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SADRŽAJ

DYNAMICS OF UNIVERSITY STUDENT PREFERENCES[1]

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Branko Urošević[3]

Abstract. We give review of literature and announce some new results related to modeling selection of educational institution (or study program) by prospective students, in the context of signaling games. There are two types of students (low and high productivity) and two types of schools (lower and higher quality). Prospective students select a school based on the expected market value of a degree. Although the announced model is quite simple, several interesting equilibria are possible.

1. Educational signaling games

There is a rich literature on the signaling value of education. In his seminal papers (Spence 1973, 1974), Spence introduced the concept of education as a job market signaling tool. He demonstrated that there is a signaling value to education even if education does not enhance productivity. The signaling works well when there is a negative correlation between the cost of education and worker productivity. In the basic Spence (1973) model, it is assumed that there are two types of workers, that have either high or low level of productivity. Employer sets the salary levels based on the education level achieved. Since the cost of education is smaller for the more productive workers, there is an incentive for the more productive type to receive higher level of education, based on the cost-benefit analysis. This provides a signal to the employer, and there is a feedback loop in which employer learns about the productivity of workers based on their education level. Employer adjusts the salary accordingly, which, in equilibrium, makes students select an appropriate education level, until equilibrium is reached.

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[1]JEL codes: I20, I23, J01; keywords: job market signaling, university selection, enrolment dynamics, completion rate.

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We announce a new model (cf. Anić, Božin, Urošević, 2011) in which we study dynamics of enrollment decisions of prospective students. In our model there are two types of students, those with high productivity and those with low productivity. There are, also, two types of universities (schools or programs), one that has higher educational standards and is, therefore, more difficult to complete and another with lower educational standards that is easier to complete. More productive students are more likely to complete either school than are the less productive ones. As it is customary in signaling games, the market cannot directly assess the quality of an individual. Instead, it attaches value to a particular university degree. As a result, everyone that has graduated from a particular school at a particular year is treated identically by the market. Importantly, market sets the salary levels based on the expected productivity of graduates, i.e. based on the respective alumnae populations of the two schools. These expectations are updated in each period (generation of students) and, thus, depend on the relative numbers of those of high and low productivity that had completed each of the schools up to that point in time. We find that schools that have a better way to distinguish between high and low productivity students are schools that have a competitive edge in the long run. In this model multiple long run equilibria are possible. It is possible that either of the two schools loses all of their students and, thus, ceases to exist. Under certain conditions a unique separating long run equilibrium exists in which more productive students enroll into more challenging university while less productive students enroll into less challenging university.[4] Such equilibrium would be preferred by the aggregate employer since they would, after sufficient number of generations, know with certainty the type of students that enroll in each university and would set the initial salaries accordingly. Finally, under certain conditions a long-run equilibrium exist in which high productivity students enroll in the more challenging university while low productivity students alternate enrollment in two universities indefinitely.

Like in Spence (1973) in our model there are two types of students. Unlike that paper, which is static, our model focuses on student enrollment dynamics.

Subsequent work has extended the basic Spence model along several dimensions (see, e.g., a comprehensive survey by Riley (2001)). Signaling plays a role in situations when agents have asymmetric information. For in-

[4]While in the short run it may be possible to have the situation in which all low productivity students enroll into more challenging school while high productivity students enroll into less challenging school, such state cannot be sustainable in the long run.

stance, in the Spence's basic model, employer does not know the productivity level of a worker, and has to infer it indirectly, while workers do know their productivity level, and education level serves as a signaling message from workers to employers. In this way, the informed agent may have incentive to send messages (signals) that reveal some of his private characteristics to the uninformed party. Such signaling games can and often do have multiple equilibria. In our model, several equilibria are observed as well.[5]

Following Spence (1973) we assume that productivity of students is unaffected by educational choices. Many important papers make the same assumption. Weiss (1983) shows that if worker type separation is achieved already at the beginning of study, then workers may be offered jobs before they complete their course of studies. Swinkels (1999) proposes a model where workers may complete just part of their studies. Similar investigations have been done in Noldeke and Van Damme (1990). In all of these models, students are assumed to be able to complete their studies with certainty, although with different costs. In our model, we assume, instead, that students complete a study program (obtain a university degree) only with some probability. The probability of completion depends on student productivity type and the difficulty of the program in which they enrolled. This makes the signal (i.e. the university diploma) available only after the full course of study.

Alos-Ferrer et al (2008) relax the assumption that worker output and productivity do not depend on the level of education achieved. Instead, the authors assume that productivity is subject to randomness or noise. The effect of noise in signaling was also studied in paper by Feltovich et al (2002). In our robustness analysis we also consider the extended model with signal noise.

Jovanovic (1979) shows that wage dynamics can depend on employer learning, after a worker has started her job. Thus, on-the-job training in effect limits the predictive value of education level for worker wages to just a few years after the initial employment (see, also, Farber and Gibbons (1996)).[6] In contrast to some of these papers, our paper focuses, solely, on the job market for new graduates. In our model, the initial salary offered corresponds to the observed productivity of workers graduating from the

[5]A way to eliminate unintuitive equilibria by restricting certain out-of-equilibrium beliefs was proposed in Cho and Kreps (1987), based on work of Riley (1979). Restrictions on equilibria have also been studied in the Banks and Sobel (1987) paper.

[6]In the models of wage dynamics, wages correspond to the expected worker productivity.

same University over a period of time.

While employers set wages, prospective students select the quality of their education. The student decision in the basic Spence's model is informed by the cost-benefit analysis. Other factors have also been considered in the literature, like social status (see, e.g., Akerlof (1997)). Like in Spence (1973), in our paper students also use their expected wages as a utility function. In contrast to that model, in our paper not all of the students that enroll into a program actually complete it. Probability of completing the program, therefore, substantially influences the students' decision-making process. It is well known that enrollment decisions of students can exhibit certain cyclicality over time (see Dellas and Sakellaris (2003)). In their model, countercyclical behavior of enrollment rates is due to cyclical movements in the economic determinants of the enrollment decisions, i.e. due to business cycles. Their paper also presents empirical evidence of the cyclicality of university enrollment. This cyclical nature of student enrollment decisions has also been observed in our paper. In contrast to that paper, however, in our model cyclicality can emerge solely from the feedback of the past generations' enrollment decisions and the market value of university degree, independently of any business cycle considerations.

One of the key assumptions in our model that drives a wedge with the Spence's and other models in this literature is that we introduce uncertainty that a given student will complete a program she enrolled in. It is well known that rates of completion of students vary across countries, schools and programs (for empirical evidence see OECD (2007, 2010)) and Long et al (2006)). In our paper the rates of completion dramatically affect the dynamics of student enrollment decisions.

We contribute to the literature on job market signaling value of education in several ways. First, we introduce a model where education costs are not given explicitly, but are embedded in program completion rates. The signal is defined not as level of education attained, but, rather, as a completion of a particular program of study. The value of a particular degree is endogenous in the model and changes dynamically over time. This is consistent with the anecdotal evidence that market perception of a particular school evolves in a course of time.

We show that the signaling value of education lies in the level of distinction that each program make between the two types of students rather than in some absolute measure of difficulty of a program. The level of distinction is controlled by school policies, via differential in completion rates for different types of students. Schools that better differentiate between high productivity and low productivity students become more attractive to the high productivity students, and increase the market value of their degrees. This, in turn, may eventually attract, also, lower productivity students. As the number of low productivity students at a school increases, the market value of the degree is reduced. In this way, cyclicality of enrollment may be obtained.

We also study dynamics of feedback loop that makes students adjust their educational choices, and employers adjust salary levels offered to newly minted graduates. Factors that influence the enrollment dynamics are examined in detail. In the basic model we find four possible long run equilibria. The conditions for each of these cases to occur are analyzed, as well as the sensitivity of the results to the initial conditions.

Finally, we show that the qualitative features of the basic model are rather robust. We show that the key features of enrollment dynamics are adequately captured by the basic model, and are preserved when we relax some of these assumptions making the model more realistic. Change in assumptions considered include: introduction of weights in employers' assessment of worker's productivity skewed towards more recent graduates, and introduction of uncertainty in students' assessment of the future salary. Neither change seems to qualitatively alter the results.

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PARCIJALNE DIFERENCIJALNE JEDNAČINE U KONTINUUM TEORIJI KRIVOLINIJSKIH DISLOKACIJA

Milan Cajić[1]

Apstrakt. U radu će biti prikazana primena parcijalnih diferencijalnih jednačina u kontinuum teoriji krivolinijskih dislokacija. Ovde prikazana teorija je samoodrživa i oslanja se na gustinu dislokacija kao osnovni parametar za opisivanje defekta u kristalnim materijalima. Prikazane su kinematičke evolucijske jednačine u kojima je gustina dislokacija u skalarnom obliku za slučaj jedne ravni klizanja. U drugom delu rada prikazani su rezultat numeričke simulacije primenom metode konačnih elemenata i izvođenje slabe forme parcijalnih diferencijalnih jednačina, uz definisane granične i početne uslove. Rezultat će biti prikazan na konfiguracionom modelu cilindra.

Uvod

Kristalna struktura kao karakteristika čvrstih tela podrazumeva da su atomi ili atomske grupe od kojih je telo sastavljeno raspoređeni u pravilnoj strukturi u prostoru. Ipak i takva jedna pravilna struktura kristala nije idealna, pa može biti narušena određenim nepravilnostima, koje nazivamo nesavršenostima kristalne rešetke. Prema geometriji i obliku razlikuju se tri tipa nesavršenosti: tačkaste, linijske i površinske. Dislokacije, o kojima će biti više reči u ovom radu, spadaju u linijske nesavršenosti. Taj vid nesavršenosti kristalne rešetke nastaje usled raznih uticaja na samo čvrsto telo, kao što su deformacije kristala pod uticajem spoljašnih sila, termička obrada, kondenzacija vakansija i dr. Razlikujemo dva osnovna tipa dislokacija – ivične i zavojne. Takođe je važno napomenuti Burgersov vektor b, koji služi za opis veličine i smera dislokacija u kristalnoj rešetki.

Prvi pomen dislokacija potiče iz zapažanja [1], [2] do kojih je došlo u XIX veku – da plastična deformacija nastaje u ravni klizanja, gde se jedan deo uzorka smiče u odnosu na drugi. Kada se saznalo za kristalnu strukturu

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metala, dokazalo se da je takvo zapažanje imalo smisla jer bi takvo klizanje, u stvari, predstavljalo smicanje jednog dela kristala u odnosu na drugi.

Kao osnivači kontinuum teorije dislokacija, nastale pedesetih godina XX veka, smatraju se Krener (Kröner) [3], Ni (Nye) [4], Bilbi (Bilby and co-workers) [5] i Kondo (Kondo) [6]. Njihova teorija se zasnivala na takozvanom Krener–Niovom tenzoru dislokacija (tzv. klasični tenzor gustine dislokacija). Bilo je, međutim, očigledno da bi se usrednjavanjem po nekoj maloj zapremini stanje defekta u kristalu moglo samo delimično opisati. Tako se klasični tenzor gustine dislokacija može koristiti samo u specijalnim slučajevima gde dislokacije formiraju glatke snopove paralelnih linija iste orijentacije (kao što su uradili u svojim radovima Acharya [7], [8], [9] i Sedláček [10]). Drugačiji pristup može se naći u radovima Kosevich-a [11] i, slično, u radovima El-Azab-a [12]. Njihov pristup se zasniva na merenju gustine dislokacija u uslovima viših dimenzija. Prvi je definisao gustinu dislokacija u prostoru koji uključuje parametre što karakterišu orijentaciju linije kao nezavisne promenljive, a drugi je upotrebio metode statističke mehanike na sisteme krivolinijskih dislokacionih linija koje evoluiraju u višim dimenzionim prostorima. Prema posmatranjima u [13], ta dva sistema nisu sposobna da razlikuju distribuciju kružnih dislokacionih petlji (u razmatranju tih pojava se umesto proizvoljne krive dislokacione linije uzima kružna petlja) od distribucije proizvoljno orijentisanih pravih linija u ravni sa istim Burgersovim vektorom. U nekim opštijim slučajevima, međutim, korišćenje klasičnog tenzora gustine dislokacija u teoriji kristalne plastčnosti i kod evolucijskih jednačina gustine dislokacija u skalarnom obliku, takozvanih statistički uskladištenih (SSD) i geometrijski neophodnih dislokacija (GND), dovodi do nezatvorenih formulacija, koje moraju biti premošćene uvođenjem fenomenoloških pretpostavki.

Nedavno je Hochrainer [14] razvio tzv. *produženu kontinuum teoriju* (Extended Continuum Theory – ECT), koja je primenjiva na mnogo opštije dislokacione konfiguracije. U toj teoriji postavljene su matematičke osnove za prenos metoda statističke mehanike na trodimenzionalni slučaj krivolinijskih dislokacija. Za kompletno izvođenje teorije pogledati reference [14] i [15].

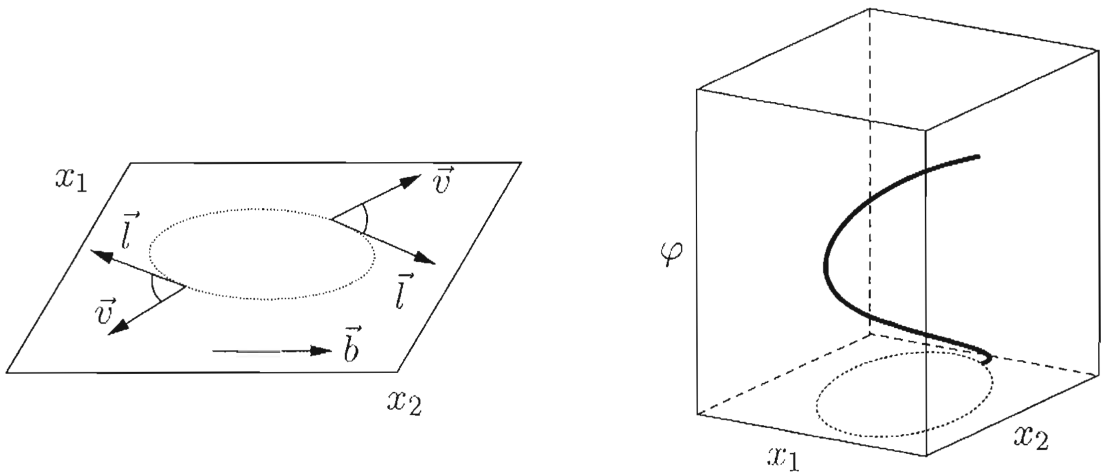
Matematički opis linijskog karaktera dislokacija

Glavna ideja *produžene kontinuum teorije* jeste razlikovanje dislokacija prema orijentaciji njihovih linija [16]. To se može postići takozvanim podizanjem krive linije (kružne petlje) iz ravni u konfiguracioni prostor gde se svaka tačka u ravni $r \in R \times R$ preslikava u tačku $(r, \varphi) \in R \times R \times S$; S označava konfiguracioni prostor orijentacije linije $S = [0 \ldots 2\pi)$. „Podizanjem" linije u konfiguracioni prostor nastaje linija helikoidnog oblika (videti sliku 1). Orijentacija φ predstavlja ugao između vektora tangente na liniju i Burgersovog vektora. Dodatni prostor zahteva uvođenje pojma generalisanog pravca linije, koji označava pravac i smer linije duž uzdignute linije i pojma generalisane brzine:

$$\vec{L}_{(r,\varphi)} = \cos\varphi\vec{e}_1 + \sin\varphi\vec{e}_2 + k\vec{e}_\varphi = \vec{l} + k\vec{e}_\varphi; \tag{1}$$

$$\vec{V}_{(r,\varphi)} = v_1\vec{e}_1 + v_2\vec{e}_2 + \theta\vec{e}_\varphi. \tag{2}$$

Vektor generalisane brzine je normalan na vektor generalisanog pravca linije.



Slika 1 – Kružna dislokaciona petlja u ravni (levo) i u konfiguracionom prostoru (desno)

U datim jednačinama $\vec{l}$ predstavlja vektor orijentacije linije u ravni (tangenta na krivu liniju), $\vec{v}$ je brzina u ravni koja je normalna na vektor orijentacije linije, dok je k mera zakrivljenosti linije i jednaka je recipročnoj vrednosti poluprečnika krivine – $k = 1/R$. Treća komponenta u (2) je od ugaone brzine θ, koja se javlja usled kretanja dislokacionih segmenata.

$$\theta = -\vec{L} \cdot \left(\hat{\nabla}\vec{v}\right)$$

U daljem tekstu oznaka $\hat{\nabla}$ predstavljaće polje divergencije u konfiguracionom prostoru – DIV, dok će oznaka ∇ označavati divergenciju u ravni – div, pa će slediti

$$\hat{\nabla} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \varphi}\right)^T \quad \nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T.$$

Evolucijske jednačine za slučaj jedne ravni klizanja

Oznakama koje smo upoznali u prethodnom poglavlju možemo da formulišemo „tenzor gustine dislokacija drugog reda" α^{II} kao

$$\frac{\alpha^{\mathrm{II}}_{(r,\varphi)}}{\partial t} = \rho_{(r,\varphi)}\vec{L}_{(r,\varphi)} \otimes \vec{b}$$

$$\text{za} \quad \alpha^{\mathrm{II}}_{(r,\varphi)} = \rho_{(r,\varphi)}\vec{L}_{(r,\varphi)} \otimes \vec{b},$$

gde $\otimes$ označava spoljašnji tenzorski proizvod, $\vec{b}$ je Burgersov vektor, a $\rho_{(r,\varphi)}$ označava skalarnu gustinu i usrednjena je vrednost, po nekoj maloj zapremini, nad dislokcionim linijama iste orijentacije φ. Bez gubljenja opštosti pretpostavljava se da je Burgersov vektor konstantan - $\vec{b} = (b_x, 0)$. U slučaju jedne ravni klizanja evolucijska jednačina za α^{II} može biti zamenjena dvema evolucijskim jednačinama za skalarnu gustinu dislokacija $\rho_{(r,\varphi)}$ i srednju vrednost krivine k, a koje su kinematički zatvorene[2] i imaju sledeći oblik:

$$\frac{\partial \rho}{\partial t} = -DIV(\rho\vec{V}) + v\rho = -\operatorname{div}(\rho\vec{v}) + \frac{\partial(\rho\theta)}{\partial\varphi} + \rho v k \tag{3}$$

$$\frac{\partial k}{\partial t} = -vk^2 + \nabla_L^2 v - \nabla_V(k). \tag{4}$$

U ovim jednačinama je indeks (r,φ) ispušten zbog pojednostavljenja jednačina. U ovde ispisanim evolucijskim jednačinama svaki od članova jednačine ima fizičko značenje. Tako u jednačini (3) prva dva sabirka predstavljaju transport gustine dislokacija u konfiguracionom prostoru, dok poslednji sabirak označava porast gustine dislokacija zavisno od ekspanzije kružne petlje. Promena krivine usled ekspanzije je sadržana u prvom članu jednačine (4), drugi član iste jednačine označava promenu krivine k usled rotacije, dok zadnji član predstavlja promenu krivine k u generalisanom pravcu kretanja.

U daljem tekstu umesto evolucijske jednačine (4) za k koristiće se evolucijska jednačina proizvoda ρk, gde je $n := \rho k$ (videti [17]), i to u sledećoj formi:

$$\frac{\partial n}{\partial t} = -DIV(n\vec{V}) + DIV(\rho\theta\vec{L}). \tag{5}$$

[2] kada se kaže kinematički zatvorene, onda se pretpostavlja da je brzina kretanja dislokacija data vrednost, tj. da brzina nije data u funkciji lokalnih napona.

Evolucijske jednačine u slaboj formi

Za izvođenje slabe forme koja se u koristi u MKE, potrebno je preformulisati evolucijske jednačine (koje imaju oblik parcijalnih diferencijalnih jednačina) svođenjem na jednu jednačinu [18]. To se u ovom slučaju radi uvođenjem novih oznaka: $\vec{F}$, $\vec{w}$ i $\hat{\sigma}$ koje zamenjuju članove u jednačinama (3) i (5) na sledeći način

$$\vec{F} := (\rho, n)^T,$$

$$\hat{\sigma} := \begin{pmatrix} \rho\vec{V}^T \\ n\vec{V}^T \end{pmatrix} = \begin{pmatrix} \rho v_1 & \rho v_2 & \rho\theta \\ n v_1 & n v_2 & n\theta \end{pmatrix} \Rightarrow DIV\hat{\sigma} = \begin{pmatrix} DIV(\rho\vec{V}) \\ DIV(n\vec{V}) \end{pmatrix},$$

$$\vec{w} := \begin{pmatrix} \rho v k \\ \hat{\nabla}(\rho\theta\vec{L}) \end{pmatrix} = \begin{pmatrix} n v \\ \nabla\rho\vec{l}\theta + n\dfrac{\partial\theta}{\partial\varphi} \end{pmatrix}.$$

Konačno, nakon sređivanja levih i desnih strana jednačina (3) i (5) dobija se kompaktna forma jednačine:

$$\frac{\partial\vec{F}}{\partial t} = -DIV\hat{\sigma} + \vec{w}. \tag{6}$$

Za konačno dobijanje slabe forme potrebno je da se kompaktna forma jednačine integrali po zapremini. U ovom slučaju to je zapremina cilindra koji predstavlja konfiguracioni prostor za koji se posmatra promena gustine dislokacije (dužina dislokacionih linija po jedinici zapremine) tokom vremena, a po zakonu koji se opisuje evolucijskim jednačinama (3) i (5). Zapremina cilindra biće označena $\hat{V}$, površina $\partial\hat{V}$ i normala na površ $\underline{\hat{N}}$. U izvođenju je potrebna i težinska funkcija $\vec{\eta} = (\eta_1\eta_2)^T$, pa se nakon integraljenja dobija

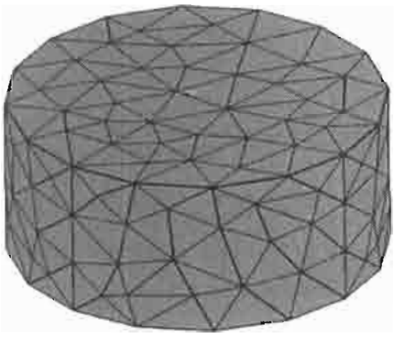
$$\int\limits_{\hat{V}} \frac{\partial\vec{F}}{\partial t}\cdot\vec{\eta}d\hat{V} = -\int\limits_{\hat{V}} DIV(\hat{\sigma})\cdot\vec{\eta}d\hat{V} + \int\limits_{\hat{V}} \vec{w}\cdot\vec{\eta}d\hat{V}.$$

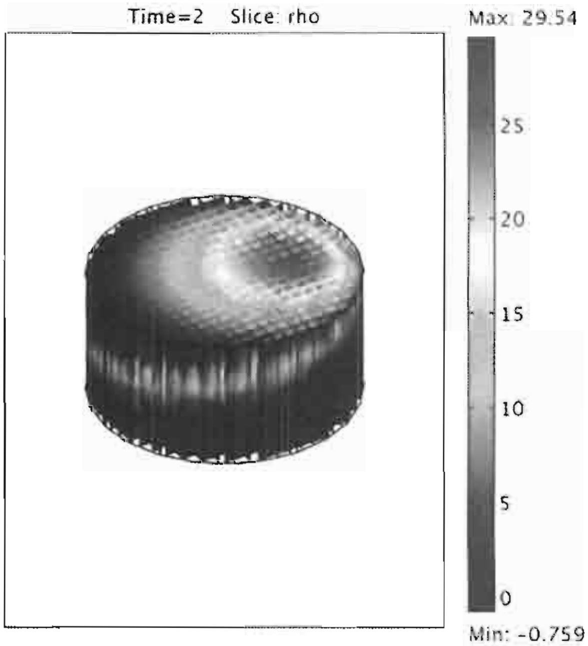
Nakon što se primeni teorema o divergenciji i Grinova formula za specijalan slučaj graničnih uslova – brzina širenja dislokacija v na graničnoj površi $\partial\hat{V}_x$ jednaka je nuli, dobija se slaba forma evolucijskih jednačina u konačnom obliku:

$$\int\limits_{\hat{V}} \frac{\partial\vec{F}}{\partial t}\cdot\vec{\eta}d\hat{V} = \int\limits_{\hat{V}} \hat{\sigma}\cdot\hat{\nabla}\vec{\eta}d\hat{V} + \int\limits_{\hat{V}} \vec{w}\cdot\vec{\eta}d\hat{V}. \tag{7}$$

Numerička simulacija metodom konačnih elemenata

Da bi se simulirala dislokaciona konfiguracija opisana evolucijskim jednačinama (3) i (5), to jest, slabom formom tih jednačina (7), potrebno je najpre definisati dimenzije domena (u ovom slučaju cilindra). Diskretizacija domena vrši se tetraedralnim konačnim elementima (slika 2). Za određivanje vremenskog koraka korišćena je implicitna BDF (backward differentiation formulas) metoda. Definisani vremenski korak je $\Delta t = 0,01s$, a vreme simulacije je $t = 2s$. Brzina dislokacije zadaje se u početnom trenutku i definiše se njeno opadanje do trenutka kada dostigne nulu na granici cilindra. Nakon implementiranja slabe forme jednačine (7) u naš kod i zadavanja početnih uslova za gustinu dislokacija $\rho_0 = 1$ i $n_0 = 10$, startuje se simulacija. Kao rezultat posmatra se parametar gustine dislokacija.





Slika 2 - Diskretizaciona mreža (levo) i rezultat numeričke simulacije (desno); na skali je prikazana gustina dislokacija u nijansama boja od najmanje do najveće gustine

Zaključak

Poznato je da se parcijalne diferencijalne jednačine koriste za matematički opis mnogih fizičkih fenomena. U ovom radu takav oblik matematičkog zapisivanja iskorišćen je u vidu evolucijskih jednačina za opisivanje ekspanzije kružnih petlji koje fizički predstavljaju dislokacione krive linije, to jest same dislokacije u kristalnim materijalima. Ovde je prikazan samo slučaj kada se posmatra jedna ravan klizanja u kristalu, dok se za slučaj više ravni koristi dodatni matematički aparat. Cilj numeričke simulacije jeste da se pokaže da tzv. produžena teorija kontinuuma može ispravno da predvidi ekspanziju kružnih petlji, što je dobijanjem helikoidnog oblika u datoj simulaciji i dokazano. Da bi se uspostavila veza između brzine dislokacija i lokalnih napona, potrebno je uvesti konstitutivne jednačine. Takav slučaj nije obrađivan u ovom saopštenju i može se pronaći u radovima navedenim u literaturi.

Mikroskala na kojoj se nalaze i same dislokacije se u pogledu plastičnosti materijala razlikuje od makroskale, pa se raznim fenomenološkim pretpostavkama pokušava pronaći način opisivanja ove pojave na nižim skalama, dok se interakcija mnogo čestica (makromolekula ili samih dislokacija) pokušava opisati metodama statističke mehanike. Naučna oblast koja se bavi takvim fizičkim fenomenima naziva se kristalna plastičnost.

Zahvalnost. Zahvalnost za pruženu mogućnost da izložim svoj rad dugujem Matematičkom fakultetu u Beogradu. Takođe zahvaljujem prof. Katici Hedrih, Matematičkom institutu SANU i Ministarstvu nauke Republike Srbije za pruženu pomoć u okviru projekta OI174001. Izloženi rezultati urađeni su u okviru IAESTE programa studentskih praksi, na Institutu za tehničku mehaniku, na katedri za mehaniku kontinuuma Univerziteta Karlsrue u Nemačkoj. Zahvaljujem i asistentu Stefanu Vulfinghofu (Stephan Wulfinghoff) i profesoru Tomasu Belkeu (Thomas Böhlke), sa istoimenog univerziteta, na bezrezervnoj pozdršci koju su mi pružili tokom moje prakse u Nemačkoj.

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TEACHING LINEAR ALGEBRA USING A COMBINATORIAL APPROACH

Dragoš Cvetković[1]

Abstract. A combinatorial approach to matrix theory has been used in the books [2], [5]. We present some specially devised slides which support teaching linear algebra in this way. Several comments on teaching and development of the field are included.

Sadržaj. Izlažu se osnovne ideje zasnivanja teorije matrica kombinatornim sredstvima, tj. sredstvima teorije grafova. Takav pristup je primenjen u knjigama:

Brualdi R.A., Cvetković D., A Combinatorial Approach to Matrix Theory and Its Applications, CRC Press, Boca Raton, 2008, i

Cvetković D., Kombinatorna teorija matrica sa primenama u elektrotehnici, hemiji i fizici, Naučna knjiga, Beograd, 1980; II izdanje, 1987; III izdanje, Zavod za udžbenike, Beograd, 2011.

Predstavljaju se specijalno izradjeni slajdovi za visokoškolsku nastavu linearne algebre (definicija i osobine determinanata, operacije sa matricama, sistemi linearnih algebarskih jednačina i dr.).

Kombinatorno zasnivanje teorije matrica je inspirisano tehnikom grafova toka signala (signal flow graphs) koja se koristi u elektrotehnici. Grafovski pristup omogućava tretiranje široke klase problema u linearnoj algebri, počevši od najelementarnijih pa sve do složenih istraživačkih problema.

1. Introduction

Linear algebra today is closely related to graph theory. Graph theory (see, for example, [8], [7], [16]) is a part of combinatorics and this explains the title of the book [2] (*A Combinatorial Approach to Matrix Theory and Its Application*) as well the title of this paper. Our introduction to the subject is a quotation of a part from *Preface* to the book [2].

"Matrix theory is a fundamental area of mathematics with application not only to many branches of mathematics but also to science and engi-

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neering. Its connections to many different branches of mathematics include: (i) algebraic structures such as groups, fields and vector spaces, (ii) combinatorics including graphs and other discrete structures, and (iii) analysis including systems of linear differential equations and functions of a matrix argument.

Generally, elementary (and some advanced) books on matrices ignore or only touch on the combinatorial or graph-theoretical connections with matrices. This is unfortunate in that these connections can be used to shed light on the subject, and to clarify and deepen one's understanding. In fact, a matrix and a (weighted) graph can each be regarded as different models of the same mathematical concept.

Most researchers in matrix theory, and most users of its methods, are aware of the importance of graphs in linear algebra. This can be seen from the great number of papers in which graph-theoretic methods for solving problems in linear algebra are used. Also, electrical engineers apply these methods in practical work. But, in most instances, the graph is considered as an auxiliary, but nonetheless very useful, tool for solving important problems.

The present book differs from most other books on matrices in that the combinatorial, primarily graph-theoretic, tools are put in the forefront of the development of the theory. Graphs are used to explain and illuminate basic matrix constructions, formulas, computations, ideas, and results. Such an approach fosters a better understanding of many ideas of matrix theory and, in some instances, contributes to easier descriptions of them. The approach taken in this book should be of interest to mathematicians, electrical engineers, and other specialists in sciences such as chemistry and physics.

Each of us has written a previous book which is related to the present book (see [3], [5]):

I. R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, 1991; reprinted 1992.

II. D. Cvetković, *Combinatorial Matrix Theory, with Applications to Electrical Engineering, Chemistry and Physics*, (in Serbian), Naučna knjiga, Beograd, 1980; II edition 1987.

This joint book came about as a result of a proposal from the second-named author (D.C.) to the first-named author (R.A.B.) to join in reworking and translating (parts of) his book (II). While that book mainly the theoretical parts of it—has been used as a guide in preparing this book, the material has been rewritten in a major way with some new organization and with substantial new material added throughout."

2. Comments

The third edition of the book [5] appears more than thirty years after the first edition. The book remained essentially in the same form as before. The long period between the two editions requires several comments.

For the author, the starting point was his observation in early 1970's [4] that the concept of a determinant can be defined by graph theoretical means.

As explained in concluding remarks to the first edition of [5], the author found the motivation for writing this book in the following four groups of papers, from electrical engineering, mathematics and chemistry[2]:

"1. Electrical engineers have developed a series of methods for solving the systems of linear algebraic equations, which appear in the theory of electrical circuits, in the feedback theory etc. These methods use flow graphs (C.L. Coates *[13], [28], [10]*), signal flow graphs (S.J. Mason *[67],[68],[93]*) and Chan graphs (S.P. Chan H.N. Mai *[11],[9]*) and they are described in Sections 6.2-6.4, respectively. It is remarkable that the mentioned graphs (and especially, Mason's signal flow graphs) give a better insight into the physical system under description than the corresponding system of equations does.

2. There are many mathematical papers in which results from the matrix theory are obtained or proved by graph theoretical means[3]. The founder of the modern graph theory, Hungarian mathematician D.König was the first who used graphical methods in the matrix theory *[59], [60]*.

3. In the theory of graph spectra (see Sections 7.4, 8.3, 10.2, and references *[15], [19], [82], [25], [26]*) and in its applications to chemistry and to other branches of sciences (12.4, 11.5, 13.1, *[2], [35]*), the results of matrix theory are used for investigations of graphs. Although we have here just an inverse procedure, compared with that of this book, a great number of results contributed to realising how, in other direction, graphs can be used in matrix theory.

4. Some problems in electrical engineering and in engineering in general, lead to the need of considering systems of equations whose matrix is sparse and entries are given numerically. Special methods of treating such matrices use graph theoretical means to a great extent *[85], [86], [89]*. Some

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[2]English translation; references, given in this quotation in italic and only a few included into the actual list of references, and section numbers refer to [5].

[3]Nowadays we can say that these papers belong to *combinatorial matrix theory* and that was not clear to the author when preparing the book [5].

of these techniques are shortly described in Chapter 9.

A synthesis of ideas from these four groups of papers led to this book."

A combinatorial proof of the Cayley-Hamilton theorem appeared between the first and the second edition (see [15], [17], [14]). The proof has been included into the second edition what is in accordance with the spirit of the book.

Distinguished American mathematicians R.A. Brualdi (one of the editors-in-chief of the international journal Linear Algebra and Its Applications) and H.J. Ryser published in 1991 the book "Combinatorial matrix theory" (see [3]).

As already said, R. Brualdi (University of Wisconsin, Madison, USA) and this author have published the book [2] based on previous books. [4]

3. Teaching experience

Together with some colleagues from the Faculty of Electrical Engineering, University of Belgrade, Belgrade, Serbia, the author was using the combinatorial approach in teaching determinants and systems of linear algebraic equations for decades since late 1970's. The author considers this approach as appropriate especially at electrical and other engineering departments since there signal flow graphs appear in teaching control theory and the theory of electrical circuits (see, for example, [9]).

The book [7] (*Matematika I - Algebra*), together with several other its editions, has been used by first year students at the Faculty of Electrical Engineering, Belgrade, since 1990. The graph theoretical definition of the determinant and the Coates formula for solving systems of linear algebraic equations are included. Theorems describing properties of determinants are given with two proofs each: one is based on the classical definition of the determinant while the other uses the graph theoretical definition. In most cases the latter proof are shorter and more transparent.

After the book [2] has been published some courses, based on this book, for doctoral students have been organized in several countries (e.g. in

[4]So far, the following reviews of the book [2] have been published: Bóna M., Newsletter, ACM SIGACT News, 41(2010), no. 2, 19-22; Bumcrot R.J., MR 2009k: 05002; Driessche P. van den, Linear Algebra Appl., 430(2009), 2501-2502; Fonseca C.M. da, Zbl. 1155.15003, 2010; Gutman I., MATCH, 61(2009), 819-820; Gutman I., Kovačević-Vujčić V., YUJOR, 18(2008), no. 2, 273-274; Solomon A., Contemporary Physics, 30(2009), No. 3, 483.

Belgrade and Novi Sad in Serbia, in Koper in Slovenia, in Rio de Janeiro in Brazil).

The authors of [2] have produced some slides to support teaching matrix theory with a combinatorial approach, in particular, in the way described in the book [2]

Brualdi R.A., Cvetković D., *A Combinatorial Approach to Matrix Theory and Its Application*, CRC Press, Boca Raton, 2008.

Slides are contained in ten PDF-files. Each file is devoted to a theme described in the book [2]. Selected themes do not cover the whole book; we have taken only those subjects which require extensive graph drawings in teaching. The idea is that a teacher of a course in matrix theory can select some of our slides and combine them with his/her own slides or other tools (board, oral explanations, etc.). It is also recommended that the students, after hearing the teacher's introduction, study the slides on their own computers.

Themes: 1. Product of matrices, 2. Powers of matrices, 3. Determinants, 4. Inverses, 5. Coates formula, 6. Signal flow graphs, 7. Cayley-Hamilton theorem, 8. Jordan canonical form, 9. Non-negative matrices, 10. Control theory.

There is also a Serbian version for the subject 3. Determinants.

The slides are produced in Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia, within the scientific project "Graph Theory and Mathematical Programming with Applications to Chemistry and Engineering". The slides are distributed free of charge and can be downloaded from web sites of the book publisher and of the Mathematical Institute, Belgrade

http://www.mi.sanu.ac.rs/projects/results144015.htm#monographs.

We are obliged to Vladimir Baltić, University of Belgrade, Faculty of Organizational Sciences, for his excellent work on implementation of the slides. He was using LATEX and the package WinGCLC for drawing graphs, which was developed by Predrag Janičić, University of Belgrade, Mathematical Faculty, http://www.matf.bg.ac.yu/janicic/gcls/index.html.

4. Some characteristic details

The book [5] in its third edition is accompanied with a compact disc containing the described slides together with hints for teachers and students.

CD also contains all the pictures from the book in an electronic form. Finally, CD contains a scan of the second edition of the book.

Here we reproduce a few characteristic features from the book relying on the suggestive effect of graph pictures and indicating only briefly the subject. For a detailed explanation the reader should consult [5] or [2].

Fig. 1 first gives the König digraph G_1 of a 2×3 matrix and the König digraph G_2 of a 3×3 matrix. Further we find the composition and the product of these digraphs with an indication how the product of matrices is constructed.

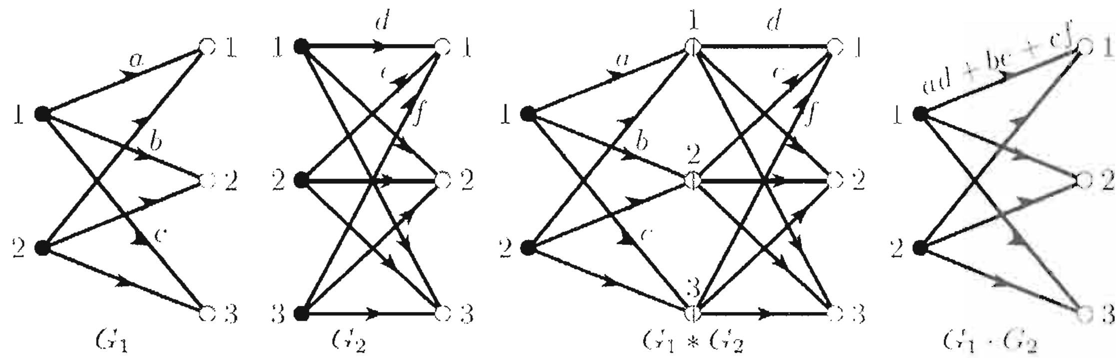

Figure 1 – König digraphs for a matrix product

The multiple composition of a König digraph with itself in Fig. 2 verifies the formula

$$\left\| \begin{array}{cc} a & b \\ 0 & c \end{array} \right\|^k = \left\| \begin{array}{cc} a^k & a^{k-1}b + a^{k-2}bc + \cdots + bc^{k-1} \\ 0 & c^k \end{array} \right\|.$$

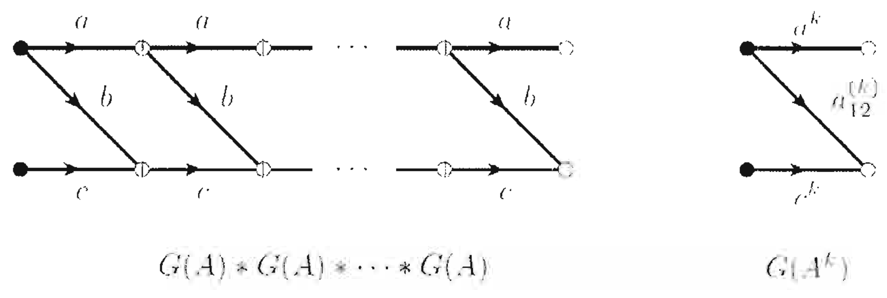

Figure 2 – Determining the k-th power of a 2×2 matrix

The determinant of a square matrix A of order n is defined by the formula

$$\det A; = (-1)^n \sum_{F \in \mathcal{F}} (-1)^{p(F)} C(F),$$

where summation goes over certain subgraphs F of the digraph G_A, with $p(F)$ being the number of cycles in F and $C(F)$ the weight of F.

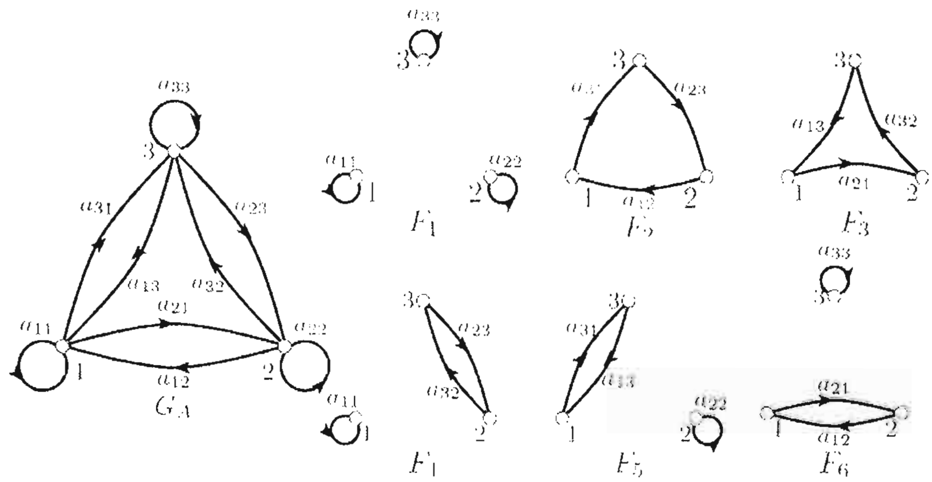

Figure 3 – Calculating a determinant of order 3

Fig. 3 clearly indicates the usual terms in the development of a determinant of order 3 while Fig. 4 visualizes the property of determinants to change sign when changing the order of two rows.

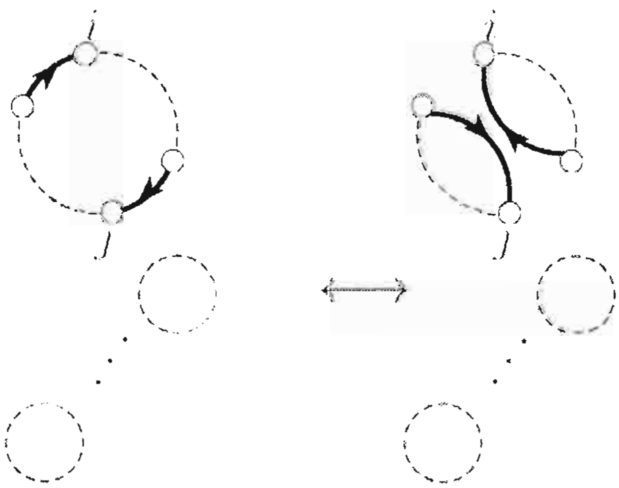

Figure 4 – Explanation of the fact that a determinant changes its sign when replacing two rows

The Coates digraph of the system

$$\begin{array}{ccccccccl}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & & & = b_1, \\
 & & a_{22}x_2 & & & + & a_{24}x_4 & = 0, \\
a_{31}x_1 & & & + & a_{33}x_3 & & & = b_3, \\
 & & a_{42}x_2 & + & a_{43}x_3 & + & a_{44}x_4 & = 0,
\end{array}$$

is given in Fig. 5

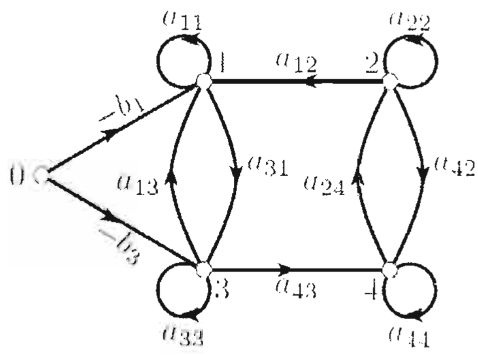

Figure 5 – Coates digraph of a 4×4 system of linear algebraic equations

with relevant subgraphs in the next two figures.

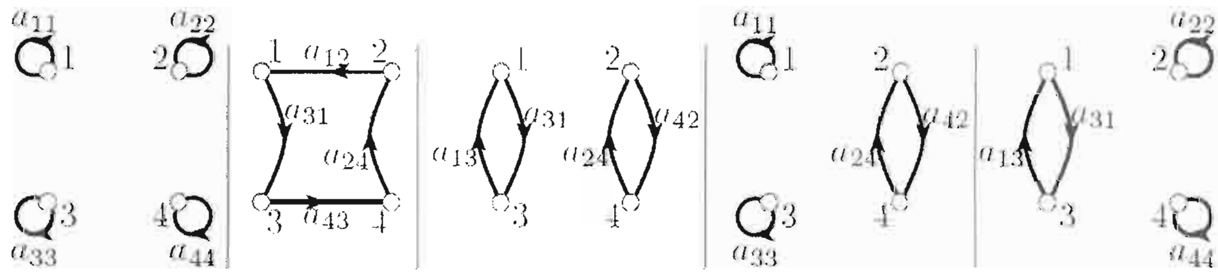

Figure 6 – Subgraphs determining the denominator of the solution

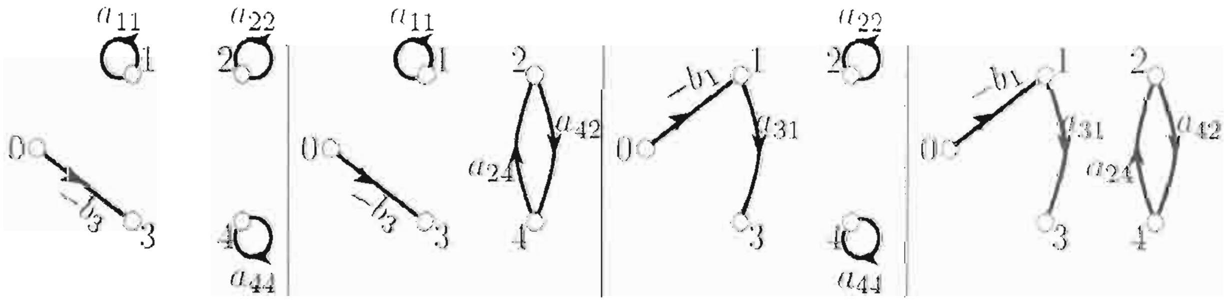

Figure 7 – Subgraphs determining the numerator of the solution

According to the Coates formula we have

$$x_3 =$$

$$= \frac{b_3a_{11}a_{22}a_{44} - b_3a_{11}a_{42}a_{24} - b_1a_{31}a_{22}a_{44} + b_1a_{31}a_{24}a_{42}}{a_{11}a_{22}a_{33}a_{44} - a_{12}a_{24}a_{43}a_{31} + a_{13}a_{31}a_{42}a_{24} - a_{11}a_{33}a_{42}a_{24} - a_{22}a_{44}a_{13}a_{31}}.$$

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p-ADIČNI BROJEVI, ULTRAMETRIČKA ANALIZA I PRIMENE

Branko Dragović[1]

1. Uvod

Kao sredstvo za rešavanje Diofantovih jednačina, a po analogiji s meromorfnim funkcijama, p-adične brojeve je uveo Kurt Hensel (1861–1941) 1897. godine. U današnje doba se uvođenje polja p-adičnih brojeva $\mathbb{Q}_p$ obično posmatra kroz kompletiranje polja racionalnih brojeva $\mathbb{Q}$ u odnosu na p-adično rastojanje indukovano p-adičnom normom. Na taj način polje $\mathbb{Q}$ je gusto ne samo u odnosu na polje realnih brojeva $\mathbb{R}$ već i u odnosu na sva polja $\mathbb{Q}_p$, gde je p bilo koji prost broj (o p-adičnim brojevima i odgovarajućoj analizi videti knjigu [1]).

Rezultati merenja u eksperimentima su racionalni brojevi i klasična analiza sa realnim brojevima je veoma pogodna za opisivanje eksperimenata i za razne druge primene matematike. Šta je, međutim, s primenom p-adične analize? Ovaj rad je uglavnom i namenjen odgovoru na postavljeno pitanje.

Za početak primene p-adičnih brojeva, i odgovarajuće analize, obično se uzima 1987. godina, kada je uspešno konstruisana amplituda za rasejanje p-adičnih struna. Od tada su napravljeni razni modeli sa p-adičnim brojevima (pregled ranijih istraživanja može se naći u [2] i [3]). Ta oblast istraživanja nazvana je *p-adična matematička fizika*. Osim što su objavljeni mnogi radovi, održavaju se međunarodni skupovi iz p-adične matematičke fizike, a postoji i interdisciplinarni međunarodni časopis na engleskom jeziku "p-Adic Numbers, Ultrametric Analysis and Applications".

2. Merenja, brojevi i njihova primenljivost

Primena matematike je usko povezana s opisivanjem rezultata merenja. Merenje je poređenje neke nepoznate veličine s određenom veličinom iste

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vrste. Najprostije merenje je prebrojavanje elemenata nekog konačnog skupa i tom prilikom kao rezultat se dobija neki prirodan broj. Merenja mogu biti direktna ili indirektna. Kada nije reč o prebrojavanje objekata, već uobičajeno fizičko merenje, ono se praktično svodi na merenje dužine.

