Section 2

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Title of the talk ON THE EMBEDDING OF MODELS

The main problem that we consider in what follows may be briefly formulated thus:

Characterize those axiomatic classes of models such that any model can be embedded in some model belonging to one of these elasses. Note that here operations and relations are represented by operations defined by some terms, relations defined by some formulae respectivily. A special case of such embedding is the usual isomorphic embedding.

We mention some known results of this kind for Ω-algebras:

Eeach Ω-algebra can be embedded in same semigroup [1] , and in some so called entropic groupoid [3] , i.e. a groupoid satisfying the law $(x*y)*(z*u)=(x*z)*(y*u)$. Further in [5] is given a necessary and sufficient condition for embedding any Ω-algebra in some O'-algebra belonging to the variety $\Sigma(O)$, where O' is some simple expansion of the language O. The condition reads:

There is a term $\Gamma(x,y;o_1,\dots,o_n)$ in the language O' such that the operation o defined by

$$x \circ y \stackrel{def}{=\!=} \Gamma(x,y;\ o_1,\dots,o_n)$$

satisfies no algebraic law (except $x=x$), while the operations $o_1,\dots,o_n$ satisfy the laws $\Sigma(O)$ anly.

In this communication we consider a similar problem for the model having beside operations some relations in addition. We restrict ourselves to embedding of relational models, i.e. models without operations, in relational-operational models of the axiomatic class $\Sigma(L)$ where $\Sigma(L)$ is a consistent set of universal formulae.

First of all we give some definitions and explain the notation.

- If A is a predicate formula whith free variables $x_1, \dots, x_n$, then we use the symbol $A[x_1, \dots, x_n]$. The similar notation is used for terms.
- Let $\Sigma(L)$ be a set of universal formulae and Γ a set of symbols(if the set of constant symbols in L is non-empty, Γ may be empty as well; otherwise it is non-empty). Further, let, T be the set of all terms in the language L built up from Γ. By $\Sigma(L)_\Gamma$ we denote the set of all formulae resulting from $\Sigma(L)$ by replacing, in all possible ways, the free variables in the matrices corresponding to the formulae in $\Sigma(L)$ by elements in T.
- Let Σ, $\underline{F}$ be consistent sets of propositional formulae in P, where P is a set of propositional letters. We say that $\underline{F}$ is <u>free</u> for Σ iff every realization τ of the set $\underline{F}$ can be expanded to some model σ of the set Σ. We recall that $p^\top$, $p^\perp$ denote the formulae p, $\neg$p respectivily.
- Let $\Sigma(L)$ be a consistent set of universal formulae in the language L, L' a simple expansion of L and $\underline{F}$ a consistent set of open formulae in L'. $\underline{F}$ is <u>free for</u> $\Sigma(L)$ iff for every set Γ of constant symbols the set $\underline{F}_{T'}$ is free for $\Sigma(L)_{T'}$, in the sentential sense, where T' is the set of all terms in L' built up from Γ.
- The set $\underline{F}$ of propositional formulae built up from propositional letters in P is the set of <u>general components</u> if the formulae in $\underline{F}$ may take any value in $\{\top, \perp\}$, i.e. iff the following condition is satisfied[1]

$$(\forall \sigma : \underline{F} \to \{\top,\perp\})(\exists \tau : P \to \{\top,\perp\})(\forall F \in \underline{F})\ \tau F = \sigma F$$

where τF is the value of F determined in the usual way. The simplest set of general components is of course P, and any consistent set Q consisting of some propositional letters and some negations of

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[1] The signs $\forall$, $\exists$ we used for meta-quantifiers.

propositional letters. But there are some, other sets of general components different from the preceding ones. For example, if $P=\{p,q,r\}$ then $\underline{F} = \{p, p\Leftrightarrow q, p\Leftrightarrow\neg r\}$ is also a set of general components.

- The set $\underline{F}$ of open formulae in the language L is the set of general components iff for every non-empty set S and every mapping[2] σ : $\underline{F}\rightarrow Rel(S)$ there is a model of L with domain S such that for every F in $\underline{F}$ the folowing condition holds (provided that a relation of length n corresponds to a formula with n free variables) :

$$\underline{M} \models F \Leftrightarrow \sigma F$$

- The open formulae φ, ψ in the language L are relationally independent iff $\{\varphi\}_T$, $\{\psi\}_T$ are disjoint for any set of symbals Γ and the corresponding set T of terms.

We now formulate two theorems about embedding of any model of the languge L_1 without operation symbols in some model of the axiomatic class $\sum(L)$.

Theorem 1 . Let L, L_1, L' be first-order languages without equality symbols and L' a simple expansion of L. Further, let $\sum(L)$ be a consistent set of universal predicate formulae in L. Then any model of the longuage L_1 can be embadded in some model of L' satisfying the axioms $\sum(L)$ if and anly if the condition (C_1) holds, where:

(C_1) For any relation symbol $\alpha \in L_1$ of length n there is some open formula $A[x_1,...,x_n]$ in the language L' such that to different relation symbols there correspond relationally independent formulae, and that the set of all such formulae is free for $\sum(L)$ and is a set of general components.

The next theorem is an immediate consequence of the preceding one.

Theorem 2. Let L be, a first-order language without equality symbol and $\sum(L)$ a consistent set of universal formulae in L. Then for any language L_1 (without equality and operation symbols) there is a simple expansion L' of L such that any model of L_1 can be

2) Rel (S) is the set of all relations of S.

embedded in some model of L' satisfying the axioms $\sum(L)$ if and only if the condition (C_2) holds, where:

(C_2) For every language L_1 (without equality or operation symbols) there exists some simple expansion L' of the language L such that to each relation symbol $\alpha \in L_1$ of length n there corresponds some open formula $A[x_1, \dots, x_n]$ in L' provided that to different relation symbols there correspond relationaly independent formulae, and that the set of all such formulae is free for $\sum(L)$ and is a set of general components.

Finally we mention that the condition (C_2) is satisfied for various axiomatic classes $\sum(L)$. For example, (C_2) holds for classes $\sum(L)$ (of consistent universal formulae) having the following two properties:

(i) The set $\sum(L)$ has at least one free formulae of the form $A[x]$ (in the language L or some simple expansion L' of L).

(ii) The language L contains at least one operation symbol of length 2.

For example, if $L = \{\alpha, *\}$ and the only member of $\sum(L)$ is the formula:

$$(1) \qquad (\forall x,y)(\alpha(x*y) \Rightarrow \alpha(y*x))$$

then the preceding two conditions are satisfied and, therefore, any relational model can be embedded in some model satisfying the axiom (1).

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