Fundamental Order and the Number of Countable Models

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A C K N O W L E D G M E N T

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A B S T R A C T

Let T be a complete, first order theory in a countable language. We investigate certain nonisolation properties of types in small superstable theories. Also, we prove that T has $2^{\aleph_0}$ nonisomorphic countable models in the following cases.

(A) T is strictly stable and the order type of rationals cannot be embedded into O(T) and there is no strictly stable group interpretable in T^{eq}.

(B) T is superstable, the generic of every simple group definable in T^{eq} is orthogonal to all NENI types and

$$\sup\{U(p) \mid p \in S(T)\} \geq \omega^{\omega}.$$

R É S U M É

Soit T une théorie complète du premier ordre dans un langage dénombrable. Nous étudions quelques propriétés des types non isolés dans les théories petites superstables. Nous démontrons que T a $2^{\aleph_0}$ modèles deux à deux non isomorphes dans les cas suivants:

(A) T est strictement stable, O(T) ne contient pas de sous-ordre du type des rationnels, et il n'y a pas de groupe strictement stable interprété dans T^{eq}.

(B) T est superstable, le generique d'une groupe simple et définissable dans T^{eq} est orthogonal aux tout les type NENI et

$$\sup\{U(p) \mid p \in S(T)\} \geq \omega^{\omega}.$$

CONTENTS

INTRODUCTION

The problem of determining the number of nonisomorphic countable models of a first order theory in a countable language dates back to 1961; Vaught conjectured that a complete such theory has either finitely many, $\aleph_0$, or $2^{\aleph_0}$ nonisomorphic countable models. Since then, the only result proved for an arbitrary theory is Morley's theorem asserting that if the number is $\geq\aleph_2$, then it is $2^{\aleph_0}$; the proof turned out to be a result in descriptive set theory and so far no further progress in that direction has been made. However, Baldwin and Lachlan proved in 1971 that an uncountably categorical theory is either totally categorical or has $\aleph_0$ countable models. Lachlan in 1974 proved that if a superstable theory has finitely many countable models, then it is $\aleph_0$-categorical and $\aleph_0$-stable.

It seems that the right approach to the problem is through the stability hierarchy. The most important result is that Vaught's conjecture is true for ω-stable theories, see [SHM]. The next step would naturally be to prove it for superstable theories. Saffe and Buechler initiated the proof of the weakly minimal case, which is the simplest, properly superstable case. This was finished by Newelski, who proved that the existence of a nonisolated U-rank 1 type of infinite multiplicity implies the maximal number of countable models. Recently, Newelski and Buechler completed the proof of the U-rank 2 case and Buechler proved the unidimensional case.

In the proof of the ω-stable case it was crucial that there are enough strongly regular types, whose nonisolation properties are determined by whether they are ENI or NENI. In this approach the class of types which are orthogonal to all NENI types forms the 'right' class of nonisolated types to work with. In between lines of [SHM] one reads that this class contains precisely those types $p \in S(A)$ for which

(*) there is a finite $B \supseteq A$ and a nonforking extension $q \in S(B)$ of p, such that for all finite $C \supseteq B$ and isolated $r \in S(C)$, $q \perp^{a} r$.

Essentially, in all many model arguments in [SHM] it is this technical property which is used.

Nevertheless, almost nothing is known about nonisolation properties of types of limit ordinal U-rank. The only result in this direction is from [CHL], stating that such types are not present in $\aleph_0$-stable $\aleph_0$-categorical theories. For example, it is not known whether the height of the fundamental order has any impact on the number of nonisomorphic countable models.

Our main task is to investigate nonisolation properties of superstable types over finite domains in general. We introduce two notions of nonisolations, eventual-strong (or esn for short) and internal. Eventual strong nonisolation is defined by (*) above. It happens that the only essential difference from the ω-stable case is a necessary passage to T^{eq}. We prove the following result.

Theorem A Let T be a complete, small, superstable theory in a countable language and let $\mathcal{M}$ be a monster model of T. In T^{eq}, consider the class of all nonalgebraic types having a finite domain. Then the subclass consisting of all esn types is the largest subclass which does not contain NENI types, and which is closed under: conjugation by automorphisms of $\mathcal{M}^{eq}$, nonforking extensions and restrictions, and domination.

Internal nonisolation is introduced for regular types only. It is preserved under nonorthogonality and parallelism (over finite domains) and it states that the dependence is a definable relation on the set of realizations of a type in question. It can be viewed as a strong negation of the NENI property; it implies eventual strong nonisolation. The use of internal nonisolation is essential for the study of NENI types. One certainly expects that the properties of NENI types involving definability translate into properties of internally isolated types involving ∞-definability instead.

We study internally isolated types of U-rank $\omega^{\alpha+1}$. Here we generalize the method from the proof from [CHL] that $\aleph_0$-stable $\aleph_0$-categorical theories have finite rank. We prove that sufficiently large 'finitely generated subspaces' of internally isolated types have big definable groups of automorphisms. This fact turns out to be decisive in the proof of the following.

Theorem B If T is a complete, superstable theory in a countable language, $U(T)=\sup\{U(p)\mid p\in S(T)\}\geq\omega^{\omega}$, and the generic of every simple, ∞-definable group in T^{eq} is esn, then $I(T,\aleph_0)=2^{\aleph_0}$.

It seems that the last assumption in Theorem B is actually a theorem (it is known to be true for types of finite U-rank).

Conjecture The generic type of a simple, superstable group is esn.

In the strictly stable case we restrict ourselves to theories with no dense forking chains. This class of theories was recently introduced by Pillay, see [HLPTW], and reasonably approximates superstable theories; for example, we have regular decomposition of (finitary) types. We prove that the major complication in determining the number of countable models of these theories is caused by the presence of big groups in T^{eq} which witness almost orthogonality.

Theorem C If T is a complete, strictly stable theory in a countable language, has no dense forking chains and no strictly stable groups definable in T^{eq}, then $I(T,\aleph_0)=2^{\aleph_0}$.

Let us briefly describe the contents of the thesis:

Following the Introduction is the first chapter called Review of Basic Stability Theory in which we overview stability theory techniques that are used in chapters 2 and 3. In the second chapter we deal with NDFC theories. In 2.1 we develop notions of dimension and U_α-rank through partial orders. In that way we have defined not only $U_\alpha(p)$, as in [HLPTW], but also $U_\alpha(p|q)$ where $p \leq q$. This is done because of Proposition 2.1.7, which follows Lascar's idea that the amount of forking, i.e. the complexity of the corresponding interval in O(T), is accurately measured by the canonical base. In 2.2 we prepare some material for the proof of Theorem C, which is presented in 2.3. At the end of 2.3 we note that strongly nonisolated types can be present due to the dimensional discontinuity property (didip, cf. [Sh]).

In the third chapter we deal with small superstable theories. In 3.1 we investigate esn and internally nonisolated types and prove Theorem A. 3.2 contains the study of internally isolated types of U-rank $\omega^{\alpha+1}$ and in 3.3 we prove Theorem B.

1. REVIEW OF BASIC STABILITY THEORY

Definition A complete, first-order theory which has infinite models is $\varkappa$-stable if for all $M\models T$ if $|M|\leq\varkappa$ then $|S(M)|\leq\varkappa$. T is stable if it is $\varkappa$-stable for some infinite cardinal $\varkappa$.

Throughout this chapter we fix a complete, stable theory T in a countable language which has infinite models. We operate in a monster model $\mathcal{M}$.

1.1 Definability, fundamental order, forking, multiplicity, T^{eq}, strong types, canonical basis

A set $\mathcal{A}\subseteq\mathcal{M}$ is definable over A if there exists a formula with parameters from A such that $\mathcal{A}$ is the solution set of it; we also say that $\mathcal{A}$ is A-definable. A $*$-type is an infinitary type with indexed set of variables. By an automorphism we mean an automorphism of $\mathcal{M}$, and an A-automorphism is one that fixes A pointwise. the corresponding groups we denote by $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}_A(\mathcal{M})$.

Definition $p\in S(A)$ is definable if for all L-formulas $\phi(\bar{x},\bar{y})$ there exists a formula $d\phi(\bar{y})$, possibly with parameters from A such that for all $\bar{a}\subseteq A$ $\phi(\bar{x},\bar{a})\in p$ iff $\models d\phi(\bar{a})$.

The following fact is from [Sh1].

Definability Lemma Any complete type is definable.

Let $\phi(\bar{x},\bar{y})$ be an L-formula. We say that $\phi(\bar{x},\bar{y})$ is represented in p if there is $\bar{a}\subseteq A$ such that $\phi(\bar{x},\bar{a})\in p$. Therefore, when speaking about $\phi(\bar{x},\bar{y})$ being represented in p, we make distinction between tuples $\bar{x}$ and $\bar{y}$ of variables, where $\bar{x}$ is reserved for the tuple of variables and $\bar{y}$ for a tuple of parameters. To make this distinction visible we write $\phi(\bar{x};\bar{y})$. The class of p is the set of all $\phi(\bar{x};\bar{y})$ which are represented in p; we denote it [p]. The set of all classes of types whose domain is a model of T, ordered by a reverse inclusion, is called the fundamental order of T and is denoted by O(T). For $p\in S(A)$ consider the set

$$\{[q]\,|\,p\subseteq q \text{ and dom}(q) \text{ is a model}\} \subseteq O(T).$$

A maximal element, in O(T), of this set is called a bound of p. The existence and uniquenes of bounds is provided by the following theorem (Theorem 5.1 in [LP]).

Theorem on the bound Every complete type has a unique bound.

The bound of p is denoted by bnd(p). If $p=tp(\bar{b}/A)$ then we write $bnd(\bar{b}/A)$ in place of bnd(p) and if $A=\emptyset$ we also write $bnd(\bar{b})$.

If $A\subseteq B$ and $p\in S(B)$ then we say that p does not fork over A if $bnd(p) = bnd(p|A)$. Otherwise, p forks over A. If $q\in S(A)$ and $p\subseteq q$ then we say that q is a nonforking extension of p if q does not fork over A; otherwise q is a forking extension of p. A formula $\varphi(x)$ forks over A if any complete type whose domain contains A and which contains $\varphi(x)$ forks over A.

Definition A is independent from B over C, or $A \underset{C}{|} B$ if for all $\bar{a}\subseteq A$, $tp(\bar{a}/BC)$ does not fork over C. If this is not the case then we say that A forks with B over C and denote it by $A \underset{C}{\not|} B$. A family $\{B_i\,|\,i\in I\}$ is independent over A if $A_i \underset{A}{|} \bigcup\{A_j\,|\,j\in I\setminus\{i\}\}$ is true for all $i \in I$.

The following properties of the independence relation are A.2 to A.5 from [M].

Existence	For any A,B,C there is an A-automorphism f of $\mathcal{M}$ such that $f(B) \mathop{\vert}\limits_{A} C$.
Monotonicity	$B \mathop{\vert}\limits_{A} C$ and $A \subseteq A' \subseteq A \cup C$, $C' \subseteq A \cup C$ implies $B \mathop{\vert}\limits_{A'} C'$.
Transitivity	$A \subseteq A'$ $B \mathop{\vert}\limits_{A'} C$ and $B \mathop{\vert}\limits_{A} A'$ implies $B \mathop{\vert}\limits_{A} C$.
Symmetry	$B \mathop{\vert}\limits_{A} C$ implies $C \mathop{\vert}\limits_{A} B$.

The following is A.10 in [M]: for all A,B there is a countable $C \subseteq B$ such that $A \mathop{\vert}\limits_{C} B$.

<u>The Open Mapping Theorem</u> Suppose that $A \subseteq B$ and let F be the set of all types from $S(B)$ which do not fork over A. Then the restriction map, which carries p to $p|A$, is onto, continous and open.

Proof Theorem 5.12 in [LP].

<u>Proposition 1.1.1</u> If $A \subseteq B$ and $p \in S(B)$ then p forks over A if and only if there exists a formula $\phi(x,\bar{b}) \in p$ which forks over A.

Proof [Sh] Theorem III.1.1(5).

Suppose that $p \in S(A)$ and f is an automorphism of the monster model. Then by $f(p)$ we mean $\{\phi(\bar{x},f(\bar{a})) \mid \phi(\bar{x},\bar{a}) \in p\}$. Note that $f(p) \in S(f(A))$.

<u>Proposition 1.1.2</u> If $\lambda \geq \aleph_1$, M is λ-saturated and λ-homogeneous, $A \subseteq M$, $|B| < \lambda$, $p,q \in S(M)$ and $\mathrm{bnd}(p)=\mathrm{bnd}(q)$ then there exists an A-automorphism f of the monster model which maps M onto M such that $f(p)=q$.

Proof [B] Theorem III.2.36.

Proposition 1.1.3 Suppose that $\beta\leq\gamma\in O(T)$ and $bnd(\bar{a}/B)=\gamma$.

(a) If $M\supseteq B$ is $(|B|+\aleph_0)^+$-saturated, then there exists $q\in S(M)$ which extends p such that $bnd(q)=\beta$.

(b) If $bnd(\bar{b}/C)=\beta$ then there exists D such that $tp(\bar{b}C)=tp(\bar{a}D)$ and $\bar{a} \underset{D}{\mid} B$.

(c) If $\beta\geq\beta_1\geq\beta_2\geq\ldots$ is a descending chain in the fundamental order, then there are $B\subseteq B_1\subseteq B_2\subseteq\ldots$ such that for all $i<j\in\omega$ $bnd(\bar{a}/B_i)=\beta_i$ and $\bar{a} \underset{B_i}{\mid} B_j$.

Proof (a) follows immediately from Theorem 2.3 in [LP]. To prove (b) let $\lambda=(|B|+|C|+\aleph_0)^+$ and let $M\supseteq B$ be λ-saturated and λ-homogeneous. By part (a) there exists an extension $q\in S(M)$ of p such that $bnd(q)=\beta$. After possibly replacing M by a B-automorphic image, we can assume that $\bar{a}\models q$. Further, let $N\supseteq C$ be such that $tp(N)=tp(M)$ and $\bar{b} \underset{C}{\mid} N$. Let f be an automorphism of the monster model which maps N onto M and let $\bar{b}'=f(\bar{b})$. Therefore $bnd(\bar{b}'/M)=bnd(\bar{a}/M)$ and by Proposition 1.1.2 there is $g\in Aut(\mathcal{M})$ which maps M onto M and such that $g(\bar{b}')\models tp(\bar{a}/M)$. Let h be an M-automorphism such that $h(g(\bar{b}'))=\bar{a}$. Let $D=h(g(f(C)))$. Since $h(g(f(\bar{b})))=\bar{a}$ we have $bnd(\bar{a}/D)=bnd(\bar{b}/C)=\beta$ and since $bnd(\bar{a}/M)=\beta$ we have $\bar{a} \underset{D}{\mid} M$ and by monotonicity $\bar{a} \underset{D}{\mid} C$.

(c) The finite case follows by induction from (a) and (b) and the infinite from the finite by compactness.

Definition A finite equivalence relation is a definable equivalence relation on $\mathcal{M}^m$, for some m, which has only finitely many classes.

The set of all A-definable finite equivalence relations on $\mathcal{M}^n$ we denote by $FE^n(A)$ and if the meaning of n is clear from the context, then we simply write FE(A). The following is Theorem III.2.8 in [Sh].

The Finite Equivalence Relation Theorem Suppose that $A\subseteq B$, $p\in S(A)$ and $q_1,q_2\in S(B)$ are distinct nonforking extensions of p. Then there is an A-definable equivalence relation E such that

$$q_1(\bar{x}) \wedge q_2(\bar{y}) \vdash \neg E(\bar{x},\bar{y}).$$

If $p \in S(A)$ then the multiplicity of p is the smallest cardinal λ such that p does not have λ^+ pairwise contradictory nonforking extensions. p is stationary if its multiplicity is 1. If $p \in S(A)$ is stationary, then for all $B \supseteq A$ there exists a unique nonforking extension of p to B, which we denote by $p|B$.

Following Makkai, see [M], we consider T^{eq} as a many sorted theory built as follows. The language of T^{eq} is that of T and for each 2n-ary formula $E(\bar{x},\bar{y})$ such that $T \vdash$ 'E is an equivalence relation' we have a sort S_E and an n-ary operation symbol p_E. The axioms of T^{eq} are those of T and those saying that 'p_E is a surjective map of n-tuples onto S_E and $p_E(\bar{x})=p_E(\bar{y})$ iff $E(\bar{x},\bar{y})$'. We will refer to $p_E(\bar{x})$ as the name for an E-class to which $\bar{x}$ belongs.

