x_1)$ depends only on x_1 ($p(x_2)$ only on x_2 and $p(x_3)$ only on x_3) provided that n is large and x_1, x_2, x_3 are much smaller than the contour length L of the chain.

Generalizing the relation (7.1) to *three dimensions* we may find the probability density $p(x_1, x_2, x_3)$, now per unit volume, that the tie point P of the freely jointed chain occurs in the infinitesimal volume element $\mathrm{d}v$ (see Figure 7.2). Under the restrictions considered it is the product of the independent probability densities according to

$$\begin{aligned} p(x_1, x_2, x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 &= p(x_1)p(x_2)p(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ &= \frac{b^3}{\pi^{3/2}}\exp(-b^2r^2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \quad , \end{aligned} \tag{7.4}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$ is the square of the distance between the tie points O and P for this detached Gaussian chain and parameter b denotes a measure of a representative length. As for the one-dimensional case the maximum of the Gaussian function (7.4) occurs at $r = 0$. The Gaussian distribution function, as given in (7.4), represents a sufficiently accurate solution to the stochastic problem in question.

EXAMPLE 7.1 In this example consider a freely jointed Gaussian chain with tie points O and P and calculate all possible conformations of the chain at any given value of the end-to-end distance $r = |\mathbf{r}|$, *irrespective of direction*. Furthermore, show that the measure of the representative length b controls the mean-square value $\overline{r^2}$ according to

$$b^2 = \frac{3}{2\overline{r^2}} \tag{7.5}$$

(compare also with Flory [1956, Section X-1b]). The mean-square value $\overline{r^2}$ is defined to be $\int_0^\infty r^2 p(r)\mathrm{d}v / \int_0^\infty p(r)\mathrm{d}v$, with the probability density p given by eq. $(7.4)_2$.

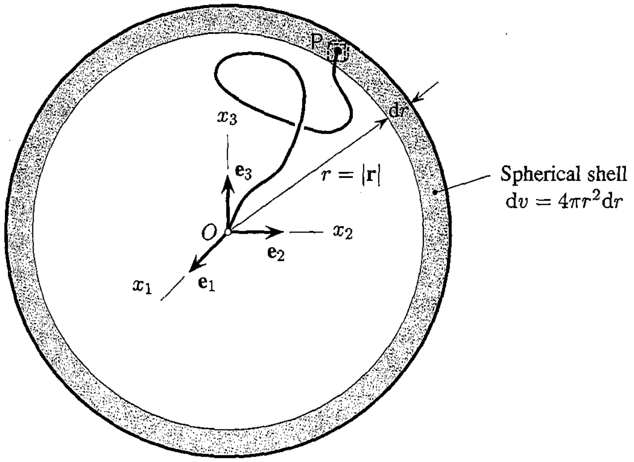

Figure 7.3 Spherical shell which defines all possible conformations of a representative Gaussian chain OP irrespective of direction.

Solution. The restriction to a particular direction in space as considered in the previous analysis is not appropriate anymore, so we take into account all directions of the vector **r** equally. Doing so, the tie point P does not move within the infinitesimal rectangular block, rather within an infinitesimal volume $\mathrm{d}v$ of a spherical shell which is

$$\mathrm{d}v = 4\pi r^2 \mathrm{d}r \quad . \tag{7.6}$$

The infinitesimal volume is defined between the inner radius r and the outer radius $r + \mathrm{d}r$ from the other tie point, which is fixed at the origin O of the coordinate system (see Figure 7.3). The required probability $p(r)\mathrm{d}r$ that the chain length lies in the interval between r and $r + \mathrm{d}r$, is, by means of $(7.4)_2$,

$$p(r)\mathrm{d}v = p(r)4\pi r^2 \mathrm{d}r = \frac{b^3}{\pi^{3/2}} 4\pi r^2 \exp(-b^2 r^2)\mathrm{d}r \quad . \tag{7.7}$$

The function (7.7) represents the Gaussian distribution of the distance r for a set of free chains. In the r-distribution function, no restriction on the direction of the vector **r** is involved. The maximum of function (7.7), i.e. the most probable value of r, is obtained by differentiation of $p(r)$ with respect to r. This maximum occurs at $r = 1/b$.

In order to compute the important mean-square value of r, i.e. $\overline{r^2}$, we find after

some manipulations the analytical solution

$$\overline{r^2} = \frac{\int\limits_0^\infty r^2 p(r) 4\pi r^2 \mathrm{d}r}{\int\limits_0^\infty p(r) 4\pi r^2 \mathrm{d}r} = \frac{\int\limits_0^\infty r^4 \exp(-b^2 r^2)\mathrm{d}r}{\int\limits_0^\infty r^2 \exp(-b^2 r^2)\mathrm{d}r}$$

$$= \frac{3\sqrt{\pi}/8b^5}{\sqrt{\pi}/4b^3} = \frac{3}{2b^2} \quad ; \tag{7.8}$$

which proves (7.5). We know from eq. (7.3) that the mean-square value $\overline{r^2}$ depends on the (statistical) segments n and their lengths l (recall that nl is the contour length L of the chains). Hence, we conclude that b is a measure of a representative length. ■

The entropy of a single chain. On the basis of the statistical concept of thermodynamics we now determine the entropy η_i of a representative single Gaussian chain i whose ends are located at specified points in space. The chain is assumed to be taken out (detached) from the network. We apply **Boltzmann's equation** (or what *Einstein* called the **Boltzmann principle**) relating thermodynamic entropy and the probability of a thermodynamic state (molecular conformations). Hence, from the statistical point of view, the entropy η_i of a single chain is defined to be proportional to the logarithm of the probability density $p(r)$ and varies with the end-to-end distance r according to Boltzmann's equation, which is given in the form

$$\eta_i = a + k \ln p(r) \quad , \tag{7.9}$$

where a denotes a constant entropy with respect to a reference level, which need not be specified here in more detail. The universal constant of proportionality $k = 1.38 \cdot 10^{-23}\mathrm{Nm/K}$ denotes **Boltzmann's constant**.

The famous relation between entropy and probability was published by the Austrian physicist *Boltzmann* in 1877 (at this time he worked as a professor for physics in Graz). The term 'Boltzmann's constant' for k and the mathematical formulation of the principle in the form of $S = k \log W$ due to to *Planck*. This form was carved on *Boltzmann's* gravestone at the 'Zentralfriedhof' in Vienna in 1933.

Substituting $(7.7)_2$ into (7.9) we obtain finally

$$\eta_i = c - k b^2 r^2 \quad , \tag{7.10}$$

where the constant $c = a + k\ln(b^3/\pi^{3/2})$ incorporates the constant entropy a. The measure of the representative length b for this Gaussian chain detached from the network is given by

$$b^2 = \frac{3}{2\overline{r^2}_{\mathrm{out}}} \tag{7.11}$$

(see eq. (7.5)), where $\overline{r^2}_{\mathrm{out}}$ denotes the mean-square value of the end-to-end distance of this un-cross-linked free chain *out* of the network. It is an *intrinsic* property of the chain molecule and is independent of volume changes. As can be seen from expression (7.10) the entropy tends to its largest value for $r \rightarrow 0$, as expected (see the considerations in the previous section).

The elasticity of a molecular network. In order to determine the elasticity of a molecular network, we choose, without loss of generality, the case of a homogeneous deformation state of a rubber block given by the principal stretches λ_a, $a = 1, 2, 3$. Further, we introduce two crucial assumptions:

(i) there is no change in volume on deformation, the material is idealized as totally incompressible *(incompressibility assumption)*, i.e. $\lambda_1 \lambda_2 \lambda_3 = 1$,

(ii) changes in the length and orientation of lines marked on chains in a network are identical to changes in lines marked on the corresponding dimensions of the macroscopic rubber sample *(affine motion assumption)*.

Thus, we refer to Figure 7.4, which shows the affine motion of a representative Gaussian chain with one end at the origin.

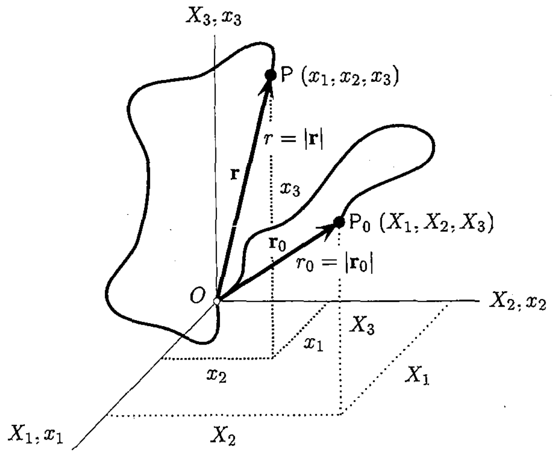

Figure 7.4 Undeformed and deformed configurations of a representative Gaussian chain.

In the undeformed configuration of a network the end-to-end distance of the chain $O\mathrm{P}_0$ is characterized by the vector $\mathbf{r}_0$, with material coordinates X_A, $A = 1, 2, 3$, and length $r_0 = |\mathbf{r}_0|$. Since we consider the affine motion assumption all N chains deform like the representative chain shown in Figure 7.4 and vector $\mathbf{r}_0$ becomes $\mathbf{r}$ after deformation, with spatial coordinates x_a, $a = 1, 2, 3$, and length $r = |\mathbf{r}|$. In the course of the motion the tie point P_0 is displaced to P. Because of the affine motion assumption we may write

$$x_1 = \lambda_1 X_1 \ , \qquad x_2 = \lambda_2 X_2 \ , \qquad x_3 = \lambda_3 X_3 \ . \tag{7.12}$$

The change in the end-to-end distance of the chain due to the deformation produces a difference of entropy between the state before deformation (with $\lambda_a = 1$, $a = 1, 2, 3$), i.e. $c - kb^2(X_1^2 + X_2^2 + X_3^2)$ (see eq. (7.10)), and the state after deformation, i.e. $c - kb^2(\lambda_1^2 X_1^2 + \lambda_2^2 X_2^2 + \lambda_3^2 X_3^2)$. Hence, the entropy change in the chain caused by the deformation of that individual chain i, denoted by $\Delta\eta_i$, is therefore

$$\begin{aligned}\Delta\eta_i &= [c - kb^2(\lambda_1^2 X_1^2 + \lambda_2^2 X_2^2 + \lambda_3^2 X_3^2)] - [c - kb^2(X_1^2 + X_2^2 + X_3^2)] \\ &= -kb^2[(\lambda_1^2 - 1)X_1^2 + (\lambda_2^2 - 1)X_2^2 + (\lambda_3^2 - 1)X_3^2] \ . \end{aligned} \tag{7.13}$$

The constant c has no physical relevance since we are only concerned with the change of entropy.

In order to reform a network all the detached chains are transferred back into the rubber specimen and cross-linked. Of course, the end-to-end distances of (detached) chains *out* of the network are not the same as the end-to-end distances of (cross-linked) chains *in* the network. Therefore, we introduce the mean-square value of the end-to-end distance r for the whole assembly of chains *in* the specimen, denoted by $\overline{r^2}_{\mathrm{in}}$. In contrast to the mean-square value $\overline{r^2}_{\mathrm{out}}$, $\overline{r^2}_{\mathrm{in}}$ is not an intrinsic property of the chain molecule. Since some constraints must be applied to the detached chains in order to reform a network, the mean-square value $\overline{r^2}_{\mathrm{in}}$ differs from $\overline{r^2}_{\mathrm{out}}$. In particular, the value $\overline{r^2}_{\mathrm{in}}$ depends on the volume of the rubber and changes by heating (or cooling) (compare also with McCRUM et al. [1997, Chapter 3]).

Our next aim is to compute the entropy change of a network of such chains generated by the macroscopic deformation state. It is the sum of the entropy changes of all N chains in a unit volume in the network, which we denote by $\Delta\eta$. Since we have assumed affine motion, all the chains have the same given intrinsic property b and imposed λ_a, $a = 1, 2, 3$. With eq. (7.13) we may write

$$\begin{aligned}\Delta\eta &= \sum_{i=1}^{N} \Delta\eta_i \\ &= -kb^2[(\lambda_1^2 - 1)\sum_1^N X_1^2 + (\lambda_2^2 - 1)\sum_1^N X_2^2 + (\lambda_3^2 - 1)\sum_1^N X_3^2] \ . \end{aligned} \tag{7.14}$$

We claim that

$$\sum_1^N r_{0\mathrm{in}}^2 = N\overline{r^2_{0\mathrm{in}}} \tag{7.15}$$

holds, where $\overline{r^2_{0\mathrm{in}}}$ is the mean-square value of the end-to-end distance $r_{0\mathrm{in}}$ of the assembled chains *in* the specimen in the undeformed state.

Since the vector $\mathbf{r}_0$ has no preferred direction in the undeformed state (which is isotropic), we may write $\sum_1^N X_1^2 = \sum_1^N X_2^2 = \sum_1^N X_3^2$. But since $\sum_1^N X_1^2 + \sum_1^N X_2^2 + \sum_1^N X_3^2 = \sum_1^N r_{0\mathrm{in}}^2$ we deduce that $\sum_1^N X_1^2 = \sum_1^N X_2^2 = \sum_1^N X_3^2 = 1/3 \sum_1^N r_{0\mathrm{in}}^2$, and finally, from (7.15)

$$\sum_1^N X_1^2 = \sum_1^N X_2^2 = \sum_1^N X_3^2 = \frac{1}{3} N \overline{r^2_{0\mathrm{in}}} \quad . \tag{7.16}$$

Combining (7.16) and (7.11) with (7.14) we find the entropy change of the network independent of the parameter b, i.e.

$$\Delta\eta = -\frac{1}{2} Nk \frac{\overline{r^2_{0\mathrm{in}}}}{\overline{r^2}_{\mathrm{out}}} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad . \tag{7.17}$$

The term $\overline{r^2_{0\mathrm{in}}}/\overline{r^2}_{\mathrm{out}}$ accounts for the different end-to-end distances of chains in the network and detached from the network.

We have learnt in the previous section that for an ideal rubber the internal energy e does not change with deformation at all. Hence, from the Legendre transformation (4.152) it follows that for an isothermal process the change in the Helmholtz free-energy function Ψ is $\Delta\Psi = -\Theta\Delta\eta$. As a consequence of the Gaussian statistical theory of a molecular network, using the fundamental expression eq. (7.17), we find finally that

$$\Psi = \frac{1}{2} Nk\Theta \frac{\overline{r^2_{0\mathrm{in}}}}{\overline{r^2}_{\mathrm{out}}} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \quad . \tag{7.18}$$

According to (6.4) we have assumed that the free energy is zero in the undeformed configuration. This important result shows that, within the scope of the Gaussian statistical theory, the only quantities pertaining to the molecular network are the total number of chains N contained in the network (per unit volume) and $\overline{r^2_{0\mathrm{in}}}/\overline{r^2}_{\mathrm{out}}$.

On comparison with relation (6.128) it emerges that (7.18) represents the simple *neo-Hookean model* with the physical parameter μ, known as the shear modulus, which is proportional to the concentration of network chains N, given as

$$\mu = Nk\Theta \frac{\overline{r^2_{0\mathrm{in}}}}{\overline{r^2}_{\mathrm{out}}} \tag{7.19}$$

(see, in addition, FLORY [1956] or TRELOAR [1975, p. 114]). A simple method for determining the number of chains N per unit volume is to measure the shear modulus μ for rubber.

For high chain extensions the end-to-end distance r is close to or equal to the contour length L. Therefore, condition (7.2) cannot be satisfied anymore and the Gaussian statistical theory becomes increasingly inadequate for the finite strain domain. In order to account for the finite extensibility of chains some significant refinements must be taken into account (SPERLING [1992, Section 9.10]). The more accurate **non-Gaussian statistical theory** is required (see, for example, TRELOAR [1975, Chapter 6] and MARK and ERMAN [1988, Chapter 13]). Within the non-Gaussian statistical theory the finite extensibility of chains is considered in the form of correction terms leading to a more realistic form of the distribution function which is valid over the whole range of r-values up to the maximum or fully extended length.

One example of such a refined theory is based on the **Langevin distribution function**. The exact treatment of the freely jointed chain is considered by KUHN and GRÜN [1942], JAMES and GUTH [1943], and summarized by FLORY [1953, Appendix B of Chapter X]. In this type of refined theory the Gaussian distribution function is included as a special case.

Other examples are phenomenologically motivated and based on mathematical arguments (see the material models introduced in Section 6.5).

EXAMPLE 7.2 At a given temperature Θ we consider a thermodynamic process in a closed system within some closed time interval $t \in [0, T]$, in which the values 0 and T denote the initial (reference) and the final time, respectively. Assume that the closed system is *thermally isolated* and *conservative*. During the thermodynamic process a unit cube of ideal (incompressible) rubber deforms homogeneously to a parallelepiped with sides of length λ_1, λ_2 and $\lambda_3 = (\lambda_1\lambda_2)^{-1}$, i.e. the principal stretches. The cross-linked network of the rubber cube consists of N Gaussian chains per unit volume.

Based on the Gaussian statistical theory find the heat generated (or destroyed) and the induced total entropy change of the rubber block due to the homogeneous deformation. Specify the problem with the values $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1/6$ and $N = 3.0 \cdot 10^{21}\mathrm{m}^{-3}$. The temperature is assumed to be $\Theta = 293.15\mathrm{K}$ (= 20°C), where K denotes the *'Kelvin temperature'* and °C the *'Celsius temperature'*. The term $\overline{r^2_{0\mathrm{in}}}/\overline{r^2}_{\mathrm{out}}$ is equal to 1.

Solution. Since rubber is incompressible, the change in internal energy, i.e. $\dot{\mathcal{E}}$, during deformation is zero. Hence, the first law of thermodynamics (4.122) reads

$$\dot{\mathcal{E}}(t) = \mathcal{P}_{\mathrm{int}}(t) + \mathcal{Q}(t) = 0 \quad . \tag{7.20}$$

The expressions for the *thermal power* $\mathcal{Q}(t) = \int_{\Omega_0} R \mathrm{d}V$ (the system is thermally isolated, i.e. thermal energy can not enter or leave the boundary ($Q_{\mathrm{N}} = 0$), no heat transfer) and the *stress power* $\mathcal{P}_{\mathrm{int}} = \mathrm{D}/\mathrm{D}t \int_{\Omega_0} \Psi \mathrm{d}V$ (the system is conservative) are adopted from (4.118) and (4.116), respectively. After integration over time interval $t \in [0, T]$ we find from the last equation (7.20), by means of (7.18), the particularized first law of thermodynamics, i.e.

$$\int_{t=0}^{t=T} R\mathrm{d}t = -\Psi = -\frac{1}{2}Nk\Theta\frac{\overline{r^2_{0\text{in}}}}{\overline{r^2}_{\text{out}}}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) \ . \tag{7.21}$$

We used the fact that according to assumption (6.4) the strain-energy function Ψ for the unit cube of rubber vanishes in the reference configuration (normalization condition). The term $\int_{t=0}^{t=T} R\mathrm{d}t$ in relation (7.21) represents the heat per unit reference volume within the closed time interval $t \in [0, T]$ (thermal work). All work done which appears as the strain energy is transformed to heat.

The total entropy change of the network $\Delta\eta$ induced by the thermodynamic process is, in accord with (7.17), given by

$$\Delta\eta = -\frac{\Psi}{\Theta} = -\frac{1}{2}Nk\frac{\overline{r^2_{0\text{in}}}}{\overline{r^2}_{\text{out}}}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) \ . \tag{7.22}$$

By substituting the given values into (7.21) and (7.22) we find using Boltzmann's constant $k = 1.38 \cdot 10^{-23}\text{Nm/K}$ that $\int R\mathrm{d}t = -[3.0 \cdot 10^{21}\, 1.38 \cdot 10^{-23}\, 293.15\,(4+9+1/36-3)]/2 = -60.85\text{N/m}^2$. The negative sign means that energy in the form of heat is *destroyed* within the solid body. The entropy change gives $\Delta\eta = -60.85/293.15 = -0.208\text{N/m}^2\text{K}$ which shows clearly that entropy is decreasing as the rubber block is deformed. ■

EXERCISES

1. A drunken man starts to walk on a flat field at a starting point O. He makes one step per second each step of length 0.5 m. The path of the walk of course meanders randomly (the man is drunken), which means that there is no correlation between the directions of successive steps.

   By applying relation (7.3) compute the average distance $\bar{r}$ from point O he has moved after three minutes.

2. A rubber band of initial cross-sectional area A_0 is applied to a mass m. At a certain temperature Θ the mass causes a 200% increase in length.

   Compute the number of chains N per unit volume for the assumption that the material is modeled as neo-Hookean and $\overline{r^2_{0\text{in}}} = \overline{r^2}_{\text{out}}$ (Boltzmann's constant $k = 1.38 \cdot 10^{-23}\text{Nm/K}$).

3. Two rubber bands, A and B, with identical material compositions and length l are tied together at their ends. Then the assembled band is stretched up to a total length of $6l$ and fixed at this position. By assuming that the rubber band A is at temperature Θ_A, and the rubber band B at Θ_B, find the displacement of the knot at which the two bands are tied together.

7.3 Thermodynamic Potentials

To characterize continuous media within the context of thermodynamics we need to define *two* material functions, namely

(i) the **thermodynamic potential** characterizing *all* thermodynamic properties of a system, and

(ii) the **heat flux vector** describing heat transfer.

A thermodynamic potential is a function from which we may derive state variables characterizing a certain thermodynamic state of a system. In the following we define four common thermodynamic potentials. All of them are scalar-valued functions and assumed to be objective. In addition, the potentials are supposed to be at least twice differentiable with respect to all associated components.

For a supplementary account of the relevant topic see the classical work of TRUESDELL and TOUPIN [1960]; see also the texts by, for example, MALVERN [1969], ZIEGLER [1983] and HAUPT [1993b].

Associated thermodynamic potentials. One example of a thermodynamic potential is the uniquely defined *Helmholtz free-energy function* $\Psi = \Psi(\mathbf{F}, \Theta)$, measured per unit reference volume (in thermodynamics the Helmholtz free-energy function is frequently denoted by f or F). The value of the free energy is determined by the changes of two independent variables, i.e. the deformation gradient $\mathbf{F}$ and a non-mechanical variable given by the temperature Θ.

In the following we consider homogeneous materials, which means that the associated functions are independent of position in the medium. With the free energy which describes here non-isothermal thermoelastic processes, we may deduce directly physical expressions from the Clausius-Planck form of the second law of thermodynamics (4.153). For *all* admissible thermoelastic processes the identity $\mathcal{D}_{\text{int}} = \mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} - \eta\dot{\Theta} = 0$ holds, which means the internal dissipation $\mathcal{D}_{\text{int}}$ is zero. By applying the chain rule, time differentiation of the free energy $\Psi(\mathbf{F}, \Theta)$ gives the hypothetical change of the thermodynamic state. We obtain

$$\dot{\Psi} = \dot{\Psi}(\mathbf{F}, \Theta) = \mathbf{P} : \dot{\mathbf{F}} - \eta\dot{\Theta} = \left(\frac{\partial\Psi(\mathbf{F}, \Theta)}{\partial\mathbf{F}}\right)_{\Theta} : \dot{\mathbf{F}} + \left(\frac{\partial\Psi(\mathbf{F}, \Theta)}{\partial\Theta}\right)_{\mathbf{F}}\dot{\Theta} \ , \tag{7.23}$$

which holds at every point of the continuum body and for all times. As usual, the subscripts in (7.23) indicate variables that are being held constant during the partial differentiation of Ψ. For convenience, in the following we will sometimes omit the subscripts. The coupled equation $(7.23)_3$ is known as the **Gibbs relation** for elastic solids (*Gibbs* postulated the equation only for the case of a fluid).

By comparing terms, we may evaluate physical expressions imposed on requirement (7.23) which must hold for any given $(\mathbf{F}, \Theta)$. Since $\dot{\mathbf{F}}$ and $\dot{\Theta}$ can be chosen arbitrarily

$$\mathbf{P} = \left(\frac{\partial \Psi(\mathbf{F}, \Theta)}{\partial \mathbf{F}} \right)_{\Theta} \qquad \text{and} \qquad \eta = - \left(\frac{\partial \Psi(\mathbf{F}, \Theta)}{\partial \Theta} \right)_{\mathbf{F}} \quad , \tag{7.24}$$

which are the general forms of constitutive equations for the first Piola-Kirchhoff stress $\mathbf{P}$ and the entropy η describing thermoelastic materials. Note that for the case of any *isothermal process* ($\Theta = \text{const}$) the free-energy function Ψ is identified with the isothermal strain-energy function (compare with eq. (6.1)). Consequently, $\Psi(\mathbf{F}) = \Psi(\mathbf{F}, \Theta)|_{\Theta=\text{const}}$. By (7.24), the stress and the entropy are determined by the free energy Ψ, which has the status of a **potential** for the stress, the entropy and their respective conjugate thermodynamic variables.

From physical expressions (7.24) we deduce the *stress* and *entropy functions* depending on the deformation gradient $\mathbf{F}$ and the temperature Θ, i.e.

$$\mathbf{P} = \mathbf{P}(\mathbf{F}, \Theta) \quad , \qquad \eta = \eta(\mathbf{F}, \Theta) \quad . \tag{7.25}$$

Eqs. (7.24) and (7.25), also known as **thermal equations of state**, are crucial in specifying material behavior and are completely determined once the free energy $\Psi = \Psi(\mathbf{F}, \Theta)$ is given.

Alternative constitutive equations for the stress and the entropy may be found by analogy with the treatment carried out in Section 6.1. The second Piola-Kirchhoff stress $\mathbf{S}$ follows from relation $(3.65)_2$, by means of $(7.24)_1$ and the analogue of (6.11), as

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = \mathbf{F}^{-1} \left(\frac{\partial \Psi(\mathbf{F}, \Theta)}{\partial \mathbf{F}} \right)_{\Theta} = 2 \left(\frac{\partial \Psi(\mathbf{C}, \Theta)}{\partial \mathbf{C}} \right)_{\Theta} \quad , \tag{7.26}$$

where $\partial \Psi(\mathbf{C}, \Theta)/\partial \mathbf{C}$ is a symmetric tensor. From $(7.24)_2$, using the relation $(6.9)_1$, we obtain the alternative constitutive equation for the entropy, i.e.

$$\eta = - \left(\frac{\partial \Psi(\mathbf{C}, \Theta)}{\partial \Theta} \right)_{\mathbf{C}} \quad . \tag{7.27}$$

In order to describe the thermodynamic state of a system by an alternative thermodynamic potential we require that the entropy function $(7.25)_2$ is uniquely invertible with respect to Θ for each fixed $\mathbf{F}$, so that we have locally the condition $\partial\eta/\partial\Theta \neq 0$.

We assume that the inversion of eq. $(7.25)_2$ is given by $\Theta = \Theta(\mathbf{F}, \eta)$ and we postulate an associated thermodynamic potential e, which is the *internal-energy function* per unit reference volume introduced on p. 157. Knowing that the internal energy is related to the free energy Ψ through the Legendre transformation (4.152), we have the (canonical) representation

$$e = e(\mathbf{F}, \eta) = \Psi(\mathbf{F}, \Theta(\mathbf{F}, \eta)) + \Theta(\mathbf{F}, \eta)\eta \quad , \tag{7.28}$$

also known as the **caloric equation of state**. It is an equation that determines the internal energy as a function of the deformation gradient $\mathbf{F}$ and a non-mechanical variable, i.e. the entropy η.

With potential (7.28) we may deduce fundamental physical expressions from the entropy principle based on the Clausius-Planck inequality (4.141). For *all* admissible thermoelastic processes the second law of thermodynamics reduces to the identity $\mathcal{D}_{\text{int}} = \mathbf{P} : \dot{\mathbf{F}} - \dot{e} + \Theta\dot{\eta} = 0$. Hence, the *Gibbs relation* is obtained by determining the total rate of change of $e = e(\mathbf{F}, \eta)$ and by use of the chain rule. Thus,

$$\dot{e} = \dot{e}(\mathbf{F}, \eta) = \mathbf{P} : \dot{\mathbf{F}} + \Theta\dot{\eta} = \left(\frac{\partial e(\mathbf{F}, \eta)}{\partial \mathbf{F}} \right)_{\eta} : \dot{\mathbf{F}} + \left(\frac{\partial e(\mathbf{F}, \eta)}{\partial \eta} \right)_{\mathbf{F}} \dot{\eta} \quad , \tag{7.29}$$

whence, for arbitrary choices of $\dot{\mathbf{F}}$ and $\dot{\eta}$, we have the physical expressions

$$\mathbf{P} = \left(\frac{\partial e(\mathbf{F}, \eta)}{\partial \mathbf{F}} \right)_{\eta} \qquad \text{and} \qquad \Theta = \left(\frac{\partial e(\mathbf{F}, \eta)}{\partial \eta} \right)_{\mathbf{F}} \tag{7.30}$$

for the first Piola-Kirchhoff stress $\mathbf{P}$ and the temperature Θ. Note that for the case of an *isentropic process* ($\eta = \text{const}$) there exists a state function e, whose partial derivative with respect to $\mathbf{F}$ gives the corresponding first Piola-Kirchhoff stress $\mathbf{P}$.

Comparing the physical expression (6.1), i.e. $\mathbf{P} = \partial \Psi(\mathbf{F})/\partial \mathbf{F}$, with eqs. $(7.24)_1$ and $(7.30)_1$ we recognize that the strain energy $\Psi = \Psi(\mathbf{F})$ serves as the free energy $\Psi(\mathbf{F}) = \Psi(\mathbf{F}, \Theta)|_{\Theta=\text{const}}$ or as the internal energy $e(\mathbf{F}) = e(\mathbf{F}, \eta)|_{\eta=\text{const}}$, depending on the process considered, *isothermal* or *isentropic*.

Knowing that $e = e(\mathbf{F}, \eta)$, we deduce from (7.30) the *stress* and *temperature functions* which depend on $\mathbf{F}$ and η. We obtain the thermal equations of state in the form

$$\mathbf{P} = \mathbf{P}(\mathbf{F}, \eta) \quad , \qquad \Theta = \Theta(\mathbf{F}, \eta) \quad . \tag{7.31}$$

The two thermodynamic potentials Ψ and e introduced are commonly applied in *solid mechanics*. They are suitable for modeling so-called **thermoelastic materials** (no 'memory effects' occur).

For the sake of completeness two additional potentials are reviewed briefly. These are the **Gibbs free energy** (or in the literature sometimes called the **Gibbs function** or **chemical potential**), denoted by g (or sometimes in the literature by G), and the

enthalpy, denoted by h (or sometimes by H). The two thermodynamic potentials g and h are used frequently in *fluid dynamics*.

We postulate that the *Gibbs free energy* $g = g(\mathbf{P};\Theta)$ is a function of the first Piola-Kirchhoff stress $\mathbf{P}$ and the temperature Θ. Performing a Legendre transformation by analogy with (7.28) we may express the *enthalpy* h by means of the thermal equation of state $(7.31)_2$ as

$$h = h(\mathbf{P};\eta) = g(\mathbf{P};\Theta(\mathbf{F};\eta)) + \Theta(\mathbf{F};\eta)\eta \ , \tag{7.32}$$

which is a function of the first Piola-Kirchhoff stress $\mathbf{P}$ and the entropy η. Here, we have used two definitions of g and h which are associated with the free energy Ψ and the internal energy e by the transformations

$$g = g(\mathbf{P};\Theta) = \Psi - \mathbf{P} : \mathbf{F} \ , \qquad h = h(\mathbf{P};\eta) = e - \mathbf{P} : \mathbf{F} \ . \tag{7.33}$$

In order to find the *Gibbs relations* we determine the total rates of change of these two thermodynamic potentials by applying the chain rule, i.e.

$$\dot{g} = \dot{g}(\mathbf{P};\Theta) = \left(\frac{\partial g(\mathbf{P};\Theta)}{\partial \mathbf{P}}\right)_{\Theta} : \dot{\mathbf{P}} + \left(\frac{\partial g(\mathbf{P};\Theta)}{\partial \Theta}\right)_{\mathbf{P}} \dot{\Theta} \ , \tag{7.34}$$

$$\dot{h} = \dot{h}(\mathbf{P};\eta) = \left(\frac{\partial h(\mathbf{P};\eta)}{\partial \mathbf{P}}\right)_{\eta} : \dot{\mathbf{P}} + \left(\frac{\partial h(\mathbf{P};\eta)}{\partial \eta}\right)_{\mathbf{P}} \dot{\eta} \ . \tag{7.35}$$

Using the second law of thermodynamics in the forms of (4.153) and (4.141) (with $\mathcal{D}_{\text{int}} = 0$) and the material time derivatives of transformations (7.33) we arrive simply at $\dot{g} = \dot{g}(\mathbf{P};\Theta) = -\dot{\mathbf{P}} : \mathbf{F} - \eta\dot{\Theta}$ and $\dot{h} = \dot{h}(\mathbf{P};\eta) = -\dot{\mathbf{P}} : \mathbf{F} + \Theta\dot{\eta}$. Hence, by comparing with Gibbs relations (7.34) and (7.35) we obtain expressions for the deformation gradient $\mathbf{F}$, the entropy η and the temperature Θ. Thus,

$$\mathbf{F} = -\left(\frac{\partial g(\mathbf{P};\Theta)}{\partial \mathbf{P}}\right)_{\Theta} \quad \text{and} \quad \eta = -\left(\frac{\partial g(\mathbf{P};\Theta)}{\partial \Theta}\right)_{\mathbf{P}} \ , \tag{7.36}$$

$$\mathbf{F} = -\left(\frac{\partial h(\mathbf{P};\eta)}{\partial \mathbf{P}}\right)_{\eta} \quad \text{and} \quad \Theta = \left(\frac{\partial h(\mathbf{P};\eta)}{\partial \eta}\right)_{\mathbf{P}} \ . \tag{7.37}$$

The Gibbs free energy g and the enthalpy h have the status of a potential from which we may derive $\mathbf{F}$, η and $\mathbf{F}$, Θ, respectively.

In order to characterize the properties of a thermoelastic material, the considered thermodynamic potential must be supplemented by a suitable constitutive equation for the Piola-Kirchhoff heat flux $\mathbf{Q}$, necessary to determine heat transfer. It may be introduced as a function of the deformation gradient, temperature and temperature gradient, i.e.

$$\mathbf{Q} = \mathbf{Q}(\mathbf{F};\Theta;\text{Grad}\Theta) \ , \tag{7.38}$$

satisfying the classical heat conduction inequality (see relation (4.140), i.e. the version in the material description). For a more specific constitutive assertion see, for example, the phenomenological Duhamel's law of heat conduction (4.144) on p. 169 (given in terms of material coordinates). A material for which the heat flux $\mathbf{Q} = \mathbf{o}$ and the heat source $R = 0$ vanish for any point and time is known as an **adiabatic material**.

EXERCISES

1. Using constitutive equations (7.24), (7.30) and (7.36), (7.37) obtain four relations combining $\mathbf{P}$, η, $\mathbf{F}$, Θ in the forms

$$\left(\frac{\partial \mathbf{P}}{\partial \Theta}\right)_{\mathbf{F}} = -\left(\frac{\partial \eta}{\partial \mathbf{F}}\right)_{\Theta} \quad \text{and} \quad \left(\frac{\partial \mathbf{P}}{\partial \eta}\right)_{\mathbf{F}} = \left(\frac{\partial \Theta}{\partial \mathbf{F}}\right)_{\eta} \ , \tag{7.39}$$

$$\left(\frac{\partial \mathbf{F}}{\partial \eta}\right)_{\mathbf{P}} = -\left(\frac{\partial \Theta}{\partial \mathbf{P}}\right)_{\eta} \quad \text{and} \quad \left(\frac{\partial \mathbf{F}}{\partial \Theta}\right)_{\mathbf{P}} = \left(\frac{\partial \eta}{\partial \mathbf{F}}\right)_{\Theta} \ . \tag{7.40}$$

   These identities are known as the **thermodynamic Maxwell** (or **reciprocal**) **relations** and are very valuable in thermodynamic analysis.

2. Each of the four (most common) thermodynamic potentials introduced, i.e. Ψ, e, g, h, is related to any other by a Legendre transformation. Show that the identity

$$e - \Psi - h + g = 0$$

   is satisfied.

7.4 Calorimetry

Two centuries ago **calorimetry** became a branch of experimental physics. In the present day calorimetry deals with both the *measurement* of the amount of heat generated (or destroyed) within a given body during a change of state, and with its *formulation* within the theory of continuum thermodynamics.

Specific heat capacity and latent heat. Firstly, we introduce the **specific heat capacity at constant deformation** ($\mathbf{F}$ = const) per unit reference volume, which is usually denoted by $c_{\mathbf{F}}$. It is the energy required to produce unit increase in the temperature of a unit volume of the body keeping the *deformation fixed*.

The specific heat capacity $c_{\mathbf{F}} = c_{\mathbf{F}}(\mathbf{F},\Theta) > 0$ for all $(\mathbf{F},\Theta)$ is, in general, defined to be a positive function of the form

$$c_{\mathbf{F}} = c_{\mathbf{F}}(\mathbf{F},\Theta) = -\Theta\left(\frac{\partial^2\Psi(\mathbf{F},\Theta)}{\partial\Theta\partial\Theta}\right)_{\mathbf{F}} > 0 \ . \tag{7.41}$$

It is proportional to the second derivative of the free energy Ψ. For a general compressible material $c_{\mathbf{F}}$ depends on the deformation gradient $\mathbf{F}$ as well as on the temperature Θ. The positiveness of $c_{\mathbf{F}}$ may be related to the stability of the material (see, for example, ŠILHAVÝ [1997, Section 17.3]).

Using $(7.24)_2$ and the Legendre transformation $\Psi = e - \eta\Theta$, the specific heat capacity may be represented by the alternative convenient form

$$\begin{aligned} c_{\mathbf{F}} = c_{\mathbf{F}}(\mathbf{F},\Theta) &= \Theta \left(\frac{\partial \eta(\mathbf{F},\Theta)}{\partial \Theta} \right)_{\mathbf{F}} \\ &= \Theta \frac{\partial \eta(\mathbf{F},\Theta)}{\partial \Theta} + \frac{\partial \Psi(\mathbf{F},\Theta)}{\partial \Theta} + \eta(\mathbf{F},\Theta) = \left(\frac{\partial e(\mathbf{F},\Theta)}{\partial \Theta} \right)_{\mathbf{F}} \quad . \end{aligned} \tag{7.42}$$

Hence, the specific heat capacity at constant deformation $c_{\mathbf{F}}$ may also be expressed through the internal energy e which, in general, depends on $\mathbf{F}$ and Θ.

Secondly, we introduce the **latent heat** which is denoted by the *symmetric* tensor $\boldsymbol{v}$. It is a *spatial field* defined to be

$$\begin{aligned} \boldsymbol{v} &= -\Theta \frac{\partial^2 \Psi(\mathbf{F},\Theta)}{\partial \mathbf{F} \partial \Theta} \mathbf{F}^{\mathrm{T}} = -\Theta \mathbf{F} \left(\frac{\partial^2 \Psi(\mathbf{F},\Theta)}{\partial \mathbf{F} \partial \Theta} \right)^{\mathrm{T}} \\ \text{or} \qquad v_{ab} &= -\Theta F_{aA} \frac{\partial^2 \Psi}{\partial F_{bA} \partial \Theta} \quad . \end{aligned} \tag{7.43}$$

Note that the latent heat $\boldsymbol{v}$ is proportional to the mixed second derivative of the free energy Ψ.