Pri merenju dužine traži se odgovor na to koliko je puta određena duž sadržana u zadanoj duži. U praksi to nije moguće odrediti tačno, već samo približno. Da bi se što približnije odredila tražena vrednost (x), merenja se ponavljaju i nalazi se srednja aritmetička vrednost (x_{sr}) i greška merenja (Δx). Dobijeni rezultat se zapisuje $x = x_{sr} \pm \Delta x$. Dakle, prava vrednost dužine x je u intervalu $(x_{sr} - \Delta x, x_{sr} + \Delta x)$, a x_{sr} je neki racionalan broj (jer ne može se merenjem dobiti broj koji sadrži beskonačno mnogo cifara različitih od nule). Finijim merenjima može se smanjiti greška Δx i povećati broj sigurnih cifara kod x_{sr}. Treba naglasiti da je proces merenja u potpunosti u saglasnosti sa svojstvima realnih brojeva, jedino ne može kao rezultat da se dobije broj koji sadrži beskonačno mnogo cifara. Prema tome, rezultati fizičkih merenja su racionalni brojevi čija se metrička svojstva izražavaju pomoću standardne apsolutne vrednosti.

Na $\mathbb{Q}$ postoje dve vrste norme: standardna apsolutna vrednost $|\cdot|_\infty$ i p-adična norma $|\cdot|_p$. Saglasno teoremi Ostrovskog, na $\mathbb{Q}$ nema drugih netrivijalnih normi. Ove norme indukuju odgovarajuća rastojanja i p-adično rastojanje $d_p(x,y) = |x-y|_p$ je ultrametričko (ne-Arhimedovo) rastojanje, tj. zadovoljava jaku nejednakost trougla: $d_p(x,y) \leq max\{d_p(x,z), d_p(z,y)\}$. Na primer, p-adično rastojanje između dva cela broja $m, n \in \mathbb{Z}$ je $d_p(m,n) = |m-n|_p \leq 1$. Svaki p-adičan broj $x \in \mathbb{Q}_p$ može se jednoznačno predstaviti redom

$$x = p^{\nu(x)} \sum_{n=0}^{+\infty} x_n p^n, \quad \nu(x) \in \mathbb{Z}, \quad x_n \in \{0, 1, \dots, p-1\},$$

koji je konvergentan u odnosu na p-adično rastojanje.

Postoje dva glavna vida p-adične analize: 1) preslikavanje p-adičnih u p-adične brojeve i 2) preslikavanje p-adičnih u kompleksne brojeve.

Realni i p-adični brojevi se objedinjuju u adele. Adel α je beskonačan niz:

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \dots, \alpha_p, \dots), \quad \alpha_\infty \in \mathbb{R}, \quad \alpha_p \in \mathbb{Q}_p,$$

gde $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ za sve proste brojeve p sem za njih konačno mnogo. Skup svih adela $\mathbb{A}$ može se predstaviti na sledeći način:

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p,$$

gde je $\mathcal{P}$ bilo koji konačan skup prostih brojeva.

Kompleksno-značne adelične funkcije daju mogućnost povezivanja p-adičnog i realnog sadržaja istog objekta. Evo dva prosta primera, gde su objekti racionalni brojevi:

$$|x|_\infty \times \prod_p |x|_p = 1, \quad x \in \mathbb{Q}^\times,$$

$$\chi_\infty(x) \times \prod_p \chi_p(x) = 1, \quad x \in \mathbb{Q},$$

$$\chi_\infty(x) = \exp(-2\pi i x), \quad \chi_p(x) = \exp 2\pi i \{x\}_p.$$

Pošto rezultati merenja nisu p-adični brojevi, postavlja se pitanje na koji način p-adični brojevi mogu učestvovati u opisivanju prirode. Pre odgovora na ovo pitanje, primetimo da već postoji slučaj upotrebe brojeva koji se ne dobijaju u direktnim merenjima. To je slučaj s kompleksnim brojevima, koji su nezaobilazni u kvantnoj teoriji. Naime, talasna funkcija i amplituda prelaza su kompleksne funkcije realnih argumenata i sadrže svu informaciju o kvantnom sistemu.

Moguća primena p-adičnih brojeva jeste u sistemima čija se unutrašnja svojstva mogu predstaviti p-adičnim brojevima. Ta p-adična svojstva nisu direktno fizički merljiva, ali su svojstvena datom sistemu i odgovaraju delu realnosti nedostupnom direktnom posmatranju. Takav slučaj je problem fizike na Plankovoj skali. Naime, saglasno kvantnoj gravitaciji, nemoguće je meriti rastojanja kraća od Plankove dužine $\ell_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}$ m, jer greška pri merenjima rastojanja ne može biti manja od ove dužine.

p-Adični brojevi su takođe pogodni za primenu u teoriji informacija kada prostor informacija ima hijerarhijsku strukturu. Takav je slučaj s prostorom kodona u genetskom kodu.

p-Adični brojevi mogu služiti i za kompletnije opisivanje fizičkog sistema, koji se već opisuje u nekoj meri realnim brojevima. To se onda radi pomoću adela, pri čemu ne moraju da se uključe u opisivanje p-adični brojevi za sva p, već za samo njih konačno mnogo. To bi bio slučaj narušene adelične simetrije (narušavanje simetrija se u fizici često koristi).

3. Primene u fizici i srodnim naukama

Do sada su uglavnom pravljeni matematički modeli nekih fizičkih sistema s p-adičnim brojevima u cilju sagledavanja mogućih primena. Ti modeli se mogu razvrstati u sledeće oblasti istraživanja:

- p-adična i adelična teorija struna,
- p-adična i adelična kvantna mehanika i teorija polja,
- p-adična i adelična gravitacija i kosmologija,
- p-adična i adelična struktura prostora na Plankovoj skali,
- p-adični i adelični dinamički sistemi,
- p-adični stohastički procesi,
- p-adični aspekti teorije informacija,
- p-adični aspekti genomike i proteomike,
- p-adični talasići (wavelets) i
- p-adični neuređeni sistemi i spinska stakla.

Takođe vrše se istraživanja p-adičnog i ultrametričkog modeliranja u obradi slika, lingvistici, sistematici, ekonomiji, finansijama ...

Kratak pregled navedenih primena p-adičnih brojeva i adela u matematičkoj fizici i srodnim naukama može se naći u preglednom članku [4].

3.1. p-Adične strune

Primena p-adičnih brojeva u fizici započela je 1987. godine uvođenjem p-adičnih struna, i to konstrukcijom odgovarajućih amplituda za njihovo rasejanje. To je bilo urađeno po analogiji sa običnim strunama. Za rasejanje otvorenih običnih struna simetrična Veneziano amplituda se zadaje na sledeći način:

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} |1-x|_\infty^{b-1}\, d_\infty x = g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)},$$

gde je $a = -s/2-1$, $b = -t/2-1$, $a = -u/2-1$, i $a+b+c = 1$, a $a, b, c \in \mathbb{C}$ su Mandelstamove kinematičke promenljive. U gornjem integralu promenljiva x ima realne vrednosti i odnosi se na granicu površi koju struna opisuje svojim kretanjem u prostoru. Funkcija ζ je Rimann-ova zeta funkcija.

Analogna Veneziano amplituda za rasejanje dveju otvorenih p-adičnih struna definiše se istim, po formi, integralnim izrazom

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x = g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}},$$

gde je sada promenljiva x p-adično-značna. Kinematičke promenljive a, b, c ostaju iste kao u realnom slučaju. Posle izračunavanja integrala s p-adičnom merom Haar-a amplituda se izražava preko elementarnih funkcija.

Dakle obična i p-adična struna se razlikuju po površi koju opisuju svojim kretanjem: obična struna ima površ koja se opisuje realnim brojevima, a površ kod p-adične strune se opisuje p-adičnim brojevima. Ove površi su u oba slučaja nedostupne direktnom posmatranju i merenju.

Nije teško primetiti da proizvod amplituda za običnu strunu i sve p-adične strune jeste neka konstanta, tj.

$$A(a,b) = A_\infty(a,b)\prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = \text{const.}$$

Na ovaj način obične i p-adične strune ulaze u teoriju struna ravnopravno. Ako postoje obične strune, tada bi trebalo da postoje i p-adične strune. Pošto običnim strunama odgovara obična materija, p-adičnim strunama bi trebalo da odgovara neka p-adična materija. Moguće je da tamna materija i tamna energija, kojih možda ima i oko 96% od ukupne materije u svemiru, imaju p-adičnu prirodu [5].

3.2. p-Adičnost genetskog koda

Druga veoma važna primena p-adičnih brojeva jeste primena u strukturi genetskog koda. Genetski kod je veza između kodona i amino-kiselina, odnosno to je preslikavanje skupa od 64 kodona na skup od 20 amino-kiselina i jednog stop signala. Pošto nekim amino-kiselinama odgovara više kodona, kaže se da je genetski kod degenerisan (izrođen).

Kodoni su sastavljeni od po tri nukleotida, a nukleotida ima četiri tipa: citozin (C), adenin (A), uracil (U) – u RNK, timin (T) – u DNK, guanin (G). Svakom od 64 kodona, pridružujemo prirodan broj na sledeći način: kodon $x_0x_1x_2 \equiv x_0 + x_1\, 5 + x_2\, 5^2$ gde je $x_k = 1,2,3,4$ – cifre koje se odnose na nukleotide po pravilu: $C=1$, $A=2$, $T=U=3$, $G=4$. Među kodonima se posmatra 5-adično i zatim 2-adično rastojanje. U odnosu na najmanje 5-adično rastojanje, koje iznosi $\frac{1}{25}$, kodoni se grupišu u kvadriplete (po četiri kodona). Ovi kvadripleti u odnosu na najmanje 2-adično rastojanje, koje je $\frac{1}{2}$, razdvajaju se u dva dubleta. Kod genetskog koda mitohondrija kičmenjaka, koji može da se smatra osnovnim genetskim kodom, nekoj amino-kiselini može da odgovara jedan, dva ili tri pomenuta dubleta. Na taj način p-adično rastojanje, između ostalog, prirodno opisuje strukturu skupa kodona i degenerisanost genetskog koda (za detalje videti [6] i [7]).

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ON MATHEMATICAL METHODS OF MECHANICS WITH APPLICATIONS (MMMA) OR MECHANICS BETWEEN TWO FIRES: MATHEMATICS AND ENGINEERING

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Abstract. A review in subjective choice of the mathematical methods of mechanics with examples of applications for solving problems of mechanical real system dynamics abstracted to the theoretical models of mechanical discrete or continuum systems, as well as hybrid systems, is presented. Paper present serried of methods and scientific results authored by author's professors Mitropolyski, Andjelić and Rašković, as well as author's of this paper original scientific research results in area of Theoretical and Applied Mechanics and Nonlinear Dynamics. Resulrs in construction of analytical dynamics of hereditary discrete system obtained in collaboration with O. A. Gorosho are presented. Also, some selections of results author's postgraduate students and doctorantes in area of nonlinear dynamics is presented.

Keywords: vector method, coupled rotations, no intersecting axes, coupled rotations, nonlinear dynamics, mass moment vectors, linear momentum, angular momentum, derivatives, deviational mass moment vector, rotator, gyrorotor, eccentricity, basic vectors of position vector tangent space, angular velocity of the tangent space basic vectors, rheonomic constraint, mobility, velocity of basic vector extension, $n+R$-dimensional tangent spaces, asymptotic approximation of solution, Krilov-Bogolyubov-Mitropolyski asymptotic averaged method, method of variation of constants, hereditary system, rheological element, rheological and relaxational kernels, standard hereditary element, integro-differential equation, fractional order derivative, material particles, rheonomic coordinate, rheological pendulum, rheonomic coordinate, covariant coordinate, thermo-rheological and piezo-rheological hereditary elements, discrete continuum method, space fractional order structure, chains, coupled, homogeneous, eigen main plane nets, eigen main chains, eigen main coordinates, fractional order oscillator, eigen circular frequency, fractional order properties characteristic number, boundary conditions, transfer of signals, multi-frequency, trigonometric method, beam, transversal, longitudinal.

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I. Introduction

Subject of this paper is focused to the mathematical methods on mechanics with applications for solving dynamics of model abstraction of real engineering system dynamics by following contents:

1* Vector method based on the mass moment vectors and vector rotators and their definitions and properties, visualizations and applications in rigid body dynamics with coupled rotations around axes without intersections (see References I.).

2* Angular velocity of the basic vectors rotation of a tangent space of the vector positions of material particles of mechanical system dynamics with geometrical, stationary and rheonomic constraints. Extensions of dimensions of tangent space of the vector positions of material particles of mechanical system dynamics from three dimensional real spaces to configuration space of independent generalized curvilinear coordinate systems. Reductions of numbers of coordinates and extensions of tangent space of vector passions (see References II.).

3* Asymptotic methods of nonlinear mechanics Krilov-Bogolyubov--Mitropolyskiy and applications in nonlinear dynamics of deformable bodies and dynamics of hybrid systems complex structures. Specially in the paper we made a comparison between obtained first approximation of the Goerge Duffing nonlinear differential equation obtained by Krilov Bobolyubov-Mitropolyskiy asymptotic method and method of variation constants with averaged procedure (see References III).

4* Integro-differential equations and their applications in development of analytical mechanics of discrete hereditary systems by Gorosko and Hedrih (Stevanović) (see References IV.).

5* Differential fractional order equations and theirs applications in development of analytical dynamics of discrete and continuous fractional order systems with like single and multi-frequency modes, or fractional order modes (see References (see References IV.).

6* Discrete continuum method. Main nets, main chains and mail fractional order oscillators with main fractional order modes of plane as well as space coupled mechanical chains (cases ideal elastic, hereditary and fractional order). Applications in mechanics of continuum (models of longitudinal and transversal oscillations of beams), in biomechanics (mechanical models of double helix DNA chains), to systems with coupled pendulums, as well is in signals transfer (see References IV).

7* Lissajous' curves, orthogonal asynchronous and synchronous oscillations, asynchronization and synchronization of subsystem in hybrid system dynamics.

8* Ito's stochastic differential equations and applications to stochastic oscillations of mechanical systems with hereditary properties.

9* Phase space method and its applications to nonlinear dynamics. Examples of nonlinear system dynamics and phase trajectories types.

10* Mathematical analogy and phenomenological mapping. Mathematical models in applications to disparate physical models dynamics.

Taking into account that length of this paper is bounded by Editorial propositions previous listed methods and applications are focused ti the necessary presentations and explanations of choused examples.

II. Vector method and applications

Vector method [4], based on mass moment vectors and vector rotators coupled for pole and oriented axes, is used for obtaining vector expressions for kinetic pressures on the shaft bearings of a rigid body dynamics with coupled rotations around no intersecting axes [16-19]. This method is very effective and suitable in applications. Mass inertia moment vectors and corresponding deviational vector components for pole and oriented axis are defined by K. Hedrih in 1991 [1]. A complete analysis of obtained vector expressions for derivatives of linear momentum and angular momentum give us a series of the kinematical vectors rotators around both directions determined by axes of the rigid body coupled rotations around no intersecting axes[16-19]. These kinematical vectors rotators are defined for a system with two degrees of freedom as well as for rheonomic system with two degrees of mobility and one degree of freedom and coupled rotations around two coupled no intersecting axes as well as their angular velocities and intensity.

As an example of defined dynamics [16-19]., we take into consideration a heavy gyrorotor-disk with one degree of freedom and coupled rotations when one component of rotation is programmed by constant angular velocity. For this system with nonlinear dynamics, series of graphical presentation of three parameter transformations in relations with changes of eccentricity and angle of inclination (skew position) of heavy rigid body- in relation to self rotation axis are presented, as well as in relation with changing orthogonal distance between no intersecting axes of coupled rotations. Some graphical visualization of vector rotators properties are presented, too.

Using K. Hedrih's (See Refs. [1-9]) mass moment vectors and vector rotators, some characteristics members of the vector expressions of derivatives of linear momentum and angular momentum for the gyro rotor coupled rotations around two no intersecting axes obtain physical and dynamical visible properties of the complex system dynamics [16-18].

Between them there are vector terms that present deviational couple effect containing vector rotators which directions are same as kinetic pressure components on corresponding gyro rotor shaft bearings [10-15] and [18-20].

II.1. Mass moment vectors for the axis to the pole

The monograph [4], IUTAM extended abstract [1] and monograph paper [5] contain definitions of three mass moment vectors coupled to a axis passing through a certain point as a reference pole. Now, we start with necessary definitions of mass momentum vectors.

Definitions of selected mass moment vectors for the axis and the pole, which are used in this paper are:

1* Vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass linear moment for the axis, oriented by the unit vector $\vec{n}$, through the point – pole O, in the form:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} \stackrel{def}{=} \iiint\limits_V [\vec{n},\vec{\rho}]dm = [\vec{n},\vec{\rho}_C]M, \quad dm = \sigma dV; \tag{1}$$

where $\vec{\rho}$ is the position vector of the elementary body mass particle dm in point N, between pole O and mass particle position N.

2* Vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment for the axis, oriented by the unit vector $\vec{n}$, through the point – pole O, in the form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \stackrel{def}{=} \iiint\limits_V [\vec{\rho}[\vec{n},\vec{\rho}]]dm. \tag{2}$$

For special cases, the details can be seen in [1-9]. In the previously cited references, the spherical and deviational parts of the mass inertia moment vector and the inertia tensor are analysed. In monograph [4] knowledge about the change (rate) in time and, the derivatives of the mass moment vectors of the body mass linear moment, the body mass inertia moment for the pole and a corresponding axis for different properties of the body, is shown, on the basis of results from the first author's References [6-9].

The relation

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O_1)} + [\vec{\rho}_O, \vec{\mathfrak{S}}_{\vec{n}}^{(O_1)}] + [\vec{M}_C^{(O_1)}, [p\vec{n}, \vec{\rho}_O]] + [\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]M \tag{3}$$

is ***the vector form of the theorem*** *for the relation of material body mass inertia moment vectors, and , for two parallel axes through two corresponding points, pole and pole (for details detail Refs. [ ] by K. Hedrih).* We can see that all the members in the last expression-relation (3) have the similar structure. These structures are: $[\vec{\rho}_O[\vec{n},\vec{r}_C]]M$, $[\vec{r}_C,[\vec{n},\vec{\rho}_O]]M$ and $[\vec{\rho}_O,[\vec{n},\vec{\rho}_O]]M$.

In the case when the pole O_1 is the centre C of the body mass, the vector $\vec{r}_C$ (the position vector of the mass centre with respect to the pole O_1) is equal to zero, whereas the vector $\vec{\rho}_O$ turns into $\vec{\rho}_C$ so that the last expression (3) can be written in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C,[\vec{n},\vec{\rho}_C]]M. \tag{4}$$

This expression (4) *represents the vector form of the theorem of the rate change of the mass inertia moment vector for the axis and the pole, when the axis is translated from the pole at the mass centre C to the arbitrary point, pole O.*

The *Huygens-Steiner theorems* (see Refs. [4] and [5]) for the body mass axial inertia moments, as well as for the mass deviational moments, emerged *from this theorem* (4) on the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at point O for the axis oriented by the unit vector $\vec{n}$ passing trough the mass center C, and when the axis is moved by translate to the other point O.

Mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ for the axis to the pole is possible to decompose in two parts: first $\vec{n}(\vec{n},\vec{\mathfrak{J}}_{\vec{n}}^{(O)})$ collinear with axis and second $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ normal to the axis. So we can write:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{n}(\vec{n},\vec{\mathfrak{J}}_{\vec{n}}^{(O)}) + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = J_{\vec{n}}^{(O)}\vec{n} + \vec{\mathfrak{D}}_{\vec{n}}^{(O)}. \tag{5}$$

Collinear component $\vec{n}(\vec{n},\vec{\mathfrak{J}}_{\vec{n}}^{(O)})$ to the axis corresponds to the axial mass inertia moment $J_{\vec{n}}^{(O)}$ of the body. Second component, $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$, orthogonal to the axis, we denote by the $\vec{\mathfrak{D}}_{O}^{(\vec{n})}$, and it is possible to obtain by both side double vector products by unit vector $\vec{n}$ with mass moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ in the following form:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},n]] = \vec{\mathfrak{J}}_{\vec{n}}^{(O)}(\vec{n},\vec{n}) - \vec{n}(\vec{n},\vec{\mathfrak{J}}_{\vec{n}}^{(O)}) = \vec{\mathfrak{J}}_{\vec{n}}^{(O)} - J_{O}^{\vec{n}}\vec{n}. \tag{6}$$

In case when rigid body is balanced with respect to the axis the mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ is collinear to the axis and there is no deviational part. In this case axis of rotation is main axis of body inertia. When axis of rotation is not main axis then mass inertial moment vector for the axis contains deviation part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$. That is case of rotation unbalanced rotor according to axis and bodies skew positioned to the axis of rotation.

II.2. Model of a rigid body coupled multi-rotation around multi-axes without intersections

Let us to consider rigid body coupled multi-rotations around axes without intersections, first oriented by unit vector $\vec{n}_1$ with fixed position and second and next oriented by unit vectors $\vec{n}_j$, $j = 2, 3, \ldots, K$, which are rotating around fixed axis as well as around series of previous axes and with corresponding angular velocities $\vec{\omega}_j = \omega_j \vec{n}_j$, $j = 1, 2, 3, \ldots, K$. See Figure 1. Axes of rotations are without intersections. Rigid body is positioned on the moving rotating axis oriented by unit vector $\vec{n}_K$. Rigid body rotates around rotating self rotation axis with angular velocity $\vec{\omega}_K = \omega_K \vec{n}_K$ and around series of the previous axes in order and in whole around fixed axis oriented by unit vector $\vec{n}_1$ with angular velocity $\vec{\omega}_1 = \omega_1 \vec{n}_1$. The shortest orthogonal distances between axes are defined by length $\overline{O_{(j)} \tilde{O}_{(j+1)}}$, $j = 1, 2, 3, \ldots, K$ and each of these is perpendicular to both close axes that each is to the direction of component angular velocities $\vec{\omega}_j = \omega_j \vec{n}_j$ and $\vec{\omega}_{j+1} = \omega_{j+1} \vec{n}_{j+1}$. These vectors are

$$\vec{r}_{0(j)(j+1)} = \overrightarrow{O_{(j)1} O_{(j+1)}} : \vec{r}_{0(j)(j+1)} = \frac{r_{0(j)(j+1)}}{\sin \alpha_{(j)(j+1)}} \left[\vec{n}_{(j)}, \vec{n}_{(j+1)} \right] =$$

$$= r_0 \frac{[\vec{n}_{(j)}, \vec{n}_{(j+1)}]}{|[\vec{n}_{(j)}, \vec{n}_{(j+1)}]|} = r_{0(j)(j+1)} \vec{u}_{0(j)(j+1)} \qquad (7)$$

and it can be seen on Fig. 1.

In the considered rigid body coupled rotations around no intersecting numerous axes, an elementary mass around point N is denoted as dm, with position vector $\vec{\rho}$, and with origin in the point O_K on the movable self rotation axis, and with $\vec{r}$ vector positions of the same body elementary mass with origin in the point O_1, where point O_1 is fixed on the axis oriented by unit $\vec{n}_1$. Both points are on the ends of the corresponding shortest orthogonal distance between two in the neighborhood axes of body coupled multi-rotations. Position vector of elementary mass with origin in pole O_1 and its velocity are in the following forms:

$$\vec{r}_k = \sum_{k=1}^{K-1} \left(\vec{r}_{0(k),(k+1)} + \vec{r}_{0(k+1),(k+1)} \right) + \vec{\rho},$$

$$\vec{v}_K = \sum_{k=1}^{K-1}\left[\sum_{j=1}^{k}\vec{\omega}_j, \vec{r}_{0(k),(k+1)} + \vec{r}_{0(k+1),(k+1)}\right] + \left[\sum_{j=1}^{K}\vec{\omega}_j, \vec{\rho}\right] \tag{8}$$

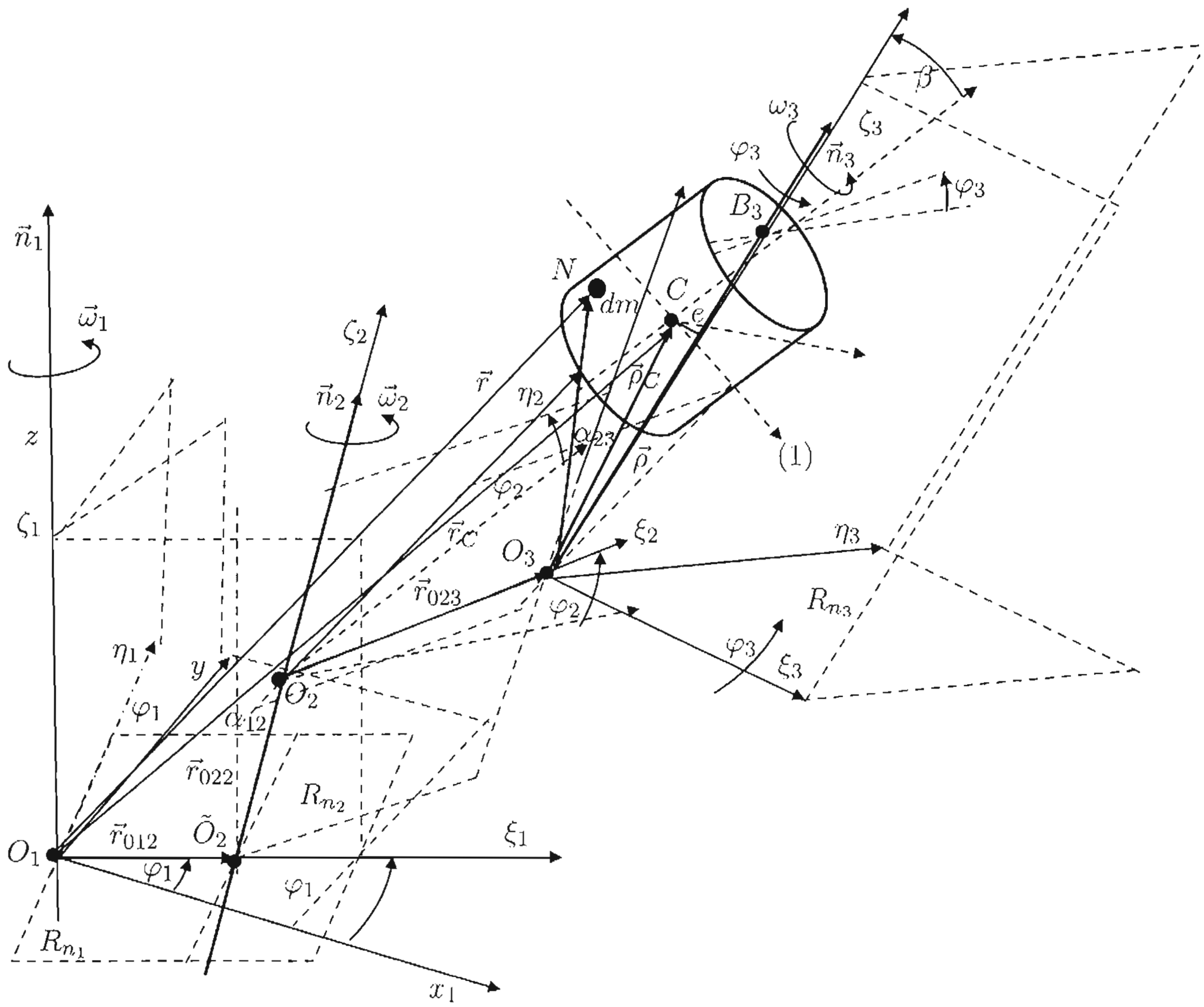


Figure 1 – Arbitrary position of rigid body multi-coupled rotations around finite numbers of axes without intersections

For the case of three coupled rotations around three axes without intersections position vector of elementary mass with origin in pole O_1 and its velocity are in the following forms (see Fig. 1):

$$\vec{r} = \vec{r}_{012} + \vec{r}_{022} + \vec{r}_{023} + \vec{\rho} \quad \text{and}$$
$$\vec{v} = [\vec{\omega}_1, \vec{r}_{012} + \vec{r}_{022}] + [\vec{\omega}_1 + \vec{\omega}_2, \vec{r}_{023}] + [\vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3, \vec{\rho}]. \tag{9}$$

II.2.1. Linear momentum of a rigid body coupled multi-rotations around axes without intersections

By using basic definition of linear momentum and expression for velocity of elementary body mass (9), we can write linear momentum in the following vector form:

$$\vec{\mathfrak{K}} = \sum_{k=1}^{K-1} \sum_{j=1}^{k} \omega_j \left[\vec{n}_j, \vec{r}_{0(k),(k+1)} + \vec{r}_{0(k+1),(k+1)}\right] M + \sum_{j=1}^{K} \omega_j \vec{\mathfrak{S}}_{\vec{n}_j}^{(O_K)} \tag{10}$$

where $\vec{\mathfrak{S}}_{\vec{n}_j}^{(O_K)} = \iiint_V [\vec{n}_j, \vec{\rho}]\, dm$, $j = 1, 2, 3, \ldots, K$, are corresponding body mass linear moments of the rigid body for the axes oriented by direction of component angular velocities of coupled multi-rotations through the movable pole O_K on self rotating axis. First terms in the form of the first sum in expression (10) presents translation part of linear momentum. This part is equal to zero in case when axes intersect in one point. Second sum in expression (10) for linear momentum present linear momentum of pure rotation, as relative motion around all axes with intersection in the pole O_K on self rotation axis. These K terms are different from zero in all cases.

Example 1. Expresion of llinear momentum of a rigid body coupled rotations around two no intersecting axes, we can write in the following form:

$$\vec{\mathfrak{K}} = \vec{\mathfrak{K}}_{\vec{n}_1}^{(O_{1-2})} + \vec{\mathfrak{K}}_{\vec{n}_1}^{(O_2)} + \vec{\mathfrak{K}}_{\vec{n}_2}^{(O_2)} = [\vec{\omega}_1, \vec{r}_{012}]M + \omega_1 \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)} + \omega_2 \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}. \tag{11}$$

Example 2. Expresion of llinear momentum of a rigid body coupled rotations around three axes without intersections, we can write in the following form:

$$\begin{aligned}\vec{\mathfrak{K}} = {} & \omega_1 \left[\vec{n}_1, \vec{r}_{012} + \vec{r}_{022} + \vec{r}_{023}\right] M + \omega_2 \left[\vec{n}_2, \vec{r}_{023}\right] M + \\ & + \omega_1 \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_3)} + \omega_2 \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_3)} + \omega_3 \vec{\mathfrak{S}}_{\vec{n}_3}^{(O)_3}. \end{aligned} \tag{12}$$

II.2.2. Angular momentum of a rigid body coupled multi-rotations around axes without intersections

By using basic definition of angular momentum and expression for velocity of rotation of elementary body mass and its position vector (9), we can write vector expression for angular momentum.

Example 1. Expresion of angular momentum of a rigid body coupled rotations around two axes without intersection, we can write in the following form:

$$\begin{aligned}\vec{\mathfrak{L}}_{(O_1)} = {} & \omega_1 \vec{n}_1 r_0^2 M + \omega_1 \left[\vec{\rho}_C, [\vec{n}_1, \vec{r}_0]\right] M + \omega_1 \left[\vec{r}_0, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right] + \\ & + \omega_2 \left[\vec{r}_0, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right] + \omega_1 \vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)} + \omega_2 \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}. \end{aligned} \tag{13}$$

Example 2. Expresion of angular momentum of a rigid body coupled rotations around three axes without intersections, we can write in the following form:

$$\vec{\mathfrak{L}}_{(O_1)} = \omega_1\left[\vec{r}_{O_3}, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_{1-2-2})}\right] + \omega_1\left[\vec{r}_{O_3}, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_{2-3})}\right] + \omega_2\left[\vec{r}_{O_3}, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_{2-3})}\right] +$$
$$+\omega_1\left[\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_{1-2-2})}\right] + \omega_1\vec{\mathfrak{J}}_{\vec{n}_1}^{(O_3)} + \omega_2\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_3)} + \omega_3\vec{\mathfrak{J}}_{\vec{n}_3}^{(O_3)} + \omega_1\left[\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_{2-3})}\right] + \quad (14)$$
$$+\omega_2\left[\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_{2-3})}\right] + \omega_1\left[\vec{r}_{O_3}, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_3)}\right] + \omega_2\left[\vec{r}_{O_3}, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_3)}\right] + \omega_3\left[\vec{r}_{O_3}, \vec{\mathfrak{S}}_{\vec{n}_3}^{(O_3)}\right].$$

II.2.3. Derivative of linear momentum and angular momentum of rigid body coupled rotations around two axes without intersection

Example 1. By using expressions for linear momentum (13), the derivative of linear momentum of rigid body coupled rotations around two axes without intersection, we can write the following vector expression:

$$\frac{d\vec{\mathfrak{R}}}{dt} = \dot{\omega}_1[\vec{n}_1, \vec{r}_0]M + \omega_1^2[\vec{n}_1, [\vec{n}_1, \vec{r}_0]]M + \dot{\omega}_1\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)} + \omega_1^2\left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right] +$$
$$+\dot{\omega}_2\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)} + \omega_2^2\left[\vec{n}_2, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right] + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]. \quad (15)$$

After analysis structure of linear momentum derivative terms, we can see that there is possibility to introduce pure kinematic vectors, depending on component angular velocities and component angular accelerations of component coupled rotations, that are useful to express derivatives of linear moment in following form

$$\frac{d\vec{\mathfrak{K}}}{dt} = \vec{\mathfrak{R}}_{01}\left|[\vec{n}_1, \vec{r}_0]\right|M + \vec{\mathfrak{R}}_{011}\left|\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right| + \vec{\mathfrak{R}}_{022}\left|\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right| + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]. \quad (16)$$

We can see that in previous vector expression (16), for derivative of linear momentum, are introduced the following three vector rotators:

$$\vec{\mathfrak{R}}_{01} = \dot{\omega}_1\vec{u}_{01} + \omega_1^2\vec{v}_{01}, \quad \vec{\mathfrak{R}}_{01} = \dot{\omega}_1\left[\vec{n}_1, \frac{\vec{r}_0}{r_0}\right] + \omega_1^2\left[\vec{n}_1, \left[\vec{n}_1, \frac{\vec{r}_0}{r_0}\right]\right],$$
$$\vec{\mathfrak{R}}_{011} = \dot{\omega}_1\vec{u}_{011} + \omega_1^2\vec{v}_{011}, \quad \vec{\mathfrak{R}}_{011} = \dot{\omega}_1\frac{\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right|} + \omega_1^2\left[\vec{n}_1, \frac{\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right|}\right], \quad (17)$$
$$\vec{\mathfrak{R}}_{022} = \dot{\omega}_1\vec{u}_{022} + \omega_1^2\vec{v}_{022}, \quad \vec{\mathfrak{R}}_{022} = \dot{\omega}_2\frac{\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right|} + \omega_2^2\left[\vec{n}_2, \frac{\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right|}\right].$$

Also, we can see that in vector expression for derivative of angular momentum appear the following vector rotators: $\vec{\mathfrak{R}}_1 = \dot{\omega}_1\vec{u}_1 + \omega_1^2\vec{v}_1$ and $\vec{\mathfrak{R}}_2 = \dot{\omega}_1\vec{u}_2 + \omega_1^2\vec{v}_2$ expressed by following expressions:

$$\vec{\mathfrak{R}}_1 = \dot{\omega}_1 \frac{\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_1}}{\left|\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_1}\right|} + \omega_1^2 \left[\vec{n}_1, \frac{\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_1}}{\left|\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_1}\right|}\right] = \dot{\omega}_1\vec{u}_1 + \omega_1^2\vec{v}_1,$$

$$\vec{\mathfrak{R}}_2 = \dot{\omega}_2 \frac{\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}}{\left|\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}\right|} + \omega_2^2 \left[\vec{n}_2, \frac{\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}}{\left|\vec{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}\right|}\right] = \dot{\omega}_2\vec{u}_2 + \omega_2^2\vec{v}_2. \tag{18}$$

Example 2. By using expressions for linear momentum the derivative of linear momentum of rigid body coupled rotations around three axes without intersections, not difficult to obtain corresponding expression. Also, as in previous example, after analysis structure of linear momentum derivative terms, we can see that there is possibility to introduce pure kinematic vectors, depending on component angular velocities and component angular accelerations of component coupled three rotations. We can see that in vector expression for derivative of linear momentum series of the vector rotators appear. Some of these vector rotators are listed here:

$$\vec{\mathfrak{R}}_{011} = \dot{\omega}_1\vec{u}_{011} + \omega_1^2\vec{v}_{011}, \quad \vec{\mathfrak{R}}_{011} = \dot{\omega}_1 \frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}\right|} + \omega_1^2 \left[\vec{n}_1, \frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}\right|}\right],$$

$$\vec{\mathfrak{R}}_{022} = \dot{\omega}_2\vec{u}_{022} + \omega_2^2\vec{v}_{022}; \quad \vec{\mathfrak{R}}_{022} = \dot{\omega}_2 \frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}\right|} + \omega_2^2 \left[\vec{n}_2, \frac{\vec{\mathfrak{S}}^{(O3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O3)}_{\vec{n}_2}\right|}\right]; \tag{19}$$

$$\vec{\mathfrak{R}}_{033} = \dot{\omega}_3\vec{u}_{033} + \omega_1^2\vec{v}_{033}, \quad \vec{\mathfrak{R}}_{033} = \dot{\omega}_3 \frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}\right|} + \omega_3^2 \left[\vec{n}_2, \frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}\right|}\right].$$

Also, we can see that in vector expression for derivative of angular momentum of rigid body coupled rotations around three axes without intersections appear the following vector rotators: $\vec{\mathfrak{R}}_1 = \dot{\omega}_1\vec{u}_1 + \omega_1^2\vec{v}_1$, $\vec{\mathfrak{R}}_2 = \dot{\omega}_2\vec{u}_2 + \omega_2^2\vec{v}_2$ and $\vec{\mathfrak{R}}_3 = \dot{\omega}_3\vec{u}_3 + \omega_1^2\vec{v}_3$ expressed by following expressions:

$$\vec{\mathfrak{R}}_1 = \dot{\omega}_1 \frac{\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_1}\right|} + \omega_1^2 \left[\vec{n}_1, \frac{\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_1}\right|}\right] = \dot{\omega}_1\vec{u}_1 + \omega_1^2\vec{v}_1,$$

$$\vec{\mathfrak{R}}_2 = \dot{\omega}_2 \frac{\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_2}\right|} + \omega_2^2 \left[\vec{n}_2, \frac{\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{D}}^{(O_3)}_{\vec{n}_2}\right|}\right] = \dot{\omega}_2\vec{u}_2 + \omega_2^2\vec{v}_2 \tag{20}$$

$$\vec{\mathfrak{R}}_3 = \dot{\omega}_3 \frac{\vec{\mathfrak{D}}_{\vec{n}_3}^{(O_3)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_3}^{(O_3)}\right|} + \omega_3^2 \left[\vec{n}_3, \frac{\vec{\mathfrak{D}}_{\vec{n}_3}^{(O_3)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_3}^{(O_3)}\right|}\right] = \dot{\omega}_3 \vec{u}_3 + \omega_3^2 \vec{v}_3$$

II.3. Concluding remarks

By using theorems of changes of linear momentum and angular momentum with respect to time, one may write two vector equations of dynamic equilibrium of rigid body coupled multi-rotations about axes without intersection as the follows:

$$\frac{d\vec{\mathfrak{K}}}{dt} = \vec{G} + \vec{F}_{AN} + \vec{F}_{BN} + \vec{F}_{Am} + \sum_{i=1}^{i=P} \vec{F}_i \tag{21}$$

$$\frac{d\vec{\mathfrak{L}}_{O_1}}{dt} = \left[\vec{r}_0 + \vec{\rho}_C, \vec{G}\right] + \left[\vec{r}_0 + \vec{\rho}_{A,}, \vec{F}_{AN}\right] + \left[\vec{r}_0 + \vec{\rho}_{B,}, \vec{F}_{BN}\right] + \\ + \left[\vec{r}_0 + \vec{\rho}_{A,}, \vec{F}_{Am}\right] + \sum_{i=1}^{i=P} \left[\vec{r}_0 + \vec{\rho}_{i,}, \vec{F}_i\right] \tag{22}$$

where $\vec{F}_i$, $\underline{i = 1, 2, 3, \ldots, p}$ are external active forces, $\vec{G}$ is weight of a rotor, $\vec{F}_A$ and $\vec{F}_B$ are forces of bearing reactions at fixed axis. From previous analysis, we can conclude that vector rotators appear into expressions of the kinetic reactions of the shaft bearings of the structures of the rigid body multi-coupled rotations and that is very important to analyze their intensity as well as their relative angular velocity and angular acceleration around axes of coupled multi-rotations.

II.4. Recommendation foe next research and for solving three main mathematical tasks

a* Generalization of the expressions for derivatives of linear momentum and angular momentum for rigid body coupled nulti-rotations around finite numbers axes without intersections; b* expressions for kinetic pressures on bearing to series of the axes of coupled rotations and corresponding numbers of coupled nonlinear differential equations depending of number of system degree of freedom with corresponding solutions and c* build a algorithm for using obtained results as a standard software program for analysis nonlinear dynamic phenomena in rigid body coupled rotation around finite number axes without intersections. These defined tasks need a team interdisciplinary research, and will be very useful for engineering practice in analysis and simulation numerous engineering system dynamics with coupled rotation and for vibrodiagnostic

Based on the scientific knowledge, not only on the experimental measurement of the different signals without and no professional conclusen based only on the quazi-professional expert opinions.

III. Tangent spaces of position vectors and angular velocities of their basic vectors in different coordinate systems

Angular velocity of the basic vectors rotation of a tangent space of the vector positions of material particles of mechanical system dynamics with geometrical, stationary and rheonomic constraints are obtained. Extensions of dimensions of tangent space of the vector positions of material particles of mechanical system dynamics from three dimensional real spaces to configuration space of independent generalized curvilinear coordinate systems is identified. Reductions of numbers of coordinates and extensions of tangent space of vector passions are analyzed (see References II [21-34]).

Angular velocities of the basic vectors of tangent spaces of the position vectors of mass particles of the discrete rheonomic mechanical system are obtained in different coordinate systems [22]. Starting from real three dimensional coordinate systems of Descartes orthogonal three dimensional system type with fixed coordinates axis as a reference, by different coordinate transformations for each position vector of corresponding mass particle in discrete rheonomic mechanical system, basic vectors of position vector tangent three dimensional spaces are obtained in different curvilinear coordinate systems suitable to the corresponding geometrical scleronomic or rheonomic constraints applied to the considered rheonomic system. For each basic vector of the basic triedar of position vector tangent space of each mass particle of the discrete rheonomic mechanical system, angular velocity vectors of basic vector rotations are determined.

Then, after consideration and analysis of the number and properties of the geometrical scleronomic and rheonomic constraints applied to the mass particles of the considered discrete rheonomic mechanical system, number of system degree of mobility as well as number of system degree of freedom are determined. Corresponding number of independent coordinates are chosen and corresponding rheonomic coordinates are introduced. By use extended set of the generalized coordinates contained corresponding number of independent coordinates and corresponding number of rheonomic coordinates, position vectors of the mass particles of the discrete rheonomic mechanical system, are separated into two subsets.

First subset contain position vectors of the mass particle, keep their three dimensional tangent space each with three basic vectors.

Second subset contain position vectors of the mass particle, each depending, in general case, of the all generalized coordinates, independent and rheonomic. Then, each of the position vectors are with $n + R$-dimensional tangent spaces and with basic vectors.

III.1. Introduction

Let consider a discrete system with N mass particles wirh mass m_α, and with corresponsing position in real three dimensional space determined by geometrical points $N_{(\alpha)}$, $\alpha = 1, 2, 3, \dots, N$ (see Figure 2). For begining we take that positions of the material points, as well as corresponding geometrical points coordinates are determined by coordinates in fixed orthogonal Descartes coordinate system with three coordinates as denoted by $N_{(\alpha)}(x_{(\alpha)}; y_{(\alpha)}; z_{(\alpha)})$. $\alpha = 1, 2, 3, \dots, N$, where O is fixed coordinate origin, and Ox, Oy and Oz fixed oriented coordinate strain lines-coordinate axes. Coordinates of the position vector of each material point are equal to coordinate of the geometrical point which determine mass particle position in the space. For Descartes coordinate system for position of the each mass particle we can write: $\vec{\rho}_{(\alpha)}(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}) = x_{(\alpha)}\vec{i} + y_{(\alpha)}\vec{j} + z_{(\alpha)}\vec{k}$, $\alpha = 1, 2, 3, \dots, N$.

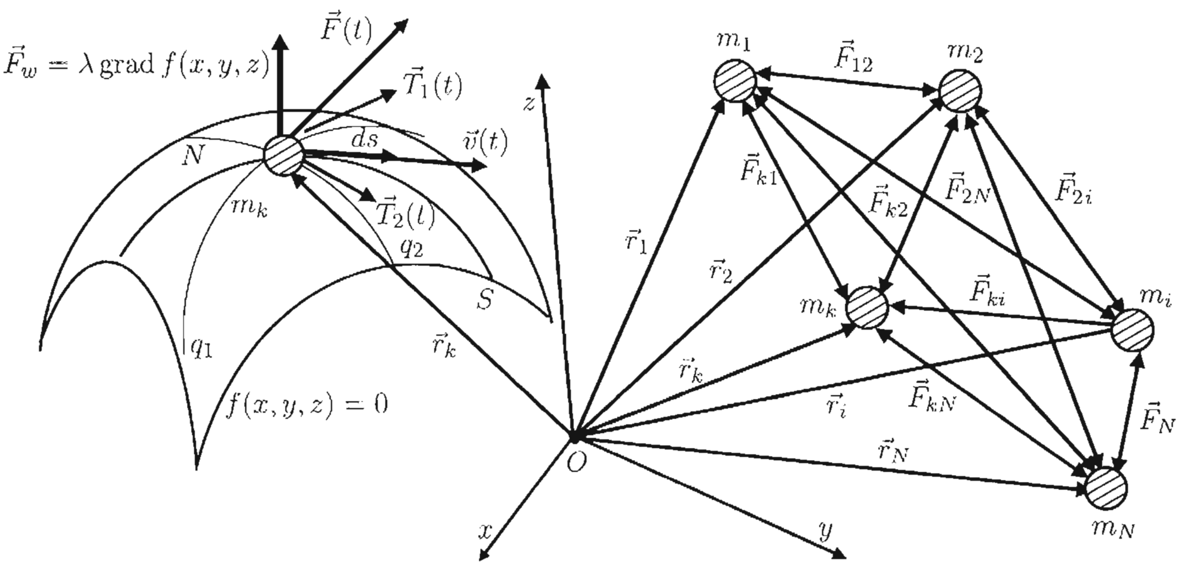


Figure 2 – Discrete material system with N mass particles and geometrical rheonomic constraints

Let, now, to consider previus discrete system with N mass particles with mass m_α, and with corresponsing position in real three dimensional space determined by same geometrical points $N_{(\alpha)}$, $\alpha = 1, 2, 3, \dots, N$ in generalized coordinate system of curvilinear coordinates $(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)})$ $\alpha = 1, 2, 3, \dots, N$ corresponding to mass particle positions. For same geometrical points coordinates in considered three coordinate systems are: $N_{(\alpha)}(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}$. $\alpha = 1, 2, 3, \dots, N$ and $N_{(\alpha)}(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)})$ $\alpha = 1, 2, 3, \dots, N$. Formule of coordinate transformation from previous coordinate system with fixed axes and new curvilinear coordinate system are:

$$\begin{aligned} x_{(\alpha)} &= x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right) \\ y_{(\alpha)} &= y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right) \\ z_{(\alpha)} &= z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right). \end{aligned} \tag{1}$$

Position vectors of each mass particle and corresponding geometrical points are invariant geometrical objects in both coordinate systems, but their coordinates in considered coordinate systems are not equal to coordinates of the corresponding geometrical point. In generalized coordinate system geometrical points $N_{(\alpha)}$, $\alpha = 1, 2, 3, \dots, N$ have following coordinates: $(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)})$, $\alpha = 1, 2, 3, \dots, N$ and coordinate of position vectors of these geometrical points are $(\rho^1_{(\alpha)}, \rho^2_{(\alpha)}, \rho^3_{(\alpha)})$, $\alpha = 1, 2, 3, \dots, N$. For position vectors we can write:

$$\begin{gathered} \vec{\rho}_{(\alpha)}(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}) = \\ = x_{(\alpha)}\vec{i} + y_{(\alpha)}\vec{j} + z_{(\alpha)}\vec{k} \overset{\text{def=inv.vektor}}{=} \vec{\rho}_{(\alpha)}(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}) \end{gathered} \tag{2}$$

$$\begin{gathered} \vec{\rho}_{(\alpha)}(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}) = \rho^1_{(\alpha)}(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)})\vec{g}_{(\alpha)1} + \\ + \rho^2_{(\alpha)}(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)})\vec{g}_{(\alpha)2} + \rho^3_{(\alpha)}(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)})\vec{g}_{(\alpha)3} \\ \alpha = 1, 2, 3, \dots, N. \end{gathered} \tag{3}$$

For first example in polar-cylindrical coordinate system geometrical points have the following coordinates: $N_{(\alpha)}(r_{(\alpha)}, \varphi_{(\alpha)}, z_{(\alpha)})$ $\alpha = 1, 2, 3, \dots, N$ and position vectors $\vec{\rho}_{(\alpha)}(r_{(\alpha)}, \varphi_{(\alpha)}, z_{(\alpha)})$ of corresponding geometrical point are: $r_{(\alpha)}, 0, z_{(\alpha)}$ and we can write:

$$\begin{gathered} \vec{\rho}_{(\alpha)}(r_{(\alpha)}, \varphi_{(\alpha)}, z_{(\alpha)}) = r_{(\alpha)}\vec{r}_{o(\alpha)} + 0 \cdot \vec{c}_{0(\alpha)} + z_{(\alpha)}\vec{k} \equiv r_{(\alpha)}\vec{r}_{o(\alpha)} + z_{(\alpha)}\vec{k} \\ \alpha = 1, 2, 3, \dots, N \end{gathered} \tag{4}$$

where $\vec{r}_{o(\alpha)}$, $\vec{c}_{0(\alpha)}$ and $\vec{k}$, $\alpha = 1, 2, 3, \dots, N$ are basic unit vectors of tangent space of corresponding position vector in polar-cylindrical coordinate system.