There is a natural one to one correspondence between models of T and T^{eq} and passage from one to the other preserves all essential properties of a theory. Further in the text, $\mathcal{M}^{eq}$ will denote the monster model of T^{eq} corresponding to $\mathcal{M}$.

Let E be an A-definable equivalence relation on $\mathcal{M}$. Then the equivalence classes of E have names in $\mathcal{M}^{eq}$ in the following sense. Let $\bar{a} \subseteq A$ and let $\phi(\bar{x},\bar{y},\bar{a})$ be a formula defining E. By compactness there exists a formula $\psi(\bar{u}) \in tp(\bar{a})$ such that $\psi(\bar{u})$ proves that $\phi(\bar{x},\bar{y},\bar{u})$ is an equivalence relation. Consider the formula $F(\bar{x}\bar{u},\bar{y}\bar{v})$ defined by $\bar{u}=\bar{v} \wedge ((\phi(\bar{x},\bar{y},\bar{u}) \wedge \psi(\bar{u})) \vee \neg\psi(\bar{u}))$. Clearly, F defines an equivalence relation and $E(\bar{x},\bar{y})$ iff $F(\bar{x}\bar{a},\bar{y}\bar{a})$. Therefore the set $\{z \in S_F \mid (\exists\bar{x}) z=p_F(\bar{x}\bar{a})\}$, which is A-definable, can be considered as the set of names of E-classes and we write $\bar{b}/E$ for $p_F(\bar{b}\bar{a})$. If $B \subseteq \mathcal{M}^n$ then by B/E we denote $\{\bar{b}/E \mid \bar{b} \in B\}$ and if $p=tp(\bar{b}/A)$ then by p/E we mean the type of $\bar{b}/E$ over A/E.

Equality is an equivalence relation on n-tuples of $\mathcal{M}$, so when operating in $\mathcal{M}^{eq}$ we sometime identify an n-tuple with its name by simply omitting its bar.

$\bar{a}$ is algebraic over B if there exists a formula $\phi(\bar{x})\in tp(\bar{a}/B)$ which has only finitely many solutions in $\mathcal{M}$, and we refer to such a ϕ as a witness for algebraicity. $\bar{a}$ is definable in B if there is $\phi(\bar{x})\in tp(\bar{a}/B)$ such that $\bar{a}$ is the only solution of ϕ in $\mathcal{M}$. A is algebraic in B if every tuple of A is so and similarly for definable. The set of all elements which are algebraic in B is the algebraic closure of B and is denoted by acl(B); B is algebraically closed if B=acl(B). Similarly, the definable closure is defined and denoted by dcl(A). If we are considering in T^{eq} then we write $acl^{eq}(A)$ and $dcl^{eq}(A)$. We say that $tp(\bar{b}/A)$ is algebraic if $\bar{b}\in acl(A)$.

The following is a consequence of the finite equivalence relation theorem, see [M] B.4.

<u>Theorem 1.1.4</u> (in T^{eq}) Every type over an algebraically closed set is stationary. If $B\subseteq B'\subseteq acl(B)$, $A\subseteq A'\subseteq acl(A)$ and $C\subseteq C'\subseteq acl(C)$ then $A \underset{C}{|} B$ if and only if $A' \underset{C'}{|} B'$.

<u>Definition</u> For any $\bar{c}$ and A, the strong type of $\bar{c}$ over A, denoted by $stp(\bar{c}/A)$ is $\{E(\bar{x},\bar{c})\,|\,E\in FE(A)\}$.

Clearly, $stp(\bar{c}/A)\vdash tp(\bar{c}/A)$ and the converse is true if $tp(\bar{c}/A)$ is stationary, in particular if A is algebraically closed in $\mathcal{M}^{eq}$ or if A=M. By a strong type extending $p\in S(A)$ we mean a strong type of a realization of p over A. Also, we say that $stp(\bar{c}/A)=stp(\bar{b}/A)$ if $tp(\bar{c}/acl^{eq}(A))=tp(\bar{b}/acl^{eq}(A))$.

Strong types are in general not preserved by automorphisms of the monster model. By a strong A-automorphism of $\mathcal{M}$ we mean an A-automorphism which preserves strong types over A.

Let $r=tp(\bar{c}/A)$ be stationary. Hrushovski's 'quantifier' $d_r\bar{x}$ is defined as follows. Let $\phi(\bar{x},\bar{y})$ be an L-formula. By $(d_r\bar{x})\phi(\bar{x},\bar{y})$ we denote the formula defining ϕ in $r|\mathcal{M}$. Since r is stationary $(d_r\bar{x})\phi(\bar{x},\bar{y})$ is over A. If r is a strong type over A then $(d_r\bar{x})\phi(\bar{x},\bar{y})$ is over $acl^{eq}(A)$. $d_r\bar{y}$ is read 'for a generic $\bar{y}$ realizing r'.

Suppose that $p,q \in S(A)$ are stationary, $\bar{a} \models p$ $\bar{b} \models q$ and $\bar{a} \underset{A}{|} \bar{b}$. Then $tp(\bar{a}\bar{b}/A)$ does not depend on the particular choice of $\bar{a}$ and $\bar{b}$ and we denote it by $p \otimes q$. If $p=q$ then we write p^2 instead of $p \otimes p$. Similarly, we define $p_1 \otimes p_2 \otimes \ldots \otimes p_n$, p^n and p^α for ordinal α.

Definition A Morley sequence in $p \in S(A)$ is an independent set over A of realizations of p.

If I is an infinite Morley sequence in $p \in S(A)$ then I is indiscernible over A and all members of I realize the same strong type over A. Thus if p is stationary and $|I| = \varkappa$ then $I \models p^\varkappa$.

Let p,q be stationary. They are **parallel** if the corresponding nonforking extensions to global types are equal. It is clear that parallelism is an equivalence relation on the set of stationary types. p **is based on A** if there exists a stationary type over B parallel to p.

Theorem 1.1.5 (in T^{eq}) Suppose that p is stationary. Consider all definably closed subsets of $\mathcal{M}^{eq}$ on which p is based. Among them there exists a minimal one which is the intersection of all of them.

Proof Theorem III.6.10 in [Sh].

For a stationary type p the minimal definably closed set on which p is based is called **the canonical base of** p and is denoted by Cb(p). Since all strong types are stationary they have canonical bases, and we write Cb(b/A) instead of Cb(stp(b/A)). Since stp(a/A) is based on $acl^{eq}(A)$ which is definably closed by the previous theorem we have $Cb(a/A) \subseteq acl^{eq}(A)$. If tp(a/A) is stationary and based on B then $Cb(a/A) \subseteq dcl^{eq}(B)$. By [M] B.2' every stationary type p is based on an infinite Morley sequence I in p, hence $Cb(p) \subseteq dcl^{eq}(I)$.

2.2 Orthogonality, regular types, ranks and superstability, coordinatization, small theories

<u>Definition</u> Let $p \in S(A)$, $q \in S(B)$ and $C \subseteq A$.

(a) p and q are almost orthogonal, $p \perp^{a} q$, if whenever a and b realize nonforking extensions of p and q to AB, then $a \underset{AB}{|} b$.

(b) $p \perp^{a} C$ if p is almost orthogonal to every type in S(C).

(c) p is orthogonal to q, $p \perp q$, if whenever p' and q' are nonforking extension of p and q then $p' \perp^{a} q'$.

(d) $p \perp D$ iff p is orthogonal to all types in S(D).

(e) $p \not\perp q$ means that p is not orthogonal to q.

Suppose that $p \in S(B)$, $stp(B)=stp(B')$ and $p' \in S(B')$ is a conjugate of p. Then $p \perp \emptyset$ if and only if $p \perp p'$. This fact is Theorem V.3.4 in [Sh].

<u>Definition</u> (a) B is dominated by C over A if for all D $D \underset{A}{|} C$ implies $B \underset{A}{|} C$.

<u>Definition</u> Let $p,q \in S(A)$ and $r \in S(B)$ be stationary.

(a) p is dominated by q if there are $a \models p$ and $b \models q$ such that b dominates a over A.

(b) p is eventually dominated by r, p r, if for some $C \supseteq AB$ $p|C$ is dominated by $r|C$ over C. $p \,\square\, r$ means p r and r p.

The pre-weight of B over A, p-wt(B/A), is the supremum of all cardinals $\varkappa$ for which there is a family $\{C_i \mid i \in I\}$ which is independent over A such that $|I|=\varkappa$ and B forks with C_i over A for all $i \in I$. The weight of B over A, wt(B/A), is the supremum of all p-wt(B/A') with A' ranging over all sets such that $B \underset{A}{|} A'$. For

$p=tp(B/A)$ we define $wt(p)=wt(B/A)$. If B is a set of tuples each having weight one over A, then forking over A is an equivalence relation on B. Also, nonorthogonality is an equivalence relation on the set of stationary, weight one types.

Definition $p\in S(A)$ is regular if whenever q is a forking extension of p then $p\perp q$.

Theorem 1.2.1 Every regular type has weight one.

Proof This is Theorem V.3.1 in [Sh].

If $p\in S(A)$ is a regular type and B is any set then $\dim(p,B)$ is the size of a maximal subset of $B\cap p(\mathcal{M})$ which is independent over A; it is well defined by D.5 from [M]. If the meaning of p is clear from the context, then we simply write $\dim(B)$.

Definition T is superstable if there is no infinite strictly decreasing sequence of bounds in $O(T)$.

As an immidiate consequence of superstability we have that every type does not fork over a finite subset of its domain.

Definition Define U-rank of complete types as follows.

(a) $U(p)\geq 0$ for all p.

(b) $U(p)\geq\alpha+1$, where α is an ordinal if there is a forking extension q of p such that $U(q)\geq\alpha$.

(c) $U(p)\geq\lambda$ where λ is a limit ordinal if $U(p)\geq\xi$ for all $\xi<\lambda$.

(d) $U(p)=\alpha$ if α is the smallest ordinal such that $U(p)\geq\alpha$, if no such ordinal exists then $U(p)=\infty$.

If $U(p)$ has ordinal value then we say that p is superstable. Hence T is superstable if and only if every type is superstable. Also, $U(p)=0$ if and only if p is algebraic. The following is Theorem 8 in [L1].

U-rank inequalities

$$U(a/Ab) + U(b/A) \le U(ab/A) \le U(a/Ab) \oplus U(b/A)$$

As an immidiate corollary we have that a superstable type of U-rank ω^{α} is regular (Corollary 2 in [L2]).

Definition R-rank is defined for all, possibly incomplete, types as follows.

(a) $R(\phi(x,\bar{a}))\ge 0$ if $\phi(x,\bar{a})$ is consistent.

(b) $R(\phi(x,\bar{a}))\ge\alpha+1$ if there exists a formula $\psi(x,\bar{b})$ which forks over $\bar{a}$ such that $R(\psi(x,\bar{b}))\ge\alpha$ and $\models \forall x(\psi(x,\bar{b})\rightarrow\phi(x,\bar{a}))$.

(c) $R(\phi(x,\bar{a}))\ge\lambda$, where λ is a limit ordinal, if for all $\alpha<\lambda$ $R(\phi(x,\bar{a}))\ge\alpha$.

(d) $R(p) = \sup\{R(\phi(x,\bar{a}))\,|\,\phi(x,\bar{a})\in p\}$.

R-rank was introduced by Shelah (he calls it $R(p,L,\infty)$) in [Sh]). The basic properties of R were proved there, such as: if $q\subseteq p$ are complete types then $R(p)\le R(q)$ and p is a forking extension of q if and only if $R(p)<R(q)$; T is superstable if and only if every complete type has ordinal R-rank.

Both U and R are well behaved with respect to algebraicity by Proposition 4.42 in [L]. If $a\in acl(bA)$ then:

$$R(a/A) \le R(ab/A) = R(b/A) \quad \text{and} \quad U(a/A) \le U(ab/A) = U(b/A).$$

Theorem 1.2.2 If p is a superstable, stationary type then it is domination equivalent to a finite product of regular types. If, in addition, $p\in S(M)$ and M is $\aleph_1$-saturated then there are regular $q_i\in S(M)$ such that $p \,\square\, q_1\otimes q_2\otimes\ldots\otimes q_n$.

Proof For T superstable this is Theorem V.3.9 in [Sh]; the proof of the same fact from [M] loosens the superstability assumption.

Theorem 1.2.3 (in T^{eq}) Let $U(p) = \omega^{\alpha_1}\cdot n_1 + \omega^{\alpha_2}\cdot n_2 + \ldots + \omega^{\alpha_k}\cdot n_k$ where $\alpha_1\ge\alpha_2\ge\ldots\ge\alpha_k$.

(a) There exists a type q such that $U(q)=\omega^{\alpha_k}$ and $q \not\perp p$.

(b) $wt(p) \leq n_1+n_2+\ldots+n_k$.

Proof Propositions 4, 5 and 6 in [L2].

Definition If q is a regular, stationary type then p is q-semiregular if $p \square q^m$ for some m, and there is an $\aleph_1$-saturated model $M \supseteq A \cup dom(q)$ such that for all $\bar{a}$ realizing a nonforking extension of p to M there is a sequence $b_1 b_2 \ldots b_n$ with $tp(b_i/M) \square q$ and $a \in acl(b_1 b_2 \ldots b_n M)$. p is semiregular if there exists a regular type q such that p is q-semiregular.

The following is a version of Shelah's theorem on semiregular types, V.4.11 in [Sh].

Theorem 1.2.4 (in T^{eq}) If $tp(a/A)$ is superstable then there is $c \in acl(aA)$ such that $U(c/A)=\omega^{\alpha}\cdot n$, where α is the smallest ordinal exponent in the Cantor normal form of $U(a/A)$ and $wt(c/A)=n$. $stp(c/A)$ is semiregular and whenever $C \supseteq A$ and $U(c/C) \geq \omega^{\alpha}$ then $stp(c/C) \not\perp stp(c/A)$.

Proof Without any loss let $A=\emptyset$. Let $r=tp(b/B)$ be stationary such that $U(r)=\omega^{\alpha}$, $a \mathrel{|} B$ and $a \not\downarrow_B b$. Let $c \in Cb(Bb/a) \setminus acl(\emptyset)$. Then $c \in acl(a) \cap dcl(I)$ where $I=B_1 b_1 B_2 b_2 \ldots B_k b_k$ is a Morley sequence in $stp(Bb/a)$. $D=B_1 B_2 \ldots B_k$ is a Morley sequence in $stp(B/a)$, which does not fork over $\emptyset$, so $a \mathrel{|} D$ and hence $c \mathrel{|} D$. Therefore:

$$U(c) = U(c/D) \leq U(b_1 b_2 \ldots b_k/D) < \omega^{\alpha+1},$$

and we conclude that $U(c)=\omega^{\alpha}\cdot n$; otherwise, Theorem 1.2.3(a) would contradict the minimality of α.

Since $a \mathrel{|} D$ and $a \not\downarrow B_i b_i$ we have $a \not\downarrow_D b_i$ and, by minimality of α, $U(b_i/D) \geq \omega^{\alpha}$. Thus, $U(b_i/D)=\omega^{\alpha}$ and $b_i \models r|D$, for all $i \leq k$. Let $m \leq k$ be the greatest integer such that $\{b_1 b_2 \ldots b_m\}$ is independent over D. We show that $m=n$, that implies $wt(c) \geq n$ and by Theorem 1.2.3(b) $m=n$.

$U(b_1 b_2 \ldots b_m/D)=\omega^{\alpha}\cdot m$ follows immidiately, as well as $U(b_{m+1} b_{m+2} \ldots b_k/b_1 b_2 \ldots b_m D) < \omega^{\alpha}$. Hence:

$U(b_1b_2..b_m/D) \oplus U(b_{m+1}b_{m+2}..b_k/b_1b_2..b_mD) < \omega^\alpha \cdot m \oplus \omega^\alpha = \omega^\alpha \cdot (m+1)$,

and $\quad U(b_1b_2...b_k/D) < \omega^\alpha \cdot (m+1)$.