Structural thermoelastic heating (or cooling). We define the general relation for the **structural thermoelastic heating (or cooling)** $\mathcal{H}_{\mathrm{e}}$ as the double contraction of the latent heat, as given in $(7.43)_1$, and the symmetric part of the spatial velocity gradient $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, i.e. the rate of deformation tensor $\mathbf{d}$. Thus, with property (1.95)

$$\begin{aligned} \mathcal{H}_{\mathrm{e}} = \boldsymbol{v} : \mathbf{d} &= -\Theta \frac{\partial^2 \Psi(\mathbf{F},\Theta)}{\partial \mathbf{F} \partial \Theta} \mathbf{F}^{\mathrm{T}} : \frac{1}{2} [\dot{\mathbf{F}}\mathbf{F}^{-1} + (\dot{\mathbf{F}}\mathbf{F}^{-1})^{\mathrm{T}}] \\ &= -\Theta \frac{\partial^2 \Psi(\mathbf{F},\Theta)}{\partial \mathbf{F} \partial \Theta} : \dot{\mathbf{F}} \quad . \end{aligned} \tag{7.44}$$

The scalar quantity $\mathcal{H}_{\mathrm{e}}$ represents the thermoelastic coupling effect. This so-called *Gough-Joule effect* occurs, for example, during an adiabatic stretching of a rubber band which typically changes its temperature, as pointed out in Section 7.1. In some problems the thermoelastic coupling effect is neglected due to the fact that this change of temperature is small.

For a thermoelastic process ($\mathcal{D}_{\mathrm{int}} = 0$) the rate of change of the entropy, as derived in (4.142), may be written by means of $(7.25)_2$ and $(7.24)_2$ and the chain rule as

$$\begin{aligned} \Theta \dot{\eta}(\mathbf{F},\Theta) &= -\mathrm{Div}\mathbf{Q} + R = \Theta \frac{\partial \eta(\mathbf{F},\Theta)}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \Theta \frac{\partial \eta(\mathbf{F},\Theta)}{\partial \Theta} \dot{\Theta} \\ &= -\Theta \frac{\partial^2 \Psi(\mathbf{F},\Theta)}{\partial \mathbf{F} \partial \Theta} : \dot{\mathbf{F}} - \Theta \frac{\partial^2 \Psi(\mathbf{F},\Theta)}{\partial \Theta \partial \Theta} \dot{\Theta} \quad , \end{aligned} \tag{7.45}$$

with $\mathrm{Div}\mathbf{Q}$ and R denoting the material divergence of the Piola-Kirchhoff heat flux $\mathbf{Q}$ and the heat source per unit time and per unit reference volume, respectively. Hence, from (7.45) we obtain finally the (coupled) energy balance equation in **temperature form** (the local evolution of the temperature Θ appears explicitly). Using definitions $(7.41)_2$ and $(7.44)_3$ we have

$$c_{\mathbf{F}} \dot{\Theta} = -\mathrm{Div}\mathbf{Q} - \mathcal{H}_{\mathrm{e}} + R \qquad \text{or} \qquad c_{\mathbf{F}} \dot{\Theta} = -\frac{\partial Q_A}{\partial X_A} - \mathcal{H}_{\mathrm{e}} + R \quad . \tag{7.46}$$

On comparison with the associated energy balance equation in *entropy form*, that is eq. (4.142), we recognize that the structural thermoelastic heating (or cooling) $\mathcal{H}_{\mathrm{e}}$ appears only explicitly in the temperature form (7.46).

Consider the case of a process during which $\mathrm{Div}\mathbf{Q}$ vanishes. We deduce from (7.46) that the heat source R per unit time and per unit reference volume is given by

$$R = c_{\mathbf{F}} \dot{\Theta} + \mathcal{H}_{\mathrm{e}} \quad , \tag{7.47}$$

which we may regard as defining the *theory of calorimetry*. For a historical study see TRUESDELL [1980, Chapter 2C].

Within the theory of *finite thermoelasticity*, in general, we observe three different types of thermomechanical coupling effects, namely

(i) the influence of a change in temperature on the stress (thermal stress),

(ii) structural thermoelastic heating (or cooling) – Gough-Joule effect, eq. (7.44), and

(iii) geometric coupling (influence of a change in deformation on heat conduction) (see, for example, eq. (4.144)).

EXERCISE

1. By means of physical expression $(7.36)_2$, show an alternative version of the energy balance equation in temperature form (7.46), i.e.

$$c_{\mathbf{P}} \dot{\Theta} = -\mathrm{Div}\mathbf{Q} - \mathcal{H}_{\mathrm{e}\mathbf{P}} + R \qquad \text{or} \qquad c_{\mathbf{P}} \dot{\Theta} = -\frac{\partial Q_A}{\partial X_A} - \mathcal{H}_{\mathrm{e}\mathbf{P}} + R \quad ,$$

where $c_{\mathbf{P}} = c_{\mathbf{P}}(\mathbf{P},\Theta) > 0$ denotes the **specific heat capacity at constant stress** ($\mathbf{P} = \mathrm{const}$), defined to be $c_{\mathbf{P}}(\mathbf{P},\Theta) = -\Theta(\partial^2 g(\mathbf{P},\Theta)/\partial\Theta\partial\Theta)|_{\mathbf{P}}$. In words: $c_{\mathbf{P}}$ is the energy required to produce unit increase in the temperature of a unit volume of the body keeping the *stress fixed*. Alternatively to eq. $(7.44)_3$ the term $\mathcal{H}_{\mathrm{e}\mathbf{P}}$ represents the structural thermoelastic heating (or cooling) which is defined to be $\mathcal{H}_{\mathrm{e}\mathbf{P}} = -\Theta(\partial^2 g(\mathbf{P},\Theta)/\partial\mathbf{P}\partial\Theta) : \dot{\mathbf{P}}$.

7.5 Isothermal, Isentropic Elasticity Tensors

The following presentation of the isothermal and isentropic elasticity tensors is based on the concept introduced in Section 6.6. It is an extension of the purely mechanical framework to thermodynamics by one thermal variable, i.e. the *temperature* Θ or its conjugate quantity, the *entropy* η.

Isothermal elasticity tensor and stress-temperature tensor. Suppose that a body admits the *right Cauchy-Green tensor* $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and the *temperature* Θ as independent mechanical and thermal variables and suppose the existence of the Helmholtz free-energy function in the form of $\Psi = \Psi(\mathbf{C}, \Theta)$ (and equivalently $\Psi = \Psi(\mathbf{F}, \Theta)$).

Then, according to relation $(7.26)_3$, we can find the second Piola-Kirchhoff stress tensor $\mathbf{S}$ of a point at a certain time t, which may be seen as a nonlinear tensor-valued tensor function of the two variables $\mathbf{C}$ and Θ. By analogy with Section 6.6, we now compute the change in $\mathbf{S}$. According to considerations (1.247) and (1.248), we obtain the total differential

$$\mathrm{d}\mathbf{S} = \mathbb{C} : \frac{1}{2}\mathrm{d}\mathbf{C} + \mathbf{T}\mathrm{d}\Theta \ , \tag{7.48}$$

which gives expressions for a purely mechanical part, $\mathbb{C}$, and a (mixed) mechanical-thermal part, $\mathbf{T}$, in the material description. In the first term in (7.48) we have introduced the definition of the *fourth-order tensor* $\mathbb{C}$, which is proportional to the second partial derivative of Ψ with respect to $\mathbf{C}$. By analogy with eqs. (6.154) and (6.157) we write

$$\mathbb{C} = 2\frac{\partial \mathbf{S}(\mathbf{C}, \Theta)}{\partial \mathbf{C}} = 4\frac{\partial^2 \Psi(\mathbf{C}, \Theta)}{\partial \mathbf{C} \partial \mathbf{C}} \qquad \text{or} \qquad C_{ABCD} = 4\frac{\partial^2 \Psi}{\partial C_{AB} \partial C_{CD}} \ , \tag{7.49}$$

evaluated at $(\mathbf{C}, \Theta)$, with the major symmetries $\mathbb{C} = \mathbb{C}^{\mathrm{T}}$ or $C_{ABCD} = C_{CDAB}$. Here we call $\mathbb{C}$ the **isothermal elasticity tensor** in the *material description* or the **referential tensor of isothermal elasticities**, which is defined by keeping the *temperature fixed* during the process. The **isothermal elasticity tensor** in the *spatial description* $\mathbb{c} = J^{-1}\chi_*(\mathbb{C})$ (or the **spatial tensor of isothermal elasticities**) is defined as a push-forward (and Piola) transformation of $\mathbb{C}$ on each large index by analogy with relation (6.159).

The second term in eq. (7.48) is the **referential stress-temperature tensor** or the **referential thermal coefficient of stress**, denoted by $\mathbf{T}$, which is proportional to the mixed second partial derivative of Ψ with respect to $\mathbf{C}$ and Θ. It is a symmetric *second-order tensor* defined as

$$\mathbf{T} = \frac{\partial \mathbf{S}(\mathbf{C}, \Theta)}{\partial \Theta} = 2\frac{\partial^2 \Psi(\mathbf{C}, \Theta)}{\partial \mathbf{C} \partial \Theta} \qquad \text{or} \qquad T_{AB} = 2\frac{\partial^2 \Psi}{\partial C_{AB} \partial \Theta} \ . \tag{7.50}$$

The spatial counterpart, denoted by $\mathbf{t}$, results via a standard push-forward (and Piola) transformation $\mathbf{t} = J^{-1}\chi_*(\mathbf{T}^\sharp)$ of the (contravariant) referential stress-temperature tensor $\mathbf{T}^\sharp = \partial \mathbf{S}^\sharp / \partial \Theta$ by the motion χ. By analogy with relation (3.66) we find, by means of eq. $(7.50)_2$, that

$$\begin{aligned} \mathbf{t} &= J^{-1}\mathbf{F}\mathbf{T}\mathbf{F}^{\mathrm{T}} = 2J^{-1}\mathbf{F}\frac{\partial^2 \Psi(\mathbf{C}; \Theta)}{\partial \mathbf{C} \partial \Theta}\mathbf{F}^{\mathrm{T}} \\ \text{or} \qquad t_{ab} &= J^{-1}F_{aA}F_{bB}T_{AB} = J^{-1}F_{aA}F_{bB}\frac{\partial^2 \Psi}{\partial C_{AB} \partial \Theta} \ . \end{aligned} \tag{7.51}$$

Note that the symbols $\mathbf{T}$ and $\mathbf{t}$ have already been used and must not be confused with the traction vectors.

We now express the symmetric **spatial stress-temperature tensor $\mathbf{t}$** in terms of the *latent heat* $\boldsymbol{v}$, as defined in eq. (7.43). Knowing the transformation $(\partial \Psi(\mathbf{F}; \Theta)/\partial \mathbf{F})^{\mathrm{T}} = 2(\partial \Psi(\mathbf{C}; \Theta)/ \partial \mathbf{C})\mathbf{F}^{\mathrm{T}}$, which is in accord with eq. (6.11), we find from $(7.51)_2$ that

$$\begin{aligned} \mathbf{t} &= J^{-1}\frac{\partial^2 \Psi(\mathbf{F}; \Theta)}{\partial \mathbf{F} \partial \Theta}\mathbf{F}^{\mathrm{T}} = J^{-1}\mathbf{F}\left(\frac{\partial^2 \Psi(\mathbf{F}; \Theta)}{\partial \mathbf{F} \partial \Theta}\right)^{\mathrm{T}} = -J^{-1}\frac{\boldsymbol{v}}{\Theta} \\ \text{or} \qquad t_{ab} &= J^{-1}F_{aA}\frac{\partial^2 \Psi}{\partial F_{bA} \partial \Theta} = -J^{-1}\frac{v_{ab}}{\Theta} \ . \end{aligned} \tag{7.52}$$

Both tensor quantities $\mathbf{T}$ and $\mathbf{t}$ measure the change of the stress in a process in which the temperature is raised by one unit keeping the *deformation fixed*.

Finally we compute the change in the function for the entropy $\eta = \eta(\mathbf{C}; \Theta)$. With constitutive equation $(7.24)_2$ and the equivalence $\Psi(\mathbf{F}; \Theta) = \Psi(\mathbf{C}; \Theta)$, we obtain

$$\mathrm{d}\eta = -2\frac{\partial^2 \Psi(\mathbf{C}; \Theta)}{\partial \mathbf{C} \partial \Theta} : \frac{1}{2}\mathrm{d}\mathbf{C} - \frac{\partial^2 \Psi(\mathbf{C}; \Theta)}{\partial \Theta \partial \Theta}\mathrm{d}\Theta \ . \tag{7.53}$$

By applying the definitions of the referential stress-temperature tensor $\mathbf{T}$ and the specific heat capacity at constant deformation $c_{\mathbf{F}}$, the entropy change may be expressed as

$$\mathrm{d}\eta = -\mathbf{T} : \frac{1}{2}\mathrm{d}\mathbf{C} + \frac{c_{\mathbf{F}}}{\Theta}\mathrm{d}\Theta \ , \tag{7.54}$$

where the relations $(7.50)_2$ and $(7.41)_2$ are to be used.

Isentropic elasticity tensor and stress-entropy tensor. Consider a body which admits the *right Cauchy-Green tensor* $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and the *entropy* η as independent mechanical and thermal variables and consider the existence of the internal-energy function per unit reference volume in the form of $e = e(\mathbf{C}; \eta)$ (and equivalently $e = e(\mathbf{F}; \eta)$).

The **isentropic elasticity tensor** in the *material description* or the **referential tensor of isentropic elasticities**, denoted by $\mathbb{C}^{\mathrm{ise}}$, is derived from the internal energy e in the same way that the isothermal elasticity tensor is derived from the free energy Ψ. Hence, by analogy with the above, the change in $\mathbf{S}$ is given by

$$\mathrm{d}\mathbf{S} = \mathbb{C}^{\mathrm{ise}} : \frac{1}{2}\mathrm{d}\mathbf{C} + \mathbf{T}^{\mathrm{ise}}\mathrm{d}\eta \ , \tag{7.55}$$

in which we have introduced the definition of the *fourth-order tensor*

$$\mathbb{C}^{\text{ise}} = 2\frac{\partial \mathbf{S}(\mathbf{C};\eta)}{\partial \mathbf{C}} = 4\frac{\partial^2 e(\mathbf{C};\eta)}{\partial \mathbf{C}\partial \mathbf{C}} \qquad \text{or} \qquad C^{\text{ise}}_{ABCD} = 4\frac{\partial^2 \Psi}{\partial C_{AB}\partial C_{CD}} \quad , \tag{7.56}$$

evaluated at $(\mathbf{C};\eta)$. The isentropic elasticity tensor $\mathbb{C}^{\text{ise}}$ is defined by keeping the *entropy fixed* during the process.

The second term in eq. (7.55) denotes the **referential stress-entropy tensor** $\mathbf{T}^{\text{ise}}$, which is proportional to the mixed second partial derivative of e with respect to $\mathbf{C}$ and η. It is a symmetric *second-order tensor* defined as

$$\mathbf{T}^{\text{ise}} = \frac{\partial \mathbf{S}(\mathbf{C};\eta)}{\partial \eta} = 2\frac{\partial^2 e(\mathbf{C};\eta)}{\partial \mathbf{C}\partial \eta} \qquad \text{or} \qquad T^{\text{ise}}_{AB} = 2\frac{\partial^2 e}{\partial C_{AB}\partial \Theta} \quad . \tag{7.57}$$

The **isentropic elasticity tensor** in the *spatial description* $\mathbb{c}^{\text{ise}} = J^{-1}\chi_*(\mathbb{C}^{\text{ise}})$ (or the **spatial tensor of isentropic elasticities**) and the (contravariant) **spatial stress-entropy tensor** $\mathbf{t}^{\text{ise}} = J^{-1}\chi_*(\mathbf{T}^{\text{ise}\sharp})$ are derived from $\mathbb{C}^{\text{ise}}$ and $\mathbf{T}^{\text{ise}}$ by analogy with the above. In the above expressions, $\mathbb{C}^{\text{ise}}$, $\mathbf{T}^{\text{ise}}$ and $\mathbf{t}^{\text{ise}}$ are the *isentropic* quantities analogous to the isothermal quantities $\mathbb{C}$, $\mathbf{T}$ and $\mathbf{t}$.

EXAMPLE 7.3 Obtain the fundamental relationship

$$\mathbb{C}^{\text{ise}} = \mathbb{C} + \frac{\Theta}{c_{\mathrm{F}}}\mathbf{T}\otimes\mathbf{T} \tag{7.58}$$

between the referential tensors of *isentropic* and *isothermal elasticities*, with the *specific heat capacity at constant deformation* c_{F} and the *referential stress-temperature tensor* $\mathbf{T}$, as defined in eqs. $(7.41)_2$ and $(7.50)_2$, respectively.

Solution. If we postulate the existence of the Helmholtz free-energy function $\Psi = \Psi(\mathbf{C};\Theta)$ we may derive the second Piola-Kirchhoff stress tensor $\mathbf{S}$ in the differential form according to (7.48). As can be seen from relation (7.48), changes in stress, $\mathrm{d}\mathbf{S}$, are associated with changes in both the deformation and the temperature.

A thermodynamic process in which the entropy $\eta = \eta(\mathbf{C};\Theta)$ is constant (fixed), $\mathrm{d}\eta = 0$, necessarily implies a change in temperature. An explicit expression may be deduced from (7.54) in the form

$$\mathrm{d}\Theta = \frac{\Theta}{c_{\mathrm{F}}}\mathbf{T} : \frac{1}{2}\mathrm{d}\mathbf{C} \quad . \tag{7.59}$$

This result substituted back into eq. (7.48) leads to

$$\begin{aligned} \mathrm{d}\mathbf{S} &= \mathbb{C} : \frac{1}{2}\mathrm{d}\mathbf{C} + \mathbf{T}\mathrm{d}\Theta \\ &= \underbrace{\left(\mathbb{C} + \frac{\Theta}{c_{\mathrm{F}}}\mathbf{T}\otimes\mathbf{T}\right)}_{\mathbb{C}^{\text{ise}}} : \frac{1}{2}\mathrm{d}\mathbf{C} \quad , \end{aligned} \tag{7.60}$$

which furnishes the desired expression for the isentropic elasticity tensor in the material description for which the entropy is held constant during the thermodynamic process.

This important relation expresses the isentropic elasticity tensor $\mathbb{C}^{\text{ise}}$ in the material description as a function of the free energy $\Psi = \Psi(\mathbf{C},\Theta)$ (compare with eqs. $(7.49)_2$, $(7.50)_2$). Note that in terms of the internal-energy function $e = e(\mathbf{C},\eta)$, relation (7.60) reduces to $(7.56)_2$. ■

Some numerical aspects. The distinction between isothermal and isentropic elasticities plays a crucial role in the analysis of *nonlinear numerical stability*.

Coupled thermomechanical problems in solid mechanics may be solved numerically within one time step leading to simultaneous (monolithic) solutions of all the fields involved in the problem which have the feature of a good stability characteristic. However, this approach leads to large *non-symmetric* systems which are inefficient to solve and are associated with high computational cost. This type of fundamental numerical solution strategy goes back to NICKELL and SACKMAN [1968] and ODEN [1969, 1972].

Alternatively, the coupled system of nonlinear differential equations is often solved using the classical **staggered solution technique** (or also known as the **fractional-step method** or **staggered method**) (see, for example, YANENKO [1971], MARCHUK [1982, and references therein]). In this method, the key idea is to partition the monolithic system of equations into smaller (symmetric) sub-systems by making use of the physical meaning of the problem considered. Within the concept of a staggered solution technique the system can be solved sequentially with much lower computational cost. For each sub-system we can apply existing algorithms and solution strategies.

The classical (merely standard) staggered solution technique for a coupled thermomechanical problem in solid mechanics is based on the solution of a mechanical (isothermal) problem at a *fixed temperature* of the system (elastodynamic phase), which involves the *isothermal elasticity tensor* (7.49), followed by the solution of a heat conduction problem at a *fixed configuration* in the temperature form (7.46). This classical partition is referred to as the **isothermal operator split**, which was used within the context of coupled thermomechanical problems (see, for example, ARGYRIS et al. [1979, 1981, 1982], MIEHE [1988] and SIMO and MIEHE [1992]). However, this type of staggered solution technique is associated with the crucial restriction of conditional stability (see ARMERO and SIMO [1992, 1993]).

It emerges that an alternative partition of a strongly coupled thermomechanical problem leads to a so-called **unconditionally stable (time-stepping) solution technique**, characterized as independent from the chosen time step. This technique allows solutions in an efficient numerically accurate way. The analysis is based on the solution of a mechanical problem at a *fixed entropy* of the system (elastodynamic phase), which involves the *isentropic elasticity tensor* (7.56), followed by the solution of a heat conduction problem at a *fixed configuration* (thermal phase) in the entropy form (4.142).

This alternative methodology is referred to as the **isentropic operator split** when *dissipative materials* are involved (damage, viscous or plastic effects may occur). Likewise for *perfectly thermoelastic materials* the split is referred to as the **adiabatic operator split**. Within this solution technique it is possible to show that a defined *Lyapunov functional* for the coupled system of evolution equations – regarded as the canonical free-energy function for thermoelasticity, first introduced by DUHEM [1911, Vol.2, pp. 220-231], is decreasing along the flows for each of the two sub-problems involved. This approach was proposed by ARMERO and SIMO [1992] for linear and nonlinear thermoelasticity and by ARMERO and SIMO [1993] for finite thermoplasticity. For a successful application to rubber thermoelasticity see the papers by MIEHE [1995b] and HOLZAPFEL and SIMO [1996b].

It must be emphasized that this class of staggered solution technique can be applied not only to coupled thermomechanical problems in solid mechanics but also to coupled problems of, for example, fluid flow in a porous medium, magnetohydrodynamics in fluid mechanics or to stress-diffusion problems. All that must be done is to replace the *temperature* Θ and the *entropy* η by the associated field variables of the coupled problem considered.

EXERCISES

1. Recall the spatial stress-temperature tensor (7.51). By means of eq. (3.66) derive the alternative expression

$$\mathbf{t} = \frac{\partial \boldsymbol{\sigma}(\mathbf{C},\Theta)}{\partial \Theta} \qquad \text{or} \qquad t_{ab} = \frac{\partial \sigma_{ab}}{\partial \Theta} \ ,$$

where $\boldsymbol{\sigma}$ denotes the symmetric Cauchy stress tensor. In addition, derive the spatial stress-entropy tensor which has the form $\mathbf{t}^{\mathrm{ise}} = \partial \boldsymbol{\sigma}(\mathbf{C},\eta)/\partial \eta$.

2. Suppose that a body admits the (spatial) *left Cauchy-Green tensor* $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and the *temperature* Θ as independent variables and consider a free-energy function in the form of $\Psi = \Psi(\mathbf{b},\Theta)$. By analogy with (6.38) we may find the associated constitutive equation in the form of $\boldsymbol{\tau} = 2\mathbf{b}(\partial\Psi(\mathbf{b},\Theta)/\partial\mathbf{b})$, where $\boldsymbol{\tau}$ denotes the symmetric Kirchhoff stress tensor (note that this type of constitutive equation represents *isotropic* thermoelastic response only).

Recall the kinematic relation (2.171) and obtain from the Oldroyd stress rate (5.59) of the Kirchhoff stress, using the chain rule,

$$\pounds_{\mathrm{v}}(\boldsymbol{\tau}^{\sharp}) = J(\mathbb{c} : \mathbf{d} + \mathbf{t}\dot{\Theta}) \ , \qquad (7.61)$$

with the definitions

$$J\mathbb{c} = 4\mathbf{b}\frac{\partial^2\Psi(\mathbf{b},\Theta)}{\partial\mathbf{b}\partial\mathbf{b}}\mathbf{b} \ , \qquad J\mathbf{t} = 2\frac{\partial^2\Psi(\mathbf{b},\Theta)}{\partial\mathbf{b}\partial\Theta}\mathbf{b} = 2\mathbf{b}\frac{\partial^2\Psi(\mathbf{b},\Theta)}{\partial\mathbf{b}\partial\Theta} \qquad (7.62)$$

of the (fourth-order) isothermal elasticity tensor $\mathbb{c}$ in the spatial description and the (second-order) spatial stress-temperature tensor $\mathbf{t}$.

For explicit derivations of eqs. (7.61) and (7.62) see the work of MIEHE [1995b]. Therein, the definitions of the tensor variables $\mathbb{c}$ and $\mathbf{t}$ exclude the factor J.

7.6 Entropic Elastic Materials

We consider so-called **entropic elastic materials**, which have the property that the change in internal energy with deformation is small or even zero (recall Section 7.1). The underlying concept of **entropic elasticity** is used particularly for the thermomechanical description of rubber-like materials such as elastomers (see, for example, FLORY [1961], CHADWICK [1974], CHADWICK and CREASY [1984], MÜLLER [1985], KRAWITZ [1986], HADDOW and OGDEN [1990] and OGDEN [1992b]).

From experimental observations it is known that the bulk modulus for rubber-like materials considerably exceeds the shear modulus. For an *ideal* rubber the internal energy e is assumed to be a function of the temperature Θ alone, $e = e(\Theta)$, which is a typical characteristic of incompressible materials (see, for example, TRELOAR [1975, p. 34] and ANTHONY et al. [1942] for more details). This assumption leads to the **purely** (or **strictly**) **entropic theory** of rubber thermoelasticity (for a theoretical treatment see the work of CHADWICK [1974]). Consequently, the change in internal energy with deformation at a given *reference temperature* Θ_0 is constant. We assume that

$$e_0(\mathbf{F}) = 0 \ . \qquad (7.63)$$

Here and elsewhere the subscript $(\bullet)_0$ characterizes quantities at a reference temperature Θ_0 so that, for example, $e_0(\mathbf{F}) = e(\mathbf{F},\Theta_0)$.

Alternatively, a thermoelastic material obeys the **modified entropic theory** if its internal energy e is expressible as the sum of $e(\Theta)$, already known from the purely entropic theory, and the internal energy $e_0(J) = e(J,\Theta_0)$. The additional contribution $e_0(J)$ to the internal energy depends on the deformation only through the volume ratio J at a given reference temperature Θ_0 (for a theoretical treatment see the work of CHADWICK [1974] and CHADWICK and CREASY [1984]). Consequently, we conclude that

$$e_0(\mathbf{F}) = e_0(J) \ . \qquad (7.64)$$

For notational simplicity we often use the same letter for different functions.

Change in temperature, internal energy, entropy. For both the *purely* and the *modified* entropic theories of rubber thermoelasticity we may conclude from eq. $(7.42)_4$ that the specific heat capacity $c_\mathbf{F}$ is a function of the temperature Θ only. Thus, we write

$$c_\mathbf{F}(\mathbf{F},\Theta) = c(\Theta) \quad . \tag{7.65}$$

Having this in mind we assume a **temperature change**

$$\vartheta = \Theta - \Theta_0 \tag{7.66}$$

between a selected reference state with reference temperature Θ_0 (the choice is quite arbitrary) and the current state with absolute temperature Θ. Hence, by means of (7.65) the change in internal energy for a process is determined by integrating eq. $(7.42)_4$ with respect to the temperature, i.e.

$$e(\mathbf{F},\Theta) - e_0(\mathbf{F}) = \int\limits_{\hat{\Theta}=\Theta_0}^{\hat{\Theta}=\Theta} c(\hat{\Theta})\mathrm{d}\hat{\Theta} \quad . \tag{7.67}$$

Recall from eqs. (7.64) and (7.63) that $e_0(\mathbf{F}) = e_0(J)$ for the modified entropic theory while $e_0(\mathbf{F})$ is assumed to be zero for the purely entropic theory.

Similarly, by means of (7.65), the entropy change simply results from $(7.42)_2$ by the integration

$$\eta(\mathbf{F},\Theta) - \eta_0(\mathbf{F}) = \int\limits_{\hat{\Theta}=\Theta_0}^{\hat{\Theta}=\Theta} c(\hat{\Theta})\frac{\mathrm{d}\hat{\Theta}}{\hat{\Theta}} \quad , \tag{7.68}$$

where $\eta_0(\mathbf{F}) = \eta(\mathbf{F},\Theta_0)$ denotes the entropy at a reference temperature Θ_0.

General structure of the thermodynamic potential. Using eqs. (7.67) and (7.68) and the Legendre transformation $\Psi(\mathbf{F},\Theta) = e(\mathbf{F},\Theta) - \Theta\eta(\mathbf{F},\Theta)$, the thermodynamic potential in the form of the uniquely defined free energy Ψ may be expressed in terms of the internal energy e_0, the entropy η_0 and an additional function T for the purely thermal contribution. Thus,

$$\Psi(\mathbf{F},\Theta) = e_0(\mathbf{F}) - \Theta\eta_0(\mathbf{F}) + T(\Theta) \quad , \tag{7.69}$$

$$T(\Theta) = -\int\limits_{\hat{\Theta}=\Theta_0}^{\hat{\Theta}=\Theta} c(\hat{\Theta})(\Theta - \hat{\Theta})\frac{\mathrm{d}\hat{\Theta}}{\hat{\Theta}} \quad . \tag{7.70}$$

Another commonly used alternative form of the thermodynamic potential may be found by considering only state functions which are assumed to characterize the reference state at a given reference temperature Θ_0. These functions are interrelated by means of the Legendre transformation according to

$$\Psi_0(\mathbf{F}) = e_0(\mathbf{F}) - \Theta_0\eta_0(\mathbf{F}) \quad . \tag{7.71}$$

Consider the thermodynamic potential (7.69) and substitute for the entropy η_0 (at a given temperature Θ_0) the expression which follows from transformation (7.71). Then, by means of the temperature change (7.66), we arrive, after some simple algebra, at the expression

$$\Psi(\mathbf{F},\Theta) = \Psi_0(\mathbf{F})\frac{\Theta}{\Theta_0} - e_0(\mathbf{F})\frac{\vartheta}{\Theta_0} + T(\Theta) \quad , \tag{7.72}$$

which is due to CHADWICK [1974]. Instead of the entropy $\eta_0(\mathbf{F})$ this alternative form uses the isothermal free energy $\Psi_0(\mathbf{F})$, which is the change in strain energy for a deformation from the reference configuration to the current configuration at a fixed (constant) reference temperature Θ_0. The purely thermal contribution $T(\Theta)$ is given by eq. (7.70).

Note that for a thermoelastic material that obeys the modified entropic theory we require $e_0(\mathbf{F}) = e_0(J)$ in eqs. (7.69) and (7.72). However, within the purely entropic theory the characterization of a specific thermoelastic material is given basically by the entropy η_0 (or the isothermal free energy Ψ_0) only, since $e_0(\mathbf{F}) = 0$.

The structure of the free-energy function $\Psi(\mathbf{F},\Theta)$ introduced in (7.69) and (7.72) is general in the sense that it may be used for the description of any entropic elastic material. A specification is accomplished with the choice of particular functions for the internal energy e_0 and the entropy η_0 (or the isothermal free energy Ψ_0) at a reference temperature Θ_0.

In order to perform the integration (7.70) we need an expression for the specific heat capacity as a function of the temperature. However, for some cases the specific heat capacity is assumed to be a positive *constant* over a given temperature range, and we write this as $c_0 > 0$. Then, the integrations in eqs. (7.67), (7.68) and (7.70) can be performed explicitly. The purely thermal contribution (7.70), for example, takes on the standard form

$$T(\Theta) = c_0\left[\vartheta - \Theta\ln\left(\frac{\Theta}{\Theta_0}\right)\right] \quad , \tag{7.73}$$

which will enable us to determine the thermal contribution $T(\Theta)$ for incompressible materials with sufficient accuracy.

EXERCISES

1. An incompressible material with a certain volume, constant specific heat capacity c_0 and with temperature Θ is thrown into a very large lake with reference temperature Θ_0. After some time thermodynamic equilibrium is reached. Assume that the lake will absorb all the heat rejected by the material without any change in its temperature.

Determine the change in entropy for the material which changes its temperature by $\vartheta = \Theta - \Theta_0$.

2. Assume that a thermoelastic material obeys the purely entropic theory ($e_0(\mathbf{F}) = 0$). The change in the strain energy from the reference to the current configuration at a given (constant) temperature Θ_0 is given by $\Psi_0(\mathbf{F})$.
   (a) Recall eq. $(7.44)_3$ and show that for this type of thermoelastic material the structural thermoelastic heating (or cooling) $\mathcal{H}_e$ is governed by the relation
$$\mathcal{H}_e = -\dot{\Psi}_0(\mathbf{F})\frac{\Theta}{\Theta_0} \quad . \tag{7.74}$$
   (b) By adopting the energy balance equation (7.46) and eq. (7.74) show that for every *adiabatic* process (in which the heat flux $Q_N = -\mathbf{Q}\cdot\mathbf{N}$ and the heat source R are zero for all points of the material and for all times) the temperature evolution from the reference to the current configuration is given explicitly by
$$\Theta = \Theta_0 \exp\left(\frac{\Psi_0(\mathbf{F})}{\Theta_0 c_0}\right) \quad , \tag{7.75}$$
   where the strain energy vanishes in the reference configuration according to agreement (6.4). For convenience assume that the specific heat capacity is a constant c_0.

3. Consider a thermoelastic material with a given isothermal free energy $\Psi_0(\mathbf{F})$ and a constant specific heat capacity c_0. Assume that the material obeys the purely entropic theory.
   (a) Recall the physical expression $(7.24)_2$ and show that the evolution (change) of the entropy η is given by
$$\dot{\eta}(\mathbf{F},\Theta) = -\frac{\dot{\Psi}_0(\mathbf{F})}{\Theta_0} + c_0\frac{\dot{\Theta}}{\Theta} \quad .$$
   (b) Consider an *isentropic* process in which the entropy possessed by the given thermoelastic material remains constant. Deduce that
$$c_0\dot{\Theta} = \dot{\Psi}_0(\mathbf{F})\frac{\Theta}{\Theta_0} \quad ,$$
   which gives the same temperature evolution as in (7.75). Interpret this result.

7.7 Thermodynamic Extension of Ogden's Material Model

In this section we particularize the general structure of the thermodynamic potential (7.69) (or its equivalent form (7.72)) introduced previously. Since the bulk modulus for rubber-like materials greatly exceeds the shear modulus it is most advantageous to employ the concept of decoupled (volumetric-isochoric) finite (hyper)elasticity, already introduced within the context of isothermal compressible hyperelasticity (see Section 6.4). This concept is based on a multiplicative split of the deformation gradient (or the corresponding right Cauchy-Green tensor) defined in eq. (6.79).

Our approach is purely phenomenological, providing a set of constitutive equations appropriate for numerical realization using the finite element method. The basic idea of the constitutive model presented for the isotropic thermoelastic behavior of elastomeric (rubber-like) materials incorporating large strains is due to CHADWICK [1974], while aspects for its computational implementation are addressed by MIEHE [1995b], HOLZAPFEL and SIMO [1996b] and REESE and GOVINDJEE [1998b].

Structure of the Helmholtz free-energy function. A useful constitutive model for the isothermal and isotropic behavior of compressible (rubber-like) materials proposed by OGDEN [1972b] was presented in Section 6.5 on p. 244.

Very briefly we recall Ogden's strain-energy function expressed in terms of the volume ratio J, the modified principal stretches $\overline{\lambda}_a = J^{-1/3}\lambda_a$, $a = 1,2,3$, and a given (fixed) reference temperature Θ_0 (typically room temperature). The *decoupled* representation of the strain-energy function $\Psi_0 = \Psi(\lambda_1,\lambda_2,\lambda_3,\Theta_0)$ reads
$$\Psi_0 = \Psi_{\mathrm{vol}\,0} + \Psi_{\mathrm{iso}\,0} \quad , \tag{7.76}$$
where $\Psi_{\mathrm{vol}\,0} = \Psi_{\mathrm{vol}}(J,\Theta_0)$ and $\Psi_{\mathrm{iso}\,0} = \Psi_{\mathrm{iso}}(\overline{\lambda}_1,\overline{\lambda}_2,\overline{\lambda}_3,\Theta_0)$ are assumed to be objective scalar-valued functions characterizing the volumetric elastic response and the isochoric elastic response of the hyperelastic material.

By recalling $(6.137)_1$ and (6.139) we may specify the two response functions $\Psi_{\mathrm{vol}\,0}$ and $\Psi_{\mathrm{iso}\,0}$. Having in mind the notation introduced, we write
$$\Psi_{\mathrm{vol}}(J,\Theta_0) = \kappa(\Theta_0)\mathcal{G}(J) \quad , \qquad \Psi_{\mathrm{iso}}(\overline{\lambda}_1,\overline{\lambda}_2,\overline{\lambda}_3,\Theta_0) = \sum_{a=1}^{3}\overline{\omega}(\overline{\lambda}_a,\Theta_0) \quad , \tag{7.77}$$
$$\overline{\omega}(\overline{\lambda}_a,\Theta_0) = \sum_{p=1}^{N}\frac{\mu_p(\Theta_0)}{\alpha_p}(\overline{\lambda}_a^{\alpha_p} - 1) \quad , \qquad a = 1,2,3 \quad . \tag{7.78}$$
In addition, we must satisfy the (consistency) condition
$$2\mu_0 = \sum_{p=1}^{N}\mu_p(\Theta_0)\alpha_p \qquad \text{with} \qquad \mu_p(\Theta_0)\alpha_p > 0 \quad , \qquad p = 1,\dots,N \quad , \tag{7.79}$$

where the parameter μ_0 denotes the shear modulus in the reference configuration at Θ_0.

The strain energy $\Psi_{\mathrm{vol}}(J, \Theta_0)$ is associated with the volumetric elastic response and is, in general, decomposed into a bulk modulus $\kappa(\Theta_0)$ at a fixed reference temperature Θ_0 and a scalar-valued scalar function $\mathcal{G}(J)$. One example for $\mathcal{G}$ was introduced in eq. $(6.137)_2$. The strain energy $\Psi_{\mathrm{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \Theta_0)$, however, is associated with the isochoric elastic behavior in the space of principal directions of $\overline{\mathbf{C}}$, i.e. the Valanis-Landel hypothesis (see VALANIS and LANDEL [1967]). The parameters $\mu_p(\Theta_0)$ denote the (constant) shear moduli at Θ_0 and α_p are dimensionless constants, $p = 1, \ldots, N$.

Next, we derive a relatively simple but very efficient thermodynamic potential for rubber-like materials which is basically a thermodynamic extension of Ogden's model. We employ the modified entropic theory with assumption (7.64). Applying the thermodynamic potential in the form of (7.72) and using the strain energy (7.76) together with relations (7.77) and (7.78), we obtain the non-isothermal free energy Ψ for isotropic thermoelastic material response, i.e.

$$\Psi(\lambda_1, \lambda_2, \lambda_3, \Theta) = \Psi_{\mathrm{vol}}(J, \Theta) + \Psi_{\mathrm{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \Theta) \quad . \tag{7.80}$$

This decoupled structure is based on the definitions

$$\Psi_{\mathrm{vol}} = \kappa(\Theta)\mathcal{G}(J) - e_0(J)\frac{\vartheta}{\Theta_0} + T(\Theta) \quad , \qquad \Psi_{\mathrm{iso}} = \sum_{a=1}^{3} \overline{\omega}(\overline{\lambda}_a, \Theta) \quad , \tag{7.81}$$

$$\overline{\omega}(\overline{\lambda}_a, \Theta) = \sum_{p=1}^{N} \frac{\mu_p(\Theta)}{\alpha_p}(\overline{\lambda}_a^{\alpha_p} - 1) \quad , \qquad a = 1, 2, 3 \quad , \tag{7.82}$$

$$\kappa(\Theta) = \kappa(\Theta_0)\frac{\Theta}{\Theta_0} \quad , \tag{7.83}$$

$$\mu_p(\Theta) = \mu_p(\Theta_0)\frac{\Theta}{\Theta_0} \quad , \qquad p = 1, \ldots, N \quad , \tag{7.84}$$

and on the condition (7.79). In $(7.81)_1$ the purely thermal contribution $T(\Theta)$ is given in eq. (7.70).