For second example in spherical coordinate system geometrical points have the following coordinates: $N_{(\alpha)}(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)})$ $\alpha=1,2,3,\dots,N$ and position vectors $\vec{\rho}_{(\alpha)}(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)})$ of corresponding geometrical point are: $\rho_{(\alpha)},0,0$ and we can write:

$$\vec{\rho}_{(\alpha)}(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)})=\rho_{(\alpha)}\vec{\rho}_{0(\alpha)}+0\cdot\vec{c}_{0(\alpha)}+0\cdot\vec{\nu}_{0(\alpha)}\equiv\rho_{(\alpha)}\vec{\rho}_{0(\alpha)} \qquad (5)$$
$$\alpha=1,2,3,\dots,N$$

where $\vec{\rho}_{0(\alpha)}$; $\vec{c}_{0(\alpha)}$ and $\vec{\nu}_{0(\alpha)}$, $\alpha=1,2,3,\dots,N$ are basic unit vectors of tangent space of corresponding position vector in polar-cylindrical coordinate system.

III.2. Basic vectors of the position vector three-dimensional tangent space in generalized curvilinear coordinate systems

In real two-dimensional coordinate systems, position vector tangent spaces are three-dimensional and the basic vectors of the tangent spaces of each position vector of each mass particle we denote with $\vec{g}_{(\alpha)i}$, $\alpha=1,2,3,\dots,N$, $i=1,2,3$ (see Figure 3). These vectors are in tangent directions to the corresponding curvilinear coordinate line and in general are not unit vectors. Basic vectors it is possible to obtain by following way (for detail see Refs. [22], [23], [24], [25], [26], [27], [28], [33] and [34]):

$$\vec{g}_{(\alpha)i}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q^i_{(\alpha)}},\quad \alpha=1,2,3,\dots,N,\quad i=1,2,3 \qquad (6)$$

or by formula coordinate transformation and by following expressions:

$$\vec{g}_{(\alpha)1}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q^1_{(\alpha)}}=$$
$$=\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\vec{i}+\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\vec{j}+\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\vec{k}$$
$$\vec{g}_{(\alpha)2}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q^2_{(\alpha)}}=$$
$$=\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\vec{i}+\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\vec{j}+\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\vec{k}$$
$$\vec{g}_{(\alpha)3}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q^3_{(\alpha)}}= \qquad (7)$$

$$= \frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\vec{i} + \frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\vec{j} + \frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\vec{k}.$$

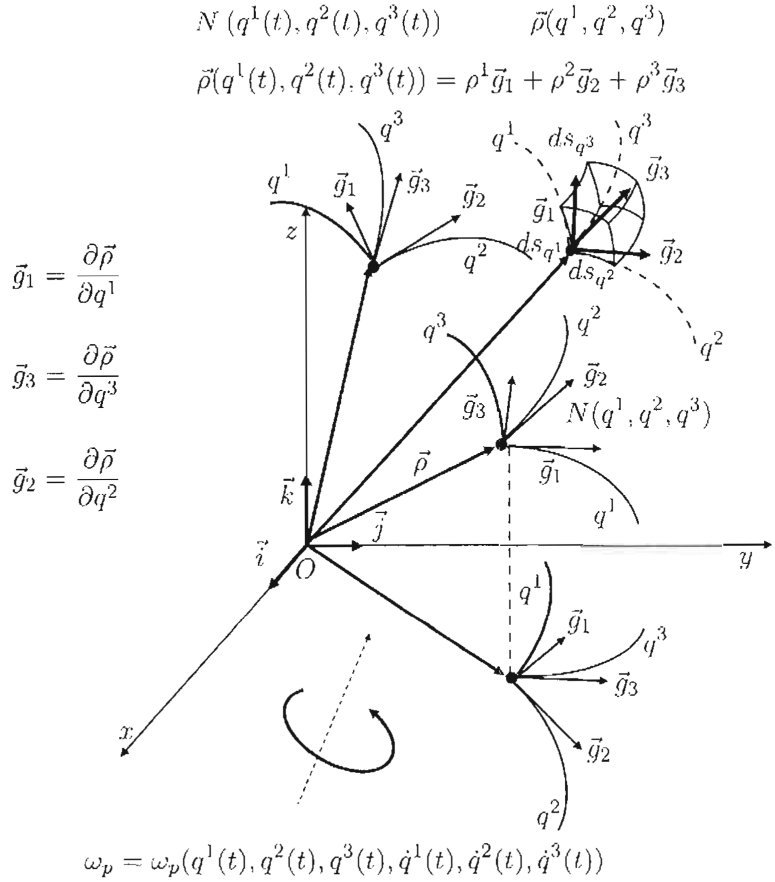


Figure 3 – A position vectors and its three-dimensional space with corresponding curvilinear coordinate system and tangent space with corresponding three basic vectors of the position vector tangent spaces along mass particle motion through time

Contravariant coordinates of the position vectors it is possible to obtain by following formulas:

$$\rho^1_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right) = \frac{1}{\Delta}\left[\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}} - \right.$$

$$\left. - \frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\right] x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right) -$$

$$- \frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}} - \right.$$

$$\left. - \frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\right] y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right) -$$

$$-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\right]z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)$$

$$\rho^2_{(\alpha)}(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)})=-\frac{1}{\Delta}\left[\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\right]x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)+$$
$$+\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\right]y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)-$$
$$-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^3_{(\alpha)}}\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\right]z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)$$

$$\rho^3_{(\alpha)}(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)})=\frac{1}{\Delta}\left[\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\right]x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)-$$
$$-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\frac{\partial z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\right]y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)+$$
$$+\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}-\right.$$
$$\left.-\frac{\partial x_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^2_{(\alpha)}}\frac{\partial y_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)}{\partial q^1_{(\alpha)}}\right]z_{(\alpha)}\left(q^1_{(\alpha)},q^2_{(\alpha)},q^3_{(\alpha)}\right)$$

where

$$\Delta_{(\alpha)} = \begin{vmatrix} \dfrac{\partial x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^1} & \dfrac{\partial y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^1} & \dfrac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^1} \\ \dfrac{\partial x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2} & \dfrac{\partial y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2} & \dfrac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^2} \\ \dfrac{\partial x_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3} & \dfrac{\partial y_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3} & \dfrac{\partial z_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)}{\partial q^3} \end{vmatrix} \neq 0$$

III.3. Change of the basic vectors of the position vector three-dimensional tangent space in generalized curvilinear coordinate systems

Without loosing generality, we consider change of basic vectors of a position vector of one mass particle during mass particle motion through real space and described in three-dimensional space. Also, we focused our attention to the orthogonal curvilinear coordinate system. For that case change (first derivative with respect to time) with time of the basic vectors of tangent space of a position vector are (see Figure 3):

$$\begin{aligned} \frac{d\vec{g}_1}{dt} &= \overset{*}{\vec{g}_1} + [\vec{\omega}_{p1}, \vec{g}_1] = \vec{g}_1\left(\Gamma^1_{11}\dot{q}^1 + \Gamma^1_{12}\dot{q}^2 + \Gamma^1_{13}\dot{q}^3\right) + \\ &+ \vec{g}_2\left(\Gamma^2_{11}\dot{q}^1 + \Gamma^2_{12}\dot{q}^2 + \Gamma^2_{13}\dot{q}^3\right) + \vec{g}_3\left(\Gamma^3_{11}\dot{q}^1 + \Gamma^3_{12}\dot{q}^2 + \Gamma^3_{13}\dot{q}^3\right) \\ \frac{d\vec{g}_2}{dt} &= \overset{*}{\vec{g}_2} + [\vec{\omega}_{p2}, \vec{g}_2] = \left(\Gamma^1_{21}\vec{g}_1 + \Gamma^2_{21}\vec{g}_2 + \Gamma^3_{21}\vec{g}_3\right)\dot{q}^1 + \\ &+ \left(\Gamma^1_{22}\vec{g}_1 + \Gamma^2_{22}\vec{g}_2 + \Gamma^3_{22}\vec{g}_3\right)\dot{q}^2 + \left(\Gamma^1_{23}\vec{g}_1 + \Gamma^2_{23}\vec{g}_2 + \Gamma^3_{23}\vec{g}_3\right)\dot{q}^3 \\ \frac{d\vec{g}_3}{dt} &= \overset{*}{\vec{g}_3} + [\vec{\omega}_{p3}, \vec{g}_3] = \left(\Gamma^1_{31}\vec{g}_1 + \Gamma^2_{31}\vec{g}_2 + \Gamma^3_{31}\vec{g}_3\right)\dot{q}^1 + \left(\Gamma^1_{32}\vec{g}_1 + \Gamma^2_{32}\vec{g}_2 + \right. \\ &\left. + \Gamma^3_{32}\vec{g}_3\right)\dot{q}^2 + \left(\Gamma^1_{33}\vec{g}_1 + \Gamma^2_{33}\vec{g}_2 + \Gamma^3_{33}\vec{g}_3\right)\dot{q}^3 \end{aligned} \tag{8}$$

After analysis of the obtained derivatives of the basic vectors of position vector tangent spaces in three-dimensional orthogonal curvilinear coordinate systems we can separate two sets of the terms in obtained expressions (8). First set correspond to the relative derivative of the corresponding basic vectors in the following forms:

$$\begin{aligned} \overset{*}{\vec{g}_1} &= \vec{g}_1\left(\Gamma^1_{11}\dot{q}^1 + \Gamma^1_{12}\dot{q}^2 + \Gamma^1_{13}\dot{q}^3\right) \\ \overset{*}{\vec{g}_2} &= \vec{g}_2\left(\Gamma^2_{21}\dot{q}^1 + \Gamma^2_{22}\dot{q}^2 + \Gamma^2_{23}\dot{q}^3\right) \\ \overset{*}{\vec{g}_3} &= \vec{g}_3\left(\Gamma^3_{31}\dot{q}^1 + \Gamma^3_{32}\dot{q}^2 + \Gamma^3_{33}\dot{q}^3\right). \end{aligned} \tag{9}$$

These vectors present vector forms of extensions of the corresponding basic vectors and in scalar form it is possible to express relative change of the intensity – dilatation of the basic vectors in direction of its previous kinetic state. In differential form is possible to write:

$$\begin{aligned}
d\varepsilon_1 &= \frac{d\,|\vec{g}_1|}{|\vec{g}_1|} = \Gamma^1_{11}dq^1 + \Gamma^1_{12}dq^2 + \Gamma^1_{13}dq^3 \\
d\varepsilon_2 &= \frac{d\,|\vec{g}_2|}{|\vec{g}_2|} = \Gamma^2_{31}dq^1 + \Gamma^2_{32}dq^2 + \Gamma^2_{33}dq^3 \\
d\varepsilon_3 &= \frac{d\,|\vec{g}_{31}|}{|\vec{g}_3|} = \Gamma^3_{31}dq^1 + \Gamma^3_{32}dq^2 + \Gamma^3_{33}dq^3.
\end{aligned} \tag{10}$$

From analysis of the obtained derivatives of the basic vectors of position vector tangent spaces in three-dimensional orthogonal curvilinear coordinate systems we can separate second set of the terms in obtained expressions (8). Second set correspond to the rotation change of the corresponding basic vectors in the following forms:

$$\begin{aligned}
[\vec{\omega}_{p1}, \vec{g}_1] &= \vec{g}_2\left(\Gamma^2_{11}\dot{q}^1 + \Gamma^2_{12}\dot{q}^2 + \Gamma^2_{13}\dot{q}^3\right) + \vec{g}_3\left(\Gamma^3_{11}\dot{q}^1 + \Gamma^3_{12}\dot{q}^2 + \Gamma^3_{13}\dot{q}^3\right) \\
[\vec{\omega}_{p2}, \vec{g}_2] &= \vec{g}_1\left(\Gamma^1_{21}\dot{q}^1 + \Gamma^1_{22}\dot{q}^2 + \Gamma^1_{23}\dot{q}^3\right) + \vec{g}_3\left(\Gamma^3_{21}\dot{q}^1 + \Gamma^3_{22}\dot{q}^2 + \Gamma^3_{23}\dot{q}^3\right) \\
[\vec{\omega}_{p3}, \vec{g}_3] &= \vec{g}_1\left(\Gamma^1_{31}\dot{q}^1 + \Gamma^1_{32}\dot{q}^2 + \Gamma^1_{33}\dot{q}^3\right) + \vec{g}_2\left(\Gamma^2_{31}\dot{q}^1 + \Gamma^2_{32}\dot{q}^2 + \Gamma^2_{33}\dot{q}^3\right)
\end{aligned} \tag{11}$$

where we introduce notation $\vec{\omega}_{p1}$, $\vec{\omega}_{p2}$ and $\vec{\omega}_{p3}$ for vectors of the angular velocities of the corresponding basic vectors of the position vector tangent space. When curvilinear coordinate system is not orthogonal and angles between three basic vectors are changeable with time these angular velocities are different for each basic vector. When basic vectors are orthogonal and without change orthogonal relation, all three angular velocity are same.

For the case of the discrete mechanical system N mass particles for each vector position of each mass particle is necessary, by analogous way as presented in previous part, is possible to determine change of the basic vectors of tangent space of position vectors.

After analysis of the obtained derivatives of the basic vectors of position vector tangent spaces for each mass particle, in three-dimensional orthogonal curvilinear coordinate systems, we can separate two sets of the terms in obtained expression and corresponding for other two sets of the basic vectors. First set correspond to the relative derivative of the corresponding basic vectors. These vectors present vector forms of extensions of the corresponding basic vectors and in scalar form it is possible to express relative changes of the intensities – dilatations of the basic vectors in direction of their previous kinetic state.

From analysis of the obtained derivatives of the basic vectors of position vector tangent spaces for each mass particle in three-dimensional orthogonal curvilinear coordinate systems, we can separate second sets of the terms in obtained expressions. Second set correspond to the rotation change of the corresponding basic vectors. We introduce notation $\vec{\omega}_{(\alpha)p1}$, $\vec{\omega}_{(\alpha)p2}$ and $\vec{\omega}_{(\alpha)p3}$ for vectors of the angular velocities of the corresponding basic vectors of the position vector tangent spaces. When basic vectors are orthogonal and without change orthogonal relation, all three angular velocity are same, for each vector position.

For example 1*: in polar-cylindrical curvilinear coordinate system by expressions (8), (9), (10) and (11) we can write (see Figure 4.a*):

$$\frac{d\vec{g}_1}{dt} = \frac{d\vec{g}_r}{dt} = \dot{\varphi}\big(-\vec{i}\sin\varphi + \vec{j}\cos\varphi\big) = \frac{d\vec{r}_0}{dt} = \dot{\varphi}\vec{c}_0 = \frac{1}{r}\dot{\varphi}\vec{g}_\varphi = [\vec{\omega}_{\mathrm{Pr}}, \vec{g}_r]$$

$$\frac{d\vec{g}_2}{dt} = \frac{d\vec{g}_\varphi}{dt} = \dot{r}\vec{c}_0 + r\frac{d\vec{c}_0}{dt} = \dot{r}\vec{c}_0 - r\dot{\varphi}\,\vec{r}_0 = \overset{*}{\vec{g}}_\varphi + [\vec{\omega}_{P\varphi}, \vec{g}_\varphi]$$

$$\frac{d\vec{g}_3}{dt} = \frac{d\vec{g}_z}{dt} = \frac{d\vec{k}}{dt} = 0$$

$$\overset{*}{\vec{g}}_r = 0, \quad d\varepsilon_r = \frac{d\,|\vec{g}_\varphi|}{|\vec{g}_\varphi|} = \frac{dr}{r}, \quad \vec{\omega}_{\mathrm{Pr}} = \dot{\varphi}\vec{k}$$

$$\overset{*}{\vec{g}}_\varphi = \dot{r}\vec{c}_0 = \frac{\dot{r}}{r}\vec{g}_\varphi, \quad \vec{\omega}_{P\varphi} = \dot{\varphi}\vec{k}, \quad \overset{*}{\vec{g}}_z = 0, \quad \vec{\omega}_{Pz} = 0.$$

Angular velocities of the basic vectors of each position vector tangent space of mass particle motion in polar-cylindrical curvilinear coordinate systems are: $\vec{\omega}_{(\alpha)P} = \dot{\varphi}_{(\alpha)}\vec{k}$, $\alpha = 1, 2, 3, \ldots, N$.

For example 2*: in spherical curvilinear coordinate system by expressions (8), (9), (10) and (11), we can write (see Figure 4.b*):

$$\frac{d\vec{g}_1}{dt} = \frac{d\vec{g}_\rho}{dt} = \dot{\varphi}\vec{c}_0\cos\psi + \dot{\psi}\vec{\nu}_0 = \frac{1}{\rho}\dot{\varphi}\vec{g}_\varphi + \frac{1}{\rho}\dot{\psi}\vec{g}_\psi$$

$$\frac{d\vec{g}_2}{dt} = \frac{d\vec{g}_\varphi}{dt} = \vec{c}_0(\dot{\rho}\cos\psi - \rho\dot{\psi}\sin\psi) - \dot{\varphi}(\vec{\rho}_0\cos\psi - \vec{\nu}_0\sin\psi)\rho\cos\psi$$

$$\frac{d\vec{g}_3}{dt} = \frac{d\vec{g}_\psi}{dt} = \dot{\rho}\vec{\nu}_0 + \rho(-\dot{\varphi}\vec{c}_0\sin\psi - \dot{\psi}\vec{\rho}_0)$$

$$\overset{*}{\vec{\rho}}_\rho = 0, \quad [\vec{\omega}_{P\rho}, \vec{\rho}_\rho] = \dot{\psi}\vec{\nu}_0 + \dot{\varphi}\vec{c}_0\cos\psi$$

$$\vec{\omega}_{P\rho} = \dot{\psi}\,\vec{c}_0 + \dot{\varphi}\vec{k} = -\frac{\dot{\psi}}{\rho\cos\psi}\,\vec{g}_\varphi + \dot{\varphi}\left(\vec{g}_\rho\sin\psi + \frac{1}{\rho}\,\vec{g}_\psi\cos\psi\right)$$

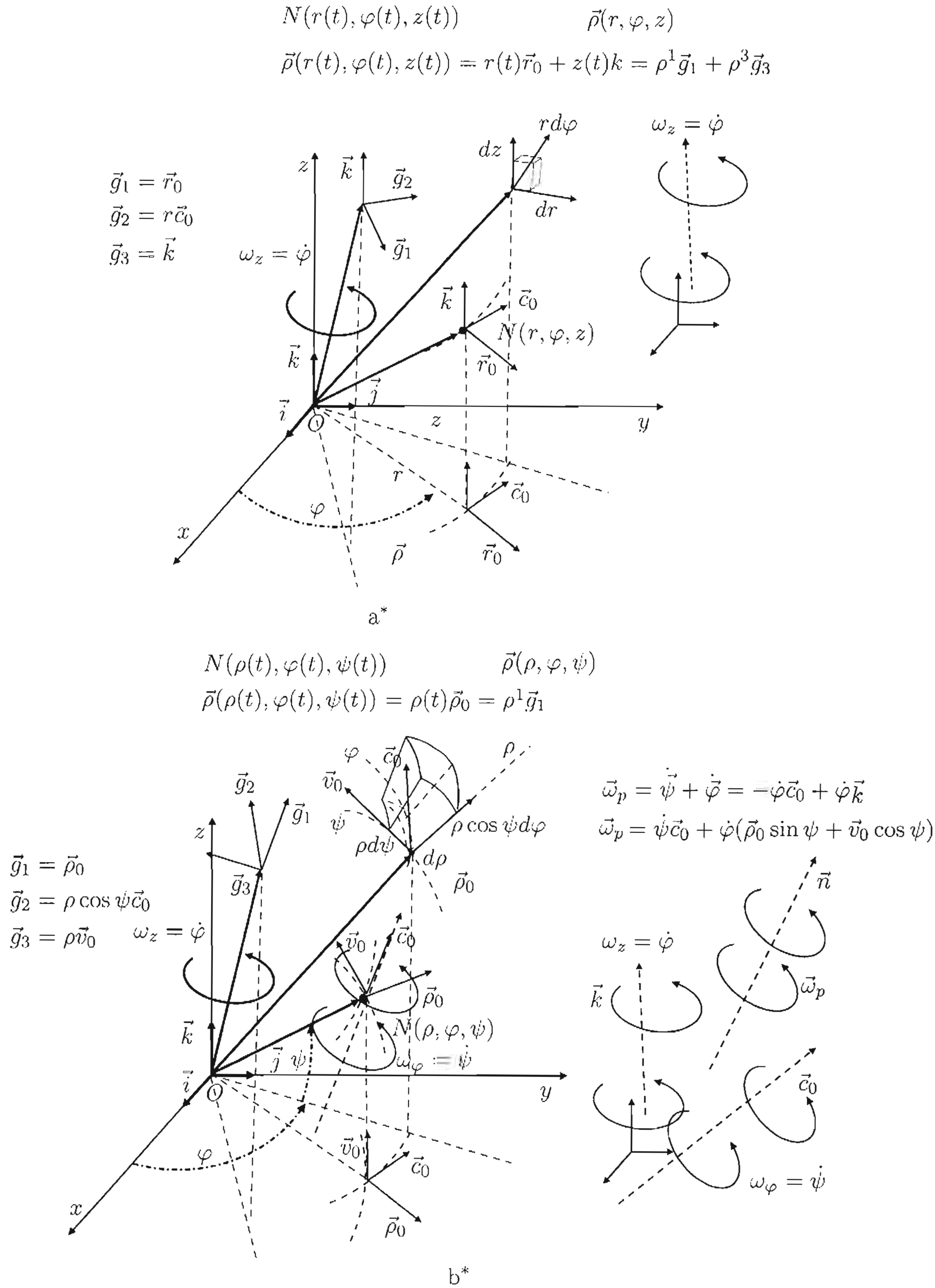


Figure 4 – A position vectors and its three-dimensional spaces with corresponding curvilinear coordinate system and tangent space with corresponding three basic vectors of the position vector tangent spaces along mass particle motion through time va* polar-cylindrical curvilinear coordinate system; b* spherical curvilinear coordinate system

$$\overset{*}{\vec{g}}_{\varphi} = \vec{c}_0(\dot{\rho}\cos\psi - \rho\dot{\psi}\sin\psi) = \frac{1}{\rho\cos\psi}(\dot{\rho}\cos\psi - \rho\dot{\psi}\sin\psi)\vec{g}_{\varphi}$$

$$[\vec{\omega}_{P\varphi}, \vec{g}_{\varphi}] = \left[-\dot{\psi}\,\vec{c}_0 + \dot{\varphi}(\vec{\rho}_0\sin\psi + \vec{\nu}_0\cos\psi), \vec{c}_0\rho\cos\psi\right] =$$
$$= -\dot{\varphi}(\vec{\rho}_0\cos\psi - \vec{\nu}_0\sin\psi)\rho\cos\psi$$

$$\vec{\omega}_{P\varphi} = -\dot{\vec{\psi}} + \dot{\vec{\varphi}} = \dot{\psi}\,\vec{c}_0 + \dot{\varphi}\vec{k} = -\frac{\dot{\psi}}{\rho\cos\psi}\,\vec{g}_{\varphi} + \dot{\varphi}\left(\vec{g}_{\rho}\sin\psi + \frac{1}{\rho}\,\vec{g}_{\psi}\cos\psi\right)$$

$$\frac{d\vec{g}_3}{dt} = \frac{d\vec{g}_{\psi}}{dt} = \dot{\rho}\vec{\nu}_0 + \rho(-\dot{\varphi}\vec{c}_0\sin\psi - \dot{\psi}\vec{\rho}_0) \quad \overset{*}{\vec{g}}_{\psi} = \dot{\rho}\vec{\nu}_0 = \frac{\dot{\rho}}{\rho}\vec{g}_{\psi},$$

$$[\vec{\omega}_{P\psi}, \vec{g}_{\psi}] = -\dot{\varphi}\vec{g}_{\varphi}tg\psi - \dot{\psi}\vec{g}_{\psi} = \left[-\dot{\psi}\,\vec{c}_0 + \dot{\varphi}(\vec{\rho}_0\sin\psi + \vec{\nu}_0\cos\psi), \rho\vec{\nu}_0\right]$$

$$\vec{\omega}_{P\psi} = -\dot{\vec{\psi}} + \dot{\vec{\varphi}} = \dot{\psi}\,\vec{c}_0 + \dot{\varphi}\vec{k} = -\frac{\dot{\psi}}{\rho\cos\psi}\,\vec{g}_{\varphi} + \dot{\varphi}\left(\vec{g}_{\rho}\sin\psi + \frac{1}{\rho}\,\vec{g}_{\psi}\cos\psi\right).$$

Angular velocities of the basic vectors of each position vector tangent space of mass particle motion in spherical curvilinear coordinate systems are:

$$\vec{\omega}_{(\alpha)P} = -\dot{\vec{\psi}}_{(\alpha)} + \dot{\vec{\varphi}}_{(\alpha)} = \dot{\psi}_{(\alpha)}\,\vec{c}_{0(\alpha)} + \dot{\varphi}_{(\alpha)}\vec{k}$$

$$\vec{\omega}_{(\alpha)P} = -\dot{\psi}_{(\alpha)}\,\vec{c}_{0(\alpha)} + \dot{\varphi}_{(\alpha)}\left(\vec{\rho}_{0(\alpha)}\sin\psi_{(\alpha)} + \vec{\nu}_{0(\alpha)}\cos\psi_{(\alpha)}\right), \quad \alpha = 1, 2, 3, \dots, N$$

III.4. Dimensional extension of the position vector tangent spaces of the reheonomic mechanical system in generalized curvilinear coordinate systems

Considered discrete mechanical system is constrained by G geometrical stationary constraints in the form:

$$f_{\beta}\left(\overset{1}{q}_{(1)}, \overset{2}{q}_{(1)}, \overset{3}{q}_{(1)}, \overset{1}{q}_{(2)}, \overset{2}{q}_{(2)}, \overset{3}{q}_{(2)}, \dots, \overset{1}{q}_{(N)}, \overset{2}{q}_{(N)}, \overset{3}{q}_{(N)}\right) = 0, \qquad (12)$$
$$\beta = 1, 2, 3 \dots, G$$

and by R geometrical rheonomic constraints in the form (see Ref. [23]):

$$f_{\gamma}\left(\overset{1}{q}_{(1)}, \overset{2}{q}_{(1)}, \overset{3}{q}_{(1)}, \overset{1}{q}_{(2)}, \overset{2}{q}_{(2)}, \overset{3}{q}_{(2)}, \dots, \overset{1}{q}_{(N)}, \overset{2}{q}_{(N)}, \overset{3}{q}_{(N)}, \phi_{\gamma}(t)\right) = 0, \qquad (13)$$
$$\gamma = 1, 2, 3 \dots, R$$

Considered system is rheonomic system with $p = 3N - G$ degree of the system mobility, and with $n = 3N - G - R$ degrees of the freedom. For the n generalized independent coordinates we take q^i, $i = 1, 2, 3, \dots, n$. Also, we introduce additional subsystem of the R rheonomic coordinates $q^{0\gamma} =$

$q^{n+\gamma} = \phi_\gamma(t)$, $\gamma = 1, 2, 3 \dots R$ which correspond to number of rheonomic constraints. Then we have extended system of the generalized curvilinear coordinates q^i, $i = 1, 2, 3, \dots, n, \dots, n+\gamma, \dots, n+R$. Then we know that subsystem of R rheonomic coordinates $q^{0\gamma} = q^{n+\gamma} = \phi_\gamma(t)$, $\gamma = 1, 2, 3 \dots R$ contain known rheonomic coordinates as functions of the time. But, force of the rheonomic constraints change are unknown (see Ref. [21]).

Let now taking into account that first n coordinates of the position vectors of the mass particles are independent generalized coordinates. Extended system of the generalized coordinates containing independent coordinates q^i, $i = 1, 2, 3, \dots, n$ and rheonomic coordinates $q^{0\gamma} = q^{n+\gamma} = \phi_\gamma(t)$, $\gamma = 1, 2, 3 \dots R$, then it is possible to list in the form:

$$\begin{aligned} q^1 &= q^1_{(1)}, \\ q^2 &= q^2_{(1)}, \\ q^3 &= q^3_{(1)}, \\ q^4 &= q^1_{(2)}, \\ q^5 &= q^2_{(2)}, \\ q^6 &= q^3_{(2)}, \\ &\dots\dots\dots\dots \\ q^{n-2} &= q^1_{(n)}, \\ q^{n-1} &= q^2_{(n)}, \\ q^n &= q^3_{(n)} \end{aligned} \tag{14}$$

$$q^{0\gamma} = q^{n+\gamma} = \phi_\gamma(t), \quad \gamma = 1, 2, 3 \dots R \tag{15}$$

On the basic of the listed system (14)-(15), we can conclude that in considered case, we use coordinates of the positions vectors of the first $K \leqslant \dfrac{n}{3} = \dfrac{1}{3}(3N - G - R)$ mass particle as generalized independent coordinates.

Then on the basis of previous for the coordinates of the geometrical point which correspond to the mass particle positions at arbitrary moment of the motion, we can write:

$$\begin{gathered} N_1\big(q^1 = q^1_{(1)}, q^2 = q^2_{(1)}, q^3 = q^3_{(1)}\big), \\ N_2\big(q^4 = q^1_{(2)}, q^5 = q^2_{(2)}, q^6 = q^3_{(2)}\big), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ N_K\big(q^{3K-2} = q^1_{(K)}, q^{3K-1} = q^2_{(K)}, q^{3K} = q^3_{(K)}\big) \\ N_{K+j}\Big(q^1_{(K+j)}\big(q^1, q^2, \dots, q^n, q_{n+1}, \dots, q^{n+R}\big), \end{gathered} \tag{16}$$

$$q^2_{(K+j)}\left(q^1,q^2,\ldots,q^n,q_{n+1},\ldots,q^{n+R}\right),$$
$$q^3_{(K+j)}\left(q^1,q^2,\ldots,q^n,q_{n+1},\ldots,q^{n+R}\right)\Big) \quad j=1,2,3,\ldots,(N-K)$$
$$\vec{\rho}_1\left(q^1=q^1_{(1)},q^2=q^2_{(1)},q^3=q^3_{(1)}\right),$$
$$\vec{\rho}_2\left(q^4=q^1_{(2)},q^5=q^2_{(2)},q^6=q^3_{(2)}\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{17}$$
$$\vec{\rho}_K\left(q^{3K-2}=q^1_{(K)},q^{3K-1}=q^2_{(K)},q^{3K}=q^3_{(K)}\right)$$
$$\vec{\rho}_{K+j}\left(q^1,q^2,\ldots,q^n,q_{n+1},\ldots,q^{n+R}\right),$$
$$j=1,2,3,\ldots,(N-K)$$

III.5. Concluding remarks

We cam see that in extended system of generalized coordinates, we can identified two sets of the position vectors of the mass particles: one set (16) contain K, $K\le\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles depending of three generalized coordinates, and second set (17) contain the $N-K\ge\frac{G+R}{3}$, $K\le\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles depending of all $p=3N-G$ generalized coordinates in general case, or more then of three generalized coordinates.

Also we can conclude that in extended system of generalized coordinates, we can identified two sets of the position vectors of the mass particles, one set (16) contain the K, $K\le\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles with three-dimensional tangent space and each with three basic vectors of this tangent spaces, and second set (17) contain the $N-K\ge\frac{G+R}{3}$, $K\le\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles with extended dimension of the tangent space and to each tangent space correspond $p=3N-G$ basic vectors in general case, or more then three basic vectors of the tangent space.

III.6. Open directions for next research and applications

As a possible open directions for next research and application are: an analysis of expressions for generalized Corilis forces introduced by changing position of motion observer from fixed coordinate system to rotate curvilinear coordinate system correspond to vector position tangent space; applications of the previous results for solving problems of the numerous coordinate system properties used in astro-dynamics; extension and proof of extension Lagrange differential equations to the description of the rheonomic system dynamics and necessary generalizations.

IV. Krilov-Bogolyubov-Mitropolyskiy asymptotic method of nonlinear mechanics, method of constant variation and averaging method

The different first approximations of solutions of nonlinear differential equations have very large applications in engineering practice for fast evaluations of the kinetic parameters of engineering dynamics (see Reference [35-37] by Hedrih (Stevanović) and [38-40] by by Hedrih (Stevanović) and Simonović). Some time these first approximation are used for engineering practice with enough precisions and not necessary to use second and higher approximation. One of the main reason that in this part we take into consideration a comparison between first approximation obtained by different method, as well as used different starting known solution for obtaining first approximation.

Let compare three first approximations of the solution of a nonlinear differential equation with small nonlinearity, describing dynamics of nonlinear oscillator with one degree of freedom (see Figure 5.a*), in the form (see Reference [35-37] by Hedrih (Stevanović)):

$$\ddot{x}_1(t) + 2\delta_1\dot{x}_1(t) + \omega_1^2 x_1(t) = -\tilde{\omega}_{N1}^2 x_1^3(t), \tag{1}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$, in which $\delta_1 = \varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2 = \varepsilon\tilde{\tilde{\omega}}_{N1}^2$, and ε and ε_1 are small parameters.

We use three different approach and three methods fo obtaining first approximation of the previous nonlinear differential equation (1). First is starting known analytical solution of a corresponding linearized differential equation which correspond to nonlinear differential equation (1).

IV.1.1* In first case, for starting known solution, we can take solution of the linear differential equation in the following form:

$$\ddot{x}_1(t) + 2\delta_1\dot{x}_1(t) + \omega_1^2 x_1(t) = 0, \quad \text{for } \delta_1 \neq 0,\ \varepsilon = 0,\ \omega_1^2 > \delta_1^2 \tag{2}$$

with known analytical solution in the form:

$$x_1(t) = R_{01}e^{-\delta_1 t}\cos(p_1 t + \alpha_{01}), \quad \text{for } \delta_1 \neq 0,\ \varepsilon = 0,\ \omega_1^2 > \delta_1^2 \tag{3}$$

in which circular frequency of damped vibration is in the form $p_1 = \sqrt{\omega_1^2 - \delta_1^2}$ and, R_{01} and α_{01} are integral constant depending of initial conditions. Amplitude of this oscillation is in the form $R_{01}e^{-\delta_1 t}$ and decreasing with time.

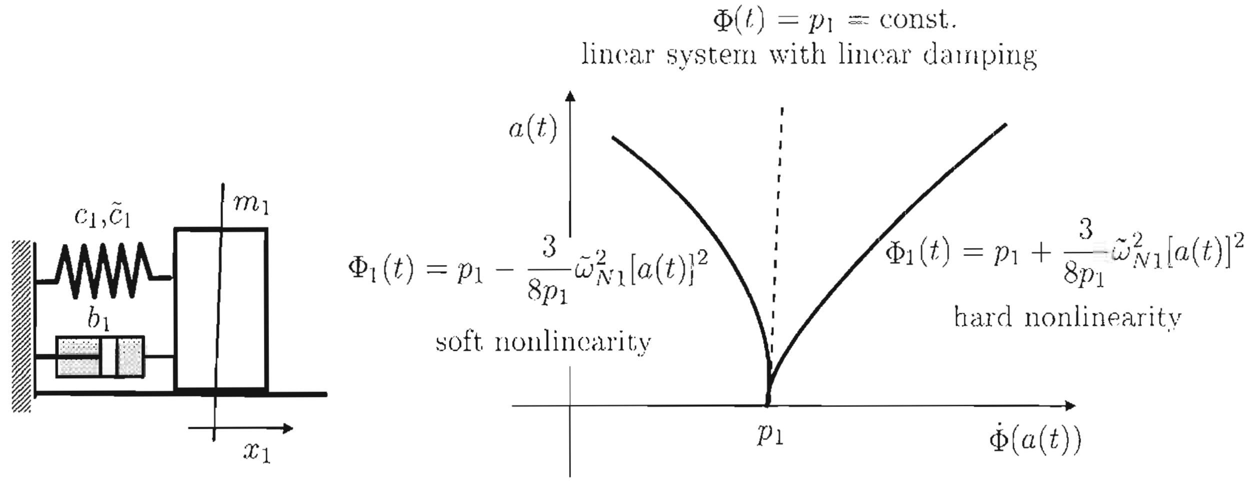


Figure 5 – a* Nonlinear system with one degree of freedom. b* Amplitude-frequency characteristic for free vibrations of the system damped nonlinear dynamics with of soft and hard nonlinearity

IV.1.2* For finding first approximation of the nonlinear differential equation (1), we take starting known analytical solution (3) of linearized differential equation in the form (2) and as a possible first approximation of the solution we take into consideration the following

$$x_1(t) = R_1(t)\, e^{-\delta_1 t} \cos \Phi_1(t), \quad \text{for } \delta_1 \neq 0,\, \varepsilon \neq 0,\, \omega_1^2 > \delta_1^2 \tag{4}$$

in which $a(t) = R_1(t)e^{-\delta_1 t}$ amplitude and full phase $\Phi_1(t) = p_1 t + \phi_1(t)$ contain unknown functions of time $R_1(t)$ and $\phi_1(t)$ which need to determine. For this fist approach, we applied Lagrange method of variation of constants to the known solution (3) of the linearized differential equation corresponding to nonlinear differential equation (see Reference [35-36] by Hedrih (Stevanović)). After obtaining system of differential equation along unknown functions of time $R_1(t)$ and $\phi_1(t)$ we applied average to the obtained members along one period of the

$$T_a = \frac{2\pi}{p_1} = \frac{2\pi}{\sqrt{\omega_1^2 - \delta_1^2}}$$

damping vibrations.

Then, after differentiation along time of the proposed approximation of the solution (4) we obtain:

$$\dot{x}_1(t) = -\delta_1 R_1(t)e^{-\delta_1 t} \cos \Phi_1(t) - R_1(t)e^{-\delta_1 t} p_1 \sin \Phi_1(t) +$$

$$+\dot{R}_1(t)e^{-\delta_1 t}\cos\Phi_1(t) - R_1(t)e^{-\delta_1 t}\dot{\phi}_1(t)\sin\Phi_1(t) \tag{5}$$

to which we introduce condition that this first derivative (5) of the proposed first approximation of the solution (4) have same form as solution (3) of the corresponding linearized differential equation (2), by other words, this condition express the following: that this first derivative (5) of the proposed first approximation of the solution (4) have same form as in the case that unknown function of time $R_1(t)$ and $\phi_1(t)$ are constant.

After applying introduced previous condition we obtain first derivative of the proposed first approximation (4) of the solution in the following form:

$$\dot{x}_1(t) = -\delta_1 R_1(t)\, e^{-\delta_1 t}\cos\Phi_1(t) - R_1(t)\, e^{-\delta_1 t} p_1 \sin\Phi_1(t) \tag{6}$$

and the following condition

$$\dot{R}_1(t)\,\cos\Phi_1(t) - R_1(t)\,\dot{\phi}_1(t)\sin\Phi_1(t) = 0 \tag{7}$$

that unknown functions of time $R_1(t)$ and $\phi_1(t)$ must to satisfy.

Second derivative of the proposed first approximation (4) of the solution is in the following form:

$$\begin{aligned}\ddot{x}_1(t) = {} & \left(\delta_1^2 R_1(t) - R_1(t)p_1^2 - \delta_1\dot{R}_1(t) - R_1(t)p_1\dot{\phi}_1\right)e^{-\delta_1 t}\cos\Phi_1(t) + \\ & +\left(\delta_1 R_1(t)\dot{\phi}_1 + 2\delta_1 R_1(t)\,p_1 - \dot{R}_1(t)\,p_1\right)e^{-\delta_1 t}\sin\Phi_1(t).\end{aligned} \tag{8}$$

Alter introducing first (6) and second (8) derivatives of the proposed first approximation (4) of the solution into nonlinear differential equation (1) and taking into account condition (7) we obtain the system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in following form:

$$\begin{aligned}&\dot{R}_1(t)\,\cos\Phi_1(t) - R_1(t)\,\dot{\phi}_1(t)\sin\Phi_1(t) = 0\\ &R_1(t)p_1\dot{\phi}_1\cos\Phi_1(t) + \dot{R}_1(t)\,p_1\sin\Phi_1(t) = \tilde{\omega}_{N1}^2 e^{-2\delta_1 t}\left[R_1(t)\right]^3\cos^3\Phi_1(t)\end{aligned} \tag{9}$$

Previous obtained system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ present a no homogeneous algebra system along derivatives of unknown function of time $\dot{R}_1(t)$ and $\dot{\phi}_1(t)$ with determinate in the form:

$$\begin{aligned}\Delta &= \begin{vmatrix}\cos\Phi_1(t) & -R_1(t)\sin\Phi_1(t)\\ p_1\sin\Phi_1(t) & R_1(t)p_1\cos\Phi_1(t)\end{vmatrix} = \\ &= R_1(t)p_1\left[\cos^2\Phi_1(t) + \sin^2\Phi_1(t)\right] = R_1(t)p_1\end{aligned} \tag{10}$$

with following solutions:

$$\dot{R}_1(t) = \frac{\Delta_1}{\Delta} = \frac{\tilde{\omega}_{N1}^2}{p_1}e^{-2\delta_1 t}\left[R_1(t)\right]^3\cos^3\Phi_1(t)\sin\Phi_1(t)\dot{\phi}_1(t) =$$

$$= \frac{\Delta_2}{\Delta} = \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t} [R_1(t)]^2 \cos^4 \Phi_1(t), \tag{11}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$.

Then, after obtaining previous system of differential equation (11) along unknown functions of time, $R_1(t)$ and $\phi_1(t)$, we applied average to the obtained members along full phase $\Phi_1(t) = p_1 t + \phi_1(t)$ in interval $\Phi \in [-, 2\pi]$ correspond to one period of the

$$T_a = \frac{2\pi}{p_1} = \frac{2\pi}{\sqrt{\omega_1^2 - \delta_1^2}}$$

damping vibrations:

$$\dot{R}_1(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t} [R_1(t)]^3 \cos^3 \Phi_1(t) \sin \Phi_1(t) d\Phi_1(t)$$

$$\dot{\phi}_1(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t} [R_1(t)]^2 \cos^4 \Phi_1(t) d\Phi_1(t), \tag{12}$$

for $\delta_1 \neq 0$, $\varepsilon = 0$, $\omega_1^2 > \delta_1^2$.

Then, we obtain a system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in first averaged approximation:

$$\dot{R}_1(t) = 0$$

$$\dot{\phi}_1(t) = \frac{3}{8} \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t} [R_1(t)]^2 \quad \text{for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2. \tag{13}$$

After integration of the previous system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in first averaged approximation for known initial values in first approximation

$$t = 0, \quad R_1(0) = R_{01} \quad \text{i} \quad \phi_1(0) = \phi_{01} = -\frac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2 + \alpha_{01}$$

we obtain:

$$R_1(t) = R_{01} = \text{const.}$$

$$\phi_1(t) = -\frac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2 (e^{-2\delta_1 t} - 1) + \phi_{01} = \tag{14}$$

$$= -\frac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01},$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$ where

$$\alpha_{01} = \phi_{01} + \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2,$$

and full phase is in the form:

$$\Phi_1(t) = p_1 t + \phi_1(t) = p_1 t - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 (e^{-2\delta_1 t} - 1) + \phi_{01} = \\ = p_1 t - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}, \tag{15}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$.

Then first averaged approximation of the solution of the nonlinear differential equation with hard cubic small nonlinearity (1)

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos\left(p_1 t - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}\right), \tag{16}$$

for $\delta_1 \neq 0$, $\varepsilon = 0$, $\omega_1^2 > \delta_1^2$.

In the case that, we have a nonlinear differential equation with sogt cubic small nonlinearity in the following form:

$$\ddot{x}_1(t) + 2\delta_1 \dot{x}_1(t) + \omega_1^2 x_1(t) = +\tilde{\omega}_{N1}^2 x_1^3(t) \tag{17}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$ on the basis of the previous obtained first averaged approximation of the solution we can write:

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos\left(p_1 t + \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}\right), \tag{18}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$ where for known initial values in first approximation $t = 0$, $R_1(0) = R_{01}$ and

$$\phi_1(0) = \phi_{01} = \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 + \alpha_{01}$$

we obtain:

$$\alpha_{01} = \phi_{01} - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2. \tag{19}$$

For the case that for $\delta_1 = 0$ we can use the system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in first averaged approximation (13) and before integration put $\delta_1 = 0$, and after that applied integration, or find limes of the solutions (16) and 18) for $\delta_1 \to 0$, and taking into account that is

$$\lim_{\delta_1 \to 0} \frac{(e^{-2\delta_1 t} - 1)}{\delta_1} = -2t,$$

obtain first averaged approximation of the solution of nonlinear differential equations (1) as well as (17)

$$\ddot{x}_1(t) + 2\delta_1 \dot{x}_1(t) + \omega_1^2 x_1(t) = \mp \tilde{\omega}_{N1}^2 x_1^3(t) \tag{20}$$

in the following form:

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos\left(p_1 t \mp \frac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}\right) \tag{21}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$. where $\alpha_{01} = \phi_{01} \pm \dfrac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2$.

$$x_1(t) = R_{01} \cos\left\langle \left(\omega_1 \pm \frac{3}{8\omega_1} \tilde{\omega}_{N1}^2 R_{01}^2\right) t + \phi_{01} \right\rangle, \tag{22}$$

for $\delta_1 = 0$, $\varepsilon_1 = 0$ $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$.

IV.2.1* In second case, for taking starting known solution for obtaining approximation of solution of the nonlinear differential equation (1), we can take solution of the linear differential equation in the following form:

$$\ddot{\tilde{x}}(t) + p^2 \tilde{x}(t) = 0 \tag{23}$$

with known analytical solution in the form:

$$\tilde{x}(t) = a \cos(p_1 t + \alpha_0) \tag{24}$$

in which circular frequency of harmonic vibration is in the form $p = \sqrt{\omega_1^2 - \delta_1^2}$ and, a and α_0 are integral constant depending of initial conditions. Amplitude of this oscillation is in the form a and is constant and no depending of time.

IV.2.2* For that case, we must transform nonlinearn differential equation (1) taking into account the following generalized coordinate transformation:

$$\tilde{x} = x e^{\delta_1 t} \quad \text{or} \quad x = \tilde{x} e^{-\delta_1 t}. \tag{25}$$

After generalized coordinate transformation an transformation of differential nonlinear equation (1), we obtain:

$$\frac{d^2\tilde{x}}{dt^2}+p^2\tilde{x}=-\tilde{\omega}_{N1}^2\tilde{x}^3(t)e^{-2\delta_1 t} \tag{26}$$

where

$$\varepsilon\tilde{f}(\tilde{x}e^{-\delta_1 t})e^{\delta_1 t}=-\tilde{\omega}_{N1}^2\tilde{x}^3(t)e^{-2\delta_1 t}. \tag{27}$$

Let start with general form of the nonlinear differential equation in the form:

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega^2x=\varepsilon f\left(x,\frac{dx}{dt}\right). \tag{28}$$

For small parameter $\varepsilon=0$ we obtain linear differential equation

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega^2x=0 \tag{29}$$

with solution:

$$x=ae^{-\delta t}\cos\psi=e^{-\delta t}\tilde{x} \tag{30}$$

with amplitude $ae^{-\delta t}$ with phase $\psi=pt+\alpha$, where $p=\sqrt{\omega^2-\delta^2}$ and also:

$$\frac{da}{dt}=0\quad\frac{d\psi}{dt}=p=\text{const.} \tag{31}$$

In which a and α are determined bay their initial values. By generalized coordinate transformation (25) nonlinear differential equation (28) take the following form:

$$\left(\frac{d^2\tilde{x}}{dt^2}+\left(\omega^2-\delta^2\right)\tilde{x}\right)=\varepsilon\tilde{f}\left(\tilde{x}e^{-\delta t},\frac{d\tilde{x}}{dt}e^{-\delta t}\right)e^{\delta t} \tag{32}$$

or

$$\frac{d^2\tilde{x}}{dt^2}+p^2\tilde{x}=\varepsilon\tilde{f}\left(\tilde{x}e^{-\delta t},\frac{d\tilde{x}}{dt}e^{-\delta t}\right)e^{\delta t}=\varepsilon\tilde{\tilde{f}}\left(\tilde{x},\frac{d\tilde{x}}{dt},\tau\right). \tag{33}$$

In beginning, we supposed that $\delta=\varepsilon_1\tilde{\delta}$, and that ε_1 is same order of small value as ε and that $\tau=\varepsilon t$ is slow changing time and that for one period $T_a=\dfrac{2\pi}{p}$ change of the system dynamics is small, and that function

$$\varepsilon\tilde{f}\left(\tilde{x}e^{-\delta t},\frac{d\tilde{x}}{dt}e^{-\delta t}\right)=\varepsilon\tilde{\tilde{f}}\left(\tilde{x},\frac{d\tilde{x}}{dt},\tau\right)$$

satisfy all necessary conditions for application of the asymptotic method Ktilov-Bogolyubov-Mitropolyskiy for application f the method with slow changing system dynamics parameters and with sloww changing time (see Reference [41-48] by Yu. A. Mitropolyskiy).

Then, the n-the asymptotic approximation of the two parametric family of a one frequency solution of differential equation (33) we suppose in the form:

$$xe^{\delta t} = \tilde{x} = a\cos\psi + \varepsilon U_1(a,\psi,\tau) + \varepsilon^2 U_2(a,\psi,\tau) + \cdots \tag{34}$$

where $U_1(a,\psi,\tau)$,$U_2(a,\psi,\tau)$, ... periodic functions of $\psi = pt + \phi(t)$, with period 2π, and no containing first harmonic of ψ, and where amplitude and phase a and ψ are unknown functions which are determined by system of differential equations corresponding order n-th asymptotic approximation along amplitude and phase in the form:

$$\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a,\tau) + \varepsilon^2 A_2(a,\tau) + \cdots \\ \frac{d\psi}{dt} &= p + \varepsilon B_1(a,\tau) + \varepsilon^2 B_2(a,v) + \cdots \end{aligned} \tag{35}$$

where $A_1(a,\tau)$, $A_2(a,\tau)$, ... , and $B_1(a,\tau)$,$B_2(a,v)$, ... Are unknowne functions of amplitude and slowchanging time.