Since $c \mid D$ and $c \in dcl(b_1b_2...b_kD)$ we have:

$$\omega^\alpha \cdot n = U(c) = U(c/D) \leq U(b_1b_2...b_k/D) < \omega^\alpha \cdot (m+1) \quad \text{and} \quad m=n.$$

Suppose that $C \supseteq A$ and $U(c/C) \geq \omega^\alpha$ and assume that $b_1b_2...b_kD$ are chosen so that $b_1b_2...b_kD \underset{C}{\mid} C$. $c \in dcl(b_1b_2...b_kD)$ and $U(c/C) \geq \omega^\alpha$ implies $U(b_1b_2...b_k/DC) \geq \omega^\alpha$. Therefore, for some $1 \leq i \leq k$ we must have $U(b_i/DC) = \omega^\alpha$, i.e. $tp(b_i/DC)$ does not fork over B_i. But $U(b_i/cD) < \omega^\alpha$, hence $b_i \underset{DC}{\not\mid} c$ and, since $tp(b_i/B_i) \square r$ by the remark after the definition of orthogonality, we have $tp(c/C) \not\perp r$ and hence $tp(c/C) \not\perp tp(c)$.

Now, let q be regular and $q \not\perp tp(c)$. With similar reasoning as in the previous paragraph we get $q \not\perp r$, hence $tp(c) \square r^n$, and $tp(c)$ is r-semiregular.

<u>Corollary 1.2.5</u> (in T^{eq}) If $tp(a/A)$ is regular and superstable then there exists $b \in dcl(aA)$ such that $U(b/A) = \omega^\alpha$, where α is the smallest ordinal exponent in the Cantor normal form of $U(a/A)$.

Proof By the Theorem there is $c \in acl(aA)$ such that $U(c/A) = \omega^\alpha$. Each aA-conjugate of c forks with c over A, by regularity of $tp(a/A)$. Hence if d is (a name for) the set of all aA-conjugates of c then $d \in dcl(aA)$ and $\omega^\alpha \leq U(d/A) < \omega^\alpha \cdot 2$. By minimality of α $U(d) = \omega^\alpha$.

<u>Definition</u> Let p,q be types, possibly incomplete. p is q-internal if there exists an $\aleph_1$-saturated model $M \supseteq dom(p) \cup dom(q)$ such that whenever $a \models p$ and $tp(a/M)$ does not fork over $dom(p)$ then there are $b_1, b_2, ..., b_k$ realizing q such that $a \in dcl(b_1b_2...b_kM)$.

The notion of internality is introduced by Hrushovski in [H1]. The following two propositions are from there.

<u>Proposition 1.2.6</u> Suppose that $p = stp(a/A)$ is q-internal where q is over A. Let $\mathcal{P}$ and $\mathcal{Q}$ be the sets of all realizations of p in q

in $\mathcal{M}$. Then there are integers m,k and an A-definable function $f:\mathcal{P}^k \times \mathcal{Q}^m \to \mathcal{M}$ such that whenever $a'\bar{c} \models p^{k+1}$ then there is $\bar{b} \in \mathcal{Q}^m$ such that $a'=f(\bar{c},\bar{b})$ ($\mathcal{X}^n$ denotes the Cartesian power of $\mathcal{X}$).

Proof Without loss of generality $A=\emptyset$. Let M and $\bar{b} \subseteq \mathcal{Q}$ be such that $a \mathop{|} M$ and $a \in dcl(\bar{b}M)$. Let $a\bar{b}a_1\bar{b}_1a_2\bar{b}_2\ldots a_k\bar{b}_k$ be a Morley sequence in $stp(a\bar{b}/M)$, long enough so that $a \in dcl(\bar{b}a_1\bar{b}_1a_2\bar{b}_2\ldots a_k\bar{b}_k)$. Since $tp(a/M)$ does not fork over $\emptyset$, $\{a,a_1,a_2,\ldots,a_k\}$ is independent. Thus if $\bar{c}=a_1a_2\ldots a_k$ and $\bar{b}=\bar{b}_1\bar{b}_2\ldots\bar{b}_k$ we have $a\bar{c} \models p^{k+1}$, $\bar{b} \in \mathcal{Q}^m$ and $a=f(\bar{c},\bar{b})$ for some definable function $f:\mathcal{P}^k \times \mathcal{Q}^m \to \mathcal{M}$. Since we can move $a\bar{c}M$ by an M-automorphism to any $a'\bar{c}'M' \models tp(a\bar{c}M)$ the conclusion follows.

<u>Proposition 1.2.7</u> (in T^{eq}) (a) If $stp(a/B) \not\perp q$ and $U(q)<\omega^{\alpha}$ then there is $c \in dcl(aB)\setminus acl(B)$ such that $U(c/B)<\omega^{\alpha}$.

(b) If $A \subseteq B$, $p=tp(a/B)$, $q \in S(A)$ and $p \not\perp q$ then there exists $c \in dcl(aB)$ such that $r=tp(d/B)$ is q-internal and nonorthogonal to q.

Proof (a) Let $A=dom(q)$ and $B \cup A \subseteq C$, $a \mathop{|}\limits_{B} C$, $b \models q$, $b \mathop{|}\limits_{A} C$ and $a \mathop{\not|}\limits_{C} b$. Pick $c \in Cb(bC/aB)\setminus acl(B)$. $a \mathop{|}\limits_{B} C$ implies $c \mathop{|}\limits_{B} C$ and since $c \mathop{\not|}\limits_{B} bC$, we conclude that $c \mathop{\not|}\limits_{BC} b$ i.e. $tp(b/B) \not\perp q$. c is definable over B from a finite Morley sequence in $stp(bC/aB)$, say $C_1b_1C_2b_2\ldots C_kb_k$. Clearly, $U(b_i/C_1C_2\ldots C_k)<\omega^{\alpha}$ for all $i \le k$, so $U(c/C_1C_2\ldots C_k)<\omega^{\alpha}$. $tp(C/aB)$ does not fork over B, so $a \mathop{|}\limits_{B} C_1C_2\ldots C_k$. It follows that $c \mathop{|}\limits_{B} C_1C_2\ldots C_k$ and $U(c/B)<\omega^{\alpha}$. After replacing c by (the name for) the set of all conjugates of $tp(c/aB)$ we have also $c \in dcl(aB)$.

(b) Let C, b and $c \in Cb(bC/aB)\setminus acl(B)$ be as in the proof of part (a). Then $tp(c/B)$ is q-internal since $c \in dcl(C_1b_1C_2b_2\ldots C_kb_k)$, $c \mathop{|}\limits_{A} C_1C_2\ldots C_k$, and $tp(b_i/C_1C_2\ldots C_k)$ is a nonforking extension of q for all $i \le k$. The same is true for any aB-conjugate of c, hence it is true for $tp(d/B)$, where d is the name for the set of all of them. Finally, $tp(c/B) \not\perp q$ implies $tp(d/B) \not\perp q$.

$p \in S(A)$ is isolated if there exists a formula $\phi(x) \in p$ such that if $\models \phi(a)$ then $a \models p$. The topological context of isolation is clear; $p \in S(A)$ is isolated if and only if it is an isolated point in the topological space $S(A)$. Therefore, as a consequence of the Open Mapping Theorem we have that if q is a nonforking extension of p and q is isolated then p is isolated, too.

Isolation is transitive in the sense that if $tp(\bar{a}/\bar{b})$ and $tp(\bar{b}/A)$ are isolated, then so is $tp(\bar{a}\bar{b}/A)$ and vice versa.

For a fixed set A CB_A-rank, the Cantor-Bendixson rank, is defined for $p \in S(A)$ as follows. For ordinals α, $CB_A(p)=\alpha$ if p is an isolated point in $S(A) \setminus \{q \mid CB_A(q)<\alpha\}$; otherwise $CB_A(p)=\infty$. If the meaning of A is clear from the context then we write CB(p).

B is atomic over A if for all $\bar{b} \subseteq B$ $tp(\bar{b}/A)$ is isolated. B is almost atomic over A if for all $\bar{b} \subseteq B$ there exists a finite $A_0 \subseteq A$ such that for all finite $A_1 \supseteq A_0$ $tp(\bar{b}/A_1)$ is isolated. M is prime over A if for each $N \supseteq A$ there is an elementary embedding of M into N. T is small if $|S(\emptyset)|=\aleph_0$. If T is small and A is finite then $|S(A)| \le \aleph_0$ and $CB_A(p)$ has ordinal value for all $p \in S(A)$.

<u>Proposition 1.2.8</u> If T is small and A is countable then there exists a countable model M which is almost atomic over A; if, in addition, A is finite, then there exists a prime model over A which is atomic over A and is unique up to an A-isomorphism.

<u>Proposition 1.2.9</u> If T is small and superstable then for all countable A and N there exists a countable model $M \supseteq A \cup N$ which is almost atomic over $A \cup N$ and is dominated by A over N.

<u>Definition</u> (a) A type $p \in S(A)$ is almost strongly regular (or aSR) via $\varphi(x)$ if p is nonalgebraic and for any stationary type $q \in S(B)$ whose domain contains A, if $\varphi(x) \in q$ then either $p \perp q$ or q is a nonforking extension of p.

(b) p is not-so-strongly-regular (or sR) via $\varphi(x)$ if p is stationary and it is a nonforking extension of a type which is aSR via $\varphi(x)$. p is sR if it is sR via some $\varphi(x)$.

We note that each sR type is regular.

Theorem 1.2.10 Suppose that T is small and superstable. If $q \not\perp N$ then there is a sR type $p \in S(N)$ such that $q \not\perp p$.

Proof Theorem D.17 in [M].

Definition A type $p \in S(A)$ is NENI if it is stationary and whenever B is finite then $p|AB$ is isolated.

1.3 Stable groups

By a stable group we mean a type-definable subset $G \subseteq \mathcal{M}$ such that there exists a definable function, possibly with parameters from $\mathcal{M}$, whose domain is G^2, whose range is G and which satisfies the group laws; in that case we denote the function by $\cdot$ and, as usual, we define $^{-1}$. We also say that G is ∞-definable; it is definable if G is a definable set.

Let $\mathcal{H} = \{H_i \mid i \in I\}$ be a family of definable groups. We say that $\mathcal{H}$ is uniformely definable if there exists a formula $\varphi(x, \bar{y})$ without parameters, and a family $\mathcal{A} = \{\bar{a}_i \mid i \in I\}$ of tuples of elements of $\mathcal{M}$ such that for all $i \in I$ H_i is defined by $\varphi(x, \bar{a}_i)$. The following is from [BS].

The Baldwin-Saxl Condition Let G be a stable group and let $\mathcal{H} = \{H_i \mid i \in I\}$ be a uniformely definable family of groups. Then there exists an integer n such that for all $\mathcal{H}' = \{H_i \mid i \in I'\} \subseteq \mathcal{H}$ there are $i_1 i_2 \ldots i_n \in I'$ such that: $G \cap \bigcap \mathcal{H}' = G \cap H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_n}$.

Proposition 1.3.1 If G is a stable group then there exists a definable group $H \supseteq G$ such that the multiplications of G and H coincide on G. Moreover, if G is ∞-definable over A, then G is

the intersection of $\leq\aleph_0$ A-definable groups.
Proof This iresult is from [H1], see also Theorem 5.18 in [P].

Let G be a stable group and let A be definable. We say that A is left-generic of G if there are $g_1,g_2,\dots,g_n\in G$ such that:

$$G\subseteq g_1A\cup g_2A\cup\dots\cup g_nA$$

A formula is called left-generic iff the set defined by it is so and a type is called left-generic iff all the formulas in it are left-generic. Similarly, we define the concepts of a right-generic set, right-generic formula and that of a right-generic type. the following is 5.7 in [P].

Proposition 1.3.2 Every left-generic set is at the same time right-generic and vice versa. The same holds for formulas and types. By both left and right translation G acts transitively on the set of generic types in S(G).

According to Proposition 1.3.2 we say that a set, formula or type is generic if it is, equivalently, left- or right-generic. Elements realizing generic types are called generic elements. The following is derived from Chapter 5 in [P].

Proposition 1.3.3 Let G be stable. Then there exists a generic type, generic types do not fork over $\emptyset$ and every element of G is a product of two generic elements.

If $p\in S(G)$ is generic, then the set of all elements of G for which p is fixed by a translation by them, forms an ∞-definable subgroup of G whose generic type is a generic of G, which is called the principal generic.

Let $A\subseteq G$ and let K be a definable group. By A/K we denote the set $\{a_H \mid a\in A\}$ of right cosets of A modulo $H=G\cap K$. Note that if A is definable then A/K is definable in G^{eq}, and if A is ∞-definable then A/K is ∞-definable, as well.

A superstable group is a stable group whose generic type has

R-rank; by 1.3.3 this is equivalent to: every type of an element of a group has R-rank. Generic types have maximal possible R-rank and U-rank, and by U(G) we mean the U-rank of a generic type. The following is III.8.1 in [BeL].

U-rank inequalities for groups

If $H \leq G$ are superstable and H is definable, then:

$$U(G/H) + U(H \cap G) \leq U(G) \leq U(G/H) \oplus U(H \cap G)$$

Proposition 1.3.4 If G is an ∞-definable superstable group then G is an intersection of $\leq \aleph_0$ definable supergroups of G which has the same R-rank and the same U-rank as G does.

Proof This result is from [H1], see 5.19 in [P].

Let $A \leq B$ be ∞-definable and superstable. A/B is not in general an object in T^{eq}, but by previous proposition it is closely approximated by elements of $\mathcal{M}^{eq}$; if $B \leq C$, where C is definable and $U(C)=U(B)$ then $U(A/C)$ does not depend on the particular choice of C. Thus when we write $U(A/B)$ we mean by that $U(A/C)$.

A **stable field** is a type-definable set endowed with two definable binary operations which satisfy the field laws. The following is from Theorem 1 in [CS].

Theorem 1.3.5 A superstable field is algebraically closed.

The following proposition is a version of Hrushovski's analysis from [H1], see also [H4]. The corollary is from [H4].

Proposition 1.3.6 Let G be ∞-definable and let H_i an ∞-definable group of automorphisms of G, for $1 \leq i \leq n$. Suppose that a generic type of G is nonorthogonal to a regular type p. Then there exists a definable group H, such that $G \cap H$ is a normal, H_i-invariant subgroup of G for $1 \leq i \leq n$, $G/G \cap H$ is p-internal and its generic type is nonorthogonal to p.

Proof Fact 1 in [H4], see also Theorem 2.26 in [P].

Corollary 1.3.7 If the generic of a field F is nonorthogonal to a regular type p, then F is p-internal.

Proof Let F^+ denote the additive group of the field and $F^{\cdot}$ the multiplicative one. Since the generic of F^+ is nonorthogonal to p it has an $F^{\cdot}$-invariant subgroup H such that F/H is p-internal and infinite. Since $F^{\cdot}$ acts transitively on $F^+\backslash\{0\}$ we must have $H=\{0\}$ and F^+ is p-internal.

Theorem 1.3.8 If G is an ∞-definable superstable group and $U(G)=\omega^{\alpha}\cdot n+\xi$ where $\xi<\omega^{\alpha}$, then there existss an ∞-definable abelian subgroup H of G such that $U(H)\geq\omega^{\alpha}$.

Proof Theorem VI.1.2 in [Be].

Definition Let $A\subseteq G$ be type definable. A is α-indecomposable if for every definable group $H\leq G$ $U(A/H\cap A)<\omega^{\alpha}$ implies that $A/H\cap A$ has exactly one element. (Here by $U(A/H\cap A)<\omega^{\alpha}$ we mean that $U(a_H)<\omega^{\alpha}$ holds for all $a\in A$)

Definition Suppose that $U(G)=\omega^{\alpha}\cdot n+\xi$ where $\xi<\omega^{\alpha}$. G is α-connected if there does not exist a proper subgroup $H<G$ such that $U(G/H)<\omega^{\alpha}$.

Proposition 1.3.9 If G is an ∞-definable superstable group and $U(G)=\omega^{\alpha}\cdot n+\xi$ where $\xi<\omega^{\alpha}$, then there is a unique ∞-definable α-connected subgroup H of G such that $U(H)=\omega^{\alpha}\cdot n$.

Proof See IV.4.6 in [Be].

The unique subgroup H from the previous proposition is called the α-connected component of G.

Proposition 1.3.10 Suppose that $p\in S(A)$ is a stationary type of an element of G, $U(p)=\omega^{\alpha}\cdot n$ and let $\mathcal{P}$ be the set of all realizations of p in G. Then $\mathcal{P}$ is α-indecomposable.

Proof Without loss of generality assume that G is saturated.

Suppose that $A \subseteq B$ and that $H \leq G$ is a subgroup definable over B. Let $g \models p|B$. Then $g_H \in dcl(g^\frown B)$ and since $U(g/B)=\omega^{\alpha}\cdot n$ we have that either $U(g_H/B)\geq\omega^{\alpha}$ or $U(g_H/B)=0$. Thus, either $U(\mathcal{P}/H)\geq\omega^{\alpha}$ or $U(\mathcal{P}/H)=0$. We *claim* that if the second possibility holds then $\mathcal{P}/H$ has exactly one element; that completes the proof of the proposition.