The first contribution $\Psi_{\mathrm{vol}}(J, \Theta)$ to the thermoelastic response defined by $(7.81)_1$ is due to volume changes and purely thermal causes. The bulk modulus $\kappa(\Theta)$ in $(7.81)_1$ depends linearly on the absolute temperature Θ (see eq. (7.83)). Note that the *energetic contribution* $e_0(J)$ to the function $\Psi_{\mathrm{vol}}(J, \Theta)$ occurs only within a modified entropic theory. An empirical expression was proposed by CHADWICK [1974] and has the form

$$\frac{e_0(J)}{\Theta_0} = 3\alpha_0\kappa(\Theta_0)\mathcal{G}(J) \qquad \text{with} \qquad \mathcal{G}(J) = \gamma^{-1}(J^{\gamma} - 1) \quad . \tag{7.85}$$

This relationship is based upon experimental considerations, where $\gamma > 0$ is a positive non-dimensional parameter and the quantity $\alpha_0 = \alpha(\Theta_0)$ denotes the so-called **linear expansion coefficient** relative to a selected reference state with reference temperature Θ_0. The empirical response function $\mathcal{G}(J)$, which is extensively studied by WOOD [1964] and CHADWICK [1974], is able to fit experimental data which are obtained from isothermal compression tests performed with different temperature values. For an instructive example on the basis of $\gamma = 1$ and the neo-Hookean model the reader is referred to the paper by OGDEN [1992b, Example 1] in which the symbol α_0 is used for the *volume* coefficient of thermal expansion. Observe that within a purely entropic theory the energetic contribution $e_0(J)$ vanishes (recall Section 7.6). Consequently, for this case the stress is proportional to the absolute temperature Θ, since $\Psi = \Psi_0(\Theta/\Theta_0) + T$.

The second contribution $\Psi_{\mathrm{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \Theta)$ to the thermoelastic response defined by $(7.81)_2$ is due to isochoric deformations. Here, we consider a representation of Ψ_{iso} in terms of the modified principal stretches $\overline{\lambda}_a$, $a = 1, 2, 3$. The shear moduli $\mu_p(\Theta)$, $p = 1, \ldots, N$, in (7.82) depend linearly on the absolute temperature Θ (see eq. (7.84)). A physical interpretation of this fact was presented within the context of Gaussian statistical theory of molecular networks which is valid for the region of small strains (compare with relation (7.19)).

The thermodynamic potential (7.80) describes the stress-strain-temperature behavior of rubber-like materials within the finite strain domain. As for the isothermal case, the thermodynamic extension of the *Mooney-Rivlin model* and the *neo-Hookean model* results from (7.82) by setting $N = 2$, $\alpha_1 = 2$, $\alpha_2 = -2$ and $N = 1$, $\alpha_1 = 2$, respectively. Observe that for an isothermal deformation process ($\Theta = \Theta_0$) the second and third term in $(7.81)_1$ vanish, the material parameters (7.83) and (7.84) change into constants, and consequently the free-energy function $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3, \Theta)$ changes into the strain-energy function $\Psi_0 = \Psi(\lambda_1, \lambda_2, \lambda_3, \Theta_0)$, as presented by eqs. (7.76), (7.77) and (7.78).

Consistent linearization. Subsequently, we point out the consistent linearization process of the thermodynamic potential given by (7.80). In particular, as a first step, we compute the thermoelastic stress response, characterized by the *second Piola-Kirchhoff stress tensor* $\mathbf{S}$, followed by a second step which determines the *isothermal elasticity tensor* $\mathbb{C}$ in the material description and the *referential stress-temperature tensor* $\mathbf{T}$. The formulation is presented exclusively within the concept of spectral decomposition and is characterized by a geometric setting relative to the reference configuration.

In order to deduce the stress tensor $\mathbf{S} = 2\partial\Psi(\lambda_1, \lambda_2, \lambda_3, \Theta)/\partial\mathbf{C}$ we follow the procedure as shown in Example 6.7 on p. 245. By means of the decomposed structure (7.80) we may find the purely *volumetric* and purely *isochoric* stress contributions $\mathbf{S}(\lambda_1, \lambda_2, \lambda_3, \Theta) = \mathbf{S}_{\mathrm{vol}} + \mathbf{S}_{\mathrm{iso}}$, which are defined to be

$$\mathbf{S}_{\mathrm{vol}} = 2\frac{\partial\Psi_{\mathrm{vol}}(J, \Theta)}{\partial\mathbf{C}} = Jp\mathbf{C}^{-1} \quad , \tag{7.86}$$

$$\mathbf{S}_{\text{iso}} = 2\frac{\partial\Psi_{\text{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \Theta)}{\partial\mathbf{C}} = \sum_{a=1}^{3} \underbrace{\frac{1}{\lambda_a}\frac{\partial\Psi_{\text{iso}}}{\partial\lambda_a}}_{S_{\text{iso}\,a}} \hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a \quad , \tag{7.87}$$

for the general case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ (see the analogues of eqs. $(6.140)_1$, (6.143)).

The constitutive equation for the hydrostatic pressure p, essential for relation $(7.86)_2$, may be specified in terms of the free energy $(7.81)_1$ as

$$p = \frac{\partial\Psi_{\text{vol}}(J, \Theta)}{\partial J} = \kappa(\Theta)\frac{\mathrm{d}\mathcal{G}(J)}{\mathrm{d}J} - \frac{\mathrm{d}e_0(J)}{\mathrm{d}J}\frac{\vartheta}{\Theta_0} \quad , \tag{7.88}$$

where the term $\mathrm{d}\mathcal{G}(J)/\mathrm{d}J$ was particularized in eq. $(6.141)_1$ or $(6.141)_2$ depending on whether the scalar-valued function $(6.137)_2$ or (6.138) is used.

In addition, with free energy $(7.81)_2$ (and (7.82)) we may compute the three principal isochoric stress functions $S_{\text{iso}\,a}$, $a = 1, 2, 3$, in the form

$$\begin{aligned} S_{\text{iso}\,a} &= \frac{1}{\lambda_a}\frac{\partial\Psi_{\text{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \Theta)}{\partial\lambda_a} = \frac{1}{\lambda_a^2}\left(\overline{\lambda}_a\frac{\partial\Psi_{\text{iso}}}{\partial\overline{\lambda}_a} - \frac{1}{3}\sum_{b=1}^{3}\overline{\lambda}_b\frac{\partial\Psi_{\text{iso}}}{\partial\overline{\lambda}_b}\right) \\ &= \frac{1}{\lambda_a^2}\sum_{p=1}^{N}\mu_p(\Theta)\left(\overline{\lambda}_a^{\alpha_p} - \frac{1}{3}\sum_{b=1}^{3}\overline{\lambda}_b^{\alpha_p}\right) \quad , \qquad a = 1, 2, 3 \quad , \end{aligned} \tag{7.89}$$

which are needed for $(7.87)_2$ (compare with the derivation which led to eqs. $(6.144)_2$ and (6.145)).

The derived set of expressions (7.86)–(7.89) completely defines the constitutive model for rubber-like materials, allowing thermoelastic deformations with strain changes unrestricted in magnitude. It is a straightforward thermodynamic extension of Ogden's model known from the isothermal regime, i.e. (6.140), (6.141) and (6.143)–(6.145).

Alternative stress measures follow directly from eqs. (7.86) and (7.87) by means of suitable transformations. For example, the stress response expressed by the Cauchy stress tensor $\boldsymbol{\sigma}$ simply results from a push-forward (and Piola) transformation $\boldsymbol{\sigma} = J^{-1}\chi_*(\mathbf{S}^\sharp) = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}}$ of $\mathbf{S}$.

As a second step in the consistent linearization process, we compute the change in the stress tensor $\mathbf{S}$, i.e. $\mathrm{d}\mathbf{S} = \mathbb{C} : \mathrm{d}\mathbf{C}/2 + \mathbf{T}\mathrm{d}\Theta$, with the definitions of the isothermal elasticity tensor $\mathbb{C}$ in the material description, i.e. (7.49), and the referential stress-temperature tensor $\mathbf{T}$, i.e. (7.50). Based on the decomposed structure of the derived stress response (7.86) and (7.87), we may obtain the decoupled representation

$$\mathbb{C} = \mathbb{C}_{\text{vol}} + \mathbb{C}_{\text{iso}} \qquad \text{and} \qquad \mathbf{T} = \mathbf{T}_{\text{vol}} + \mathbf{T}_{\text{iso}} \tag{7.90}$$

for the tensors $\mathbb{C}$ and $\mathbf{T}$, where the first expression represents the familiar additive split of the fourth-order (isothermal) elasticity tensor $\mathbb{C}$ (compare with the considerations on p. 254 and subsequently). The second expression consists of second-order tensors only. Analogously to $(7.90)_1$, it is composed of a purely volumetric contribution $\mathbf{T}_{\text{vol}}$ and a purely isochoric contribution $\mathbf{T}_{\text{iso}}$.

The explicit forms of the isothermal elasticity tensors $\mathbb{C}_{\text{vol}}$ and $\mathbb{C}_{\text{iso}}$ are adopted from isothermal finite elasticity and are based on eqs. $(6.166)_4$ and (6.196), respectively. We bear in mind that the underlying free energies Ψ_{vol} and Ψ_{iso} depend on both the three principal stretches $\lambda_a = J^{1/3}\overline{\lambda}_a$, $a = 1, 2, 3$, and the temperature Θ.

The isothermal elasticity tensor $\mathbb{C}_{\text{vol}}$, as given by eq. $(6.166)_4$, requires the constitutive equation for the hydrostatic pressure p (which is given in (7.88)) and the scalar function $\tilde{p}$. With reference to specification (7.88), we obtain finally from (6.167) the explicit form

$$\tilde{p} = \kappa(\Theta)\left(\frac{\mathrm{d}\mathcal{G}(J)}{\mathrm{d}J} + J\frac{\mathrm{d}^2\mathcal{G}(J)}{\mathrm{d}J\mathrm{d}J}\right) - \left(\frac{\mathrm{d}e_0(J)}{\mathrm{d}J} + J\frac{\mathrm{d}^2 e_0(J)}{\mathrm{d}J\mathrm{d}J}\right)\frac{\vartheta}{\Theta_0} \quad . \tag{7.91}$$

Considering the isothermal elasticity tensor $\mathbb{C}_{\text{iso}}$ in the spectral form, as given by eq. (6.196), we have just to take care of the coefficients

$$\frac{1}{\lambda_b}\frac{\partial S_{\text{iso}\,a}}{\partial\lambda_b} \qquad \text{and} \qquad \frac{S_{\text{iso}\,b} - S_{\text{iso}\,a}}{\lambda_b^2 - \lambda_a^2} \quad ; \tag{7.92}$$

which depend on the material model in question. The three values $S_{\text{iso}\,a}$, $a = 1, 2, 3$, denote the principal values of the second Piola-Kirchhoff stress tensor $\mathbf{S}_{\text{iso}}$ and are given by eq. $(7.89)_2$. Hence, the second term in (7.92) is already determined. In order to determine the first term recall the closed form solution (6.197) and just consider that the shear moduli μ_p, $p = 1, \ldots, N$, are temperature dependent according to relation (7.84).

Finally we compute the referential stress-temperature tensor $\mathbf{T}$ defined by $(7.50)_1$. By recalling the constitutive equations $(7.86)_2$ and $(7.87)_2$ we may compute the purely volumetric and purely isochoric contributions to the referential stress-temperature tensor. They are defined in the sense that

$$\mathbf{T}_{\text{vol}} = \frac{\partial\mathbf{S}_{\text{vol}}}{\partial\Theta} = Jp'\mathbf{C}^{-1} \quad ; \tag{7.93}$$

$$\mathbf{T}_{\text{iso}} = \frac{\partial\mathbf{S}_{\text{iso}}}{\partial\Theta} = \sum_{a=1}^{3} S'_{\text{iso}\,a}\hat{\mathbf{N}}_a \otimes \hat{\mathbf{N}}_a \tag{7.94}$$

for $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, where the values p' and $S'_{\text{iso}\,a}$, $a = 1, 2, 3$, depend on the material model in question. They are the derivatives of the hydrostatic pressure p and the three principal isochoric stress functions $S_{\text{iso}\,a}$, $a = 1, 2, 3$, with respect to the temperature Θ.

From $(7.88)_2$ and $(7.89)_3$ we find, using the relations for the bulk modulus (7.83) and the shear moduli (7.84), which are temperature dependent quantities, that

$$p' = \frac{\partial p}{\partial \Theta} = \frac{\kappa(\Theta_0)}{\Theta_0} \frac{\mathrm{d}\mathcal{G}(J)}{\mathrm{d}J} - \frac{\mathrm{d}e_0(J)}{\mathrm{d}J} \frac{1}{\Theta_0} \quad , \tag{7.95}$$

$$S'_{\mathrm{iso}\,a} = \frac{\partial S_{\mathrm{iso}\,a}}{\partial \Theta} = \frac{S_{\mathrm{iso}\,a}}{\Theta} \quad , \qquad a = 1, 2, 3 \quad . \tag{7.96}$$

Observe the similar structure of the referential stress-temperature tensors (7.93) and (7.94) to the volumetric-isochoric stress response, as defined in relations (7.86) and (7.87). The *spatial* counterparts $\mathbb{c}$ and $\mathbf{t}$ of the tensors defined in (7.90) result via a standard push-forward (and Piola) transformation.

Heat conduction. The considered thermoelastic problem, for which $(\mathcal{D}_{\mathrm{int}} = 0)$, is governed essentially by *Cauchy's first equation of motion* (see, for example, the local forms (4.53) or (4.63)) and by the *balance of (mechanical and thermal) energy* in entropy or temperature form (see eq. $(7.45)_1$ or (7.46)). Hence, in regard to the energy balance equation we need an additional constitutive equation for the heat flux vector governing heat transfer. One example which satisfies the heat conduction inequality is Duhamel's law of heat conduction (see the considerations on p. 170). For the class of thermally isotropic materials we may express the constitutive equation as

$$\mathbf{Q}(\mathbf{C}, \Theta, \mathrm{Grad}\Theta) = -k_0(\Theta)\mathbf{C}^{-1}\mathrm{Grad}\Theta \tag{7.97}$$

(i.e. Fourier's law of heat conduction, which should be compared with eq. (4.148)), where $\mathbf{Q}$ is the Piola-Kirchhoff heat flux and $k_0 \geq 0$ denotes the coefficient of thermal conductivity associated with the reference configuration. Note that this coefficient is, in general, not a constant. In fact, for vulcanized elastomers, k_0 decreases linearly with increasing temperature (see SIRCAR and WELLS [1981]) according to

$$k_0(\Theta) = k_0(\Theta_0)[1 - \xi(\Theta - \Theta_0)] \quad , \tag{7.98}$$

where $k_0(\Theta_0)$ denotes the coefficient of thermal conductivity at the reference temperature Θ_0 and ξ is a **softening parameter**.

The solution of the coupled thermomechanical problem may be performed by adopting the staggered solution technique. Within a time step this technique leads to a decomposition of the coupled problem (compare with the statements on p. 331). As a result we must solve two smaller, in general, symmetric decoupled sub-problems on a staggered basis. For algorithmic aspects of the entropic theory of rubber thermoelasticity see, for example, MIEHE [1995b] and HOLZAPFEL and SIMO [1996b].

EXERCISES

1. Consider the scalar-valued function $\mathcal{G}(J) = \beta^{-2}(\beta \ln J + J^{-\beta} - 1)$ according to eq. $(6.137)_2$ (due to *Ogden*) and the energetic contribution $e_0(J)/\Theta_0 = 3\alpha_0\kappa(\Theta_0)\gamma^{-1}(J^\gamma - 1)$, with $\gamma > 0$, according to eq. (7.85) (due to *Chadwick*).

   Use $(7.88)_2$, (7.91) and the relation for the bulk modulus (7.83) in order to obtain the constitutive equations

   $$Jp = \kappa(\Theta_0)\left(\beta^{-1}(1 - J^{-\beta})\frac{\Theta}{\Theta_0} - 3\alpha_0 J^\gamma(\Theta - \Theta_0)\right) \quad ,$$

   $$J\tilde{p} = \kappa(\Theta_0)\left(J^{-\beta}\frac{\Theta}{\Theta_0} - 3\alpha_0\gamma J^\gamma(\Theta - \Theta_0)\right) \quad ,$$

   which completely determine the isothermal elasticity tensor $\mathbb{C}_{\mathrm{vol}}$ in the material description.

2. Consider the extension of the isothermal Ogden material to the non-isothermal domain (7.80)–(7.84) (including the quantity $e_0(J)$ which represents an energetic (volumetric) contribution to the free energy). Recall the definition of the structural thermoelastic heating (or cooling) $\mathcal{H}_{\mathrm{e}}$, i.e. eq. $(7.44)_3$, and show that $\mathcal{H}_{\mathrm{e}}$ may be written in the decoupled structure of the form $\mathcal{H}_{\mathrm{e}} = \mathcal{H}_{\mathrm{e\,vol}} + \mathcal{H}_{\mathrm{e\,iso}}$, with the definitions

   $$\mathcal{H}_{\mathrm{e\,vol}} = (\dot{e}_0(J) - \dot{\Psi}_{\mathrm{vol}\,0})\frac{\Theta}{\Theta_0} \quad , \qquad \mathcal{H}_{\mathrm{e\,iso}} = -\dot{\Psi}_{\mathrm{iso}\,0}\frac{\Theta}{\Theta_0} \quad .$$

   The response functions $\Psi_{\mathrm{vol}\,0} = \Psi_{\mathrm{vol}}(J, \Theta_0)$, $\Psi_{\mathrm{iso}\,0} = \Psi_{\mathrm{iso}}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \Theta_0)$ are given by (7.77) with (7.78).

   The analogue of the decoupled structure of $\mathcal{H}_{\mathrm{e}}$ was derived by MIEHE [1995b]. In his work the structural thermoelastic heating $\mathcal{H}_{\mathrm{e}}$ is, however, based on the multiplicative split of the (spatial) left Cauchy-Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and is defined with the opposite sign.

7.8 Simple Tension of Entropic Elastic Materials

The aim of this section is to illustrate the ability and performance of Ogden's model for the non-isothermal domain as outlined in the last section. We set up the basic equations required to describe the realistic physical stress-strain-temperature response of rubber-like materials. In particular, here we consider the simple tension of an entropic elastic material, a class of material defined in Section 7.6. A representative example concerned with the adiabatic stretching of a rubber band will contribute to a deeper insight in the coupled thermomechanical phenomena.

Thermoelastic deformation. Before we start our studies with the simple tension of entropic elastic materials it is most beneficial to point out briefly some aspects of the **thermoelastic deformation** of a continuum body.

Consider a fixed reference configuration of a body with the geometrical region Ω_0 corresponding to a fixed reference time $t = 0$. The position of a typical point may be identified by the position vector $\mathbf{X}$ (with material coordinates (X_1, X_2, X_3)) relative to a fixed origin O (see Figure 7.5). The reference configuration is assumed to be *stress-free* and possesses a homogeneous (uniform) reference temperature value $\Theta_0 (> 0)$. A map of the reference configuration Ω_0 to a current configuration (with the new region) Ω is characterized by the macroscopic motion $\mathbf{x} = \chi(\mathbf{X}, t)$ for all $\mathbf{X} \in \Omega_0$ and for all times t. The motion carries a typical point $\mathbf{X} \in \Omega_0$ to a point $\mathbf{x} \in \Omega$ which is characterized by the spatial coordinates (x_1, x_2, x_3).

As a measure of the thermoelastic deformation we use the deformation gradient $\mathbf{F}(\mathbf{X}, t)$ and the volume ratio $J(\mathbf{X}, t) = \det\mathbf{F}(\mathbf{X}, t) > 0$. Very often it is convenient to decompose the (local) motion $\chi(\mathbf{X}, t) = \chi_\mathrm{M}[\chi_\Theta(\mathbf{X}, t)]$ into two successive motions χ_M and χ_Θ according to

$$\mathbf{F} = \frac{\partial \chi(\mathbf{X}, t)}{\partial \mathbf{X}} = \mathbf{F}_\mathrm{M}\mathbf{F}_\Theta \qquad \text{and} \qquad J = J_\mathrm{M} J_\Theta > 0 \ , \tag{7.99}$$

with the definitions

$$\mathbf{F}_\mathrm{M} = \frac{\partial \chi_\mathrm{M}(\mathbf{X}_\Theta, t)}{\partial \mathbf{X}_\Theta} \ , \qquad J_\mathrm{M} = \det\mathbf{F}_\mathrm{M} > 0 \ , \tag{7.100}$$

$$\mathbf{F}_\Theta = \frac{\partial \chi_\Theta(\mathbf{X}, t)}{\partial \mathbf{X}} \ , \qquad J_\Theta = \det\mathbf{F}_\Theta > 0 \ . \tag{7.101}$$

The multiplicative decomposition (7.99) separates the total thermoelastic deformation into a purely *mechanical* contribution $\mathbf{F}_\mathrm{M}$, J_M and a purely *thermal* contribution $\mathbf{F}_\Theta$, J_Θ, which represents the *Duhamel-Neumann hypothesis* for the nonlinear theory (see, for example, CARLSON [1972, p. 310]).

The two successive motions establish a new **intermediate (imagined) configuration** with geometrical region Ω_Θ, as illustrated in Figure 7.5. The new configuration is assumed to be *isolated* from the body so that a *thermal stress-free* deformation may occur. Hence, a relative temperature field $\vartheta = \Theta - \Theta_0$ causes a (free) thermal expansion (or contraction) about the reference configuration Ω_0 characterized by the associated variables $\mathbf{F}_\Theta$ and J_Θ. The intermediate configuration with the region Ω_Θ is given by the macroscopic motion $\mathbf{X}_\Theta = \chi_\Theta(\mathbf{X}, t)$ which carries points $\mathbf{X}$ located at Ω_0 to points $\mathbf{X}_\Theta$ in the intermediate configuration Ω_Θ. A typical point $\mathbf{X}_\Theta$ is characterized by the coordinates $(X_{\Theta 1}, X_{\Theta 2}, X_{\Theta 3})$.

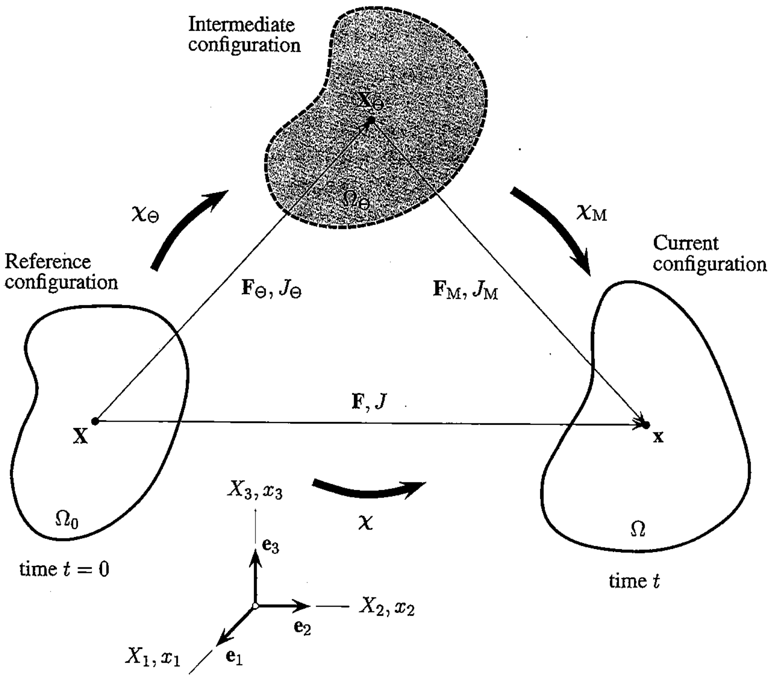

Figure 7.5 Multiplicative decomposition of the thermoelastic deformation into a purely *mechanical* contribution $\mathbf{F}_\mathrm{M}$, $J_\mathrm{M} = \det\mathbf{F}_\mathrm{M}$ and a purely *thermal* contribution $\mathbf{F}_\Theta$, $J_\Theta = \det\mathbf{F}_\Theta$.

According to the multiplicative split (7.99) we have defined, additionally, a macroscopic motion $\mathbf{x} = \chi_\mathrm{M}(\mathbf{X}_\Theta, t)$ at a constant (fixed) temperature Θ along with the *stress-producing* deformation gradient $\mathbf{F}_\mathrm{M}$ and the volume change J_M. A so-called **mechanically incompressible material** for which $J_\mathrm{M} = \det\mathbf{F}_\mathrm{M} = 1$, keeps the volume constant during a motion χ_M.

EXAMPLE 7.4 Consider a mechanically incompressible and thermoelastic material under a non-isothermal deformation process. The thermoelastic material is assumed to be thermally isotropic so that the deformation gradient $\mathbf{F}_\Theta$ may be given by an isotropic tensor according to

$$\mathbf{F}_\Theta = F(\Theta)\mathbf{I} \quad , \qquad F(\Theta) = \exp[\int\limits_{\Theta_0}^{\Theta} \alpha(\hat{\Theta})\mathrm{d}\hat{\Theta}] > 0 \quad , \tag{7.102}$$

where $F(\Theta)$ is a scalar-valued scalar function determining the volume change relative to the reference configuration. In eq.(7.102)$_2$ we have assumed a particularization of $F(\Theta)$, where $\alpha = \alpha(\Theta)$ denotes the temperature dependent expansion coefficient.

Determine an expression for the total volume change J of the material due to the non-isothermal deformation process. Linearize the result by assuming a constant value $\alpha_0 = \alpha(\Theta_0)$ for the expansion coefficient at a reference temperature Θ_0.

Solution. Since the considered thermoelastic material is mechanically incompressible (no volume change during an isothermal process) we introduce the constraint condition through $J_{\mathrm{M}} = 1$. Thus, the total volume change J within a non-isothermal deformation process from region Ω_0 to Ω is characterized by eq. (7.99)$_2$ which degenerates, using eqs. (7.101)$_2$ and (7.102), to

$$J = J_\Theta(\Theta) = \det\mathbf{F}_\Theta = \exp[\int\limits_{\Theta_0}^{\Theta} 3\alpha(\hat{\Theta})\mathrm{d}\hat{\Theta}] > 0 \quad . \tag{7.103}$$

With the linear expansion coefficient α_0, relation (7.103)$_3$ gives

$$J = \exp[3\alpha_0(\Theta - \Theta_0)] \quad , \tag{7.104}$$

and linearization leads to the approximate solution

$$J \approx 1 + 3\alpha_0(\Theta - \Theta_0) \quad . \tag{7.105}$$

It defines the total volume change J of a mechanically incompressible and thermally isotropic material within the infinitesimal strain theory. The approximate solution (7.105) is a well-known relation in linear continuum mechanics. It may be viewed as the volume of a unit cube at temperature Θ with values α_0 and Θ_0 which correspond to a reference state. ■

Entropic elasticity for a stretched piece of rubber. In the following, attention will be confined to the thermoelastic description of isotropic and entropic elastic materials (such as elastomers) at finite strains. The stress-deformation-temperature response of a piece of rubber under simple tension, in particular, is examined. We assume that the material under consideration obeys the modified entropic theory of rubber thermoelasticity. Furthermore, the material is assumed to be mechanically incompressible, which motivates the use of a multiplicative split of the thermoelastic deformation, as introduced above. Consequently, the total volume change is given in (7.103)$_1$, i.e. $J = J_\Theta(\Theta)$ ($\det\mathbf{F}_{\mathrm{M}} = 1$).

Now we consider a thin sheet of rubber stretched in one direction from the reference (undeformed) state to $\lambda_1 = \lambda$ (simple tension). Then, obeying condition $J = J_\Theta(\Theta) = \lambda_1\lambda_2\lambda_3$, we deduce by symmetry that $\lambda_2 = \lambda_3 = (J/\lambda)^{1/2}$. Hence, the Helmholtz free-energy function per unit reference volume

$$\Psi(\lambda, \Theta) = \Psi(\lambda, (J/\lambda)^{1/2}, \Theta) \tag{7.106}$$

is given in terms of one independent mechanical variable and one thermal variable, i.e. the stretch ratio λ and the temperature Θ. For notational convenience we use the same Greek letter Ψ for different free-energy functions.

Since the material is isotropic it is appropriate to use the thermodynamic extension of Ogden's model, as discussed in Section 7.7 (see eqs. (7.80)–(7.84)). Recall that, in general, one contribution to the free energy is due to volume changes and purely thermal causes (compare with relation (7.81)$_1$). Since we study a mechanically incompressible material we need only to consider the purely thermal contribution. We use, without loss of generality, the standard form (7.73), i.e. $T(\Theta) = c_0[\vartheta - \Theta\ln(\Theta/\Theta_0)]$, where the specific heat capacity $c_0 > 0$ is a positive constant.

Hence, the free energy relative to the reference configuration, which is stress-free and free of thermal expansion (or contraction), results from eqs. (7.80)–(7.82) and (7.84), using the temperature change $\vartheta = \Theta - \Theta_0$ and the specified kinematic relations, in the form

$$\begin{aligned} \Psi(\lambda, \Theta) = {} & \frac{\Theta}{\Theta_0}\sum_{p=1}^{N}\frac{\mu_p(\Theta_0)}{\alpha_p}\left[\lambda^{\alpha_p} + 2\left(\frac{J}{\lambda}\right)^{\alpha_p/2} - 3\right] \\ & + c_0\left[(\Theta - \Theta_0) - \Theta\ln\left(\frac{\Theta}{\Theta_0}\right)\right] \quad , \end{aligned} \tag{7.107}$$

where the additional condition (7.79) must be enforced. Note that the volume change due to thermal expansion governed by $J = J_\Theta(\Theta)$ is of considerable importance, which will be pointed out in more detail within the Example 7.5 below.

Next, we derive the associated thermal equations of state, namely the *stress* and *entropy functions*, as given in eqs. (7.24) and (7.25). For simple tension we may write

$$P = \frac{\partial\Psi(\lambda, \Theta)}{\partial\lambda} \quad \text{and} \quad \eta = -\frac{\partial\Psi(\lambda, \Theta)}{\partial\Theta} \quad , \tag{7.108}$$

with given free energy Ψ substituted from (7.107),

$$P = \frac{\Theta}{\Theta_0}\sum_{p=1}^{N}\frac{\mu_p(\Theta_0)}{\lambda}\left[\lambda^{\alpha_p} - \left(\frac{J}{\lambda}\right)^{\alpha_p/2}\right] \quad , \tag{7.109}$$

$$\eta = -\frac{1}{\Theta_0}\sum_{p=1}^{N}\frac{\mu_p(\Theta_0)}{\alpha_p}\left[\lambda^{\alpha_p} + (2+3\alpha_0\Theta\alpha_p)\left(\frac{J}{\lambda}\right)^{\alpha_p/2} - 3\right] + c_0 \ln\left(\frac{\Theta}{\Theta_0}\right) \quad . \qquad (7.110)$$

Relations (7.109) and (7.110) specify the (non-zero) nominal stress $P_1 = P$ (also called the first Piola-Kirchhoff stress) $P_2 = P_3 = 0$, and the entropy η which are generated by the stretch ratio λ and the temperature Θ.

In order to describe the coupled thermomechanical problem of the piece of stretched rubber completely we must add a constitutive equation for the heat flux vector governing heat transfer. For the class of thermally isotropic materials we may adopt Fourier's law of heat conduction and refer to eq. (7.97). Having in mind the specified kinematic relations, the inverse of the right Cauchy-Green tensor, $\mathbf{C}^{-1}$, as needed in eq. (7.97), may be given in the form of its matrix representation

$$[\mathbf{C}^{-1}] = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda/J & 0 \\ 0 & 0 & \lambda/J \end{bmatrix} \quad , \qquad (7.111)$$

where the diagonal elements are the eigenvalues of $\mathbf{C}^{-1}$.

EXAMPLE 7.5 Consider the stretching of a mechanically incompressible piece of rubber, for example a rubber band, obeying the modified entropic theory of rubber thermoelasticity. The rubber band is elongated rapidly so that no time remains for isothermal removal of heat. Hence, the homogeneous deformation process is viewed as an adiabatic process for which the heat flux on the boundary surface is zero and, in addition, heat sources are zero (thermal energy cannot be generated or destroyed within the material). The non-isothermal deformation process is assumed to be reversible.

In the present example attention is paid to the effects of *structural thermoelastic heating (or cooling)* and to the *stress-strain-temperature response* of the rubber band. In particular, show how the nominal stress P depends on the temperature change $\vartheta = \Theta - \Theta_0$ at a fixed elongation, i.e. a fixed stretch λ, and derive the temperature evolution of the rubber band with stretch. Finally, discuss the results of this classical example of rubber thermoelasticity, which demonstrates one of the great differences between *rubber* and *'hard' solids*, namely the distinctive effects of temperature.

As known from Section 6.5 the Ogden model excellently replicates the finite extensibility domain of rubber-like materials. Hence, as a basis for the constitutive model take the thermodynamic extension of Ogden's model with three pairs of constants ($N = 3$) characterized by the Helmholtz free-energy function (7.107). The constants α_p, $\mu_p(\Theta_0)$, $p = 1, 2, 3$, at the reference temperature Θ_0 are those given by OGDEN [1972a], listed in eq. (6.121) of this text. The volume change due to thermal expansion (or contraction) $J = J_\Theta(\Theta)$ is assumed to be governed by relation (7.104). The specific heat capacity and the linear expansion coefficient are given by $c_0 = 1.83 \cdot 10^3 \mathrm{Nm/kgK}$ and $\alpha_0 = 22.333 \cdot 10^{-5}\mathrm{K}^{-1}$ (see CYR [1988]), respectively. We assume that for the reference state the specimen and its environment constitute a system, which is in thermodynamic equilibrium. The homogeneous temperature field is given by the value $\Theta_0 = 293.15\mathrm{K}$ (= 20°C).

Solution. The relationship between the nominal stress P and the temperature change $\vartheta = \Theta - \Theta_0$ is given explicitly by the theoretical solution (7.109), with volume change (7.104), and by experimental results due to ANTHONY et al. [1942]. The respective diagram for various fixed values of λ is depicted in Figure 7.6.

The figure is supplemented by a curve indicating the evolution of the temperature change ϑ with respect to the stretch ratio λ. Since the adiabatic process is assumed to be reversible the energy balance equation (4.142) reduces to $\Theta\dot{\eta} = 0$. Consequently, the entropy ($\eta = \text{const}$) possessed by the material remains constant, which indicates that the deformation process is also isentropic. The condition $\eta = \text{const}$ gives the relation between the temperature change ϑ and the stretch λ, which is based upon the constitutive equation for the entropy, as derived in eq. (7.110), with eq. (7.104). The analytical solution is completed by experimental data due to JOULE [1859].

The qualitative fit to the experimental data given by JOULE [1859] and ANTHONY et al. [1942] is satisfying; nevertheless, the physical properties of the vulcanized rubber strip used in *Joule's* and *Anthony's* experiments are only partly documented and may differ from those given above.

One of the very first works reporting simple observations regarding thermal effects due to deflections of vulcanized rubber bands was published by GOUGH [1805]. The first work which explored experimentally the crucial physical properties of elastic india-rubber was presented by JOULE [1859]. The phenomena of structural thermoelastic heating (or cooling) of stressed rubber-like materials is called the *Gough-Joule effect*. A wide range of publications followed over more than one century, which mostly deal with theoretical formulations of more or less approximate theories. The history of these investigations is discussed in the classical book by FLORY [1953, Section XI.1a], an overview is given by TRELOAR [1975, Chapters 2, 5], CHADWICK [1974] and PRICE [1976, and references therein]. For a detailed account of the relevant results see also OGDEN [1992b].

Numerical analyses for this type of adiabatic (isentropic) simple tension test within the finite element context were given by MIEHE [1995b] and HOLZAPFEL and SIMO [1996b]. The algorithmic solution procedure reduces merely to *one* step, which is concerned with solving a mechanical problem with *isentropic elasticities* as indicated explicitly through expression (7.58). During the solution process the entropy constraint condition $\Theta\dot{\eta} = 0$ must be enforced, which means that the entropy at each time-step

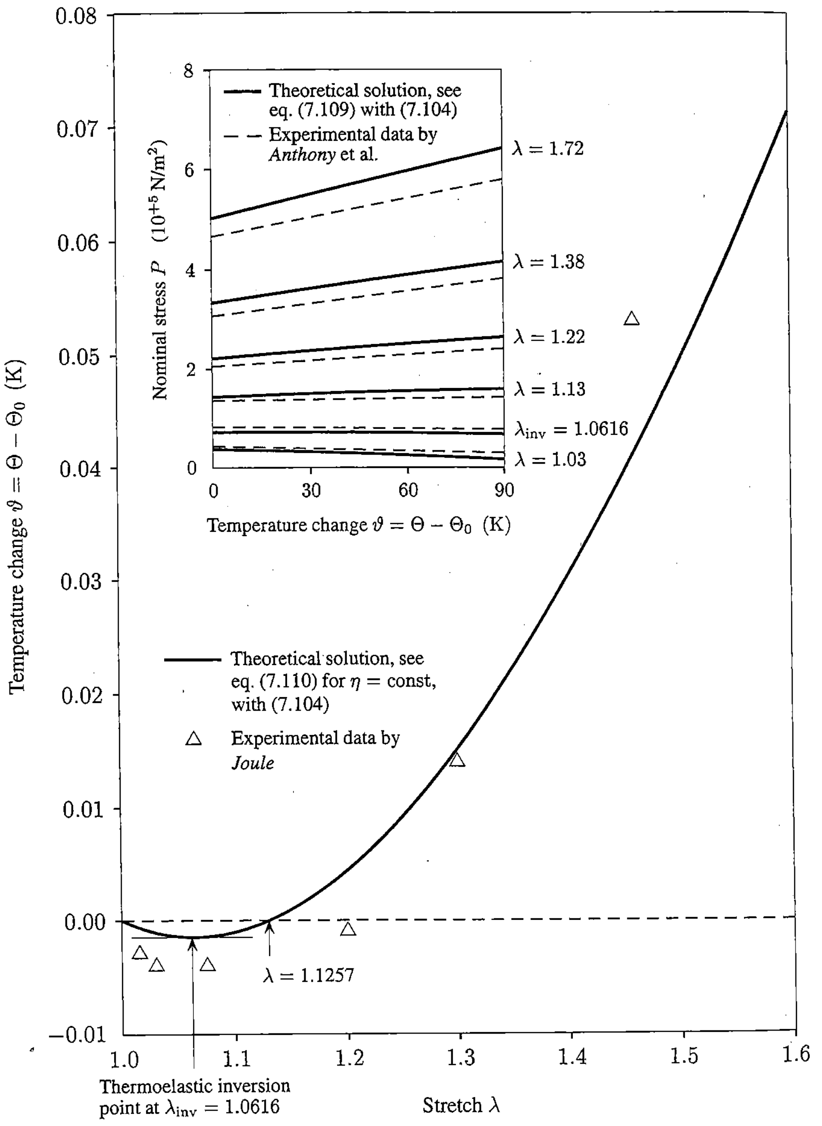

Figure 7.6 Temperature rise of a rubber band due to adiabatic stretching – showing the *thermoelastic inversion point* – and nominal stress P versus temperature change $\vartheta = \Theta - \Theta_0$ for various stretch ratios λ for a three-term Ogden elastic material.

is unchanged. As a direct result the temperature and the displacement field follow according to Figure 7.6.

Thermoelastic inversion point. For a small stretch ratio λ the rubber band indicates an initial *cooling effect* (see Figure 7.6), which increases first with deformation and changes to a *heating effect* at a certain minimum point, which entered into the literature as the so-called **thermoelastic inversion point**. The associated extension, labeled as λ_{inv}, characterizes a configuration of the sample in which the acting force is independent of the temperature.