Introducing, on the basis of previous formulated condition we can write:

$$\int_0^{2\pi} U_j(a,\psi,\tau)e^{i\psi}d\psi = 0 \quad j = 1,2,\ldots m \tag{36}$$

Then, we calculate first and second derivatives of the n-th supposed asymptotic approximation in the following forms:

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= -ap\sin\psi + \varepsilon\left\{A_1(a,\tau)\cos\psi - aB_1(a,\tau)\sin\psi + \varepsilon\frac{\partial U_1}{\partial\tau} + p\frac{\partial U_1}{\partial\psi}\right\} + \\ &\quad + \varepsilon^2\left\{A_1(a,\tau)\cos\psi - aB_2(a,\tau)\sin\psi + A_1(a,\tau)\frac{\partial U_1}{\partial a} + \right. \\ &\quad \left. + B_1(a,\tau)\frac{\partial U_1}{\partial\psi} + p\frac{\partial U_2}{\partial\psi} + \varepsilon\frac{\partial U_2}{\partial\tau}\right\} + \varepsilon^3 \ldots \end{aligned} \tag{37}$$

$$\begin{aligned} \frac{d^2\tilde{x}}{dt^2} &= -ap^2\cos\psi + \varepsilon\left\{-2pA_1\sin\psi - 2paB_1\cos\psi + \frac{dA_1(a,\tau)}{d\tau}\frac{d\tau}{dt}\cos\psi - \right. \\ &\quad \left. -a\frac{dB_1(a,\tau)}{d\tau}\frac{d\tau}{dt}\sin\psi + p^2\frac{\partial^2 U_1}{\partial\psi^2} + \frac{\partial^2 U_1}{\partial t^2} + p\frac{\partial^2 U_1}{\partial t\partial\psi}\right\} + \end{aligned}$$

$$+\varepsilon^2\left\{\left(A_1\frac{dA_1}{da}-aB_1^2-2paB_2\right)\cos\psi-\left(2pA_2+2A_1B_1+\right.\right. \quad (38)$$
$$\left.+A_1a\frac{dB_1}{da}\right)\sin\psi+2pA_1\frac{\partial^2U_1}{\partial a\partial\psi}+2pB_1\frac{\partial^2U_1}{\partial\psi^2}+p^2\frac{\partial^2U_2}{\partial\psi^2}+\frac{\partial^2U_2}{\partial t^2}+$$
$$\left.+\varepsilon p\frac{\partial^2U_2}{\partial\tau\partial\psi}+\varepsilon A_1\frac{\partial^2U_1}{\partial\tau\partial a}+\cdots\right\}+\varepsilon^3\ldots$$

After introducing previous asysmptotic aproximation of the solution (34) and their first and second derivatives into nonlinear differential equation (33) and applying metod of equal coeficinet of the small parameters un left and right side of transformation of nonlinear diferential equation, we obtain series of the relation between unknown functions $U_1(a,\psi,\tau)$, $U_2(a,\psi,\tau)$, $\ldots$, $A_1(a,\tau)$, $A_2(a,\tau)$, $\ldots$, and $B_1(a,\tau)$, $B_2(a,v)$, $\ldots$ For the reason that we need only first asymptotic approximation of the solution, we take into account the following relation obtained from coefficients with first step of the small parameter ε

$$-2pA_1(a.\tau)\sin^2\psi=\tilde{f}(e^{-\delta t}\left\{a\cos\psi\right\},e^{-\delta t}\left\{-ap\sin\psi\right\})e^{\delta t}\sin\psi$$
$$-2paB_1(a.\tau)\cos^2\psi=\tilde{f}(e^{-\delta t}\left\{a\cos\psi\right\},e^{-\delta t}\left\{-ap\sin\psi\right\})e^{\delta t}\cos\psi. \quad (39)$$

Takin into acount development of the previous expressions along full pahse $\psi=pt+\phi(t)$ we obtain relations equations for obtaining unknown functions $A_1(a,\tau)$ and $B_1(a,\tau)$ in the following form (see Reference [41-48] by Yu. A. Mitropolyskiy):

$$A_1(a.\tau)=-\frac{1}{2\pi p}\int_0^{2\pi}\tilde{\tilde{f}}\left(e^{-\tilde{\delta}\tau}\{a\cos\psi\},e^{-\tilde{\delta}\tau}\{-ap\sin\psi\}\right)e^{\tilde{\delta}\tau}\sin\psi d\psi$$
$$B_1(a.\tau)=-\frac{1}{2\pi pa}\int_0^{2\pi}\tilde{\tilde{f}}\left(e^{-\tilde{\delta}\tau}\{a\cos\psi\},e^{-\tilde{\delta}\tau}\{-ap\sin\psi\}\right)e^{\tilde{\delta}\tau}\cos\psi d\psi \quad (40)$$

Then taking into account that

$$\varepsilon\tilde{f}(\tilde{x}e^{-\delta t},\frac{d\tilde{x}}{dt}e^{-\delta t})=\varepsilon\tilde{\tilde{f}}(\tilde{x},\frac{d\tilde{x}}{dt},\tau)$$

and differential equation in the form (33) and introducing (27) in previous obtained expression (33) for obtaining functions $A_1(a,\tau)$ and $B_1(a,\tau)$ we can write:

$$\varepsilon A_1(a.t)=0$$

$$\varepsilon B_1(a.\tau) = \frac{1}{2\pi p}\tilde{\omega}_{N1}^2 a^2 e^{-2\tilde{\delta}\tau}\frac{3}{4} = \frac{3}{8\pi p}\tilde{\omega}_{N1}^2 a^2 e^{-2\delta t} \tag{41}$$

where δ_1 ($\delta_1 = \varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2 = \varepsilon\tilde{\tilde{\omega}}_{N1}^2$, for ε and ε_1 same order small values.

Then, system of differential equation (35) along a and ψ in the first asymptotic approximation is possible to write in the following form:

$$\frac{da}{dt} = 0$$

$$\frac{d\psi}{dt} = p + \frac{3}{8\pi p}\tilde{\omega}_{N1}^2 a^2 e^{-2\delta t} \tag{42}$$

and first asymptotic approximation of the solution in the form

$$xe^{\delta t} = \tilde{x} = a\cos\psi \quad \text{or in the form:} \quad x = \tilde{x}e^{-\delta t} = ae^{-\delta t}\cos\psi. \tag{43}$$

In the previous first asymptotic approximation full phase is in the form:

$$\psi(t) = pt - \frac{3}{16\delta_1 p}\tilde{\omega}_{N1}^2 R_{01}^2\left(e^{-2\delta_1 t} - 1\right) + \psi_{01} =$$
$$= p_1 t - \frac{3}{16\delta_1 p}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}, \tag{44}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$, $p = \sqrt{\omega_1^2 - \delta_1^2}$.

We can see and conclude that first approximation of the solution of considered nonlinear differential equation (1) obtained by application different methods, first method of variation constant with average along full phase, and asymptotic method by Krilov -Bogolyubov-Mitropolyskiy (see Reference [41-48] by Yu. A. Mitropolyskiy) give us same results, but with different methods and proof.

IV.3.1* In third case, for taking starting known solution for obtaining approximation of solution of the nonlinear differential equation (1), we can take solution of the linear differential equation in the following form:

$$\ddot{x} + \omega_1^2 x = 0 \tag{45}$$

with known analytical solution in the form:

$$x(t) = a\ \cos(\omega_1 t + \alpha_9) \tag{46}$$

in which circular frequency of harmonic vibration is in the form ω_1 and, a and α_0 are integral constant depending of initial conditions. Amplitude of this oscillation is in the form a and is constant and no depending of time.

IV.3.2* For that case, for finding first approximation of the nonlinear differential equation (1), we take starting known analytical solution (46) of linearized differential equation in the form (45) and as a possible first approximation of the solution we take into consideration the following

$$x(t) = a(t)\ \cos\Phi(t).,\quad \text{for}\quad \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{47}$$

in which $a(t)$ amplitude and full phase $\Phi_1(t) = \omega_1 t + \phi_1(t)$ contain unknown functions of time $a(t)$ and $\phi_1(t)$ which must to determine. For this third approach, we applied known Krilov-Bogolyubov-Mitropolyskiy asymptotic method of average to find first asymptotic approximation of the solution of nonlinear differential equation (1).

Then we start with nonlinear differential equation

$$\ddot{x} + \omega_1^2 x = \varepsilon f(x, \dot{x}), \tag{48}$$

and suppose first asymptotic approximation in the form:

$$x(t) = a(t)\ \cos\Phi(t). \tag{49}$$

where unknown functions $a(t)$ and $\Phi(t)$ are determined from the system of differential equations of first asymptotic approximation (see Reference [41-48] by Yu. A. Mitropolyskiy) in the following form:

$$\begin{aligned}\frac{da(t)}{dt} &= \varepsilon A(a),\\ \frac{d\Phi(t)}{dt} &= \omega_1 + \varepsilon B(a),\end{aligned} \tag{50}$$

where

$$\begin{aligned}A(a) &= -\frac{1}{2\pi\omega_1}\int_0^{2\pi} f(a\cos\Phi,\ -a\omega_1\sin\Phi)\sin\Phi\, d\Phi,\\ B(a) &= -\frac{1}{2\pi\, a\omega_1}\int_0^{2\pi} f(a\cos\Phi,\ -a\omega_1\sin\Phi)\cos\Phi\, d\Phi.\end{aligned} \tag{51}$$

For our nonlinear differential equation (1)

$$f(x, \dot{x}) = -\left(2\delta\,\dot{x} + \omega_N^2 x^3\right), \tag{52}$$

where $\delta_1 = \varepsilon_1\delta$, $\tilde{\omega}_{N1}^2 = \varepsilon\omega_N^2$.

Taking into acount initial values in first aproximation $t = 0$: $a(0) = a_o$, $\Phi(0) = \Phi_o$ we obtain that

$$
\begin{aligned}
a(t) &= a_o e^{-\varepsilon\delta t} = a_o e^{-\delta_1 t},\\
\Phi(t) &= \omega_1 t - \frac{3}{16\delta\omega_1}\omega_N^2 a_o^2 (e^{-2\varepsilon\delta t} - 1) + \Phi_o =\\
&= \omega_1 t - \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2 (e^{-2\delta_1 t} - 1) + \Phi_o
\end{aligned}
\tag{53}
$$

and first asymptotic approximation of the solution of a nonlinear differential equation (1) around harmonic starting known analytical solution, we can write in the following form:

$$
x(t) = a_o e^{-\delta_1 t} \cos\left[\omega_1 t - \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2 (e^{-2\delta_1 t} - 1) + \Phi_o\right]. \tag{54}
$$

From this obtained first asymptotic approximation (54) of the solutions of nonlinear differential equation (1) with starting known analytical harmonic solution (46) in the case for damping coefficient tends to zero $\delta_1 \to 0$, and taking into account that is

$$
\lim_{\delta_1 \to 0} \frac{(e^{-2\delta_1 t} - 1)}{\delta_1} = -2t,
$$

we obtain first asymptotic approximation of the solution for conservative nonlinear system vibrations in the form as in the previous two case obtained first approximation of solution of same nonlinear differential equation (1) by use different method and different starting known analytical solution.

IV.4. Concluding remarks

Let we made a general review of the obtained results for approximately solving of the nonlinear differential equation with small cubic nonlinearity in the form: :

$$
\ddot{x}_1(t) + 2\delta_1\dot{x}_1(t) + \omega_1^2 x_1(t) = \mp\tilde{\omega}_{N1}^2 x_1^3(t) \tag{55}
$$

in which hard or soft, refers to $\mp$ sign approximately, $\delta_1 = \varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2 = \varepsilon\tilde{\tilde{\omega}}_{N1}^2$, and ε and ε_1 are small parameters.

By use first two methods, starting known analytical solutions in the form (3) and we obtained same first approximation of the solution in the following form:

$$x_1(t) = R_{01}e^{-\delta_1 t}\cos\left(p_1 t \mp \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}\right), \tag{56}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$ where

$$\alpha_{01} = \phi_{01} \pm \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2.$$

For the case that damping coefficient tends to zero, from this first approximation (56), we obtain first approximation of the solution for conservative nonlinear system dynamics in the following form:

$$x_1(t) = R_{01}\cos\left\langle\left(\omega_1 \pm \frac{3}{8\omega_1}\tilde{\omega}_{N1}^2 R_{01}^2\right)t + \phi_{01}\right\rangle, \tag{57}$$

for $\delta_1 = 0$, $\varepsilon_1 = 0$ $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$.

We can see that circular frequency of nonlinear dynamic of conservative system is not isochroous and depends of initial conditions initial amplitude.

For the case that coefficient of the cubic nonlinearity tends to zero, from this first approximation (56), we obtain known analytical solution of the linear no conservative system dynamics in the following form:

$$x_1(t) = R_{01}e^{-\delta_1 t}\cos(p_1 t + \alpha_{01}), \tag{58}$$

for $\delta_1 \neq 0$, $\varepsilon = 0$, $\omega_1^2 > \delta_1^2$ $\tilde{\omega}_{N1}^2 = 0$.

From the third case we start by harmonic known analytical solution in the form (46), we obtain the following first asymptotic approximation of the solution of same nonlinear differential equation:

$$x(t) = a_o e^{-\delta_1 t}\cos\left[\omega_1 t \mp \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2 (e^{-2\delta_1 t} - 1) + \Phi_o\right]. \tag{59}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$.

This asymptotical approximation is different them in previous case (56) and this is normally because we take different starting analytical known solution if different basic linear differential equations as a two different linearizations of the considered same nonlinear differential equation.

For the case that damping coefficient tends to zero, from this first approximation (59), we obtain first approximation of the solution for conservative nonlinear system dynamics in the following form:

$$x(t) = a_o e^{-\delta_1 t}\cos\left[\left(\omega_1 \pm \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\right)t + \Phi_o\right], \tag{60}$$

for $\delta_1 = 0, \varepsilon_1 = 0$ $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$ same as in the previous cases (57).

For the case that coefficient of the cubic nonlinearity tends to zero, from this first approximation (59), we can not obtain known analytical solution of the linear no conservative system dynamics in the form (58) but we obtain:

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos(\omega_1 t + \alpha_{01}), \tag{61}$$

for $\delta_1 \neq 0$, $\varepsilon = 0$, $\omega_1^2 > \delta_1^2$ $\tilde{\omega}_{N1}^2 = 0$. not acceptable, because in this case starting solution was harmonic. In this case if we need harmonic solution we must annulled parameters of cubic nonlinearity and of dumping in the same time.

Then we can conclude that first two approach (56) for obtaining first approximation are more general and more suitable for use in the considered approximation of the solution them (61).

IV.5. Open directions for next research and applications

Directions for next research in area of approximation must be focused to find analytical forms of approximations of solutions of nonlinear differential equations. Present in science, there are numerous numerical approach and numerical experiments over the nonlinear differential equations for numerical slowing nonlinear one, or coupled system of nonlinear differential equations, but these are only particular solutions without proof that these solutions are right, and general.

For practical applications in mechanics and engineering system dynamics analytical forms of the approximations of solutions are necessary for easier quantitative estimation larger class of the nonlinear dynamic phenomena and nonlinear dynamics of the stem behavior.

V. Analytical mechanics of hereditary discrete system vibrations

V.1. Introduction

Integro-differential equations and their applications in development of analytical mechanics of discrete hereditary systems are used by Gorosko and Hedrih (Stevanović) (see References IV [49-64])

Research results in area of mechanics of hereditary discrete systems, obtained by Gorosko and Hedrih (Stevanović (see References IV [49-64]) are generalized and presented in the monograph [49] which contains first completed presentation of the analytical dynamics of hereditary discrete systems. Two classes of dynamically defined and undefined hereditary systems are defined and considered by introducing corresponding restrictions. Main results of mechanics of hereditary discrete systems are presented with new applications important to engineering.

Approximation of expressions for the coefficients of damping and corresponding decrements as well as for circular frequency of oscillations of hereditary oscillatory systems are obtained with high accuracy in the first and second approximations.

Analogy between hereditary interactions and reactive forces in systems of automatic control is identified and a possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control is pointed out (see References [50] and [51] by Gorosko and Hedrih (Stevanović)).

The Lagrange's mechanics of hereditary systems are extended and generalized to thermo-rheological [4, 8] and piezo-rheological [4, 9] discrete mechanical systems as well as to discrete mechanical systems with standard light creep elements.

Analytical dynamics as general science of mechanical system motions was founded by Lagrange (*Joseph Louis Lagrange* (1736-1813)) in the period of his work at Name Berlin TypeAcademy. The Lagrange's book "*Mécanique Analytique*" contains basic analytical methods of mechanics and was published in placecountry-regionFrance in 1788. Introduced analytical methods in Mechanics by Lagrange are main and first base of analytical mechanics in general. Lagrange's equations of second kind and Lagrange's equations of first kind with unwoven Lagrange's multipliplicators of constraints are main fundament of Analytical Dynamics.

Analytical dynamics is largely applied and used in engineering system dynamics and in natural sciences as well as for investigation of mechanical system dynamics and in physics of the microworld.

Mechanics of hereditary continuum is presented by series of fundamental publications and monographs.

In current literature term "*hereditary*" and "*rheological*" systems are equivalent. Mechanics of discrete hereditary systems up to a ten years before was presented only by separate single papers and containing only solutions of partial problems.

Research results in area of mechanics of hereditary discrete systems, obtained by authors of this paper, are generalized and presented in the monograph [49], published in 2001 by Gorosko and Hedrih (Stevanović)), which contains first presentation of analytical dynamics of hereditary discrete systems. We can conclude that this monograph contains *complete foundation of analytical dynamics theory of discrete hereditary systems* and by using these results, numerous examples are obtained and solved (see Refs. [49-66]). In this analytical mechanics of hereditary discrete systems, modified Lagrange's differential equations second kind in the form differential and integro-differential forms with kernels of relaxation or rhelogogy are derived.

V.2. Models of hereditary elements in analytical dynamics of hereditary discrete systems

Hereditary system is each system which contains mutual hereditary interaction between material particles in the form of one or more coupling constraints with hereditary properties.

Simple visco-elastic element is Voigt's type element (Woldemar Voigt, 1859–1919). In the state of extension resultant force appears by two components, one by visco and one by elastic properties in the deformation of viscoeelastic element and constitutive stress-strain relation given as relation between force and extension of element in the following form:

$$P(t) = cy(t) + \mu\dot{y}(t). \tag{1}$$

In Mechanics of hereditary continuum in the case of axial (in one direction) stressed and deformed Voigh's type body stress strain constitutive relation is expressed by following relation: $\sigma(t) = E\varepsilon(t) + \mu\dot{\varepsilon}(t)$. More acceptable and precise and better compatible with experimental data with real hereditary body properties is model of the standard visco-elastic body (Kelvin and Poyting-Thompson's body). Constitutive stress strain relation given as relation between force and extension of element in the following form:

$$n\dot{P}(t) + P(t) = nc\dot{y}(t) + \tilde{c}y(t). \tag{2}$$

In mechanics of hereditary system constants n, c and $\tilde{c}$ obtain special names: time of relaxation, rigidity coefficients, one momenteneous and prologues one.

For generalized hereditary element model relation between force and deformation is possible to describe by differential equation high order derivative in the following form:

$$\sum_{r=1}^{m} a_k \frac{d^k P}{dt_k} + P(t) = b_0 x + \sum_{k=1}^{m} b_k \frac{d^k x}{dt^k}. \tag{3}$$

For more complex viscose elements (represented by the Jeffreys' body (J-body) and Lethersich's body) stress-strain state is described by differential equation in the form:

$$n\dot{P}(t) + P(t) = b_1 \dot{y}(t) + nb_2 \ddot{y}(t). \tag{4}$$

Equivalency and analogy of hereditary interactions and reactive forces in systems of automatic control gives possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control. For example, automaton with transfer function presented in the following form

$$W(p) = \frac{b_0 + b_1 p + \cdots + b_n p^n}{1 + a_1 p + \cdots + a_n p^n} \tag{5}$$

presents a hereditary interaction (3) between material particles of the discrete mechanical system with one degree of freedom.

Parameters of the automaton of arbitrary structures are defined in an experimental way and it is possible to obtain amplitude-phase characteristic. In our opinion there are real possibilities and perspective to use method of amplitude-phase characteristic for experimental obtaining of mechanical characteristic of the hereditary discrete mechanical systems. It is possible to solve some difficulties with identification coefficient of the momenteneous rigidity which appear in the mechanical investigation of the hereditary forms and shortened longtime experiments.

V.3. Integral models of the stress-strain state of the hereditary elements

There are three mathematical forms for description of constitutive relations of hereditary properties of hereditary interaction [49], in the building of hereditary system's mechanics. These forms are:

1* Differential equation, expressed in the form of dependence reaction force P of the rheological coordinate x, usually presented as deformation or relative displacement of the hereditary constraint in the form (3).

2* Integral equation, expressed in the form of dependence reaction force P of the rheological coordinate y , usually presented as deformation or relative displacement of the hereditary constraint:

$$P(t) = c\left[y(t) - \int_0^t \Re(t-\tau)y(\tau)d\tau\right] \tag{6}$$

where

$$\Re(t-\tau) = \frac{c-\tilde{c}}{nc}e^{-\frac{1}{n}(t-\tau)}$$

is relaxation kernel, and $\beta = \dfrac{1}{n}$ is coefficient of the element relaxation.

This integral relation (6) can be obtained by solving equation (2) with respect to the force P. By this integral equation, the relaxation of the reaction force P depending on the rheological coordinate y, is presented and expressed.

For the case of the generalized standard hereditary element (3) integral equation is possible to obtain in the form (6) in which relaxation kernel $\Re(t-\tau)$ presents sum by sum of exponents.

3* Integral equation, expressed in the form of dependence rheological coordinate y, usually presented deformation or relative displacement of the hereditary constraint and reaction force P:

$$y(t) = \frac{1}{c}\left[P(t) + \int_0^t \Re(t-\tau)P(\tau)d\tau\right] \tag{7}$$

where

$$\Re(t-\tau) = \frac{c-\tilde{c}}{nc}e^{-\frac{\tilde{c}}{nc}(t-\tau)}$$

is kernel of rheology and $\beta_1 = \dfrac{\tilde{c}}{nc}$ is the coefficient of the creep or retardation or rheology.

V.4. Three forms of equations of motions of a hereditary oscillator

Simple model of a hereditary discrete system is hereditary oscillator with one degree of freedom which contains one material particle with mass m and one standard hereditary element P with material viscoelastic properties defined by following coefficients: n, c and $\tilde{c}$ constitutive stress-strain relation expressed by relation (2) between force $P(t)$ and generalized and rheological coordinate $y(t)$. Then by using principle of dynamical equilibrium of the oscillator it is possible to obtain equation of the oscillator motion in the following form:

$$m\ddot{y}(t) + P(t) = F(t) \tag{8}$$

where $P(t)$ is resistive reaction of the rheological element, $F(t)$ external forced excitation. Using constitutive relation (2) or (10) for stressed and deformed standard hereditary (rheological) element for eliminating resistive reaction of the rheological element $P(t)$ from last equation (8) we obtain three corresponding forms of the equation of motion of the rheological – hereditary oscillator with one degree of freedom listed as follow: one in differential form:

$$nm\dddot{y}(t) + m\ddot{y}(t) + nc\dot{y}(t) + \tilde{c}y(t) = n\dot{F}(t) + F(t) \tag{9}$$

and two in integro-differential forms

$$m\ddot{y}(t) + c\left[y(t) - \int_{-\infty}^{t} \Re(t-\tau)y(\tau)d\tau\right] = F(t) \tag{10}$$

$$m\ddot{y}(t)+c\left[y(t) + \int_{-\infty}^{t} \Re(t-\tau)m\ddot{y}(\tau)d\tau\right] = F(t)+c\int_{-\infty}^{t} \Re(t-\tau)F(\tau)d\tau \tag{11}$$

For the case of the weak singular hereditary oscillator equation of the dynamic equilibrium (oscillator motion) in the differential form is not possible to obtain, but in the integro-differential forms is possible.

V.5. Thermo-rheological pendulum

V.5.1. Light standard thermo-rheological hereditary element

When standard hereditary element is modified by two temperatures $T_K(t)$ and $T_M(t)$, which are introduced by thermo-modification of visco-elastic properties by temperature $T_K(t)$, and by thermo-modification of elasto-viscosic properties by temperature $T_M(t)$, than constitutive relation between stress and strain state of the thermo-rheological hereditary element (see Ref. [49]) is:

$$n\dot{P}(t) + P(t) + n\dot{F}_M(t) + F_K(t) = nc\dot{\rho}(t) + \tilde{c}\left[\rho(t) - \rho_0\right] \tag{12}$$

in which

$$F_M(t) = c_M \alpha_M T_M(t), \quad F_K(t) = c_K \alpha_K T_K(t) \tag{13}$$

are thermo-elastic forces, and $\rho(t)$ is rheological coordinate, c_M, c_K are coefficients of thermo-elastic rigidity, α_M, α_K are coefficients of thermo-elastic dilatations, n is time of relaxation, and c, $\tilde{c}$ an instantaneous rigidity and a prolonged one of an element.

Constitutive relation (12) of the thermo-rhelogical hereditary element from differential form, we can rewrite in two integro-differential forms.

V.5.2. Light standard piezo- and thermo-rheological hereditary element

When standard hereditary element is modified by two polarization voltages $U_K(t)$ and $U_M(t)$, which are introduced by piezo-modification of visco-elastic properties of subelement of piezoceramics, by $U_K(t)$ and by piezo-modification of elasto-viscosic properties by $U_M(t)$, and thermo-modified by two temperatures $T_K(t)$ and $T_M(t)$, than constitutive relation between stress and strain state of the piezo-rheological hereditary hybrid element is in the form (12) in which

$$\begin{aligned} F_M(t) &= c_{UM} \alpha_{UM} U_M(t) + c_{TM} \alpha_{TM} T_M(t) \\ F_K(t) &= c_{UK} \alpha_{UK} U_K(t) + c_{TK} \alpha_{TK} T_K(t) \end{aligned} \tag{14}$$

are thermoelastic and piezo-elastic forces, and $\rho(t)$ is rheological coordinate, c_{TM}, c_{TK} are coefficients of thermo-elastic rigidity, α_{TM}, α_{TK} are coefficients of thermo-elastic dilatations, c_{UM}, c_{UK} are coefficients of piezo-elastic rigidity, α_{UM}, α_{UK} are coefficients of piezo-elastic dilatations, n is time of relaxation, and c, $\tilde{c}$ an instantaneous rigidity and a prolonged one of an hybrid element.

V.5.3. Pendulum with standard thermo-rheological hereditary element

The thermo-rheological hereditary pendulum has two degrees of freedom, one degree of motion freedom defined by angular coordinate ϑ and one degree of deformations freedom defined by changeable length of thread as a coordinate $\rho(t)$.

Let us compose the equations of the thermo-rheological pendulum synamics with thread in which the standard thermo-rheological hereditary element with constitutive stress-strain relation (12) is incorporated. Now, by introducing force $P(t)$ of the extension of the thermorheological hereditary thread from constitutive relation (12) presented into integral form, the equations of the pendulum motion are in the forms (for detail see Reference [66] by Hedrih (Stevanović)):

$$\ddot{\rho} - (\rho_o + \rho(t))\dot{\theta}^2 + g\cos\theta + \frac{c}{m}\left[\rho(t) - \int_0^t \rho(\tau)R(t-\tau)d\tau\right] =$$

$$= \frac{1}{m}F_M(t) - \frac{c}{m(c-\tilde{c})}\int_0^t \left[F_M(\tau) - F_K(\tau)\right]R(t-\tau)d\tau P(t) \tag{15}$$

$$(\rho_0 + \rho(t))^2\ddot{\theta} + 2(\rho_0 + \rho(t))\dot{\theta}\dot{\rho}(t) + g(\rho_0 + \rho(t))\sin\theta = \mathfrak{M}(t). \tag{16}$$

This system is a system with one integro-differential and one differential equation of the thermo-rheological hereditary pendulum with motion in vertical plane.

If the thermo-rheological pendulum is in the horizontal plane, from second differential equation of the previous system, we can obtain the relation between the length of the pendulum thread and of the angular velocity in the following form:

$$\dot{\theta}(t) = \dot{\theta}(0)\left[\frac{\rho_0 + \rho(0)}{\rho_0 + \rho(t)}\right]^2. \tag{17}$$

By introducing this previous expression (17) in the first equation of the system (16) (for the case of horizontal plane) the following integro-differential equation for the pendulum length thread is obtained:

$$\ddot{\rho}(t) - \left[\dot{\theta}(0)\right]^2\frac{\left[\rho_0 + \rho(0)\right]^4}{\left[\rho_0 + \rho(t)\right]^3} + \frac{c}{m}\left[\rho(t) - \int_0^t \rho(\tau)R(t-\tau)d\tau\right] =$$

$$= \frac{1}{m}F_M(t) - \frac{c}{m(c-\tilde{c})}\int_0^t \left[F_M(\tau) - F_K(\tau)\right]R(t-\tau)d\tau P(t). \tag{18}$$

V.6. Concluding remarks

Solution of obtained integrdo-differential equation (18) is mathematical problem in analytical mechanics of hereditary discrete system dynamics.

Also, solutions of the similar integro-differential equations are tasks for mathematics in the function of applications in mechanics and engineering system dynamics with hereditary properties.

In the basis of the construction of Lagrange's mechanics of hereditary discrete systems, the classical mechanics principles are used [49]. These principles are: Principle of the work of forces along corresponding possible system displacements, as well as Principle of dynamical equilibrium.

Initial conditions of hereditary system dynamics are very important, containing the history of rheological interactions of the system. Than, it is important to take into account stress-strain history of viscoelastic elements -- interactions between hereditary system material particles.

Analogy between hereditary interactions and reactive forces in the systems of automatic control gives possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control.

For description of properties of dynamics of a hereditary system by using relaxational or rheological kernel (resolvent), these kernels are expressed by exponential or fractional-exponential forms [49]. Descriptions of hereditary properties of the system by using differential forms (2) and integral form (6) and (7) with exponential kernels are equivalent. For the case of fractional-exponential forms of the kernel (6) and (7) in the integral form corresponding equivalent differential forms not exist.

The Lagrange's mechanics of hereditary systems is extended and generalized to the thermo-rheological [49, 66] and piezo-rheological [49] mechanical systems.

V.7. Open directions for next research and applications

Directions for next research in area of mechanics of hereditary discrete system must be focused to find analytical forms of solutions or approximations of solutions of integro-differential equations and to build mathematical theory of the material memory of the history of previous stress and strains in the material before starting system motion and its observation. Mathematical theory for slowing problems with determinations of the initial conditions of the hereditary system is second main task in this area. Present in science, there are numerous numerical approach and numerical experiments over the integro-differential equations and numerical procedure expressed by software tools but for advances in area of analytical dynamics of hereditary systems it is necessary analytical approach, solutions and qualitative methods for ecaluations of the system solution stability.

For practical applications in mechanics and engineering system dynamics analytical forms of the approximations of solutions of intgro-differential equations are necessary for easier quantitative estimation larger class of the dynamic phenomena hereditary system behavior. All real constructions and ehgineering structures are with hereditary properties.

VI. Analytical mechanics of fractional order discrete system vibrations

VI.1. Introduction

Differential fractional order equations and theirs applications in development of analytical dynamics of discrete and continuous fractional order systems with like single and multi-frequency modes, or fractional order modes are important results for applications in different area of science and practice (see References (see References [53-59] and [62]).

Discrete continuum method is based on the continuum discretization and coupling by standard light hereditary, or fractional order elements (see References [63] and [64]). Main nets, main chains and mail fractional order oscillators with main fractional order modes of plane as well as space coupled mechanical chains (cases with ideal elastic, hereditary and fractional order). Fractional order standard light elements have applications in mechanics of continuum (models of longitudinal and transversal oscillations of beams), in biomechanics (mechanical models of double helix DNA chains [59]), to systems with coupled pendulums, as well is in signals transfer (see References IVI.).

VI.2. Standard light fractional order element

Standard light coupling element of negligible mass is in the form of axially stressed rod without bending, and which has the ability to resist deformation under static and dynamic conditions. ***Standard light fractional order creep element*** for which the constitutive stress-strain relation for the restitution force as the function of element elongation is given by fractional order derivatives in the form (see References [54] and [55] by Hedrih (Stevanović)):

$$P(t) = -\left\{c_0 x(t) + c_\alpha \mathfrak{D}_t^\alpha \left[x(t)\right]\right\} \tag{1}$$

where $\mathfrak{D}_t^{\alpha}[\bullet]$ is operator of the α^{th} derivative with respect to time t in the following form:

$$\mathfrak{D}_t^{\alpha}[x(t)] = \frac{d^{\alpha}x(t)}{dt^{\alpha}} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}}d\tau \qquad (2)$$

where c, c_{α} are rigidity coefficients – momentary and prolonged one, and α a rational number between 0 and $1, 0 < \alpha < 1$.

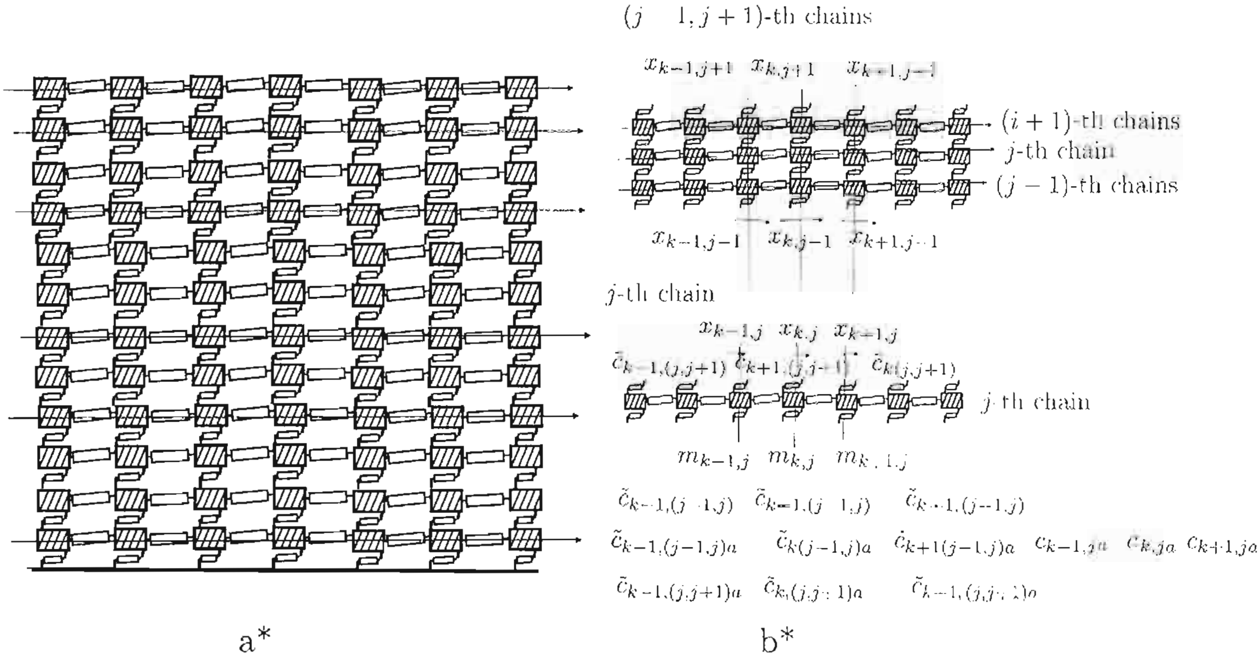


Figure 6 – Discrete continuum fractional order model in a plane – Hybrid multi chain fractional order plane system. (a*) Hybrid multi-chain system, in one plane and in the form of coupled chains by standard light fractional order elements and in the cantilever form of boundary conditions; (b*) three coupled chains $(j-1)$-th , j-th and $(j+1)$-th chains, $j = 1, 2, 3, 4, \dots, M$, as part subsystem of the hybrid multi chain system with coupling elements and kinetic parameters: masses, stiffnesses and fractional order parameter of the fractional order element and generalized coordinates of the system. Attached a separate j-th chain of the hybrid chain system with notation of the generalized coordinates $x_{i,j}$, $k = 1, 2, 3, 4, \dots, N$, $j = 1, 2, 3, 4, \dots, M$

VI.3. Governing equations of the fractional order multi-chain plane system model

Coupled governing fractional order differential equations of the multi chain fractional order plane system vibrations, according notation in Figures 6.a* and b*, and determined standard light fractional elements by constitutive relation (1) and (2), used for coupling of the mass particles, are in the following form:

$$\begin{gathered} m_{k,j}\ddot{x}_{k,j} = -c_{k-1,j}(x_{k,j} - x_{k-1,j}) + c_{k,j}(x_{k+1,j} - x_{k,j}) - \\ -c_{\alpha,k-1,j}\mathfrak{D}_t^{\alpha}\left[x_{k,j} - x_{k-1,j}\right] + c_{\alpha,k,j}\mathfrak{D}_t^{\alpha}\left[x_{k+1,j} - x_{k,j}\right] - \\ -\tilde{c}_{k,(j-1,j)}(x_{k,j} - x_{k,j-1}) + \tilde{c}_{k,(j,j+1)}(x_{k,j+1} - x_{k,j}) - \\ -\tilde{c}_{\alpha,k,(j-1,j)}\mathfrak{D}_t^{\alpha}\left[x_{k,j} - x_{k,j-1}\right] + \tilde{c}_{\alpha,k,(j,j+1)}\mathfrak{D}_t^{\alpha}\left[x_{k,j+1} - x_{k,j}\right] \\ k = 1,2,3,4,\dots,N, \quad j = 1,2,3,4,\dots,M. \end{gathered} \tag{3}$$

For the homogeneous plane system corresponding to the system (3) of fractional order differential equations let us introduce the coordinate transformation, in accordance with trigonometric method (see References [67] by Rašković and [64], [60] and [54] by Hedrih (Stevanović)) in the following form:

$$x_{k,j} = \sum_{s=1}^{s=M} \xi_{k(s)} \sin j\phi_s, \quad k = 1,2,3,4,\dots,N, \quad j = 1,2,3,4,\dots,M \tag{4}$$

where $\xi_{k(s)}$ are normal coordinates of the main chains of the hybrid plane system as well as generalized coordinates of the s-th main chain from the sets and for the corresponding linear system are in the form:

$$\begin{gathered} \xi_{k(s)} = C_{k(s)} \cos(\tilde{\omega}_{k(s)} t + \alpha_{k(s)}) \quad \text{and} \\ \tilde{\omega}^2_{k(s)} = \tilde{u}_{k(s)} \frac{\kappa c}{m} = 4\frac{\kappa c}{m} \sin^2\frac{\phi_s}{2}, \quad \tilde{u}_{k(s)} \equiv \tilde{u}_{(s)} \end{gathered}$$

and ϕ_s are eigen characteristic numbers of the hybrid system obtained by using the trigonometric method (see Refs. [67] by Rašković (1965) and [54], [55], [57], [59], [60] and [62] by Hedrih (Stevanović) (2002, (2004), (2007), (2009) and (2010)) depending of boundary conditions of the transversal chains in the form of the longitudinal chain connections.

$$\begin{gathered} \frac{m}{c}\ddot{\xi}_{k(s)} - \xi_{k-1(s)} + \left(2 + \kappa\tilde{u}_{k(s)}\right)\xi_{k(s)} - \xi_{k+1(s)} + \\ +\mathfrak{D}_t^{\alpha}\left[- \xi_{k-1(s)} + \left(2\kappa_{\alpha} + \tilde{\kappa}_{\alpha}\tilde{u}_{k(s)}\right)\xi_{k(s)} - \xi_{k+1(s)}\right] = 0 \\ k = 1,2,3,4,\dots,N, \quad s = 1,2,3,\dots,M. \end{gathered} \tag{5}$$

Let us introduce the following coordinate transformation:

$$\xi_{k(s)} = \sum_{r=1}^{r=N} \xi_{k(s)(r)} = \sum_{r=1}^{r=N} \eta_{(s)(r)} \sin k\vartheta_r, \quad s = 1,2,3,\dots,M. \tag{6}$$

Taking into account that $\sin k\vartheta_r$ is different them zero in arbitrary cases from (3), we can obtain the transformed system of the governing fractional order differential equations (5) of the eigen main chains as well as the transformed basic governing system of fractional order differential equations (3), with respect to the new introduced coordinates $\eta_{(s)(r)}$ containing $M \times N$ independent subsystems for each pair of the $(s)(r)$ from the sets $s = 1, 2, 3, \dots, M$ and $r = 1, 2, 3, 4, \dots, N$ in the following forms:

$$\ddot{\eta}_{(s)(r)} + \omega^2_{(s)(r)}\eta_{(s)(r)} + \omega^2_{\alpha(s)(r)}\mathfrak{D}^{\alpha}_{t}\left[\eta_{(s)(r)}\right] = 0, \qquad (7)$$
$$s = 1, 2, 3, \dots, M, \quad r = 1, 2, 3, 4, \dots, N,$$

where

$$\omega^2_{\alpha(s)(r)} = \frac{c}{m}\left\langle 2(\kappa_\alpha - 1) + u_{(s)(r)} + (\tilde{\kappa}_\alpha - \kappa)\tilde{u}_{k(s)}\right\rangle, \qquad (8)$$
$$s = 1, 2, 3, \dots, M, \quad r = 1, 2, 3, 4, \dots, N.$$

This last system of fractional order differential equations (7) represents $M \times N$ independent partial fractional order differential equations describing independent fractional order oscillators each with one degree of freedom and eigen normal coordinate $\eta_{(s)(r)}$, $s = 1, 2, 3, \dots, M$, $r = 1, 2, 3, 4, \dots, N$ of the considered fractional order hybrid system and containing N sets of the M eigen main chains normal coordinates $\eta_{(s)(r)}$, $s = 1, 2, 3, \dots, M$, $r = 1, 2, 3, 4, \dots, N$. Then, we can conclude that simultaneously with determination of the normal coordinates of the eigen main chains, we determine as well as normal coordinates of the considered hybrid fractional order plane system vibrations with $M \times N$ degrees of freedom. Also, we can conclude that normal coordinates for the linear system, correspond to the normal coordinates of the corresponding fractional order system and expressions for generalized coordinate transformation to the eigen normal coordinates of the basic linear system vibrations, we can use for the corresponding coordinate transformation of the corresponding fractional order hybrid system vibrations to the eigen normal coordinates.

VI.2. Eigen factional order signals and eigen main chain signals in the fractional order multi-chain plane system model

Type of the obtained fractional order differential equations in the system (7) is same as in numerous author's papers, but with different coefficients. These coefficients are sets of eigen circular frequencies $\omega^2_{(s)(r)}$ and fractional order material properties characteristic numbers $\omega^2_{\alpha(s)(r)}$ in the form:

$$\omega^2_{(s)(r)} = \frac{c}{m}\left[4\kappa\sin^2\frac{\phi_s}{2} + 2(1-\cos\vartheta_r)\right], \tag{9}$$

$$s = 1,2,3,\dots,M, \quad r = 1,2,3,4,\dots,N$$

$$\omega^2_{\alpha(s)(r)} = \frac{c}{m}\left\langle 2(\kappa_\alpha - 1) + 2(1-\cos\vartheta_r) + 4\tilde{\kappa}_\alpha\sin^2\frac{\phi_s}{2}\right\rangle, \tag{10}$$

$$s = 1,2,3,\dots,M, \quad r = 1,2,3,4,\dots,N$$

which depend of boundary multi-chain system conditions determining characteristic numbers: ϕ_s and ϑ_r (see Refs. [67] by Rašković (1965) and [54], [55], [57], [59], [60] and [62] by Hedrih (Stevanović)).

Now, taking into account solutions of the fractional order differential equation (see Reference [52] by Bačlić and Atanacković (2000)), for the system of the fractional order differential equations (7), we can write corresponding solutions of $\eta_{(s)(r)}$, $s = 1,2,3,\dots,M$, $r = 1,2,3,4,\dots,N$, in the form of fractional order like one frequency time functions which are eigen main modes and normal coordinate $\eta_{(s)(r)}$ of the fractional order plane system in the following expansion:

$$\eta_{(s)(r)}(t) = \eta_{(s)(r)}(0)\sum_{k=0}^{\infty}(-1)^k\omega^{2k}_{\alpha(s)(r)}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega^{2j}_{\alpha(s)(r)}t^{-\alpha j}}{\omega^{2j}_{(s)(r)}\Gamma(2k+1-\alpha j)} +$$
$$+\dot{\eta}_{(s)(r)}(0)\sum_{k=0}^{\infty}(-1)^k\omega^{2k}_{\alpha(s)(r)}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp1)^j\omega^{-2j}_{\alpha(s)(r)}t^{-\alpha j}}{\omega^{2j}_{(s)(r)}\Gamma(2k+2-\alpha j)} \tag{11}$$

$s = 1,2,3,\dots,M$, $r = 1,2,3,4,\dots,N$.

This last expressions (11) determines eigen normal fraction order like one frequency modes of the hybrid fractional order system corresponding to one eigen circular frequency and corresponding eigen fractional order properties characteristic number (10) of material fractional order properties of standard light elements. Also, we can conclude that the expressions (11) are mathematical descriptions of the main normal fractional order like one frequency signals.

Now, taking into account coordinate transformation:

$$\xi_{k(s)} = \sum_{r=1}^{r=N}\xi_{k(s)(r)} = \sum_{r=1}^{r=N}\eta_{(s)(r)}\sin k\vartheta_r \tag{12}$$

$$x_{k,j} = \sum_{s=1}^{s=M} \xi_{k(s)} \sin j\phi_s = \sum_{s=1}^{s=M} \sum_{r=1}^{r=N} \eta_{(s)(r)} \sin k\vartheta_r \sin j\phi_s \tag{13}$$

and solutions (11), we obtain expressions for the like multi-frequency fractional order generalized coordinate in the following form:

1* eigen normal coordinates for obtaining eigen main chanins and generalized coordinate of the eigen main chains,

$$\begin{gathered}\xi_{p(s)} = \sum_{r=1}^{r=N} \eta_{(s)(r)}(0) \sin p\vartheta_r \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha(s)(r)}^{2k} t^{2k} \\ \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha(s)(r)}^{2j} t^{-\alpha j}}{\omega_{(s)(r)}^{2j} \Gamma(2k+1-\alpha j)} + \sum_{r=1}^{r=N} \dot{\eta}_{(s)(r)}(0) \sin p\vartheta_r \\ \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha(s)(r)}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha(s)(r)}^{-2j} t^{-\alpha j}}{\omega_{(s)(r)}^{2j} \Gamma(2k+2-\alpha j)} \\ p = 1, 2, 3, \ldots, N, \quad s = 1, 2, 3, \ldots, M.\end{gathered} \tag{14}$$

2* generalized coordinate of the hybrid fractional order system:

$$\begin{gathered}x_{p,q} = \sum_{s=1}^{s=M} \sum_{r=1}^{r=N} \sin p\vartheta_r \sin q\phi_s \sum_{r=1}^{r=N} \eta_{(s)(r)}(0) \sin p\vartheta_r \\ \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha(s)(r)}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha(s)(r)}^{2j} t^{-\alpha j}}{\omega_{(s)(r)}^{2j} \Gamma(2k+1-\alpha j)} + \\ + \sum_{s=1}^{s=M} \sum_{r=1}^{r=N} \sin p\vartheta_r \sin q\phi_s \sum_{r=1}^{r=N} \dot{\eta}_{(s)(r)}(0) \sin p\vartheta_r \\ \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha(s)(r)}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha(s)(r)}^{-2j} t^{-\alpha j}}{\omega_{(s)(r)}^{2j} \Gamma(2k+2-\alpha j)} \\ p = 1, 2, 3, \ldots, N, \quad Q = 1, 2, 3, \ldots, M.\end{gathered} \tag{15}$$

VI.3. Eigen factional order signals and eigen main chain signals in the fractional order multi-chain space system model

For the homogeneous fractional order multi-chain space system dynamics, presented in Figure 7, by the similar way as in previous chapter VI.2. is possible to write system of coverning fractional order differential equations by use coordinate notation from Figure 7, and corresponding material system coordinates. Then let introduce the following coordinate transformation:

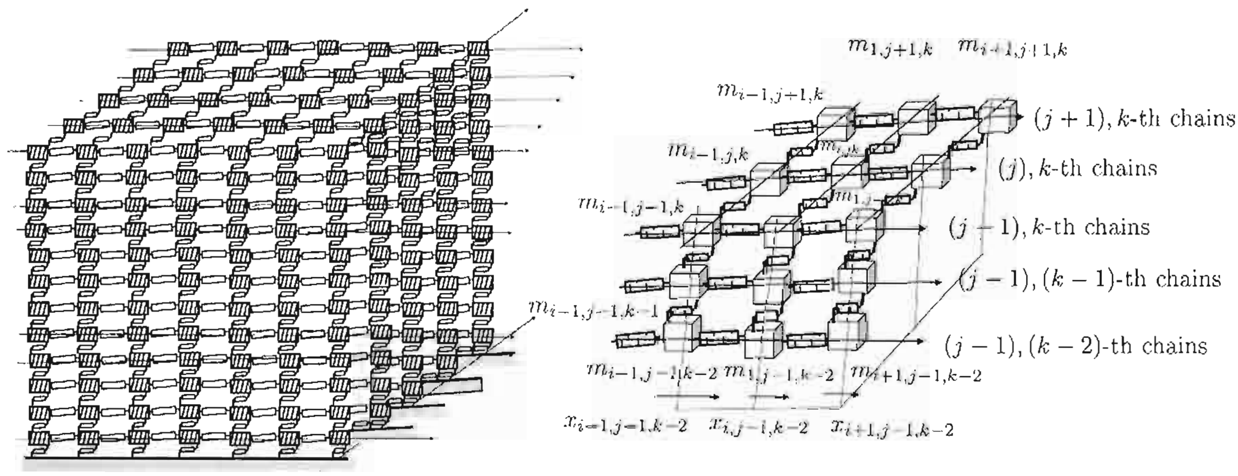
$$x_{(i),(j),(k)} = \sum_{s=1}^{s=K} \xi_{(i),(j)}^{(s)} \sin k\phi_s, \tag{16}$$

$$i = 1, 2, 3, 4, \dots, N, \quad j = 1, 2, 3, 4, \dots, M, \quad k = 1, 2, 3, \dots, K$$

where

$$\xi_{(i),(j)}^{(s)}, \quad i = 1, 2, 3, 4, \dots, N, \quad j = 1, 2, 3, 4, \dots, M$$

for each $s = 1, 2, 3, \dots, K$, are normal coordinates of the main plane subsystems $R^{(s)}$, $s = 1, 2, 3, \dots, K$ in the form of the independent K plane nets each consisting of the coupled M chains each with N degrees of freedom.



a* b*

Figure 7 – Discrete continuum fractional order space model - Hybrid multi chain fractional order space system. (a*) Hybrid multi-chain system, in space and in the form of coupled chains by standard light fractional order elements and in the cantilever form of boundary conditions. (b*) The coupled chains $(j-1)$, k-th, j, k-th and $(j+1)$, k-th chains, and $(j-1)$, k-th, $(j-1)$, $(k-1)$-th , and $(j-1)$, $(k-2)$-th, $j = 1, 2, 3, 4, \dots, M$, $k = 1, 2, 3, \dots, K$ as part subsystem of the hybrid multi chain system with coupling elements and kinetic parameters: masses, stiffnesses and fractional order parameter of the fractional order element and generalized coordinates of the system, with notation of the generalized coordinates $x_{i,j,k}$, $k = 1, 2, 3, 4, \dots, N$, $j = 1, 2, 3, 4, \dots, M$, $k = 1, 2, 3, \dots, K$.