So, suppose that $U(\mathcal{P}/H)=0$. Let $\mathcal{H}$ be the set of all conjugates of H under $Aut_A(G)$ and let $K=\bigcap\mathcal{H}$. By the Baldwin-Saxl condition, there is $\{H_1,H_2,\ldots,H_k\}\subseteq\mathcal{H}$ such that $H_1\cap H_2\cap\ldots\cap H_k=K$. It follows that $U(\mathcal{P}/K)=0$ and $|\mathcal{P}/H|\leq|\mathcal{P}/K|$. K is definable over A, being fixed by all A-automorphisms of G. Thus, K induces an equivalence relation with finitely many classes on $\mathcal{P}$ and since p is stationary we must have $|\mathcal{P}/K|=1$. Then $|\mathcal{P}/H|=1$ as well, and the *claim* is proved.

The following is a superstable version of Zilber's Theorem on Indecomposables [Z1]. Proposition 6 in [H4] is a version in stable context.

<u>Zilber's Theorem on Indecomposables</u>

Suppose that $U(G)=\omega^{\alpha}\cdot n+\xi$ where $\xi<\omega^{\alpha}$ and that $\mathcal{A}=\{A_i\,|\,i\in I\}$ is a family of ∞-definable, α-indecomposable subsets of G each of them containing the unit element. Then the group $H\leq G$ generated by $\bigcup\mathcal{A}$ is ∞-definable and α-connected. Moreover, there exists a finite set $\{i_1,i_2,\ldots,i_k\}\subseteq I$ such that $H=(A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_k})^2$.

<u>Corollary 1.3.11</u> Suppose that $U(G)=\omega^{\alpha}\cdot n+\xi$ where $\xi<\omega^{\alpha}$ and that A is a definable, α-indecomposable subset of G. Then the group H generated by $A\cdot A^{-1}$, which is the smallest subgroup of G such that A is included in a single (right) coset of H, is definable; and α-connected; moreover, there exists $n\in\mathbb{N}$ such that $H=(A\cdot A^{-1})^n$.

Proof Let $\mathcal{A}=\{a^{-1}A\,|\,a\in A\}$. Every member of $\mathcal{A}$ is α-indecomposable, being a translate of an α-indecomposable set, and contains the unit element. The conclusion follows from the previous Theorem.

<u>Theorem 1.3.12</u> Suppose that A is an ∞-definable, α-connected

abelian group and G is an infinite, α-connected group of automorphisms of A and $F=End_G(A)$. Then F is an infinite, α-connected, definable field and A is definably an F-vector space.

Proof This is Zilber's theorem from [Z2].

Theorem 1.3.13 Suppose that G is a solvable group of automorphisms of an abelian group A, that both A and G are ∞-definable and α-connected and that A is G-minimal. Also suppose that $\omega^{\alpha} \leq U(A), U(G) < \omega^{\alpha+1}$. Then in T^{eq} there is a definable field F such that $\omega^{\alpha} \leq U(F) < \omega^{\alpha+1}$.

Proof This is a version of Nesin's theorem, see Theorem 3.8. in [P].

The following theorem is crucial for the proofs of theorems A and C. It is from [H1] (cf. Theorem 2.20 in [P]).

Theorem 1.3.14 Suppose that $p,q \in S(A)$, p is stationary and q-internal. Let $\mathcal{P},\mathcal{Q}$ denote, respectively, the sets of all realizations of p and q in $\mathcal{M}$. Then the group $G=Aut_{AQ}(\mathcal{P})$ of all AQ-automorphisms of $\mathcal{P}$ is ∞-definable over A. Moreover, if $U(p)=\omega^{\alpha}\cdot m$ then G is α-connected.

Sometimes we write $Aut_q(p)$ instead of $Aut_{AQ}(\mathcal{P})$.

2. THEORIES WITH NO DENSE FORKING CHAINS

2.1 Dimension and U_α-rank

Let $(P,\leq)$ be a partial order. For $p,q\in P$ we denote by $[p,q]_P$ the interval $\{x\in P\,|\,p\leq x\leq q\}$ ordered by (the restriction of) $\leq$, $(\leq,p]_P$ denotes $\{x\in P\,|\,x\leq p\}$ ordered by $\leq$, and similarly we define $(<,p)_P$, $(p,<)_P$ and $[p,\leq)_P$.

<u>Definition</u> Let $(P,\leq)$ be a nonempty partial order. Inductively, define the dimension $\dim(P)$ which is -1, an ordinal or ∞:

(1) $\dim(P)=-1$ if $|P|=1$.

(2) $\dim(P)\geq\alpha+1$ if there is an infinite decreasing chain $p_0>p_1>p_2>\ldots$ such that for every $i\in\omega$ $\dim([p_{i+1},p_i]_P)\geq\alpha$.

(3) $\dim(P)\geq\lambda$ where λ is a limit ordinal if $\dim(P)\geq\alpha$ for every $\alpha<\lambda$.

(4) $\dim(P)=\alpha$ iff α is the greatest ordinal for which $\dim(P)\geq\alpha$ holds; $\dim(P)=\infty$ iff $\dim(P)\geq\alpha$ holds for all ordinals α.

<u>Definition</u> Let α be an ordinal and let $(P,\leq)$ be a partial order. Inductively we define U_α-rank of $(P,\leq)$:

(1) $U_\alpha(P)\geq 0$ if $P\neq\emptyset$.

(2) $U_\alpha(P)\geq\beta+1$ iff there exists $p\in P$ such that $U_\alpha((\leq,p]_P)\geq\beta$ and $\dim([p,\leq)_P)\geq\alpha$.

(3) $U_\alpha(P)\geq\lambda$ where λ is a limit ordinal iff $U_\alpha(P)\geq\beta$ for all

ordinals $\beta<\lambda$.

(4) $U_\alpha(P)=\xi$, where ξ is an ordinal, if ξ is the greatest ordinal for which $U_\alpha(P)\geq\xi$. If no such ordinal exists then $U_\alpha(P)=\infty$

<u>Lemma 2.1.1</u> Let $(P,\leq_P)$ and $(Q,\leq_Q)$ be partial orders.

(a) If $f:P\to Q$ is strictly increasing then $\dim(P)\leq\dim(Q)$ and $U_\alpha(P)\leq U_\alpha(Q)$.

(b) If $\alpha\geq 1$ then $U_\alpha(P)+U_\alpha(Q) \leq U_\alpha(P\oplus Q)$ where $P\oplus Q$ is the set $P\times\{0\}\cup Q\times\{1\}$ ordered by $\{((p,0),(p',0)\,|\,p\leq p'\} \cup \{((q,0),(q',0)\,|\,q\leq q'\} \cup \{((p,0),(q,1))\,|\,p\in P, q\in Q\}$.

Proof: (a) is easy induction on $\dim(P)$ and $U_\alpha(P)$; we prove only (b). Q is embedded in $P\oplus Q$, so if $U_\alpha(Q)=\infty$ the conclusion follows by part (a).

Let $\xi=U_\alpha(Q)$. We use induction on ξ. For $\xi=0$ it is obvious and for $\xi=1$ it follows from the definition of U_α. Suppose that $\xi=\eta+1$ and let $q\in Q$ be such that $U_\alpha((\leq,q]_Q)=\eta$ and $U_\alpha([q,\leq)_Q)=1$. By the induction hypothesis

$$U_\alpha(P) + \xi = U_\alpha(P) + U_\alpha((\leq,q]_Q) \leq U_\alpha(P\oplus(\leq,q]_Q) = U_\alpha((\leq,q]_{P\oplus Q}).$$

On the other hand $1 = U_\alpha([q,\leq)_Q) = U_\alpha([q,\leq)_{P\oplus Q})$, and from the definition of U_α we get $U_\alpha(P)+U_\alpha(Q) = U_\alpha(P)+\xi+1 \leq U_\alpha(P\oplus Q)$.

The case when ξ is a limit ordinal is similar.

<u>Lemma 2.1.2</u> Let $(P,\leq)$ be a partial order. Then $\dim(P)=\infty$ if and only if there exists an embedding of rationals into $(P,\leq)$.

Proof: $\leftarrow$)is clear so we prove only $\rightarrow$). Assume that $\dim(P)=\infty$. Let α be an such that for all $p,q\in P$ $\dim([p,q]_P)\geq\alpha$ implies $\dim([p,q]_P)=\infty$. Since $\dim(P)\geq\alpha+1$ there is an infinite decreasing chain $p_0>p_1>p_2>\ldots$ such that for all $i\in\omega$ $\dim([p_{i+1},p_i]_P)\geq\alpha$, thus $\dim([p_{i+1},p_i]_P)=\infty$. Applying the same reasoning to each $[p_{i+1},p_i]_P$ for $i\in\omega$ in place of P we get infinite descending chains $p^i_0>p^i_1>p^i_2>\ldots$in $[p_{i+1},p_i]_P$ so that $\dim([p^i_{j+1},p^i_j]_P)=\infty$. Continuing in this way we get a chain in P isomorphic to the rationals.

Lemma 2.1.3 If $(P_i,\leq_i)$ for $i\leq n$ are nonempty partial orders then:

$$\dim(P_1xP_2x\ldots xP_n)=\max\{\dim(P_1),\dim(P_2),\ldots,\dim(P_n)\}.$$

(where $P_1xP_2x\ldots xP_n$ is ordered by the product order, i.e.

$(p_1,p_2,\ldots,p_n)\leq(p_1',p_2',\ldots,p_n')$ iff $p_1\leq_1 p_1'$ $p_2\leq_2 p_2'$ $\cdots$ $p_n\leq_n p_n'$).

Proof: Assume $n=2$. $\dim(P_1xP_2)\geq\max\{\dim(P_1),\dim(P_2)\}$ follows immediately from Lemma 2.1.1, so we prove the reverse inequality. Actually, we show by induction on ordinals α that $\dim(P_1xP_2)\geq\alpha$ implies $\max\{\dim(P_1),\dim(P_2)\}\geq\alpha$. For $\alpha=-1$ or 0 the claim is obvious, so we distinguish the following two cases:

Case 1 $\alpha=\beta+1$

Assume that $\dim(P_1xP_2)\geq\beta+1$. Then there is an infinite decreasing sequence $(p_0,p_0')>(p_1,p_1')>(p_2,p_2')>\cdots$ such that for all $i\in\omega$ $\dim([(p_{i+1},p_i),(p_{i+1}',p_i')]_{P_1xP_2})\geq\beta$. By the induction hypothesis for each $i\in\omega$ either $\dim([p_{i+1},p_i]_{P_1})\geq\beta$ or $\dim([p_{i+1}',p_i')]_{P_2})\geq\beta$ holds. Therefore either for infinitely many $i\in\omega$ $\dim([p_{i+1},p_i]_{P_1})\geq\beta$ or for infinitely many $i\in\omega$ $\dim([p_{i+1}',p_i')]_{P_2})\geq\beta$. Thus either $\dim(P_1)\geq\beta+1$ or $\dim(P_2)\geq\beta+1$ holds.

Case 2 α is a limit ordinal

Let $\alpha=\bigcup\{\alpha_\xi\,|\,\xi<\varkappa\}$ where $\varkappa=cf(\alpha)$. By the induction hypothesis for each $\xi<\varkappa$ at least one of $\dim(P_1)\geq\alpha_\xi$ and $\dim(P_2)\geq\alpha_\xi$ holds. Thus at least one of sets $\{\xi<\varkappa\,|\,\dim(P_1)\geq\alpha_\xi\}$ and $\{\xi<\varkappa\,|\,\dim(P_2)\geq\alpha_\xi\}$ is cofinal in $\varkappa$ and that means that either $\dim(P_1)\geq\alpha$ or $\dim(P_2)\geq\alpha$.

Thus we proved the lemma for $n=2$. The general case follows rather easily from this one.

Definition Let $A\subseteq B$, $p\in S(A)$ and $p\subseteq q\in S(B)$.

(a) $\dim(p|q)=\dim([bnd(q),bnd(p)]_{O(T)})$.

(b) $U_\alpha(p|q)=U_\alpha([bnd(q),bnd(p)]_{O(T)})$.

(c) $\dim(p)=\dim(p|r)$ where r is any algebraic extension of r, also $U_\alpha(p)=U_\alpha(p|r)$.

Further in the text, we will write $\dim(\bar{a}/B)$ instead of $\dim(tp(\bar{a}/B))$ and $\dim(\bar{a})$ instead of $\dim(\bar{a}/\emptyset)$. Similarly for U_α-rank.

If we allow $*$-types and not just types in the previous definitions, we get the notions of dim and U_α-rank of $*$-types as well.

Note that U_0 is the usual U-rank and $\dim(p)=0$ means exactly that p has ordinal U-rank. Also, $U_\alpha(p|q)=0$ implies $\dim(p|q)<\alpha$.

<u>Lemma 2.1.4</u> If $A\subseteq B$ and $C\subseteq acl(DA)$ then:

$$\dim(C/A|C/B) \le \dim(D/A|D/B) \quad \text{and} \quad U_\alpha(C/A|C/B) \le U_\alpha(D/A|D/B).$$

Proof By induction on $\dim(C/A|C/B)$. Suppose that $\beta_1>\beta_2>\ldots$ is an infinite descending chain between bnd(C/B) and bnd(C/A) such that $\dim(C/A|C/B) > \dim([\beta_{i+1},\beta_i])=\xi_i$. By Proposition 1.1.3(c) there is an increasing sequence of sets $A\subseteq M_1\subseteq M_2\subseteq\ldots$ such that for all $i<j$ $bnd(C/M_i)=\beta_i$, $C \mathop{|}_{M_1} A$ and $C \mathop{|}_{M_i} M_j$. Moreover, assume that $\cup M_i \mathop{|}_{CA} DB$. Then, by the induction hypothesis $\dim(C/M_i|C/M_j) \le \dim(D/M_i|D/M_j)$ for all $j<i$, and hence $\dim(D/M_i|D/M_j)\ge\xi_i$. From the independence assumptions we derive $D \mathop{|}_{M_o} A$, $D \mathop{|}_{M_i} M_j$ and $D \mathop{|}_{B} M_i$, for all $i<j$. Therefore $bnd(D/A) \ge bnd(D/M_o) > bnd(D/M_1) >\ldots. \ge bnd(D/B)$. If $\dim(C/A|C/B)=\xi+1$ then we could choose β_i's so that $\xi_i=\xi$, and if $\dim(C/A|C/B)$ is a limit ordinal then it can be chosen so that ξ_i's form a cofinal sequence. In both cases the conclusion follows.

A similar argument works for U_α.

<u>Lemma 2.1.5</u> If $p\subseteq q\subseteq r$ then $U_\alpha(q|r) + U_\alpha(p|q) \le U_\alpha(p|r)$. If r is algebraic then $U_\alpha(q) + U_\alpha(p|q) \le U_\alpha(p)$.

Proof Follows from Lemma 2.1.1(b).

<u>Definition</u> T has no dense forking chains if the order type of rationals can not be embedded into O(T).

As an immediate consequence of Lemma 2.1.2 we have that if T has no dense forking chains and $p \leq q$ then $\dim(p|q) < \infty$.

Theorem 2.1.6 Suppose that T has no dense forking chains.

(a) For any a,b and $A \subseteq B$ and $\alpha \geq 0$,

$$\dim(ab/A|ab/B) = \sup\{\dim(b/aA|b/aB), \dim(a/A|a/B)\}.$$

(b) (U_α-rank inequalities)

$$U_\alpha(b/aA) + U_\alpha(a/A) \leq U_\alpha(ab/A) \leq U_\alpha(b/aA) \oplus U_\alpha(a/A).$$

(c) Every type decomposes as a product of regular types.

Proof (a) is Lemma 10, (b) is Proposition 11 and (c) is Theorem 14 from [HLPTW].

We note the following instance of Theorem 2.1.6(a) and Lemma 2.1.4 that we will use often in this chapter: if $\dim(C_i/A) \leq \alpha$ for $1 \leq i \leq n$ and $B \subseteq \mathrm{acl}(C_1 C_2 \ldots C_n A)$ then $\dim(B/A) \leq \alpha$.