To gain more insight in the interesting thermoelastic inversion phenomena the following observations are emphasized:

(i) An analytical investigation of the reversible adiabatic (isentropic) process – characterized by constitutive relation (7.110), with η = const – leads, within an adequate linearization process, to an explicit expression for the stretch at the inversion point, denoted by λ_{inv}. Specifically, from $\partial\Theta/\partial\lambda = 0$ we obtain the approximate solution

$$\lambda_{\text{inv}}^3 \approx 1 + 3\alpha_0\Theta_0 \tag{7.112}$$

(see JAMES and GUTH [1943] among many others). With given data thermoelastic inversion takes place for an extension of about $\lambda_{\text{inv}} = 1.0616$ at which the maximum temperature drop occurs (see Figure 7.6).

A second point of interest is at $\vartheta = 0$, where a heating process takes place. Note that this behavior is contrary to that for a metallic spring, which cools continuously on stretching within the elastic domain. An approximate solution for λ at $\vartheta = 0$ may be derived from eq. (7.110) using an adequate linearization process. We obtain a quadratic equation in λ, i.e. $\lambda^2 + \lambda - 2\lambda_{\text{inv}}^3 \approx 0$ (see also JAMES and GUTH [1943]). Remarkably, the stretch ratio at $\vartheta = 0$ is only influenced by the location of the thermoelastic inversion point; with the given data the numerical value is $\lambda = 1.1257$ (see Figure 7.6).

(ii) In view of eq. (7.109) it is crucial to remark that for a fixed λ the stress is a nonlinear function with respect to the temperature. The nonlinearity is clearly caused by the volume thermal expansion term $J = J_\Theta(\Theta)$ characterized by eq. (7.104).

As a direct consequence the term J_Θ governs the change in the slope $\partial P/\partial\Theta$ of the stress-temperature curves, as seen in Figure 7.6. If the elongation is small the slope is negative and the deformation behavior is dominated by thermal expansion (energetic contribution). For larger strain ranges the slope $\partial P/\partial\Theta$ becomes positive. It emerges that the reversal in the slope of the weak nonlinear stress-temperature plot is indicated at $\lambda = \lambda_{\text{inv}}$, which can be shown using eq. (7.109) along with condition $\partial P/\partial\Theta = 0$.

At the thermoelastic inversion point thermal expansion and entropy contraction balance. Hence, thermoelastic inversion is governed clearly by *thermal expansion*, which is expressed by relation (7.104), i.e. $J = \exp[3\alpha_0(\Theta - \Theta_0)]$. As can be seen from (7.112), thermoelastic inversion depends basically on the linear expansion coefficient $\alpha_0 \neq 0$. Alternatively, it is interesting to note that the inversion point also occurs at the same value of extension by fixing the load of a strip of rubber and increasing the temperature. A connection of the mentioned observations may be found within the context of the thermodynamic Maxwell relations (7.39) and (7.40), which motivates the inclusion of the two diagrams within one figure (Figure 7.6) (see also the theoretical study by OGDEN [1992b]).

By setting $\alpha_0 = 0$ (which gives $e_0(J) = 0$) we may recover the purely entropic theory of rubber thermoelasticity as a special case. With the assumed constant specific heat capacity c_0 the temperature evolution is then given by an explicit function, i.e. $\Theta = \Theta_0 \exp[\Psi_0/(\Theta_0 c_0)]$ (compare with Exercise 2 on p. 336). Here, $\Psi_0 = \Psi(\lambda, \Theta_0)$ denotes the strain energy for the deformation at a fixed reference temperature Θ_0, i.e. eq. (7.107) with $\Theta = \Theta_0$. ■

One-dimensional finite thermoelasticity. We consider here the one-dimensional case of finite thermoelasticity. A rod (with uniform cross-section) is imagined as being stretched to λ in the direction of its (x-)axis, with the associated kinematic relation $x = \lambda X$ (uniform non-isothermal deformation). The rod capable of supporting finite thermoelastic deformations admits the *stretch ratio* λ and the *temperature* Θ as independent mechanical and thermal variables.

From the Helmholtz free-energy function $\Psi = \Psi(\lambda, \Theta)$ (measured per unit reference volume), which is assumed to exist, we are able to deduce the stress and entropy functions $P = P(\lambda, \Theta)$ and $\eta = \eta(\lambda, \Theta)$, respectively. They are nonlinear scalar-valued functions which depend on the two scalar variables λ and Θ and have the same form as (7.108). Note that the constitutive relation $(7.108)_1$ determines the nominal stress P by keeping the temperature Θ fixed while $(7.108)_2$ determines the entropy η at a fixed stretch λ.

For a more profound understanding of the constitutive relations it is beneficial to note the differential mathematical relationship between the nominal stress P and the entropy η. For the continuous function Ψ the well-known property

$$\frac{\partial}{\partial \Theta}\left(\frac{\partial \Psi}{\partial \lambda}\right)_\Theta = \frac{\partial}{\partial \lambda}\left(\frac{\partial \Psi}{\partial \Theta}\right)_\lambda \tag{7.113}$$

holds. Using eqs. $(7.108)_1$ and $(7.108)_2$, (7.113) can be rewritten in an expression which relates the stress and the entropy function according to

$$\left(\frac{\partial P}{\partial \Theta}\right)_\lambda = -\left(\frac{\partial \eta}{\partial \lambda}\right)_\Theta \quad . \tag{7.114}$$

This identity shows that the change in stress P with the temperature Θ of, for example, a rod at fixed stretch λ is equal to the negative value of the change in entropy η with the stretch ratio λ of that rod at fixed temperature Θ. Relation (7.114) characterizes the **thermodynamic Maxwell relation** for the one-dimensional case, fundamental in rubber thermoelasticity (see WALL [1965, p. 314], TRELOAR [1975, p. 30] and CARLSON [1972, p. 304]). Compare also the identities (7.39) and (7.40), valid for the three-dimensional case.

Furthermore, it is important to note that for polymers in the 'rubbery' state the term on the left-hand side of eq. (7.114) is found to be positive at large extensions λ, and negative at small extensions, as seen clearly in Figure 7.6 (recall Example 7.5 on p. 348).

All that remains is the computation of the isothermal elasticity tensor in the material description (consistent linearized tangent moduli) and the referential stress-temperature tensor. With this aim in view we derive the change in the second Piola-Kirchhoff stress $S = S(\lambda, \Theta)$, defined as $S = \lambda^{-1} P = \lambda^{-1}(\partial \Psi(\lambda, \Theta)/\partial \lambda)$ (compare with relations (6.50), with (6.47), valid for three dimensions). Knowing that the Green-Lagrange strain, denoted by E, is $(\lambda^2 - 1)/2$ (compare with relation (2.69), valid for three dimensions), we may find

$$\begin{aligned} \mathrm{d}S &= \left(\frac{\partial S}{\partial \lambda}\right)_\Theta \mathrm{d}\lambda + \left(\frac{\partial S}{\partial \Theta}\right)_\lambda \mathrm{d}\Theta \\ &= C\mathrm{d}E + T\mathrm{d}\Theta \quad , \end{aligned} \tag{7.115}$$

where $\mathrm{d}E = \lambda \mathrm{d}\lambda$. Here we have introduced the definitions

$$C = \lambda^{-1}\frac{\partial S(\lambda, \Theta)}{\partial \lambda} \qquad \text{and} \qquad T = \frac{\partial S(\lambda, \Theta)}{\partial \Theta} \tag{7.116}$$

of the *isothermal elasticity tensor* C in the material description (fixed temperature during the process) and the *referential stress-temperature tensor* T.

The scalar-valued functions $(7.115)_2$ and (7.116) are the one-dimensional counterparts of the tensor-valued functions (7.48) and $(7.49)_1$, $(7.50)_1$, respectively. Observe that by means of $S = \lambda^{-1} P$ and the thermodynamic Maxwell relation (7.114) we may find an equivalent of the referential stress-temperature tensor $(7.116)_2$ in the form

$$\lambda T = \left(\frac{\partial P}{\partial \Theta}\right)_\lambda = -\left(\frac{\partial \eta}{\partial \lambda}\right)_\Theta \quad . \tag{7.117}$$

In order to complete our presentation of one-dimensional finite thermoelasticity we have to add a constitutive equation for the heat flux. For one dimension, Fourier's law of heat conduction in a coupled thermomechanical regime reads

$$Q(\lambda, \Theta, \Theta') = -k_0(\Theta)\lambda^{-2}\Theta' \ . \qquad (7.118)$$

The temperature gradient along the axis of the considered rod is denoted by Θ' while Q denotes the Piola-Kirchhoff heat flux, which is, in the one-dimensional case, a scalar-valued function.

EXERCISES

1. Consider the Helmholtz free-energy function $\Psi = \Psi(\lambda, \Theta)$ in terms of the stretch ratio λ and the temperature Θ characterizing a *one-dimenional* constitutive problem of finite thermoelasticity. According to relations (7.108) we may derive constitutive equations for the nominal stress $P = P(\lambda, \Theta)$ and the entropy $\eta = \eta(\lambda, \Theta)$. Assume that the specific heat capacity is a positive constant $c_0 > 0$.
   (a) Using the chain rule, show that the change in entropy may be expressed as

   $$\mathrm{d}\eta = -T\mathrm{d}E + \frac{c_0}{\Theta}\mathrm{d}\Theta \ , \qquad (7.119)$$

   where $\mathrm{d}E = \lambda\mathrm{d}\lambda$. The referential stress-temperature tensor T is given in eq. $(7.116)_2$ (or by the equivalent of eq. (7.117)).

   (b) Using eqs. $(7.115)_2$ and (7.119) show that the isentropic elasticity tensor C^{ise} in the material description (for a fixed entropy η during a process) is governed by the relationship

   $$C^{\mathrm{ise}} = C + \frac{\Theta}{c_0}T^2 \ . \qquad (7.120)$$

   Note that relations (7.119), (7.120) are the one-dimenional counterparts of relations (7.54), (7.58).

2. Suppose that a rod (considered as a one-dimensional structure) admits the stretch ratio λ and the entropy η as independent variables and consider the existence of the internal-energy function per unit reference volume in the form of $e = e(\lambda, \eta)$. Derive the second Piola-Kirchhoff stress $S = S(\lambda, \eta)$ and show that its change is $\mathrm{d}S = C^{\mathrm{ise}}\mathrm{d}E + T^{\mathrm{ise}}\mathrm{d}\eta$, with the definitions

   $$C^{\mathrm{ise}} = \lambda^{-1}\frac{\partial S(\lambda, \eta)}{\partial \lambda} \ , \qquad T^{\mathrm{ise}} = \frac{\partial S(\lambda, \eta)}{\partial \eta}$$

   of the *isentropic elasticity tensor* C^{ise} in the material description and the *referential stress-entropy tensor* T^{ise}, evaluated at (λ, η) (compare with Section 7.5 for the three-dimensional case, in particular, with relations $(7.56)_1$ and $(7.57)_1$).

3. Consider an adiabatic (isentropic) stretching of a mechanically incompressible rubber band and study the homogeneous deformation process in the large strain domain up to a stretch ratio $\lambda = 8$. Take the material properties and assumptions according to Example 7.5 (see p. 348). As a basis for the constitutive model take the thermodynamic extension of Ogden's model, with $N = 3$, and compare with the coupled thermomechanical Mooney-Rivlin model, by setting $N = 2$, and the neo-Hookean and Varga models, by setting $N = 1$.

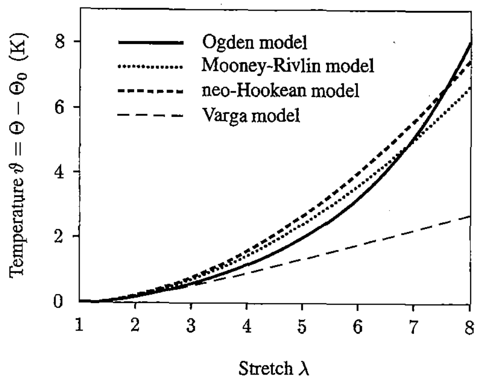

Figure 7.7 Temperature rise ϑ of a rubber band due to adiabatic stretching for the large strain domain. Comparison between four different thermoelastic models.

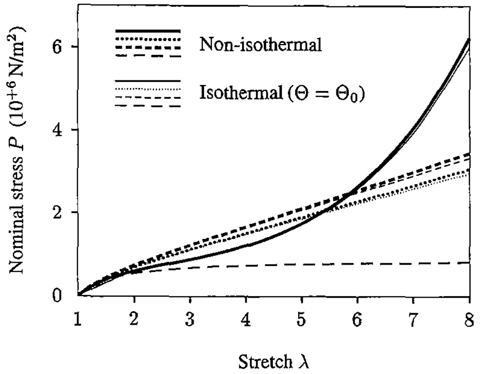

Figure 7.8 Nominal stress P versus stretch ratio λ. Comparison between a coupled thermomechanical (non-isothermal) and a decoupled (isothermal) deformation process.

In particular, for $N = 2$, assume $\alpha_1 = 2$, $\alpha_2 = -2$ and $\mu_1(\Theta_0) = 0.875\mu_0$, $\mu_2(\Theta_0) = -0.125\mu_0$, so that $\mu_1(\Theta_0)/\mu_2(\Theta_0) \doteq -7$ (see ANAND [1986]), with the shear modulus $\mu_0 = 4.225 \cdot 10^5 \mathrm{N/m}^2$ in the reference configuration. For the neo-Hookean model ($N = 1$) assume $\alpha_1 = 2$ and $\mu_1(\Theta_0) = \mu_0$ and for the Varga model ($N = 1$), $\alpha_1 = 1$ and $\mu_1(\Theta_0) = 2\mu_0$.

(a) Based on the Ogden, Mooney-Rivlin, neo-Hookean and Varga material models derive the temperature evolution $\vartheta = \Theta - \Theta_0$ of the rubber band due to the stretch ratio λ.

(b) Derive a relationship between the nominal stress P and the stretch ratio λ for a non-isothermal deformation process. Show the difference compared with the (classical) solution, which is based on the isothermal theory (set $\Theta = \Theta_0$). Note that for an isothermal deformation process the free energy Ψ reduces to the strain energy, as given in (6.119) and eqs. (6.127)–(6.129).

Figures 7.7 and 7.8 show a comparison between the thermoelastic Ogden, Mooney-Rivlin, neo-Hookean and Varga material models. Observe the sharp upturn in the stress at high elongations ($\lambda \geq 6$) for the Ogden material model as indicated in Figure 7.8. This fact may partly be explained physically within more advanced theories obeying Langevin distribution function (non-Gaussian statistical theory), as discussed by, for example, TRELOAR [1975, Chapter 6]. The observed rise of stress is mainly caused by limited extensibility of the polymer chains themselves and by strain-induced crystallization (see FLORY [1976] for further insight).

4. A bicycle wheel with spokes made of rubber bands is mounted with the axle horizontal. The spokes are in tension so that the rim is kept in place. An electric heat plate is placed on the right-hand side of the wheel (see Figure 7.9) so that the spokes on one side of the wheel are heated. As a consequence, the bicycle wheel starts to rotate counterclockwise, as long as heat is induced. Explain this effect.

   This amusing device may be viewed as a **Carnot thermal engine**, in which rubber alone constitutes the working substance. The Carnot thermal engine is alternately subjected to two adiabatic and two isothermal processes. Investigate a hypothetical thermodynamic Carnot cycle for rubber-like materials. In particular, discuss the temperature change ϑ of a rubber band which occurs during the four processes as a function of the stretch λ and the entropy η. For a detailed exposition of the relevant results the reader is referred to the paper by HOLZAPFEL and SIMO [1996b].

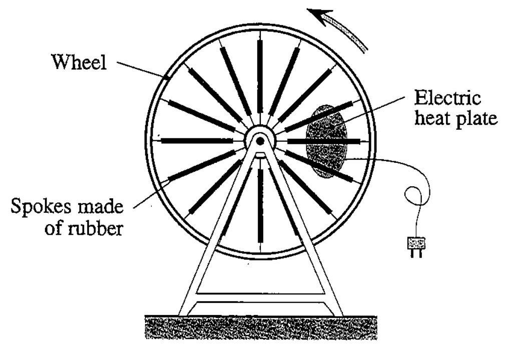

Figure 7.9 A wheel with rubber spokes starts to rotate counterclockwise when heated on one side.

7.9 Thermodynamics with Internal Variables

In this section we link together *finite elasticity* and *non-equilibrium thermodynamics*. We consider the thermodynamics of continuous media within the large strain regime and apply the theory of **finite thermoviscoelasticity**.

We use a thermodynamic approach with internal variables which leads to a very general description of materials involving irreversible (dissipative) effects, such as damage, relaxation and/or creep and plastic deformations. It generalizes finite thermoelasticity, as outlined in Section 7.3, in the sense that additional thermodynamic variables (known as internal variables) are incorporated with the aim of representing the irreversible mechanism of the (inelastic) structural material behavior. A general discussion of constitutive models with internal variables was emphasized in Section 6.9.

A fully coupled three-dimensional thermomechanical model for viscous materials is examined. It is particularly suited for the thermoviscoelastic behavior of dissipative elastomeric (rubber-like) materials under varying temperatures at finite strains. Constitutive equations for the stress, the entropy and the internal variables are specified.

Finite thermoviscoelasticity. We define a Helmholtz free-energy function (measured per unit reference volume) as

$$\Psi = \Psi(\mathbf{F}, \Theta, \xi_1, \ldots, \xi_m) \ . \tag{7.121}$$

The thermodynamic state is completely characterized through the set of independent variables $(\mathbf{F}, \Theta, \xi_1, \ldots, \xi_m)$, i.e. the deformation gradient $\mathbf{F}$, the absolute temperature

Θ and m additional internal variables $\boldsymbol{\xi}_\alpha$, $\alpha = 1,\ldots,m$. For the case of thermoviscoelasticity the tensor variables $\boldsymbol{\xi}_\alpha$ represent the thermoviscoelastic contribution to the material response. Note that the variables $\boldsymbol{\xi}_\alpha$ may also represent *damage* and/or *plastic* mechanisms.

A material which is characterized by the free energy (7.121) for any point and time we call a **thermoviscoelastic material**. The behavior of a thermoviscoelastic material is assumed to be governed by $\alpha = 1,\ldots,m$ relaxation (retardation) processes with given relaxation (retardation) times $\tau_\alpha(\Theta) \in (0,\infty)$, $\alpha = 1,\ldots,m$, which are, in general, temperature dependent.

The next aim is to derive the complete set of the constitutive equations for the first Piola-Kirchhoff stress tensor $\mathbf{P}$ and the entropy η (per unit reference volume) in the general form. For that purpose we follow the standard methods presented in Sections 6.9, 7.3 and use the Clausius-Planck form of the second law of thermodynamics (4.153), i.e. the internal dissipation inequality $\mathcal{D}_{\text{int}} = \mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} - \eta\dot{\Theta} \geq 0$ (the Clausius-Planck form assumes non-negative entropy production due to conduction of heat, i.e. $-(1/\Theta)\mathbf{Q} \cdot \text{Grad}\Theta \geq 0$ (compare with Section 4.6)). By means of the chain rule, time differentiation of the free energy $\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)$ gives the hypothetical change in the thermodynamic state, that is the **Gibbs relation** for thermoviscoelastic materials, namely

$$\begin{aligned}\dot{\Psi} &= \dot{\Psi}(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) = \mathbf{P} : \dot{\mathbf{F}} - \eta\dot{\Theta} - \mathcal{D}_{\text{int}} \\ &= \left(\frac{\partial\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)}{\partial\mathbf{F}}\right)_{\Theta,\boldsymbol{\xi}_\alpha} : \dot{\mathbf{F}} + \left(\frac{\partial\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)}{\partial\Theta}\right)_{\mathbf{F},\boldsymbol{\xi}_\alpha} \dot{\Theta} \\ &\quad + \sum_{\alpha=1}^{m} \left(\frac{\partial\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)}{\partial\boldsymbol{\xi}_\alpha}\right)_{\mathbf{F},\Theta} : \dot{\boldsymbol{\xi}}_\alpha \ ,\end{aligned} \tag{7.122}$$

where $\dot{\boldsymbol{\xi}}_\alpha$, $\alpha = 1,\ldots,m$, denote the internal strain rates.

Since the rates $\dot{\mathbf{F}}$ and $\dot{\Theta}$ can be chosen arbitrarily we find the constitutive equations

$$\mathbf{P} = \left(\frac{\partial\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)}{\partial\mathbf{F}}\right)_{\Theta,\boldsymbol{\xi}_\alpha} , \qquad \eta = -\left(\frac{\partial\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)}{\partial\Theta}\right)_{\mathbf{F},\boldsymbol{\xi}_\alpha} \tag{7.123}$$

for the first Piola-Kirchhoff stress $\mathbf{P}$ and the entropy η, and we deduce a remainder inequality

$$\mathcal{D}_{\text{int}} = \sum_{\alpha=1}^{m} \boldsymbol{\Xi}_\alpha : \dot{\boldsymbol{\xi}}_\alpha \geq 0 \ , \qquad \boldsymbol{\Xi}_\alpha = -\left(\frac{\partial\Psi(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)}{\partial\boldsymbol{\xi}_\alpha}\right)_{\mathbf{F},\Theta} . \tag{7.124}$$

The inequality $\mathcal{D}_{\text{int}} \geq 0$ characterizes the internal dissipation in the viscous material which generates heat in an irreversible manner.

The defined tensor variables $\boldsymbol{\Xi}_\alpha$, $\alpha = 1,\ldots,m$, correspond to the internal tensor variables $\boldsymbol{\xi}_\alpha$ according to the internal constitutive equations $(7.124)_2$, which is the thermodynamic extension of relation (6.233) (or (6.244)). By analogy with a linear solid, $\boldsymbol{\Xi}_\alpha$, $\alpha = 1,\ldots,m$, are to be interpreted as (internal) non-equilibrium stresses (compare with Section 6.10).

From physical expressions (7.123) and $(7.124)_2$ we deduce functions for the first Piola-Kirchhoff stress $\mathbf{P}$, the entropy η and the internal (stress-like) variables $\boldsymbol{\Xi}_\alpha$ which depend on $\mathbf{F}$, Θ, $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m$. Thus,

$$\mathbf{P} = \mathbf{P}(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) \ , \qquad \eta = \eta(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) \ , \tag{7.125}$$

$$\boldsymbol{\Xi}_\alpha = \boldsymbol{\Xi}_\alpha(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) \ , \qquad \alpha = 1,\ldots,m \ . \tag{7.126}$$

The fundamental inequality $(7.124)_1$, which characterizes internal dissipation, must be satisfied by a suitable set of evolution equations for the internal strain rates $\dot{\boldsymbol{\xi}}_\alpha$, $\alpha = 1,\ldots,m$, described generally in the form

$$\dot{\boldsymbol{\xi}}_\alpha(t) = \boldsymbol{\mathcal{A}}_\alpha(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) \ , \qquad \alpha = 1,\ldots,m \ . \tag{7.127}$$

The equations of evolution (rate equations) (7.127) describe the way in which an irreversible process evolves.

Note that the non-equilibrium stresses characterize the current 'distance from equilibrium' and vanish at the state of *thermodynamic equilibrium*. In view of (7.127) we may write the equations $\boldsymbol{\mathcal{A}}_\alpha(\mathbf{F},\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) = \mathbf{O}$, $\alpha = 1,\ldots,m$, as time t goes to infinity; further, $\boldsymbol{\Xi}_\alpha = -\partial\Psi/\partial\boldsymbol{\xi}_\alpha|_{t\to\infty} \equiv \mathbf{O}$, $\alpha = 1,\ldots,m$. This implies that with reference to $(7.124)_1$ the internal dissipation vanishes ($\mathcal{D}_{\text{int}} = 0$). Then equilibrium is reached and the values for stress and entropy remain constant. They are governed by the potential relations as derived in Section 7.3 (see eqs. (7.24)).

The limiting case of thermodynamic equilibrium states that the thermodynamic process is reversible and the continuum responds *fully thermoelastically*. All associated thermodynamic potentials, as outlined in Section 7.3 for finite thermoelasticity, are approximated asymptotically.

In summary: the response of a thermoviscoelastic material is defined through the constitutive equations (7.123) and $(7.124)_2$ (or (7.125) and (7.126)), the internal dissipation $(7.124)_1$ and the evolution equations as outlined generally in (7.127). In addition, these equations are supplemented by a suitable constitutive equation for the heat flux vector, necessary to determine heat transfer. The Piola-Kirchhoff heat flux $\mathbf{Q}$ may be introduced as a function of the deformation gradient, temperature, temperature gradient and internal variables, i.e.

$$\mathbf{Q} = \mathbf{Q}(\mathbf{F},\Theta,\text{Grad}\Theta,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) \ , \tag{7.128}$$

and must satisfy the inequality $\mathbf{Q} \cdot \text{Grad}\Theta \leq 0$.

Structural thermoviscoelastic heating (or cooling). The specific heat capacity $c_{\mathbf{F}}$ at constant deformation per unit reference volume was introduced and discussed in Section 7.4. Within the theory of thermodynamics with internal variables we define the specific heat capacity to have a positive value which depends on the deformation gradient $\mathbf{F}$, the temperature field Θ and, additionally, on the internal variables ξ_α, $\alpha = 1,\dots,m$. Thus, we write

$$c_{\mathbf{F}} = c_{\mathbf{F}}(\mathbf{F},\Theta,\xi_1,\dots,\xi_m) = -\Theta\left(\frac{\partial^2\Psi(\mathbf{F},\Theta,\xi_1,\dots,\xi_m)}{\partial\Theta\partial\Theta}\right)_{\mathbf{F},\xi_\alpha} > 0 \tag{7.129}$$

for all $(\mathbf{F},\Theta,\xi_1,\dots,\xi_m)$. Here the specific heat capacity is the energy required to produce unit increase in the temperature of a unit volume of the body keeping the *deformation* and the *internal variables fixed*. For notational convenience we shall use the same symbol $c_{\mathbf{F}}$ for the specific heat capacity introduced in Section 7.4.

Recall that within the theory of finite thermoelasticity we observe, in general, three different thermomechanical coupling effects (see p. 327). In addition, finite thermoviscoelasticity incorporates *viscous dissipation* according to $(7.124)_1$ and **structural thermoviscoelastic heating (or cooling)**, denoted $\mathcal{H}_{\text{in}}$ and defined as

$$\mathcal{H}_{\text{in}} = -\Theta\sum_{\alpha=1}^{m}\frac{\partial^2\Psi(\mathbf{F},\Theta,\xi_1,\dots,\xi_m)}{\partial\xi_\alpha\partial\Theta}:\dot{\xi}_\alpha \quad . \tag{7.130}$$

For the case in which the quantities ξ_α, $\alpha = 1,\dots,m$, represent plastic contributions, eq. (7.130) defines **structural inelastic (plastic) heating (or cooling)**.

By analogy with the derivation which led to the energy balance equation in temperature form (7.46) we proceed now by determining the change in entropy. With the equations of state $(7.125)_2$ and $(7.123)_2$ and by means of the chain rule we deduce that

$$\begin{aligned}\dot{\eta}(\mathbf{F},\Theta,\xi_1,\dots,\xi_m) &= \frac{\partial\eta}{\partial\mathbf{F}}:\dot{\mathbf{F}} + \frac{\partial\eta}{\partial\Theta}\dot{\Theta} + \sum_{\alpha=1}^{m}\frac{\partial\eta}{\partial\xi_\alpha}:\dot{\xi}_\alpha \\ &= -\frac{\partial^2\Psi}{\partial\mathbf{F}\partial\Theta}:\dot{\mathbf{F}} - \frac{\partial^2\Psi}{\partial\Theta\partial\Theta}\dot{\Theta} - \sum_{\alpha=1}^{m}\frac{\partial^2\Psi}{\partial\xi_\alpha\partial\Theta}:\dot{\xi}_\alpha\end{aligned} \tag{7.131}$$

(the arguments of the functions have been omitted for simplicity). By multiplying this equation with the temperature Θ and using eqs. $(7.44)_3$, $(7.129)_2$ and (7.130), we find that

$$\Theta\dot{\eta} = \mathcal{H}_{\text{e}} + c_{\mathbf{F}}\dot{\Theta} + \mathcal{H}_{\text{in}} \quad . \tag{7.132}$$

On comparison with relation (4.142) we obtain finally the energy balance equation in *temperature form*, i.e.

$$c_{\mathbf{F}}\dot{\Theta} = -\text{Div}\mathbf{Q} + \mathcal{D}_{\text{int}} - \mathcal{H}_{\text{e}} - \mathcal{H}_{\text{in}} + R \quad . \tag{7.133}$$

From the energy balance equation (7.46) we know that, within the theory of finite thermoelasticity, the evolution of the temperature Θ is influenced by the material divergence $\text{Div}\mathbf{Q}$ of the Piola-Kirchhoff heat flux $\mathbf{Q}$, the structural thermo*elastic* heating $\mathcal{H}_{\text{e}}$ and the heat source R. However, relation (7.133) indicates that due to viscous effects the quantity $c_{\mathbf{F}}\dot{\Theta}$ depends additionally on the internal dissipation $\mathcal{D}_{\text{int}}$ and the structural thermo*viscoelastic* heating $\mathcal{H}_{\text{in}}$, as defined in eqs. (7.124) and (7.130), and this is particularly important to the thermomechanical behavior of viscous materials.

A constitutive model for finite thermoviscoelasticity. Many materials which behave elastically at ordinary (room) temperatures display pronounced inelastic characteristics at elevated temperatures, solid polymers being important examples. The molecular network of vulcanized rubber exhibits (nearly) no stress relaxation in the low temperature range of the 'rubbery' state. However, in the temperature range of 100 to 150^{0}C stress relaxation-experiments of vulcanized rubber at constant deformation (see TOBOLSKY et al. [1944]) indicate a rapid stress-decay which may be explained by chemical rupture of the three-dimensional network. The phenomenon of stress relaxation of rubber is strongly influenced by temperature changes but is independent of both the deformed state of the network and the absence or presence of carbon-black fillers in the rubber (see TOBOLSKY et al. [1944], TOBOLSKY [1960, Chapter 5] and LEE et al. [1966]).

In the remaining part of this chapter we consider a three-dimensional constitutive model for dissipative continuous media capable of accommodating thermoviscoelastic changes at finite strains. The model is considered to be an extension of the three-dimensional viscoelastic model (proposed in Section 6.10) to the thermodynamic regime satisfying the entropy inequality principle. The mathematical structure of the thermoviscoelastic material model is based on the concept of internal state variables and is well-suited for numerical realization using the finite element method. The material model is based on a common assumption for viscous materials, namely that the evolution equations for the internal variables are linear. This simplification enables us to characterize states close to thermodynamic equilibrium. For thermoviscoelastic models that account for finite perturbations away from thermodynamic equilibrium the reader is referred to LION [1997b] and REESE and GOVINDJEE [1998b].

We choose a geometric setting relative to the reference configuration and postulate for non-isothermal processes a Helmholtz-free energy function Ψ (measured per unit reference volume) in the form

$$\Psi(\mathbf{C},\Theta,\mathbf{\Gamma}_1,\dots,\mathbf{\Gamma}_m) = \Psi_\infty(\mathbf{C},\Theta) + \sum_{\alpha=1}^{m}\Upsilon_\alpha(\mathbf{C},\Theta,\mathbf{\Gamma}_\alpha) \quad , \tag{7.134}$$

valid for some closed time interval $t \in [0,T]$ of interest. We require that

$$\Psi_\infty(\mathbf{I},\Theta_0) = 0 \quad , \qquad \sum_{\alpha=1}^{m}\Upsilon_\alpha(\mathbf{I},\Theta_0,\mathbf{I}) = 0 \quad , \tag{7.135}$$

where $\Theta_0 (> 0)$ is a given homogeneous reference temperature relative to a selected stress-free reference configuration. The set of independent variables $(\mathbf{C}, \Theta, \mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_m)$, i.e. the (symmetric) right Cauchy-Green tensor $\mathbf{C}$, the absolute temperature Θ and the (symmetric) internal variables $\mathbf{\Gamma}_\alpha$, $\alpha = 1, \ldots, m$ (not accessible to direct observation), completely characterizes the thermodynamic state. The internal variables $\mathbf{\Gamma}_\alpha$ are considered as inelastic (viscous) strains akin to the strain measure $\mathbf{C}$.

The first term $\Psi_\infty(\mathbf{C}, \Theta)$ in (7.134) characterizes the *equilibrium state* of the solid. We employ the subscript $(\bullet)_\infty$ to designate functions which represent the hyperelastic behavior of sufficiently slow processes. An efficient free energy Ψ_∞ describing the stress-strain-temperature response of rubber-like materials at finite strains, which is based on the concept of entropic elasticity, may be adopted from Section 7.7 (see eq. (7.80)).

The second term $\sum_{\alpha=1}^{m} \Upsilon_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha)$ in (7.134) represents the configurational free energy ('dissipative' potential) and characterizes the *non-equilibrium state* of the solid (relaxation and/or creep behavior). The potential Υ_α has to satisfy the thermodynamic restrictions imposed on the second law of thermodynamics (namely the non-negativeness of the internal dissipation) for any thermodynamic process. For a general form of Υ_α and a detailed discussion of this issue the reader is referred to HOLZAPFEL and SIMO [1996c].

Following arguments analogous to those which led from (7.121) to relations (7.123) and (7.124) we find, using the property (6.11) and $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$, physical expressions for the (symmetric) second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the entropy η (per unit reference volume) in the forms

$$\mathbf{S} = \mathbf{S}_\infty(\mathbf{C}, \Theta) + \sum_{\alpha=1}^{m} \mathbf{Q}_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha) \quad , \tag{7.136}$$

$$\eta = \eta_\infty(\mathbf{C}, \Theta) + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha) \quad , \tag{7.137}$$

and a remainder inequality

$$\mathcal{D}_{\text{int}} = -\sum_{\alpha=1}^{m} 2\frac{\partial \Upsilon_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha)}{\partial \mathbf{\Gamma}_\alpha} : \frac{1}{2}\dot{\mathbf{\Gamma}}_\alpha \geq 0 \quad , \tag{7.138}$$

where $\dot{\mathbf{\Gamma}}_\alpha$, $\alpha = 1, \ldots, m$, denote the internal strain rates. The inequality governs the non-negativeness of the internal dissipation $\mathcal{D}_{\text{int}}$ in the thermoviscoelastic material.

We have introduced the definitions

$$\mathbf{S}_\infty = 2\frac{\partial \Psi_\infty(\mathbf{C}, \Theta)}{\partial \mathbf{C}} \quad , \qquad \eta_\infty = -\frac{\partial \Psi_\infty(\mathbf{C}, \Theta)}{\partial \Theta} \quad , \tag{7.139}$$

$$\mathbf{Q}_\alpha = 2\frac{\partial \Upsilon_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha)}{\partial \mathbf{C}} \quad , \qquad \eta_\alpha = -\frac{\partial \Upsilon_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha)}{\partial \Theta} \tag{7.140}$$

of the contributions to the stress and the entropy, with $\alpha = 1, \ldots, m$.

As a result of the mathematical structure given in (7.134) the second Piola-Kirchhoff stress $\mathbf{S}$ and the entropy η are decomposed into *equilibrium* and *non-equilibrium parts* according to eqs. (7.136) and (7.137), respectively. The stress contribution $\mathbf{S}_\infty$ and the entropy contribution η_∞ are associated with the fully thermoelastic response which we describe within the framework of finite thermolasticity introduced in Section 7.3. In particular, we may adopt the relations $(7.26)_3$ and (7.27) by using Ψ_∞ instead of Ψ. According to definitions (7.140) the second terms in eqs. (7.136) and (7.137), i.e. $\sum_{\alpha=1}^{m} \mathbf{Q}_\alpha$ and $\sum_{\alpha=1}^{m} \eta_\alpha$, contribute to the stresses and the entropy and are responsible for the viscous response of the material.

The variables $\mathbf{Q}_\alpha$, $\alpha = 1, \ldots, m$, denote non-equilibrium stresses and are related (conjugate) to the (right Cauchy-Green) strain-like variables $\mathbf{\Gamma}_\alpha$. Hence, $\mathbf{Q}_\alpha$ are internal tensor variables with the internal constitutive equations

$$\mathbf{Q}_\alpha = -2\frac{\partial \Upsilon_\alpha(\mathbf{C}, \Theta, \mathbf{\Gamma}_\alpha)}{\partial \mathbf{\Gamma}_\alpha} \quad , \qquad \alpha = 1, \ldots, m \quad , \tag{7.141}$$

which restrict the configurational free energy Υ_α in view of eq. $(7.140)_1$ (compare also with the discussion in Section 6.10, in particular, eq. (6.244)). Hence, considering (7.138), the internal dissipation takes on the form $\mathcal{D}_{\text{int}} = \sum_{\alpha=1}^{m} \mathbf{Q}_\alpha : \dot{\mathbf{\Gamma}}_\alpha / 2 \geq 0$, and vanishes at the state of thermodynamic equilibrium, since $\mathbf{Q}_\alpha = -2\partial \Upsilon_\alpha / \partial \mathbf{\Gamma}_\alpha |_{t \to \infty} \equiv \mathbf{O}$, $\alpha = 1, \ldots, m$.

In order to describe thermoviscoelastic processes the proposed constitutive model must be complemented by suitable *equations of evolution* (rate equations). In particular, we want to specify the evolution of the non-equilibrium stresses $\mathbf{Q}_\alpha$, $\alpha = 1, \ldots, m$, involved. A simple set of linear evolution equations for $\mathbf{Q}_\alpha$ is assumed to have the form

$$\dot{\mathbf{Q}}_\alpha + \frac{\mathbf{Q}_\alpha}{\tau_\alpha} = \dot{\mathbf{S}}_\alpha - \mathbf{Q}_{\alpha_{\text{cpl}}} \quad , \qquad \alpha = 1, \ldots, m \quad , \tag{7.142}$$

where (7.142) is valid for some semi-open time interval $t \in (0, T]$ and for small perturbations away from the equilibrium state (for small strain rates). As usual, we start from a stress-free reference configuration which requires the values $\mathbf{Q}_\alpha |_{t=0} = \mathbf{O}$ for the internal variables to be zero at initial time $t = 0$. The first-order differential equations (7.142) require additional data in the form of initial conditions $\mathbf{Q}_{\alpha\,0^+}$ at time $t = 0^+$.

The term $\mathbf{Q}_{\alpha_{\text{cpl}}}$, $\alpha = 1, \ldots, m$, represents thermomechanical coupling effects which come from temperature dependent material parameters. Note that this term vanishes for purely mechanically based theories (see eq. (6.252)). The second-order tensors

$\mathbf{Q}_{\alpha_\text{cpl}}$ have to be determined such that evolution equations (7.142) are dissipative and compatible with the internal constitutive equations (7.141).

The second Piola-Kirchhoff stresses $\mathbf{S}_\alpha$, $\alpha = 1, \ldots, m$, in evolution equations (7.142) correspond to the free energies $\Psi_\alpha(\mathbf{C}, \Theta)$ (with $\Psi_\alpha(\mathbf{I}, \Theta_0) = 0$ at the reference configuration). They are responsible for the viscoelastic contribution and are related to the α-relaxation (retardation) process. We characterize the material variables $\mathbf{S}_\alpha$ by the constitutive equations

$$\mathbf{S}_\alpha = 2\frac{\partial \Psi_\alpha(\mathbf{C}, \Theta)}{\partial \mathbf{C}} \quad , \qquad \alpha = 1, \ldots, m \quad . \tag{7.143}$$

The stresses $\mathbf{S}_\alpha$ only depend on the *external variables* $\mathbf{C}$ and Θ. For the case of solid polymers that are composed of identical polymer chains we may replace Ψ_α by the free energy Ψ_∞ and adopt a relation similar to that given by eq. (6.256), namely $\Psi_\alpha(\mathbf{C}, \Theta) = \beta_\alpha^\infty \Psi_\infty(\mathbf{C}, \Theta)$, where $\beta_\alpha^\infty \in [0, \infty)$ are given non-dimensional free-energy factors. Note that Ψ_α must be related to the configurational free energy Υ_α in such a way that the second law of thermodynamics is satisfied for any thermodynamic process (see HOLZAPFEL and SIMO [1996c]).