ϕ_s, $s = 1, 2, 3, \dots, K$ are eigen characteristic numbers of the hybrid system and according trigonometric method (see Refs. [67] by Rašković

(1965) and [54], [55], [57], [59], [60] and [62] by Hedrih (Stevanović)) depending on boundary conditions of the transversal coupled chains in the form of the normal direction of chain connections between parallel plane nets, determined by direction of increasing indices $k = 1, 2, 3, \ldots, K$. Each of these K main and independent plane nets are with $N \times M$ degree of freedom with $N \times M$ normal coordinates $\xi^{(s)}_{(i),(j)}$, $i = 1, 2, 3, 4, \ldots, N$, $j = 1, 2, 3, 4, \ldots, M$ for each $s = 1, 2, 3, \ldots, K$ and are like multi frequency fractional order time functions form K independent subsets of circular frequencies $\omega^{(s)}_{(i),(j)}$ and corresponding fractional order characteristic numbers $\omega^{(s)}_{\alpha,(i),(j)}$ $i = 1, 2, 3, 4, \ldots, N$, $j = 1, 2, 3, 4, \ldots, M$ for each $s = 1, 2, 3, \ldots, K$.

Taking into account that $\sin k\phi_s$ is different them zero in arbitrary cases from system of governing fractional order differential equations, we can obtain the transformed basic governing system of fractional order differential equations with respect to the coordinates $\xi^{(s)}_{(i),(j)}$, $i = 1, 2, 3, 4, \ldots, N$, $j = 1, 2, 3, 4, \ldots, M$ containing K independent subsystems of coupled fractional order differential equations of like multi-frequency $N \times M$-frequency main plane nets for each s from the set of $s = 1, 2, 3, \ldots, K$ in the following forms:

$$\begin{gathered}
\frac{m}{c}\ddot{\xi}^{(s)}_{(i),(j)} - \xi^{(s)}_{(i-1),(j)} + (2 + \kappa\tilde{u}^{(s)}_{(i),(j)})\xi^{(s)}_{(i),(j)} - \xi^{(s)}_{(i+1),(j)} + \\
+\kappa\left[-\xi^{(s)}_{(i),(j-1)} + 2\xi^{(s)}_{(i),(j)} - \xi^{(s)}_{(i),(j+1)}\right] + \\
+\kappa_\alpha \mathfrak{D}^\alpha_t\left[-\xi^{(s)}_{(i-1),(j)} + (2 + \kappa\tilde{u}^{(s)}_{(i),(j)})\xi^{(s)}_{(i),(j)} - \xi^{(s)}_{(i+1),(j)}\right] + \\
+\kappa_\alpha \mathfrak{D}^\alpha_t\left[-\xi^{(s)}_{(i),(j-1)} + 2\xi^{(s)}_{(i),(j)} - \xi^{(s)}_{(i),(j+1)}\right] = 0 \\
i = 1, 2, 3, 4, \ldots, N, \quad j = 1, 2, 3, 4, \ldots, M, \quad s = 1, 2, 3, \ldots, K.
\end{gathered} \tag{17}$$

Previous obtained K subsets of fractional order differential equations describing dynamics of the main subsystems is expressed by new coordinates $\xi^{(s)}_{(i),(j)}$, $i = 1, 2, 3, 4, \ldots, N$, $j = 1, 2, 3, 4, \ldots, M$ containing K independent subsystems for each s from the set of $s = 1, 2, 3, \ldots, K$.

These subsystems present K mathematical descriptions of dynamics of independent eigen main plane (or surface) nets containing coupled chains with corresponding subset of the eigen circular frequencies and corresponding fractional order characteristic numbers of main plane (surface) nets. Through these eigen main plane nets is possible to transfer subset of the signals with frequencies from the corresponding subset of the eigen circular frequencies. These signals are fractional order like $N \times M$ frequency signals.

Next approach is similar as in the previous chapter VI.2. for system containing coupled chains in one plane. Then, due to the limited length of the paper, we will not to present all derivatives and suppose writer to follow previpus chapter to obtain independent subsystems of the fractional order differential equations describing main independent fractional order oscillators each with one degree of freedom, in the form:

$$\ddot{\zeta}^{(s)(r)(p)} + \omega^2_{(s)(r)(p)}\zeta^{(s)(r)(p)} + \omega^2_{\alpha(s)(r)(p)} D^{\alpha}_{t}\left[\zeta^{(s)(r)(p)}\right] = 0, \tag{18}$$

$$s = 1,2,3,\dots,K, \quad r = 1,2,3,\dots,M, \quad p = 1,2,3,\dots,N$$

where

$$\xi^{(s)}_{(i),(j)} = \sum_{r=1}^{r=M} \eta^{(s)(r)}_{(i)} \sin j\vartheta_r, \tag{19}$$

$$i = 1,2,3,4,\dots,N, \quad j = 1,2,3,4,\dots,M, \quad s = 1,2,3,\dots,K$$

$$\eta^{(s)(r)}_{(i)} = \sum_{p=1}^{p=N} \zeta^{(s)(r)(p)}\eta^{(s)(r)}_{(i)} \sin i\psi_p, \tag{20}$$

$$i = 1,2,3,4,\dots,N, \quad s = 1,2,3,\dots,K, \quad r = 1,2,3,\dots,M.$$

Previous system (18) contains $N \times M \times K$ independent fractional order differential equations each only along one coordinate $\zeta^{(s)(r)(p)}$, $s = 1,2,3,\dots,K$, $r = 1,2,3,\dots,M$, $p = 1,2,3,\dots,N$. These coordinates $\zeta^{(s)(r)(p)}$, $s = 1,2,3,\dots,K$, $r = 1,2,3,\dots,M$, $p = 1,2,3,\dots,N$, are normal coordinates of the hybrid discrete fractional order space system containing parallel coupled chains in the parallel planes and in the parallel lines in these planes.

Number of fractional order partial oscillators is equal to the product $N \times M \times K$ and equal to the number of the system degrees of freedom.

VI.3. Concluding remarks

Then we can conclude that through eigen main plane (surface) nets $R^{(s)}$, $s = 1,2,3,\dots,K$, it is possible to transfer like $N \times M$-eigen frequency fractional order signals as independent on other subsets of plane like $N \times M$ eigen frequency fractional order signals in other eigen main plane nets $R^{(s)}$, $s = 1,2,3,\dots,K$.

Each eigen main fractional order plane nets $R^{(s)}$, $s = 1,2,3,\dots,K$ is possible to decompose into M independent eigen chains, in total there are $M \times K$ main independent chains of all space system, with normal coordinates $\eta_{(i)}^{(s)(r)}$ $i = 1,2,3,4,\dots,N$, $s = 1,2,3,\dots,K$, $r = 1,2,3,\dots,M$ of these independent eigen chains. Then we can conclude that through each independent eigen main chain is possible to transfer like N-frequency fractional order signal, as well as that coordinate $\eta_{(i)}^{(s)(r)}$ are N-frequency fractional order time functions with corresponding main chains sub set of N-frequencies and corresponding characteristic fractional order properties numbers.

From the last obtained system (18) containing $N \times M \times K$ independent fractional order differential equations each along one normal coordinate $\zeta^{(s)(r)(p)}$, $s = 1,2,3,\dots,K$, $r = 1,2,3,\dots,M$, $p = 1,2,3,\dots,N$ we can conclude that each of these normal coordinates of the system is like one frequency fractional order time function with eigen circular frequency from the set $\omega_{(s)(r)(p)}^2$, $s = 1,2,3,\dots,K$, $r = 1,2,3,\dots,M$, $p = 1,2,3,\dots,N$ and fractional order characteristic numbers $\omega_{\alpha(s)(r)(p)}^2$, $s = 1,2,3,\dots,K$, $r = 1,2,3,\dots,M$, $p = 1,2,3,\dots,N$ describing fractional order properties of the system eigen vibrations. Number of fractional order partial oscillators is equal to the product $N \times M \times K$ and equal to the number of the system's degrees of freedom. Then we obtain normal coordinates for the transfer of one frequency fractional order signals through space fractional order vibration structure.

Then, we can conclude that simultaneously with determination of the normal coordinates of the eigen main nets and eigen main chains we determine as well as normal coordinates of the considered hybrid fractional order system with $N \times M \times K$ degree of freedom. Also, we can conclude that normal coordinates for the linear system, corresponding to normal coordinates of the corresponding fractional order space system and expressions for generalized coordinate transformations to the eigen normal coordinates of the basic linear system we can use for the corresponding coordinate transformation of the corresponding fractional order space hybrid system to the eigen normal coordinates for the considered hybrid system.

VI.4. Open directions for next research and applications

Directions for next research in area of mechanics of fractional order discrete system must be focused to find analytical forms of solutions or approximations of solutions of fractional order differential equations different types and integrals.

Also, applications of the fractional derivatives and fractional integrals for describing constitutive relations of different types and sources of material.

Research in this area nust be focused also to the experimental investigation of the material constants and parameters defined by fractional order derivatives and operators.

For practical applications in mechanics and engineering system dynamics analytical forms of the approximations of solutions of fractional order differential equations are necessary for easier quantitative estimation larger class of the dynamic phenomena fractional order system behavior. All real constructions and engineering structures are with plastic properties.

VII. Analysis of the trigger of coupled singularities in nonlinear dynamic of no ideal system with Coulomb's type friction

VII.1. Free vibrations of the heavy mass particle along rotate rough curvilinear line with Coulomb friction

For beginning let to consider free vibrations of the heavy mass particle along rotate rough curvilinear line with Coulomb's type friction, see Figure 8.a*. For the case that curvilinear line is in the vertical rotate plane Oxz around vertical Oz axis, we can take that equation of the curve-linear line is: $z = f(x)$, or $f_1(x, z) = z - f(x) = 0$ and with the following properties $f(-x) = f(x)$ and that coordinate pole is in the zero point $f(0) = 0$ in which line have minimum (see Figure 8.a*). Also we take that curvilinear line rotate around vertical Oz axis with constant angular velocity $\vec{\Omega} = \Omega\vec{k}$.

Heavy mass particle, mass m, moving along rought curvilinear line with Coulomb's type sliding friction coefficient μ, is loaded by proper weight mg, as a active conservative force and by four no ideal constraint reactios, one F_N - normal ideal constrain reaction, second F_{BN} in binormal direction and two additional, $F_{\mu 1}$ first tangential component of the no ideal constraint reaction induced by friction and proportional to the normal component reaction F_N, $F_{\mu 1} = -\mu F_N \, sign \, \vec{v}_{rel}$, and $F_{\mu 2}$ second tangential component of the no ideal constraint reaction induced by friction caused by pressures in the binormal direction and proportional to the binormal component of the inertia force

$$F_{BN}, \quad F_{\mu 2} = -\mu F_{BN} \, sign \, \vec{v}_{rel},$$

cauzed by curvilinear line rotation around vertical Oz axis with constant angular velocity $\vec{\Omega} = \Omega\vec{k}$.

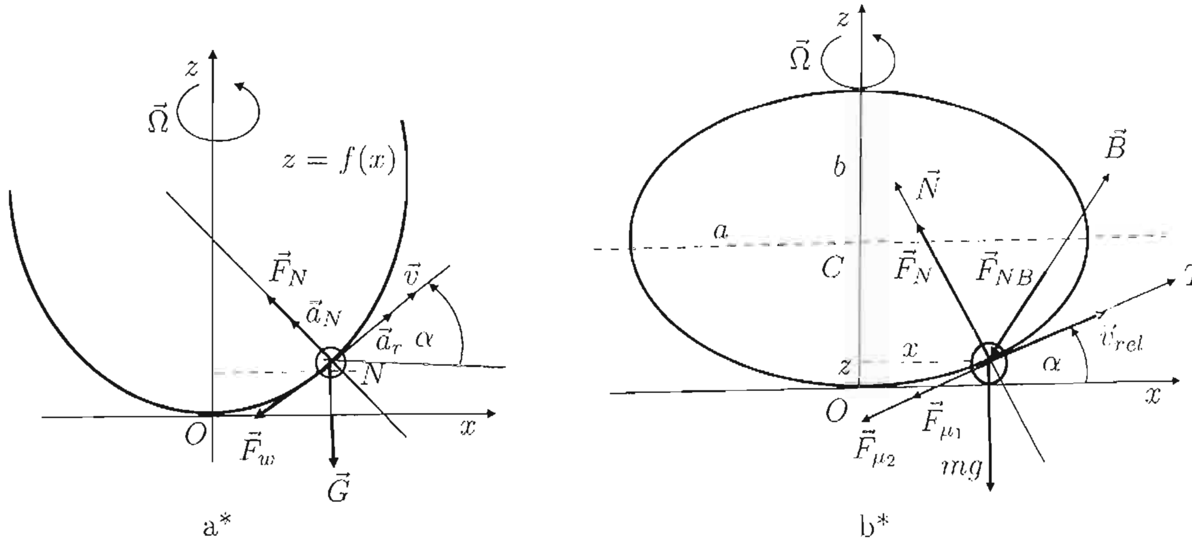


Figure 8 – Heavy material particle motion along rough curvilinear line with Coulomb friction

Force of the inertia of mass particle realtive motion along the curvilinear line which rotate around vertical Oz axis with constant angular velocity $\vec{\Omega} = \Omega \vec{k}$, have two components. One component is force of the inertia of the circle rotation around vertical axis in the form $\vec{F}_{jp} = m\Omega^2 x\,\vec{u}$, and second is Coriolis inertia force of the system and we can write:

$$\vec{F}_{NB} = -\vec{F}_{jC} = 2m\left[\vec{\Omega}\vec{v}_{rel}\right] = 2m\Omega v_{rel}\cos\alpha\vec{B}.$$

Corresponding force of Coulomb's type friction is ubthe form:

$$\vec{F}_{\mu 2} = -\mu\left|\vec{F}_{NB}\right|\frac{\vec{v}_{rel}}{|\vec{v}_{rel}|} = -2\mu m\Omega v_{rel}\cos\alpha\frac{\vec{v}_{rel}}{\left|\vec{v}_{rel}\right|} = -2\mu m\Omega\vec{v}_{rel}\cos\alpha.$$

By use principle of dynamical equilibrium we obtain expression for the intensity of normal and binormal components of curvilinear constraint reactionscorresponding differential double non-linear equation of the heavy mass particle motion along rotate arbitrary curvilinear rogh line, with angular velocity of rotation $\vec{\Omega}$, and defined by function $z = f(x)$, for the case that the coefficient of the Coulomb's type sliding friction is μ, s in the following form:

$$\frac{d}{dt}(\dot{x}\sqrt{1+z'^2}) \pm \mu\dot{x}^2\frac{z''}{\sqrt{1+z'^2}} + \Omega^2\frac{x}{\sqrt{1+z'^2}}(1 \mp \mu z') +$$
$$+\frac{g}{\sqrt{1+z'^2}}(z' \pm \mu) \pm 2\mu\Omega\dot{x} = 0. \tag{1}$$

For the case of heavy mass particle motion along no ideal arrbitrary rough curvilinear line without rotation, differential equation is in the form:

$$\frac{d}{dt}(\dot{x}\sqrt{1+z'^2}) + g\frac{z'}{\sqrt{1+z'^2}} \pm \mu\frac{1}{\sqrt{1+z'^2}}(\dot{x}^2 z'' + g) = 0. \tag{2}$$

Let consider special case of the rough curvilinear line with friction along normal surface contact (without last term $\pm 2\mu\Omega\dot{x}$ in (1)) and let introduce new variable in the following form: $u = \dot{x}^2$, then previous differential doube equation (1) of the mass particle motion along rough line is possible to transform in the following form:

$$\frac{du}{dx} + 2u\frac{z''(z' \pm \mu)}{(1+z'^2)} =$$

$$= -2\Omega^2\frac{x}{(1+z'^2)}(1 \mp \mu z') - \frac{2g}{(1+z'^2)}(z' \pm \mu) \quad \begin{cases} \text{for} & v_{rel} > 0 \\ \text{for} & v_{rel} < 0 \end{cases} \tag{3}$$

Previous differential double equation (3) of the material particle motion along rought curvilinear line according new helping coordinate u is ordinary double differential equation first order with changeable coeficients and type in following form:

$$\frac{du}{dx} \pm P(x)u = Q(x),$$

with following solution:

$$[\dot{x}(x)]^2 = e^{-2\int\frac{z''(z'\pm\mu)}{(1+z'^2)dx}}\left[-2\int\left[\Omega^2\frac{x}{(1+z'^2)}(1 \mp \mu z') + \right.\right.$$

$$\left.\left. + \frac{g}{(1+z'^2)}(z' \pm \mu)\right] e2^{\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}, dx + C\right]. \tag{4}$$

From previous first integral, the following equation of the phase trajectories in the phase plane $(x, \dot{x})$ we obtain:

$$v_{rel}^2(x) = (1+z'^2)e^{-2\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}\left[-2\int\left[\Omega^2\frac{x}{(1+z'^2)}(1 \mp \mu z') + \right.\right.$$

$$\left.\left. + \frac{g}{(1+z'^2)}(z' \pm \mu)\right] e2^{\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}dx + C\right] \tag{5}$$

where C integral constant depending of initial conditions, angular coordinate and angular velocity at initial moment, or starting terminate mass particle positions for next phase trajectory branch.

For reason to compare properties of kinetic parameters of main considered system dynamics and corresponding fictive (neglecting terms with aquare of velocity $\dot{x}^2$) for comparison we transform corresponding differential double equations in the form of the system of first order differential double equations and for obtaining singularities for main system and fictive systems use conditions that right hand side all equations must be equal to zero (nul). Then we obtain the following conditions:

*For main system dynamics:

$$\frac{dx}{dt} = v = 0$$

$$\frac{dv}{dt} = -\dot{x}^2 \frac{z'z''}{(1+z'^2)} \mp \mu \dot{x}^2 \frac{z''}{(1+z'^2)} - \Omega^2 \frac{x(1 \mp \mu z')}{(1+z'^2)} - \frac{g(z' \pm \mu)}{(1+z'^2)} = 0. \quad (6)$$

*For corresponding fictive systems

$$\frac{dx}{dt} = v = 0$$

$$\frac{dv}{dt} = -\Omega^2 \frac{x(1 \mp \mu z')}{(1+z'^2)} - \frac{g(z' \pm \mu)}{(1+z'^2)} = 0 \quad (7)$$

and

$$\frac{dx}{dt} = v = 0$$

$$\frac{dv}{dt} = -\frac{(\Omega^2 x + g z')}{(1+z'^2)} = 0. \quad (8)$$

We can see that for listed main system and for fictive system, conditions for obtaining singularities are same. Depending of the curvilinear line form $z = f(x)$, we obtained two nonlinear algebra equations in the following forms:

$$-\Omega^2 \frac{x(1 \mp \mu z')}{(1+z'^2)} - \frac{g(z' \pm \mu)}{(1+z'^2)} = 0 \quad \text{and} \quad \frac{(\Omega^2 x + g z')}{(1+z'^2)} = 0 \quad (9)$$

from which we can obtain, one or more roots.

If corresponding algebra double equation (9) have one root for $\mu = 0$ then words are about one equilibrium position with "one side left" and "one side right" bifurcation of the equilibrium position and one fictive triger of coupled singlarities caused by Coulomb's type friction between mass particle and rough curvilinear line.

If corresponding algebra double equation (9) have odd number of roots for $\mu = 0$ then words are about trigger of coupled singularities in a dynamics of a basic non-linear system correspond to the system with friction. In this case corresponding algebra double equation (9) for $\mu \neq 0$ have corresponding odd number of roots for erach of the sets of the sign $\pm$, but all these roots are slected in two subsets, first one an "one side righ" singularities and other "one side left" singularities correspond to the "one side left" and "one side right" relative equilibrium positions. Then, each roots of the corresponding algebra double equation (9) for $\mu = 0$, have two corresponding roots obtained from corresponding algebra double equation (9) for $\mu \neq 0$ and then there are present new fictive triggers of coupled one side singularities. Then we have trigger of the coupled trigers of coupled one side left, one central and one side right singularities, which are present in the system with Coulomb's type friction and with a corresponding nonlinear system with ideal constraints and with minimum a triger of coupled singularities in its nonlinear dynamics.

Example 1. For the case that line is a circle shaped by

$$(z-R)^2 + x^2 = R^2, \quad z = R - \sqrt{R^2 - x^2},$$

for $\mu = 0$ and $\mu \neq 0$, from (9) there are two corresponding algebra equations, one of which for $\mu \neq 0$ is algebra double equations:

$$\Omega^2 x + g\frac{x}{\sqrt{R^2 - x^2}} = 0$$

and

$$\Omega^2 x\left(1 \mp \frac{\mu x}{\sqrt{R^2 - x^2}}\right) + g\left(\frac{x}{\sqrt{R^2 - x^2}} \pm \mu\right) = 0. \tag{10}$$

From first algebra equation for $\mu = 0$ of previous system (10) is visible that $x = 0$ is a root correspond to the equilibrium position, but there are also pair of the roots:

$$x_{1,3} = \pm\sqrt{R^2 - \frac{g^2}{\Omega^4}} \quad \text{for} \quad \left|\frac{g}{R\Omega^2}\right| \leq 1.$$

In this case for $\mu = 0$ in system dynamics minimum a trigger of couped three singularities exists. Also, we can conclude that second algebra equation for $\mu \neq 0$ have minimum two roots. First approximation of the minimum values of first two roots are

$$x_{1,3} \approx \pm 2\mu\frac{g}{\Omega^2},$$

which correspond to the "one side right" and "one side left" equilibrium positions and with $x = 0$ build a trigger of coupled two one side singularities appeared as a result of bifurcation by introducing Coulomb's type friction. By qualitative analyzing of the second algebra double equation from system (10) in the form:

$$x^4(1+\mu^2) - x^2\left(R^2 - (1+4\mu^2)\frac{g^2}{\Omega^4}\right) - x\left(\pm 2\mu\frac{gR^2}{\Omega^2}\right) - 4\mu^2\frac{g^2R^2}{\Omega^4} = 0,$$

we conclude that, also, one triger of coupled three triggers of coupled one side singularities appear.

Example 2. For the case that line is an ellipse shaped by

$$\left(\frac{z-R}{a}\right)^2 + \frac{x^2}{b^2} = 1, \quad z = R \pm a\sqrt{1-\frac{x^2}{b^2}},$$

for $\mu = 0$ and $\mu \neq 0$, from (9) there are two corresponding algebra equations, one of which for $\mu \neq 0$ is algebra double equations (see Figure 8.b*):

$$\Omega^2 x \pm ag\frac{\frac{x}{b^2}}{\sqrt{1-\frac{x^2}{b^2}}} = 0$$

and

$$\Omega^2 x\left(1 - \mu a\frac{\frac{x}{b^2}}{\sqrt{1-\frac{x^2}{b^2}}}\right) + g\left(\pm a\frac{\frac{x}{b^2}}{\sqrt{1-\frac{x^2}{b^2}}} \pm \mu\right) = 0. \qquad (11)$$

From first algebra equation for $\mu = 0$ of previous system (11) is visible that $x = 0$ is a root correspond to the equilibrium position, but there are also pair of two roots:

$$x_{1,3} = \pm b\sqrt{1-\left(\frac{ag}{b^2\Omega^2}\right)^2} \quad \text{for} \quad \frac{ag}{b^2\Omega^2} < 1.$$

In this case for $\mu = 0$ in system dynamics, minimum a trgger of couped three singularities exists. Also, we can conclude that second algebra equation for $\mu \neq 0$ have minimum two roots. First approximation of the minimum values of first two roots are

$$x_{1,3} \approx \pm 2\mu\frac{g}{\Omega^2},$$

which correspond to the "one side right" and "one side left" equilibrium positions and with $x = 0$ build a trigger of coupled two one side singularities appeared as a result of bifurcation by introducing Coulomb's type friction. By qualitative analyzing of the second algebra double equation from system (11) in the form:

$$\left(x \pm \mu \frac{g}{\Omega^2}\right)^2 (R^2 - x^2) = x^2 \left(\frac{g}{\Omega^2} \mp \mu x\right)^2,$$

we conclude that appear also one triger of coupled three triggers of coupled one side singularities.

VII.2. Theorem of trigger of coupled singularities

Previous considered differential double equations of the heavy mass particle along rough curvilinear line with Coulomb's type friction is possible to express in the following generalized form of differential double equation with double signs:

$$\ddot{x} \pm b_\mu \dot{x}^2 + g\left[k, F(x, \mp x_\mu)\right] f(x, \pm x_\mu) = 0 \quad \text{for} \quad \dot{x} \gtrless 0 \tag{12}$$

where b_μ coefficient depending of Coulomb's type coefficient of friction, and x_μ parameter in coordinate dimension depending of Coulomb's type coefficient of friction and with a corresponding governing differential equation for ideal system dynamics for $x_\mu = 0$ in the following form:

$$\ddot{x} + g[k, F(x)]f(x) = 0. \tag{13}$$

VII.2.1. Theorem on the existence of a trigger of the coupled singularities and the separatrix in the form of number eight in the conservative system

By using nonlinear dynamic analysis of systems with described nonlinear phenomenon of the trigger of coupled singularities and corresponding families of phase portraits and potential energies (see Reference [68-88]) as well as the corresponding experimental investigations of such non-linear dynamics in mechanical engineering systems with coupled rotation motions (see Refs. [72] and [73]) it was easy to define and to prove a series of the theorem of the existence of a trigger of coupled singularities in non-linear dynamical conservative and no conservative systems with periodical structure.

Theorem. *In the system whose dynamics can be described with the use of non-linear differential equation in the form* (see Refs. [72] and [73]):

$$\ddot{x} + g\left[k, F(x)\right] f(x) = 0 \tag{14}$$

and whose potential energy is in the form:

$$\mathbf{E} = m\int_0^x g\big[k, F(x)\big] f(x)dx = \mathbf{G}\big[k, F(x)\big] \tag{15}$$

in which the functions $f(x)$ and $g(x)$ are:

$$F(x) = \int_0^x f(x)dx \quad \text{and} \quad G(k,x) = \int_0^x g(k,x)dx \tag{16}$$

and satisfy the following conditions:

$$\begin{gathered} f(-x) = -f(x) \qquad g(k, x+nT_0) = g(k,x) \\ f(x+nT_0) = f(x) \qquad g(k,-x) = g(k,x) \\ f(0)=0 \qquad g\big[k, F(x_r)\big] = 0, \qquad \text{for} \qquad k \in (k_1,k_2)\cup(k_2,k_3)\dots \\ f(x_s) = 0, \qquad x_s = sT_0, \qquad s = 1,2,3,4,\dots \qquad x_r = \pm x_0 \pm rT_0, \\ r = 0,1,2,3,4,\dots \qquad |x_0| < \frac{T_0}{2} \\ g\left[k, F(x)\right] \neq 0, \quad \text{for} \quad k \notin (k_1,k_2)\cup(k_2,k_3)\dots \end{gathered} \tag{17}$$

and both functions $f(x)$ and $g(x)$ have one maximum or minimum in the interval between two zero roots:

*a^* for parameters values $k \notin (k_{1\neq}, k_2)\cup(k_2,k_3)\dots$, outside of the intervals $(k_{1\neq},k_2)\cup(k_2,k_3)\dots$, **the trigger of singularities in the local area does not exist.***

*b^* for parameters values $k \in (k_{1\neq}, k_2)\cup(k_2,k_3)\dots$, inside of the intervals $(k_{1\neq},k_2)\cup(k_2,k_3)\dots$, **the series of triggers of coupled singularities in the local domains exist.***

We can see that for the case a* the second derivative of the potential energy can be positive or negative:

$$\left.\frac{d^2\mathbf{E_p}}{dx^2}\right|_{x=x_s} = \begin{cases} \text{for } \dfrac{df(0)}{dx} > 0 & \mathbf{E}_{\mathbf{p}\,\text{min}}\ x_0 \text{ stable equilibrium } p. \\ \text{for } \dfrac{df(x_s)}{dx} > 0 & \mathbf{E}_{\mathbf{p}\,\text{min}}\ x_s \\ & s = 2p,\ p = 1,2,3,4,\dots \\ & \text{stable equilibrium } p. \\ \text{for } \dfrac{df(x_s)}{dx} < 0 & \mathbf{E}_{\mathbf{p}\,\text{max}}\ x_s\ s = 2p-1, \\ & p = 1,2,3,4,\dots \\ & \text{unstable equilibrium } p \end{cases} \tag{18}$$

and equilibrium positions can be stable and unstable with corresponding singular points alternatively change, periodically, with period T_0 from stable center to unstable saddle point, and corresponding phase portrait is without trigger of coupled singularities and without separatrix in the form of number eight.

Also we can see that for the case b* the second derivative of the potential energy can be positive or negative

$$\left.\frac{d^2\mathbf{E_p}}{dx^2}\right|_{x=x_s} = m\left.\left\{g\big[k, F(x)\big]\frac{df(x)}{dx}\right\}\right|_{x=x_s} < 0 \quad \mathbf{E}_{\mathbf{p}\max}$$

$$x_s \quad \text{unstable equilibrium position}$$

$$\left.\frac{d^2\mathbf{E_p}}{dx^2}\right|_{x=x_r} = m\left.\left\{\frac{dg[k, F(x)]}{dF(x)}[f(x)]^2\right\}\right|_{x=x_r} > 0 \quad \mathbf{E}_{\mathbf{p}\min} \tag{19}$$

$$x_r \quad \text{stable equilibrium positions}$$

and equilibrium positions can be stable and unstable with corresponding singular points alternatively change, periodically, with period T_0.from stable center to unstable saddle points, and corresponding phase portrait is with triggers of coupled singularities and with series of the separatrix in the form of numbers eight. Then, the triggers of coupled singularities exist in the phase portrait in the intervals defined by:

$$x \in \left(-\frac{T_0}{2} + sT, \frac{T_0}{2} + sT\right) \quad s = 0, 1, 2, 3, 4, \ldots .$$

Integral energy of the system is in the form:

$$\dot{x}^2 + 2G\left[k, F(x)\right] = \dot{x}_0^2 + 2G\left[k, F(x(t_0))\right] = \text{const.} \tag{20}$$

Equation of homoclinic orbit in the form number "eight" trough homoclinic point $(0, 0)$ is:

$$\dot{x}^2 + 2G\big[k, F(x)\big] = 2G\big[k, F(0)\big] = h_{0hc} = \text{const.} \tag{21}$$

for $g[k, F(x_r)] = 0$, for $k \in (k_1, k_2) \cup (k_2, k_3) \ldots$ in which the functions $F(x)$ and $G(k, x)$ are in the form (16) and satisfy the conditions (17).

In Figure 9B* and B* equivalent of potential energy $\mathbf{E}_p(\phi)$ graph of basic ideal mechanical system correspond to no ideal wich Coulomb's type friction is presented. In Figures 9 B*$\mathbf{b}$*, B*$\mathbf{c}$* and B*$\mathbf{d}$* the sets of the homoclinic phase trajectory layering, for $\alpha_0 = 0$ and different values of the

$$k = \frac{1}{\lambda} = \left|\frac{g}{\ell\Omega^2}\right| \leq 1$$

and axis eccentricity are presented. Homoclinic orbits in the form of number eight appear and disappear with changing parameter

$$k = \frac{1}{\lambda} = \left| \frac{g}{\ell \Omega^2} \right| \begin{matrix} \leq \\ \geq \end{matrix} 1$$

Two sets of the of the singular points: $\phi_s = s\pi$, $s = 1, 2, 3, 4, \ldots$ and $\phi_s = \arccos \dfrac{g}{\ell \Omega^2} \pm 2s\pi$, $s = 1, 2, 3, 4, \ldots$ fort $k = \dfrac{1}{\lambda} = \left| \dfrac{g}{\ell \Omega^2} \right| \leqslant 1$ exists together with homoclinic orbits – separatrix in the form of number eight.

In Figure 9 A*a* equivalent of potential energy $\mathbf{E}_p(\phi)$ graph of basic ideal mechanical system correspond to no ideal wich Coulomb's type friction is presented in function of coordinate ϕ. In Figures 9. (A*b*), (A*c*) and (A*d*) series of the phase trajectory portraits, for $\alpha_0 = 0$, and different values of the

$$k = \frac{1}{\lambda} = \left| \frac{g}{\ell \Omega^2} \right| \begin{matrix} \leq \\ \geq \end{matrix} 1$$

and eccentricity of axis of circle rotation are presented. Two sets (A*b*) and (A*c*) of the of the singular points in phase portraits are visible: $\phi_s = s\pi$, $s = 1, 2, 3, 4, \ldots$ and $\phi_s = \arccos \dfrac{g}{\ell \Omega^2} \pm 2s\pi$, $s = 1, 2, 3, 4, \ldots$ for $k = \dfrac{1}{\lambda} = \left| \dfrac{g}{\ell \Omega^2} \right| \leq 1$. One set (A***d***) of singular points in phase trajectory portrait is visible: $\phi_s = s\pi$, $s = 1, 2, 3, 4, \ldots$ for

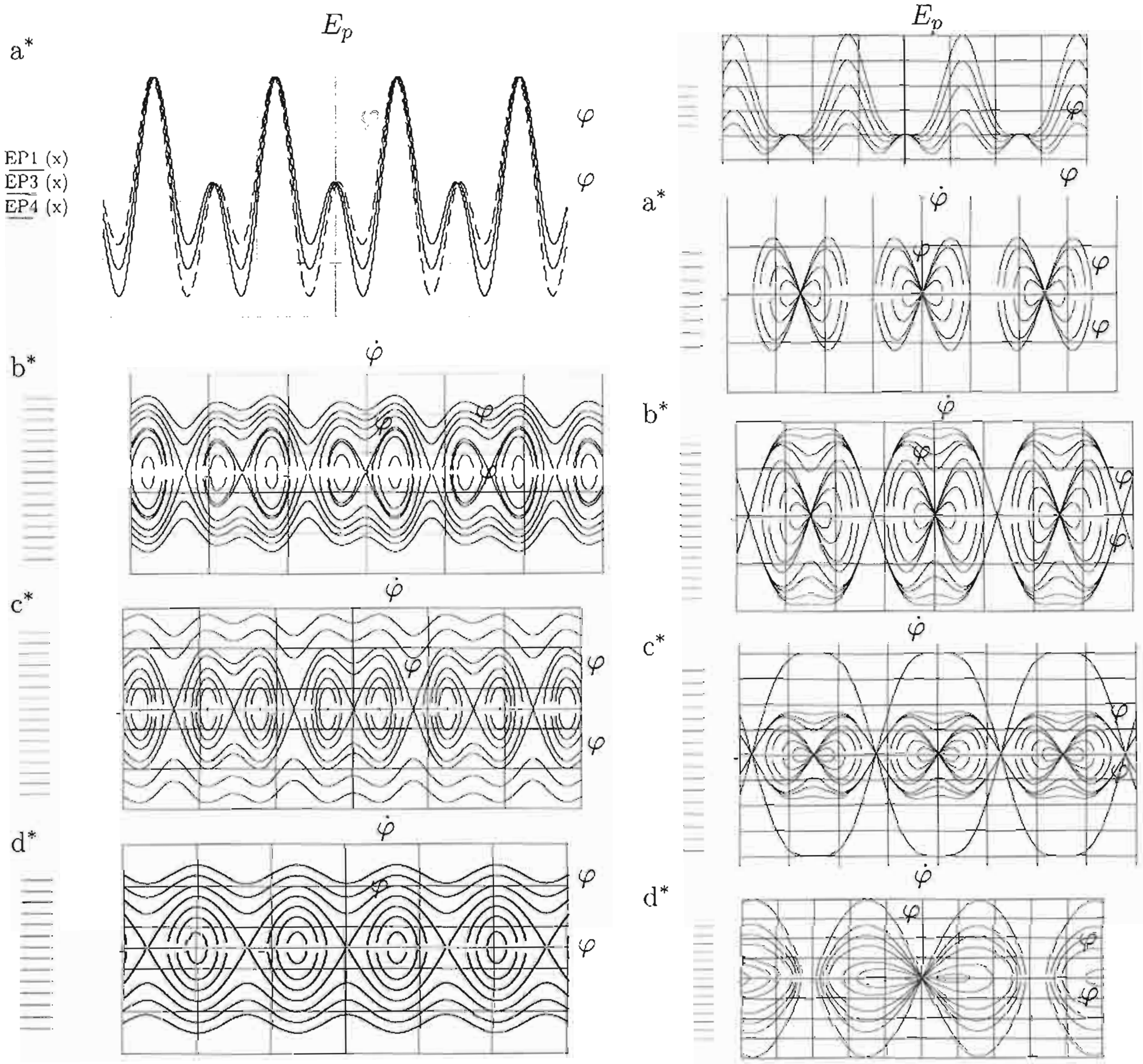
$$k = \frac{1}{\lambda} = \left| \frac{g}{\ell \Omega^2} \right| \geq 1.$$

VII.2.2. Triggers of coupled singularities in non-linear dynamics of coupled double rotor systems with Coulomb's type friction

In this part, we start with a new model of the non-linear dynamics of two coupled rigid rotors with mass particle debalances and no ideal surfaces between rotor shafts and cylindrical bearing where appear Coulomb's type friction (for detail see Reference [88]).

In Figure 10 a* the structure of the coupled double rotor system with Coulomb's type friction into contact surfaces between rotor shafts and cylindrical bearings is presented. In Figure 10. b* decomposition of this system with plan of the Coulomb's type friction forces is presented.

Governing nonlinear differential double equation of the coupled double rotor system dynamics with Coulomb's type friction into contact surfaces



$\ddot{\varphi} + \Omega^2(\lambda - \cos\varphi) - \Omega^2\varepsilon\cos\varphi = 0$ $\qquad$ $\ddot{\varphi} + \Omega^2(\lambda - \cos\varphi)\sin\varphi - \Omega^2\varepsilon\cos\varphi = 0$

A* $\qquad$ B*

Figure 9 – **A*** Equivalent of potential energy $\mathbf{E}_p(\phi)$ graph of basic ideal mechanical system (**A*****a***) correspond to no ideal wich Coulomb's type friction and phase trajectory portrait (**A*****b***), (**A*****c***) and (**A*****d***) for $\alpha_0 = 0$ and different values of the $k = \dfrac{1}{\lambda} = \left|\dfrac{g}{\ell\Omega^2}\right| \substack{\leq \\ \geq} 1$ and axis eccentricity. Two sets (**A*****b***) and (**A*****c***) of the of the singular points: $\phi_s = s\pi$, $s = 1, 2, 3, 4, \ldots$ and $\phi_s = \arccos\dfrac{g}{\ell\Omega^2} \pm 2s\pi$, $s = 1, 2, 3, 4, \ldots$ for $k = \dfrac{1}{\lambda} = \left|\dfrac{g}{\ell\Omega^2}\right| \leq 1$. One set (**A*****d***) of singular points $\phi_s = s\pi$, $s = 1, 2, 3, 4, \ldots$ for $k = \dfrac{1}{\lambda} = \left|\dfrac{g}{\ell\Omega^2}\right| \geq 1$. **B*** Equivalent of potential energy $\mathbf{E}_p(\phi)$ graph of basic ideal mechanical system (**B*****a***) correspond to no ideal wich Coulomb's type friction and homoclinic phase trajectory layering (**B*****b***), (**B*****c***) and (**B*****d***) for $\alpha_0 = 0$ and different values of the $k = \dfrac{1}{\lambda} = \left|\dfrac{g}{\ell\Omega^2}\right| \leq 1$ and axis eccentricity.

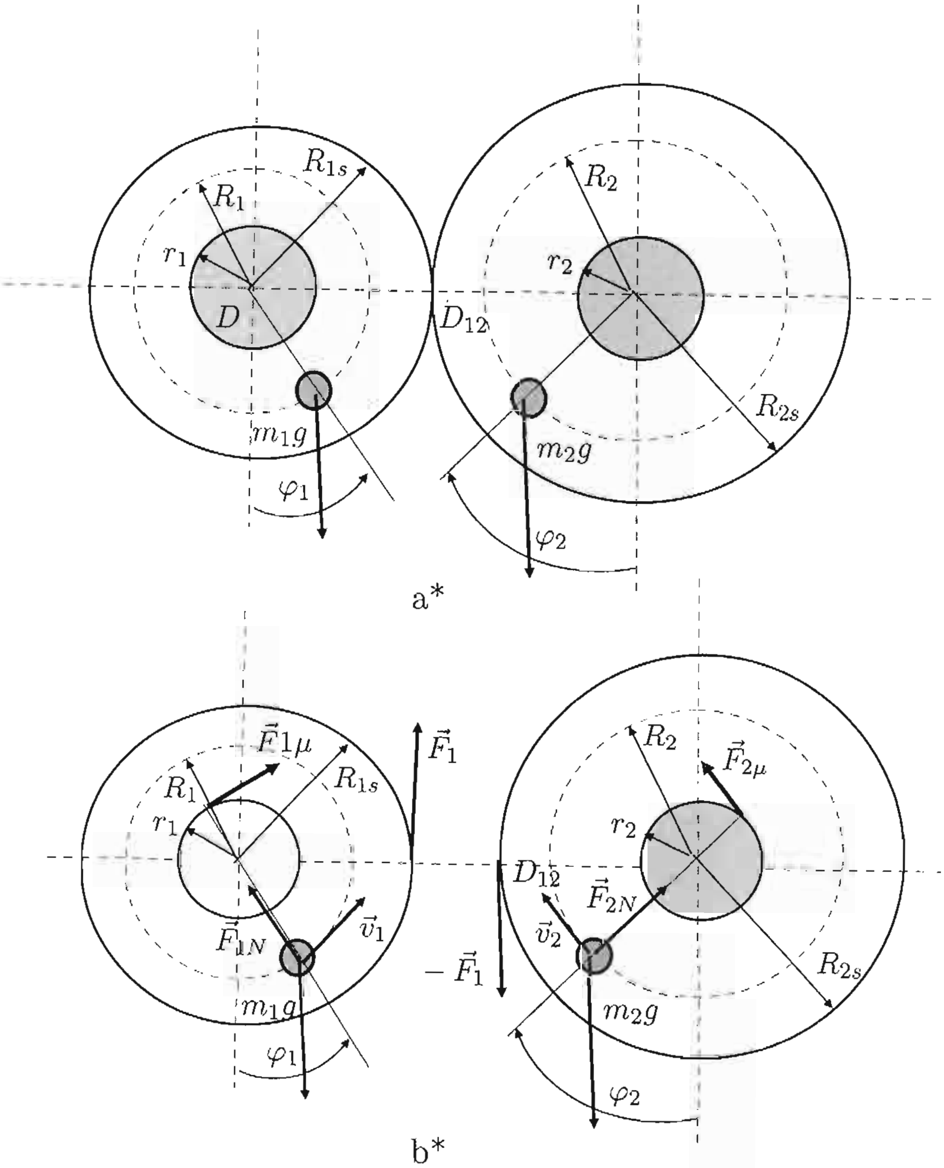


Figure 10 - Coupled double rotor system (a*) with Coulomb's type friction into contact surfaces between discs and shafts; Decomposition (b*) of the system with plan of the Coulombs type friction forces

between rotor shafts and cylindrical bearings take the following form:

$$
\begin{aligned}
& mR^3\left[\frac{1}{k}+\lambda p^2\right]\ddot{\phi} \pm \mu m r R^2\left[\cos k\phi - k\lambda p^2\cos\phi\right]\ddot{\phi} + \\
& + mgR\sin\phi\left(\frac{R}{k}\pm\mu r\cos k\phi\right) - \lambda p m g R_2 \sin k\phi (R \mp \mu r\cos\phi) \mp \\
& \mp \mu r m g\left[\cos\phi + \frac{R}{g}\dot{\phi}^2\right]\left(\frac{R}{k}\pm\mu r\cos k\phi\right) \mp \\
& \mp \mu\lambda r m g\left[\cos k\phi + \frac{pR}{g}k^2\dot{\phi}^2\right](R\mp\mu r\cos\phi) = 0.
\end{aligned}
\tag{22}
$$

For $k = 2$ and for ideal constraints (by neglecting friction) previous differential equation obtain the following form:

$$\ddot{\phi} + \frac{g}{R(1+2\lambda p^2)}(1 - 4\lambda p \cos\phi)\sin\phi = 0. \tag{23}$$

Obtained differential equations (23) is in the class (13) and than on the basis of the listed theorem of trigger of coupled singularities in chapter VII.2.1. we can conclude that non-linear dynamics in basic system when condition $\frac{1}{4\lambda p} \leq 1$ is satisfied appear a trigged of coupled singularities and in the phase trajectory portrait appear a homoclinic orbit in the form of number eight. Sets of singularities are: first $\phi_s = s\pi$ and second

$$\phi_s = \arccos\frac{1}{4\lambda p} \quad \text{for} \quad 4\lambda p \geq 1,$$

presented in Figure 9.

VII.3. Concluding remarks

Systems with coupled multi-step rotors are important for engineering applications, then it is important to investigate ideal as well as no ideal system nonlinear dynamics. Also, no stability in the working processes of like that system dynamics caused higher level of noise and vibrations. Present Coulomb's type frictions in these kind of system dynamics caused new instability and more higher level of noise and vibrations. This is reason that is important to investigate non-linear phenomena in dynamics of other corresponding ideal as well as no ideal system dynamics. Also, system with vibro-impacts are important for engineering practice. Vibro-impacts are strong nonlinearity with discontinuities in the system kinetic parameters and alternations of the forced and velocities directions in coparison before and after impacts..

These processes and their singularities are described by one or more differential double equations and corresponding boundary and impact conditions.

VII.4. Open directions for next research and applications

For previous listed reasons, open directions for research are:

- investigate of phase trajectory portraits of the models of the coupled multi-rotor system dynamics, corresponding singular points, trigger of coupled singularities and non-linear phenomena in vibrations;

- nonlinear double differential equations describig different engineering model dynamics with frictions and no ideal constraints;
- vibro-impact system dynamics;
- Dynamics of hybrid system with complex structure.

VIII. General concluding remarks

For limited length of paper, now we made only some comments concerning the following

∗ Lissajous' curves, orthogonal asynchronous and synchronous oscillations, asynchronization and synchronization of subsystem in hybrid system dynamics.

Series of **Lissajous curves as well as new series of the generalized Lissajous curves** obtained by software MathCad as a results of the coupled orthogonal multi-frequency oscillations are suitable for to build a method of asynchronization/synchronization for applications to the discrete continuum for synchronization some parts of discrete continuum. By this method based on attractors of asynchronization/synchronization of the component oscillations of the subsystems of hybrid system is possible and suitable for two obtain conditions of the integrity of the dynamical system. Generalized **Lissajous curves** can be used as attractors of asynchronization/synchronization of the component oscillations which are coupled as that these oscillations are orthogonal. By changing some parameters of the coupled oscillators synchronization and by use current software tools as it is MathCad (or MathLab or Mathematica), the visualization of the transformation of the generalized **Lissajous curve**, up to its degeneration into part of straight line, can be obtain as results of the orthogonal coupling of oscillatory multi-frequency signals. If this degeneration is not possible, then these oscillators it is not possible to synchronized and corresponding parameter is not parameter of synchronization. If as results of the change of some parameters of the coupled oscillators synchronization is transformation of the generalized **Lissajous curve** into one unique line then it is possible to obtain system parameters of the attractor of partially synchronization or asynchronization of the coupled oscillators.

Also, there are some models of the discrete continuum in plane or in the space, which mass particles moves oscillatory as result of coupled two in plane, or three in space, orthogonal multi-frequency oscillations Trajectories of this mass particles are generalized **Lissajous curves**. Applications of the knowledge about generalized **Lissajous curves** is important for constructions of some processing machines with working processes based on the motion of the coupled orthogonal multy frequency vibrations.

Ito's stochastic differential equations and applications to stochastic oscillations of mechanical systems with hereditary properties are also actual mathematical task for open possibilities in engineering practice, as well as in other area of scence.

Mathematical analogy and phenomenological mapping by use mathematical models in applications to disparate physical models dynamics open very large interactions between different area of science and easier transfer knowledge from one area of science to other.

Also, one of main education task of Serbian mathematicians and other university professor to fined minimum volume of the classical and new current mathematical knowledge necessary to be in the programs of Ph.D study enough for mathematical background of new Ph.D specialist for their next two decade research and possibility to accept new and future mathematical discovery and be competent to applied these new mathematical knowledge in research and practice, as well as to define new mathematical tasks appear from his research and to directed to mathematicians for future research.

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O MATEMATIČKIM METODAMA MEHANIKE I NJIHOVIM PRIMENAMA (MMMP) ILI MEHANIKA IZMEĐU DVE VATRE: MATEMATIKA I INŽENJERSTVO

Katica R. (Stevanović) Hedrih[1]

Apstrakt. Prikazuje se izbor matematičkih metoda mehanike na primerima primena za rešavanje zadataka mehanike realnih sistema koji su apstrahovani do teorijskih modela mehaničkih diskretnih ili kontinualnih sistema, kao i hibridnih sistema. Predavač usmerava deo predavanja i na metode i rezultate u ovoj oblasti svojih učitelja Mitropoljskog, Anđelića i Raškovića, kao i na sopstvene, originalne rezultate i dostignuća u oblasti teorijske i primenjene mehanike i nelinearne dinamike, te i na rezultate svojih magistranata i doktoranata u oblasti primena matematičkih metoda u mehanici.