Proposition 2.1.7 Let $A \subseteq B$, $p \in S(A)$ and $p \leq q \in S(B)$. Then:

$$\dim(p|q) \geq \sup\{\dim(\bar{c}/A) \mid \bar{c} \in Cb(q)\}.$$

Proof: Without loss of generality, assume that $A = \emptyset$ and we operate in $\mathcal{M}^{eq}$. Let $\bar{c} \in Cb(q)$ and let $I = \bar{a}_1 \bar{a}_2 \ldots \bar{a}_n$ be a Morley sequence in (a stationarization of) q long enough so that $\bar{c} \in \mathrm{dcl}(I)$. Let $C = \mathrm{acl}(\bar{c})$ and we show that $\dim(p|q) \geq \dim(C)$; since $\dim(\bar{c}) \leq \dim(C)$ (by Lemma 2.1.4) this will imply the conclusion of the Proposition.

Let $I_k = \bar{a}_1 \bar{a}_2 \ldots \bar{a}_{k-1}$ for $k \leq n$, let $P = \{\beta \in O(T) \mid \beta < \mathrm{bnd}(C)\}$ and for $\beta \in P$ let D_β be such that $\mathrm{bnd}(C/D_\beta) = \beta$. For $k \leq n$, $\beta \in P$, define $p^k_\beta = \mathrm{bnd}(\bar{a}_k / I_k E_\beta)$ where E_β satisfies $tp(E_\beta/C) = tp(D_\beta/C)$ and $E_\beta \underset{C}{\downarrow} I$. We note that p^k_β does not depend on the particular choice of E_β. Actually, since C is algebraically closed, $tp(D_\beta/C)$ is stationary so it has a unique nonforking extension over CI, thus

$tp(E_\beta/CI)$ is uniquely determined and hence $tp(I/CE_\beta)$ is uniquely determined, too.

For natural $k\le n$ let $P_k=\{p^k_\beta \mid \beta\in P\}$ with the inherited order from $O(T)$. Now we show that $P_k\subseteq[\mathrm{bnd}(q),\mathrm{bnd}(p)]_{O(T)}$. From the definition of p^k_β we have $p^k_\beta\le \mathrm{bnd}(p)$, and $\mathrm{bnd}(q)\le p^k_\beta$ follows from:

$$\mathrm{bnd}(q) \le \mathrm{bnd}(\bar{a}_k/I_kC) = \mathrm{bnd}(\bar{a}_k/I_kCE_\beta) \le \mathrm{bnd}(\bar{a}_k/I_kE_\beta) = p^k_\beta.$$

The first inequality above is true since I is a Morley sequence in q. From $E_\beta \underset{C}{\mid} I$ we have $E_\beta \underset{I_kC}{\mid} \bar{a}_k$ and the first equality follows. The second inequality is clear and hence $P_k\subseteq[\mathrm{bnd}(q),\mathrm{bnd}(p)]_{O(T)}$.

Further, order $P_1xP_2x\ldots xP_n$ with the product order and define a mapping $f:P\to P_1xP_2x\ldots xP_n$ by $f(\beta)=(p^1_\beta,p^2_\beta,\ldots,p^n_\beta)$. We show that f is strictly increasing. Assume that $\beta,\gamma\in P$ and $\gamma\le\beta$. By Proposition 1.1.3(b) there are E_β and E_γ such that:

$$tp(E_\beta/C)=tp(D_\beta/C),\quad tp(E_\gamma/C)=tp(D_\gamma/C),\quad E_\beta \underset{E_\gamma}{\mid} C \quad\text{and}\quad E_\beta E_\gamma \underset{C}{\mid} I.$$

Then $p^k_\beta=\mathrm{bnd}(\bar{a}_k/I_kE_\beta)$ and $p^k_\gamma=\mathrm{bnd}(\bar{a}_k/I_kE_\gamma)$. From the independence assumptions we derive $I \underset{E_\gamma}{\mid} E_\beta$, and thus $\bar{a}_k \underset{I_kE_\gamma}{\mid} E_\beta$. We have:

$$(!)_k \qquad p^k_\beta = \mathrm{bnd}(\bar{a}_k/I_kE_\beta) \ge \mathrm{bnd}(\bar{a}_k/I_kE_\beta E_\gamma) = \mathrm{bnd}(\bar{a}_k/I_kE_\gamma) = p^k_\gamma.$$

Thus $p^k_\beta\ge p^k_\gamma$ and f is increasing. Now, if $\gamma<\beta$ then $C \underset{E_\beta}{\not\mid} E_\gamma$ and since $C\subseteq acl(I)$ we have $I \underset{E_\beta}{\not\mid} E_\gamma$ so for some $j\le n$ we have $\bar{a}_j \underset{I_jE_\beta}{\not\mid} E_\gamma$ and $\mathrm{bnd}(\bar{a}_j/I_jE_\beta E_\gamma)<\mathrm{bnd}(\bar{a}_j/I_jE_\beta)$. We conclude that in $(!)_j$ the strict inequality holds and $p^k_\gamma<p^k_\beta$. This proves that f is strictly increasing.

By Lemma 2.1.1 we have $\dim(P)\le\dim(P_1xP_2x\ldots xP_n)$ and by Lemma 2.1.3 we have $\dim(P_1xP_2x\ldots xP_n)=\dim(P_k)$, for some $k\le n$. Therefore $\dim(P)\le\dim(P_k)$. But $P_k\subseteq[\mathrm{bnd}(q),\mathrm{bnd}(p)]_{O(T)}$ thus $\dim(P_k)\le\dim(p|q)$ and we have:

$$\dim(C) = \dim(P) \le \dim(P_k) \le \dim(p|q)$$

completing the proof of the Proposition.

2.2 Strictly stable theories with no dense forking chains

Throughout this section we assume that T is strictly stable and has no dense forking chains and we operate in $\mathcal{M}^{eq}$. Consider all complete types whose domain is finite. Let $\alpha \geq 1$ be the smallest ordinal such that at least one of the types considered has dimension α and let ξ be the smallest possible U_α-rank of such a type. We say that a type is an (α,ξ)-type if its domain is finite, its dimension is α and its U_α-rank is ξ.

<u>Lemma 2.2.1</u> If $p=tp(\bar{a}/B)$ is an (α,ξ)-type then there exists $\bar{c} \in dcl(\bar{a}B) \setminus acl(B)$ such that $\dim(\bar{c}/B)=0$. In particular, every (α,ξ)-type is nonorthogonal to a type of dimension 0.

Proof: Without loss of generality assume that $B=\emptyset$. Since $\alpha>0$, there exists an infinite sequence $\beta_1>\beta_2>\ldots..$ below bnd(p) in O(T). Let $r=tp(\bar{a}/C)$ be such that $bnd(r)=\beta_2$. Note that $\beta_3>\beta_4>\ldots.$ is an infinite descending sequence below bnd(r) so that $\dim(r)\geq 1$. If we replace C by a large enough finite subset of Cb(r) we can assume that C is finite, r is a forking extension of p and $\dim(r)\geq 1$.

By the minimality assumptions on α and ξ we have $\dim(r)=\alpha$ and $U_\alpha(r)=U_\alpha(p)=\xi$. By Lemma 2.1.5 $U_\alpha(r)+U_\alpha(p|r)\leq U_\alpha(p)$ and it follows that $U_\alpha(p|r)=0$. Thus, $\dim(p|r)<\alpha$. By Proposition 2.1.7 we have

$$\sup\{\dim(\bar{d}) \mid \bar{d}\in Cb(r)\} \leq \dim(p|r).$$

Therefore $\sup\{\dim(\bar{d}) \mid \bar{d}\in Cb(r)\}<\alpha$ and by the minimality assumption on α we have $\dim(\bar{d})=0$ for all $\bar{d}\in Cb(r)$. Let $\bar{d}\in Cb(r)$ be such that $\bar{a} \not\downarrow \bar{d}$ and let $\bar{c}'\in Cb(\bar{d}/\bar{a})\setminus acl(\emptyset)$. $\bar{c}'$ is definable in a finite Morley sequence $\bar{d}_1\bar{d}_2\ldots\bar{d}_k$ in $stp(\bar{d}/\bar{a})$. Also $\dim(\bar{d}_i)=0$, so $\dim(\bar{d}_1\bar{d}_2\ldots\bar{d}_k)=0$ and $\dim(\bar{c}')=0$. Let c be the name for the set of all $\{\bar{a}\}$-conjugates of $\bar{c}'$. Since $\bar{c}'\in acl(\bar{a})$ this set is finite so $c\in dcl(\bar{a})$; also, every $\{\bar{a}\}$-conjugate of $\bar{c}'$ has dimension 0 so that

$\dim(c)=0$. Finally, from $\bar{c}'\in acl(c)\setminus acl(\emptyset)$ we have $c\notin acl(\emptyset)$ completing the proof of the Lemma.

Proposition 2.2.2 If T is strictly stable and has no dense forking chains then $I(T,\aleph_0) \geq \aleph_0$.

Proof Let B be finite, let $p=tp(\bar{a}/B)$ be an (α,ξ)-type, let $A=\{\bar{d}\in dcl(\bar{a}B) \mid \dim(\bar{d}/B)=0\}$ and let $q=tp(\bar{a}/AB)$. We show that q is nonisolated.

Suppose, on the contrary, that $\varphi(\bar{x},\bar{b})$ is a formula over AB which isolates q; here $\varphi(\bar{x},\bar{y})$ is an L-formula $\bar{b}\subseteq AB$ and without any loss of generality we assume that $B\subseteq\bar{b}$. Clearly $\dim(\bar{b}/B)=0$ holds, so that $\dim(\bar{a}/\bar{b})>0$ by Theorem 2.1.6(a). By the minimality assumptions on α and ξ we must have $\dim(\bar{a}/\bar{b})=\alpha$ and $U_\alpha(\bar{a}/\bar{b})=\xi$. By Lemma 2.2.1 there exists $\bar{c}\in A\setminus acl(\bar{b})$. Choose $\bar{a}_1\models stp(\bar{a}/\bar{b})$ such that $\bar{a}_1 \underset{b}{|} \bar{c}$. Hence $\models \varphi(\bar{a}_1,\bar{b})$ holds and thus $tp(\bar{a}_1/AB)=q$. From the independence assumption on $\bar{a}_1$ we derive $\bar{c}\notin acl(\bar{a}_1\bar{b})$; on the other hand $\bar{c}\in dcl(\bar{a}\bar{b})$, so that $tp(\bar{a}_1/\bar{c}B)\neq tp(\bar{a}/\bar{c}B)$ and $tp(\bar{a}_1/AB)\neq tp(\bar{a}/AB)=q$. This is a contradiction.

Let $r\in S(dcl(\bar{a}B))$ be a nonforking extension of q. Then, by The Open Mapping Theorem r must be nonisolated too, and $r|B\bar{a}$ is nonisolated as well. We have found a nonisolated type over a finite domain, hence there exists a nonisolated type over $\emptyset$. To complete the proof of the Proposition, we repeat the proof from the superstable case, cf [La].

Suppose that T is small, otherwise $I(T,\aleph_0)=2^{\aleph_0}$. Let $tp(\bar{d})$ be nonisolated and let $\bar{d}_1\bar{d}_2\ldots$ be an infinite Morley sequence in $tp(\bar{d})$. For each n let M_n be prime over $\bar{d}_1\bar{d}_2\ldots\bar{d}_n$. By Theorem 2.1.6(c) $m=wt(\bar{d})<\omega$. We show that in M_n there is no Morley sequence in $tp(\bar{d})$ of length $m\cdot n+1$, which clearly implies the conclusion of the Proposition.

If $\bar{e}\models tp(\bar{d})$ and $\bar{e}\in M_n$ then $tp(\bar{e}/\bar{d}_1\bar{d}_2\ldots\bar{d}_n)$ is isolated, hence by the Open Mapping Theorem $\bar{e}$ forks with $\bar{d}_1\bar{d}_2\ldots\bar{d}_n$. On the other hand $wt(\bar{d}_1\bar{d}_2\ldots\bar{d}_n) = m\cdot n$, hence there is no independent set of realizations of $tp(\bar{d})$ of size $m\cdot n+1$ in M_n.

<u>Lemma 2.2.3</u> There exists a finite set $\bar{a}B$ such that if

$$A=\{\bar{c}\in dcl(\bar{a}B)\,|\,\dim(\bar{c}/B)=0\} \quad \text{then:}$$

(1) $tp(\bar{a}/B)$ is an (α,ξ)-type,

(2) $stp(\bar{a}/A)$ is semiregular and superstable, and

(3) for every finite $A_o\subseteq A$ and $\bar{b}$ for which $\dim(\bar{b}/A_o)=0$ we have $\bar{a} \underset{A}{|} \bar{b}$.

Proof: Consider all (α,ξ)-types. Among them let $p=tp(\bar{a}/B)$ be such that if $A=\{\bar{c}\in dcl(\bar{a}B)|\dim(\bar{c}/B)=0\}$ then $\dim(\bar{a}/AB)=\beta$ is minimal possible and $U_\beta(\bar{a}/AB)=\eta$ is minimal possible as well.

Claim 1 $\beta=0$.

Proof $stp(\bar{a}/AB)$ is based on $\bar{a}B$ which is finite, so by the minimality assumptions on α we have that $\dim(\bar{a}/AB)$ is either α or 0. Suppose that $\beta=\alpha$. Then $\dim(\bar{a}/AB)=\alpha$ and from $U_\alpha(\bar{a}/AB)\leq U_\alpha(\bar{a}/B)=\xi$ and minimality of ξ we get $U_\alpha(\bar{a}/AB)=\xi$. By Lemma 2.2.1 $tp(\bar{a}/AB)$ is nonorthogonal to a superstable type. By Proposition 1.2.7(a) there is $\bar{a}'\in dcl(\bar{a}B)\backslash acl(AB)$ such that $tp(\bar{a}'/AB)$ is superstable. Now, we show that $tp(\bar{a}'/B)$ and $A'=\{\bar{c}\in dcl(\bar{a}'B)|\dim(\bar{c}/B)=0\}$ contradict the minimality of β.

From $\bar{a}'\notin acl(AB)$ we have $\dim(\bar{a}'/B)>0$ and from $\bar{a}'\in dcl(\bar{a}B)$ and the minimality assumption on α we have $\dim(\bar{a}'/B)=\alpha$. Further, let $B'=Cb(A/\bar{a}'B)$. Each $\bar{d}\in B'$ is algebraic in a finite Morley sequence in $stp(A/B)$, so it is algebraic in a finite Morley sequence in $stp(\bar{d}'/B)$ for some $\bar{d}'\in A$. But $\dim(\bar{d}'/B)=0$ hence $\dim(\bar{d}/B)=0$, and we have just shown that $B'\subseteq A'$. Since $B'=Cb(A/\bar{a}'B)$ we have $A \underset{B'}{|} \bar{a}'B$ and thus $\bar{a}' \underset{BB'}{|} A$. Thus:

$$\dim(\bar{a}'/A'B) \leq \dim(\bar{a}'/B'B) = \dim(\bar{a}'/ABB') \leq \dim(\bar{a}'/AB) = 0.$$

(The first inequality here follows from $B'\subseteq A'$, the first equality from $\bar{a}' \underset{BB'}{|} A$). We conclude that $\dim(\bar{a}'/A'B)=0$ and the proof of the claim is complete.

Because $\dim(\bar{a}/AB)=0$, Theorem 1.2.4 applies and there is $\bar{a}'\in acl(\bar{a}B)$ such that $stp(\bar{a}'/AB)$ is semiregular. We show that with $\bar{a}'B$ in place of $\bar{a}B$ the conditions (1) and (2) are satisfied. Let

$A' = \{\bar{d}\in dcl(\bar{a}'B) \mid dim(\bar{d}/B)=0\}$ and let $B'=Cb(A/\bar{a}'B)$. Then $A \underset{B'}{|} \bar{a}'B$ and hence $A \underset{BB'}{|} \bar{a}'$. As in the proof of Claim 1 we get $B'\leq A'$. We have:

$$U(\bar{a}'/A'B) \leq U(\bar{a}'/B'B) \leq U(\bar{a}'/AB) \leq U(\bar{a}/AB).$$

The first inequality follows from $B'\leq A'$, the second from $A \underset{BB'}{|} \bar{a}'$, and the third from $\bar{a}'\in acl(\bar{a}AB)$. By the minimality assumption on η we must have $U(\bar{a}'/A'B)=U(\bar{a}/AB)$, hence $U(\bar{a}'/A'B)=U(\bar{a}'/B'B)$ so $\bar{a}' \underset{BA'}{|} A$ and $stp(\bar{a}'/A'B)$ is semiregular. Since $\bar{a}'\notin acl(AB)$, $tp(\bar{a}'/B)$ is an (α,ξ)-type. Therefore, if we replace $\bar{a}$ by $\bar{a}'$ we have that conditions (1) and (2) are valid.