The α-relaxation process is associated with the relaxation time $\tau_\alpha \in (0, \infty)$, which, in general, depends on the absolute temperature Θ (see TOBOLSKY et al. [1944]). The temperature dependence on the relaxation time may be related to the **activation energy** E_a of the relaxation process and expressed according to the **Arrhenius equation**,

$$\tau = A \exp\left(\frac{E_\text{a}}{\mathcal{R}\Theta}\right) \quad , \tag{7.144}$$

where A represents a constant for the reacting substance and $\mathcal{R} = 8.31\text{Nm}/\text{Kmol}$ denotes the **gas constant** (or **universal gas constant**). For an explicit derivation of the empirical exponential function (7.144) the reader is referred to SPERLING [1992, Appendix 10.1]. The form of the functional dependence of the relaxation time on temperature predicts the physical observation that viscoelastic effects occur faster as temperature increases. The Arrhenius equation in the form of (7.144) is representative of most polymer relaxations (LEE et al. [1966] and TOBOLSKY et al. [1944]).

EXAMPLE 7.6 The purpose of this example is to illustrate the introduced phenomenological constitutive model for thermoviscoelastic materials by means of simple particularizations. The free-energy functions Υ_α and Ψ_α are considered to be of the quadratic forms

$$\Upsilon_\alpha = \mu_\alpha(\Theta)\operatorname{tr}[(\mathbf{E} - \boldsymbol{\Lambda}_\alpha)^2] \quad , \qquad \Psi_\alpha = \mu_\alpha(\Theta)\operatorname{tr}(\mathbf{E}^2) \quad , \tag{7.145}$$

(with $\alpha = 1, \ldots, m$), where $\mu_\alpha > 0$ is a *temperature dependent* (Lamé-type) shear modulus characterizing the thermoviscoelastic behavior of the α-relaxation process with given relaxation time $\tau_\alpha > 0$. The functions (7.145) are of *Saint-Venant Kirchhoff-type* (compare with eq. (6.151)), in which, for simplicity, only the shear moduli are attached to the functions. The elastic strains are described by the (symmetric) Green-Lagrange strain tensor $\mathbf{E}$ defined by (2.69), while

$$\boldsymbol{\Lambda}_\alpha = \frac{1}{2}(\boldsymbol{\Gamma}_\alpha - \mathbf{I}) \quad ; \qquad \alpha = 1, \ldots, m \tag{7.146}$$

denote the inelastic (viscous) strain measures expressed by the (symmetric) second-order tensors $\boldsymbol{\Lambda}_\alpha$.

Consider a coupling term $\mathbf{Q}_{\alpha_\text{cpl}}$, $\alpha = 1, \ldots, m$, and a viscous dissipation $\mathcal{D}_\text{int}$ of the forms

$$\mathbf{Q}_{\alpha_\text{cpl}} = 2\dot{\mu}_\alpha(\Theta)\boldsymbol{\Lambda}_\alpha \quad ; \qquad \mathcal{D}_\text{int} = \sum_{\alpha=1}^{m} \frac{1}{\hat{\eta}_\alpha}|\mathbf{Q}_\alpha|^2 \geq 0 \quad ; \tag{7.147}$$

with the relationship $\hat{\eta}_\alpha = 2\mu_\alpha\tau_\alpha$, motivated by the linear theory of viscoelasticity. The parameter $\hat{\eta}_\alpha > 0$ characterizes the viscosity of the α-relaxation process (usually denoted in the literature by the symbol η, but to avoid confusion with the entropy η we place an accent over the symbol), while $|\mathbf{Q}_\alpha| = (\mathbf{Q}_\alpha : \mathbf{Q}_\alpha)^{1/2}$ denotes the *norm* of the tensor $\mathbf{Q}_\alpha$, which is a non-negative real number. Hence, the non-equilibrium stresses generate a non-negative dissipation such that the inequality is satisfied.

Obtain all the relevant thermodynamic relations, in particular the constitutive equations for the stress and the entropy. Specify the evolution equations introduced in (7.142) and discuss the state of thermodynamic equilibrium. For a nice rheological interpretation of this type of thermoviscoelastic model, given by the thermomechanical description (7.145)–(7.147), the reader is referred to the exercises below in this section.

Solution. By adopting $(7.140)_1$ and particularization $(7.145)_1$ we may derive the non-equilibrium stresses $\mathbf{Q}_\alpha$, which characterize the current 'distance from equilibrium'. By means of property $(1.252)_2$, relation (2.69) and the chain rule we obtain

$$\begin{aligned}
\mathbf{Q}_\alpha &= 2\frac{\partial \Upsilon_\alpha(\mathbf{C}, \Theta, \boldsymbol{\Gamma}_\alpha)}{\partial \mathbf{C}} = 2\mu_\alpha(\Theta)\frac{\partial \operatorname{tr}[(\mathbf{E} - \boldsymbol{\Lambda}_\alpha)^2]}{\partial(\mathbf{E} - \boldsymbol{\Lambda}_\alpha)} : \frac{\partial(\mathbf{E} - \boldsymbol{\Lambda}_\alpha)}{\partial \mathbf{C}} \\
&= 2\mu_\alpha(\Theta)(\mathbf{E} - \boldsymbol{\Lambda}_\alpha) \quad ; \qquad \alpha = 1, \ldots, m \quad .
\end{aligned} \tag{7.148}$$

It is important to note that a straightforward differentiation of Υ_α with respect to $\boldsymbol{\Gamma}_\alpha$ gives the same result as for $\mathbf{Q}_\alpha$, as can be seen by recalling (7.141) and using property $(1.252)_2$, the relation (7.146) and the chain rule.

In order to compute the entropy generated by the relaxation process, we use $(7.140)_2$ and particularization $(7.145)_1$. Thus, we obtain

$$\begin{aligned}
\eta_\alpha &= -\frac{\partial \Upsilon_\alpha(\mathbf{C}, \Theta, \boldsymbol{\Gamma}_\alpha)}{\partial \Theta} = -\frac{\partial \mu_\alpha(\Theta)[\operatorname{tr}(\mathbf{E} - \boldsymbol{\Lambda}_\alpha)^2]}{\partial \Theta} \\
&= -\mu_\alpha'(\Theta)[\operatorname{tr}(\mathbf{E} - \boldsymbol{\Lambda}_\alpha)^2] \quad ; \qquad \alpha = 1, \ldots, m \quad ;
\end{aligned} \tag{7.149}$$

with the common notation $\mu'_\alpha(\Theta) = \mathrm{d}\mu_\alpha(\Theta)/\mathrm{d}\Theta$.

Hence, the stress and entropy response at time t follow from eqs. (7.136) and (7.137) as

$$\mathbf{S} = \mathbf{S}_\infty + \sum_{\alpha=1}^{m} 2\mu_\alpha(\Theta)(\mathbf{E} - \mathbf{\Lambda}_\alpha) \ , \tag{7.150}$$

$$\eta = \eta_\infty - \sum_{\alpha=1}^{m} \mu'_\alpha(\Theta)\mathrm{tr}[(\mathbf{E} - \mathbf{\Lambda}_\alpha)^2] \ . \tag{7.151}$$

Before proceeding to examine the evolution equations it is first necessary to determine the second Piola-Kirchhoff stresses $\mathbf{S}_\alpha$. From (7.143) and $(7.145)_2$ we find, by means of property $(1.252)_2$, relation (2.69) and the chain rule, that

$$\mathbf{S}_\alpha = 2\frac{\partial \Psi_\alpha(\mathbf{C}, \Theta)}{\partial \mathbf{C}} = 2\mu_\alpha(\Theta)\frac{\partial \mathrm{tr}(\mathbf{E}^2)}{\partial \mathbf{E}} : \frac{\partial \mathbf{E}}{\partial \mathbf{C}} = 2\mu_\alpha(\Theta)\mathbf{E} \ , \tag{7.152}$$

with $\alpha = 1, \ldots, m$. Hence, from (7.142) and with help of the product rule and assumption $(7.147)_1$ we obtain finally the evolution equations for the non-equilibrium stresses, namely

$$\dot{\mathbf{Q}}_\alpha + \frac{\mathbf{Q}_\alpha}{\tau_\alpha} = 2\mu_\alpha(\Theta)\dot{\mathbf{E}} + 2\dot{\mu}_\alpha(\Theta)(\mathbf{E} - \mathbf{\Lambda}_\alpha) \ , \qquad \alpha = 1, \ldots, m \ , \tag{7.153}$$

which are valid for some semi-open time interval $t \in (0, T]$.

On comparing the given viscous dissipation $(7.147)_2$ with (7.138) (by means of (7.141)), we conclude that $\mathbf{Q}_\alpha : (\mathbf{Q}_\alpha/\hat{\eta}_\alpha - \dot{\mathbf{\Gamma}}_\alpha/2) = 0$. Hence, for the case in which $\mathbf{Q}_\alpha$ is different from zero we find that the expressions in parentheses must vanish. Thus, using (7.146), we obtain

$$\mathbf{Q}_\alpha = \hat{\eta}_\alpha \dot{\mathbf{\Lambda}}_\alpha \ , \qquad \alpha = 1, \ldots, m \ , \tag{7.154}$$

which is viewed as the classical Newtonian constitutive equation for the shear stress applied to simple shear (see eq. (5.100)). In eq. (7.154), $\mathbf{Q}_\alpha$ may be interpreted as the non-equilibrium shear stress, $\dot{\mathbf{\Lambda}}_\alpha$ as the shear rate (of a dashpot) and $\hat{\eta}_\alpha$ has the characteristic of the Newtonian shear viscosity.

The state of thermodynamic equilibrium requires that $\mathbf{Q}_\alpha = \mathbf{O}$ for $t \to \infty$, and that the internal dissipation $\mathcal{D}_{\mathrm{int}}$ vanishes (see eq. $(7.147)_2$). In view of (7.154) this implies that $\dot{\mathbf{\Lambda}}_\alpha|_{t\to\infty} = \mathbf{O}$. For this limiting case the thermodynamic process is reversible and the material response is fully thermoelastic. ■

Note that the vast majority of polymers exhibit the well-known **Newtonian shear thinning phenomenon** (pseudoplasticity) which means that with respect to Newtonian characteristics the shear rate $\dot{c}$ increases faster than the shear stress σ_{12} increases (see, for example, BARNES et al. [1989, Chapter 2]). A model for both Newtonian and non-Newtonian materials is the extensively used **power law model** (see, for example, ROSEN [1979, and references therein]). This empirical model is frequently written in the form

$$\sigma_{12} = \sigma_{21} = m\dot{c}^n \ , \tag{7.155}$$

which, in this form, is only valid for simple shear. Here, n is the **power law factor** (or the **flow behavior index**) and m is a (temperature dependent) parameter called the **viscosity index**. Shear thinning occurs if $n < 1$.

This model is used for a large number of engineering applications because it can be fitted to experimental results for various materials and reduces to a Newtonian fluid for n equal to 1, in which case m is known as the *viscosity* of the Newtonian fluid (compare with eqs. (7.154) or (5.100)). For typical parameter values see BARNES et al. [1989, p. 22]. For an overview of different types of rheological models the reader is referred to, for example, ROSEN [1979] and SCHOFF [1988, p. 455].

EXERCISES

1. The rheological model as illustrated in Figure 7.10 (referred to briefly as a *thermomechanical device*) is a suitable spring-and-dashpot model representing quantitatively the mechanical behavior of real thermoviscoelastic materials.

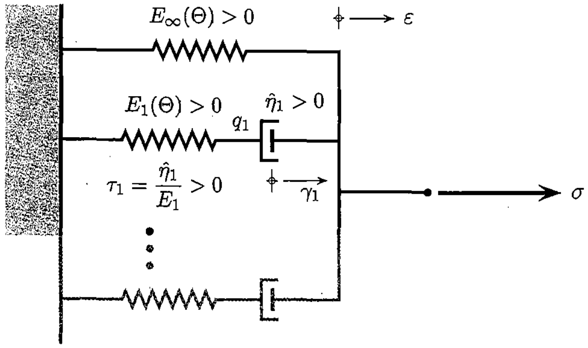

Figure 7.10 Rheological model with temperature dependent moduli.

For simplicity the thermomechanical device is assumed to have unit area and unit length so that stresses and strains are to be interpreted as forces and extensions (contractions), respectively.

It is considered to be a one-dimensional *generalized Maxwell model* with springs of *Hookean* type and dashpots of *Newtonian* type. The temperature dependent Young's moduli and the Newtonian shear viscosities are given by $E_\infty(\Theta) > 0$, $E_\alpha(\Theta) > 0$ and $\hat{\eta}_\alpha > 0$, $\alpha = 1, \ldots, m$, respectively. We now define the free energy $\psi(\varepsilon, \Theta, \gamma_1, \ldots, \gamma_m) = \psi_\infty(\varepsilon, \Theta) + \sum_{\alpha=1}^{m} \upsilon_\alpha(\varepsilon, \Theta, \gamma_\alpha)$, with the quadratic forms

$$\psi_\infty(\varepsilon, \Theta) = \frac{1}{2} E_\infty(\Theta)\varepsilon^2 \ , \tag{7.156}$$

$$\upsilon_\alpha(\varepsilon, \Theta, \gamma_\alpha) = \frac{1}{2} E_\alpha(\Theta)(\varepsilon - \gamma_\alpha)^2 \ , \qquad \alpha = 1, \ldots, m \ , \tag{7.157}$$

and the requirements $\psi_\infty(0, \Theta_0) = 0$ and $\upsilon_\alpha(0, \Theta_0, 0) = 0$, $\alpha = 1, \ldots, m$. The energy function ψ depends on the external variable ε (measuring the total linear strain), the absolute temperature Θ and the inelastic (viscous) strains γ_α, $\alpha = 1, \ldots, m$. The free energy υ_α is responsible for the α-relaxation process of the α-Maxwell element with given relaxation time $\tau_\alpha > 0$.

(a) Based on assumptions (7.156) and (7.157) obtain the explicit constitutive equations for the total linear stress $\sigma = \partial\psi/\partial\varepsilon$ and the total entropy $\eta = -\partial\psi/\partial\Theta$ in the forms

$$\sigma = \sigma_\infty + \sum_{\alpha=1}^{m} \underbrace{E_\alpha(\Theta)(\varepsilon - \gamma_\alpha)}_{q_\alpha} \ , \tag{7.158}$$

$$\eta = \eta_\infty - \sum_{\alpha=1}^{m} \frac{1}{2} E'_\alpha(\Theta)(\varepsilon - \gamma_\alpha)^2 \ , \tag{7.159}$$

with the common notation $E'_\alpha(\Theta) = \mathrm{d}E_\alpha(\Theta)/\mathrm{d}\Theta$, $\alpha = 1, \ldots, m$, and the physical expressions for the equilibrium parts $\sigma_\infty = E_\infty(\Theta)\varepsilon$, $\eta_\infty = -E'_\infty(\Theta)\varepsilon^2/2$.

The non-equilibrium stresses q_α act on each dashpot of the α-Maxwell element and are related to the associated inelastic (viscous) strains γ_α. Compute the internal constitutive equations $q_\alpha = -\partial\upsilon_\alpha(\varepsilon, \Theta, \gamma_\alpha)/\partial\gamma_\alpha$, $\alpha = 1, \ldots, m$, and obtain the current 'distance from equilibrium' q_α as specified in eq. (7.158).

(b) From Figure 7.10 using equilibrium derive the total linear stress σ and establish eq. (7.158). Interpret the result as a superposition of the equilibrium stress σ_∞ and the non-equilibrium stresses q_α, $\alpha = 1, \ldots, m$.

The considered thermoviscoelastic model presents a thermodynamic extension of the viscoelastic constitutive model introduced in Example 6.10 (see p. 286). Remarkably, the equilibrium equation (7.158) and the constitutive equation for the entropy (7.159) constitute the one-dimensional linear counterparts of eqs. (7.150) and (7.151), respectively. Hence, the purely phenomenological thermoviscoelastic model presented in Example 7.6 can be viewed as a nonlinear multi-dimensional generalization of the linear rheological model, as illustrated in Figure 7.10.

2. Recall the proposed one-dimensional thermoviscoelastic model from the previous exercise. The dashpots in the rheological model (Figure 7.10) characterize the dissipation mechanism. According to a Newtonian viscous fluid we may relate q_α to the strain rates $\dot{\gamma}_\alpha$ by the linear constitutive equations $q_\alpha = \hat{\eta}_\alpha\dot{\gamma}_\alpha$, $\alpha = 1, \ldots, m$.

   (a) Consider the time derivative of the non-equilibrium stresses $q_\alpha = E_\alpha(\Theta)(\varepsilon - \gamma_\alpha)$ and obtain the physically based evolution equations for the internal variables, namely

   $$\dot{q}_\alpha + \frac{q_\alpha}{\tau_\alpha} = E_\alpha(\Theta)\dot{\varepsilon} + \dot{E}_\alpha(\Theta)(\varepsilon - \gamma_\alpha) \ , \qquad \alpha = 1, \ldots, m \ , \tag{7.160}$$

   where the relations $\tau_\alpha = \hat{\eta}_\alpha/E_\alpha$, $\alpha = 1, \ldots, m$, should be used.

   (b) Knowing that q_α and $\dot{\gamma}_\alpha$ are the stresses and the strain rates acting on each dashpot, derive the rate of work dissipated within the considered thermomechanical device and derive the non-negative expression

   $$\mathcal{D}_{\mathrm{int}} = \sum_{\alpha=1}^{m} \frac{q_\alpha^2}{\hat{\eta}_\alpha} \geq 0 \ . \tag{7.161}$$

   Discuss the thermostatic limit and, in particular, specify in which parts of the device the stresses and the entropy remain.

Note that the evolution equations (7.160) and the internal dissipation (7.161) constitute the one-dimensional linear counterparts of eqs. (7.153) and $(7.147)_2$.

8 Variational Principles

The last chapter deals with the formulation of the field equations in the form of variational principles and methods.

The variational approach in various forms is often taken as the *cornerstone* for the development of discretization techniques such as the well established finite element methodology. The finite element method is today becoming widely used in industrial applications because of its predictive capability and general effectiveness in providing approximate solutions for the underlying initial boundary-value problems. On the finite element method, which is one of the most powerful numerical techniques, a large amount of literature is available. See, for example, the books by ODEN [1972], STRANG and FIX [1988b], ZIENKIEWICZ and TAYLOR [1989, 1991], BREZZI and FORTIN [1991], REDDY [1993], BATHE [1996], CRISFIELD [1991, 1997], BELYTSCHKO et al. [2000], HUGHES [2000], and WRIGGERS [2001]. For a description of a finite element program solving problems of continua in the *nonlinear* context (with available computer software) the reader is referred to, for example, ZIENKIEWICZ and TAYLOR [1991], CRISFIELD [1997] and BONET and WOOD [1997]. It is pointed out, however, that this reference list is by no means complete on this subject.

Variational principles are particularly powerful tools for the evaluation of continuous bodies and belong to the fundamental principles in mathematics and mechanics. It is important to note that the finite element method need not necessarily depend upon the existence of a variational principle. However, good approximate solutions are often related to the weak forms of field equations, which are consequences of the stationarity condition of a functional.

In this chapter we discuss (and compare) the most important variational principles leading to the finite element method. We focus attention solely on isothermal processes, for which the temperature remains constant. Coupling between mechanical and thermal quantities is not considered; hence, only the mechanical balance principles enter the variational formulations.

We start by explaining the notion of virtual displacements and variations and continue with the principle of virtual work, which is fundamental for a large number of efficient finite element formulations. The powerful concept of linearization is reviewed briefly and the principle of virtual work in both the material and spatial descriptions is linearized explicitly. We present some of the basic ideas of two and three-field variational principles particularly designed to capture kinematic constraints such as incompressibility.

The reader who wishes for additional information on the rich area of variational principles should consult the books by TRUESDELL and TOUPIN [1960], VAINBERG [1964], DUVAUT and LIONS [1972], ODEN and REDDY [1976] and WASHIZU [1982].

8.1 Virtual Displacements, Variations

Consider a continuum body $\mathcal{B}$ with a typical particle $P \in \mathcal{B}$ at a given instant of time t. As usual, points $X \in \Omega_0$ and $x \in \Omega$ characterize the positions $\mathbf{X}$ and $\mathbf{x}$ of that particle in the reference configuration Ω_0 at time $t = 0$ and the current configuration Ω at a subsequent time $t > 0$. In the following we indicate the displacement vector field of P as $\mathbf{u}$, pointing from the reference configuration of the continuum body into the current configuration, i.e. from $\mathbf{X}$ to $\mathbf{x}$ (see Figure 8.1).

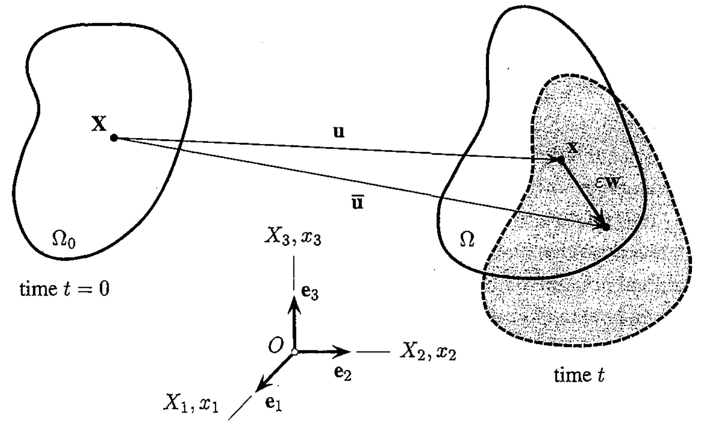

Figure 8.1 Virtual configuration in the neighborhood of $\mathbf{u}$, given by $\bar{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{w}$.

Next, consider some arbitrary and entirely new vector field $\mathbf{w}$ at point x which yields a **virtual**, slightly modified deformed configuration in the neighborhood of $\mathbf{u}$. The virtual configuration is characterized by the modified displacement vector field $\bar{\mathbf{u}}$ according to

$$\bar{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{w} \quad , \tag{8.1}$$

where ε is a scalar parameter.

The displacement vector field is regarded as a continuous and differentiable function of space and time. It may be written in the spatial or material form, i.e. $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{U}(\mathbf{X}, t)$, as introduced in Section 2.2. In order to keep the notation as simple as possible, we agree not to use this distinction any longer, we write subsequently $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{X}, t)$. It will be clear from the text if the displacement field actually depends on spatial or material coordinates. In addition, within this chapter, the position and time arguments will often be omitted, for convenience.

Virtual displacement field. Following *Lagrange* we know that the difference between two neighboring displacement fields, i.e. $\bar{\mathbf{u}}$ and $\mathbf{u}$, is called the **(first) variation** of the displacement field $\mathbf{u}$, denoted by $\delta\mathbf{u}$. We write

$$\delta\mathbf{u} = \bar{\mathbf{u}} - \mathbf{u} = \varepsilon\mathbf{w} \quad . \tag{8.2}$$

In mechanics $\delta\mathbf{u}$ is also known as the **virtual displacement field**. The variation of $\mathbf{u}$ is assumed to be an *arbitrary*, *infinitesimal* (since $\varepsilon \to 0$) and a **virtual change**, i.e. an imaginary (not a 'real') change. Note that $\mathrm{d}\mathbf{u}$ also characterizes an *infinitesimal change* of $\mathbf{u}$. However, $\mathrm{d}\mathbf{u}$ refers to an **actual change**. The variation of the time-dependent displacement vector field $\mathbf{u}$ is performed at *fixed* instant of time.

The virtual displacement field $\delta\mathbf{u}$ is totally independent of the actual displacement field $\mathbf{u}$ and may be expressed in terms of spatial coordinates or material coordinates. Omitting the time argument t, we have

$$\delta\mathbf{u}(\mathbf{x}) = \delta\mathbf{u}(\chi(\mathbf{X})) = \delta\mathbf{U}(\mathbf{X}) \quad . \tag{8.3}$$

For simplicity, we agree, by analogy with the relevant notation introduced above, to write $\delta\mathbf{u}(\mathbf{x}) = \delta\mathbf{u}(\mathbf{X})$ for the virtual displacement field $\delta\mathbf{u}$.

We discuss briefly the two fundamental commutative properties of the δ-process. For the gradient of $\delta\mathbf{u}$ we find by means of relation $(8.2)_1$ that

$$\mathrm{grad}(\delta\mathbf{u}) = \mathrm{grad}\bar{\mathbf{u}} - \mathrm{grad}\mathbf{u} \quad , \tag{8.4}$$

while, on the other hand, the variation of $\mathrm{grad}\mathbf{u} = \partial\mathbf{u}/\partial\mathbf{x}$ yields by analogy with $(8.2)_1$

$$\delta(\mathrm{grad}\mathbf{u}) = \mathrm{grad}\bar{\mathbf{u}} - \mathrm{grad}\mathbf{u} \quad . \tag{8.5}$$

On comparing eq. (8.4) with (8.5) we find finally the commutative property

$$\delta(\mathrm{grad}\mathbf{u}) = \mathrm{grad}(\delta\mathbf{u}) \quad , \tag{8.6}$$

which shows that the variation of the derivative of a function $(\bullet)$ is equal to the derivative of the variation of that function $(\bullet)$. In an analogous manner we may derive the second characteristic commutative property of the δ-process, namely, that the order of variation and definite integral is interchangeable.

By analogy with the transformation (2.48) we may relate the gradient with respect to the current position of a particle to the material gradient, defined on region Ω_0. For subsequent use we note the relation

$$\mathrm{grad}\delta\mathbf{u} = \mathrm{Grad}\delta\mathbf{u}\,\mathbf{F}^{-1} \qquad \text{or} \qquad \frac{\partial \delta u_a}{\partial x_b} = \frac{\partial \delta u_a}{\partial X_A} F^{-1}_{Ab} \ . \tag{8.7}$$

For more details on the variation see COURANT and HILBERT [1968a, 1968b]. A clearly arranged summary of the calculus of variations may also be found in the book by FUNG [1965, Chapter 10].

First variation of a function in material description. In the following let $\mathcal{F} = \mathcal{F}(\mathbf{u})$ be a smooth (possibly time-dependent) vector function. The single argument of $\mathcal{F}$ is the displacement vector variable $\mathbf{u}$ given in the *material* description. We agree that the value of $\mathcal{F}$, which characterizes some physical quantity, is either a scalar, vector or tensor. (Note the abuse of notation in regard to Section 2.3 where $\mathcal{F} = \mathcal{F}(\mathbf{X}, t)$ characterizes a smooth material field).

In order to obtain the *first variation* of the vector function $\mathcal{F}$ we must evaluate simply the directional derivative (or Gâteaux derivative) of $\mathcal{F}(\mathbf{u})$ at any fixed $\mathbf{u}$ in the direction of $\delta\mathbf{u}$, which we denote as $D_{\delta\mathbf{u}}\mathcal{F}(\mathbf{u})$ (recall the concept introduced in Section 1.8). We may consider the definition

$$\delta\mathcal{F}(\mathbf{u}, \delta\mathbf{u}) = D_{\delta\mathbf{u}}\mathcal{F}(\mathbf{u}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\mathbf{u} + \varepsilon\delta\mathbf{u})|_{\varepsilon=0} \tag{8.8}$$

and say that $\delta\mathcal{F}(\mathbf{u}, \delta\mathbf{u})$ is the first variation of the function $\mathcal{F}(\mathbf{u})$ in the direction of the virtual displacement field $\delta\mathbf{u}$. It is the ordinary differentiation of $\mathcal{F}(\mathbf{u} + \varepsilon\delta\mathbf{u})$ with respect to the scalar parameter ε.

Note that the variational operator $\delta(\bullet)$ and the Gâteaux operator $D(\bullet)$ are *linear*. The usual properties of differentiation are valid, i.e. the chain rule, product rule and so forth.

EXAMPLE 8.1 Show that the first variation $\delta\mathbf{F}$ of the deformation gradient $\mathbf{F}$ may be expressed as

$$\delta\mathbf{F} = \mathrm{Grad}\delta\mathbf{u} \qquad \text{or} \qquad \delta F_{aA} = \frac{\partial \delta u_a}{\partial X_A} = \mathrm{Grad}_{X_A}\delta u_a \ . \tag{8.9}$$

In addition, verify that the first variation $\delta\mathbf{F}^{-1}$ of the inverse of the deformation gradient $\mathbf{F}^{-1}$ is given by

$$\delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathrm{grad}\delta\mathbf{u} \qquad \text{or} \qquad \delta F^{-1}_{Aa} = -F^{-1}_{Ab}\frac{\partial \delta u_b}{\partial x_a} = -F^{-1}_{Ab}\mathrm{grad}_{x_a}\delta u_b \ . \tag{8.10}$$

Solution. With the definition (2.39) of the deformation gradient and rule (8.8), we may compute the directional derivative of $\mathbf{F}$ in the direction of the virtual displacement field $\delta\mathbf{u}$ at the position $\mathbf{u}(\mathbf{x})$ of the current configuration, i.e.

$$\begin{aligned}\delta\mathbf{F} = D_{\delta\mathbf{u}}\mathbf{F} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{F}(\mathbf{u} + \varepsilon\delta\mathbf{u})|_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathbf{F} + \varepsilon\mathrm{Grad}\delta\mathbf{u})|_{\varepsilon=0} = \mathrm{Grad}\delta\mathbf{u} \ .\end{aligned} \tag{8.11}$$

Alternatively, knowing that the operators $\delta(\bullet)$ and $\partial(\bullet)$ commute we obtain simply result (8.9) by thinking of $\delta(\bullet)$ as a *linear* operator. Applying relation $(2.45)_2$ we conclude that

$$\delta\mathbf{F} = \delta(\mathrm{Grad}\mathbf{u} + \mathbf{I}) = \delta(\mathrm{Grad}\mathbf{u}) = \mathrm{Grad}\delta\mathbf{u} \ . \tag{8.12}$$

In order to show eq. (8.10) we start with the variation of the identity $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}$ which gives $\delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}\delta\mathbf{F}\mathbf{F}^{-1}$ (compare with eq. $(2.145)_1$). Substituting (8.9) and using transformation (8.7) we find that

$$\delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}(\mathrm{Grad}\delta\mathbf{u}\,\mathbf{F}^{-1}) = -\mathbf{F}^{-1}\mathrm{grad}\delta\mathbf{u} \ , \tag{8.13}$$

which is the desired expression (8.10). ■

Finally we establish the variation of the Green-Lagrange strain tensor $\mathbf{E}$. By recalling the definition (2.69) of $\mathbf{E}$, we obtain with the product rule that $\delta\mathbf{E} = [\delta(\mathbf{F}^{\mathrm{T}})\mathbf{F} + \mathbf{F}^{\mathrm{T}}\delta\mathbf{F}]/2$. With relation (8.9) and property (1.84) we arrive at

$$\delta\mathbf{E} = \frac{1}{2}[(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u})^{\mathrm{T}} + \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}] = \mathrm{sym}(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}) \ , \tag{8.14}$$

or, in index notation,

$$\delta E_{AB} = \frac{1}{2}\left(F_{aB}\frac{\partial \delta u_a}{\partial X_A} + F_{aA}\frac{\partial \delta u_a}{\partial X_B}\right) \ , \tag{8.15}$$

which is an important relation used in subsequent studies. The notation $\mathrm{sym}(\bullet)$ is used to indicate the symmetric part of a tensor (compare with eq. $(1.112)_1$). Since $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$, we find additionally that $\delta\mathbf{E} = (\delta\mathbf{C})/2$.

First variation of a function in spatial description. Let $f = f(\mathbf{u})$ be a smooth (possibly time-dependent) vector function in the *spatial* description. Note that the value of the function $f(\mathbf{u}) = \chi_*(\mathcal{F}(\mathbf{u}))$, which we consider as the push-forward of $\mathcal{F}$, is either a scalar, vector or tensor.

In order to obtain the first variation of f we formally apply the important concept of *pull-back* and *push-forward* operations introduced in Section 2.5. The variation of f is obtained by the following three steps:

(i) compute the *pull-back* of f to the reference configuration, which results in the associated function $\mathcal{F}(\mathbf{u}) = \chi_*^{-1}(f(\mathbf{u}))$;

(ii) apply the concept of variation to $\mathcal{F}$, as introduced in eq. (8.8), and

(iii) carry out the *push-forward* of the result to the current configuration.

This concept is actually the same as for the computation of the *Lie time derivative* introduced in Section 2.8. Instead of the direction $\mathbf{v}$ used for the Lie time derivative we take here the virtual displacement field $\delta\mathbf{u}$.

Consequently, for the *first variation* of a vector function f given in the spatial description we merely write, with reference to eq. (2.189),

$$\delta f(\mathbf{u}, \delta\mathbf{u}) = \chi_*(D_{\delta\mathbf{u}}\chi_*^{-1}(f)) = \chi_*(D_{\delta\mathbf{u}}\mathcal{F}) \quad . \tag{8.16}$$

Since $D_{\delta\mathbf{u}}\mathcal{F}(\mathbf{u}) = \delta\mathcal{F}(\mathbf{u}, \delta\mathbf{u})$ according to $(8.8)_1$, we obtain

$$\delta f(\mathbf{u}, \delta\mathbf{u}) = \chi_*(\delta\mathcal{F}(\mathbf{u}, \delta\mathbf{u})) \quad . \tag{8.17}$$

Therefore, the first variation of function $f = f(\mathbf{u})$ is the push-forward of the first variation of the associated function $\mathcal{F}(\mathbf{u}) = \chi_*^{-1}(f(\mathbf{u}))$ in the direction of the virtual displacement field $\delta\mathbf{u}$. If $f = f(\mathbf{u})$ is a scalar-valued function, then $f = f(\mathbf{u}) = \mathcal{F}(\mathbf{u})$ and the variation of f coincides with the variation of its associated function $\mathcal{F}$; thus, $\delta f = \delta\mathcal{F}$.

Note that in our terminology the introduced operator δ is used for the variation of a function in both the material and spatial descriptions.

EXAMPLE 8.2 Show that the first variation $\delta\mathbf{e}$ of the Euler-Almansi strain tensor $\mathbf{e}$ may be expressed as

$$\delta\mathbf{e} = \frac{1}{2}(\mathrm{grad}^{\mathrm{T}}\delta\mathbf{u} + \mathrm{grad}\delta\mathbf{u}) = \mathrm{sym}(\mathrm{grad}\delta\mathbf{u}) \quad , \tag{8.18}$$

or, in index notation, as

$$\delta e_{ab} = \frac{1}{2}\left(\frac{\partial\delta u_b}{\partial x_a} + \frac{\partial\delta u_a}{\partial x_b}\right) \quad . \tag{8.19}$$

Solution. Compare Example 2.15 on p. 107. From rule (8.16) we obtain the variation of the spatial tensor $\mathbf{e}$, i.e.

$$\delta\mathbf{e} = \mathbf{F}^{-\mathrm{T}}(D_{\delta\mathbf{u}}(\mathbf{F}^{\mathrm{T}}\mathbf{e}\mathbf{F}))\mathbf{F}^{-1} = \mathbf{F}^{-\mathrm{T}}(D_{\delta\mathbf{u}}\mathbf{E})\mathbf{F}^{-1} \quad , \tag{8.20}$$

which is the push-forward of the directional derivative of the associated Green-Lagrange strain tensor $\mathbf{E} = \chi_*^{-1}(\mathbf{e})$ in the direction of $\delta\mathbf{u}$.

By means of rule $(8.8)_1$, relation $(8.14)_1$ and transformation (8.7) we find from $(8.20)_2$ that

$$\begin{aligned} \delta\mathbf{e} = \mathbf{F}^{-\mathrm{T}}\delta\mathbf{E}\,\mathbf{F}^{-1} &= \frac{1}{2}(\mathbf{F}^{-\mathrm{T}}\mathrm{Grad}^{\mathrm{T}}\delta\mathbf{u} + \mathrm{Grad}\delta\mathbf{u}\,\mathbf{F}^{-1}) \\ &= \frac{1}{2}(\mathrm{grad}^{\mathrm{T}}\delta\mathbf{u} + \mathrm{grad}\delta\mathbf{u}) \quad . \end{aligned} \tag{8.21}$$

According to eq. (1.112) the variation of the spatial tensor $\mathbf{e}$ is the symmetric part of the tensor $\mathrm{grad}\delta\mathbf{u}$, which gives the desired result $(8.18)_2$. ■

EXERCISES

1. Show that the first variations of the volume ratio J and the inverse right Cauchy-Green tensor $\mathbf{C}^{-1}$ are

$$\delta J = J\mathrm{div}\delta\mathbf{u} \quad , \qquad \delta\mathbf{C}^{-1} = -\mathbf{F}^{-1}(\mathrm{grad}^{\mathrm{T}}\delta\mathbf{u} + \mathrm{grad}\delta\mathbf{u})\,\mathbf{F}^{-\mathrm{T}} \quad .$$

2. Show that the first variations of the spatial line, surface and volume elements are

$$\begin{aligned} \delta(\mathrm{d}\mathbf{x}) &= \mathrm{grad}\delta\mathbf{u}\mathrm{d}\mathbf{x} \quad , \\ \delta(\mathrm{d}\mathbf{s}) &= (\mathrm{div}\delta\mathbf{u}\,\mathbf{I} - \mathrm{grad}^{\mathrm{T}}\delta\mathbf{u})\mathrm{d}\mathbf{s} \quad , \\ \delta(\mathrm{d}v) &= \mathrm{div}\delta\mathbf{u}\mathrm{d}v \quad , \end{aligned}$$

where $\mathbf{I}$ denotes the second-order unit tensor. Compare eqs. $(2.175)_3$, $(2.180)_3$ and $(2.182)_2$.

8.2 Principle of Virtual Work

In the following two sections we study variational principles with only one field of unknowns, called **single-field variational principles**. In particular, we introduce work and stationary principles in which the displacement vector $\mathbf{u}$ is the only unknown field.

These principles are fundamental and will become essential in establishing finite element formulations.

Initial boundary-value problem. The finite element method requires the formulation of the balance laws in the form of variational principles.

As one of the most fundamental balance laws we recall Cauchy's first equation of motion (i.e. balance of mechanical energy) discussed in Section 4.3. Knowing that the spatial velocity field $\mathbf{v}$ may be expressed as the time rate of change of the displacement field $\mathbf{u}$, we may write Cauchy's first equation of motion, i.e. (4.53), as

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} \quad . \tag{8.22}$$

From the fundamental standpoint adopted in Section 4.3, the Cauchy stress tensor is governed by the symmetry condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ deriving from the *balance of angular momentum*. The spatial mass density of the material is $\rho = J^{-1}\rho_0$, which describes *continuity of mass*. The body force $\mathbf{b}$ per unit current volume which acts on a particle in region Ω is considered to be a prescribed (given) force while the term $\rho\ddot{\mathbf{u}}$ characterizes the inertia force per unit current volume. Note that when we write eq. (8.22) we mean $\mathrm{div}\boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t) = \rho(\mathbf{x}, t)\ddot{\mathbf{u}}(\mathbf{x}, t)$ at every point $\mathbf{x} \in \Omega$ and for all times t.