Sadržaj predavanja:

1. Vektorska metoda, vektori momenata masa i vektori rotatori, njihove osobine, vizuelizacija i primene u dinamici spregnutih rotacija krutih tela. Devijacioni spregovi.

2. Asimptotske metode nelinearne mehanike Krilov–Bogoljubov–Mitropoljski i njihove primene u nelinearnoj dinamici deformabilnih tela i dinamici hibridnih sistema složenih struktura.

3. Integro-diferencijalne jednačine i njihova primena u razvoju analitičke dinamike diskretnih naslednih sistema Gorosko-Hedrih.

4. Diferencijalne jednačine frakcionog (necelog) reda i njihove primene u razvoju analiticke dinamike diskretnih i kontinualnih sistema frakcionog reda. Glavne koordinate i glavni, slični višefrekventnim, kao i jednofrekventnim modovima, modovi frakcionog reda.

5. Metoda diskretnog kontinuuma – glavne površinske mreže, glavni lanci i glavni modovi sistema ravanskih i prostornih sistema spregnutih

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mehaničkih lanaca (slučajevi idealno elastičnih, naslednih ili frakcionog reda) – primene u mehanici kontinuuma (modeli oscilacija deformabilnih tela), u biomehanici (oscilacije modela dvostrukih helikoidnih DNK lanaca), primene na spregnute sisteme klatna.

6. Lissajous-ove krive, ortogonalne asinhrone oscilacije, asinhronizacija i sinhronizacija podsistema hibridnih sistema.

7. Ugaone brzine obrtanja baznih vektora tangentnog prostora vektora položaja materijalnih tačaka mehaničkog sistema s integrabilnim vezama – trodimenzionalni tangentni prostori i njihova proširenja.

8. Itô-ova stohastička diferencijalna jednačina i primena na slučajne oscilacije mehaničkih sistema s naslednim svojstvima.

9. Metoda fazne ravni i njene primene na nelinearnu dinamiku sistema. Primeri sistema i faznih trajektorija.

10. Tenzorski račun i njegove primene u mehanici diskretnih i kontinualnih sistema.

11. Metoda funkcije kompleksne promenljive i primene u teoriji elastičnosti i mehanici fluida.

12. Naučno računanje i fenomeni nelinearne dinamike: čudni atraktor, bazen atrakcije, rezonantni skok, triger spregnutih singulariteta, homokliničke orbite oblika osmice. Teorema o trigeru spregnutih singulariteta.

13. Matematička analogija, fenomenološko preslikavanje i matematički modeli u primeni na modele fizički disparatnih sistema.

Tokom predavanja u okviru svakog od datih podnaslova predavač će izložiti i sopstvene originalne doprinose, pored prikaza naznačenih metoda. Izlaganje je osmišljeno tako da ga mogu razumeti i matematičari i mehaničari, i bilo bi dobro da mlađi istraživači iz oblasti matematike i mehanike, kao i doktoranti, u većem broju prisustvuju predavanju te da aktivnim slušanjem i pitanjima učestvuje u radu simpozijuma.

BUHBERGEROV[1] ALGORITAM I VIZUELIZACIJA MONOMIJALNIH IDEALA

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Apstrakt. Razvijena je Java aplikacija za računanje Grebnerove[9] baze pomoću Buhbergerovog algoritma, korak po korak, uz vizuelizaciju monomijalnih ideala.

1. Uvod

Ideja rada je da se napravi veb-aplikacija u Javi koja će rešavati problem određivanja Grebnerove baze, korak po korak, pomoću Buhbergerovog algoritma. Realizovan je algoritam deljenja polinoma po dve ili tri promenljive u tri monomijalna poretka: leksikografskom, gradiranom leksikografskom i gradiranom obrnutom leksikografskom poretku. Svaki korak u određivanju Grebnerove baze je predstavljen i vizuelno kao skup multistepena monoma koji pripadaju idealu generisanom vodećim monomima Grebnerove baze. Algoritam deljenja je realizovan u programskim jezicima Java i Python. Dvodimenzionalni prikaz je realizovan korišćenjem Java paketa Swing. Trodimenzionalni prikaz je realizovan korišćenjem biblioteke JOGL.

2. Kratak matematički pregled

U ovoj sekciji kratko ćemo se podsetiti matematičkih pojmova koji su nam potrebni za realizaciju aplikacije.

Neka je $\mathbb{K}[x_1, x_2, \dots, x_n]$ prsten polinoma n promenljivih. Proizvod

$$X^\alpha = \prod x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0,$$

nazivamo *monomom*, a multi-indeks α *multi-stepenom monoma. Polinom*

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[9] Gröbner

po promenljivim $x_1, x_2, \ldots, x_n$ s koeficijentima u polju $\mathbb{K}$ predstavlja se kao izraz oblika $\sum_\alpha a_\alpha X^\alpha$, gde je samo konačno mnogo koeficijenata $a_\alpha \in \mathbb{K}$ različito od 0. Preciznije, linearna kombinacija monoma s koeficijentima u polju $\mathbb{K}$ je polinom po n promenljivih. *Totalni stepen monoma* X^α, u oznaci $|\alpha|$, definišemo na sledeći način $|\alpha| = \sum_{k=1}^{n} \alpha_k$.

Monomijalni poredak je relacija totalnog poretka $\succeq$ na skupu $\mathbb{N}_0^n$ ako važi:

1. $(\forall \alpha, \beta, \gamma \in \mathbb{N}_0^n) \alpha \succeq \beta \Rightarrow \alpha + \gamma \succeq \beta + \gamma$,
2. relacija $\succeq$ je relacija dobrog uređenja, tj. svaki neprazan podskup ima minimalni element.

Leksikografski poredak na skupu $\mathbb{N}_0^n$ definišemo na sledeći način:

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_{LEX} (\beta_1, \beta_2, \ldots, \beta_n) \Leftrightarrow$$
$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n) \vee (\exists k \leq n)(\forall i < k) \alpha_i = \beta_i \wedge \alpha_k > \beta_k$$

Znači, $\alpha \succeq_{LEX} \beta$ ako i samo ako je $\alpha = \beta$ ili je prva koordinata različita od nule u vektoru razlike $\alpha - \beta \in \mathbb{Z}_0^n$ pozitivna. Dati poredak se prenosi na skup monoma prstena polinoma sa n promenljivih, tj. važi $X^\alpha \succeq_{LEX} X^\beta \Leftrightarrow \alpha \succeq_{LEX} \beta$. *Relacija leksikografskog poretka* $\succeq_{LEX}$ *je relacija monomijalnog poretka.*

Gradirani leksikografski poredak na skupu $\mathbb{N}_0^n$ definišemo kao:

$$\alpha \succeq_{GRLEX} \beta \Leftrightarrow$$
$$|\alpha| = \sum_{k=1}^{n} \alpha_k > \sum_{k=1}^{n} \beta_k = |\beta| \vee (|\alpha| = |\beta| \wedge \alpha \succeq_{LEX} \beta).$$

Dakle, prvo upoređujemo totalne stepene, a zatim prelazimo na leksikografski poredak. Dati poredak se prenosi na skup monoma prstena polinoma sa n promenljivih, tj. važi $X^\alpha \succeq_{GRLEX} X^\beta \Leftrightarrow \alpha \succeq_{GRLEX} \beta$. *Relacija gradiranog leksikografskog poretka* $\succeq_{GRLEX}$ *je relacija monomijalnog poretka.*

Obrnuti leksikografski poredak na skupu $\mathbb{N}_0^n$ definišemo na sledeći način:

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_{RELEX} (\beta_1, \beta_2, \ldots, \beta_n) \Leftrightarrow$$
$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n) \vee (\exists k \leq n)(\forall i > k) \alpha_i = \beta_i \wedge \alpha_k < \beta_k.$$

Znači, $\alpha \succeq_{RELEX} \beta$ ako i samo ako je $\alpha = \beta$ ili je poslednja koordinata različita od nule u vektoru razlike $\alpha - \beta \in \mathbb{Z}_0^n$ negativna. Dati poredak se prenosi na skup monoma prstena polinoma sa n promenljivih, tj. važi $X^\alpha \succeq_{RELEX} X^\beta \Leftrightarrow \alpha \succeq_{RELEX} \beta$. *Relacija obrnutog leksikografskog poretka*

$\succeq_{RELEX}$ *nije relacija monomijalnog poretka.* Relacija obrnutog leksikografskog poretka $\succeq_{RELEX}$ jeste relacija totalnog poretka za koju važi uslov (1) iz definicije monomijalnog poretka, ali nije dobro uređenje.

Gradirani obrnuti leksikografski poredak na skupu $\mathbb{N}_0^n$ definišemo sa:

$$\alpha \succeq_{GREVLEX} \beta \Leftrightarrow$$
$$|\alpha| = \sum_{k=1}^{n} \alpha_k > \sum_{k=1}^{n} \beta_k = |\beta| \vee (|\alpha| = |\beta| \wedge \alpha \succeq_{RELEX} \beta).$$

Dakle, prvo upoređujemo totalne stepene a zatim prelazimo na obrnuti leksikografski poredak. Poredak se prenosi na skup monoma prstena polinoma sa n promenljivih, tj. važi $X^\alpha \succeq_{GREVLEX} X^\beta \Leftrightarrow \alpha \succeq_{GREVLEX} \beta$. *Relacija gradiranog obrnutog leksikografskog poretka* $\succeq_{GREVLEX}$ *je relacija monomijalnog poretka.*

Jedno od osnovnih pitanja koje se postavlja pri proučavanju ideala prstena polinoma više promenljivih jeste pod kojim uslovima polinom f pripada idealu $I =< f_1, f_2, \dots, f_k >$, koji je generisan konačnim skupom polinoma $\{f_1, f_2, \dots, f_k\}$. Neka su polinomi $f, f_1, f_2, \dots, f_k$ dati u istom monomijalnom poretku $\succeq$. Maksimalan term polinoma f u datom monomijalnom poretku označavamo sa $LT(f)$ i nazivamo *vodećim termom*, a maksimalan monom označavamo sa $LM(f)$ i nazivamo *vodećim monomom.* Primenom *algoritma deljenja*:

Input: $f, f_1, f_2, \dots, f_k$

Output: $a_1, a_2, \dots, a_k, r$

$a_1 := 0; a_2 := 0; \dots ; a_k := 0; r := 0$

$h := f;$

while $h \neq 0$ if $(\exists i) LT(f_i) \mid LT(h)$

then for min index i such that $LT(f_i) \mid LT(h)$

$a_i := a_i + LT(h)/LT(f_i)$

$h := h - LT(h)/LT(f_i) \cdot f_i$

else

$r := r + LT(h)$

$h := h - LT(h),$

polinom f možemo zapisati u obliku $f = \sum_{i=1}^{k} a_i \cdot f_i + r$. Pritom treba istaći da u polinomu r ne postoji term koji je deljiv sa $LT(f_1), LT(f_2), \dots, LT(f_k)$.

Osobine ovog algoritma su sledeće:

1. polinom r nije jedinstveno određen;
2. ako je $r = 0$, onda $f \in I =< f_1, f_2, \dots, f_k >$;
3. $r \neq 0$ ne implicira da f ne pripada I.

Da bi se ovaj problem rešio treba pokušati da se nađe baza ideala I takva da važi sledeće:

1. polinom r je jedinstveno određen;
2. $r = 0$ ako i samo ako $f \in I$.

Takva baza ideala I naziva se *Grebnerova baza.*

Neka su f i g polinomi prstena $\mathbb{K}[x_1, x_2, \dots, x_n]$. *S-polinom polinoma f i g* definišemo sa

$$S(f,g) = \frac{NZS(LM(f), LM(g))}{LT(f)} \cdot f - \frac{NZS(LM(f), LM(g))}{LT(g)} \cdot g,$$

gde $NZS(LM(f), LM(g))$ označava monom čiji je multistepen jednak maksimumu po komponentama multistepena monoma $LM(f)$ i $LM(g)$.

Neka je ideal I generisan skupom polinoma $G = \{g_1, g_2, \dots, g_k\}$. Skup G je Grebnerova baza ideala I ako i samo ako je ostatak pri deljenju $S(g_i, g_j)$ sa G jednak nuli za svako i i j.

Proizvoljnu bazu $\{f_1, f_2, \dots, f_k\}$ ideala I *Buhbergerovim algoritmom*:

Input: $F = \{f_1, f_2, \dots, f_k\}$ skup generatora ideala I

Output: $G = \{g_1, g_2, \dots, g_m\}$ Grebnerova baza ideala I

$G := F$;

repeat

$G' := G$

for $\{p, q\} \subseteq G', p \neq q$ do

$S := S(p, q)$

$h := REM(S; G')$

if $h \neq 0$ then $G := G \cup \{h\}$

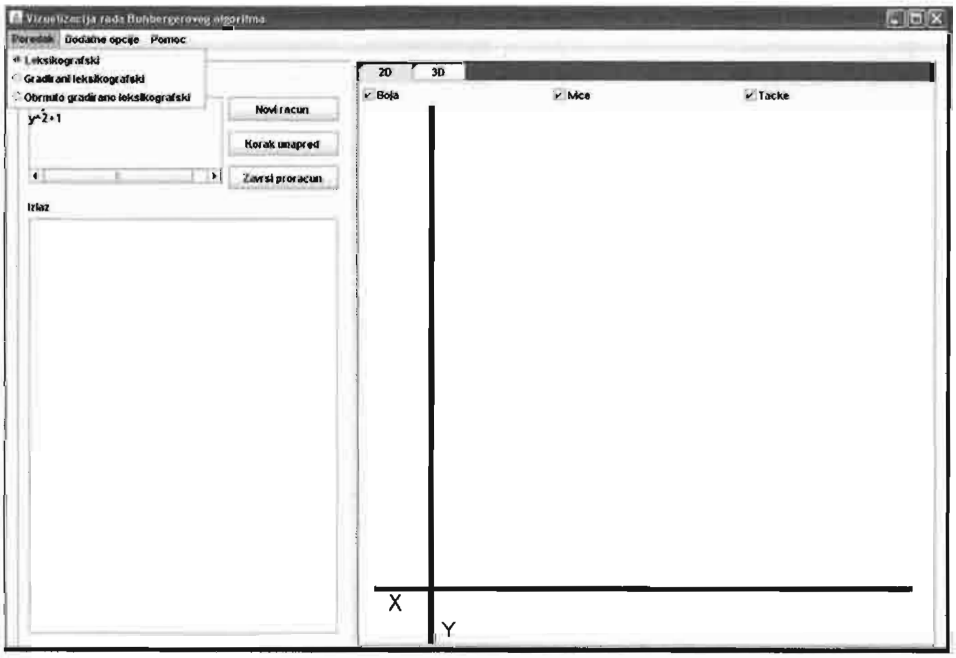
until G=G'

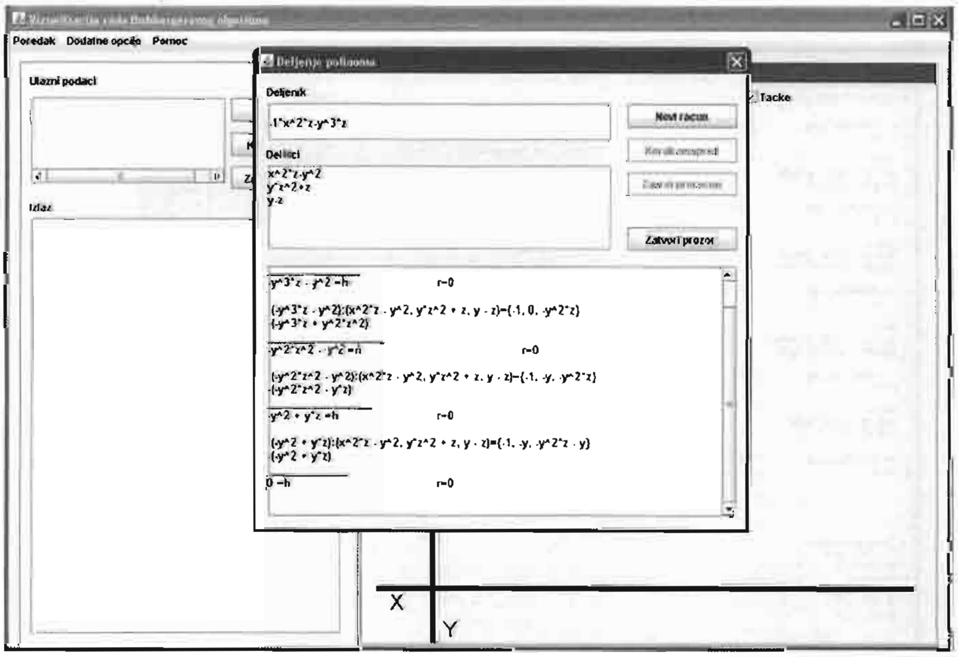
možemo proširiti do Grebnerove baze.

3. Programska realizacija

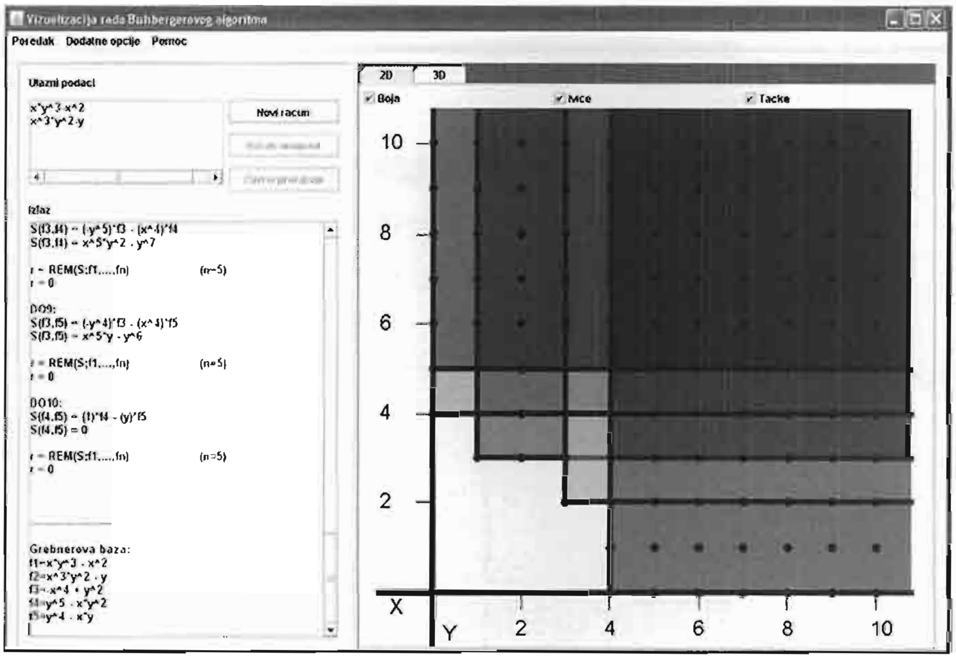
Prilikom kreiranja aplikacije naglasak je stavljen na vizuelno predstavljanje rada Buhbergerovog algoritma. Rukovanje polinomima je izvedeno pomoću klasa biblioteke Sympy.

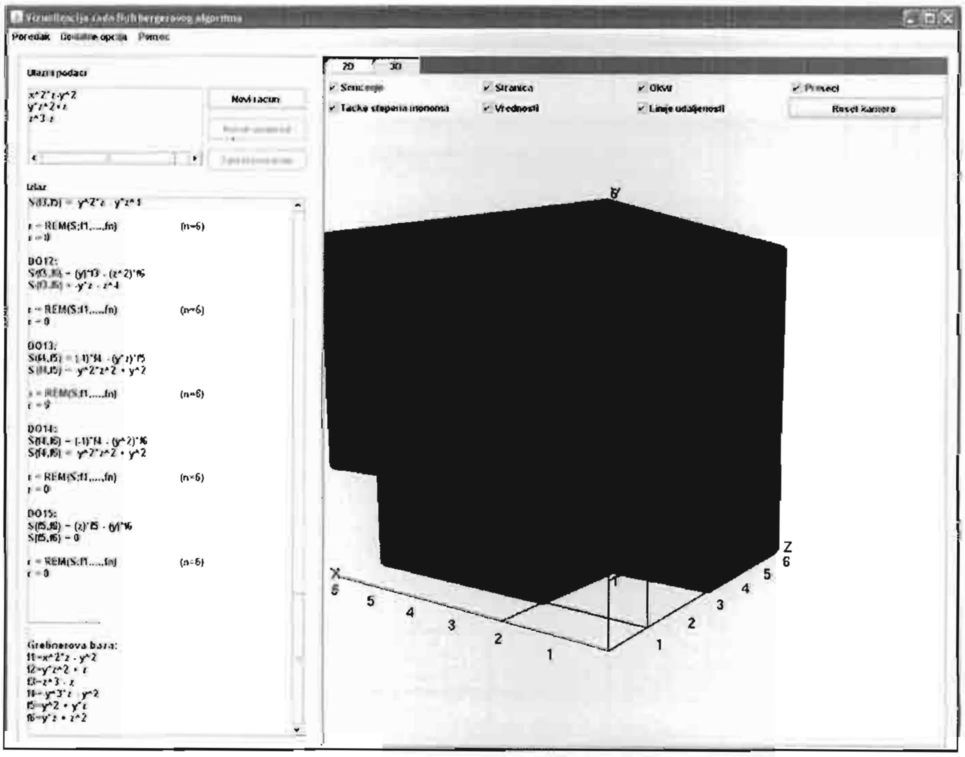
Algoritam deljenja je realizovan po navedenom pseudokodu u programskim jezicima Java i Python. Korišćena je programska biblioteka Simpy, u kojoj su realizovane osnovne strukture podataka i operacije za rad s polinomima. Ulaz algoritma su osnovni polinom i niz polinoma delilaca, svi u određenom monomijalnom poretku. Izlaz algoritma je niz polinoma koeficijenata i polinom ostatak u istom monomijalnom poretku. Rezultat zavisi od redosleda polinoma delilaca u ulaznom nizu. Algoritam na početku inicijalizuje vrednosti koeficijenata i ostatka na nula polinome. Promenljiva h se inicijalizuje na osnovni polinom. U svakom koraku se ažuriraju vrednosti polinoma h, tako da je algoritam konačan. Evo prikaza grafičkog interfejsa:





Buhbergerov algoritam je realizovan korak po korak uz prateću vizuelizaciju u Javi. Ostatak pri deljenju polinoma nizom polinoma preuzet je iz algoritma deljenja korišćenjem komunikacije Sympy Java. Ulaz algoritma jesu polinomi koji čine proizvoljnu bazu datog ideala. Izlaz algoritma je Grebnerova baza dobijena prema Buhbergerovom algoritmu. Vizuelizacijom formiranja monomijalnog ideala vodećih termova dobija se u 2D/3D geometrijsko tumačenje razloga zaustavljanja rada Buhbergerovog algoritma.





Grafičko predstavljanje objekata u Javi bilo je najveći izazov. Trenutno je dostupno vrlo malo biblioteka koje bi mogle pomoći u tome. Zbog toga je paket za grafičko predstavljanje objekata u Javi kreiran od početka. Sam sistem je načinjen tako da koristi openGL api. Svaki od matematičkih elemenata koji su predstavljeni u okviru aplikacije prikazan je kao neka vrsta grafičke primitive. Takođe su implementirani algoritmi za ispitivanje postojanja i određivanja pozicije elemenata koji se iscrtavaju. Formirani su algoritmi za utvrđivanje preseka granica skupova čiji su elementi multistepeni odgovarajućih monoma i algoritam za ispitivanje da li je celokupan skup definisan nekim od multistepena monoma uključen u skup nekih drugih multistepena. Osim zbog potrebe da se odrede elementi za iscrtavanje, ti algoritmi su ugrađeni i kako bi se smanjio skup elemenata koji se prikazuju.

4. Primer u prstenu $\mathbb{K}[x, y]$

Ovde je dat primer koji se može realizovati korak po korak pomoću opisane Java aplikacije. Svaki od koraka se ispisuje u navedenom obliku i propraćen je odgovarajućom slikom.

Neka su dati polinomi $f_1 = xy + 1$ i $f_2 = y^2 + 1$ u leksikografskom poretku $\succeq_{LEX}$.

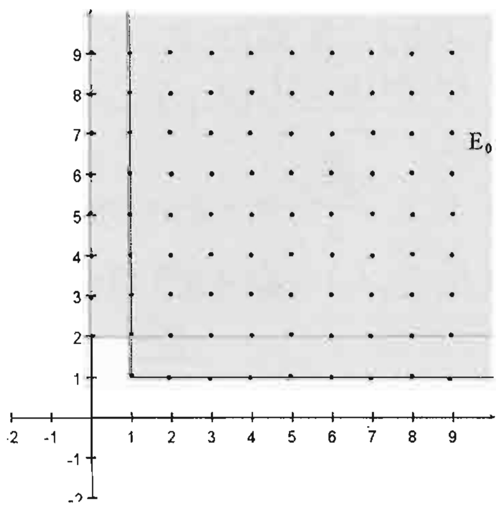
1. Formirati skup $E_0 \subseteq \mathbb{N}_0^2$ takav da važi

$$(\alpha, \beta) \in E_0 \Leftrightarrow x^\alpha y^\beta \in \langle LT(f_1), LT(f_2)\rangle.$$

2. Primenom Buhbergerovog algoritma odrediti standardnu Grebnerovu bazu G ideala $I = \langle f_1, f_2\rangle$.

3. Formirati skup $E_1 \subseteq \mathbb{N}_0^2$ takav da važi

$$(\alpha, \beta) \in E_1 \Leftrightarrow x^\alpha y^\beta \in \langle LT(I)\rangle.$$



- $x^\alpha y^\beta \in \langle LT(f_1), LT(f_2)\rangle = \langle xy, y^2\rangle \Leftrightarrow (\alpha,\beta) \in \{(1,1) + $
  $+ (p,q) \mid p,q \in \mathbb{N}_0\} \cup \{(0,2) + (p,q) \mid p,q \in \mathbb{N}_0\}$
- $F = \{f_1, f_2\}$, $G = \{f_1, f_2\}$

Korak 1:

$$S(f_1,f_2) = \frac{LCM(xy,y^2)}{xy}(xy+1) - \frac{LCM(xy,y^2)}{y^2}(y^2+1) =$$

$$= \frac{xy^2}{xy}(xy+1) - \frac{xy^2}{y^2}(y^2+1) = xy^2 + y - xy^2 - x = -x + y$$

$$S(f_1,f_2) : (f_1,f_2) = (-x+y) : (xy+1, y^2+1) = (0,0)$$

$$\frac{0}{-x+y}$$

$$f_3 = REM(S(f_1,f_2); f_1, f_2) = -x + y, \quad G = \{f_1, f_2, f_3\}$$

Korak 2:

- $S(f_1,f_2) = -x + y$

$$S(f_1,f_2) : (f_1,f_2,f_3) = (-x+y) : (xy+1, y^2+1, -x+y) = (0,0,1)$$

$$\frac{-x+y}{0}$$

$$REM(S(f_1,f_2); f_1, f_2, f_3) = 0$$

- $S(f_1,f_3) = \dfrac{LCM(xy,x)}{xy}(xy+1) - \dfrac{LCM(xy,x)}{-x}(-x+y) =$

$$= \frac{xy}{xy}(xy+1) - \frac{xy}{-x}(-x+y) = xy + 1 - xy + y^2 = y^2 + 1$$

$$S(f_1,f_3) : (f_1,f_2,f_3) = (y^2+1) : (xy+1, y^2+1, -x+y) = (0,1,0)$$

$$\frac{y^2+1}{0}$$

$REM(S(f_1,f_3); f_1, f_2, f_3) = 0$

- $S(f_2,f_3) = \dfrac{LCM(y^2,x)}{y^2}(y^2+1) - \dfrac{LCM(y^2,x)}{-x}(-x+y)$

$$= \frac{xy^2}{y^2}(y^2+1) - \frac{xy^2}{-x}(-x+y) = xy^2 + x - xy^2 + y^3 = x + y^3$$

$$S(f_2,f_3) : (f_1,f_2,f_3) = (x+y^3) : (xy+1, y^2+1, -x+y) = (0,y,1)$$

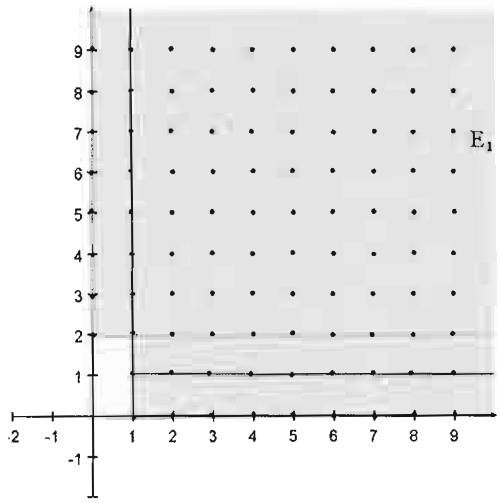
$$\frac{y^3+y}{x-y}$$

$$\frac{-x+y}{0}$$

$REM(S(f_2, f_3); f_1, f_2, f_3) = 0$

$G = \{f_1, f_2, f_3\}$ je Grebnerova baza ideala I.

- $x^\alpha y^\beta \in \langle LT(I) \rangle = \langle xy, y^2, -x \rangle \Leftrightarrow (\alpha, \beta) \in E_0 \cup \{(1,0) + (p,q) \mid p, q \in \mathbb{N}_0\}$



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RAZVOJ I ARHITEKTURA APLIKACIJA ZA *Android* OPERATIVNI SISTEM I *Eurobank EFG Glider* MOBILNA APLIKACIJA

Nikola Milenković[1]

Apstrakt. Poslednjih nekoliko godina svedoci smo ogromnog buma mobilnih uređaja (pametnih telefona i tableta). U isto vreme se razvijaju i aplikacije za te uređaje, kojih ima sve više.

Veliki broj kompanija poseduje svoju mobilnu aplikaciju, a takođe i veliki broj developera, zbog toga što je razvoj za takve uređaje maksimalno olakšan *API*-jem, koji nudi operativni sistem za mobilne telefone.

U ovom radu će biti reči o mobilnoj aplikaciji *Eurobank EFG Glider* koja je napravljena za *Android* operativni sistem, i koja se može besplatno skinuti sa *Android Marketa*.

Takođe će biti i reči o arhitekturi aplikacije i daljim planovima razvoja, kao i o prednostima koje takav vid bankarstva donosi.

1. Android

Šta je *Android*? *Android* je operativni sistem namenjen prenosivim uređajima kao što su mobilni telefoni i tableti. Zasnovan je na jezgru *Linux* operativnog sistema, a razvija ga kompanija *Google*.

Zbog licence koju ima *Linux*, sav softver koji je zasnovan na istom mora biti besplatan i otvoren, pa ni *Android* nije izuzetak. *Android* je moguće preuzeti sa sajta www.android.com kao virtuelni disk koji je moguće pokrenuti pod virtuelnom mašinom, a takođe je tu dostupan i izvorni kod samog *Android*-a.

Prva verzija operativnog sistema je postala dostupna u septembru 2008. godine. Pre toga je postojala beta verzija koja je kompanijama i programerima davala uvid u mogućnosti *Android* operativnog sistema i *Android* platforme.

Postoji više verzija *Android* operativnog sistema koje se danas koriste.

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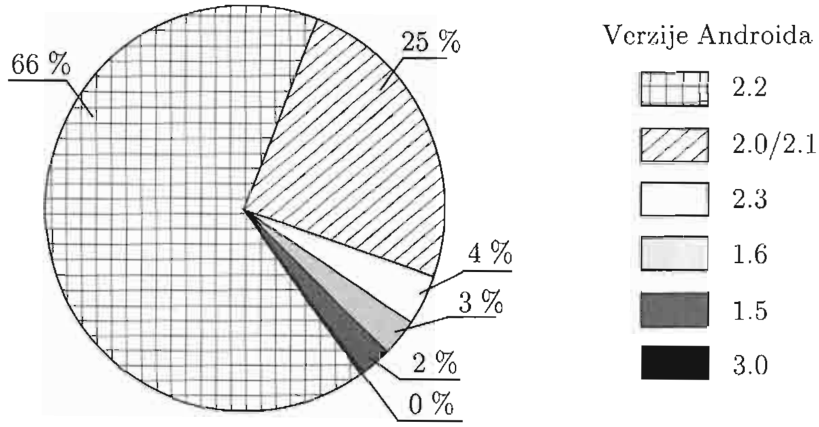
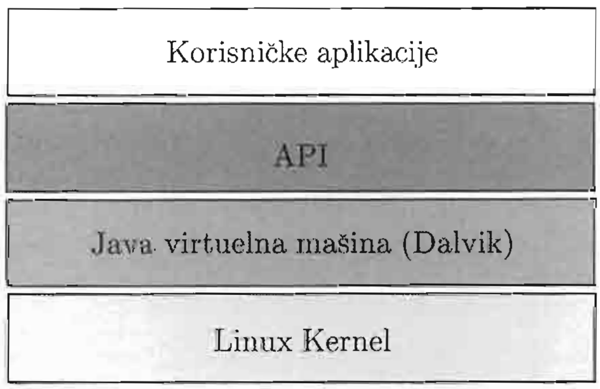
Verzija 2.x je rezervisana za pametne telefone, dok je verzija 3.x rezervisana za tablet uređaje. Najzastupljenija je verzija 2.2, što se vidi na slici. Različite verzije stvaraju problem programerima, kao i korisnicima, a rešenje tog problema *Google* vidi u verziji 2.4, koja će biti namenjena i pametnim telefonima i tablet uređajima.



Slika 1 – Zastupljenost verzija Androida

Arhitektura Android uređaja

Android operativni sistem sastoje se od sledeća četiri sloja:



Slika 2 – Arhitektura Android operativnog sistema

Linux Kernel je sloj koji direktno komunicira s hardverom uređaja i pruža *Java* virtuelnoj mašini interfejs ka tom hardveru. Osim toga, *kernel* se bavi i menadžmentom resursa, tj. dodeljivanjem memorije, procesorskim vremenom i kontrolom pristupa uređajima.

***Java* virtuelna mašina** (poznata još i pod imenom *Dalvik*) jeste modifikacija *Java* virtuelne mašine za *PC* računare. Namenjena je specijalno *Android* operativnom sistemu i svi programi koje korisnik piše se izvršavaju na njoj. Takođe, vodi računa o pravima koja korisnik ima za pristup određenim podacima i hardveru.

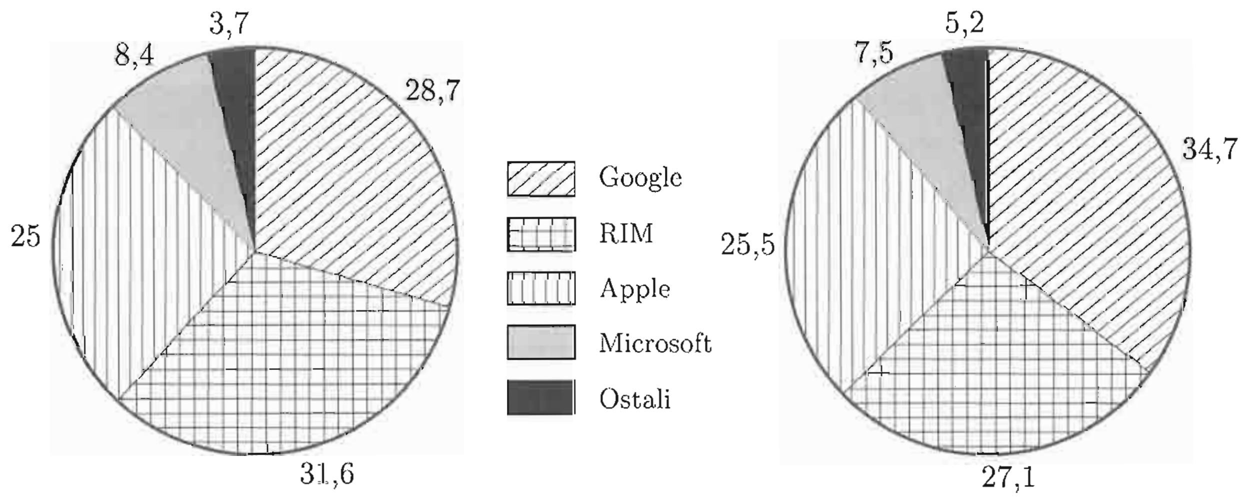
API (*Application Programming Interface*) jeste interfejs koji pruža *Dalvik* virtuelna mašina. On programerima omogućava da koriste pogodnosti koje *Android* pruža, od interfejsa za hardver (akcelerometar, ekran, *GPS* itd.), pa do pristupa internetu, bilo preko bežične, bilo preko mobilne mreže.

Korisničke aplikacije su aplikacije koje iskorišćavaju mogućnosti koje *API* nudi. Na primer, aplikacija za mape koristi internet i *GPS* uređaj, kompas aplikacija koristi akcelerometar, igrice koriste grafičko procesiranje itd. Prilikom korišćenja *Android* uređaja korisnici uglavnom imaju kontakt s korisničkim aplikacijama.

Konkurencija

Trenutno u svetu raste potražnja za pametnim mobilnim telefonima. To su, osim *Google*-a, uvidele i druge kompanije, pa su razvile svoje operativne sisteme za pametne telefone i tablete. Te kompanije su *Microsoft* (*Windows Phone* 6 i 7), *Apple* (*iOS* za *iPhone* i *iPad*) i *RIM* (*Blackberry*).

Trenutno je *Android* najpopularniji operativni sistem za pametne telefone, što se može videti na grafikonu. Takođe se može videti i da *Android* zauzima sve veći i veći deo tržišta, najviše na račun *RIM*-a.



Slika 3 – Tržišni udeo operativnih sistema za decembar 2010. (levo) i mart 2011. (desno)

Jedan od glavnih razloga popularnosti *Android*-a jeste njegova cena. Kompanije koje proizvode telefone mogu besplatno da preuzmu operativni sistem, preurede ga prema svojim potrebama, instaliraju na svoje telefone, koje će potom prodavati po znatno nižim cenama od konkurencije.

Programiranje za Android

Programiranje za Android zahteva sledeće: [1]:

1. poznavanje OOP programiranja i *Jave*,
2. poznavanje *Android* platforme,
3. *Eclipse* razvojno okruženje,
4. *Android SDK* i dodatak za *Eclipse* koji omogućava *Android* programiranje.

S obzirom na to da se radi o *Java* programskom jeziku, nije iznenađujuće što je taj programski jezik neophodno znati za *Android* programiranje. Postoji i alternativa za one koji znaju *C#* programski jezik: *Mono Droid* [2].

Nezavisno od jezika, neophodno je znati šta sve korisniku pruža *Android* platforma. *Google* je dokumentovao [3] ceo interfejs, tako da programeri mogu lako da se snađu, čak i ako nisu nikada ranije imali kontakta s *Android* programiranjem.

Google je odabrao *Eclipse* kao okruženje u kome će se *Android* aplikacije razvijati. Pored razvojnog okruženja potrebno je imati i *Android SDK*, u kome se nalaze svi neophodni alati za kompajliranje/pokretanje/testiranje *Android* aplikacija.

3. Eurobank EFG Glider

Eurobank EFG je razvila bankarsko-poslovnu *Android* aplikaciju koja za cilj ima da pruži dodatne servise korisnicima koji koriste *Android* telefone. Aplikaciju mogu da koriste svi, ne samo klijenti banke.

Aplikaciju je moguće besplatno preuzeti sa *Android Market*-a [4], koji je inače glavni način distribucije aplikacija.

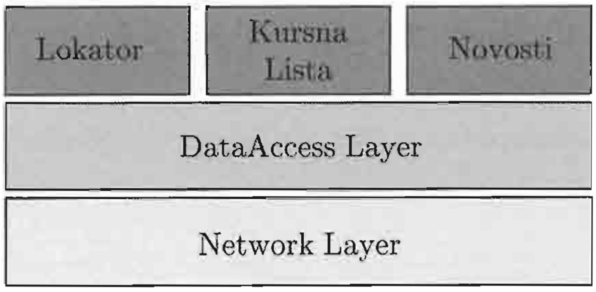
Aplikacija ima sledeće funkcionalnosti:

- Lokator bankomata i ekspozitura,
- Kalkulator za kredite,
- Kursnu listu,
- Naručivanje poziva od banke,
- Novosti,
- Podrška za srpski i engleski jezik.

Slika 4 – Izgled aplikacije

Arhitektura aplikacije

Aplikacija je napravljena iz više slojeva, kao što je to prikazano na slici 5.



Slika 5 – Arhitektura *Eurobank EFG Glider Android* aplikacije

Najniži sloj je mrežni sloj (*Network Layer*), koji preko interneta komunicira sa serverom *Eurobank EFG.* Na tom serveru se čuvaju najsvežiji podaci neophodni za rad aplikacije – dnevna kursna lista, kamatne stope za različite kredite, novosti itd. Kad korisnik zahteva da se podaci osveže, mrežni sloj izvrši upit nad serverom banke i parsirane podatke vrati sloju za podatke (*Data Access Layer*), koji ih pohrani u lokalnu bazu. Sloj za podatke u tom trenutku obavesti odgovarajuću komponentu da je došlo do promene u podacima i podaci se ponovo prikažu tako što ih određena komponenta ponovo zatraži.

Dalji razvoj

Usled sve veće popularnosti *Android* platforme, kao i rasta popularnosti pametnih telefona, *Eurobank EFG* ima u planu da nastavi razvoj svoje *Glider* aplikacije. Neke od funkcionalnosti koje su planirane u budućnosti jesu:

1. Uvid u stanje računa,
2. Blokiranje kartica,
3. Plaćanja na predefinisane račune,
4. Prebacivanje sredstava na proizvoljne račune.

Pored unapređenja postojeće *Android* aplikacije, u planu je i razvijanje aplikacije za druge platforme. Time bi aplikacija pokrila veći deo tržišta i, što je još važnije zadovoljila bi klijente koji su navikli na aplikacije na pametnim telefonima.

Literatura

[1] Meier, R., *Professional Android 2 Application Development*, Wrox, 2010.

[2] „Mono for Android", [na mreži]. Available: http://www.mono-droid.com/

[3] „Android documentation", Google [na mreži]. Available: http://developer.android.com/sdk/index.html

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REŠAVANJE DVOSTEPENOG PROBLEMA INSTALACIJE NEOGRANIČENIH KAPACITETA PRIMENOM GENETSKOG ALGORITMA

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Mentor: dr Zorica Stanimirović

Rezime. U radu je razmotren dvostepeni lokacijski problem instalacije neograničenih kapaciteta. Genetski algoritam koji rešava taj problem koristi jednu modifikaciju turnirske selekcije koja se naziva fino gradirana turnirska selekcija. U pitanju je uopštenje klasične turnirske selekcije koje čuva sve njene dobre karakteristike. Uz korišćenje tog genetskog algoritma rešavanje početnog problema može dati zavidne rezultate i za nešto veće ulazne instance. Analizirani su i upoređeni rezultati implementacije tog genetskog algoritma s rezultatima dobijenim korišćenjem CPLEX-a.

1. Definicija problema

Neka je N skup terminala, M – skup mogućih lokacija za koncentratore prvog nivoa i K – skup mogućih lokacija za koncentratore drugog nivoa. Za svaki od terminala data je cena pridruživanja tog terminala koncentratoru prvog nivoa, za svaki od lokacija koncentratora prvog nivoa data je cena instalacije koncentratora na toj lokaciji i cena pridruživanja svakom koncentratoru drugog nivoa, a za sve lokacije koncentratora drugog nivoa data je cena instalacije koncentratora na tom mestu. Cilj je minimizovati cenu instaliranja datih koncentratora i pridruživanja terminala datim koncentratorima, tako da svaki terminal bude pridružen nekom koncentratoru prvog nivoa, koji će dalje biti dodeljen nekom od koncentratora drugog nivoa.

Preciznije, matematički, dvostepeni lokacijski problem instalacije neograničenih kapaciteta formuliše se na sledeći način. Potrebno je minimizovati funkciju cilja koja je oblika

$$\sum_{i\in N}\sum_{j\in M} C_{ij}x_{ij} + \sum_{j\in M}\sum_{k\in K} B_{jk}y_{jk} + \sum_{k\in K} F_k z_k$$

a pritom imati u vidu sledeći skup ograničenja:

$$\sum_{j\in M} x_{ij} = 1, \quad \forall i \in N, \tag{1}$$

$$x_{ij} \leq \sum_{k\in K} y_{jk}, \quad \forall i \in N, \quad \forall j \in M, \tag{2}$$

$$y_{jk} \leq z_k, \quad \forall j \in M, \quad \forall k \in K, \tag{3}$$

$$\sum_{k\in K} y_{jk} \leq 1, \quad \forall j \in M, \tag{4}$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in N, \quad \forall j \in M, \tag{5}$$

$$y_{jk} \in \{0,1\}, \quad \forall j \in M, \quad \forall k \in K, \tag{6}$$

$$z_k \in \{0,1\}, \quad \forall k \in K. \tag{7}$$

gde je C_{ij} cena pridruživanja terminala i koncentratoru prvog reda na lokaciji j, B_{jk} – suma cene postavljanja koncentratora prvog nivoa na lokaciji j i cene njegovog pridruživanja koncentratoru drugog nivoa na lokaciji k, a F_k – cena postavljanja koncentratora drugog reda na mestu k. Za sve $i \in N$, $j \in M$, $k \in K$, promenljive su definisane na sledeći način:

$$x_{ij} = \begin{cases} 1, & \text{terminal } i \text{ je dodeljen koncentratoru prvog reda na lokaciji } j, \\ 0, & \text{inače,} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{koncentrator prvog reda na lokaciji } j \text{ dodeljen je} \\ & \text{koncentratoru drugog reda na lokaciji } k, \\ 0, & \text{inače,} \end{cases}$$

$$z_k = \begin{cases} 1, & \text{koncentrator drugog reda je instaliran na lokaciji } k, \\ 0, & \text{inače.} \end{cases}$$

Značenja uslova su sledeća: (1) – svaki terminal se pridružuje koncentatoru prvog reda na tačno jednoj lokaciji; (2) – ako je dati terminal pridružen koncentratoru prvog reda, taj koncentrator mora biti instaliran i pridružen bar nekom koncentratoru drugog reda; (3) – ako je koncentrator prvog reda pridružen koncentratoru drugog reda, taj koncentrator drugog reda mora biti instaliran; (4) – koncentrator prvog reda može biti dodeljen najviše jednom koncentratoru drugog reda; (5), (6), (7) – uslovi su očigledni.

Pokazano je, na primer, u [6] da je dvostepeni lokacijski problem instalacije neograničenih kapaciteta NP–težak. Dakle, za sada se ne zna da li se može konstruisati egzaktan polinomijalni algoritam, pa su iz praktičnih razloga razvijene neke od heuristika za ovaj problem. Preciznije, široko je analiziran njemu sličan (jednostepeni) lokacijski problem instalacije neograničenih kapaciteta.

Kod lokacijskog problema instalacije neograničenih kapaciteta, za dati skup terminala N i koncentratora M potrebno je minimizovati cenu pridruživanja terminala nekim koncentratorima i instalacije tih koncentratora. Kod matematičke formulacije ovog problema (uz analogne oznake) traži se minimum funkcije

$$\sum_{i \in N} \sum_{j \in M} C_{ij} x_{ij} + \sum_{k \in K} F_j y_j,$$

a pritom treba imati u vidu sledeći skup ograničenja:

$$\sum_{i \in M} x_{ij} = 1, \quad \forall i \in N, \tag{8}$$

$$x_{ij} \leq y_j, \quad \forall i \in N, \quad \forall j \in M, \tag{9}$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in N, \quad \forall j \in M, \tag{10}$$

$$y_j \in \{0,1\}, \quad \forall j \in M. \tag{11}$$

Uslovi (8) i (9) znače da svaki terminal mora biti dodeljen tačno jednom koncentratoru, dok uslov (10) znači da terminal može biti dodeljen koncentratoru samo ako je ovaj instaliran. Uslov (11) je trivijalan.

Razvijene su brojne heuristike koje rešavaju pomenuti jednostepeni lokacijski problem, pa se, zbog sličnosti s dvostepenim lokacijskim problemom instalacije neograničenih kapaciteta, i neke od tih karakteristika mogu primeniti na razmatrani problem.

2. Genetski algoritmi

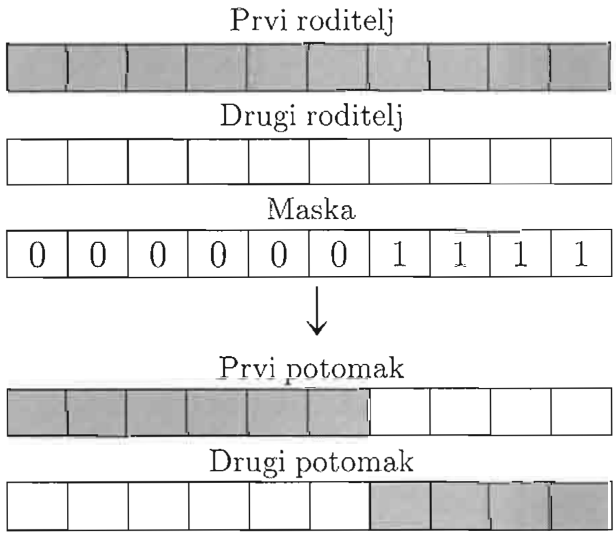
Prema Darvinovoj teoriji evolucije, jedinke unutar jedne populacije se stalno nadmeću za resurse. Jedinke koje su u većoj meri prilagođene okruženju znatno češće preživljavaju i imaju veći uticaj na formiranje novih generacija. Svaka jedinka nove generacije nastaje kombinovanjem genetskog materijala svojih roditelja. Pritom s vremena na vreme može doći do pojave mutacije, što predstavlja slučajnu promenu genetskog materijala. Idejnim tvorcem genetskih algoritama smatra se Holand, mada su se prvi radovi o njima pojavili pre njega. Oni su bazirani na Darvinovoj teoriji evolucije i imitirajući prirodni proces prilagođavanja u stanju su da reše dati problem.