We show that (3) follows from (1) and (2). Suppose that $A_0\leq A$ is finite and $\bar{b}$ is such that $dim(\bar{b}/A_0B)=0$, and $\bar{b} \underset{AB}{\not|} \bar{a}$. Because $dim(\bar{b}/A_0B)=0$ there is a finite A_1 such that $A_0\leq A_1\leq A$ and $\bar{b} \underset{BA_1}{|} A$. Let $\bar{c}\in Cb(\bar{b}/\bar{a}B)$ be such that $\bar{a} \underset{AB}{\not|} \bar{c}$. Clearly, $\bar{c}\notin acl(A)$. Also $\bar{c}$ is definable from a finite Morley sequence in $stp(\bar{b}/A_1B)$, which has dimension 0, hence $dim(\bar{c}/A_1B)=0$. Since $\bar{c}\in acl(\bar{a}B)$, the set of all $\bar{a}B$-conjugates of $\bar{c}$ is finite so let $\bar{d}\in dcl(\bar{a}B)$ be the name for that set. Then $\bar{d}\in dcl(\bar{a}B)$, $dim(\bar{d}/A_1B)=0$ and since $dim(A_1/B)=0$ we have $dim(\bar{d}/B)=0$. Thus $\bar{d}\in A$ and since $\bar{c}\in acl(\bar{d})$ we have $\bar{c}\in acl(AB)$ and hence $\bar{a} \underset{AB}{|} \bar{c}$. This is a contradiction and the Lemma is proved.

2.3 Proof of Theorem C

<u>Theorem C</u> If T is strictly stable, has no dense forking chains and no strictly stable groups definable in T^{eq}, then $I(T,\aleph_0)=2^{\aleph_0}$.

In this section we prove Theorem C. Let T be a theory with no dense forking chains and no strictly stable groups definable in it; we operate in $\mathcal{M}^{eq}$. Let $tp(\bar{a}/B)$ and A be as in the conclusion of Lemma 2.2.3. $stp(\bar{a}/B)$ is semiregular, so let q be a regular type such that $stp(\bar{a}/A)\square q^m$. We are going to construct $2^{\aleph_0}$ nonisomorphic countable models of T, so without any loss of generality we absorb B into the language. Therefore, we have:

(1) $tp(\bar{a})$ is an (α,ξ)-type,

(2) $A=\{\bar{c}\in dcl(\bar{a})\,|\,dim(\bar{c})=0\}$,

(3) $stp(\bar{a}/A)\square q^m$, and

(4) for every finite $A_0\subseteq A$ and $\bar{b}$ for which $dim(\bar{b}/A_0)=0$ we have $\bar{a} \underset{A}{|} \bar{b}$.

The absence of strictly stable groups definable in T^{eq} implies that the following, stronger version of (4) holds.

<u>Lemma 2.3.1</u> For every finite $A_0\subseteq A$ $stp(\bar{a}/A) \perp A_0$.

Proof: To prove the Lemma suppose, on the contrary, that $A_0\subseteq A$ is finite and that $stp(\bar{a}/A)$ is nonorthogonal to A_0. Let $r=tp(\bar{b}/A_0)$ be nonorthogonal to $stp(\bar{a}/A)$ which is superstable. By Proposition 1.2.7(a) we can replace $\bar{b}$ by an element from $dcl(\bar{b}A_0)$, so that r is superstable. We are going to find a strictly stable group which is definable in T^{eq}.

By Proposition 1.2.7(b) there exists $\bar{a}'\in dcl(\bar{a})$ such that $stp(\bar{a}'/A)$ is nonalgebraic and r-internal. Let $\mathcal{P}$ be the set of all realizations of $stp(\bar{a}'/A)$ and let $\mathcal{R}$ be the set of all realizations of $tp(\bar{b}/A_0)$. By Theorem 1.3.14, the group of all $A\mathcal{R}$-automorphisms of $\mathcal{P}$ is ∞-definable over acl(A). Denote this group by G. We show that G acts transitively on $\mathcal{P}$.

Notice that for all finite $B_0\subseteq\mathcal{R}$ we have $dim(B_0)=0$. So, by condition (4) above, for every $\bar{a}_0\in\mathcal{P}$ we have $\bar{a}_0 \underset{A}{|} B_0$. Therefore $stp(\bar{a}_0/A)\vdash stp(\bar{a}_0/A\mathcal{R})$, so for $\bar{a}_0,\bar{a}_1\in\mathcal{P}$ $stp(\bar{a}_0/A\mathcal{R})=stp(\bar{a}_1/A\mathcal{R})$, and G acts transitively on $\mathcal{P}$. By Proposition 1.3.1, there exists an acl(A)-definable group $G_0\supseteq G$. Thus G_0 acts transitively on $\mathcal{R}$, as well, and we have:

and let q_k be the corresponding conjugate of q (so that $r_k \square q_k^m$). By Lemma 2.3.1 $q_k \perp \emptyset$, thus if $k \neq l$ we have $q_k \perp q_l$. Let I_k be an independent set of size k of realizations of r_k; notice that $A_k \subseteq dcl(\bar{e})$ for every $\bar{e} \in I_k$. Let M_X be a countable almost atomic model over $\bigcup\{I_k \mid k \in X\}$.

Consider the set of all $\emptyset$-conjugates of $stp(\bar{a}/A)$ whose domain is a subset of M_X and which are realized in M_X. Each of them is domination equivalent to a power of a regular type; thus nonorthogonality is an equivalence relation on our set. By a class we mean an equivalence class of this relation and the class containing r_k is denoted by $\mathcal{C}_k$. For every class $\mathcal{C}$ define $\dim(\mathcal{C})$ to be the greatest natural number n, if one exists, for which there exists $r \in \mathcal{C}$ and a sequence of n realizations of r in M_X which are independent over the domain of r. Otherwise, define $\dim(\mathcal{C}) = \infty$.

Claim 1 $\quad k \leq \dim(\mathcal{C}_k) \leq 2mk+1 \quad$ for every $k \in X$.

Proof: The first inequality follows from the construction. To prove the other one, let $r \in \mathcal{C}_k$, $dom(r) = A'$ and let $I \subseteq M_X$ be an independent set over A' of size n of realizations of r. We show that $n \leq 2mk+1$, which suffices to prove the claim. Define:

$$C = \bigcup\{A_i \mid i \in X\} \cup A' \qquad D = \bigcup\{I_i \mid i \in X \setminus \{k\}\} \cup C$$

For every $\bar{c} \in C$ we have $\dim(\bar{c}) = 0$ hence, by Corollary 2.3.2 each q_k is orthogonal to every extension of $tp(C)$. On the other hand, $stp(I/A')$ is domination equivalent to a power of q_k, thus $stp(I/A') \vdash stp(I/C)$ and I is independent over C. Similarly, for every $i \in X$ $stp(I_i/A_i) \vdash stp(I_i/C)$ thus I_i is independent over C and $stp(I_i/C) \square q_i^{im}$. But distinct q_i's are orthogonal so $\bigcup\{I_i \mid i \in X \setminus \{k\}\}$ is an independent set over C of realizations of types which are orthogonal to q_k. Both $stp(I/C)$ and $stp(I_k/C)$ are domination equivalent to a power of q_k, hence both of them are orthogonal to $tp(\bigcup\{I_i \mid i \in X \setminus \{k\}\}/C)$ so $stp(I/C) \vdash stp(I/D)$ and $stp(I_k/C) \vdash stp(I_k/D)$. We conclude that $stp(I/A') \vdash stp(I/D)$ and $stp(I_k/A_k) \vdash stp(I_k/D)$. It follows that both I_k and I are independent sequences over D and that both $stp(I/D)$ and $stp(I_k/D)$ are domination equivalent to a power of q_k.

Now, let $\bar{a}_1, \bar{a}_2 \in I$ be distinct. We prove that $\bar{a}_1\bar{a}_2 \not\downarrow_D I_k$. Note that $tp(\bar{a}_1/\bar{a}_2)$ is nonisolated by the same argument as in the proof of Proposition 2.2.2. By construction, for some finite set $J \subseteq \bigcup\{I_i \mid i \in X \setminus \{k\}\}$, $tp(\bar{a}_1\bar{a}_2/JI_k)$ is isolated, hence $tp(\bar{a}_1/JI_k\bar{a}_2)$ is isolated, too. But $tp(\bar{a}_1/\bar{a}_2)$ is nonisolated so that $\bar{a}_1 \not\downarrow_{\bar{a}_2} JI_k$ and, since $A' \subseteq dcl(\bar{a}_2)$ we have $\bar{a}_1 \not\downarrow_{A'\bar{a}_2} JI_k$. Therefore $\bar{a}_1\bar{a}_2 \not\downarrow_{A'} JI_k$ and thus $\bar{a}_1\bar{a}_2 \not\downarrow_{A'} DI_k$. From the previous paragraph we have $I \downarrow_{A'} D$ and hence $\bar{a}_1\bar{a}_2 \downarrow_{A'} D$. From this and the last forking relation we derive the desired conclusion, i.e. $\bar{a}_1\bar{a}_2 \not\downarrow_D I_k$.

Over D, I_k is an independent sequence of k elements, each of which has weight m so that $wt(I_k/D)=mk$. Also, every pair of distinct elements of I forks with I_k over D. Since there are at least $\frac{n-1}{2}$ disjoint pairs of elements of I (which form an independent set over D, as I is such) we conclude that $\frac{n-1}{2} \le km$ or $n \le 2mk+1$, finishing the proof of Claim 1.

Claim 2 $\dim(C)=1$ for all other classes.

Proof: Let $C \ne C_k$ for $k \in X$. By definition $\dim(C) \ge 1$, so it remains to prove $\dim(C) \le 1$. Suppose, on the contrary that $\dim(C) \ge 2$. Let $r \in C$, $A'=dom(r)$ and let $I=\bar{a}_1\bar{a}_2 \subseteq M_X$ be a pair of independent realizations of r. Let $C=\bigcup\{A_i \mid i \in X\} \cup A'$ and let $D=\bigcup\{I_i \mid i \in X\} \cup C$. As in the proof of Claim 1 we conclude that $stp(I/A') \vdash stp(I/D)$ and that implies that $\bar{a}_1\bar{a}_2 \downarrow_{A'} D$.

Now, by construction of M_X, for some finite set $J \subseteq \bigcup\{I_i \mid i \in X\}$ $tp(\bar{a}_1\bar{a}_2/J)$ is isolated, hence $tp(\bar{a}_1/J\bar{a}_2)$ is isolated, too. But $tp(\bar{a}_1/\bar{a}_2)$ is nonisolated so that $\bar{a}_1 \not\downarrow_{\bar{a}_2} J$ and, since $A' \subseteq dcl(\bar{a}_2)$ we have $\bar{a}_1 \not\downarrow_{A'\bar{a}_2} J$ $\bar{a}_1\bar{a}_2 \not\downarrow_{A'} J$ and $\bar{a}_1\bar{a}_2 \not\downarrow_{A'} D$. This contradicts the above and the claim is proved.

Continuing the proof of Theorem C, define inductively a sequence of natural numbers u_n in the following way:

$$u_1=2 \qquad u_{n+1}= 2m\cdot u_n + 2$$

For every $Y\subseteq\omega\setminus\{0,1\}$ let $u(Y)=\{u_i \mid i\in Y\}$ and let $N_Y=M_{u(Y)}$.

If C is a class in N_Y and $\dim(C)\geq 2$ then by Claims 1 and 2, for some $n\in Y$ we have $u_n \leq \dim(C) \leq 2mu_n+1 < u_{n+1}$. Hence:

$$\{n \mid \text{for some class } C \text{ in } N_Y \;\; u_n \leq \dim(C) < u_{n+1}\} = Y,$$

and for distinct $Y,Z\subseteq\omega\setminus\{0,1\}$ N_Y and N_Z are not isomorphic. We conclude that T has $2^{\aleph_0}$ nonisomorphic countable models finishing the proof of Theorem C.

In the rest of this section we look more closely into the situation that arose after Lemma 2.3.1 and prove that it induces a 'strong' nonisolation property of tp(a/A). We introduce the notion of a strongly nonisolated type, which will play an important role in the next section in the proof of Theorem A. Then, althought we did not use it in the proof of Theorem A, we show that $tp(\bar{a}/A)$ is strongly nonisolated.

Example The following is a strictly stable, one-based theory with only $\aleph_0$ nonisomorphic countable models. $L=\{0,+,-,V_i \mid i\in\omega\}$, and consider the structure $(M,0,+,-,A_i)_{i\in\omega}$ where M is infinite abelian group of exponent 2 and $A_0\supseteq A_1\supseteq A_2\supseteq\ldots$ is an infinite descending chain of subgroups each having infinite index in the previous one. Let T=Th(M). Then T eliminates quantifiers, it is one-based and strictly stable. Consider the type $p=\{V_i(x) \mid i\in\omega\}\cup\{x\neq 0\}$. It determines a complete type q in $S(\emptyset)$. $U(q)=1$ and every M is prime over q(M), hence M depends only on dim(q,M) and T has exactly $\aleph_0$ nonisomorphic countable models. Thus T is small and since every small one-based theory is NDFC (by [H3] or [HLPTW]), T is NDFC.

Definition $p\in S(D)$ is strongly nonisolated if for every $B\supseteq D$ and every isolated type $q\in S(B)$ we have $p\perp^a q$.

Let $p \in S(D)$ be strongly nonisolated, let $q \in S(D)$ and let $p' \in S(DB)$ be a nonforking extension of p. Note that p' is strongly nonisolated, too. Also, if $wt(q)=1$ and $q \not\perp^a p$, then q must be strongly nonisolated as well.

The following lemma says basically that strongly nonisolated types can be easily omitted.

Lemma 2.3.3 Suppose that D is finite and $p=tp(\bar{a}/D)$ is strongly nonisolated. Then, if $\bar{a} \underset{D}{\mid} B$ and C is almost atomic over DB then $\bar{a} \underset{D}{\mid} C$. In particular, if $p \perp^a tp(B/D)$ and if C is almost atomic over DB, then $p \perp^a tp(C/D)$.

Proof: For every finite $\bar{c} \in C$ there is some finite $B' \subseteq B$ such that $tp(\bar{c}/DB')$ is isolated. Then since $tp(\bar{a}/D)$ is strongly nonisolated and $\bar{a} \underset{D}{\mid} B'$ we have $\bar{a} \underset{DB'}{\mid} \bar{c}$. Therefore $\bar{a} \underset{D}{\mid} \bar{c}B'$ and thus $\bar{a} \underset{D}{\mid} C$.

From now until the end of the section we operate in $\mathcal{M}^{eq}$. Suppose that $p \in S(D)$ isn't strongly nonisolated, $\dim(p)=0$ and $\bar{a} \models p$. Then there are $\bar{b}$ and $\bar{c}$ such that $tp(\bar{b}/\bar{c}D)$ is isolated, $\bar{a} \underset{D}{\mid} \bar{c}$ and $\bar{a} \underset{D\bar{c}}{\not\mid} \bar{b}$. We claim that such $\bar{b}$ and $\bar{c}$ can be found so that $stp(\bar{b}/\bar{c}D)$ is semiregular. By Proposition 1.2.7(a) replace $\bar{b}$ by an element from $dcl(\bar{b}\bar{c}D)$ whose dimension over $D\bar{c}$ is 0, so assume that $\dim(\bar{b}/D\bar{c})=0$. Further, assume that $\bar{b},\bar{c}$ satisfying the above requirements are chosen so that $U(\bar{b}/\bar{c}D)$ is minimal possible. Then by Theorem 1.2.4 there is $\bar{b}' \in acl(\bar{b}\bar{c}D)$ such that $stp(\bar{b}'/\bar{c}D)$ is semiregular. But $tp(\bar{b}'/\bar{c}D)$ is isolated and so is $tp(\bar{b}/\bar{b}'\bar{c}D)$. Also $U(\bar{b}/\bar{b}'\bar{c}D)<U(\bar{b}/\bar{c}D)$, so by minimality of $U(\bar{b}/\bar{c}D)$, we must have $\bar{a} \underset{D\bar{c}}{\not\mid} \bar{b}'$ (otherwise $\bar{b}$ and $\bar{c}\bar{b}'$ in place of $\bar{b}$ and $\bar{c}$ respectively would contradict the minimality assumption). Therefore, if we replace $\bar{b}$ by $\bar{b}'$ we have the desired conclusion.