In the following we consider boundary conditions and initial conditions for the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ required to satisfy the second-order differential equation (8.22). We assume subsequently that the boundary surface $\partial\Omega$ of a continuum body $\mathcal{B}$ occupying region Ω is decomposed into disjoint parts so that

$$\partial\Omega = \partial\Omega_{\mathrm{u}} \cup \partial\Omega_\sigma \qquad \text{with} \qquad \partial\Omega_{\mathrm{u}} \cap \partial\Omega_\sigma = \emptyset \quad . \tag{8.23}$$

Figure 8.2 illustrates the decomposition of the boundary surface $\partial\Omega$ in a two-dimensional space at time t.

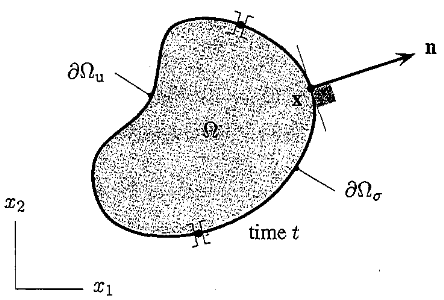

Figure 8.2 Partition of a boundary surface $\partial\Omega$.

We distinguish two classes of boundary conditions, namely the **Dirichlet boundary conditions**, which correspond to a displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, and the **von Neumann boundary conditions**, which are identified physically with the surface traction $\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n})$.

We write

$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on} \qquad \partial\Omega_{\mathrm{u}} \ , \qquad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}} \qquad \text{on} \qquad \partial\Omega_\sigma \ , \tag{8.24}$$

where the overbars $(\bar{\bullet})$ denote prescribed (given) functions on the boundaries $\partial\Omega_{(\bullet)} \subset \partial\Omega$ of a continuum body occupying the region Ω. The unit exterior vector normal to the boundary surface $\partial\Omega_\sigma$ is characterized by $\mathbf{n}$. The prescribed displacement field $\bar{\mathbf{u}}$ and the prescribed Cauchy traction vector $\bar{\mathbf{t}}$ (force measured per unit current surface area) are specified on a portion $\partial\Omega_{\mathrm{u}} \subset \partial\Omega$ and on the remainder $\partial\Omega_\sigma$, respectively. Note that in the previous section the symbol $\bar{\mathbf{u}}$ also stands for the modified displacement field.

We call the prescribed body force $\mathbf{b}$ and surface traction $\bar{\mathbf{t}}$ **loads**. We say that the continuum body is subjected to **holonomic external constraints** if $\mathbf{u} = \bar{\mathbf{u}}$ on the boundary surface $\partial\Omega_{\mathrm{u}}$. External constraints are **nonholonomic** if they are given by an inequality.

The second-order differential equations (8.22) themselves require additional data in the form of **initial conditions**. The displacement field $\mathbf{u}|_{t=0}$ and the velocity field $\dot{\mathbf{u}}|_{t=0}$ at initial time $t = 0$ are specified as

$$\mathbf{u}(\mathbf{x}, t)|_{t=0} = \mathbf{u}_0(\mathbf{X}) \ , \qquad \dot{\mathbf{u}}(\mathbf{x}, t)|_{t=0} = \dot{\mathbf{u}}_0(\mathbf{X}) \ , \tag{8.25}$$

where $(\bullet)_0$ denotes a prescribed function in Ω_0. Since we agreed to consider a stress-free reference configuration at $t = 0$, the initial values $(\bullet)_0$ are assumed to be zero in our case. However, in dynamics the configuration at $t = 0$ is sometimes not chosen as a reference configuration.

In order to achieve *compatibility* of the boundary and initial conditions we require additionally on $\partial\Omega_{\mathrm{u}}$ that

$$\bar{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{X}) \ , \qquad \dot{\bar{\mathbf{u}}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{X}) \quad . \tag{8.26}$$

Now, the problem is to find a motion that satisfies eq. (8.22) with the prescribed boundary and initial conditions (8.24), (8.25) and compatibility conditions (8.26).

This leads to the formulation in the **strong form** (or **classical form**) of the **initial boundary-value problem (IBVP)**. Given the body force, and the boundary and initial conditions, find the displacement field $\mathbf{u}$ so that (considering only mechanical variables)

$$\left.\begin{array}{lll} \mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} \ , & & \\ \quad \mathbf{u} = \bar{\mathbf{u}} & \text{on} & \partial\Omega_{\mathrm{u}} \ , \\ \quad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}} & \text{on} & \partial\Omega_\sigma \ , \\ \mathbf{u}(\mathbf{x}, t)|_{t=0} = \mathbf{u}_0(\mathbf{X}) \ , & & \\ \dot{\mathbf{u}}(\mathbf{x}, t)|_{t=0} = \dot{\mathbf{u}}_0(\mathbf{X}) \quad . & & \end{array}\right\} \tag{8.27}$$

Note that the set (8.27) of equations generally defines a *nonlinear* initial boundary-value problem for the unknown displacement field $\mathbf{u}$. In addition, we need a constitutive equation for the stress $\boldsymbol{\sigma}$ which is, in general, a nonlinear function of the displacement field $\mathbf{u}$.

If data depend on time and the acceleration is assumed to vanish, i.e. $\ddot{\mathbf{u}} = \mathbf{o}$, the considered problem is called **quasi-static**. For this case the Cauchy's equation of equilibrium is subjected to the conditions (8.24), $(8.25)_1$, and the requirement that eq. $(8.26)_1$ holds for compatibility.

If the data are independent of time the problem is referred to as **static**. For this case we consider a body in static equilibrium for which the set (8.27) of equations reduces to the associated nonlinear **boundary-value problem** (BVP) of elastostatics, i.e.

$$\left.\begin{array}{lll} \mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{o} \quad , & & \\ \mathbf{u} = \bar{\mathbf{u}} & \text{on} & \partial\Omega_{\mathrm{u}} \quad , \\ \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}} & \text{on} & \partial\Omega_{\sigma} \quad . \end{array}\right\} \tag{8.28}$$

Thus, the solution of a static problem at a point of a continuum body depends only on the data of the boundary and not on initial conditions (there is no need for initial conditions).

Principle of virtual work in spatial description. An analytical solution of the nonlinear initial boundary-value problem described is only possible for some special cases. Therefore, on the basis of variational principles, solution strategies such as the finite element method are often used in order to achieve approximate solutions.

In order to develop the principle of virtual work we start with Cauchy's first equation of motion (4.53) which we multiply with an arbitrary vector-valued function $\boldsymbol{\eta} = \boldsymbol{\eta}(\mathbf{x})$, defined on the current configuration Ω of the body. Integration over the region Ω of the body yields the scalar-valued function

$$f(\mathbf{u},\boldsymbol{\eta}) = \int\limits_{\Omega} (-\mathrm{div}\boldsymbol{\sigma} - \mathbf{b} + \rho\ddot{\mathbf{u}}) \cdot \boldsymbol{\eta}\mathrm{d}v = 0 \quad . \tag{8.29}$$

For the first argument of function f we conveniently introduce the displacement vector field $\mathbf{u}$ rather then the motion $\boldsymbol{\chi}$ for a given time t. The second argument of f is a function $\boldsymbol{\eta} = \boldsymbol{\eta}(\mathbf{x}) = \boldsymbol{\eta}(\boldsymbol{\chi}(\mathbf{X}, t))$ (at a *fixed* instant of time t), called a **test function** (or **weighting function**). It is a smooth function with $\boldsymbol{\eta} = \mathbf{o}$ on the boundary surface $\partial\Omega_{\mathrm{u}}$. Eq. (8.29) is known as the **weak form** of the equation of motion with respect to the spatial configuration. Equations in the weak form often remain valid for discontinuous problems such as *shocks* where most of the variables undergo a discontinuous variation. For this type of problem differential equations are not necessarily appropriate.

Since $\boldsymbol{\eta}$ is arbitrary, the vector equation $\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}}$ on Ω is *equivalent* to the weak form (8.29). The method used to prove this important property goes along with the **fundamental lemma** of the calculus of variations. The solution of the problem in the strong form is identical to the solution in the weak form. For further details see the books by HUGHES [2000] and MARSDEN and HUGHES [1994].

Subsequently, applying the product rule (1.290) to the term $\mathrm{div}\boldsymbol{\sigma} \cdot \boldsymbol{\eta}$, i.e.

$$\mathrm{div}\boldsymbol{\sigma} \cdot \boldsymbol{\eta} = \mathrm{div}(\boldsymbol{\sigma}\boldsymbol{\eta}) - \boldsymbol{\sigma} : \mathrm{grad}\boldsymbol{\eta} \quad , \tag{8.30}$$

and using the divergence theorem in the form of (1.301), eq. (8.29) may be written as

$$f(\mathbf{u},\boldsymbol{\eta}) = \int\limits_{\Omega} [\boldsymbol{\sigma} : \mathrm{grad}\boldsymbol{\eta} - (\mathbf{b} - \rho\ddot{\mathbf{u}}) \cdot \boldsymbol{\eta}]\mathrm{d}v - \int\limits_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{\eta} \cdot \mathbf{n}\mathrm{d}s = 0 \quad . \tag{8.31}$$

Since $\boldsymbol{\eta}$ vanishes on the part of the boundary surface $\partial\Omega_{\mathrm{u}}$ where $\bar{\mathbf{u}}$ is prescribed, the surface integral only needs to be integrated over the portion $\partial\Omega_{\sigma} \subset \partial\Omega$. By use of boundary conditions (8.24) and by formulating the initial conditions (8.25) in the weak form, we obtain the following set of scalar equations known as the **variational problem**:

$$\left.\begin{array}{l} f(\mathbf{u},\boldsymbol{\eta}) = \displaystyle\int\limits_{\Omega} [\boldsymbol{\sigma} : \mathrm{grad}\boldsymbol{\eta} - (\mathbf{b} - \rho\ddot{\mathbf{u}}) \cdot \boldsymbol{\eta}]\mathrm{d}v - \int\limits_{\partial\Omega_{\sigma}} \bar{\mathbf{t}} \cdot \boldsymbol{\eta}\mathrm{d}s = 0 \quad , \\ \displaystyle\int\limits_{\Omega} \mathbf{u}(\mathbf{x},t)|_{t=0} \cdot \boldsymbol{\eta}\mathrm{d}v = \int\limits_{\Omega} \mathbf{u}_0(\mathbf{X}) \cdot \boldsymbol{\eta}\mathrm{d}v \quad , \\ \displaystyle\int\limits_{\Omega} \dot{\mathbf{u}}(\mathbf{x},t)|_{t=0} \cdot \boldsymbol{\eta}\mathrm{d}v = \int\limits_{\Omega} \dot{\mathbf{u}}_0(\mathbf{X}) \cdot \boldsymbol{\eta}\mathrm{d}v \quad . \end{array}\right\} \tag{8.32}$$

This set of equations characterizes the **weak form** (or **variational form**) of the **initial boundary-value problem**. It is the equivalent counterpart in the strong form (8.27) which is satisfied when (8.32) is satisfied. Note that the stress boundary conditions on the portion $\partial\Omega_{\sigma}$ are part in the weak form $(8.32)_1$, so they are often referred to as **natural** boundary conditions. However, the conditions $\mathbf{u} = \bar{\mathbf{u}}$ which are prescribed over the boundary surface $\partial\Omega_{\mathrm{u}}$ are called **essential** boundary conditions of the variational problem.

Hence, variational problems are related to initial boundary-value problems which are described through differential equations and initial and boundary conditions. The differential equation is usually called the **Euler-Lagrange equation** in the weak formulation which in our case is Cauchy's first equation of motion $(8.27)_1$. Formulations in the weak form are mathematically helpful for investigations of existence, uniqueness or stability of solutions (see, for example, MARSDEN and HUGHES [1994, Sections 6.1-6.5]).

Note that the test function $\boldsymbol{\eta}$ in (8.32) is *arbitrary*. If we look upon $\boldsymbol{\eta}$ as the virtual displacement field $\delta\mathbf{u}$, defined on the *current configuration*, then the formulation in the weak form of the initial boundary-value problem (8.32) leads to the fundamental **principle of virtual work** (or **principle of virtual displacement**). Considering the symmetry of $\boldsymbol{\sigma}$ and the variation of the Euler-Almansi strain tensor $\delta\mathbf{e}$, as derived in eq. $(8.18)_2$, we arrive at the principle of virtual work in the *spatial description* expressed in terms of the virtual displacement, i.e.

$$f(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega} [\boldsymbol{\sigma} : \delta\mathbf{e} - (\mathbf{b} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u}]\mathrm{d}v - \int\limits_{\partial\Omega_\sigma} \bar{\mathbf{t}} \cdot \delta\mathbf{u}\mathrm{d}s = 0 \ , \tag{8.33}$$

with the additional initial conditions $\int_\Omega \mathbf{u}(\mathbf{x},t)|_{t=0} \cdot \delta\mathbf{u}\mathrm{d}v = \int_\Omega \mathbf{u}_0(\mathbf{X}) \cdot \delta\mathbf{u}\mathrm{d}v$ and $\int_\Omega \dot{\mathbf{u}}(\mathbf{x},t)|_{t=0} \cdot \delta\mathbf{u}\mathrm{d}v = \int_\Omega \dot{\mathbf{u}}_0(\mathbf{X}) \cdot \delta\mathbf{u}\mathrm{d}v$. An equation of type (8.33) is typically called a **variational equation**.

The smooth virtual displacement field $\delta\mathbf{u}$ is arbitrary over the region Ω and over the boundary surface $\partial\Omega_\sigma$ where the traction vector $\bar{\mathbf{t}}$ is prescribed. We require that $\delta\mathbf{u}$ vanishes on $\partial\Omega_u$, where the displacement field $\bar{\mathbf{u}}$ is prescribed (see the boundary conditions $(8.24)_1$). The virtual displacement field is assumed to be *infinitesimal*, which is not a requirement for an arbitrary test function. It is an imaginary (not a 'real') change of the continuum which is subjected to the loadings.

The principle of virtual work is the simplest variational principle and it states: the virtual stress work $\boldsymbol{\sigma} : \delta\mathbf{e}$ at fixed $\boldsymbol{\sigma}$ is equal to the work done by the body force $\mathbf{b}$ and inertia force $\rho\ddot{\mathbf{u}}$ per unit current volume and the surface traction $\bar{\mathbf{t}}$ per unit current surface along $\delta\mathbf{u}$ removed from the current configuration.

The functions

$$\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega} \boldsymbol{\sigma} : \delta\mathbf{e}\mathrm{d}v \ , \tag{8.34}$$

$$\delta W_{\text{ext}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega} \mathbf{b} \cdot \delta\mathbf{u}\mathrm{d}v + \int\limits_{\partial\Omega_\sigma} \bar{\mathbf{t}} \cdot \delta\mathbf{u}\mathrm{d}s \ , \tag{8.35}$$

are known as **internal (mechanical) virtual work** δW_{int} and **external (mechanical) virtual work** δW_{ext}.

In the first case the stress $\boldsymbol{\sigma}$ does internal work along the virtual strains $\delta\mathbf{e}$. In the second case external work is done by the loads, which are the body force $\mathbf{b}$ and the surface traction $\bar{\mathbf{t}}$, along the virtual displacement $\delta\mathbf{u}$ about region Ω and its boundary surface $\partial\Omega$, respectively. For vanishing accelerations $\ddot{\mathbf{u}}$, the internal virtual work equals the external virtual work, i.e. $\delta W_{\text{int}} = \delta W_{\text{ext}}$.

It is important to emphasize that the principle of virtual work does not necessitate the existence of a potential. No statement in regard to a particular material is invoked. Therefore, the principle of virtual work is general in the sense that it is applicable to any material including inelastic materials.

Pressure boundary loading. In the following we are concerned with an important load case, the pressure boundary loading, which is caused by liquids or gases, for example, water or wind. Pressure loads are deformation dependent and of crucial interest for finite deformation problems.

We consider a *pressure boundary condition* on the current boundary surface $\partial\Omega_\sigma \subset \partial\Omega$. In particular, we consider a traction vector $\bar{\mathbf{t}} = \boldsymbol{\sigma}\mathbf{n} = p\mathbf{n}$ per unit current surface acting in the direction of the (pointwise) outward unit vector $\mathbf{n} = \mathbf{n}(\mathbf{x})$. The unit normal vector is perpendicular to the pressure loaded surface $\partial\Omega_\sigma$ of the body with region Ω. Further, we assume that the normal pressure p is a given *constant*. An example in which a pressure boundary condition typically exists is inflation of a balloon.

The external virtual work done by the constant pressure p along the virtual displacement $\delta\mathbf{u}$ is then defined by

$$\delta W_{\text{ext}}(\mathbf{u},\delta\mathbf{u}) = p \int\limits_{\partial\Omega_\sigma} \mathbf{n} \cdot \delta\mathbf{u}\mathrm{d}s \ , \tag{8.36}$$

where $\mathrm{d}s$ denotes an infinitesimal surface element in the current configuration.

The external virtual work of the pressure boundary condition is discussed in more detail by SCHWEIZERHOF and RAMM [1984], BUFLER [1984], BONET and WOOD [1997, Section 6.5] and SIMO et al. [1991b] describing applications to axisymmetric problems.

In the following we introduce briefly a parametrization of the current boundary surface $\partial\Omega_\sigma$ which is very useful for implementation in a finite element program. The parameter plane with region Ω_ξ is characterized by ξ_1 and ξ_2 (see Figure 8.3). The parametrization of the surface on which p is prescribed is given in the form $\mathbf{x} = \boldsymbol{\gamma}(\xi_1,\xi_2,t) \subset \partial\Omega_\sigma$ $(x_a = \gamma_a(\xi_1,\xi_2,t))$ at fixed time t. Hence, the outward *unit* vector $\mathbf{n}$ may be expressed as the cross product of the displacement dependent vectors $\partial\boldsymbol{\gamma}/\partial\xi_1$ and $\partial\boldsymbol{\gamma}/\partial\xi_2$. The infinitesimal surface element $\mathrm{d}s$ follows from eq. (1.32) and the use of the chain rule. We write

$$\mathbf{n} = \frac{\dfrac{\partial\boldsymbol{\gamma}}{\partial\xi_1} \times \dfrac{\partial\boldsymbol{\gamma}}{\partial\xi_2}}{\left|\dfrac{\partial\boldsymbol{\gamma}}{\partial\xi_1} \times \dfrac{\partial\boldsymbol{\gamma}}{\partial\xi_2}\right|} \ , \qquad \mathrm{d}s = \left|\frac{\partial\boldsymbol{\gamma}}{\partial\xi_1} \times \frac{\partial\boldsymbol{\gamma}}{\partial\xi_2}\right| \mathrm{d}\xi_1\mathrm{d}\xi_2 \ . \tag{8.37}$$

These relations enable the external virtual work (8.36) to be expressed as

$$\delta W_{\text{ext}}(\mathbf{u},\delta\mathbf{u}) = p \int\limits_{\Omega_\xi} \left(\frac{\partial\boldsymbol{\gamma}}{\partial\xi_1} \times \frac{\partial\boldsymbol{\gamma}}{\partial\xi_2}\right) \cdot \delta\mathbf{u}\mathrm{d}\xi_1\mathrm{d}\xi_2 \ , \tag{8.38}$$

which is appropriate for finite element approximations.

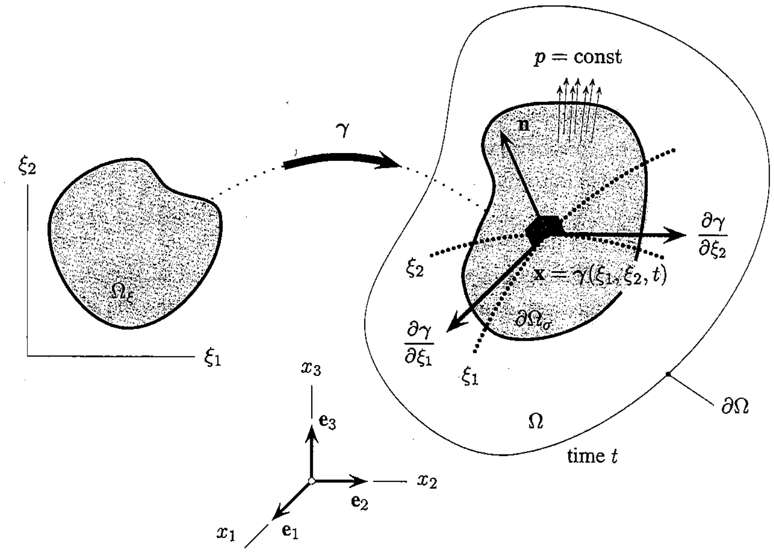

Figure 8.3 Constant pressure boundary loading and parametrization of the pressure loaded surface $\partial\Omega_\sigma$.

Principle of virtual work in material description. Now we are in a position to express the principle of virtual work in terms of material variables.

We assume a region Ω_0 of the continuum body which is bounded by a reference boundary surface $\partial\Omega_0$. This boundary surface is partitioned into disjoint parts (compare with the associated partition (8.23)) so that

$$\partial\Omega_0 = \partial\Omega_{0\,\mathrm{u}} \cup \partial\Omega_{0\,\sigma} \qquad \text{with} \qquad \partial\Omega_{0\,\mathrm{u}} \cap \partial\Omega_{0\,\sigma} = \emptyset \ . \tag{8.39}$$

As a point of departure we recall the equation of motion in the material description (4.63). We use the form

$$\mathrm{Div}\mathbf{P} + \mathbf{B} = \rho_0\ddot{\mathbf{u}} \ , \tag{8.40}$$

corresponding to (8.22). Here, $\mathbf{P}$, $\mathbf{B}$ and $\rho_0\ddot{\mathbf{u}}$ denote the first Piola-Kirchhoff stress tensor, the reference body force and the inertia force per unit reference volume, respectively.

For the Dirichlet and von Neumann boundary conditions, i.e. $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ and $\mathbf{T} = \mathbf{T}(\mathbf{X}, t, \mathbf{N})$, we write, by analogy with (8.24),

$$\mathbf{u} = \overline{\mathbf{u}} \quad \text{on} \quad \partial\Omega_{0\,\mathrm{u}} \ , \qquad \mathbf{T} = \mathbf{PN} = \overline{\mathbf{T}} \quad \text{on} \quad \partial\Omega_{0\,\sigma} \ , \tag{8.41}$$

where the unit exterior vector normal to the boundary surface $\partial\Omega_{0\,\sigma}$ is characterized by $\mathbf{N}$. The prescribed displacement field $\overline{\mathbf{u}}$ and the prescribed first Piola-Kirchhoff traction vector $\overline{\mathbf{T}}$ (force measured per unit reference surface area) are specified on the disjoint parts $\partial\Omega_{0\,\mathrm{u}}$ and $\partial\Omega_{0\,\sigma}$, respectively. The second-order differential equation (8.40) must be supplemented by initial conditions for the displacement field and the velocity field at the instant of time $t = 0$ (see eq. (8.25)).

Using the above concept, we may show the principle of virtual work in the *material description* expressed in terms of the virtual displacement, i.e.

$$\mathcal{F}(\mathbf{u}, \delta\mathbf{u}) = \int\limits_{\Omega_0} [\mathbf{P} : \mathrm{Grad}\delta\mathbf{u} - (\mathbf{B} - \rho_0\ddot{\mathbf{u}}) \cdot \delta\mathbf{u}]\mathrm{d}V - \int\limits_{\partial\Omega_{0\,\sigma}} \overline{\mathbf{T}} \cdot \delta\mathbf{u}\mathrm{d}S = 0 \ , \tag{8.42}$$

with the virtual displacement field $\delta\mathbf{u}$ (here defined on the *reference configuration*) satisfying the condition $\delta\mathbf{u} = \mathbf{o}$ on the part of the boundary surface $\partial\Omega_{0\,\mathrm{u}}$ where the displacement field $\overline{\mathbf{u}}$ is prescribed. The surface traction $\overline{\mathbf{T}}$ acts on the portion $\partial\Omega_{0\,\sigma} \subset \partial\Omega_0$. According to relation (3.1), $\mathbf{T}$ has the same direction as $\mathbf{t}$, but $\mathbf{T} \neq \mathbf{t}$. It is important to note that the description of the variational equation (8.42) is **equivalent** to that of (8.33).

EXAMPLE 8.3 Show that the material form of the principle of virtual work, as given in (8.42), can be obtained alternatively by a *pull-back* operation of relation (8.31) to the reference configuration.

Solution. In order to show (8.42) we must express the internal and external virtual work δW_{int} and δW_{ext} in eqs. (8.34) and (8.35) and the contribution $\int_\Omega \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\mathrm{d}v$ in terms of material variables.

We begin by considering the internal virtual work (8.34). With the help of identities $(8.18)_2$, (2.51) and transformation (8.7) we obtain

$$\int\limits_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\mathrm{d}v = \int\limits_\Omega \boldsymbol{\sigma} : \mathrm{grad}\delta\mathbf{u}\mathrm{d}v = \int\limits_{\Omega_0} J\boldsymbol{\sigma} : \mathrm{Grad}\delta\mathbf{u}\mathbf{F}^{-1}\mathrm{d}V \ , \tag{8.43}$$

where the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$ is to be used.

Applying property (1.95) and Piola transformation (3.8) we obtain the canonical representation of the material version, i.e.

$$\delta W_{\mathrm{int}}(\mathbf{u}, \delta\mathbf{u}) = \int\limits_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\mathrm{d}v = \int\limits_{\Omega_0} \mathbf{P} : \mathrm{Grad}\delta\mathbf{u}\mathrm{d}V \ . \tag{8.44}$$

The external virtual work δW_{ext} in the form of eq. (8.35) may be transformed by means of the relation for the body force $\mathbf{b}$, i.e. $\mathbf{b} = J^{-1}\mathbf{B}$, and the change in volume which is given by $\mathrm{d}v = J\mathrm{d}V$. In addition, we must show the equivalence of the prescribed traction vectors. With relation (3.1) and boundary conditions $(8.24)_2$ and $(8.41)_2$ we deduce that $\overline{\mathbf{t}}\mathrm{d}s = \overline{\mathbf{T}}\mathrm{d}S$. Hence,

$$\delta W_{\mathrm{ext}}(\mathbf{u}, \delta\mathbf{u}) = \int\limits_{\Omega} \mathbf{b} \cdot \delta\mathbf{u}\mathrm{d}v + \int\limits_{\partial\Omega_{\sigma}} \overline{\mathbf{t}} \cdot \delta\mathbf{u}\mathrm{d}s = \int\limits_{\Omega_0} \mathbf{B} \cdot \delta\mathbf{u}\mathrm{d}V + \int\limits_{\partial\Omega_{0\,\sigma}} \overline{\mathbf{T}} \cdot \delta\mathbf{u}\mathrm{d}S \quad . \tag{8.45}$$

The remaining term in eq. (8.42), i.e. the inertia force $\rho_0\ddot{\mathbf{u}}$ per unit reference volume over the region Ω_0, may simply be established from the third term in the associated eq. (8.31) by means of $\rho_0\mathrm{d}V = \rho\mathrm{d}v$, i.e. conservation of mass in the local form (4.6). This result together with (8.44) and (8.45) leads to the desired relation (8.42). ■

EXERCISE

1. Starting at eq. $(8.44)_2$, show that the internal virtual work δW_{int} may also be expressed as the contraction of the symmetric second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the virtual Green-Lagrange strain tensor $\delta\mathbf{E}$ defined in eq. (8.14).

   In addition, show that δW_{int} may also be given in terms of the Mandel stress tensor $\boldsymbol{\Sigma} = \mathbf{CS}$ and the variation of the symmetric Green-Lagrange strain tensor, i.e. $\delta\mathbf{C} = 2\delta\mathbf{E}$, by using eq. (3.67) as

$$\delta W_{\mathrm{int}} = \int\limits_{\Omega_0} \mathbf{P} : \mathrm{Grad}\delta\mathbf{u}\mathrm{d}V = \int\limits_{\Omega_0} \mathbf{S} : \delta\mathbf{E}\mathrm{d}V = \int\limits_{\Omega_0} \boldsymbol{\Sigma} : \frac{1}{2}\mathbf{C}^{-1}\delta\mathbf{C}\mathrm{d}V \quad . \tag{8.46}$$

8.3 Principle of Stationary Potential Energy

In the principle of virtual work, as derived in the last section, the stresses are considered without their connections to the strains. We have not taken into account a particular material.

In this section we assume a conservative mechanical system (compare with p. 159) requiring the existence of an energy functional Π for both the stresses and the loads. The assumption of the existence of Π is common in many fields in solid mechanics. The loads may depend on the motion, but they must emanate from a functional. A formulation based on energy functionals is very useful, for example, for the development of robust numerical algorithms that are based on optimization techniques.

In the following we introduce a stationary energy principle in which the displacement vector field $\mathbf{u}$ is taken to be the only fundamental unknown.

Stationary energy principle. From now on we assume that the loads do not depend on the motion of the body, which is usually the case, for example, for body forces. It means that the directions of the loads remain parallel and their values unchanged throughout the deformation of the body. We say that such loads are **'dead'**.

Instead of a vibrating body we consider a body in static equilibrium under the action of specified *'dead'* loadings and boundary conditions on $\partial\Omega_{0\,\mathrm{u}}$ and $\partial\Omega_{0\,\sigma}$, according to eq. (8.41). Then the total potential energy Π of the system is given as the sum of the internal and external potential energy, Π_{int} and Π_{ext}, i.e.

$$\Pi(\mathbf{u}) = \Pi_{\mathrm{int}}(\mathbf{u}) + \Pi_{\mathrm{ext}}(\mathbf{u}) \quad , \tag{8.47}$$

$$\Pi_{\mathrm{int}}(\mathbf{u}) = \int\limits_{\Omega_0} \Psi(\mathbf{F}(\mathbf{u}))\mathrm{d}V \quad , \qquad \Pi_{\mathrm{ext}}(\mathbf{u}) = -\int\limits_{\Omega_0} \mathbf{B} \cdot \mathbf{u}\mathrm{d}V - \int\limits_{\partial\Omega_{0\,\sigma}} \overline{\mathbf{T}} \cdot \mathbf{u}\mathrm{d}S \quad , \tag{8.48}$$

where $\Psi = \Psi(\mathbf{F})$ denotes the strain-energy function per unit reference volume, as introduced in Section 6.1.

Since the deformation gradient $\mathbf{F}$ depends on the displacement vector field $\mathbf{u}$ by the relation according to eq. $(2.45)_2$, i.e. $\mathbf{F} = \mathrm{Grad}\mathbf{u} + \mathbf{I}$, we indicate explicitly the dependence of $\mathbf{F}$ on $\mathbf{u}$ and write $\mathbf{F} = \mathbf{F}(\mathbf{u})$. For Π_{int} and Π_{ext} we will also indicate subsequently the dependence on $\mathbf{u}$. Note that for a rigid body the term Π_{int} is zero. The *'dead'* loadings, given by the external forces $\mathbf{B}$ and $\overline{\mathbf{T}}$, are distributed over the volume of the continuum body and its von Neumann boundary, respectively.

One main objective of common engineering interests is to find the state of equilibrium (the deformed configuration) for which the potential is **stationary**. The stationary position of the total potential energy Π is obtained by requiring the directional derivative with respect to the displacements $\mathbf{u}$ to vanish in all directions $\delta\mathbf{u}$. Compute

$$\delta\Pi(\mathbf{u}, \delta\mathbf{u}) = D_{\delta\mathbf{u}}\Pi(\mathbf{u}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Pi(\mathbf{u} + \varepsilon\delta\mathbf{u})|_{\varepsilon=0} = 0 \quad , \tag{8.49}$$

which is known as the **principle of stationary potential energy**, another fundamental variational principle in mechanics. In other words, we require that the first variation of the total potential energy, denoted $\delta\Pi$, vanishes. The variation of Π clearly is a function of both $\mathbf{u}$ and $\delta\mathbf{u}$. The arbitrary vector field $\delta\mathbf{u}$ is consistent with the conditions imposed on the continuum body. Thus, $\delta\mathbf{u} = \mathbf{o}$ over $\partial\Omega_{\mathrm{u}}$, where boundary displacements are prescribed.

In order to decide if the solution corresponds to a *maximum*, a *minimum* or a *saddle point* we must determine the **second variation** of the total potential energy Π, denoted by $\delta^2\Pi(\mathbf{u}, \delta\mathbf{u}, \Delta\mathbf{u}) = D^2_{\delta\mathbf{u},\Delta\mathbf{u}}\Pi(\mathbf{u})$. Here, $\Delta\mathbf{u}$ is the increment of the displacement field $\mathbf{u}$ which will be discussed later in Section 8.4. The quantity $D^2_{\delta\mathbf{u},\Delta\mathbf{u}}\Pi(\mathbf{u})$ is obtained from the directional derivative of variational equation (8.49) with respect to $\mathbf{u}$ in the direction $\Delta\mathbf{u}$ (i.e. the second directional derivative of Π with respect to $\mathbf{u}$), which is either a maximum ($D^2_{\delta\mathbf{u},\Delta\mathbf{u}}\Pi(\mathbf{u}) < 0$), a minimum ($D^2_{\delta\mathbf{u},\Delta\mathbf{u}}\Pi(\mathbf{u}) > 0$), or a saddle point ($D^2_{\delta\mathbf{u},\Delta\mathbf{u}}\Pi(\mathbf{u}) = 0$).

EXAMPLE 8.4 Show that the directional derivative of the total potential energy Π, as given by (8.47), (8.48), with respect to $\mathbf{u}$ in the arbitrary direction $\delta\mathbf{u}$ leads to

$$D_{\delta\mathbf{u}}\Pi(\mathbf{u}) = D_{\delta\mathbf{u}}\Pi_{\mathrm{int}}(\mathbf{u}) + D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}}(\mathbf{u}) \ , \tag{8.50}$$

$$D_{\delta\mathbf{u}}\Pi_{\mathrm{int}}(\mathbf{u}) = \delta W_{\mathrm{int}} \qquad \text{and} \qquad D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}}(\mathbf{u}) = -\delta W_{\mathrm{ext}} \ . \tag{8.51}$$

The internal virtual work δW_{int} and the external virtual work δW_{ext} are given through eqs. (8.44) and (8.45), respectively.

Furthermore, show that the stationary position of the total potential energy Π gives the principle of virtual work for a body in static equilibrium, as given in eq. (8.42).

Solution. Since the loads $\mathbf{B}$ and $\overline{\mathbf{T}}$ do not depend on the deformation of the body, they do not contribute to the directional derivative. Hence, from $(8.49)_1$, we find, with use of expressions (8.47) and (8.48), that

$$\begin{aligned} D_{\delta\mathbf{u}}\Pi(\mathbf{u}) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})|_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\int\limits_{\Omega_0}\Psi(\mathbf{F}(\mathbf{u}+\varepsilon\delta\mathbf{u}))\mathrm{d}V - \int\limits_{\Omega_0}\mathbf{B}\cdot(\mathbf{u}+\varepsilon\delta\mathbf{u})\mathrm{d}V \\ &\quad - \int\limits_{\partial\Omega_{0\sigma}}\overline{\mathbf{T}}\cdot(\mathbf{u}+\varepsilon\delta\mathbf{u})\mathrm{d}S]|_{\varepsilon=0} \ . \end{aligned} \tag{8.52}$$

Interchanging differentiation and integration and applying the chain rule, we obtain

$$D_{\delta\mathbf{u}}\Pi(\mathbf{u}) = \int\limits_{\Omega_0}\frac{\partial\Psi(\mathbf{F}(\mathbf{u}))}{\partial\mathbf{F}} : D_{\delta\mathbf{u}}\mathbf{F}(\mathbf{u})\mathrm{d}V - \int\limits_{\Omega_0}\mathbf{B}\cdot\delta\mathbf{u}\mathrm{d}V - \int\limits_{\partial\Omega_{0\sigma}}\overline{\mathbf{T}}\cdot\delta\mathbf{u}\mathrm{d}S \ , \tag{8.53}$$

where $D_{\delta\mathbf{u}}\mathbf{F}$ denotes the directional derivative of the deformation gradient $\mathbf{F}$ at $\mathbf{u}$ in the direction of $\delta\mathbf{u}$, derived in eq. (8.11).

In order to specify the first integral in eq. (8.53) we recall the physical expression $(6.1)_1$ and use result $(8.11)_4$ to obtain

$$\begin{aligned} \int\limits_{\Omega_0}\frac{\partial\Psi(\mathbf{F}(\mathbf{u}))}{\partial\mathbf{F}} : D_{\delta\mathbf{u}}\mathbf{F}(\mathbf{u})\mathrm{d}V &= \int\limits_{\Omega_0}\frac{\partial\Psi(\mathbf{F}(\mathbf{u}))}{\partial\mathbf{F}} : \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{F}(\mathbf{u}+\varepsilon\delta\mathbf{u})|_{\varepsilon=0}\mathrm{d}V \\ &= \int\limits_{\Omega_0}\mathbf{P} : \mathrm{Grad}\delta\mathbf{u}\mathrm{d}V \ . \end{aligned} \tag{8.54}$$

By recalling definitions $(8.44)_2$ and $(8.45)_2$ and combining (8.53) with $(8.54)_2$ we find the desired results (8.50) and (8.51). Hence, stationary condition (8.49) yields precisely the principle of virtual work (8.42) for a configuration in static equilibrium which renders stationary the functional Π.

We conclude that the total potential energy Π is stationary for arbitrary variations $\delta\mathbf{u}$, which means evaluating $\delta\Pi(\mathbf{u},\delta\mathbf{u}) = 0$ with respect to the displacements, *if and only if* the nonlinear variational equation (8.42) (for $\ddot{\mathbf{u}} = \mathbf{o}$) is satisfied (equilibrium state).

Finally, note that for the purpose of computing the stationary position of Π the magnitude of the virtual displacements need not be small, as is sometimes stated in the literature. However, in order to achieve a first-order approximation the magnitude of the virtual displacements must be small. ■

Penalty method for incompressibility. The principle of virtual work is not the appropriate variational approach to invoke kinematic constraint conditions such as incompressibility, contact boundary conditions or Kirchhoff-Love (kinematic) conditions on plates and shells often occurring in engineering applications.

A numerical analysis of nearly incompressible and incompressible materials necessitates so-called **multi-field variational principles** in which additional variables are incorporated. Multi-field variational principles, dealt with in more detail in Sections 8.5 and 8.6, lead to *mixed* or *hybrid methods* for finite elements.

Nevertheless, a single-field variational approach with the displacement $\mathbf{u}$ as the only field variable is very often used in order to approximate, for example, incompressible materials. This leads to the so-called **penalty method**, which is based on the simple (physical) idea of modeling an incompressible material as slightly compressible by using a large value of the bulk modulus. Of course, following this idea, an incompressible material can be obtained by taking the limit infinity for the bulk modulus. However, the result of this idea from the numerical point of view is that we always work with a slightly compressible material since the incompressible limit can never be achieved.

To be more precise, rather than employing the strain-energy function in the form of $\Psi = \Psi(\mathbf{F})$, it is standard to use the unique decoupled representation of the strain-energy function in the form

$$\Psi(\mathbf{C}) = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}(\overline{\mathbf{C}}) \qquad \text{with} \qquad \Psi_{\mathrm{vol}}(J) = \kappa\mathcal{G}(J) \ , \tag{8.55}$$

so that energy functional (8.47) takes on the **penalty form**

$$\Pi_{\mathrm{p}}(\mathbf{u}) = \int\limits_{\Omega_0}[\Psi_{\mathrm{vol}}(J(\mathbf{u})) + \Psi_{\mathrm{iso}}(\overline{\mathbf{C}}(\mathbf{u}))]\mathrm{d}V + \Pi_{\mathrm{ext}}(\mathbf{u}) \ , \tag{8.56}$$

with the external potential energy Π_{ext} given by eq. $(8.48)_2$.