Osnovna konstrukcija je populacija jedinki. Svaka jedinka predstavlja moguće rešenje u pretraživačkom prostoru za dati probem (prostoru svih rešenja) i predstavljena je genetskim kodom nad nekom azbukom simbola.

Najčešće se koristi binarno kodiranje, gde se genetski kôd sastoji od niza bitova. Svakoj jedinki se dodeljuje funkcija prilagođenosti koja ocenjuje kvalitet date jedinke, predstavljene kao pojedinačno rešenje u pretraživačkom prostoru. Genetski algoritam treba da bude u mogućnosti da obezbedi način da stalno, iz generacije u generaciju, poboljšava prilagođenost svake jedinke u populaciji, kao i prilagođenost cele populacije. To se postiže uzastopnom primenom genetskih operatora selekcije, ukrštanja i mutacije, čime se dobijaju sve bolja rešenja razmatranog problema. Nekad je poželjno i izvršiti izbor određenog broja najboljih jedinki i direktno ih preneti u narednu generaciju. Ta pojava se naziva elitizam.

Mehanizam selekcije favorizuje natprosečno prilagođene jedinke i njihove natprosečno prilagođene delove – gene, koji dobijaju veću šansu za sopstvenu reprodukciju pri formiranju nove generacije. Slabije prilagođene jedinke i geni imaju smanjene šanse za reprodukciju, pa postepeno izumiru.

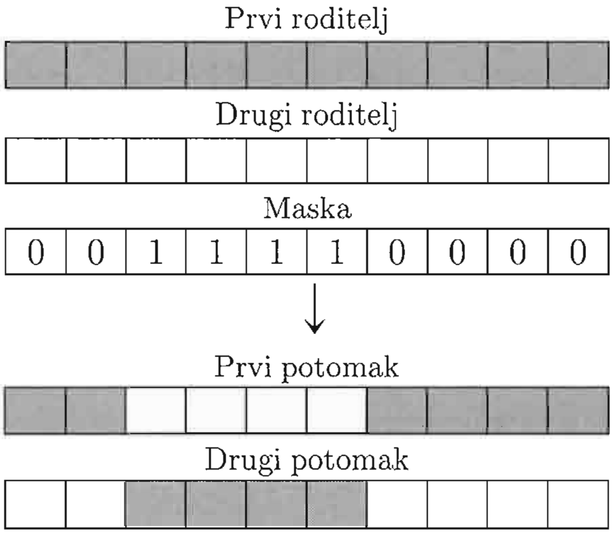
Operator ukrštanja vrši rekombinaciju gena jedinki i time doprinosi raznovrsnosti genetskog materijala. Rezultat ukrštanja je nedeterministička razmena genetskog materijala između jedinki, s mogućnošću da dobro prilagođene jedinke generišu još bolje jedinke. Ovim mehanizmom i relativno slabije prilagođene jedinke, s dobro prilagođenim genima, dobijaju svoju šansu da rekombinacijom dobrih gena proizvedu dobro prilagođene jedinke. U osnovnoj verziji algoritma se na slučajan način bira tačka ukrštanja, tj. pozicija u kodovima roditelja u odnosu na koju će simboli u ta dva koda razmeniti mesta (slika 1).



Slika 1 – Ukrštanje (jedna tačka ukrštanja)

Operator ukrštanja može biti definisan i u odnosu na dve tačke ukrštanja (slika 2).

Višestrukom primenom selekcije i ukrštanja moguće je gubljenje genetskog materijala, to jest postaju nedostupni neki regioni pretraživačkog prostora. Mutacija vrši slučajnu promenu određenog gena, s datom malom verovatnoćom p_{mut}, čime je moguće vraćanje izgubljenog genetskog materijala u populaciju. To je osnovni mehanizam za sprečavanje preuranjene konvergencije genetskog algoritma u lokalnom ekstremu.



Slika 2 – Ukrštanje (dve tačka ukrštanja)

Početna populacija se često generiše na slučajan način, što doprinosi raznovrsnosti genetskog materijala. U nekim slučajevima se povoljnije pokazalo generisanje cele početne populacije ili dela populacije nekom drugom pogodno izabranom heuristikom. Jedini preduslov je da vreme izvršavanja date heuristike bude relativno kratko. Na osnovu prethodnog, šematski bi se jednostavni genetski algoritam mogao predstaviti pseudokodom, kao na slici 3.

```
učitavanje_ulaznih_podataka();
inicijalizacija_populacije();
while (!kriterijum_završetka())
{
  for (i=0; i<br_pop; i++)
    p[i] = vrednosna_funkcija();
  funkcija_prilagođenosti();
  selekcija();
  ukrštanje();
  mutacija();
}
stampanje_izlaznih_podataka();
```

Slika 3 – Šematski prikaz prostog genetskog algoritma

Veliki je skup pozitivnih karakteristika koje genetske algoritme izdvajaju od ostalih metoda rešavanja jednog problema: algoritmi operišu nad kodovima kojima su predstavljeni parametri, a ne nad samim paremetrima; pretraga se vrši u nekoliko različitih tačaka (populacije); promena nije deterministička, već je zasnovana na zakonima verovatnoće. Razni su i načini poboljšanja osnovne verzije genetskog algoritma, a ovde će biti razmotren jedan način unapređivanja jedne vrste selekcije – turnirske selekcije.

3. Turnirska selekcija

Turnirska selekcija je jedan od najpopularnijih načina selekcije. Razvitkom paralelnih genetskih algoritama njena popularnost je u poslednjem periodu još više izražena. U turnirskoj selekciji svaki element populacije koji ima bolju funkciju prilagođenosti od nekoliko ostalih proizvoljno odabranih elemenata populacije bira se za prelazak u narednu generaciju. Parametar selekcije je veličina turnira, označimo je N. Važi da je $N \in \mathbb{N}$. Algoritam se može prikazati pseudokodom, kao na slici 4.

```
Ulaz:
populacija A (veličine br_pop)
veličina turnira N

Izlaz:
populacija A' (veličine br_pop)

Algoritam:
for (i=0; i<br_pop; i++)
  A'[i] = element populacije A koji ima
          bolju funkciju prilagođenosti
          među njenih N proizvoljno
          odabranih elemenata;
return A';
```

Slika 4 – Turnirska selekcija

Vremenska složenost ovog algoritma je $O(Nn)$ gde je n broj elemenata populacije, br_pop. Međutim, obično je N veoma malo u odnosu na n, pa je složenost linearna i iznosi $O(n)$. Kod turnirske selekcije nije potrebno sortiranje, koje je obično prisutno kod nekih od načina selekcije. Međutim, izvršavanjem algoritma se često dešava da se sam proces odvija veoma sporo ili pak dolazi do prerane konvergencije. To se može izbeći korišćenjem tzv. fino gradirane turnirske selekcije, koja predstavlja unapređenje prethodnog algoritma.

Umesto celobrojnog parametra N (veličina turnira) koristi se parametar $F \in \mathbb{R}$ – željena prosečna veličina turnira. Od tog parametra zavisi proces selekcije, pa bi prosečna veličina turnira u populaciji trebalo da bude što bliža njegovoj vrednosti. Baš kao i kod obične turnirske selekcije, element populacije je izabran ako ima bolju funkciju prilagođenosti od svojih proizvoljno odabranih konkurenata. Međutim, u ovom slučaju, tokom jednog koraka selekcije postojaće različite vrednosti turnira. Pri implementaciji ovog algoritma (slika 5) razlike veličinâ turnirâ su manje ili jednake 1 i izabrane su tako da je njihova prosečna veličina što bliža parametru F.

```
Ulaz:
populacija A (veličine br_pop)
željena prosečna veličina turnira F

Izlaz:
populacija A' (veličine br_pop)

Algoritam:
F_manje = trunc(F);
F_vece = trunc(F) + 1;
sr_pop = trunc(n * (1 - frac(F)));
for (i=0; i<sr_pop; i++)
  A'[i] = element populacije A koji ima
          bolju funkciju prilagođenosti
          među njenih F_manje  proizvoljno
          odabranih elemenata;
for (i=sr_pop; i<br_pop; i++)
  A'[i] = element populacije A koji ima
          bolju funkciju prilagođenosti
          medju njenih F_veće proizvoljno
          odabranih elemenata;
return A';
```

Slika 5 – Fina turnirska selekcija

Ovaj algoritam zadržava sve dobre karakteristike obične turnirske selekcije i otklanja pomenute nedostatke. Ovaj metod ima svoj analogon pri odvijanju prirodnih procesa. Teorijski aspekti ovog operatora selekcije se mogu pronaći u [3]. Tu je i pokazano da je klasična turnirska selekcija specijalan slučaj fino gradirane turnirske selekcije.

4. Rešenje problema

Za kodiranje promenljivih iskorišćen je binarni kôd, koji se za ovakve vrste problema donekle prirodno nameće. Najpre je izvršena inicijalizacija populacije, i to pohlepnim algoritmom ili na slučajan način. Verovatnoća odabira jednog od načina iznosi 0,5. Za svaki terminal se tako određuje koncentrator prvog reda kome se on pridružuje, a za svaki od pridruženih koncentratora prvog reda se opet na neki od tih načina bira njemu odgovarajući pridruženi koncentrator drugog reda. Na taj način odgovarajuće promenljive imaju vrednost 1, a ostale 0 (analogno kao u matematičkoj formulaciji problema). Svakom nizu promenljivih koje predstavljaju kodove pridružena je i odgovarajuća vrednost funkcije prilagođenosti.

Uzeto je da je broj elemenata populacije 160, dok je ukupan broj generacija 10000. Osnovna ideja implementacije algoritma se može prikazati i pseudokodom, kao na slici 3. Nakon učitavanja ulaznih podataka vrši se inicijalizacija populacije, a zatim se, dok god kriterijum završetka ne označi kraj algoritma, u jednoj iteraciji izvršavaju, redom, selekcija, ukrštanje i mutacija. Na kraju se vrši štampanje izlaznih podataka.

Tabela 1 – Razultat i vremena izvršavanja na manjim test instancama[1]

N	M	K	Rezultat[1]	t (CPLEX)	t (GA_1)	t (GA_2)
5	5	5	4,306	3,213	0,623	0,617
5	5	5	5,208	3,117	0,551	0,566
5	5	5	4,994	3,338	0,778	0,721
5	5	5	5,123	3,117	0,714	0,751
10	10	10	11,362	6,446	1,517	1,521
10	10	10	10,306	7,009	1,221	1,113
10	10	10	9,508	6,431	1,094	0,984
10	10	10	9,223	6,212	1,212	1,117
15	15	15	13,840	12,517	2,133	2,021
15	15	15	15,625	11,496	2,006	2,140
15	15	15	14,536	13,088	1,889	1,743
15	15	15	14,903	13,964	1,603	1,587

Ulazni podaci su matrica C, koja predstavlja cene pridruživanja terminala odgovarajućim koncentratorima prvog reda; matrica B koja predstavlja sumu instalacije koncentratora prvog reda na datom mestu i cene njegovog pridruživanja odgovarajućem koncentratoru drugog reda; niz F koji označava cene instalacije koncentratora drugog reda na datim lokacijama. Kriterijum zaustavljanja je zadovoljen ukoliko je prekoračen maksimalan broj generacija. Izlazni podaci se ispisuju na standardni izlaz. Ispisuje se vrednost minimalne cene, a zatim i podaci kom koncentratoru prvog reda su pridruženi odgovarajući terminali, a kom koncentratoru drugog reda su pridruženi odgovarajući koncetratori prvog.

[1]Na manjim test instancama CPLEX i obe verzije genetskog algoritma dali su optimalno rešenje.

Za operator selekcije iskorišćena je unapređena verzija turnirske selekcije, fino gradirana turnirska selekcija. Pokazuje se da ovo unapređenje daje bolje rezultate u razmatranom problemu. Za parametar fino gradirane turnirske selekcije uzeto je $F = 5,5$, a obične $N = 5$. Rezultati su prikazani u tabelama 1 i 2.

Tabela 2 – Razultati i vremena izvršavanja na većim test instancama[2]

N	M	K	GA_1	$t(GA_1)$	GA_2	$t(GA_2)$
50	50	50	52,388	15,994	51.004	16,301
100	100	100	104,416	32,007	102,103	33,640
150	150	150	138,413	42,221	135,201	38,113
200	200	200	208,333	60,223	206,986	62,419
300	300	300	303,813	82,211	299,410	88,188
400	400	400	416,738	131,084	414,556	125,513
500	500	500	491,693	155,011	488,220	160,106
150	100	50	97,926	32,981	93,513	34,292
200	100	50	116,320	35,598	110,516	38,220
300	150	50	147,313	51,270	146,251	53,334
400	200	100	220,833	74,336	219,318	78,216
500	300	100	296,310	91,073	293,554	96,730

Kad je u pitanju elitizam, polovina jedinki je direktno preneta u narednu generaciju. S jedne strane, to ubrzava proces algoritma, budući da odgovarajuće vrednosti promenljivih za te jedinke ne moraju biti ponovo računate.

Preostali deo nove generacije će biti dobijen na osnovu operatora ukrštanja. Za taj operator, roditelji se biraju nasumično. U radu je korišćeno trostruko jednopoziciono ukrštanje. Takođe, korišćena je prosta mutacija sa zaleđenim bitovima s verovatnoćom 0,05 pri čemu verovatnoća na zaleđenim bitovima iznosi 0,25.

5. Analiza rezultata

U tabeli 1 prikazano je poređenje rezultata dobijenih korišćenjem CPLEX-a i genetskih algoritama koji koriste običnu i fino gradiranu turnirsku selekciju. Za manje ulazne instance svi rezultati se poklapaju, dok za veće ulazne instance CPLEX nije u mogućnosti da reši problem.

[2] Na većim test instancama CPLEX nije bio u mogućnosti da reši problem.

U tabeli 2 su upoređeni GA1 i GA2. CPLEX ovoga puta nije razmatran, zbog njegove nemogućnosti da generiše rezultat. Na ovim primerima vidi se da je vreme izvršavanja približno isto (jer je i slična implementacija), dok GA2 prikazuje bolje performanse od GA1 pri generisanju rezultata. (Ovde je kao GA1 označen genetski algoritam koji koristi običnu turnirsku selekciju, dok je kao GA2 označen genetski algoritam koji koristi fino gradiranu turnirsku selekciju.)

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ODREĐIVANJE FLUKSA I UGLOVNIH PARAMETARA LJUSKASTIH OSTATAKA SUPERNOVIH

Slobodan Opsenica[1]

Apstrakt. Napisan je program u IDL-u koji za posmatrani ostatak supernove određuje fluks i uglovne parametre.

Uvod

Eksplozije supernovih dešavaju se u sledećim slučajevima [1]:

1) masivna usamljena zvezda došla je do kraja svog životnog ciklusa;

2) beli patuljak u dvojnom sistemu je pretakanjem materije sa zvezde pratioca dostigao masu od 1,44 mase Sunca;

3) dva bela patuljka u dvojnom sistemu su se sudarila.

Kao rezultat eksplozije supernove nastaje neutronska zvezda ili crna rupa (u slučaju da nije došlo do potpunog raznošenja jezgra prilikom eksplozije), kao i ostatak supernove (eng. *supernova remnant*, skr. *SNR*), koji se širi. Energija koja se oslobađa u eksploziji supernove je ogromna – pri eksploziji masivne usamljene zvezde oslobođena kinetička energija je reda veličine 10^{44} džula, a brzina izbačenog materijala dostiže oko 20.000 km/s [2].

Proces nastanka ostatka supernove odvija se na sledeći način: Magnetno polje međuzvezdane sredine skreće protone izbačene eksplozijom s puta i tera ih na kružno kretanje, zbog čega oni stvaraju tanku barijeru. Kada ljuska koja se širi probije tu barijeru, nastaje udarni talas, a samim tim i ostatak supernove. On može trajati i više od milion godina i dostići ogromne dimenzije (oko 300 parseka, tj. oko $9 \cdot 10^{15}$ km). Udarni talas sabija okolno magnetno polje i ubrzava čestice do relativističkih brzina, što su uslovi za jaku sinhrotronsku emisiju [2].

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Modeli ostatka supernove

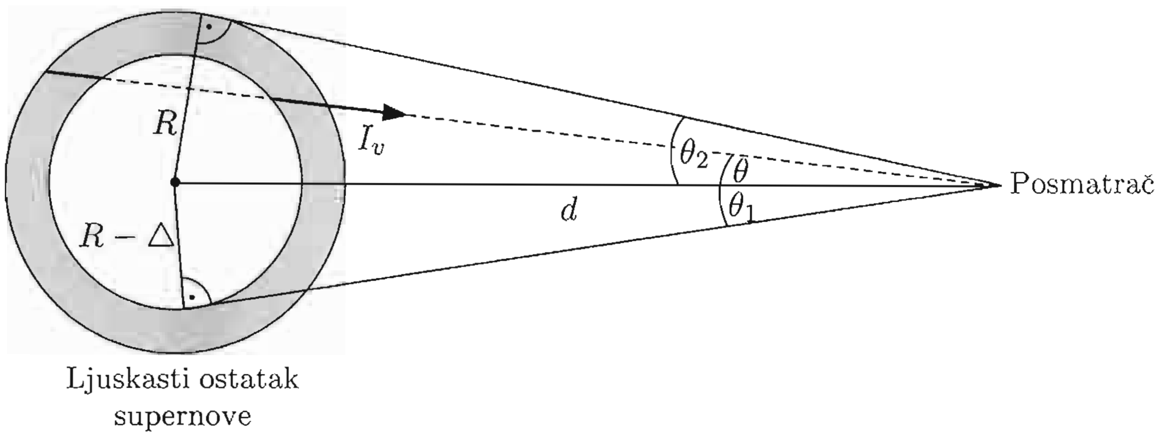
U radu su razmatrani ljuskasti ostaci supernovih. Uzeta su u obzir dva modela: model homogene optički tanke ljuske sa konstantnom emisivnošću i model ljuske s radijalnim magnetnim poljem. Na slici 1 su označene linearne i uglovne dimenzije ostatka supernove. U slučaju modela homogene optički tanke ljuske s konstantnom emisivnošću, intenzitet sinhrotronskog zračenja je određen formulom [3]:

$$l_v = \begin{cases} C_v\left(\sqrt{\sin^2\theta_2 - \sin^2\theta} - \sqrt{\sin^2\theta_1 - \sin^2\theta}\right), & 0 < \theta < \theta_1 \\ C_v\left(\sqrt{\sin^2\theta_2 - \sin^2\theta}\right), & \theta_2 \leq \theta \leq \theta_2 \end{cases}$$

gde je $\theta_1 = \sin^{-1}\dfrac{R-\triangle}{d}$, $\theta_2 = \sin^{-1}\dfrac{R}{d}$, a $C_v = 2\varepsilon_v d$. Emisivnost je označena s ε_v, a u ovom slučaju je $\varepsilon_v = \text{const.}$ Gustina fluksa zračenja u slučaju ovog modela određena je sledećim izrazom [3]:

$$S_v = \frac{2\pi}{3}C_v(\sin^2\theta_1 - \sin^2\theta_1).$$

U slučaju modela ljuske s radijalnim magnetnim poljem, intenzitet sinhrotron-



Slika 1 – Linearne i uglovne dimenzije ljuskastog ostatka supernove

skog zračenja je određen formulom [3]:

$$l_v = \begin{cases} 2C_v\sin\theta\displaystyle\int_{\mu_{1-}}^{\mu_{2-}}(1-\mu^2)^{\frac{\alpha-2}{2}}d\mu, & 0 < \theta < \theta_1 \\ C_v\sin\theta\displaystyle\int_{\mu_{2+}}^{\mu_{2-}}(1-\mu^2)^{\frac{\alpha-2}{2}}d\mu, & \theta_1 \leq \theta \leq \theta_2 \end{cases}$$

Ovde je

$$\mu_{1,2\pm} = \mp\frac{\sqrt{\sin^2\theta_{1,2} - \sin^2\theta}}{\sin\theta_{1,2}}, \quad \theta_1 = \sin^{-1}\frac{R-\triangle}{d}, \quad \theta_2 = \sin^{-1}\frac{R}{d},$$

a $C_v = \tilde{\varepsilon}_v d$. Emisivnost je određena izrazom $\varepsilon_v - \tilde{\varepsilon}_v(\sin\theta')^{\alpha+1}$, gde je θ' ugao između pravca linije sile magnetnog polja i pravca vizure, $\tilde{\varepsilon}_v$ – konstanta proporcionalna indukciji magnetnog polja B, a α – spektralni indeks koji za ostatke supernovih u proseku iznosi oko 0,5. Gustina fluksa zračenja za model ljuske s radijalnim magnetnim poljem određena je sledećim izrazom [3]:

$$S_v = \frac{2\pi\sqrt{\pi}}{3}\frac{\Gamma\left(\dfrac{\alpha+3}{2}\right)}{\Gamma\left(\dfrac{\alpha+4}{2}\right)}C_v(\sin^2\theta_2 - \sin^2\theta_1).$$

Simuliranje posmatranja ljuskastih ostataka supernovih radio-teleskopom pomoću programskog jezika IDL

Slika koja se dobija posmatranjem objekta na nebu radio-teleskopom predstavlja konvoluciju pravog (realnog) intenziteta zračenja objekta i dijagrama snage radio-teleskopa. Ono što se dobija jeste konvoluirani intenzitet:

$$I_v^{conv}(\theta_0, \varphi_0) = \frac{\displaystyle\int_0^{\pi}\int_0^{2\pi} I_v(\theta,\varphi)P_n(\theta')\sin\theta d\theta d\varphi}{\displaystyle\int_0^{\pi}\int_0^{2\pi} P_n(\theta)\sin\theta d\theta d\varphi}.$$

gde je $I_v(\theta,\varphi)$ pravi (realni) intenzitet zračenja objekta, a $P_n(\theta)$ normirani dijagram snage radio-teleskopa. Ugao θ' povezan je sa ostalim uglovnim veličinama sledećom relacijom sferne trigonometrije:

$$\cos\theta' = \cos\theta\cos\theta_n + \sin\theta\sin\theta_n\cos(\varphi - \varphi_n).$$

Ovde je pretpostavljeno da je dijagram snage antene simetričan, tj. da ne zavisi od koordinate φ. Ako se ova konvolucija simulira numerički, mora se uzeti da se dijagram snage antene proteže do nekog graničnog ugla θ_{gr}, zbog tehničkih ograničenja. Za dijagram snage antene uzeta su dva moguća slučaja: aproksimacija gausijanom i aproksimacija Beselovom funkcijom.

1) Aproksimacija gausijanom:

$$P_n(\theta) = \begin{cases} e^{-\alpha\theta^2}, & \theta \le \theta_{gr} \\ 0, & \theta > \theta_{gr} \end{cases},$$

gde je $a = \frac{4 \ln 2}{HPBW^2}$. Širina glavnog snopa dijagrama snage radio-teleskopa na polovini maksimuma snage (eng. *half power beam width*) označena je $HPBW$.

2) Aproksimacija Beselovom funkcijom:

$$P_n(\theta) = \begin{cases} \left[\dfrac{2J_1\left(\dfrac{\pi D}{\lambda}\sin\theta\right)}{\dfrac{\pi D}{\lambda}\sin\theta} \right]^2, & \theta \leq \theta_{gr} \\ 0, & \theta > \theta_{gr}, \end{cases}$$

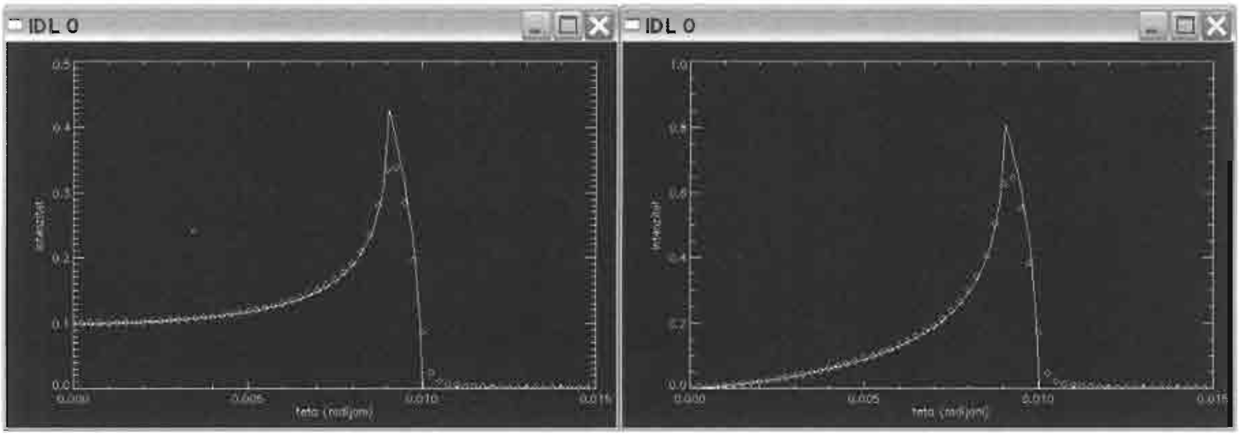
gde je $J_1(x)$ Beselova funkcija prvog reda od argumenta x, D – prečnik antene, a λ – talasna dužina posmatranog zračenja. Takođe, važi relacija: $HPBW = 1,02\lambda/D$ [2].

Konačno, izraz koji se koristi za numeričko simuliranje konvolucije intenziteta zračenja ljuskastog ostatka supernove s dijagramom snage antene glasi:

$$I_v^{conv}(\theta_0) = \frac{\iint I_v(\theta) P_n(\theta') \sin\theta d\theta d\varphi}{2\pi \int_D^{\theta_{gr}} P_n(\theta) \sin\theta d\theta}.$$

Za $I_v(\theta)$ uzima se jedan od izraza iz prethodnog poglavlja, u zavisnosti od toga za koji model ljuskastog ostatka supernove smo se odlučili. Ugao θ' je povezan sa ostalim uglovnim veličinama relacijom sferne trigonometrije: $\cos\theta' = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos\varphi$. Dvojna integracija u brojiocu gornjeg izraza vrši se po preseku objekta i dijagrama snage radio-teleskopa. Taj dvojni integral rešavan je u IDL-u pomoću funkcije `INT_2D`. Jednostruki integral u imeniocu rešavan je pomoću funkcije `QROMB`. Jednostruki integrali koji se pojavljuju u izrazu za intenzitet zračenja u slučaju modela ljuske s radijalnim magnetnim poljem rešavani su pomoću funkcija `QROMB`, `INT_TABULATED`, kao i ručno pisane funkcije koja računa određeni integral pravougaonom metodom.

U slučaju modela homogene optički tanke ljuske s konstantnom emisivnošću korisnik unosi kao ulazne podatke parametre objekta C_v, θ_1, θ_2, kao i parametar antene $HPBW$, a program vrši konvoluciju. U slučaju modela ljuske s radijalnim magnetnim poljem korisnik unosi i parametar α. Na slici 2 prikazane su simulirane konvolucije za dva razmatrana modela.

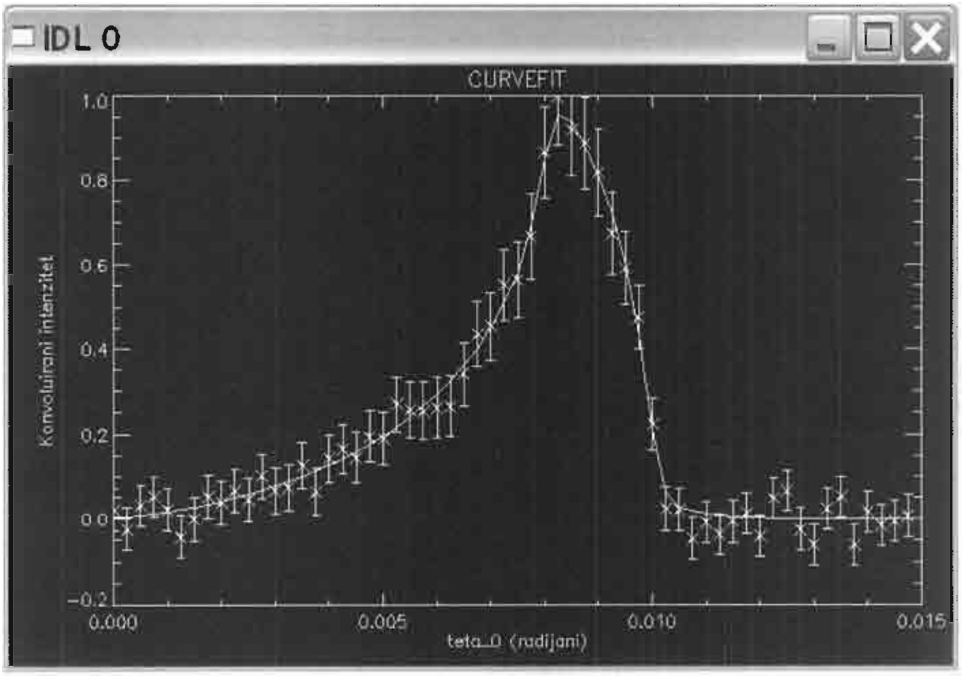


Slika 2 – Simulirani konvoluirani intenziteti zračenja za dva razmatrana modela. Punom linijom predstavljen je pravi (realni) intenzitet zračenja objekta, a dijamantima simulirana konvolucija realnog intenziteta i dijagrama snage, tj. „posmatrani“ intenzitet, u zavisnosti od ugla. Levo: model homogene optički tanke ljuske s konstantnom emisivnošću; desno: model ljuske s radijalnim magnetnim poljem

Obrnuti problem – fitovanje radio-posmatranja teorijskim modelom

Obrnuti problem je sledeći: korisnik unosi u vidu tabele posmatrani intenzitet radio-zračenja ljuskastog ostatka supernove (tj. radijalni profil zračenja ostatka), kao i $HPBW$, a program treba fitovanjem ulaznih podataka odabranim modelom da pronađe najbolje vrednosti za C_v, θ_1 i θ_2 u slučaju prvog modela (s konstantnom emisivnošću) ili za C_v, θ_1, θ_2 i α (ukoliko α već nije poznato iz spektra) u slučaju drugog modela (s radijalnim magnetnim poljem). Ovo je u IDL-u postignuto uz pomoć procedure `CURVEFIT`, koja je iterativna. Da bi moglo da se izvrši fitovanje, korisnik mora da proceni inicijalne vrednosti traženih parametara. Inicijalne vrednosti parametara θ_1 i θ_2 mogu da se procene direktno iz posmatračkih podataka (iz posmatranog radijalnog profila zračenja). Za inicijalnu vrednost spektralnog indeksa α, ukoliko taj parametar nije poznat iz spektra, može se uzeti 0,5. Uz te procenjene vrednosti, inicijalna vrednost parametra C_v može se dobiti korišćenjem posmatračkih podataka i izraza za intenzitet zračenja ostatka supernove (u zavisnosti od odabranog modela). Program uz nalaženje najboljih vrednosti traženih parametara daje i njihove greške, tj. standardne devijacije. Takođe, program računa gustinu fluksa zračenja S_v. Na slici 3 je prikazan primer fitovanog radijalnog profila zračenja ljuskastog ostatka supernove.

Ova metoda određivanja uglovnih parametara i fluksa iz radio-posmatranja ima i svojih nedostataka. Pre svega, pri korišćenju ove numeričke



Slika 3 – Fitovani radijalni profil zračenja ljuskastog ostatka supernove. Znacima „X“ prikazani su posmatrački podaci, tj. radijalni profil zračenja koji korisnik unosi. Puna linija predstavlja najbolji fit tih podataka teorijskim radijalnim profilom u skladu s odabranim modelom. U ovom primeru je uzet model ljuske s radijalnim magnetnim poljem

metode treba imati u vidu sledeće činjenice:

- realni ljuskasti ostaci supernovih nisu u opštem slučaju potpuno sferni;
- u ovoj metodi se koristi idealizovan, a ne realan dijagram snage;
- magnetno polje nije radijalno postavljeno u realnom slučaju;
- osetljivost drugog modela (s radijalnim magnetnim poljem) na parametar α je vrlo slaba, pa se ovaj parametar ne može precizno odrediti pomoću ove metode. Zato se preporučuje njegovo određivanje iz spektra, ako za to postoje mogućnosti.

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NOVA Σ–D RELACIJA PRIMENJENA NA GALAKTIČKE OSTATKE SUPERNOVIH

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1. Uvod

1.1. Ostaci supernovih

Prilikom eksplozije supernove (SN) u okolni prostor preko materijala izbačenog eksplozijom, oslobodi se kinetička energija reda $\sim 10^{51}$ erga. Ostatak supernove (OS)[2] započinje svoj život onda kad se formira udarni talas u interakciji izbačenog materijala s okolnom međuzvezdanom materijom (MZM)[3] i magnetnim poljem. Za tipične vrednosti brzine izbačenog materijala, $\sim 10^4$ km/s, i gustinu sredine od $n_H \sim 1$ cm^{-3} udarni talas se ne bi formirao. Značaj magnetnog polja je u tome što izbačene naelektrisane čestice počinju da kruže oko linija sila i tako stvaraju barijeru između izbačenog gasa SN i MZM. Na opisani način nastaju udarni talas i, samim tim, ostatak supernove. Udarni talas dalje sabija okolno magnetno polje i ubrzava čestice do ultrarelativističkih brzina, što dovodi do jake sinhrotronske emisije.

OS dakle zrače pretežno sinhrotronski, njihov spektar uglavnom je netermalni i dobro se aproksimira kao

$$S_\nu \propto \nu^{-\alpha}, \tag{1}$$

gde je S_ν gustina fluksa zračenja na frekvenciji ν, a α predstavlja spektralni indeks.

Određivanje daljina do OS u Galaksiji predstavlja važan zadatak. Detaljno proučavanje OS zavisilo je najvećim delom od posmatranja u radio-domenu. Ipak, na posmatranja radio-teleskopima utiče nekoliko selekcionih efekata (Case & Bhattacharya, 1998): (1) teška detekcija ostataka zbog malog površinskog sjaja, posebno u regionima gde je pozadina sjajna; (2) neki

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[2]engl. supernova remnant (SNR)

[3]engl. interstellar matter (ISM)

OS ne mogu biti detektovani zbog male ugaone veličine; (3) posmatranja ne pokrivaju nebo ravnomerno (Green, 1991).

Rastojanja do OS mogu direktno biti određena ukoliko se nalaze blizu regiona HI, HII regiona ili molekulskih oblaka, pomoću OB asocijacija zvezda, pulsara ili merenjem optičkih brzina i sopstvenih kretanja. Kada je direktno određivanje rastojanja do OS nemoguće, procena udaljenosti za ljuskaste ostatke vrši se pomoću Σ–D relacije, veze između površinskog sjaja objekta Σ i njegovog dijametra D.

1.2. Ukratko o ovom radu

Glavna tema ovog rada je Σ–D relacija za ostatke supernovih i njena primena na ostatke u Galaksiji. Cilj je izvođenje nove empirijske Σ–D relacije koja će pokazivati bolje slaganje s novijim teorijskim Σ–D relacijama (Berezhko & Volk, 2004).

Za postizanje pomenutog cilja korišćen je uzorak kalibratora iz novog Grinovog kataloga supernovih iz 2009. godine (Green, 2009). U novom katalogu navedena su 274 ostatka iz Galaksije, što je za 59 više nego u katalogu iz 1996. godine (Green, 1996) koji su koristili Kejs i Batačarija (Case & Bhattacharya, 1998) za izvođenje njihove Σ–D relacije. Pored toga što je dat veći broj ostataka, rastojanja do pojedinih su u novom katalogu preciznije određena, korišćenjem boljih instrumenata i metoda.

U radu je opisan metod dobijanja empirijske Σ–D relacije a treba istaći da su razni autori do sada koristili standardnu proceduru fitovanja u Σ–D ravni baziranu na vertikalnoj (paralelno y-osi) χ^2 regresiji. U ovom radu, umesto vertikalne, koristimo ortogonalnu regresiju. Ova promena ima svoje teorijsko opravdanje, a vrednost parametra β je bliža teorijskoj iz rada Bereška i Folka iz 2004. godine.

2. Σ–D relacija

2.1. Definicija i osnovni pojmovi

Σ–D relacija je veza između površinskog sjaja na određenoj frekvenciji (radio-oblast) Σ_ν i dijametra ostatka supernove D. Pomenuta veza najčešće se navodi u obliku

$$\Sigma_\nu(D) = AD^{-\beta}, \tag{2}$$

gde parametar A zavisi od karakteristika eksplozije SN i karakteristika MZM, kao što su energija eksplozije SN, masa izbačenog materijala SN, gustina MZM, jačina magnetnog polja itd., dok je parametar β nezavisan (ili zavisi veoma malo) od pomenutih karakteristika eksplozije (Arbutina & Urošević, 2005).

Kao što je već rečeno, Σ–D relacija je značajna jer opisuje evoluciju OS, ali upotrebljava se vrlo često za određivanje rastojanja do ostataka. Površinski sjaj se dobija iz posmatranja i nezavisan je od rastojanja do objekta. Uglovni dijametar θ se takođe dobija iz posmatranja. Daljina do objekta označena je r, a dijametar D. Daljina do objekta se može dobiti ako se izračuna D iz Σ–D relacije, a prethodno računa Σ iz direktno merene gustine fluksa u Jy.

Iako postoje precizniji, direktni metodi određivanja rastojanja do OS, ovaj metod je i dalje nezamenljiv za veliki broj ostataka (Urošević, 2000). Od oko 270 otkrivenih ostataka u Galaksiji (Green, 2009), samo daljine do oko 80 ostataka su procenjene direktnim metodama. Za procenu udaljenosti do drugih ostataka mora se primeniti Σ–D relacija, mada treba napomenuti da se ta relacija primenjuje samo na ljuskaste ostatke (S tip).

2.2. Teorijske relacije

Prvu teorijsku Σ–D relaciju dobio je Šklovski (Shklovsky, 1960a,b), i prvi ju je iskoristio za određivanje rastojanja do OS (Arbutina et al., 2004). Nakon Šklovskog, mnogo autora bavilo se unapređivanjem i modifikovanjem teorijskih i empirijskih Σ–D relacija. Najnovije i najznačajnije teorije koje će biti navedene u ovom radu su teorija Đurića i Sikvista (Duric & Seaquist, 1986) i Bereška i Folka (Berezhko & Volk, 2004). Odličan pregled teorijskih i empirijskih Σ–D relacija raznih autora dat je u radu Uroševića iz 2000. godine (Urošević, 2000).

2.3. Empirijske relacije

Šklovski (Shklovsky, 1960b) je prvi predložio algoritam za konstruisanje jedne empirijske Σ–D relacije. Ideja je da se preciznije odrede daljine do datih OS, ukoliko je to moguće za te ostatke; onda, znajući uglovne dijametre, možemo dobiti linearne dimenzije. Površinski sjaj se dobija direktno iz posmatranja. Na taj način dobijamo Σ–D dijagram gde svaki ostatak predstavlja jednu tačku (Urosević, 2000). Ti ostaci predstavljaju kalibracione ostatke jer povlačenjem najbolje prave kroz njih dobijamo koeficijente

A i β jednačine (2). Tako dobijena relacija, predstavlja u srednjem najbolju relaciju za sve OS, pa pomoću nje možemo dobiti rastojanja do ostalih ostataka iz merenog površinskog sjaja i uglovnog dijametra.

Prvu empirijsku Σ–D relaciju dali su Poveda i Voltjer (Poveda & Woltjer, 1968). Sikvist (Seaquist, 1968) je ispitivao zavisnost temperature po sjaju T_b od dijametra D jer je poznato da važi $\Sigma_\nu \propto T_b$. Njegov rad može se uzeti kao jedan od pionirskih, mada u njemu nisu računati nikakvi koeficijenti već je nacrtana najbolja prava za sedam kalibracionih ostataka. Prvi rad koji je bio posvećen samo empirijskoj Σ–D relaciji je Milnov (Milne, 1970). Često se baš taj rad u literaturi navodi kao početni jer je u njemu objavljen katalog svih do tada poznatih OS, kao i daljine do svakog ostatka, izračunate pomoću Σ–D relacije. Tokom narednih godina radilo se i dalje na ovoj relaciji, s težnjom da se dođe do preciznije određenog seta kalibratora, a samim tim i do bolje relacije. Bolja posmatranja su povećavala broj kalibratora i doprinela tačnijem određivanje rastojanju do njih.

Najvažnija skorija empirijska Σ–D relacija, najviše citirana u radovima koji se bave OS, jeste relacija Kejsa i Batačarije (Case & Bhattacharya, 1998).

3. Statistička obrada podataka

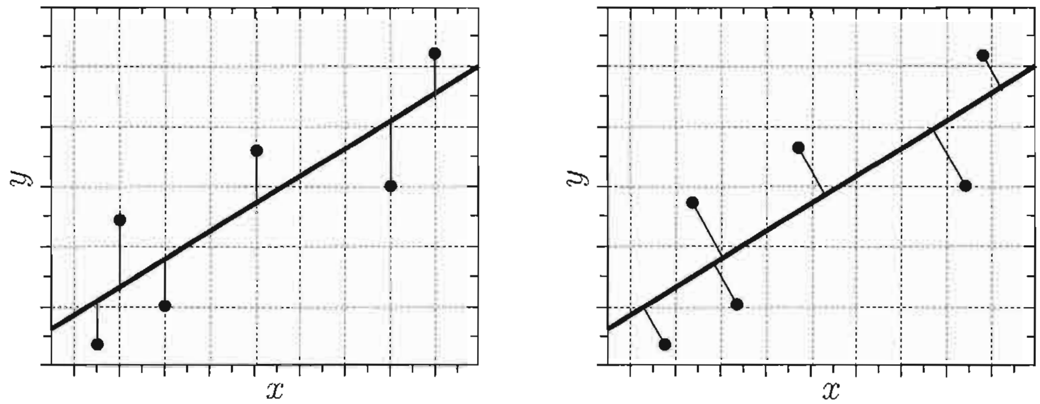
3.1. Σ–D i D-Σ relacije

Grin je u svom radu iz 2005. godine (Green, 2005) istakao, koristeći posmatranja karakteristika uzorka OS da čak i ako se zanemare selekcioni efekti i posmatrački nedostaci, ostaje problem fitovanja proizvoljne stepene zavisnosti $\Sigma \propto D^{-\beta}$. Σ–D relacija ima dve zasebne namene, prva – dobijanje dijametra (rastojanja) galaktičkih ostataka iz posmatranog površinskog sjaja i druga – parametrizacija veze između površinskog sjaja i dijametra i upoređivanje s modelima i teorijama ili različitim uzorcima OS. Za slučaj kada se vrednost Σ koristi da se predvidi D trebalo bi koristiti MNK koji minimizuje odstupanja za $\log D$. To ipak nije slučaj u radovima koji se bave dobijanjem empirijske Σ–D relacije, uglavnom se minimizuju odstupanja za $\log \Sigma$. Razlike između ova dva MNK fita mogu biti veoma velike za ostatke koji su mnogo sjajniji ili slabiji od prosečnog sjaja ostataka s poznatim rastojanjima korišćenim za izvođenje Σ–D relacije. Na primer, Kejs i Batačarija (Case & Bhattacharya, 1998) dobili su za parametar fita Σ–D relacije $\beta = 2,38 \pm 0,26$ koristeći uzorak od 36 OS (bez ostatka Cas A) i MNK koji minimizuje zbir kvadrata odstupanja $\log \Sigma$. Minimizacijom odstupanja log D, koja više odgovara pošto se relacija koristi za predviđanje dijametra iz površinskog sjaja, dobija se mnogo strmiji fit: $\beta = 3,37 \pm 0,35$. To nas navodi na zaključak da iz najboljeg fita Kejsa i Batačarije dobijamo veće prečnike i samim tim veća rastojanja za ostatke manjeg sjaja.

Ukoliko se $\Sigma \propto D^{-\beta}$ relacija koristi za opisivanje evolucije površinskog sjaja OS, potrebno je ispitivati Σ–D korelaciju. Ipak, glavni cilj ovog rada je određivanje daljina do ostataka u Galaksiji pa je korisnija D-Σ korelacija. Početna tačka naše analize je zahtev da parametri fita Σ–D i D-Σ budu invarijantni unutar procenjenih grešaka. Ovaj zahtev biće zadovoljen primenom metode ortogonalnog fita (Urošević et al., 2010).

3.2. Ortogonalna regresija

Običan metod najmanjih kvadrata zasniva se na nalaženju prave za koju je zbir vertikalnih odstupanja eksperimentalnih tačaka minimalan. Umesto zbira vertikalnih rastojanja tačaka, koristi se zbir kvadrata tih rastojanja kako bi se postiglo da taj zbir bude neprekidna i diferencijabilna funkcija koeficijenata a i b prave $y = a + bx$. Ovaj metod predstavlja tzv. ortogonalni MNK. Razlog za češće korišćenje klasičnog MNK umesto ortogonalnog MNK (slika 1) takođe je i jednostavnije analitičko dobijanje jednačina za parametre tražene najbolje prave.



a) Vertikalna regresija b) Ortogonalna regresija

Slika 1 - Računanje odstupanja tačaka od prave povučene metodom najmanjih kvadrata (MNK); označena je glavna razlika između najčešće korišćenih – vertikalnih (a) i ortogonalnih (b) odstupanja koja se koriste u ovom radu za izvođenje Σ–D relacije

Ispostavlja se da je i za ortogonalni MNK moguće izvesti jednačine za parametre tražene prave u analitičkom obliku, što predstavlja mnogo brži metod od bilo kojih drugih numeričkih ili seminumeričkih metoda.

4. Rezultati

4.1. Formiranje uzorka kalibratora daljina

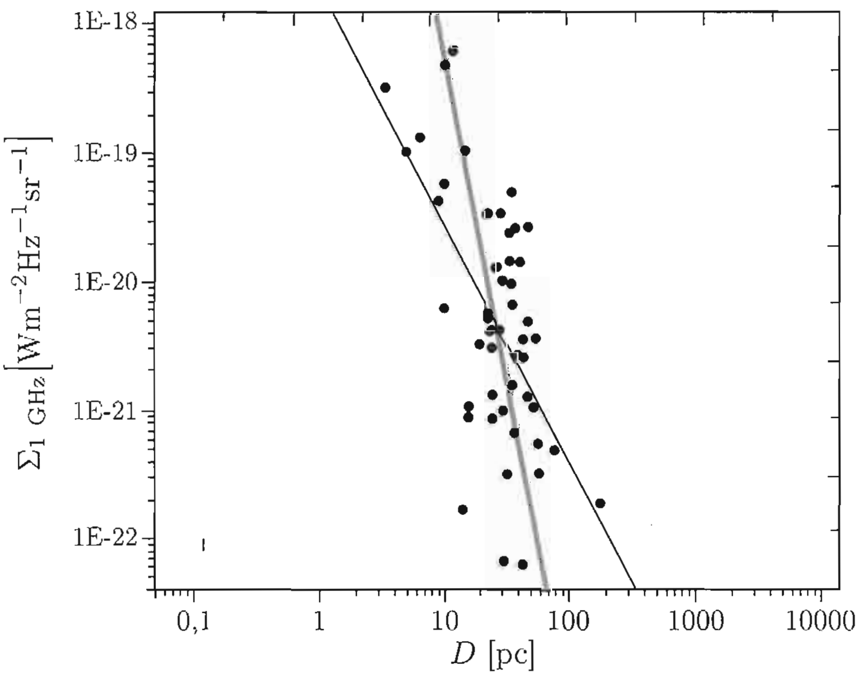
Odabir ostataka s precizno određenim daljinama predstavlja ključni zahtev prilikom izvođenja Σ–D relacije. Korišćen je najnoviji Grinov katalog (Green, 2009), koji sadrži ukupno 274 OS. Verzija kataloga koju su koristili Kejs i Batačarija (Case & Bhattacharya, 1998) za izvođenje njihove relacije sadržala je 215 ostataka. Za mnoge ostatke su u međuvremenu tačnije određene daljine, dok su neki izbačeni iz kataloga jer se ispostavilo da ne spadaju u OS.

Uzorak kalibratora korišćen u ovom radu sadrži ukupno 50 ostatka, od kojih je 31 bio prisutan u uzorku koji su koristili Kejs i Batačarija.

4.2. Primena ortogonalnog MNK na skup kalibratora

Primenom ortogonalne regresije na formirani uzorak kalibratora dobija se sledeća relacija:

$$\Sigma_{1GHz} = 5,167 \cdot 10^{-14} D^{-4,958} \ \mathrm{Wm^{-2}\,Hz^{-1}sr^{-1}}. \tag{3}$$



Slika 2 – Površinski sjaj kalibratora daljina u zavisnosti od dijametra (Σ–D relacija) za ljuskaste ostatke supernovih. Debljom linijom prikazana je prava dobijena primenom ortogonalnog MNK, a tankom linijom optimalna prava koja minimizuje vertikalna odstupanja tačaka (običan MNK).

Vidi se da je dobijena Σ–D relacija znatno strmija od ranijih i ujedno se dosta razlikuje od običnog MNK primenjenog na iste podatke.

Nova Σ–D relacija se dalje može primeniti na sve ljuskaste ostatke iz Grinovog kataloga (Green, 2009) za koje je izmerena gustina fluksa radio--zračenja na 1 GHz i ugaone dimenzije. Od ukupno 274 ostatka u tom katalogu, pomenute uslove ispunjava 202 ostatka i njihove daljine su izračunate primenom relacije (3).

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REPRODUCTIVE AND NON-REPRODUCTIVE SOLUTIONS OF MATRIX EQUATION $AXB = C$

Branko Malešević[1] **and Biljana Radičić**

Abstract. In this article we consider consistent matrix equation $AXB = C$ when a particular solution X_0 is given and we represent the new form of general solution which contains both reproductive and non-reproductive solutions (it depends on the form of particular solution X_0). We also analyse the solutions of some matrix systems using the concept of reproductivity and we give the new form of the condition for the consistency of matrix equation $AXB = C$.

Keywords: Reproductive equations, reproductive solutions, matrix equation $AXB = C$

1. Reproductive equations

The concept of the reproductive equations was introduced by S.B. Prešić [4].