Proposition 2.3.4 If $p \in S(D)$ is orthogonal to every finite subset of D and $\dim(p)=0$ then p is strongly nonisolated.

Proof: Suppose on the contrary that p is not strongly nonisolated and pick $\bar{a} \models p$ and $\bar{b},\bar{c}$ such that $\bar{a} \underset{D}{\mid} \bar{c}$, $tp(\bar{b}/\bar{c}D)$ is isolated and $\bar{a} \underset{D\bar{c}}{\not\mid} \bar{b}$. Also, by the previous remark, we can assume that $tp(\bar{b}/D\bar{c})$ is semiregular. Let $r=stp(\bar{c}/D)$ and let $\phi(\bar{y},\bar{c})$ be a formula over $\bar{c}D$ which isolates $tp(\bar{b}/\bar{c}D)$. Let $\varphi(\bar{x},\bar{b},\bar{c})$ be a formula over $\bar{b}\bar{c}D$ which forks over $\bar{c}D$ and for which $\models \varphi(\bar{a},\bar{b},\bar{c})$ holds. Let $\psi(\bar{x})$ be:

$$(d_r\bar{z})(\exists\bar{y})(\phi(\bar{y},\bar{z})\wedge\varphi(\bar{x},\bar{y},\bar{z})).$$

We claim that every type p' which is over D and which contains $\psi(\bar{x})$ is nonorthogonal to some stationarization of p.

Assume that $\models \psi(\bar{a}')$. Let $\bar{c}' \models r$ be such that $\bar{a}' \underset{D}{\mid} \bar{c}'$ and let $\bar{b}'$ be such that $\models \phi(\bar{b}',\bar{c}')\wedge\varphi(\bar{a}',\bar{b}',\bar{c}')$. Then $tp(\bar{b}'\bar{c}'/D)=tp(\bar{b}\bar{c}/D)$ so let f be an D-automorphism of the monster such that $f(\bar{b}'\bar{c}')=\bar{b}\bar{c}$ and let $\bar{a}''=f(\bar{a}')$. Therefore $\models \varphi(\bar{a}'',\bar{b},\bar{c})$ and we conclude that $\bar{a}'' \underset{D\bar{c}}{\not\mid} \bar{b}$. Finally, since $stp(\bar{b}/D\bar{c})$ is semiregular it follows that $stp(\bar{a}''/D) \not\perp stp(\bar{a}/\bar{c}D)$. Thus $stp(\bar{a}'/D) \not\perp stp(f^{-1}(\bar{a})/\bar{c}'D)$ and since $f^{-1}(\bar{a})$ realizes a nonforking extension of p to $D\bar{c}'$ the claim is proved.

To complete the proof of the Proposition, notice that $\psi(\bar{x})$ uses only finitely many parameters from acl(D), so it is over some finite $D_o \subseteq acl(D)$; by the claim p is nonorthogonal to a type over D_o and that contradicts the assumption.

3. SMALL SUPERSTABLE THEORIES

In this chapter we deal with nonisolated types and countable models of superstable theories. The main results are theorems A and B. It is convenient to assume that T is a complete, small, superstable theory in a countable language, which we do, though the smallness of T is not essentially used in section 2.

3.1 Nonisolated types in small superstable theories

Recall that p is strongly nonisolated if whenever $B \supseteq A$ and $q \in S(B)$ is isolated then $p \perp^{a} q$.

<u>Definition</u> Let A be finite and $p \in S(A)$. p is eventually strongly nonisolated, or esn for short, if a nonforking extension of p to some finite set is strongly nonisolated.

Suppose that A is finite and $p \in S(A)$ is not esn. Then

for all finite $B \supseteq A$ there is a finite $C \supseteq B$, $\bar{a}$ and $\bar{b}$ such that

(*) $\qquad \bar{a} \underset{A}{\mid} C, \quad \bar{a} \vdash p, \quad tp(\bar{b}/C)$ is isolated and $\bar{a} \underset{C}{\not\mid} \bar{b}$.

Assuming that we operate in $\mathcal{M}^{eq}$, we can require $stp(\bar{b}/C)$ to be semiregular, as in the remark after Lemma 2.3.3.

It is clear from the definition that for types over finite domains, eventual strong nonisolation is invariant under taking nonforking extensions and restrictions. From the above remark it is easily seen that in the case of regular types, the property of being esn is preserved under nonorthogonality, as well. Moreover, it is preserved under domination, that is if p q and p is esn then q is esn as well. Also, if p is a stationary type and $p \,\square\, q_1 \otimes q_2 \otimes \ldots \otimes q_n$ is a regular decomposition of p, then p is esn if and only if all the q_i's are.

Since we are operating in a small theory the following criterion is useful:

$p \in S(A)$ is strongly nonisolated if and only if:
for all finite $B \supseteq A$, if M is prime over B, $q \in S(M)$ and $q|B$ is a nonforking extension of p then q is a nonforking extension of p.

<u>Lemma 3.1.1</u> Suppose that A is finite, $p \in S(A)$, $\psi(\bar{x}) \in p$ and $M \supseteq A$. Then if $\psi(M) = \bigcup \{\bar{b} \subseteq M \mid\ \models \psi(\bar{b})\}$ and $\models \psi(\bar{a})$ we have:

$$tp(\bar{a}/\psi(M)) \vdash tp(\bar{a}/M).$$

Proof Proposition C.2'(i) in [M].

<u>Corollary 3.1.2</u> a) Suppose that A is finite, $p \in S(A)$ is not esn and $\psi(\bar{x}) \in p$. Then for any $B \supseteq A$, C, $\bar{b}$ and $\bar{a}$ can be chosen such that (*) holds and $\models \psi(\bar{b})$.

b) Eventual strong nonisolation is invariant under passing from T to T^{eq} and vice versa, that is: if p is a type of an element of T then p is esn in T iff it is esn in T^{eq}.

Proof (a) Let $D \supseteq B \supseteq A$ and $\bar{a} \models p$ be such that $\bar{a} \underset{A}{\mid} D$. Suppose that the claim is not true. Thus for all finite $E \supseteq B$ and all $\bar{b}$ if $\bar{a} \underset{A}{\mid} E$ $\models \psi(\bar{b})$ and $tp(\bar{b}/E)$ is isolated then $\bar{a} \underset{E}{\mid} \bar{b}$. Hence if M is a prime model over D, we have $\bar{a} \underset{D}{\mid} \psi(M)$. Now, by Lemma 3.1.1 we have $\bar{a} \underset{\psi(M)}{\mid} M$. Therefore, $tp(\bar{a}/M)$ is a nonforking extension of p. Since D was arbitrary we conclude that $tp(\bar{a}/B)$ is strongly nonisolated, which is a contradiction.

(b) If p is esn in T^{eq} then clearly p is esn in T. The other direction follows from part (a); assume that p is not esn in T^{eq} and let $B\subseteq\mathcal{M}$ be such that p is based on B, then by part (a) witnesses for p not being esn can be taken in $\mathcal{M}$ (for $\psi(\bar{x})$ simply take the formula `$\bar{x}\subseteq\mathcal{M}$') hence p is not esn in T, as well.

<u>Lemma 3.1.3</u> A nonisolated almost strongly regular type over a finite domain is strongly nonisolated.

Proof Let A be finite and let $p\in S(A)$ be nonisolated and aSR via $\phi(\bar{x})$. Further, let $B\supseteq A$ be finite, let p' be a nonforking extension of p to B and let M be a prime model over B. Since p is nonisolated and aSR via $\phi(\bar{x})$ it is almost orthogonal to every isolated type q such that $\phi(\bar{x})\in q$ and $\mathrm{dom}(q)\supseteq B$. Therefore we have $p'\perp^{a}tp(\phi(M)/B)$. Also, if q is a nonforking extension of p to $\phi(M)$ by Lemma 3.1.1 we have $q\vdash q|M$. This implies that p is strongly nonisolated.

<u>Definition</u> Let $p\in S(A)$ be a regular type.

(a) p is internally isolated if for every integer $n\in\mathbb{N}$ there exists a formula $\phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ over A, such that for every strong type q over A which extends p the following holds:

$$q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge\ldots\wedge q(\bar{x}_n)\wedge \varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \leftrightarrow q^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n).$$

b) p is internally nonisolated if it is not internally isolated.

<u>Lemma 3.1.4</u> Suppose that $p\in S(A)$ is a regular type.

(a) If there exists a strong type q over A which extends p and a formula $\phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ over $acl^{eq}(A)$ such that

$$q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge\ldots\wedge q(\bar{x}_n)\wedge \phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \leftrightarrow q^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$$

then p is internally isolated.

(b) p is internally isolated if and only if some extension of p to $acl^{eq}(A)$ is internally isolated, if and only if all extensions of p to $acl^{eq}(A)$ are internally isolated.

Proof (a) Suppose that q and $\phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$, which is over $acl^{eq}(A)$, satisfy the above condition. Further, let $\bar{a}\models q$ and let $e\in acl^{eq}(A)$ be such that $\phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ is over Ae. Thus $tp(e/\bar{a}A)$ is algebraic so let $\psi(y,\bar{a})$ be a formula which isolates it. Consider the formula

$$(\exists y)(\psi(y,\bar{x}_1) \wedge \phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n,y)).$$

and denote it by $\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$. It is over A. From the above assumption it is consistent with $q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge \ldots\wedge q(\bar{x}_n)$ (take e in place of $\bar{y}$); since we can move q by an A-automorphism to any other strong type extending p while φ_n remains fixed we have that the $\Leftarrow$ part of the equivalence holds. To prove $\Rightarrow$ assume that q' is a strong type extending p and

$$\models q'(\bar{a}_1)\wedge q'(\bar{a}_2)\wedge \ldots\wedge q'(\bar{a}_n)\wedge \varphi_n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n).$$

We show $\models (q')^n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n)$. Let e' be such that

$$\models \psi(e',\bar{a}_1)\wedge \phi_n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n,e').$$

From $\models \psi(e',\bar{a}_1)$ and $tp(\bar{a}_1/A)=tp(\bar{a}/A)$ we get $tp(\bar{a}_1e'/A)=tp(\bar{a}e/A)$. Hence there is an A-automorphism taking $\bar{a}_1e'$ to $\bar{a}e$. Thus it takes q' to q and if $\bar{b}_i$ is the image of $\bar{a}_i$ for $2\le i\le n$ we have

$$\models q(\bar{a})\wedge q(\bar{b}_2)\wedge \ldots\ q(\bar{b}_n)\wedge \phi_n(\bar{a},\bar{b}_2,\ldots,\bar{b}_n,e).$$

It follows that $\models q^n(\bar{a},\bar{b}_2,\ldots,\bar{b}_n)$ and hence $\models (q')^n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n)$.

(b) Follows immediately from (a).

We note the following criterion of internal isolation. If p is a regular type and q is a strong type extending p (over the same domain) then p is internally isolated if and only if for every integer n there exists a formula $\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ consistent with $q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge\ldots\wedge q(\bar{x}_n)$ such that

$$q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge \ldots\wedge q(\bar{x}_n)\wedge \varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \rightarrow q^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n).$$

<u>Lemma 3.1.5</u> Let $p\in S(A)$ be a regular type, let B be finite and let $q\in S(AB)$ be a nonforking extension of p. Then p is internally isolated if and only if q is.

Proof By Lemma 3.1.4 we can replace p and q by strong types over

A and B, respectively.

$\Rightarrow$) Suppose that p is internally isolated. For each $n\in\mathbb{N}$ we will find a formula $\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ over $acl^{eq}(A)$ which witnesses that q is internally isolated.

For every natural number n let $\phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ be a formula over A such that

$(*)_n$ $p(\bar{x}_1)\wedge p(\bar{x}_2)\wedge\ldots\wedge p(\bar{x}_n)\wedge \phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \leftrightarrow p^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$.

By superstability there exists $I=\bar{b}_1\bar{b}_2\ldots\bar{b}_m\models p^m$ such that $tp(B/AI)\perp^a p^\omega$. Let $r=stp(\bar{b}_1\bar{b}_2\ldots\bar{b}_m/BA)$. For every natural number n let $\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ be the following formula

$$(d_r\bar{y}_1\bar{y}_2\ldots\bar{y}_m)\ \phi_{n+m}(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n,\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_m).$$

Clearly, φ_n is over $acl^{eq}(A)$. We claim that:

$(**)_n$ $q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge\ldots\wedge q(\bar{x}_n)\wedge \varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \leftrightarrow q^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$.

The consistency condition is clear, so assume that

$$\models q(\bar{a}_1)\wedge q(\bar{a}_2)\wedge\ldots\wedge q(\bar{a}_n)\wedge \varphi_n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n).$$

Let $I'=\bar{b}'_1\bar{b}'_2\ldots\bar{b}'_m\models r|AB\bar{a}_1\bar{a}_2\ldots\bar{a}_n$. By the choice of φ_n we have

$$\models \phi_{n+m}(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n,\bar{b}'_1,\bar{b}'_2,\ldots,\bar{b}'_m)$$

Hence $\bar{a}_1\bar{a}_2\ldots\bar{a}_n\bar{b}'_1\bar{b}'_2\ldots\bar{b}'_m\models p^{n+m}$ and $\bar{a}_1\bar{a}_2\ldots\bar{a}_n\models (p|AI')^n$. Further, from $tp(BI/A)=tp(BI'/A)$ we get $tp(B/AI')\perp^a p^\omega$ hence $\bar{a}_1\bar{a}_2\ldots\bar{a}_n \underset{AI'}{\mid} B$. We conclude $\bar{a}_1\bar{a}_2\ldots\bar{a}_n\models (p|ABI')^n$ and $\bar{a}_1\bar{a}_2\ldots\bar{a}_n\models q^n$.

$\Leftarrow$) Suppose that q is internally isolated and let $\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n,\bar{b})$ be a formula over $\bar{b}$, where $B=\bar{b}$, such that $(**)_n$ holds. Let $r=stp(\bar{b}/A)$ and let $\phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ be the formula $(d_r\bar{y})\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n,\bar{y})$. Clearly, ϕ_n is over $acl^{eq}(A)$ and the consistency condition holds so it remains to show that

$$p(\bar{x}_1)\wedge p(\bar{x}_2)\wedge\ldots\wedge p(\bar{x}_n)\wedge \phi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \rightarrow p^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n);$$

since n is arbitrary this implies that p is internally nonisolated.

Suppose that $\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n$ realize p, $\models \phi_n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n)$ and $\bar{b}'\models r|A\bar{a}_1\bar{a}_2\ldots\bar{a}_n$. Therefore, $\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n$ realize $q'=p|\bar{b}'A$ and $\models \varphi_n(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n,\bar{b}')$. Further, since q' is a conjugate of $q|\bar{b}'A$,

$(**)_n$ holds with q' in place of q and $\bar{b}'$ in place of $\bar{b}$, hence $\bar{a}_1\bar{a}_2\ldots\bar{a}_n \models (q')^n$ and $\bar{a}_1\bar{a}_2\ldots\bar{a}_n \models p^n$, finishing the proof of the Lemma.

Lemma 3.1.6 Suppose that $p \in S(A)$ is a regular, internally nonisolated type, $\bar{a} \models p$ and $\bar{b} \in dcl^{eq}(\bar{a}A) \setminus acl^{eq}(A)$. Then $tp(\bar{b}/A)$ is internally nonisolated.

Proof Let $r = tp(\bar{b}/A)$. Since $\bar{b} \in dcl^{eq}(\bar{a}A) \setminus acl^{eq}(A)$, r is regular. Suppose that r is internally isolated and we show that p is internally isolated, too. For each $n \in \mathbb{N}$ let $\phi_n(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ be a formula over A witnessing internal isolation of r. Let f be an A-definable function such that $\models \bar{b} = f(\bar{a})$. Let $\varphi_n(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ be:

$$\forall \bar{y}_1\bar{y}_2 \ldots \bar{y}_n (\bar{y}_1 = f(\bar{x}_1) \wedge \bar{y}_2 = f(\bar{x}_2) \wedge \ldots \wedge \bar{y}_n = f(\bar{x}_n) \rightarrow \phi_n(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)).$$

Clearly φ_n is over A. We claim that it witnesses that p is internally isolated. Again, the consistency condition is trivially satisfied so assume that q is a strong type extending p and:

$$\models q(\bar{a}_1) \wedge q(\bar{a}_2) \wedge \ldots \wedge q(\bar{a}_n) \wedge \varphi_n(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n).$$

Let $\bar{b}_i = f(\bar{a}_i)$ for $1 \le i \le n$. Then $stp(\bar{b}_1/A) = stp(\bar{b}_2/A) = \ldots = stp(\bar{b}_n/A)$ and this strong type extends r. Since $\models \phi_n(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n)$ witnesses that r is internally isolated we have that $\{\bar{b}_1, \bar{b}_2 \ldots \bar{b}_n\}$ is independent over A. By regularity of p and r, $\{\bar{a}_1, \bar{a}_2 \ldots \bar{a}_n\}$ is independent over A and hence $\models q^n(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$. Therefore p is internally isolated and the proof of the Lemma is complete.