Here, $J = J(\mathbf{u}) = (\det\mathbf{C})^{1/2}$ defines the volume ratio and $\overline{\mathbf{C}} = \overline{\mathbf{C}}(\mathbf{u}) = J^{-2/3}\mathbf{C}$ the corresponding modified right Cauchy-Green tensor, as introduced in eq. $(6.79)_2$. The *strictly convex* function Ψ_{vol} describes the *volumetric elastic response* while Ψ_{iso} is associated with the *isochoric elastic response* of the hyperelastic material. We require $\Psi_{\mathrm{vol}}(J)$ and $\Psi_{\mathrm{iso}}(\overline{\mathbf{C}})$ to be zero *if and only if* $J = 1$ and $\overline{\mathbf{C}} = \mathbf{I}$, ensuring that the reference configuration is stress-free.

According to eq. $(8.55)_2$ the volumetric contribution Ψ_{vol} is characterized by a (positive) **penalty parameter** $\kappa > 0$ which is independent of the deformation. The parameter κ may be viewed as the bulk modulus. The function $\mathcal{G}$ is motivated mathematically. It is known as the **penalty function** and may adopt the simple form

$$\mathcal{G}(J) = \frac{1}{2}(J(\mathbf{u}) - 1)^2 \quad , \tag{8.57}$$

often used in numerical computations. Consequently, the meaning of the function Ψ_{vol} as used in eq. $(8.55)_1$ differs significantly from its meaning in eq. (6.85), in which Ψ_{vol} is of physical relevance.

We now derive the stationary position of Π_{p} with respect to the displacement field, which is basically a procedure according to Example 8.4. Starting with the fundamental condition (8.49) and following the steps which have led to eq. (8.54) we find, using decomposition (8.55), that

$$\begin{aligned} D_{\delta\mathbf{u}}\Pi_{\mathrm{p}}(\mathbf{u}) = \int\limits_{\Omega_0} \left(\frac{\partial\Psi_{\mathrm{vol}}(J(\mathbf{u}))}{\partial\mathbf{C}} + \frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}}(\mathbf{u}))}{\partial\mathbf{C}} \right) : D_{\delta\mathbf{u}}\mathbf{C}(\mathbf{u})\mathrm{d}V \\ + D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}}(\mathbf{u}) = 0 \quad , \end{aligned} \tag{8.58}$$

where $D_{\delta\mathbf{u}}\mathbf{C}$ denotes the directional derivative of the right Cauchy-Green tensor $\mathbf{C}$ at $\mathbf{u}$ in the direction of $\delta\mathbf{u}$, which is $2\delta\mathbf{E}$ (see Section 8.1).

A specification of the integral in eq. (8.58) implies, by means of the chain rule and relation $(6.82)_1$, that

$$\begin{aligned} \int\limits_{\Omega_0} \left(\frac{\partial\Psi_{\mathrm{vol}}}{\partial\mathbf{C}} + \frac{\partial\Psi_{\mathrm{iso}}}{\partial\mathbf{C}} \right) : D_{\delta\mathbf{u}}\mathbf{C}\mathrm{d}V &= \int\limits_{\Omega_0} \left(\frac{\mathrm{d}\Psi_{\mathrm{vol}}}{\mathrm{d}J}\frac{\partial J}{\partial\mathbf{C}} + \frac{\partial\Psi_{\mathrm{iso}}}{\partial\mathbf{C}} \right) : D_{\delta\mathbf{u}}\mathbf{C}\mathrm{d}V \\ &= \int\limits_{\Omega_0} \left(J\frac{\mathrm{d}\Psi_{\mathrm{vol}}}{\mathrm{d}J}\mathbf{C}^{-1} + 2\frac{\partial\Psi_{\mathrm{iso}}}{\partial\mathbf{C}} \right) : \delta\mathbf{E}\mathrm{d}V \end{aligned} \tag{8.59}$$

(the arguments of the functions have been omitted for simplicity).

With reference to eq. $(6.91)_1$ the term $\mathrm{d}\Psi_{\mathrm{vol}}/\mathrm{d}J$ defines the hydrostatic pressure p. Hence, recalling definitions $(6.89)_2$ and $(6.90)_1$, we conclude that the terms in parentheses of $(8.59)_2$ are associated with the volumetric and isochoric stress contributions $\mathbf{S}_{\mathrm{vol}}$ and $\mathbf{S}_{\mathrm{iso}}$, respectively.

Using the second Piola-Kirchhoff stress tensor $\mathbf{S}$, which is based on the additive decomposition $(6.88)_2$, we achieve finally the principle of virtual work (for a configuration in static equilibrium) in the form

$$D_{\delta\mathbf{u}}\Pi_{\mathrm{p}}(\mathbf{u}) = \int\limits_{\Omega_0} \mathbf{S} : \delta\mathbf{E}\mathrm{d}V + D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}}(\mathbf{u}) = 0 \quad . \tag{8.60}$$

Note that the integral in eq. (8.60), i.e. the directional derivative of the internal potential energy Π_{int} with respect to $\mathbf{u}$ in the arbitrary direction $\delta\mathbf{u}$, precisely gives the internal virtual work δW_{int} (see eq. $(8.46)_2$). The second term in eq. (8.60) is related to the external virtual work according to $(8.51)_2$.

A parametrization of the hydrostatic pressure p is simply obtained from eq. $(8.55)_2$ by means of assumption (8.57), i.e.

$$p = \frac{\mathrm{d}\Psi_{\mathrm{vol}}(J(\mathbf{u}))}{\mathrm{d}J} = \kappa\frac{\mathrm{d}\mathcal{G}(J(\mathbf{u}))}{\mathrm{d}J} = \kappa(J(\mathbf{u}) - 1) \quad . \tag{8.61}$$

In contrast to eqs. $(6.140)_2$ and (6.141), this is an artificial constitutive equation for p designed to prevent a significant volumetric response, as already pointed out.

The user-specified penalty parameter κ is merely an adjustable numerical parameter which is often chosen through numerical experiments. Clearly, with increasing κ the violation of the constraint is reduced. If we take the restriction on the value $\kappa \to \infty$, the constraint condition is exactly enforced, and then eq. (8.56) represents a functional for an incompressible material with $J = 1$.

Unfortunately, for an approximation technique such as the finite element method, the stiffness matrix becomes increasingly *ill-conditioned* for increasing κ (see, for example, BERTSEKAS [1982] and LUENBERGER [1984] for detailed studies). For that case the **reduced integration method**, and the later proposed **selective-reduced integration method** (which are equivalent to certain types of mixed finite element methods, as discussed in MALKUS and HUGHES [1978] and HUGHES [2000]) is often used to underintegrate (weaken) the penalty function.

However, penalty methods are attractive because they are based on a simple variational principle with all its computational advantages and are very effective to implement in a finite element program.

EXERCISES

1. The classical **Hamilton's variational principle** represents a generalization of the principle of stationary potential energy (8.49) to continuum dynamics. It is presented by the stationary condition

$$\delta\int\limits_{t_0}^{t_1} L(\mathbf{u},\dot{\mathbf{u}})\mathrm{d}t = D_{\delta\mathbf{u}}\int\limits_{t_0}^{t_1} L(\mathbf{u},\dot{\mathbf{u}})\mathrm{d}t = 0 \qquad \text{where} \qquad L(\mathbf{u},\dot{\mathbf{u}}) = \Pi(\mathbf{u}) - \mathcal{K}(\dot{\mathbf{u}}) \quad ,$$

with $\delta\mathbf{u}$ denoting the variation of the displacement vector field, which is a function of *position* and *time*. The functional L (in the literature sometimes introduced as $-L$) is integrated with respect to time t over the closed time interval $t \in [t_0, t_1]$ (t_0 and t_1 are two arbitrary instants of time). We now assume the restriction that at the times t_0 and t_1, the variation $\delta\mathbf{u}$ of the displacement vector field vanishes at all points of the continuum body, i.e.

$$\delta\mathbf{u}|_{t=t_0} = \delta\mathbf{u}|_{t=t_1} = \mathbf{o} \quad . \tag{8.62}$$

The scalar-valued functionals Π and $\mathcal{K}$ denote the total potential energy (4.114) and the kinetic energy (4.83) of the moving body, as usual. The potential energy of the loads exists and is given by eq. $(8.48)_2$.

Show that the vanishing variation of the functional L with the imposed restrictions (8.62) gives the principle of virtual work (8.42) for all $\delta\mathbf{u}$ which are zero on $\partial\Omega_{\mathrm{u}}$ throughout the entire closed time interval $t \in [t_0, t_1]$.

2. Consider a constant normal pressure p applied to a boundary surface enclosing a certain region. Show that there exists the associated potential

$$\Pi_{\text{ext}}(\mathbf{u}) = p \int_{\Omega} \mathrm{d}v = p \int_{\Omega_0} J(\mathbf{X}, t)\mathrm{d}V \quad ,$$

whose variation gives the external virtual work δW_{ext}, as defined in eq. (8.36), i.e. $D_{\delta\mathbf{u}}\Pi_{\text{ext}}(\mathbf{u}) = \delta W_{\text{ext}}(\mathbf{u}, \delta\mathbf{u}) = p\int_{\partial\Omega_\sigma} \mathbf{n} \cdot \delta\mathbf{u}\mathrm{d}s$.

8.4 Linearization of the Principle of Virtual Work

Variational principles such as the principle of virtual work in the forms of (8.33) or (8.42) are generally nonlinear in the unknown displacement vector field $\mathbf{u}$. Typically, the nonlinearities are due to *geometric* and *material* contributions, i.e. the kinematics of the body and the constitutive equation of the material, respectively.

As mentioned above, one main objective of engineering analysis is to find the unknown field $\mathbf{u}$ which is the solution of the associated nonlinear boundary-value problem. Usually, (exact) closed-form mathematical solutions of a set of nonlinear partial differential equations are only available for some special engineering problems; they are rather complicated and often unusable.

In order to keep the complexities of engineering problems intact, approximate numerical solutions, based on, for example, the finite element method, are required. A very common and simple numerical technique to solve nonlinear equations is to employ the reliable *incremental/iterative solution technique* of *Newton's type*. It is an efficient method with the feature of a quadratic convergence rate near the solution point. This technique requires a *consistent linearization* of all the quantities associated with the considered nonlinear problem generating efficient *recurrence update formulas*. The nonlinear problem then is replaced by a sequence of linear problems which are easy to solve at each iteration.

Linearization is a systematic process which is based on the concept of directional derivatives, see the pioneering work of HUGHES and PISTER [1978]; for the more generalized concept see the book by MARSDEN and HUGHES [1994, Chapter 4], and for an application to rods and plates the work of WRIGGERS [1988] among others. For the concepts of linearization and directional derivative and their applications in nonlinear continuum mechanics see also the textbook by BONET and WOOD [1997].

Concept of linearization. The following part of this section deals with the linearization of a nonlinear and smooth (possibly time-dependent) function $\mathcal{F} = \mathcal{F}(\mathbf{u})$ in the *material* description which is either scalar-valued, vector-valued or tensor-valued. The single argument of $\mathcal{F}$ is the displacement vector variable $\mathbf{u}$.

Consider $\mathbf{u}$, then the fundamental relationship for the linearization of the nonlinear function $\mathcal{F}$ is based on the first-order (Taylor's) expansion, which is expressed as

$$\mathcal{F}(\mathbf{u}, \Delta\mathbf{u}) = \mathcal{F}(\mathbf{u}) + \Delta\mathcal{F}(\mathbf{u}, \Delta\mathbf{u}) + o(\Delta\mathbf{u}) \quad , \tag{8.63}$$

where $\Delta(\bullet)$ denotes the linearization operator similar to $\delta(\bullet)$. The operator $\Delta(\bullet)$ is also *linear* and the usual properties of differentiation are valid. The quantity $\Delta\mathbf{u}$ denotes the **increment** of the displacement field $\mathbf{u}$, here expressed in the reference configuration.

The remainder $o(\Delta\mathbf{u})$, characterized by the *Landau order* symbol $o(\bullet)$, is a small error that tends to zero faster than $\Delta\mathbf{u} \to \mathbf{o}$, i.e. $\lim_{\Delta\mathbf{u}\to\mathbf{o}} o(\Delta\mathbf{u})/|\Delta\mathbf{u}| = \mathbf{o}$.

Within the classical solution technique of *Newton's method*, Taylor's expansion is truncated after the first derivative of $\mathcal{F}$. Hence, the first term in (8.63) is a constant part, i.e. an approximate solution for a given state $\mathbf{u}$. The second term $\Delta\mathcal{F}$ is the linear change in $\mathcal{F}$ due to $\Delta\mathbf{u}$ at $\mathbf{u}$. It is the directional derivative of $\mathcal{F}$ at given $\mathbf{u}$ (fixed) in the direction of the incremental displacement field $\Delta\mathbf{u}$, i.e.

$$\Delta\mathcal{F}(\mathbf{u}, \Delta\mathbf{u}) = D_{\Delta\mathbf{u}}\mathcal{F}(\mathbf{u}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\mathbf{u} + \varepsilon\Delta\mathbf{u})|_{\varepsilon=0} \quad , \tag{8.64}$$

where the linear Gâteaux operator $D(\bullet)$ is with respect to the incremental displacement field $\Delta\mathbf{u}$. We say that $\Delta\mathcal{F}(\mathbf{u}, \Delta\mathbf{u})$ is the **linearization** (or **linear approximation**) of $\mathcal{F}$ at $\mathbf{u}$.

Note that in regard to eq. (8.8) the *first variation* $D_{\delta\mathbf{u}}\mathcal{F}(\mathbf{u})$ of a vector function $\mathcal{F}(\mathbf{u})$ and the *linearization* $D_{\Delta\mathbf{u}}\mathcal{F}(\mathbf{u})$ of that vector function are based on the same concept of directional derivatives. By taking notice of this equivalence of variation and linearization, all relations derived in the previous Section 8.1 can be adopted here; we just use the symbol $\Delta(\bullet)$ instead of $\delta(\bullet)$.

For example, relation (8.6) and transformation (8.7) read

$$\Delta(\operatorname{grad}\mathbf{u}) = \operatorname{grad}(\Delta\mathbf{u}) \quad , \tag{8.65}$$

$$\operatorname{grad}\Delta\mathbf{u} = \operatorname{Grad}\Delta\mathbf{u}\,\mathbf{F}^{-1} \qquad \text{or} \qquad \frac{\partial \Delta u_a}{\partial x_b} = \frac{\partial \Delta u_a}{\partial X_A} F^{-1}_{Ab} \quad . \tag{8.66}$$

In addition, the linearizations of tensors $\mathbf{F}$, $\mathbf{F}^{-1}$, $\mathbf{E}$ are

$$\Delta\mathbf{F} = D_{\Delta\mathbf{u}}\mathbf{F} = \operatorname{Grad}\Delta\mathbf{u} \quad , \tag{8.67}$$

$$\Delta\mathbf{F}^{-1} = D_{\Delta\mathbf{u}}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\operatorname{grad}\Delta\mathbf{u} \quad , \tag{8.68}$$

$$\Delta\mathbf{E} = D_{\Delta\mathbf{u}}\mathbf{E} = \frac{1}{2}[(\mathbf{F}^{\mathrm{T}}\operatorname{Grad}\Delta\mathbf{u})^{\mathrm{T}} + \mathbf{F}^{\mathrm{T}}\operatorname{Grad}\Delta\mathbf{u}] = \operatorname{sym}(\mathbf{F}^{\mathrm{T}}\operatorname{Grad}\Delta\mathbf{u}) \quad , \tag{8.69}$$

which are analogous to eqs. (8.9), (8.10), (8.14).

EXAMPLE 8.5 Show that the linearization $\Delta\delta\mathbf{E}$ of the virtual Green-Lagrange strain tensor $\delta\mathbf{E} = \operatorname{sym}(\mathbf{F}^{\mathrm{T}}\operatorname{Grad}\delta\mathbf{u})$, as derived in eq. $(8.14)_2$, may be expressed as

$$\Delta\delta\mathbf{E} = \operatorname{sym}(\operatorname{Grad}^{\mathrm{T}}\Delta\mathbf{u}\,\operatorname{Grad}\delta\mathbf{u}) \quad . \tag{8.70}$$

Solution. According to the rule (8.64), we compute the directional derivative of $\delta\mathbf{E}$ in the direction of $\Delta\mathbf{u}$ at $\mathbf{u}$, i.e.

$$\begin{aligned}\Delta\delta\mathbf{E} = D_{\Delta\mathbf{u}}\delta\mathbf{E} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta\mathbf{E}(\mathbf{u}+\varepsilon\Delta\mathbf{u})|_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\operatorname{sym}[(\mathbf{F}(\mathbf{u}+\varepsilon\Delta\mathbf{u}))^{\mathrm{T}}\operatorname{Grad}\delta\mathbf{u}]|_{\varepsilon=0} \quad .\end{aligned} \tag{8.71}$$

Since the virtual displacement field $\delta\mathbf{u}$ is independent of the displacement, the term $\operatorname{Grad}\delta\mathbf{u}$ is not affected by the linearization. Knowing that $\mathrm{d}/\mathrm{d}\varepsilon\,\mathbf{F}(\mathbf{u}+\varepsilon\Delta\mathbf{u})|_{\varepsilon=0} = \operatorname{Grad}\Delta\mathbf{u}$ (see eq. (8.67)), we find the desired result (8.70). ■

In order to linearize a nonlinear smooth vector function $f = f(\mathbf{u})$ in the *spatial* description we adopt the concept for the first variation of f introduced on p. 375. By analogy with relation (8.16), we may write

$$\Delta f(\mathbf{u}, \Delta\mathbf{u}) = \chi_*(D_{\Delta\mathbf{u}}\chi_*^{-1}(f)) = \chi_*(D_{\Delta\mathbf{u}}\mathcal{F}) \tag{8.72}$$

for the **linearization** (or **linear approximation**) of f. Since $D_{\Delta\mathbf{u}}\mathcal{F}(\mathbf{u}) = \Delta\mathcal{F}(\mathbf{u}, \Delta\mathbf{u})$ according to (8.64) we obtain

$$\Delta f(\mathbf{u}, \Delta\mathbf{u}) = \chi_*(\Delta\mathcal{F}(\mathbf{u}, \Delta\mathbf{u})) \quad , \tag{8.73}$$

which is analogous to eq.(8.17). For notational simplicity, the linearization operator Δ is not particularly marked when applied to a function in the spatial description, as for the δ-process.

Note that the operators required for the *Lie time derivatives*, the *variations* and *linearizations* of spatial tensor variables are formally the same. They are based on the concept of directional derivative. For the Lie time derivative the considered direction of the derivative is $\mathbf{v}$, while for the variation and linearization it is the virtual displacement field $\delta\mathbf{u}$ and the incremental displacement field $\Delta\mathbf{u}$, respectively. Compare relations (2.189), (8.16) and (8.72).

EXAMPLE 8.6 Show that the linearization $\Delta\delta\mathbf{e}$ of the virtual Euler-Almansi strain tensor $\delta\mathbf{e}$, which is a spatial tensor field according to eq. (8.18), may be expressed as

$$\Delta\delta\mathbf{e} = \operatorname{sym}(\operatorname{grad}^{\mathrm{T}}\Delta\mathbf{u}\,\operatorname{grad}\delta\mathbf{u}) \quad . \tag{8.74}$$

Solution. Since we apply the systematic technique of linearization to material quantities, as a first step we pull-back the variation of the Euler-Almansi strain tensor $\delta\mathbf{e}$, that is the inversion of eq. $(8.20)_2$, which yields the variation $\delta\mathbf{E}$ of the associated Green-Lagrange strain tensor $\mathbf{E}$. The linearization of $\delta\mathbf{E}$ is carried out by analogy with Example 8.5. In the last step the push-forward operation on $\Delta\delta\mathbf{E}$, as given in (8.70), is

$$\begin{aligned}\chi_*(\Delta\delta\mathbf{E}) &= \mathbf{F}^{-\mathrm{T}}\operatorname{sym}(\operatorname{Grad}^{\mathrm{T}}\Delta\mathbf{u}\,\operatorname{Grad}\delta\mathbf{u})\mathbf{F}^{-1} \\ &= \operatorname{sym}(\operatorname{grad}^{\mathrm{T}}\Delta\mathbf{u}\,\operatorname{grad}\delta\mathbf{u}) \quad ,\end{aligned} \tag{8.75}$$

where the relations (8.66) and (8.7) should be used. ■

Linearization of the principle of virtual work in material description. In order to linearize the principle of virtual work in the material description we recall the nonlinear variational equation (8.42). For simplicity we consider a purely static problem, so that $\ddot{\mathbf{u}} = \mathbf{o}$. In addition, we assume the loads $\mathbf{B}$ and $\overline{\mathbf{T}}$ to be 'dead' (independent of the deformation of the body), so that the corresponding linearization of the external virtual work (8.45) vanishes, i.e. $D_{\Delta\mathbf{u}}\delta W_{\mathrm{ext}}(\mathbf{u}, \delta\mathbf{u}) = 0$. This is certainly not the case for some other types of loads, like the pressure loads discussed on p. 383 and subsequently.

Hence, the linearization of the variational equation (8.42) only affects the internal virtual work δW_{int}, on which we will focus subsequently. For our purpose we take the material (or Lagrangian) form $(8.46)_2$, i.e.

$$\delta W_{\mathrm{int}}(\mathbf{u}, \delta\mathbf{u}) = \int\limits_{\Omega_0} \mathbf{S}(\mathbf{E}(\mathbf{u})) : \delta\mathbf{E}(\mathbf{u})\mathrm{d}V \quad . \tag{8.76}$$

Note that the Green-Lagrange strain tensor $\mathbf{E}$ depends on the displacement field $\mathbf{u}$

through the relationship (2.91).

Now we may adopt rule (8.64) in order to compute the linearization of (8.76), i.e.

$$\begin{aligned} D_{\Delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\delta W_{\text{int}}(\mathbf{u}+\varepsilon\Delta\mathbf{u})|_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\int\limits_{\Omega_0}\mathbf{S}(\mathbf{E}(\mathbf{u}+\varepsilon\Delta\mathbf{u})):\delta\mathbf{E}(\mathbf{u}+\varepsilon\Delta\mathbf{u})\mathrm{d}V]|_{\varepsilon=0} \quad . \end{aligned} \tag{8.77}$$

Interchanging differentiation and integration and using the product rule results in

$$D_{\Delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega_0}[\mathbf{S}(\mathbf{E}(\mathbf{u})):D_{\Delta\mathbf{u}}\delta\mathbf{E}(\mathbf{u})+\delta\mathbf{E}(\mathbf{u}):D_{\Delta\mathbf{u}}\mathbf{S}(\mathbf{E}(\mathbf{u}))]\mathrm{d}V \quad , \tag{8.78}$$

where $D_{\Delta\mathbf{u}}\delta\mathbf{E}$ characterizes the directional derivative of $\delta\mathbf{E}$ at $\mathbf{u}$ in the direction of $\Delta\mathbf{u}$, i.e. the linearization of $\delta\mathbf{E}$ according to eq. (8.70).

In order to specify the linearization $D_{\Delta\mathbf{u}}\mathbf{S}$ of the (symmetric) second Piola-Kirchhoff stress tensor $\mathbf{S}$ in eq. (8.78), use the chain rule to obtain

$$D_{\Delta\mathbf{u}}\mathbf{S}(\mathbf{E}(\mathbf{u})) = \frac{\partial\mathbf{S}(\mathbf{E}(\mathbf{u}))}{\partial\mathbf{E}}:D_{\Delta\mathbf{u}}\mathbf{E}(\mathbf{u}) = \mathbb{C}(\mathbf{u}):D_{\Delta\mathbf{u}}\mathbf{E}(\mathbf{u}) \quad , \tag{8.79}$$

with $D_{\Delta\mathbf{u}}\mathbf{E}$ denoting the linearization of the Green-Lagrange strain tensor $\mathbf{E}$ (see relations (8.69)). It is important to emphasize that the term $\partial\mathbf{S}(\mathbf{E})/\partial\mathbf{E}$ is precisely the elasticity tensor $\mathbb{C}$ in the material description, as defined in eq. (6.155). It is a fourth-order tensor which possesses the *minor symmetries* $C_{ABCD} = C_{BACD} = C_{ABDC}$.

Hence, eq. (8.78) may be re-expressed as

$$D_{\Delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega_0}[\mathbf{S}(\mathbf{E}(\mathbf{u})):D_{\Delta\mathbf{u}}\delta\mathbf{E}(\mathbf{u})+\delta\mathbf{E}(\mathbf{u}):\mathbb{C}(\mathbf{u}):D_{\Delta\mathbf{u}}\mathbf{E}(\mathbf{u})]\mathrm{d}V \quad . \tag{8.80}$$

Finally, we use the explicit expression (8.70) and property (1.95) for the first term of the integral in eq. (8.80) and relations (8.14) and (8.69) for the second term. Since the stress tensor $\mathbf{S}$ is symmetric and the elasticity tensor $\mathbb{C}$ has minor symmetries, the linearization of the internal virtual work in the material description leads to the set of linear increments

$$\begin{aligned} D_{\Delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega_0}&(\mathrm{Grad}\delta\mathbf{u}:\mathrm{Grad}\Delta\mathbf{u}\,\mathbf{S} \\ &+\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}:\mathbb{C}:\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u})\mathrm{d}V \quad , \end{aligned} \tag{8.81}$$

or, in index notation,

$$\begin{aligned} D_{\Delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) &= \int\limits_{\Omega_0}\left(\frac{\partial\delta u_a}{\partial X_B}\frac{\partial\Delta u_a}{\partial X_D}S_{BD}+F_{aA}\frac{\partial\delta u_a}{\partial X_B}C_{ABCD}F_{bC}\frac{\partial\Delta u_b}{\partial X_D}\right)\mathrm{d}V \\ &= \int\limits_{\Omega_0}\frac{\partial\delta u_a}{\partial X_B}\left(\delta_{ab}S_{BD}+F_{aA}F_{bC}C_{ABCD}\right)\frac{\partial\Delta u_b}{\partial X_D}\mathrm{d}V \quad , \end{aligned} \tag{8.82}$$

which describes the fully nonlinear (finite) deformation case. The terms $\delta_{ab}S_{BD}$ and $F_{aA}F_{bC}C_{ABCD}$ represent the effective elasticity tensor, which has the nature of the *(tangent) stiffness matrix.*

Relations (8.81) and (8.82) are linear with respect to $\delta\mathbf{u}$ and $\Delta\mathbf{u}$ depending on $\mathbf{X}$. These relations show a clear mathematical structure in the sense that $\delta\mathbf{u}$ and $\Delta\mathbf{u}$ can be interchanged without altering the result of the integral; thus $D_{\Delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) = D_{\delta\mathbf{u}}\delta W_{\text{int}}(\mathbf{u},\Delta\mathbf{u})$. Relations (8.81) and (8.82) lead to a *symmetric* (tangent) stiffness matrix upon discretization. Note that, for example, the set of nonlinear equations associated with nonlinear heat conduction results in a different mathematical structure leading to a *non-symmetric* stiffness matrix.

The first term in eq. (8.81) comes from the current state of stress and represents the so-called **geometrical stress contribution** (in the literature sometimes called the **initial stress contribution**) to the linearization. Since S_{AB} is not the initial stress (it is in fact the current stress), the terminology is misleading. Within an *incremental/iterative solution technique* we can think of S_{AB} as the initial stress at every increment, so the term initial stress contribution has some meaning. The second term in eq. (8.81) represents the so-called **material contribution** to the linearization.

The linearized principle of virtual work (8.81) constitutes the starting point for approximation techniques such as the finite element method, typically leading to the **geometrical** (or **initial stress**) **stiffness matrix** and to the **material stiffness matrix**.

Note that for some cases it is more convenient to discretize the nonlinear variational equation as a first step and to linearize the result with respect to the positions of the nodal points as a second step.

Linearization of the principle of virtual work in spatial description. In order to linearize the principle of virtual work in the spatial description we recall the nonlinear variational equation (8.33).

As above we consider the static case ($\ddot{\mathbf{u}} = \mathbf{o}$) and assume the loads $\mathbf{b}$ and $\bar{\mathbf{t}}$ to be independent of the motion of the body. Only the linearization of the internal virtual work δW_{int} in the spatial description remains. We adopt δW_{int} in the spatial (or Eulerian) form (8.34). The idea is first to pull-back the spatial quantities to the reference configuration, so they correspond with the internal virtual work in the material description. Then they are linearized, as above, and as a last step it is necessary to push-forward the linearized terms.

Starting with the equivalence

$$\delta W_{\text{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega}\boldsymbol{\sigma}(\mathbf{u}):\delta\mathbf{e}(\mathbf{u})\mathrm{d}v = \int\limits_{\Omega_0}\mathbf{S}(\mathbf{E}(\mathbf{u})):\delta\mathbf{E}(\mathbf{u})\mathrm{d}V \quad , \tag{8.83}$$

we consider the linerization of the internal virtual work in the material description which we have derived in (8.78), i.e.

$$D_{\Delta\mathbf{u}}\delta W_{\mathrm{int}} = \int\limits_{\Omega_0} (\mathbf{S} : D_{\Delta\mathbf{u}}\delta\mathbf{E} + \delta\mathbf{E} : D_{\Delta\mathbf{u}}\mathbf{S})\mathrm{d}V \tag{8.84}$$

(the arguments have been omitted).

Hence, the push-forward operation on the second Piola-Kirchhoff stress tensor $\mathbf{S}$ yields, according to (3.64), the Kirchhoff-stress tensor $\boldsymbol{\tau}$, which is related to the Cauchy stress tensor by $\boldsymbol{\tau} = J\boldsymbol{\sigma}$. Pushing forward the linearized variation of the Green-Lagrange strain tensor, $\Delta\delta\mathbf{E}$, yields the linearized variation of the Euler-Almansi strain tensor, $\Delta\delta\mathbf{e}$, as discussed in Example 8.6. Computing the push-forward of $\delta\mathbf{E}$ results in $\delta\mathbf{e}$, as introduced in Section 8.1.

Finally, we derive the push-forward of the linearized second Piola-Kirchhoff stress tensor, i.e. the last term in (8.84), which will yield the linearized Kirchhoff stress tensor $\Delta\boldsymbol{\tau}$. We write

$$\Delta\boldsymbol{\tau} = \chi_*(D_{\Delta\mathbf{u}}\mathbf{S}) = \mathbf{F}(D_{\Delta\mathbf{u}}\mathbf{S})\mathbf{F}^{\mathrm{T}} \quad , \tag{8.85}$$

with $D_{\Delta\mathbf{u}}\mathbf{S}$ given by (8.79), i.e. $D_{\Delta\mathbf{u}}\mathbf{S} = \mathbb{C} : D_{\Delta\mathbf{u}}\mathbf{E}$. By use of (8.69) and (8.66), the term $\mathbf{F}(D_{\Delta\mathbf{u}}\mathbf{S})\mathbf{F}^{\mathrm{T}}$ in eq. $(8.85)_2$ may be written as

$$\mathbf{F}(\mathbb{C} : \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u})\mathbf{F}^{\mathrm{T}} = \mathbf{F}(\mathbb{C} : \mathbf{F}^{\mathrm{T}}\mathrm{grad}\Delta\mathbf{u}\,\mathbf{F})\mathbf{F}^{\mathrm{T}} \quad , \tag{8.86}$$

where we have also employed the minor symmetries of $\mathbb{C}$. In order to proceed it is more instructive to employ index notation. With the definition (6.159) of the spatial elasticity tensor $(\mathbb{c})_{abcd} = c_{abcd}$, eq. (8.86) is equivalent to

$$\begin{aligned} F_{aA}C_{ABCD}F_{cC}\frac{\partial\Delta u_c}{\partial x_d}F_{dD}F_{bB} &= F_{aA}F_{bB}F_{cC}F_{dD}C_{ABCD}\frac{\partial\Delta u_c}{\partial x_d} \\ &= Jc_{abcd}\frac{\partial\Delta u_c}{\partial x_d} \quad . \end{aligned} \tag{8.87}$$

Hence, the linearization of the spatial Kirchhoff stress tensor, i.e. (8.85), gives the useful relation

$$\Delta\boldsymbol{\tau} = J\mathbb{c} : \mathrm{grad}\Delta\mathbf{u} \quad . \tag{8.88}$$

Note that the increment $\Delta\boldsymbol{\tau}$ denotes the linearized tensor-valued function $\boldsymbol{\tau}$ according to the concept of directional derivative introduced in (8.72). Replacing the associated direction $\Delta\mathbf{u}$, used in the directional derivative, by the velocity vector $\mathbf{v}$, $\Delta\boldsymbol{\tau}$ and $\mathrm{grad}\Delta\mathbf{u}$ result in the Lie time derivative $\pounds_{\mathbf{v}}(\boldsymbol{\tau})$ of $\boldsymbol{\tau}$ and the spatial velocity gradient $\mathbf{l}$ (defined by $(2.141)_4$), respectively. By using the symmetries of $\mathbb{c}$ relation (8.88) reads $\pounds_{\mathbf{v}}(\boldsymbol{\tau}) = J\mathbb{c} : \mathbf{d}$, which proves (6.161).

Considering the push-forward operations derived we obtain finally, from (8.84) with relation $\mathrm{d}v = J\mathrm{d}V$ and some rearranging, the linearized internal virtual work in the spatial description, i.e.

$$D_{\Delta\mathbf{u}}\delta W_{\mathrm{int}}(\mathbf{u}, \delta\mathbf{u}) = \int\limits_{\Omega} (\mathrm{grad}\delta\mathbf{u} : \mathrm{grad}\Delta\mathbf{u}\,\boldsymbol{\sigma} + \mathrm{grad}\delta\mathbf{u} : \mathbb{c} : \mathrm{grad}\Delta\mathbf{u})\mathrm{d}v \quad , \tag{8.89}$$

or, in index notation,

$$\begin{aligned} D_{\Delta\mathbf{u}}\delta W_{\mathrm{int}}(\mathbf{u}, \delta\mathbf{u}) &= \int\limits_{\Omega} \left(\frac{\partial\delta u_a}{\partial x_b}\frac{\partial\Delta u_a}{\partial x_d}\sigma_{bd} + \frac{\partial\delta u_a}{\partial x_b}c_{abcd}\frac{\partial\Delta u_c}{\partial x_d} \right) \mathrm{d}v \\ &= \int\limits_{\Omega} \frac{\partial\delta u_a}{\partial x_b}\left(\delta_{ac}\sigma_{bd} + c_{abcd}\right)\frac{\partial\Delta u_c}{\partial x_d}\mathrm{d}v \quad , \end{aligned} \tag{8.90}$$

where $\delta_{ac}\sigma_{bd} + c_{abcd}$ represents the effective elasticity tensor in the spatial description. Relations (8.89) and (8.90) are linear with respect to the terms $\delta\mathbf{u}$ and $\Delta\mathbf{u}$. They describe the fully nonlinear (finite) deformation case and have a similar symmetric structure to the linearized eqs. (8.81) and (8.82).

Formulations according to (8.81) and (8.89) are in the literature sometimes called **total-Lagrangian** and **updated-Lagrangian**, respectively. This really means that integrals are calculated over the respective regions of the reference and the current configuration. However, it is important to emphasize that the derived material representations (8.81) and (8.82) of the linearized virtual internal work are **equivalent** to the spatial versions (8.89) and (8.90). The two representations are based on the use of change of variables, and the results are the same in both cases.

In order to recapture the small deformation (but nonlinear elastic) case (in the literature often called the materially nonlinear case) we fix the geometry, i.e. we do not distinguish between initial and current geometry. Further, we do not account for the initial stress contribution $\delta_{ac}\sigma_{bd}$ and ignore the quadratic terms in the Green-Lagrange strain tensor. In addition, for the fully linear case, the coefficients of the elasticity tensor are given and are not functions of strains anymore.

EXAMPLE 8.7 An alternative approach to derive the linearized internal virtual work (8.89) in the spatial description is to use the formal equivalence of the material time derivative of scalar-valued functions in the spatial description with the directional derivative of these functions in the direction $\Delta\mathbf{u}$ (compare with Section 2.8).

Carry out the material time derivative of the internal virtual work δW_{int} in the spatial description and use this property to obtain its linearization. This approach circumvents the extensive pull-back and push-forward operations.

Solution. We again start with δW_{int} in the spatial form (8.34) and change the domain of integration with $\mathrm{d}v = J\mathrm{d}V$ so that

$$\delta W_{\mathrm{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \delta\mathbf{e}(\mathbf{u})\mathrm{d}v = \int\limits_{\Omega_0} J\boldsymbol{\sigma}(\mathbf{u}) : \delta\mathbf{e}(\mathbf{u})\mathrm{d}V \quad , \tag{8.91}$$

where the Euler-Almansi strain tensor $\mathbf{e}$ depends on the displacement field $\mathbf{u}$ through the relationship (2.92). The Cauchy stress tensor $\boldsymbol{\sigma}$ also depends on $\mathbf{u}$, here a function of the current position $\mathbf{x}$.

Employing material time derivatives and the product rule we find from $(8.91)_2$ that

$$\dot{\overline{\delta W_{\mathrm{int}}(\mathbf{u},\delta\mathbf{u})}} = \int\limits_{\Omega_0} [\boldsymbol{\tau}(\mathbf{u}) : \dot{\overline{\delta\mathbf{e}(\mathbf{u})}} + \delta\mathbf{e}(\mathbf{u}) : \dot{\overline{\boldsymbol{\tau}(\mathbf{u})}}]\mathrm{d}V \quad , \tag{8.92}$$

where additionally the relationship between the symmetric Kirchhoff-stress tensor $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, i.e. $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, is to be used. Recall that the superposed dot denotes the material time derivative, as usual.

Firstly, we derive the material time derivative of the virtual Euler-Almansi strain tensor $\delta\mathbf{e}$. By means of (8.7) and $(2.145)_2$, we find from $(8.18)_2$ that

$$\begin{aligned} \dot{\overline{\delta\mathbf{e}}} &= \dot{\overline{\mathrm{sym}(\mathrm{grad}\delta\mathbf{u})}} = \dot{\overline{\mathrm{sym}(\mathrm{Grad}\delta\mathbf{u}\,\mathbf{F}^{-1})}} \\ &= \mathrm{sym}(\mathrm{Grad}\delta\mathbf{u}\dot{\overline{\mathbf{F}^{-1}}}) = \mathrm{sym}[\mathrm{Grad}\delta\mathbf{u}(-\mathbf{F}^{-1}\mathbf{l})] \\ &= -\mathrm{sym}(\mathrm{grad}\delta\mathbf{u}\,\mathbf{l}) \quad , \end{aligned} \tag{8.93}$$

with the spatial velocity gradient $\mathbf{l} = \mathrm{grad}\mathbf{v}$ (recall definition (2.137)).

Secondly, we focus attention on the crucial material time derivative of the Kirchhoff stress tensor $\boldsymbol{\tau}$. We start with the relation for the (objective) *Oldroyd stress rate* of the spatial stress field $\boldsymbol{\tau}$. Recall the *Oldroyd stress rate* of $\boldsymbol{\tau}$ which is identical to the Lie time derivative of $\boldsymbol{\tau}$, and with reference to eq. (5.59) expressed as $\pounds_{\mathbf{v}}(\boldsymbol{\tau}) = \dot{\boldsymbol{\tau}} - \mathbf{l}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}}$. By means of $\pounds_{\mathbf{v}}(\boldsymbol{\tau}) = J\mathbb{c} : \mathbf{d}$, i.e. eq. (6.161), we conclude that

$$\dot{\boldsymbol{\tau}} = J\mathbb{c} : \mathbf{d} + \mathbf{l}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}} \quad , \tag{8.94}$$

where $\mathbb{c}$ and $\mathbf{d} = \mathrm{sym}(\mathbf{l})$ are the spatial elasticity tensor and the rate of deformation tensor (compare with definitions (6.159) and (2.148), respectively).