Definition 1.1. *The reproductive equations* are the equations of the following form:

$$x = f(x), \tag{1}$$

where x is a unknown, S is a given set and $f : S \longrightarrow S$ is a given function which satisfies the following condition:

$$f \circ f = f. \tag{2}$$

The condition (2) is called *the condition of reproductivity.* The most important statements in relation to the reproductive equations are given by the following two theorems [4] (see also [5], [6] and [12]):

Theorem 1.1. (S.B. Prešić) For any consistent equation $J(x)$ there is an equation of the form $x = f(x)$, which is equivalent to $J(x)$ being in the same time reproductive as well. ♦

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Theorem 1.2. (S.B. Prešić) If a certain equation $J(x)$ is equivalent to the reproductive one $x = f(x)$, the general solution is given by the formula $x = f(y)$, for any value $y \in S$. ♦

The concept of the reproductive equations allows us to analyse the solutions of some matrix systems (see Application 2.1. and Application 2.2. in the following section of this paper). In [5], [6] and [9] authors considered the general applications of the concept of reproductivity.

2. The matrix equation $AXB = C$

Let $m, n \in \mathbb{N}$ and $\mathbb{C}$ is the field of complex number. The set of all matrices of order $m \times n$ over $\mathbb{C}$ is denoted by $\mathbb{C}^{m\times n}$. For the set of all matrices from $\mathbb{C}^{m\times n}$ with a rank a we use denotement $\mathbb{C}_a^{m\times n}$. Let $A = [a_{i,j}] \in \mathbb{C}^{m\times n}$. By $A_{i\rightarrow}$ we denote the i-th row of A, $i = 1, ..., m$. For the j-th column of A, $j = 1, ..., n$, we use denotement $A_{\downarrow j}$.

A solution of matrix equation

$$AXA = A \tag{3}$$

is called $\{1\}$-inverse of the matrix A and it is denoted by $A^{(1)}$. The set of all $\{1\}$-inverses of the matrix A is denoted by $A\{1\}$. For the matrix A, let regular matrices $Q \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{C}^{n\times n}$ be determined so that the following equality is true.

$$QAP = E_a = \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & 0 \end{array}\right], \tag{4}$$

where $a = rank(A)$. In [3] C. Rohde showed that the general $\{1\}$-inverse $A^{(1)}$ can be represented in the following form:

$$A^{(1)} = P\left[\begin{array}{c|c} I_a & X_1 \\ \hline X_2 & X_3 \end{array}\right]Q, \tag{5}$$

where X_1, X_2 and X_3 are arbitrary matrices of switable sizes.

Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$. In the paper [1] R. Penrose proved the following theorem related to the matrix equation

$$AXB = C. \tag{6}$$

Theorem 2.1. (R. Penrose) The matrix equation (6) is consistent iff for some choice of $\{1\}$-inverses $A^{(1)}$ and $B^{(1)}$ of the matrices A and B the condition

$$AA^{(1)}CB^{(1)}B = C \tag{7}$$

is true. The general solution of the matrix equation (6) is given by the formula

$$X = f(Y) = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}, \tag{8}$$

where Y is an arbitrary matrix corresponding dimensions. ♦

If a particular solution X_0 of the matrix equation (6) is given, the formula of general solution is given in the following theorem.

Theorem 2.2. If X_0 is a particular solution of the matrix equation (6), then the general solution of the matrix equation (6) is given by the formula

$$X = g(Y) = X_0 + Y - A^{(1)}AYBB^{(1)}, \tag{9}$$

where Y is an arbitrary matrix corresponding dimensions. The function g satisfies the condition of reproductivity (2) iff $X_0 = A^{(1)}CB^{(1)}$.

Proof. See [11] and [12] (where is given different proof). ♦

The formula (9) contains both reproductive and non-reproductive solutions. It depends on the form of particular solution X_0.

Remark 2.1. In the paper [11] authors proved that there is a matrix equation (6) and a particular solution X_0 so that:

$$X_0 \neq A^{(1)}CB^{(1)}, \tag{10}$$

for any choice of $\{1\}$-inverses $A^{(1)}$ and $B^{(1)}$. In that case the formula (9) gives general non-reproductive solutions. Otherwise, the formula (9) gives general reproductive solutions. ♦

In [2] S.B. Prešić analysed the matrix equation (3) and he proved the following theorem.[2]

Theorem 2.3. For any square matrix $A \in \mathbb{C}^{n\times n}$ and any general $\{1\}$-inverse $A^{(1)}$ the following equivalences are true:

(E_1) $AX = 0 \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = Y - A^{(1)}AY,$

(E_2) $XA = 0 \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = Y - YAA^{(1)},$

[2] with the first appearances of non-reproductive solutions (see (E_3)-(E_5))

(E_3) $AXA = A \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = A^{(1)} + Y - A^{(1)}AYAA^{(1)}$,

(E_4) $AX = A \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = I + Y - A^{(1)}AY$,

(E_5) $XA = A \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = I + Y - YAA^{(1)}$. ♦

In general case the general solutions (E_3)-(E_5) of Theorem 2.3. do not directly appear according to Penrose's theorem. In [7] M. Haverić showed that we can get Penrose's solutions from Prešić's solutions. She proved the following statement.

Theorem 2.4. For any square matrix $A \in \mathbb{C}^{n\times n}$ and any general $\{1\}$-inverse $A^{(1)}$ the following equivalences are true.

(E_1) $AX = 0 \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = Y - A^{(1)}AY$,

(E_2) $XA = 0 \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = Y - YAA^{(1)}$,

$(E_3^{'})$ $AXA = A \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = A^{(1)}AA^{(1)} + Y - A^{(1)}AYAA^{(1)}$,

$(E_4^{'})$ $AX = A \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = A^{(1)}A + Y - A^{(1)}AY$,

$(E_5^{'})$ $XA = A \iff (\exists Y \in \mathbb{C}^{n\times n})\ X = AA^{(1)} + Y - YAA^{(1)}$. ♦

Let's note that the previous two theorems are special case of Theorem 2.2.

For consistent matrix equation (6) the following equivalence is true

$$AXB = C \iff X = f(X) = X - A^{(1)}(AXB - C)B^{(1)}. \tag{11}$$

Therefore, based on Theorem 1.2., we have short proof of generality of formula (7) in Theorem 2.1.

In the following applications we analysed the solutions of some matrix systems using the concept of reproductivity.

Application 2.1. Let A, B, D and E be given complex matrices corresponding dimensions. If the following matrix system is consistent:

$$AX = B \quad \wedge \quad XD = E, \tag{12}$$

then the general solution is given by the formula ([10] A. Ben-Israel and T.N.E. Greville)

$$X = g(Y) = X_0 + (I - A^{(1)}A)Y(I - DD^{(1)}), \tag{13}$$

where Y is an arbitrary matrix corresponding dimensions.

In [12] authors proved that if the matrix system (12) is consistent, the general reproductive solution is given by the formula

$$X = f(Y) = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)} + (I - A^{(1)}A)Y(I - DD^{(1)}), \tag{14}$$

where Y is an arbitrary matrix corresponding dimensions. The proof is based on the equivalence

$$(AX = B \ \wedge \ XD = E) \iff X = f(X) \tag{15}$$

and Theorem 1.2. more detailed proof can be found in [12]. ♦

Application 2.2. Let $A \in \mathbb{C}^{n \times n}$ be a singular matrix. If the following matrix system is consistent:

$$AXA = A \quad \wedge \quad AX = XA, \tag{16}$$

then the general solution is given by the formula ([8] J.D. Kečkić)

$$X = f(Y) = Y + \bar{A}A\bar{A} - \bar{A}AY - YA\bar{A} + \bar{A}AYA\bar{A}, \tag{17}$$

where Y is an arbitrary matrix corresponding dimensions and $\bar{A}$ is commutative $\{1\}$-inverse. In [12] authors proved that (17) represents the general solution of (16) using the concept of reproductivity. ♦

The following theorem gives the new form of condition for the consistency of the matrix equation (6). In the formulation of the theorem we use the following matrices

$$\widehat{A} = T_A A, \quad \widehat{B} = BT_B \quad \text{and} \quad \widehat{C} = T_A C T_B \tag{18}$$

where T_A is a permutation matrix which permutes linearly independent rows of the matrix A at the first a positions and T_B is a permutation matrix which permutes linearly independent columns of the matrix B at the first b positions.

Therefore, the matrix $\widehat{A}$ has linearly independent rows at the first a positions and the matrix $\widehat{B}$ has linearly independent columns at the first b positions.

Let

$$\widehat{A}_{i\rightarrow} = \sum_{l=1}^{a} \alpha_{i,l}\widehat{A}_{l\rightarrow}, \quad i = a+1, ..., m \tag{19}$$

and

$$\widehat{B}_{\downarrow j} = \sum_{k=1}^{b} \beta_{k,j} \widehat{B}_{\downarrow k}, \qquad j = b+1, ..., q. \tag{20}$$

for some scalars $\alpha_{i,l}$ and $\beta_{k,j}$. Then, the following theorem is true.

Theorem 2.5. Let $A \in \mathbb{C}_a^{m\times n}$, $B \in \mathbb{C}_b^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$. Suppose that $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are determined by (18) and that (19) and (20) are satisfied. Then, the condition (7) is true for any choice of $\{1\}$-inverses $A^{(1)}$ and $B^{(1)}$ iff

$$\widehat{C} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,b} & \sum\limits_{k=1}^{b} \beta_{k,b+1} c_{1,k} & \cdots & \sum\limits_{k=1}^{b} \beta_{k,q} c_{1,k} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ c_{a,1} & \cdots & c_{a,b} & \sum\limits_{k=1}^{b} \beta_{k,b+1} c_{a,k} & \cdots & \sum\limits_{k=1}^{b} \beta_{k,q} c_{a,k} \\ \sum\limits_{l=1}^{a} \alpha_{a+1,l} c_{l,1} & \cdots & \sum\limits_{l=1}^{a} \alpha_{a+1,l} c_{l,b} & \sum\limits_{l=1}^{a}\sum\limits_{k=1}^{b} \alpha_{a+1,l} \beta_{k,b+1} c_{l,k} & \cdots & \sum\limits_{l=1}^{a}\sum\limits_{k=1}^{b} \alpha_{a+1,l} \beta_{k,q} c_{l,k} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{l=1}^{a} \alpha_{m,l} c_{l,1} & \cdots & \sum\limits_{k=1}^{b} \alpha_{m,l} c_{l,b} & \sum\limits_{l=1}^{a}\sum\limits_{k=1}^{b} \alpha_{m,l} \beta_{k,b+1} c_{l,k} & \cdots & \sum\limits_{l=1}^{a}\sum\limits_{k=1}^{b} \alpha_{m,l} \beta_{k,q} c_{l,k} \end{bmatrix},$$

where $c_{i,j}$ are arbitrary elements of $\mathbb{C}$.

Proof. The proof can be found in [12]. ♦

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TEMPUS PROJEKAT „SEE DOCTORAL STUDIES IN MATHEMATICAL SCIENCES“ – PREGLED REZULTATA

dr Zorica Stanimirović[1]

Doktorske studije, kao treći ciklus obrazovanja po Bolonjskim principima, od izuzetne su važnosti za istraživački i inovacioni potencijal jedne države i društva. Naučnoistraživačke ustanove, ali i privreda, imaju potrebu za sve većim brojem visokokvalifikovanih stručnjaka. Međutim, u implementaciji trećeg ciklusa školovanja fakulteti u državama zapadnog Balkana sreću se s brojnim problemima: nedostatkom finansijskih sredstava za adekvatno izvođenje nastave, disperzijom mladih istrazivača, atomizacijom istraživačkih grupa, nedostatkom stručnog nastavničkog kadra u pojedinim oblastima istraživanja itd. U praksi se često dešava da je broj studenata zainteresovanih za isti kurs veoma mali, te je teško oformiti grupu i realizovati nastavu na doktorskim studijama, pa se nastava svodi na povremene konsultacije.

Jedan od načina da se taj problem prevaziđe predložen je u trogodišnjem Tempus projektu "SEE Doctoral Studies in Mathematical Sciences", koji je počeo da se realizuje 2009. godine. Osnovna ideja jeste regionalna saradnja, to jest udruživanje fakulteta zapadnog Balkana kako bi se unapredile doktorske studije i povezali istraživački centri u oblasti teorijske i primenjene matematike i informatike.

Glavni ciljevi projekta su sledeći: uspostavljanje i razvoj strukturiranih doktorskih studija matematike i informatike povezivanjem prirodno-matematičkih fakulteta sa univerziteta zapadnog Balkana i njihovih EU partnera, osavremenjivanje laboratorije za primenjenu matematiku na fakultetima-partnerima zapadnog Balkana koji učestvuju u projektu, i pojačavanje programa na master studijama, posebno u oblastima matematičkog modeliranja i finansijske matematike. Predviđeno je da se projekat realizuje tokom tri godine kroz tri faze. Trenutno se on nalazi u završnoj fazi – u poslednjoj godini realizacije.

Partneri na projektu su:

- Prirodno-matematički fakultet, Univerzitet u Sarajevu (nosilac projekta),
- Matematički fakultet, Univerzitet u Beogradu,

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- Prirodno-matematički fakultet, Univerzitet u Podgorici,
- Prirodno-matematički fakultet, Univerzitet u Banjaluci,
- Prirodno-matematički fakultet, Univerzitet u Tuzli,
- Prirodno-matematički fakultet, Univerzitet u Skadru,
- Matematički institut, Univerzitet u Gracu,
- Matematički institut, Bugarska akademija nauka i umetnosti,
- Fakultet za matematiku i informatiku, Univerzitet u Sofiji,
- Društvo matematičara jugoistočne Evrope (MASEE).

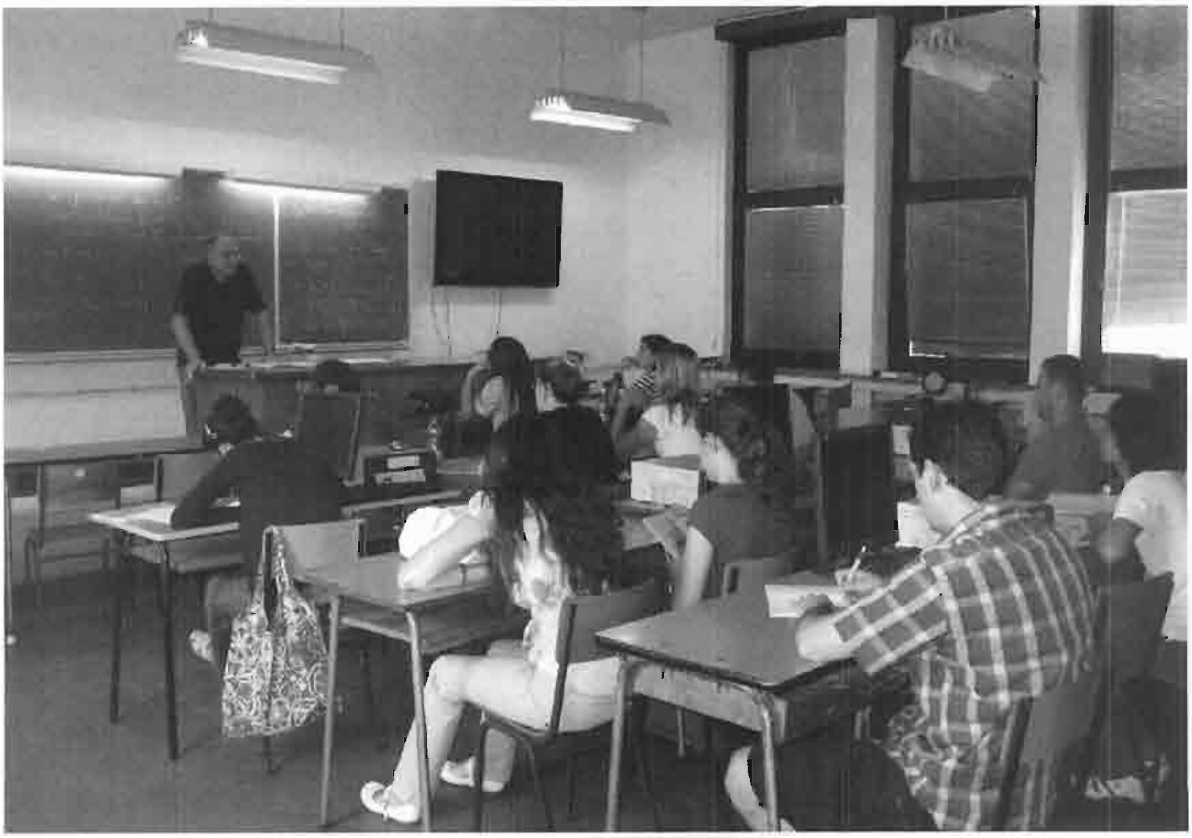
Slika 1 – Učesnici Workshop-a u Gracu, 22–29. jun 2009

U toku prve godine projekta, kroz intenzivne studijske posete, napravljen je pregled trenutne situacije na višim nivoima matematičkog obrazovanja (master i doktorske studije) fakulteta-partnera zapadnog Balkana i identifikovane su uže matematičke oblasti u kojima su ti fakulteti uspešni. Na sastanku u Gracu 22–29. juna 2009. uspostavljeni su okviri i struktura doktorskih studija i formiran Akademski savet, koji će koordinisati i upravljati realizacijom projekta. Među aktivnostima realizovanim tokom prve godine projekta izdvajaju se Workshop u Gracu "Training in Use of IT in Mathematical Modeling", 7–11. septembra 2009, i Young Researchers Workshop, u Ohridu, 16–20. septembra 2009 (u okviru MICOM International Congress on Mathematics). Na Workshop-u Ohridu, mladi istraživači – studenti doktorskih studija imali su priliku da prezentuju svoje istraživačke projekte i probleme na kojima trenutno rade, kao i razmene iskustva i ideje.

U drugoj fazi (druga godina projekta) posebna pažnja je posvećena planovima kurseva doktorskih studija. Osnovne strukture kurseva, usvojene na sastanku u Gracu dalje su razvijaju i proširuju. U okviru Workshop-a u Tuzli (4–8. novembra 2009) napravljen je detaljan plan intenzivnih kurseva, seminara, projekata koji treba da budu realizovani na način koji bi sprečio fragmentaciju i konstantno podsticao razvoj naučnog podmlatka. Na Workshop-u u Skoplju (15–20. decembra 2010) planovi kurseva su usvojeni, kao i principi njihove realizacije.

Slika 2 – Workshop u Tuzli, 4–8. novembar 2009

U toku druge faze projekta intenzivno se radilo i na unapređenju programa master studija. U drugoj godini projekta realizovana su dva intenzivna kursa za studente master studija: kurs iz Matematičkog modeliranja na Prirodno-matematičkom fakultetu u Skadru (3–31. jul 2010) i kurs iz Finansijske matematike, na Prirodno-matematičkom fakultetu u Banjaluci (7. avgust – 4. septembar 2010).



Slika 3 – Kurs Optimizacija, 29. avgust – 23. septembar 2011, Matematički fakultet, Beograd

Treća godina projekta (završna faza) usmerena je ka potpunoj implementaciji uspostavljenih planova i aktivnosti na doktorskim studijama. Pritom je naročito važno omogućiti i podsticati razmenu nastavnog osoblja između fakulteta-partnera. U toku treće godine projekta planirana je realizacija šest zajedničkih intenzivnih kurseva na doktorskim studijama u trajanju od mesec dana. Planovi tih kurseva su dizajnirani prema preporukama Akademskog odbora projekta i zasnovani na analizi istraživačkih trendova i dostignuća u regionu jugoistočne Evrope. Sprovedena je i eksterna evaluacija planova, što je dodatno doprinelo njihovom kvalitetu. Planirano je da svaki od kurseva pohađa po ukupno 24 studenta, od toga 20 studenata s prirodno-matematičkih fakulteta univerziteta zapadnog Balkana i četiri studenta sa univerziteta-partnera iz Evropske unije. Svaki od kurseva se sastoji od tri celine, a nastava se odvija na engleskom jeziku. Predavači na kursevima su profesori i istraživači sa univerziteta-partnera u projektu, ali i sa naučnoistraživačkih ustanova van projekta. Svi predavači su eminentni stručnjaci u datoj oblasti. Troškovi puta i boravka studenata i predavača finansirani su iz sredstava projekta. Nakon završetka kursa studenti polažu završni ispit i dobijaju odgovarajuće sertifikate, što im omogućava ostvarivanje izvesnog broja ESPB u matičnim institucijama.

Kursevi se realizuju na različitim fakultetima-partnerima:

- "Number Theory" - na Univerzitetu u Sarajevu, februar 2011,
- "Probability and Statistics" - Univerzitet u Podgorici, april 2011,
- "Dynamical Systems" - Univerzitet u Tuzli, maj 2011,
- "Optimization" - Univerzitet u Beogradu, septembar 2011,
- "Algebraic Combinatorics", Computability and Complexity - Univerzitet u Sofiji, oktobar 2011,
- "Data structures and High Performance Computing" - Univerzitet u Skoplju, novembar 2011.

Slika 4 - Studenti - učesnici kursa Optimizacija

U periodu od 29. avgusta do 23. septembra 2011 Matematički fakultet bio je domaćin učesnicima Tempus kursa "Optimization" na doktorskim studijama. Kurs su pohađala 24 studenta, od toga 21 student iz zemalja zapadnog Balkana, dva studenta sa Univerziteta u Gracu i jedan student sa Univerziteta u Sofiji. Predavači na kursu su bili prof. dr Franz Kappel i dr Gregory von Winckel sa Univerziteta u Gracu i Mikhail Ivanov Krastanov s Bugarske akademije nauka i umetnosti. Kurs sadrži pregled značajnog broja metoda za rešavanje različitih problema optimizacije (problemi optimizacije sa ograničenjima i bez njih, problemi kombinatorne i globalne optimizacije...). Studenti su upoznati sa osnovnim teorijskim rezultatima iz ove oblasti, kao i s numeričkim algoritmima za njihovo rešavanje. Kroz rešavanje brojnih zadataka i praktične vežbe, kao i izradu seminarskih radova studenti su pokazali zavidan nivo znanja, a polaganjem završnog ispita da su uspešno savladali gradivo ovog intenzivnog kursa.

Nakon završetka projekta planirane su brojne aktivnosti u cilju održivosti postignutih rezultata. Intenzivni kursevi različitog sadržaja, radionice, seminari, namenjeni studentima doktorskih i master studija fakulteta-partnera, biće održani u toku narednih godina. Takođe se planira integracija realizovanih kurseva na master i doktorskim studijama u odgovarajuće planove i programe studija na fakultetima-partnerima.

O UBRZANJU ČESTICA U JAKIM UDARNIM TALASIMA

Dejan Urošević[1]

Apstrakt. U ovom članku će ukratko biti objašnjeno na koji način se stvaraju ultrarelativističke čestice koje se kroz kosmička prostranstva šire i detektuju na Zemlji kao tzv. **kosmički zraci**. Trenutno najbolje razvijena i u naučnom svetu prihvaćena teorija je tzv. **difuzno udarno ubrzanje čestica**. To je mehanizam proizvodnje ultrarelativističkih čestica prilikom višestrukog prolaza visokoenergetske čestice kroz udarne talase nezavisno od toga koji je astrofizički poremećaj stvorio udarni diskontinuitet. Difuzno udarno ubrzanje je podvrsta tzv. **Fermijevih mehanizama ubrzanja čestica**, i to prvog reda. Shodno prethodno navedenom, u članku će ukratko biti objašnjeno šta su udarni talasi, Fermijevi mehanizmi, kao i osnovni koncepti na kojima je zasnovano difuzno udarno ubrzanje čestica i, naravno, koji objekti u kosmosu proizvode kosmičko zračenje.

1. Udarni talasi

Informacija o bilo kojem poremećaju koji se desio u neprekidnoj sredini (fluidu) širi se talasanjem tog istog fluida. Ako se talas kreće kroz fluid brzinom većom od brzine zvuka za tu sredinu, taj talas postaje tzv. **udarni talas**.

1.1. Matematički opis udarnih talasa

Da bi se matematički opisala neprekidna sredina, potrebno je rešiti set magnetohidrodinamičkih (MHD) relacija. Tu su tri jednačine standardne dinamike (jednačina kontinuiteta, II Njutnov zakon i zakon o održanju energije) zajedno s Maksvelovim jednačinama elektrodinamike. U najopštijem obliku, sistem ovih jednačina se svodi na rešavanje Košijevog problema na mnogostrukostima.

Uzmimo da imamo jednu trodimenzionalnu mnogostrukost (Σ) u če-

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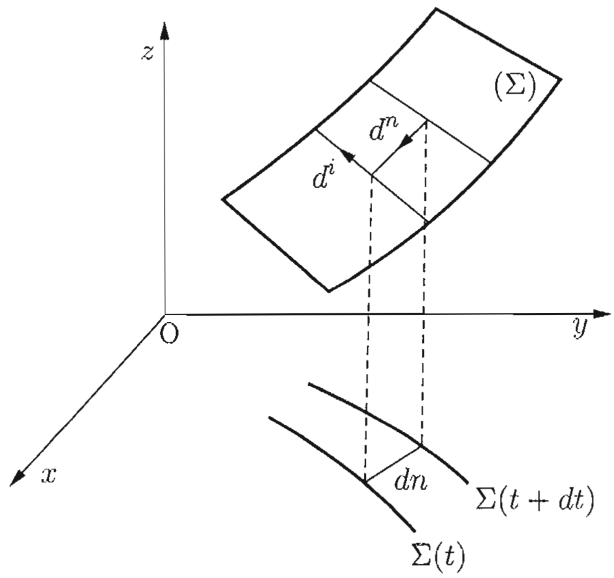
tvorodimenzionalnom prostoru (t, x, y, z). Neka je Σ (t) površina u trodimenzionalnom euklidskom prostoru (slika 1).



Slika 1 – Prikaz mnogostrukosti (Σ) (Cabannes, 1970)

Kako je već navedeno, potrebno je rešiti Košijev problem, tj. naći rešenja MHD jednačina za magnetno polje $\boldsymbol{H}$, brzinu fluida $\boldsymbol{u}$, pritisak p i gustinu ρ na mnogostrukosti (Σ). Kada je sistem neodređen, takva mnogostrukost predstavlja tzv. karakterističnu mnogostrukost. Površina $\Sigma(t)$ koja se dobija presekom $t =$ const. s karakterističnom mnogostrukošću naziva se **talasna površina** ili, prostije, **talas**. Takođe se nešto direktnije može definisati karakteristična mnogostrukost: ako dva različita rešenja MHD jednačina imaju istu vrednost na mnogostrukosti (Σ), onda je ona karakteristična mnogostrukost. Važno je napomenuti da prvi izvodi funkcija koje se traže na karakterističnoj mnogostrukosti imaju različite vrednosti.

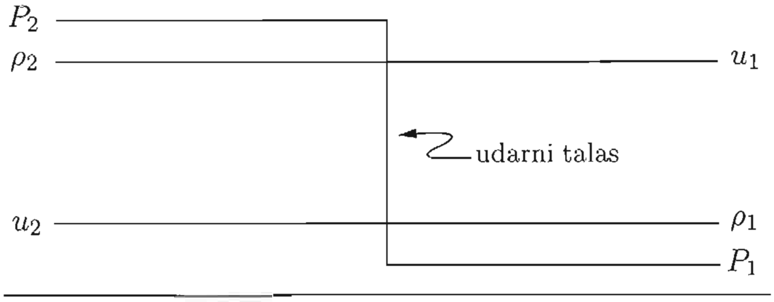
Talasna površina $\Sigma(t)$ predstavlja graničnu površ između poremećenog i neporemećenog fluida. Ako talas ima strm profil (približno vertikalan), onda je to **udarni talas**.

Jednačine koje povezuju brzinu kretanja fluida, kao i stanje termodinamičkih parametara (pritisak, gustna, temperatura) pre i posle dejstva udarnog talasa, nazivaju se Rankin–Igonoove (Rankine-Hugoniot) jednačine. Izvedene su za fluid u kome nema magnetnog polja. To su tzv. hidrodinamičke (HD) jednačine udara. Udarni talasi skokovito menjaju vrednosti brzine fluida, pritiska, temperature, gustine (i magnetnog polja u MHD okruženju). Rankin–Igonoove jednačine imaju sledeći oblik:

$$\rho_1 u_1 = \rho_2 u_2 \quad \text{– održanje mase,}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \quad \text{– održanje impulsa,}$$
$$1/2u_1^2 + h_1 = 1/2u_2^2 + h_2 \quad \text{– održanje energije,}$$

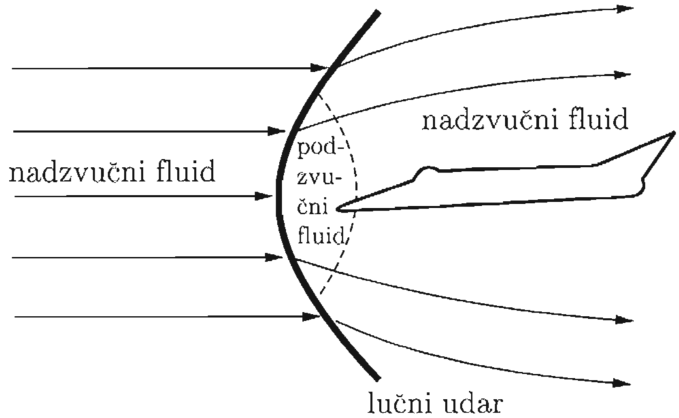
gde je h specifična entalpija ($h = \gamma/(\gamma - 1) \cdot p/\rho$), uz $\gamma = c_p/c_v$ – količnik specifičnih toplota), a indeksi 1 i 2 su za neporemećenu sredinu i za sredinu preko koje je udarni talas prešao, respektivno. Skokovite promene parametara fluida na udarnim talasima prikazane su na slici 2.



Slika 2 – Ilustracija skoka brzine fluida, gustine i pritiska na udarnom talasu (Shu, 1992)

1.2. Vrste udarnih talasa

Kada se supersonični objekat kreće kroz fluid brzinom većom od brzine zvuka, on ispred sebe „gura" tzv. **lučni udarni talas** (slika 3).



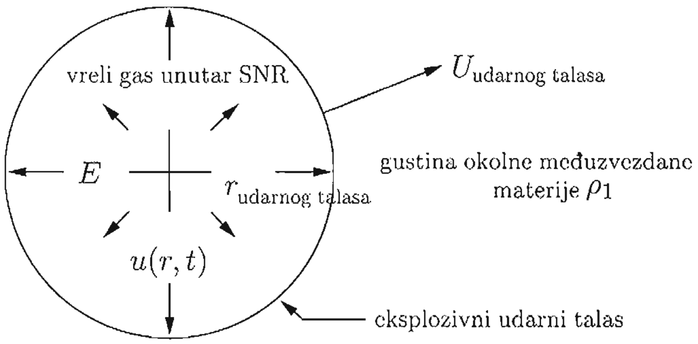
Slika 3 – Kretanjem supersoničnog aviona kroz vazduh stvara se lučni udarni talas (Shu, 1992)

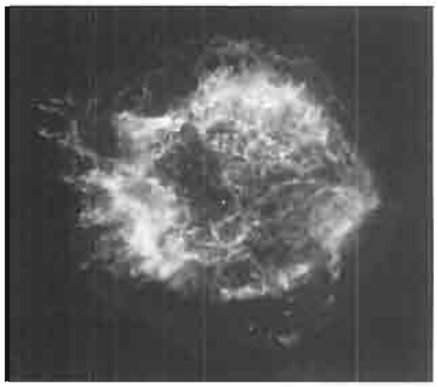
Bilo koji objekat koji se kreće brže od brizine zvuka u međuzvezdanoj sredini obrazovaće ispred sebe lučni udarni talas. Posmatrano je nekoliko lučnih udarnih talasa u blizini pulsara koji se kreću nadzvučnom brzinom (brzina zvuka u međuzvezdanoj sredini je oko 10 km/s).

Vrsta udarnih talasa za koju smo posebno zainteresovani zbog procesa ubrzavanja čestica jesu tzv. **jaki udarni talasi**. Definisani su tako da je promena brzine fluida na udarnom talasu približno jednaka brzini udarnog talasa (u sistemu reference u kojem udarni talas miruje približno je jednaka brzini neporemećene sredine koja „uleće“ u udarni diskontinuitet) tj.:

$$\Delta u = u_1 - u_2 \sim u_1.$$

Ako se veoma velika količina energije oslobodi u vrlo kratkom vremenskom intervalu, oko objekta se poremećaj širi u obliku ekstremno jakog udarnog talasa (engl. blast wave), koji ćemo mi nazvati **eksplozivni udarni talas**. Takva vrsta udarnog talasa se javlja prilikom eksplozija nuklearnih i termonuklearnih bombi, a u prirodi prilikom eksplozija supernovih. Posle eksplozije supernove eksplozivni udarni talas se širi kroz međuzvezdanu sredinu, remeti je, povećava joj temperaturu, pritisak i gustinu. Tako poremećena materija postaje intenzivno svetla u različitim delovima elektromagnetnog spektra (posebno u radio, X i vidljivom području, mada se u poslednje vreme intenzivno otkriva materija pod dejstvom udarnih talasa u γ i infracrvenom području). U astronomskoj literaturi ti objekti su poznati kao ostaci supernovih (engl. supernova remnants), u daljem tekstu SNRs (slika 4).





Slika 4 Eksplozivni udarni talas (ilustracija) i ostatak supernove Kasiopeja A (Cas A)

Ostaci supernovih, osim što imaju vrlo važan uticaj na međuzvezdanu materiju, na rađanje novih zvezda i obogaćivanje okoline težim elementima, jesu najvažniji objekti za stvaranje kosmičkog zračenja – na njihovim jakim udarnim talasima se efikasno ubrzavaju čestice do ultrarelativističkih brzina.

2. Kosmički zraci

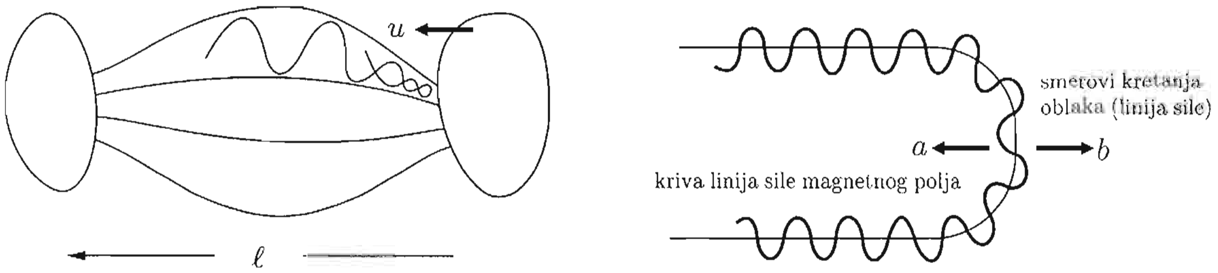
Kosmičko zračenje (engl. cosmic rays) otkrio je Viktor Hes 1912. godine. Za to otkriće je dobio Nobelovu nagradu 1936. godine. Kosmičko zračenje nije elektromagnetno zračenje, već je sastavljeno od visokoenergetskih čestica (uglavnom protona, elektrona i jezgara helijuma). Najenergetskiji kosmički zraci su jezgra čestica težih od helijuma (verovatno gvožđa) – detektovani su oni koji imaju energiju i do 10^{21} eV. Po standardnim modelima mehanizama ubrzanja čestica ostaci supernovih bez većih problema mogu da ubrzaju čestice do 10^{15} eV. Najnovija istraživanja pokazuju da ostaci supernovih mogu da proizvedu čestice sa energijama 10^{18} eV. Još uvek se pouzdano ne zna na koji način nastaju visokoenergetske čestice koje imaju energije u intervalu od 10^{18} do 10^{21} eV i često se nazivaju ultra visokoenergetske čestice.

Kosmičko zračenje u prvoj aproksimaciji ima prilično izotropnu raspodelu na nebeskoj sferi. Čestice koje iz kosmosa dospevaju do granice Zemljine atmosfere su tzv. primarni kosmički zraci, koji u interakciji s konstituentima atmosfere stvaraju lavinu tzv. sekundarnih kosmičkih zraka koje detektori na Zemljinoj površini prikupljaju. Kosmičko zračenje predstavlja značajnu kariku u energetskoj raspodeli međuzvezdane sredine. Gustina energije koju nosi kosmičko zračenje je približno jednaka gustini energije koju nosi elektromagnetno zračenje, magnetno polje, kao i gustini energije koja je sadržana u haotičnom kretanju čestica koje čine međuzvezdane materije (sva četiri „sastojka“ imaju prosečnu gustinu energije od oko 1 eV/cm^3).

Prvo objašnjenje na koji način nastaje kosmičko zračenje dao je nobelovac Enriko Fermi 1949. godine, u svom krucijalnom radu *O poreklu kosmičke radijacije* (engl. *On the Origin of the Cosmic Radiation*) gde predlaže dva osnovna koncepta na koji način čestice mogu da dobiju toliku energiju da bi postale kosmički zraci, što će detaljnije biti objašnjeno u narednoj sekciji.

3. Originalno Fermijevo ubrzanje čestica

Fermi (1949) predlaže dva modela ubrzanja čestica (tipovi A i B). Oba su zasnovana na sudaru probne visokoenergetske čestice sa oblakom međuzvezdane materije koji u sebi nosi magnetno polje, tzv. **magnetnim ogledalom**, od koga će se čestica „odbiti“ i pri tom odbijanju dobiti određenu količinu energije. Čestica će se ubrzati ako se ogledalo kreće ka njoj, a usporiće se ako se ogledalo kreće od nje. Kako je veća verovatnoća da dođe do sudara sa oblakom koji se kreće ka čestici, statistički gledano, čestice će se ubrzavati – sporo ali sigurno! Fermijev tip B se razlikuje od tipa A u tome što su kod njega linije sila magnetnog polja u oblaku krive, dok su kod tipa A linije sila u istom pravcu u odnosu na koji se kreće probna čestica (slika 5).



Slika 5 – Fermijevi mehanizmi ubrzanja. Levo, tip A, gde je predstavljeno kretanje čestica između dva magnetna ogledala koja se kreću jedno ka drugom (što se u kosmosu retko može sresti); desno: tip B

U oba slučaja Fermijevih mehanizama neto dobitak energije je drugog reda u odnosu na kinetičku energiju čestice pre sudara, te su originalni Fermijevi modeli, u stvari, mehanizmi ubrzanja drugog reda. Sam Fermi je svoj model tipa B definisao kao efikasniji za ubrzanje čestica. Relativno se lako može doći do izraza za neto energetski dobitak prilikom jednog sudara. Ako energiju čestice E koja se približava ogledalu relativistički transformišemo u odnosu na sistem u kojem magnetno ogledalo miruje, i tako dobijenu energiju (koja je u tom sistemu jednaka i pre i posle sudara) opet vratimo u laboratorijski sistem, dobijamo:

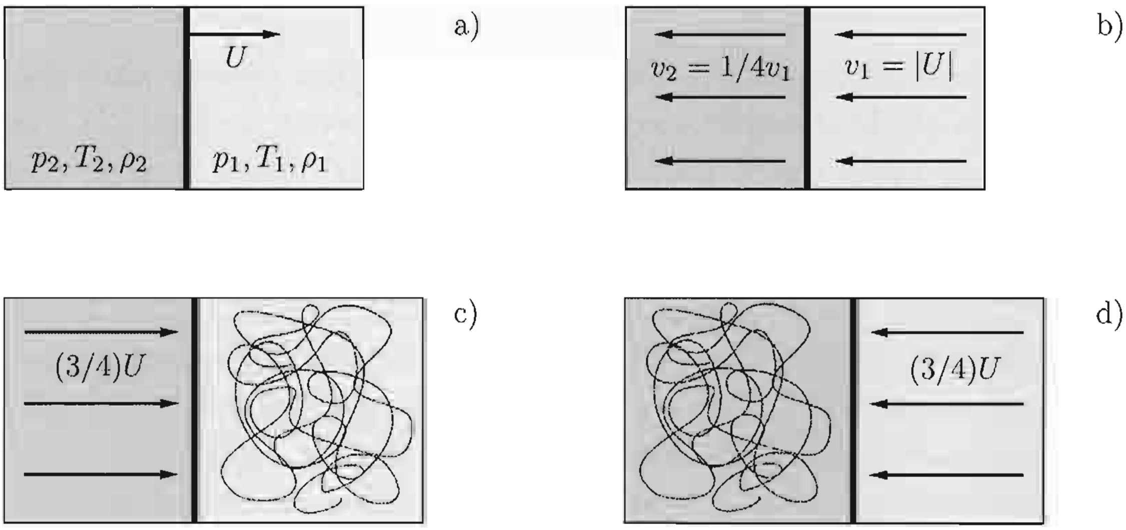
$$\Delta E/E \sim (u/c)^2,$$

gde je u brzina magnetnog ogledala, a c brzina svetlosti. Očigledno je da je energetski dobitak drugog reda od u/c.

4. Difuzno udarno ubrzanje čestica

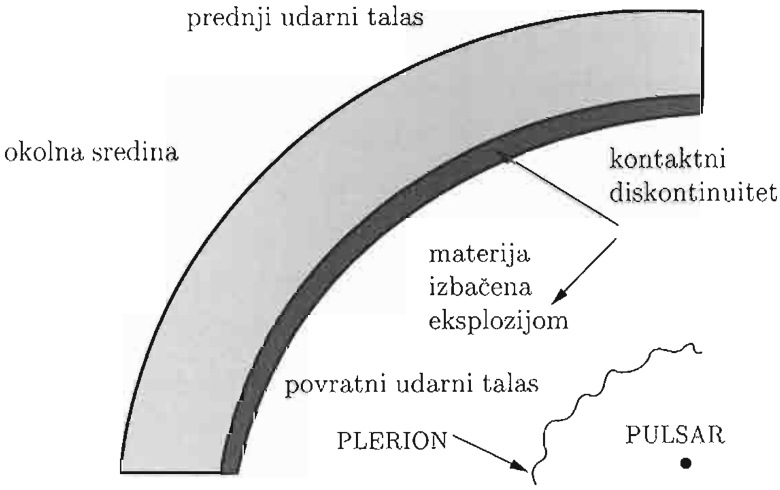
Već nekoliko puta je pomenuto da je trenutno najbolje zasnovana teorija koja opisuje mehanizme ubrzavanja čestica do ultrarelativističkih energija tzv. teorija difuznog udarnog ubrzanja (engl. diffuse shock acceleration), u daljem tekstu DSA. Ovaj tip ubrzanja takođe pripada Fermijevom tipu ubrzanja, ali je neto dobitak prilikom sudara s magnetnim ogledalom prvog reda: $\Delta E/E \sim u/c$ – samim tim Fermijev mehanizam ubrzanja prvog reda je znatno efikasniji od Fermijevih mehanizama ubrzanja drugog reda.

Velika razlika u odnosu na originalne Fermijeve modele je u tome što umesto oblaka koji u sebi ima zgušnjenje linija sila magnetnog polja, u DSA teoriji udarni talas postaje magnetno ogledalo! Udarni talasi, inače, značajno kompresuju, a nelinearnim efektima značajno pojačavaju magnetno polje prilikom svog prelaska preko međuzvezdane sredine, pa je osnovni Fermijev zahtev za nehomogenim magnetnim poljem ispunjen. S druge strane, DSA, osim što obezbeđuje prelazak probne čestice iz neporemećene u poremećenu sredinu, pri čemu ta čestica dobija energiju, stvara veliku verovatnoću da se čestica ponovo vrati u neporemećenu sredinu i opet trpi čeoni sudar sa udarnim talasom pri kome ponovo dobija energiju (slika 6). Samim tim se efikasnost ubrzavanja čestica na udarnim talasima opet značajno povećava. DSA mehanizam je zasnovan u radovima Bela (Bell 1978, a, b) i Blendforda i Ostrajkera (Blandford and Ostriker, 1978, 1980). Detaljni prikazi DSA modela su dati u preglednim radovima Drurija, i Malkova i Drurija (Drury, 1983, a, b, Malkov and Drury, 2001). S druge strane, iako je DSA opšte prihvaćen model za ubrzavanje čestica, i dalje se puno radi na modelima zasnovanim na Fermijevom ubrzanju drugog reda (npr. Scott and Chevalier, 1975, Galinsky and Shevchenko, 2007).



Slika 6 Čestica uvek trpi čeoni sudar kada prelazi kroz udarni talas iz neporemećene u poremećenu oblast, i obrnuto. (a) Širenje udarnog talasa u sistemu koji je vezan za posmatrača. (b) Kretanje fluida kroz udarni talas u sistemu koji je vezan za udarni talas. (c) U odnosu na čestice koje se haotično kreću u neporemećenoj sredini udarni talas napreduje ka njima. (d) U odnosu na čestice koje se kreću haotično u oblasti gde je poremećena sredina na njih naleće fluid iz regiona gde je sredina neporemećena i to istom brzinom (3/4U) kao u slučaju (c) – ponovo čeoni sudar!!! (Longair, 1994)

Kao astrofizički objekti koji imaju jake udarne talase, ostaci supernovih su idealna mesta gde će se ubrzavati čestice, tj. stvarati kosmički zraci. Na slici 7 je prikazan model mladog ostatka supernove koji u sebi ima dva udarna talasa (prednji i povratni, na kojima se čestice ubrzavaju DSA mehanizmom) i kontaktni diskontinuitet, kao i oblast (neposredno iza udarnog talasa) gde bi se Fermijevim mehanizmima drugog reda mogle efikasno ubrzavati čestice (svetlosivo osenčena oblast).



Slika 7 Šematski prikaz mladog ostatka supernov

Osnovni preduslov za efikasno DSA ubrzanje čestica jeste da čestica mnogo puta pređe iz neporemećene (1) u poremećenu sredinu (2), i obrnuto. Intuitivno je jasno da se prelazak iz oblasti 1 u oblast 2 može dogoditi s velikom verovatnoćom za čeoni prelazak (sudar), ali nije sasvim jasno zbog čega onda čestica ne ode daleko u oblast 2 i nikada se ponovo ne vrati na udarni talas. Uz to, važno je naglasiti da je probna čestica koja ulazi u sistem ubrzanja visokoenergetska i da je brzina udarnog talasa mnogo manja od brzine probne čestice. Ta visokoenergetska probna čestica biva ubrzavana DSA mehanizmom i postaje ultrarelativistička. Iz teorije difuzije čestica, tj. rešavanjem tzv. difuziono-advekcione jednačine za transport čestica sledi da je verovatnoća da čestica pobegne daleko od udarnog talasa u izrazito turbulentnoj sredini, kao što je ona neposredno iza udarnog talasa, veoma mala. Verovatnoća „bekstva“ daleko od udarnog talasa u regionu 2 je data izrazom:

$$\eta = 4u_2/v,$$

gde je u_2 brzina fluida u regionu 2, dok je v brzina probne čestice koja je približno jednaka brzini svetlosti. Iz ove formule jasno sledi da je velika verovatnoća da će se čestica vratiti na udarni talas i potom preći u region 1, čime će pretrpeti, opet, dodatno ubrzanje. Intuitivno, nanovo dolazimo do još jednog zanimljivog pitanja. Šta će sada naterati česticu da se ponovo vrati na udarni talas, da ne „pobegne“ daleko od njega, u okolnu sredinu, daleko od ostatka supernove? Odgovor na ovo pitanje jeste, da sama visokoenergetska čestica, koja se kreće značajno brže nego što je Alfenova brzina u regionu 1, spontano indukuje turbulencije u okolnom magnetnom polju u formi Alfenovih talasa. Proračuni kažu da su te magnetne perturbacije sasvim dovoljne da većinu probnih čestica skrenu s pravolinijske putanje i vrate ih na udarni talas. Zatim ih u regionu 2 opet hvata turbulentno magnetno polje i opet ih vraća na udarni talas. Posle N ciklusa čestica gubi „pamćenje“ o svom inicijalnom spektru, postaje **„šokirana difuzijom“** (engl. diffuse shocked) i, što je za nas najvažnije, ultrarelativistička! Zbog efekta gubljenja pamćenja o početnim parametrima čestice ceo proces se naziva difuzno udarno ubrzanje čestica – ne samo zbog toga što se proces dešava na udarnom talasu već i zato što čestica gubi „pamćenje“ i ostaje „šokirana zbog toga šta joj se desilo“!

5. Umesto zaključka

U ovom članku je detaljno prikazan samo jedan, trenutno najznačajniji mehanizam proizvodnje ultrarelativističkih čestica. Svi objekti u kosmosu koji imaju u sebi udarne talase mogu da ubrzavaju čestice. Ovde su pominjani ostaci supernovih jer su najbolji primer za DSA, ali uz njih, aktivna galaktička jezgra, pulsari, HII regioni, planetarne magline, pa i zvezde u svojim atmosferama mogu da ubrzavaju čestice, ako imaju udarne talase, DSA mehanizmom, ako nemaju, nekim drugim.

Gotovo celokupna visokoenergetska astrofizika se zasniva na proučavanju ultrarelativističkih čestica. Usporavanjem ultrarelativističkih elektrona nastaje najveći deo radio-zračenja koje stiže iz kosmosa, sinhrotronsko i netermalno zakočno X-zračenje, kao i visokoenergetsko zračenje nastalo inverznim Komptonovim efektom. γ-zračenje nastaje u interakciji ultrarelativističkih bariona s međuzvezdanom materijom. Nastanak visokoenergetskog X i γ elektromagnetnog zračenja (zajedno s nastankom radio-zračenja) neodvojivo je povezan s visokoenergetskom čestičnom astrofizikom, a bez poznavanja procesa koji dovode čestice na tako visoke energije ne možemo proniknuti u procese stvaranja zračenja, koje nosi najviše informacija neophodnih u savremenoj astronomiji. Ova oblast astrofizike je tek u povoju i samo veliki trud i rad, bilo teorijski, bilo posmatračko-eksperimentalni, može dovesti do boljeg razumevanja procesa ubrzanja čestica.

Zahvalnica. Zahvaljujem dekanu Matematičkog fakulteta, prof. dr Miodragu Mateljeviću i prodekanu za nauku dr Zorici Stanimirović što su me pozvali da održim predavanje na simpozijumu na ovu temu. Bila mi je velika čast i zadovoljstvo da kolege matematičare upoznam s ovom jako zanimljivom oblašću.

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