Substituting relations $(8.93)_5$, (8.18) and (8.94) into (8.92) and using the symmetry of $\boldsymbol{\tau}$ and the minor symmetries of $\mathbb{c}$, i.e. $c_{abcd} = c_{bacd} = c_{abdc}$, we obtain

$$\begin{aligned} \dot{\overline{\delta W_{\mathrm{int}}(\mathbf{u},\delta\mathbf{u})}} = \int\limits_{\Omega_0} [&\boldsymbol{\tau} : (-\mathrm{grad}\delta\mathbf{u}\,\mathbf{l}) + \mathrm{grad}\delta\mathbf{u} : J\mathbb{c} : \mathbf{d} \\ &+ \mathrm{grad}\delta\mathbf{u} : \mathbf{l}\boldsymbol{\tau} + \mathrm{grad}\delta\mathbf{u} : \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}}]\mathrm{d}V \quad . \end{aligned} \tag{8.95}$$

By use of property (1.95) the sum of the first and last terms in eq. (8.95) vanishes and we obtain

$$\dot{\overline{\delta W_{\mathrm{int}}(\mathbf{u},\delta\mathbf{u})}} = \int\limits_{\Omega_0} (\mathrm{grad}\delta\mathbf{u} : J\mathbb{c} : \mathbf{d} + \mathrm{grad}\delta\mathbf{u} : \mathbf{l}\boldsymbol{\tau})\mathrm{d}V \quad . \tag{8.96}$$

Applying now the formal equivalence of the material time derivatives and the directional derivatives, and replacing the spatial velocity field $\mathbf{v}$ by the linear increment $\Delta\mathbf{u}$, we may rewrite (8.96) as

$$D_{\Delta\mathbf{u}}\delta W_{\mathrm{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega_0} (\mathrm{grad}\delta\mathbf{u} : J\mathbb{c} : \mathrm{grad}\Delta\mathbf{u} + \mathrm{grad}\delta\mathbf{u} : \mathrm{grad}\Delta\mathbf{u}\,\boldsymbol{\tau})\mathrm{d}V \quad . \tag{8.97}$$

Changing the domain of integration back and using the stress transformation $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ again we arrive finally at the linearized internal virtual work in the spatial description, i.e. (8.89). ■

EXERCISES

1. Take the part $\delta_{ac}\sigma_{bd}$ of the effective elasticity tensor due to the current stresses in the spatial description.

   (a) Show that this term possesses the *major symmetry*, that means that nothing changes by the manipulation

   $$\delta_{ac}\sigma_{bd} = \delta_{ca}\sigma_{db} \quad .$$

   (b) Further, show that it does not possess the *minor symmetry*, i.e.

   $$\delta_{ac}\sigma_{bd} \neq \delta_{ab}\sigma_{cd} \quad , \qquad \delta_{ac}\sigma_{bd} \neq \delta_{dc}\sigma_{ba} \quad .$$

   However, for the case in which the term $\delta_{ac}\sigma_{bd}$ is absent, under the assumption of a hyperelastic material the minor *and* the major symmetry of $\mathbb{c}_{abcd}$ hold. Discuss the consequences in regard to finite element discretizations.

2. Show that the linearization of the *internal virtual work* δW_{int} may also be written as

   $$D_{\Delta\mathbf{u}}\delta W_{\mathrm{int}}(\mathbf{u},\delta\mathbf{u}) = \int\limits_{\Omega_0} \mathrm{Grad}\delta\mathbf{u} : \mathbb{A} : \mathrm{Grad}\Delta\mathbf{u}\mathrm{d}V \quad ,$$

   where $\mathbb{A}$ is a (mixed) fourth-order tensor useful in numerical implementations. It is known as the **(first) elasticity tensor** with the definition

   $$\mathbb{A} = \frac{\partial\mathbf{P}(\mathbf{F})}{\partial\mathbf{F}} \qquad \text{or} \qquad A_{aAbB} = \frac{\partial P_{aA}}{\partial F_{bB}} \quad ,$$

where **P** denotes the first Piola-Kirchhoff stress tensor depending on the deformation gradient **F**.

Note that the linearization of δW_{int} contains one term only (compare with relation (8.81)).

3. Carry out the linearization of the *external virtual work* δW_{ext}, assuming pressure boundary loading, as derived in eq. (8.38). Consider a constant pressure load p and show by means of the product rule and relation (2.5) that

$$D_{\Delta\mathbf{u}}\delta W_{\text{ext}}(\mathbf{u},\delta\mathbf{u}) = p\int\limits_{\Omega_\xi}\left(\frac{\partial\Delta\mathbf{u}}{\partial\xi_1}\times\frac{\partial\gamma}{\partial\xi_2}-\frac{\partial\Delta\mathbf{u}}{\partial\xi_2}\times\frac{\partial\gamma}{\partial\xi_1}\right)\cdot\delta\mathbf{u}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \quad . \tag{8.98}$$

Note that the two terms in eq. (8.98) are not symmetric in $\delta\mathbf{u}$ and $\Delta\mathbf{u}$. In finite element discretizations the associated tangent stiffness matrix is also not, in general, symmetric.

8.5 Two-field Variational Principles

So far we have considered single-field variational principles such as the principle of virtual work. It is not always the best principle to choose, particularly when constraint conditions are imposed on the deformation. In finite element analyses of problems which are associated with constraint conditions, significant numerical difficulties must be expected within the context of a **Galerkin method**, i.e. a standard *displacement-based method* in which only the displacement field is discretized. This method exhibits rather poor numerical performances such as penalty sensitivity and *ill-conditioning* of the stiffness matrix, as is well-known from

(i) the numerical analysis of rubber, which is frequently modeled as a nearly incompressible or incompressible material,

(ii) bending dominated (plate and shell) problems,

(iii) elastoplastic problems that are based on J_2-flow theory (the plastic flow is isochoric), and

(iv) Stokes' flow, which mathematically yields a problem identical to that for isotropic incompressible elasticity.

In the computational literature these devastating numerical difficulties are referred to as **locking phenomena**. Essentially, these locking difficulties arise from the overstiffening of the system and are associated with a significant loss of accuracy observed, in particular, with low-order finite elements (for fundamental studies and for more references see the books by ZIENKIEWICZ and TAYLOR [1989, 1991] and HUGHES [2000]).

To eliminate these difficulties inherent in the conventional single-field variational approach a great deal of research effort by engineers and mathematicians has been devoted to the developments of efficient so-called **mixed finite element methods** (see HUGHES [2000, and references therein]). For a more mathematically oriented presentation see the work of BREZZI and FORTIN [1991].

For these types of methods the constraints imposed on the deformation are dealt with within a variational sense resulting in effective *multi-field variational principles*. This approach has attracted considerable attention in the computational mechanics literature. Besides the usual displacement field, a mixed (finite element) method incorporates one or more additional fields (typically the internal pressure field, the volume ratio field ...) which are treated as independent variables. The basic idea within the mixed finite element method is to discretize these additional variables independently with the aim of achieving *nonlocking* and *stable* numerical solutions in the incompressible limit.

Lagrange-multiplier method. In the subsequent part we focus attention on a suitable variational approach which captures nearly incompressible and incompressible hyperelastic material response.

Rubber or rubber-like materials may show a very high resistance to volumetric changes compared with that to isochoric changes. Typically, the ratio of the bulk modulus to the shear modulus is roughly of four orders of magnitude (exhibiting an almost incompressible response), and a very careful numerical treatment is needed. A standard displacement-based method cannot be applied directly to these types of problems, because locking or instabilities will occur. Hence, almost all hyperelastic materials which show a nearly incompressible or incompressible deformation behavior are treated with a mixed finite element formulation.

The **Lagrange-multiplier method**, in which a constraint is introduced using a scalar parameter called the **Lagrange multiplier**, is often used to prevent volumetric locking. Utilizing p as the Lagrange multiplier to enforce the incompressibility constraint $J = 1$, we may formulate a functional Π_{L} in the decoupled representation

$$\Pi_{\text{L}}(\mathbf{u},p) = \Pi_{\text{int}}(\mathbf{u},p) + \Pi_{\text{ext}}(\mathbf{u}) \quad , \tag{8.99}$$

$$\Pi_{\text{int}}(\mathbf{u},p) = \int\limits_{\Omega_0}[p(J(\mathbf{u})-1)+\Psi_{\text{iso}}(\overline{\mathbf{C}}(\mathbf{u}))]\mathrm{d}V \quad , \tag{8.100}$$

where the term $p(J(\mathbf{u})-1)$ denotes the Lagrange-multiplier term, with the volume ratio $J = J(\mathbf{u}) = (\det\mathbf{C})^{1/2}$. The function $\Psi_{\text{iso}} = \Psi_{\text{iso}}(\overline{\mathbf{C}})$ characterizes the isochoric elastic response of the hyperelastic material with the corresponding modified right Cauchy-Green tensor $\overline{\mathbf{C}} = \overline{\mathbf{C}}(\mathbf{u}) = J^{-2/3}\mathbf{C}$. The loads do not depend on the motion of the body, so that the external potential energy Π_{ext} is given by the standard relation $(8.48)_2$.

Note that the Lagrange-multiplier term vanishes for the case of incompressible finite elasticity. In addition, it is important to emphasize that the solution of the Lagrange-multiplier method can be recovered from the penalty method just by taking the restriction on the value $\kappa \to \infty$ (compare with the outline given on p. 389). Hence, the penalty method may also be seen as an approximation to the Lagrange-multiplier method.

The Lagrange multiplier p plays the role of the (physical) *hydrostatic pressure*. The described formulation gives rise to a two-field mixed finite element implementation involving $\mathbf{u}$ and p as the independent field variables. Since we consider extra numbers of unknowns this technique requires additional computational effort.

The main objective is now to derive a two-field variational principle for finite elasticity by finding the stationary point of functional Π_{L}. In other words, we must compute and set to zero the directional derivatives of Π_{L} with respect to both the displacement field $\mathbf{u}$ and the hydrostatic pressure field p.

In addition to the virtual displacement field $\delta\mathbf{u}$ we need an arbitrary smooth scalar function $\delta p(\mathbf{x}) = \delta p(\boldsymbol{\chi}(\mathbf{X})) = \delta p(\mathbf{X})$, which is interpreted as the **virtual pressure field**, here defined in the reference configuration. We find the stationary conditions with respect to $\mathbf{u}$ and p as

$$D_{\delta\mathbf{u}}\Pi_{\mathrm{L}}(\mathbf{u},p) = 0 \qquad \text{and} \qquad D_{\delta p}\Pi_{\mathrm{L}}(\mathbf{u},p) = 0 \tag{8.101}$$

for all $\delta\mathbf{u}$ satisfying $\delta\mathbf{u} = \mathbf{o}$ on the boundary surface $\partial\Omega_{0\,\mathrm{u}}$ and all δp.

Firstly, we compute the directional derivative of (8.99) in the direction of an arbitrary virtual displacement $\delta\mathbf{u}$. By means of the chain rule and the relations derived in Section 8.1, i.e. $D_{\delta\mathbf{u}}J = \delta J = J\mathrm{div}\delta\mathbf{u}$ and $D_{\delta\mathbf{u}}\mathbf{C} = \delta\mathbf{C} = 2\delta\mathbf{E}$, we obtain, with reference to $(8.101)_1$, the weak form

$$\begin{aligned} D_{\delta\mathbf{u}}\Pi_{\mathrm{L}} &= \int\limits_{\Omega_0} \left(pD_{\delta\mathbf{u}}J + 2\frac{\partial\Psi_{\mathrm{iso}}}{\partial\mathbf{C}} : \frac{1}{2}D_{\delta\mathbf{u}}\mathbf{C} \right) \mathrm{d}V + D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}} \\ &= \int\limits_{\Omega_0} \left(Jp\mathrm{div}\delta\mathbf{u} + 2\frac{\partial\Psi_{\mathrm{iso}}}{\partial\mathbf{C}} : \delta\mathbf{E} \right) \mathrm{d}V + D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}} = 0 \end{aligned} \tag{8.102}$$

(the arguments have been omitted). The contribution due to the loads is given by eq. $(8.51)_2$ as $D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}} = -\delta W_{\mathrm{ext}}$, with the external virtual work δW_{ext}, i.e. (8.45).

In order to rewrite the term $Jp\mathrm{div}\delta\mathbf{u}$ of $(8.102)_2$ we use the analogue of eq. $(1.279)_2$ and invoke relation (8.7) and property (1.95) to obtain

$$\begin{aligned} \mathrm{div}\delta\mathbf{u} &= \mathbf{I} : \mathrm{grad}\delta\mathbf{u} = \mathbf{I} : \mathrm{Grad}\delta\mathbf{u}\,\mathbf{F}^{-1} \\ &= \mathbf{F}^{-\mathrm{T}} : \mathrm{Grad}\delta\mathbf{u} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} : \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u} \\ &= \mathbf{C}^{-1} : \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u} \quad , \end{aligned} \tag{8.103}$$

with the inverse right Cauchy-Green tensor $\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$. Since $\mathbf{C}^{-1}$ is symmetric we find finally, by analogy with property (1.115) and with $(8.14)_2$, that

$$\mathrm{div}\delta\mathbf{u} = \mathbf{C}^{-1} : \delta\mathbf{E} \quad . \tag{8.104}$$

This result substituted back into $(8.102)_2$ gives

$$\begin{aligned} D_{\delta\mathbf{u}}\Pi_{\mathrm{L}}(\mathbf{u},p) = \int\limits_{\Omega_0} &\left(J(\mathbf{u})p\mathbf{C}^{-1}(\mathbf{u}) + 2\frac{\partial\Psi_{\mathrm{iso}}(\overline{\mathbf{C}}(\mathbf{u}))}{\partial\mathbf{C}} \right) : \delta\mathbf{E}(\mathbf{u})\mathrm{d}V \\ &+ D_{\delta\mathbf{u}}\Pi_{\mathrm{ext}}(\mathbf{u}) = 0 \quad . \end{aligned} \tag{8.105}$$

By recalling the unique additive decomposition for the stress, i.e. eq. $(6.88)_2$ with definitions $(6.89)_2$ and $(6.90)_1$, we recognize that the two terms in parentheses of the variational equation (8.105) give precisely the second Piola-Kirchhoff stress tensor $\mathbf{S} = \mathbf{S}(\mathbf{E}(\mathbf{u}))$. Hence, the integral in (8.105) characterizes the volumetric and isochoric contributions to the internal virtual work δW_{int} (see eq. $(8.46)_2$).

Relation (8.105) is identified as the standard principle of virtual work expressed in the reference configuration (for a configuration in static equilibrium), i.e. $\delta W_{\mathrm{int}} - \delta W_{\mathrm{ext}} = 0$. The associated Euler-Lagrange equation is Cauchy's equation of equilibrium.

Secondly, we compute the directional derivative of (8.99) in the direction of an arbitrary virtual pressure δp. With reference to $(8.101)_2$, we obtain the weak form

$$D_{\delta p}\Pi_{\mathrm{L}}(\mathbf{u},p) = \int\limits_{\Omega_0} (J(\mathbf{u}) - 1)\delta p\mathrm{d}V = 0 \quad . \tag{8.106}$$

As can be seen, the pressure variable p maintains the incompressibility constraint and we find the associated Euler-Lagrange equation to be $J = 1$ (see also the study by LE TALLEC [1994]).

The variational equations (8.105) and (8.106) provide the fundamental basis for a finite element implementation.

Linearization of the Lagrange-multiplier method. In order to solve the nonlinear equations (8.105) and (8.106) for the two independent variables $\mathbf{u}$ and p on an incremental/iterative basis, a *Newton type* method is usually employed (recall Section 8.4). In preparation for an incremental/iterative solution technique a systematic linearization of (8.105) and (8.106) with respect to $\mathbf{u}$ and p, essentially the second variation of (8.99), is required.

In order to linearize variational equation (8.106) in the directions of the increments $\Delta\mathbf{u}$ and Δp we recall that $D_{\Delta\mathbf{u}}J = J\mathrm{div}\Delta\mathbf{u}$. Employing the concept of directional derivative we obtain

$$D^2_{\delta p,\Delta\mathbf{u}}\Pi_{\mathrm{L}}(\mathbf{u},p) = \int\limits_{\Omega_0} J(\mathbf{u})\mathrm{div}\Delta\mathbf{u}\,\delta p\mathrm{d}V \quad , \qquad D^2_{\delta p,\Delta p}\Pi_{\mathrm{L}}(\mathbf{u},p) = 0 \quad . \tag{8.107}$$

The linearization of the principle of virtual work (8.105) in the direction of the increment Δp gives

$$D^2_{\delta\mathbf{u},\Delta p}\Pi_{\mathrm{L}}(\mathbf{u},p) = \int\limits_{\Omega_0} J(\mathbf{u})\Delta p\mathbf{C}^{-1}(\mathbf{u}) : \delta\mathbf{E}(\mathbf{u})\mathrm{d}V = \int\limits_{\Omega_0} J(\mathbf{u})\Delta p \operatorname{div}\delta\mathbf{u}\mathrm{d}V \quad , \tag{8.108}$$

while a linearization process in the direction of $\Delta\mathbf{u}$ was already carried out in detail within the last section, leading to

$$\begin{aligned} D^2_{\delta\mathbf{u},\Delta\mathbf{u}}\Pi_{\mathrm{L}}(\mathbf{u},p) = \int\limits_{\Omega_0} (&\mathrm{Grad}\delta\mathbf{u} : \mathrm{Grad}\Delta\mathbf{u}\,\mathbf{S} \\ &+\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u} : (\mathbb{C}_{\mathrm{vol}} + \mathbb{C}_{\mathrm{iso}}) : \mathbf{F}^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u})\mathrm{d}V \quad . \end{aligned} \tag{8.109}$$

Since (8.105) is based on the additive decomposition of the stresses $\mathbf{S}$, we obtain the decoupled representation of the elasticity tensor $\mathbb{C} = \mathbb{C}_{\mathrm{vol}} + \mathbb{C}_{\mathrm{iso}}$, with the definitions

$$\mathbb{C}_{\mathrm{vol}} = 2\frac{\partial(J(\mathbf{u})p\mathbf{C}^{-1}(\mathbf{u}))}{\partial\mathbf{C}} \quad \text{and} \quad \mathbb{C}_{\mathrm{iso}} = 4\frac{\partial^2\Psi_{\mathrm{iso}}(\overline{\mathbf{C}}(\mathbf{u}))}{\partial\mathbf{C}\partial\mathbf{C}} \quad . \tag{8.110}$$

For an explicit treatment of these expressions recall Section 6.6, in particular relations $(6.166)_4$ and (6.168). Note that for the considered case the scalar quantity $\tilde{p}$ must be replaced by p in eq. $(6.166)_4$.

Since the structure of the linearized principle of virtual work is symmetric and because of the symmetry between eqs. $(8.107)_1$ and (8.108) a finite element implementation of this set of equations will lead to a symmetric (tangent) stiffness matrix.

In order to use these equations within a finite element regime so-called **interpolation functions** must be invoked separately for the displacement field $\mathbf{u}$, the pressure field p and their variations $\delta\mathbf{u}$ and δp, respectively (see SUSSMAN and BATHE [1987], ZIENKIEWICZ and TAYLOR [1989] among others). A well-considered choice of these functions is a crucial task in order to alleviate volumetric locking. It was observed that a discontinuous (constant) pressure and a continuous displacement interpolation over a typical finite element domain is computationally more efficient than with the choice of functions of the same order for $\mathbf{u}$ and p.

Perturbed Lagrange-multiplier method. The Lagrange-multiplier method results in a stiffness matrix which is not positive definite for incompressible materials. In order to overcome the numerical difficulties associated with this fact and to avoid ill-conditioning of the stiffness matrix associated with the penalty approach, regularization procedures such as the so-called **perturbed Lagrange-multiplier method** have been introduced successfully (see, for example, GLOWINSKI and LE TALLEC [1982, 1984]).

It may be viewed as a two-field variational principle in which the functional (8.99) is *perturbed* by a penalty term. Thus,

$$\begin{aligned} \Pi_{\mathrm{PL}}(\mathbf{u},p) = &\int\limits_{\Omega_0} [p(J(\mathbf{u}) - 1) + \Psi_{\mathrm{iso}}(\overline{\mathbf{C}}(\mathbf{u}))]\mathrm{d}V \\ &-\frac{1}{2}\int\limits_{\Omega_0} \frac{1}{\kappa}p^2\mathrm{d}V + \Pi_{\mathrm{ext}}(\mathbf{u}) \end{aligned} \tag{8.111}$$

(see the work of CHANG et al. [1991]), where the third (penalty) term in functional (8.111) regularizes (relaxes) the incompressibility constraint $J = 1$ involved in the first term of the integral. The Lagrange multiplier p which enforces the constraint no longer has the meaning of a pressure, in contrast to the Lagrange-multiplier method.

The positive penalty parameter κ may be viewed as a (constant) bulk modulus. Incompressible materials can be treated by replacing $1/\kappa$ with zero, so the first and the third terms in (8.111) vanish. For this incompressible limit the penalty method, the Lagrange-multiplier method and the perturbed Lagrange-multiplier method lead to identical equations.

By taking for the term $(J(\mathbf{u}) - 1)$ a more general, sufficiently smooth and strictly convex function $\mathcal{G}(J(\mathbf{u}))$ (so that $\mathcal{G}(J(\mathbf{u})) = 0$ *if and only if* $J = 1$) the functional (8.111) is identical to that proposed by BRINK and STEIN [1996]. Functional (8.111) may also be identified with a special form of the two-field variational principle given by ATLURI and REISSNER [1989]. This work deals with a general framework for incorporating volume constraints into multi-field variational principles. In addition, note that the formulation (8.111) also reduces from a mixed (finite element) formulation proposed by SUSSMAN and BATHE [1987].

EXERCISES

1. Consider the decoupled strain energy formulation proposed in (8.55), i.e. $\Psi(\mathbf{C}) = \Psi_{\mathrm{vol}}(J) + \Psi_{\mathrm{iso}}(\overline{\mathbf{C}})$, with $\Psi_{\mathrm{vol}}(J) = \kappa\mathcal{G}(J)$, and treat the displacement $\mathbf{u}$ and the hydrostatic pressure p as independent field variables (permitted to be varied). Require that $\mathrm{d}\mathcal{G}(J)/\mathrm{d}J = 0$ *if and only if* $J = 1$.
   (a) Derive the stationary condition with respect to $\mathbf{u}$ and incorporate the definition of the volumetric stress contribution, i.e. (6.89), with the constitutive equation for the hydrostatic pressure p according to $(6.91)_1$. Show that the resulting variational equation is, in accordance with the principle of virtual work, in the form (8.105).
   (b) Obtain the additional variational equation in the form

$$\int\limits_{\Omega_0} \left(\frac{\mathrm{d}\mathcal{G}(J(\mathbf{u}))}{\mathrm{d}J} - \frac{1}{\kappa}p\right)\delta p\mathrm{d}V = 0 \quad ; \tag{8.112}$$

which is relation (8.61) enforced in a weak sense. Interpret the result for the case $\kappa \to \infty$.

Note that variational equation (8.112) cannot simply be obtained by taking the first variation of the energy functional.

This type of two-field variational principle was proposed by DE BORST et al. [1988] and VAN DEN BOGERT et al. [1991].

2. Consider the augmented functional (8.111).

   (a) Derive the stationary conditions with respect to **u** and p, i.e. $D_{\delta\mathbf{u}}\Pi_{\mathrm{PL}}(\mathbf{u},p) = 0$ and $D_{\delta p}\Pi_{\mathrm{PL}}(\mathbf{u},p) = 0$, for arbitrary variations $\delta\mathbf{u}$ and δp, respectively. Show that the Euler-Lagrange equations are Cauchy's equation of equilibrium and the artificial constitutive equation $p = \kappa(J(\mathbf{u}) - 1)$.

   (b) Show that the linearization of the Euler-Lagrange equations in the weak forms gives

$$D^2_{\delta p,\Delta p}\Pi_{\mathrm{PL}}(\mathbf{u},p) = -\int\limits_{\Omega_0}\frac{1}{\kappa}\Delta p\,\delta p\mathrm{d}V \;,$$

   and three further equations which are in accord with $(8.107)_1$, (8.108) and the linearized principle of virtual work, which has basically the form of eq. (8.109). Note that, thereby, the elasticity tensor $\mathbb{C} = \mathbb{C}_{\mathrm{vol}} + \mathbb{C}_{\mathrm{iso}}$ is given explicitly by eqs. $(6.166)_4$ and (6.168).

3. Study a two-field variational principle which involves *displacements* and *stresses* as independent field variables. A formulation of this type leads to the classical **Hellinger-Reissner variational principle**, widely used in, for example, nonlinear theories of plates and shells (see HELLINGER [1914] and REISSNER [1950]).

   Assume that the constitutive relation $\mathbf{P}(\mathbf{F}) = \partial\Psi(\mathbf{F})/\partial\mathbf{F}$, as introduced in $(6.1)_1$, is invertible, which is not a valid assumption, in general. (It is important to emphasize that, in general, there does not exist a unique deformation gradient **F** corresponding to a given first Piola-Kirchhoff stress tensor **P** (see OGDEN [1977, 1997, Section 6.2.2]). Define a **complementary strain-energy function** $\Psi_{\mathrm{c}}(\mathbf{P})$ so that a *Legendre transformation* gives

$$\Psi_{\mathrm{c}}(\mathbf{P}) = \mathbf{P}:\mathbf{F} - \Psi(\mathbf{F}) \;,$$

   where **P** and **F** are the first Piola-Kirchhoff stress tensor and the deformation gradient, respectively. Hence, the functional

$$\Pi_{\mathrm{HR}}(\mathbf{u},\mathbf{P}) = \int\limits_{\Omega_0}[\mathbf{P}:\mathbf{F} - \Psi_{\mathrm{c}}(\mathbf{P})]\mathrm{d}V - \int\limits_{\Omega_0}\mathbf{B}\cdot\mathbf{u}\mathrm{d}V$$

$$-\int\limits_{\partial\Omega_{0\,\sigma}}\overline{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S - \int\limits_{\partial\Omega_{0\,\mathrm{u}}}\mathbf{T}\cdot(\mathbf{u}-\overline{\mathbf{u}})\mathrm{d}S \;,$$

   which is valid for large strains, is referred to as the **Hellinger-Reissner functional**. Here, the prescribed loads are **B** on Ω_0 and $\overline{\mathbf{T}}$ on $\partial\Omega_{0\,\sigma}$ and are assumed to be independent of the motion of the body. The third quantity prescribed is the displacement field $\overline{\mathbf{u}}$ acting on the boundary surface $\partial\Omega_{0\,\mathrm{u}}$. Note the relation $\mathbf{T} = \mathbf{PN}$ introduced in $(3.3)_2$.

   (a) Invoke the stationarity of Π_{HR} and determine the weak form of the elastic equilibrium equation. Since the principle is based on treating the displacement **u** and the stress **P** as independent field variables (permitted to be varied), evaluate separately

$$D_{\delta\mathbf{u}}\Pi_{\mathrm{HR}}(\mathbf{u},\mathbf{P}) = 0 \;, \qquad D_{\delta\mathbf{P}}\Pi_{\mathrm{HR}}(\mathbf{u},\mathbf{P}) = 0 \;.$$

   The first variations of **u** and **P** are arbitrary vector-valued and tensor-valued functions for which the restriction $\delta\mathbf{u} = \mathbf{o}$ over the boundary surface $\partial\Omega_{0\,\mathrm{u}}$ holds.

   (b) Show that the associated Euler-Lagrange equations for the volume of the body and the (Dirichlet and von Neumann) boundary conditions are

$$\mathrm{Div}\mathbf{P} + \mathbf{B} = \mathbf{o} \;, \qquad \mathbf{F}(\mathbf{P}) = \frac{\partial\Psi_{\mathrm{c}}(\mathbf{P})}{\partial\mathbf{P}} \;,$$

$$\mathbf{u} = \overline{\mathbf{u}} \quad \text{on} \quad \partial\Omega_{0\,\mathrm{u}} \;, \qquad \mathbf{T} = \mathbf{PN} = \overline{\mathbf{T}} \quad \text{on} \quad \partial\Omega_{0\,\sigma} \;.$$

   The first relation represents Cauchy's equation of equilibrium in the material description, while the second relation denotes the inverse form of the constitutive equation for a hyperelastic material (in some cases not available).

   (c) Alternatively to the functional stated above find Π_{HR} for which displacements and strains are the independent variables (instead of **u** and **P**).

8.6 Three-field Variational Principles

We are interested in a suitable variational approach in order to capture nearly incompressible and incompressible materials. It is recognized that a constant pressure interpolation over the finite element within the framework of two-field variational principles leads to unpleasant pressure oscillation. There are some remedies for this problem in the computational literature, for example, the pressure smoothing technique by HUGHES et al. [1979]. However, a certain improvement is based on the idea of introducing additional independent field variables such as the volume ratio, leading to a more efficient three-field variational principle.

Simo-Taylor-Pister variational principle. A very efficient variational principle that takes account of nearly incompressible response was originally proposed by SIMO et al. [1985] and is known as the mixed **Jacobian-pressure formulation** (for relevant applications to elastomers see SIMO [1987] and SIMO and TAYLOR [1991a]). It emanates from a three-field variational principle of HU [1955] and WASHIZU [1955]. Thereby, besides the displacement and pressure fields $\mathbf{u}$ and p, a third additional kinematic field variable, which we denote by $\tilde{J}$, is treated independently within finite element discretizations. The principle is decomposed into volumetric, isochoric and external parts and is defined by the expression

$$\Pi_{\mathrm{STP}}(\mathbf{u}, p, \tilde{J}) = \int\limits_{\Omega_0} [\Psi_{\mathrm{vol}}(\tilde{J}) + p(J(\mathbf{u}) - \tilde{J}) + \Psi_{\mathrm{iso}}(\overline{\mathbf{C}}(\mathbf{u}))]\mathrm{d}V + \Pi_{\mathrm{ext}}(\mathbf{u}) \quad . \tag{8.113}$$

Following *Simo-Taylor-Pister* the first two terms in the three-field variational principle are responsible for the nearly incompressible behavior of the material. They describe volume-changing (dilational) deformations and are expressed by J, p and the new variable $\tilde{J}$. The kinematic variable $\tilde{J}$ enters the functional as a constraint which is enforced by the Lagrange multiplier p. The Lagrange multiplier is an independent field variable which may be identified as the hydrostatic pressure.

In addition to the virtual displacement and pressure fields $\delta\mathbf{u}$ and δp, we introduce an arbitrary smooth (vector) function $\delta\tilde{J}(\mathbf{x}) = \delta\tilde{J}(\boldsymbol{\chi}(\mathbf{X})) = \delta\tilde{J}(\mathbf{X})$ for the constraint, which we call the **virtual volume change** (here defined on the reference configuration). In equilibrium, functional (8.113) must be stationary. The necessary conditions for the stationarity of functional Π_{STP} with respect to the three field variables $(\mathbf{u}, p, \tilde{J})$ are evaluated separately. We require

$$\left.\begin{aligned} D_{\delta\mathbf{u}}\Pi_{\mathrm{STP}}(\mathbf{u}, p, \tilde{J}) = 0 \quad , \qquad D_{\delta p}\Pi_{\mathrm{STP}}(\mathbf{u}, p, \tilde{J}) = 0 \quad , \\ D_{\delta\tilde{J}}\Pi_{\mathrm{STP}}(\mathbf{u}, p, \tilde{J}) = 0 \end{aligned}\right\} \tag{8.114}$$

for all $\delta\mathbf{u}$ satisfying $\delta\mathbf{u} = \mathbf{o}$ on the part of the boundary surface $\partial\Omega_{0\,\mathrm{u}}$ where displacements $\overline{\mathbf{u}}$ are prescribed and all δp, $\delta\tilde{J}$.

Differentiating functional Π_{STP} with respect to changes in $\mathbf{u}$ gives the weak form of the elastic equilibrium, i.e. the principle of virtual work in the form of (8.105). For an explicit derivation recall the manipulations of the last section.

A straightforward differentiation of Π_{STP} with respect to changes in the field variables $p, \tilde{J}$ gives the weak enforcement of the equivalence between J and $\tilde{J}$, and the constitutive equation for the volumetric changes, i.e.

$$D_{\delta p}\Pi_{\mathrm{STP}}(\mathbf{u}, p, \tilde{J}) = \int\limits_{\Omega_0} (J(\mathbf{u}) - \tilde{J})\delta p \mathrm{d}V = 0 \quad , \tag{8.115}$$

$$D_{\delta\tilde{J}}\Pi_{\mathrm{STP}}(\mathbf{u}, p, \tilde{J}) = \int\limits_{\Omega_0} \left(\frac{\mathrm{d}\Psi_{\mathrm{vol}}(\tilde{J})}{\mathrm{d}\tilde{J}} - p\right) \delta\tilde{J}\mathrm{d}V = 0 \quad . \tag{8.116}$$

For arbitrary δp, the variational equation (8.115) results in the Euler-Lagrange equation $J - \tilde{J} = 0$. It implies that the additional independent variable $\tilde{J}$ equals $J = (\det\mathbf{C}(\mathbf{u}))^{1/2}$, i.e. the kinematic constraint associated with the volumetric behavior. For arbitrary $\delta\tilde{J}$, eq. (8.116) results in the second constraint condition in the local form, that is the Euler-Lagrange equation $\mathrm{d}\Psi_{\mathrm{vol}}/\mathrm{d}\tilde{J} - p = 0$. This is the standard constitutive equation implying the volumetric stresses to be equal to the hydrostatic pressure.

A finite element procedure in which the dilatation $\tilde{J}$ and the pressure variables p are discretized by the same local interpolations as for the displacement field $\mathbf{u}$ would not give any advantage. To prevent volumetric locking an appropriate choice of the interpolation functions for the volumetric variables $p, \tilde{J}$ and their variations $\delta p, \delta\tilde{J}$ is crucial. A simple formulation arises by discretizing the dilatation and pressure variables over a typical finite element domain with the same *discontinuous* (constant) function which need not be continuous across the finite element boundaries. This approach is known as the **mean dilatation method** and is proposed in the notable work of NAGTEGAAL et al. [1974] who recognized the effect of volumetric locking in elastoplastic J_2-flow theory.

Since the interpolation functions are discontinuous, the volumetric variables $p, \tilde{J}$ can be eliminated on the finite element level, a process known as **static condensation** in the computational mechanics literature. Therefore, the variational equations (8.115) and (8.116) need not be solved on the global level leading back to a reduced *displacement-based method*.

The work of BRINK and STEIN [1996] is a comparative study of various multi-field variational principles. It emerges that under certain conditions the above three-field variational principle and some two-field principles yield the same discrete result in each step of the Newton method.

EXERCISES

1. Consider the functional (8.113) with the three independent field variables $(\mathbf{u}, p, \tilde{J})$ and the associated variation equations (8.115), (8.116), (8.105), and show that for each step of the Newton type method the problem is completely described by the

set of linearized equations

$$D^2_{\delta p,\Delta\mathbf{u}}\Pi_{\mathrm{STP}}(\mathbf{u},p,\tilde{J}) = \int\limits_{\Omega_0} J(\mathbf{u})\mathrm{div}\Delta\mathbf{u}\,\delta p\mathrm{d}V \quad ,$$

$$D^2_{\delta p,\Delta\tilde{J}}\Pi_{\mathrm{STP}}(\mathbf{u},p,\tilde{J}) = -\int\limits_{\Omega_0} \Delta\tilde{J}\delta p\mathrm{d}V \quad ,$$

$$D^2_{\delta\tilde{J},\Delta p}\Pi_{\mathrm{STP}}(\mathbf{u},p,\tilde{J}) = -\int\limits_{\Omega_0} \Delta p\,\delta\tilde{J}\mathrm{d}V \quad ,$$

$$D^2_{\delta\tilde{J},\Delta\tilde{J}}\Pi_{\mathrm{STP}}(\mathbf{u},p,\tilde{J}) = \int\limits_{\Omega_0} \frac{\mathrm{d}^2\Psi_{\mathrm{vol}}(\tilde{J})}{\mathrm{d}\tilde{J}\mathrm{d}\tilde{J}}\Delta\tilde{J}\,\delta\tilde{J}\mathrm{d}V \quad ,$$

and the linearized principle of virtual work, which has the form of eqs. (8.109) and (8.108).

2. The described functional $\Pi_{\mathrm{STP}}(\mathbf{u},p,\tilde{J})$ takes into account only the volumetric strain and stress components. Study a more general and very powerful type of a **Hu-Washizu variational principle** fundamental for various finite element methods, i.e.

$$\Pi_{\mathrm{HW}}(\mathbf{u},\mathbf{F},\mathbf{P}) = \int\limits_{\Omega_0} (\Psi(\mathbf{F}) - \mathbf{P}:\mathbf{F} - \mathbf{B}\cdot\mathbf{u} - \mathrm{Div}\mathbf{P}\cdot\mathbf{u})\mathrm{d}V$$
$$+ \int\limits_{\partial\Omega_{0\,\sigma}} \mathbf{u}\cdot(\mathbf{T}-\overline{\mathbf{T}})\mathrm{d}S - \int\limits_{\partial\Omega_{0\,\mathrm{u}}} \mathbf{T}\cdot(\mathbf{u}-\overline{\mathbf{u}})\mathrm{d}S \quad ,$$

with the three independent variables $\mathbf{u}$, $\mathbf{F}$, $\mathbf{P}$ and the prescribed quantities $\mathbf{B}$ on Ω_0, $\overline{\mathbf{T}}$ on $\partial\Omega_{0\,\sigma}$ and $\overline{\mathbf{u}}$ on $\partial\Omega_{0\,\mathrm{u}}$. The loads $\mathbf{B}$ and $\overline{\mathbf{T}}$ are assumed to be conservative and the first Piola-Kirchhoff traction vector $\mathbf{T}$ is given in eq. $(3.3)_2$.

   (a) With identity (1.290) show that the Hu-Washizu variational principle can be posed as a generalization of the principle of virtual work, i.e.

$$\Pi_{\mathrm{HW}}(\mathbf{u},\mathbf{F},\mathbf{P}) = \Pi - \int\limits_{\Omega_0} \mathbf{P}:(\mathbf{F}-\mathrm{Grad}\mathbf{u})\mathrm{d}V - \int\limits_{\partial\Omega_{0\,\mathrm{u}}} \mathbf{T}\cdot(\mathbf{u}-\overline{\mathbf{u}})\mathrm{d}S \quad ,$$

   where the total potential energy Π is given in (8.47) and (8.48).

   (b) Invoke the stationarity of Π_{HW} with respect to $\mathbf{u},\mathbf{F}$ and $\mathbf{P}$. The vector-valued and tensor-valued functions $\delta\mathbf{u}$ and $\delta\mathbf{F},\delta\mathbf{P}$ are arbitrary with the conditions $\delta\mathbf{u} = \mathbf{o}$ over the boundary surface $\partial\Omega_{0\,\mathrm{u}}$ and $\delta\mathbf{P} = \mathbf{O}$ on $\partial\Omega_{0\,\sigma}$. Show that the associated Euler-Lagrange equations for the functional Π_{HW} are

$$\mathrm{Div}\mathbf{P} + \mathbf{B} = \mathbf{o} \quad , \qquad \mathbf{P} = \frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}} \quad , \qquad \mathbf{F} = \mathrm{Grad}\mathbf{u} \quad ,$$

   with the (Dirichlet and von Neumann) boundary conditions

$$\mathbf{u} = \overline{\mathbf{u}} \qquad \text{on} \qquad \partial\Omega_{0\,\mathrm{u}} \quad , \qquad \mathbf{T} = \mathbf{PN} = \overline{\mathbf{T}} \qquad \text{on} \qquad \partial\Omega_{0\,\sigma}$$

   for the body under consideration.

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Note: Numbers in parentheses following the reference indicate the chapters in which it is cited.

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