Proposition 3.1.7 A regular, internally nonisolated type whose domain is finite is eventually strongly nonisolated.

Proof Without any loss of generality we operate in $\mathcal{M}^{eq}$. Let A be finite and let $p \in S(A)$ be an internally nonisolated, regular type. We prove that p is esn. By Lemma 3.1.5 and the remarks from the beginning of this section both internal nonisolation and eventual strong nonisolation are invariant under nonforking extensions and restrictions to finite sets, so after possibly adding a few parameters to A we may assume that p is stationary.

Let $\bar{a}' \models p$ and let $\bar{b} \in dcl(\bar{a}'A)$ be such that $U(\bar{b}/A) = \omega^\alpha$; it exists by Corollary 1.2.5. Let $q = tp(\bar{b}/A)$. Then q is a stationary, regular type which is internally nonisolated by Lemma 3.1.6. We are going to prove that q is esn; since the property of being esn is preserved under nonorthogonality of regular types it follows that p is esn, too.

Let n be the smallest integer such that for no formula $\varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ consistent with $q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge\ldots\wedge q(\bar{x}_n)$ the following holds:

$$q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge \ldots\wedge q(\bar{x}_n)\wedge \varphi_n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \Rightarrow q^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n).$$

Let $\{\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_{n-1}\}$ be an independent set of realizations of q. We will prove that $s=q|\bar{b}_1\bar{b}_2\cdots\bar{b}_{n-1}A$ is strongly nonisolated; since s is parallel to q it will follow that q is eventually strongly nonisolated. By the minimality assumption on n there is a formula $\phi(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_{n-1})$ over A such that:

$$q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge\ldots\wedge q(\bar{x}_{n-1})\wedge \phi(\bar{x}_1,\bar{x}_2,\ldots\bar{x}_{n-1}) \Leftrightarrow q^{n-1}(\bar{x}_1,\bar{x}_2,\ldots\bar{x}_{n-1}).$$

Further, assume that s is not strongly nonisolated. Let $\bar{d}$ contain $\bar{b}_1\bar{b}_2\cdots\bar{b}_{n-1}A$, $\bar{a}\models s|\bar{d}$ and $\bar{c}$ be such that $tp(\bar{c}/\bar{d})$ is isolated, semiregular, $\square\, q^m$ and $\bar{a} \underset{\bar{d}}{\not\downarrow} \bar{c}$ (they exist by the remark after the definition of esn). Let $\varphi(\bar{z},\bar{d})$ be a formula which isolates $tp(\bar{c}/\bar{d})$, let $\psi(\bar{a},\bar{d},\bar{z})\in tp(\bar{c}/\bar{d}\bar{a})$ fork over $\bar{d}$ and let $r=stp(\bar{d}/\bar{a}_1\bar{a}_2\cdots\bar{a}_{n-1}A)$. Consider the formula

$$d_r\bar{y}(\exists\bar{z})(\varphi(\bar{z},\bar{y}) \wedge \psi(\bar{x},\bar{y},\bar{z})).$$

This formula is over $acl(\bar{b}_1\bar{b}_2\cdots\bar{b}_{n-1}A)$. Denote it by $\chi(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_{n-1},\bar{e},\bar{x})$ where $\bar{e}\in acl(\bar{b}_1\bar{b}_2\cdots\bar{b}_{n-1}A)$. Further, let $\tau(\bar{u},\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_{n-1})$ be a formula isolating $tp(\bar{e}/\bar{b}_1\bar{b}_2\ldots\bar{b}_{n-1}A)$ and let $\theta(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ be

$$\phi(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_{n-1}) \wedge \exists\bar{u}(\tau(\bar{u},\bar{x}_1,\bar{x}_2\ldots\bar{x}_{n-1}) \wedge \chi(\bar{x}_1,\bar{x}_2\ldots\bar{x}_{n-1},\bar{u},\bar{x}_n)).$$

We *claim* that $\theta(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$ satisfies

$$q(\bar{x}_1)\wedge q(\bar{x}_2)\wedge \ldots\wedge q(\bar{x}_n)\wedge \theta(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) \Leftrightarrow q^n(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n).$$

So suppose $\models q(\bar{b}_1')\wedge q(\bar{b}_2')\wedge \ldots\wedge q(\bar{b}_n')\wedge \theta(\bar{b}_1',\bar{b}_2',\ldots,\bar{b}_n')$ and we try to prove that $\bar{b}_1'\bar{b}_2'\ldots\bar{b}_n'\models q^n$. Let $\bar{e}'$ be such that

$$\models \tau(\bar{e}',\bar{b}_1',\bar{b}_2',\ldots,\bar{b}_{n-1}') \wedge \chi(\bar{b}_1',\bar{b}_2',\ldots,\bar{b}_{n-1}',\bar{e}',\bar{b}_n').$$

From $\models \phi(\bar{b}_1',\bar{b}_2',\ldots,\bar{b}_{n-1}')$, we derive $\bar{b}_1'\bar{b}_2'\ldots\bar{b}_{n-1}'\models q^{n-1}$ and since

$\models \tau(\bar{e}',\bar{b}_1',\bar{b}_2',\ldots,\bar{b}_{n-1}')$ we have

$$tp(\bar{b}_1\bar{b}_2\ldots\bar{b}_{n-1}\bar{e}/A) = tp(\bar{b}_1'\bar{b}_2'\ldots\bar{b}_{n-1}'\bar{e}'/A).$$

Thus, moving the $\bar{b}_i$'s and $\bar{e}'$ by an A-automorphism of the monster model we can assume that $\bar{b}_1'=\bar{b}_1$, $\bar{b}_2'=\bar{b}_2,\ldots,\bar{b}_{n-1}'=\bar{b}_{n-1}$ and $\bar{e}'=\bar{e}$. Let $\bar{b}_n''$ be the image of $\bar{b}_n'$ under the automorphism. Therefore, we have $\models \chi(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_{n-1},\bar{e},\bar{b}_n'')$; i.e. $\models d_r\bar{y}\exists\bar{z}(\varphi(\bar{z},\bar{y}) \wedge \psi(\bar{b}_n'',\bar{y},\bar{z}))$. If $\bar{d}'\models r|\bar{b}_1\bar{b}_2\cdots\bar{b}_{n-1}\bar{b}_n''A$ and if $\models \psi(\bar{b}_n'',\bar{d}',\bar{c}') \wedge \varphi(\bar{c}',\bar{d}')$ we have $tp(\bar{d}\bar{c}/A)=tp(\bar{d}'\bar{c}'/A)$, $\bar{b}_n'' \not\downarrow_{\bar{d}'} \bar{c}'$ and $tp(\bar{c}'/\bar{d}') \sqsupseteq q^m$. It follows that $stp(\bar{b}_n''/\bar{d}') \not\perp q$. Since $U(q)=\omega^\alpha$ we must have $U(\bar{b}_n''/\bar{d}')\geq\omega^\alpha$. But $tp(\bar{b}_n''/\bar{d}')$ is an extension of q,so $U(\bar{b}_n''/\bar{d}')=\omega^\alpha$. We conclude that $tp(\bar{b}_n''/\bar{d}')$ is the nonforking extension of q. Since $\bar{d}'\supseteq\bar{b}_1\bar{b}_2\ldots\bar{b}_{n-1}A$ we have $\bar{b}_n''\models q|\bar{b}_1\bar{b}_2\ldots\bar{b}_{n-1}A$. Hence $\bar{b}_n'\models q|\bar{b}_1'\bar{b}_2'\ldots\bar{b}_{n-1}'A$ finishing the proof of the claim.

But the *claim* contradicts our choice of n and the proof of the Lemma is complete.

Example We give an example of a regular, internally isolated esn type. Let $L=\{V_i|i\in\omega\}$ and $(M,A_i)_{i\in\omega}$ be an L-structure, where the A_i's are infinite disjoint subsets of M. Let $p=\{\neg V_i(x)|i\in\omega\}$. p determines a complete, stationary, U-rank 1 type $q\in S(\emptyset)$. q is internally isolated, for if $\phi(x_1,x_2,\ldots x_n)$ is $\bigwedge_{i\neq j}(x_i\neq x_j)$, then

$$q(x_1)\wedge q(x_2)\wedge\ldots\wedge q(x_n)\wedge \phi(x_1,x_2,\ldots x_n) \leftrightarrow q^n(x_1,x_2,\ldots x_n).$$

Clearly q is orthogonal to all isolated types, so q is esn.

For the following definition, we fix some terminology. By the full equivalence relation on a set A we mean the relation A^2; we say that an equivalence relation is nontrivial if it is neither the full relation nor equality.

Definition $p\in S(A)$ is primitive if there is no nontrivial A-definable equivalence relation on the set of realizations of p.

Lemma 3.1.8 Suppose that A is finite and $p \in S(A)$ is a stationary, internally isolated, regular type. Then there exists an A-definable equivalence relation E such that $q=p/E$ is primitive.

Proof We operate in $\mathcal{M}^{eq}$. Since p is internally isolated, there is a formula $\phi(\bar{x}_1,\bar{x}_2)$ over A such that:

$$p_1(\bar{x}_1) \wedge p_1(\bar{x}_2) \wedge \phi(\bar{x}_1,\bar{x}_2) \leftrightarrow p_1^2(\bar{x}_1,\bar{x}_2).$$

Thus, $\neg\phi(\bar{x}_1,\bar{x}_2)$ is an equivalence relation on the set of realizations of p. By compactness, we can assume that it is an A-definable equivalence relation, say E, on the whole monster model. We show that our E and $q=p/E$ satisfy the conclusion of the Lemma.

Clearly, E is not the full relation and p is stationary, so q is nonalgebraic and stationary. Suppose that F is an A-definable equivalence relation on the set of realizations of q, other then the full relation. Since q is stationary, F cannot have finitely many classes, so whenever $\bar{a},\bar{b} \models q$ and $\models F(\bar{a},\bar{b})$ then $\bar{a} \not\downarrow_A \bar{b}$. Now, If $\bar{c},\bar{d} \models p$ and $\models F(\bar{c}/E,\bar{d}/E)$, then $\bar{c}/E \not\downarrow_A \bar{d}/E$, which implies $\bar{c} \not\downarrow_A \bar{d}$ and hence $\models E(\bar{c},\bar{d})$, i.e. $\bar{c}/E=\bar{d}/E$. We conclude that F is equality on the set of realizations of q, and hence q is primitive.

Lemma 3.1.9 Suppose that $p \in S(A)$ is a regular, primitive, internally isolated type.

(a) If $\bar{a}_1 \neq \bar{a}_2 \models p$ then $\bar{a}_1 \downarrow_A \bar{a}_2$.

(b) If $\bar{a} \models p$, $\bar{a} \not\downarrow_A \bar{b}$ and $wt(\bar{b}/A)=1$ then $\bar{a} \in dcl(\bar{b}A)$.

Proof (a) Forking is a definable equivalence relation on the set of realizations of p; hence it has to be the equality relation.

(b) Let $\bar{a}' \models tp(\bar{a}/\bar{b}A)$ be such that $\bar{a}' \downarrow_{\bar{b}A} \bar{a}$. Then since $wt(\bar{b}/A)=1$ we have $\bar{a}' \not\downarrow_A \bar{a}$ hence, by part (a), $\bar{a}=\bar{a}'$. Therefore $tp(\bar{a}/\bar{b}A)$ has a unique realization, so $\bar{a} \in dcl(\bar{b}A)$.

Proposition 3.1.10 A regular, trivial type over a finite domain, whose U-rank is a limit ordinal, is internally nonisolated.

Proof We show that a regular, trivial, stationary, internally isolated type of limit ordinal U-rank is not primitive; then Lemma 3.1.8 implies the conclusion of the Lemma.

Suppose that A is finite and that $p \in S(A)$ is a primitive, regular, trivial, internally isolated type of limit ordinal U-rank. Let q be a forking, nonalgebraic extension of p and let $I = \bar{a}\bar{a}_1\bar{a}_2\ldots\bar{a}_n\ldots$ be an infinite Morley sequence in q. $tp(\bar{a}/A\bar{a}_1\bar{a}_2\ldots)$ is parallel to q so $\bar{a} \not\downarrow_A \bar{a}_1\bar{a}_2\ldots$. But p is trivial and regular, so we must have $\bar{a} \not\downarrow_A \bar{a}_n$ for some $n \in \mathbb{N}$. By Lemma 3.1.9(a) we conclude $\bar{a} = \bar{a}_n$; so $tp(\bar{a}/A\bar{a}_1\bar{a}_2\ldots)$ is algebraic. We have reached a contradiction.

Proposition 3.1.11 If T is one-based then every regular type over a finite domain whose U-rank is a limit ordinal is internally nonisolated.

Proof Again we show that a regular, stationary, internally isolated type of limit ordinal U-rank is not primitive, and Lemma 3.1.8 implies the desired conclusion.

Suppose that A is finite and that $p \in S(A)$ is a primitive, regular, internally isolated type of limit ordinal U-rank. Let q be a forking, nonalgebraic extension of p. Let $\bar{a} \models q$. Then, since T is one-based, q is based on $\bar{a}$, so if $\bar{b} \models q|\bar{a}A$ we have that $\bar{a} \not\downarrow_A \bar{b}$ and $\bar{a} \neq \bar{b}$, contradicting Lemma 3.1.9(a).

Lemma 3.1.12 Suppose that A is finite and $p \in S(A)$ is a regular type. Then p is NENI if and only if it is stationary, isolated and internally isolated.

Proof $\Rightarrow$) is obvious, so we prove only $\Leftarrow$). So assume that p is a stationary, regular, isolated and internally isolated type. We show that p is NENI; namely, that for all finite $B \supseteq A$ $p|B$ is isolated.

Let $B \supseteq A$ be finite. By superstability, there exists $n \in \mathbb{N}$ and $I = \bar{a}_1\bar{a}_2\ldots\bar{a}_n \models p^n$ such that $tp(B/AI) \perp^a p$. Let $\psi(\bar{x}) \in p$ be a formula which isolates p. Further, let $\phi(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_{n+1})$ be a formula over A such that

$$\psi(\bar{y}_1)\wedge\psi(\bar{y}_2)\wedge\ldots\wedge\psi(\bar{y}_{n+1})\wedge\phi(\bar{y}_1,\bar{y}_2\ldots\bar{y}_{n+1})\Leftrightarrow p^{n+1}(\bar{y}_1,\bar{y}_2\ldots\bar{y}_{n+1});$$

ϕ exists since p is internally isolated. Let q=stp(I/B). Consider the following formula:

$$\psi(\bar{x})\wedge(d_q\bar{y})\phi(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_n,\bar{x}).$$

Denote it by $\varphi(\bar{x})$. So, $\varphi(\bar{x})$ is over $acl^{eq}(B)$ and if we prove that it isolates $p|B$, since $p|B$ is stationary the conclusion will follow.

Assume that $\models\varphi(\bar{a})$, and let $I'=\bar{b}_1\bar{b}_2\ldots\bar{b}_n\models q|\bar{a}B$. Then

$$\models\psi(\bar{a})\wedge\phi(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_n,\bar{a}).$$

Since $\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_n,\bar{a}$ realize p and $\models\phi(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_n,\bar{a})$ we conclude $\bar{a}I'\models p^{n+1}$. Further, from $stp(BI/A)=stp(BI'/A)$ and $tp(B/AI)\perp^a p$ we derive $tp(B/AI')\perp^a p$, and since $\bar{a}\models p|AI'$ we have $B \underset{AI'}{|} \bar{a}$. It follows that $\bar{a}\underset{A}{|} BI'$ and in particular $\bar{a}\models p|B$.

Theorem A In T^{eq}, consider the class of all nonalgebraic types having a finite domain. Then the subclass consisting of all esn types is the largest subclass which does not contain NENI types, and which is closed under conjugation by automorphisms of $\mathcal{M}$, nonforking extensions and restrictions, and domination.

Proof We have already mentioned that the subclass is closed under nonforking extensions and restrictions, and domination. It is easily seen that it is closed under conjugation by automorphisms, too, so it remains to prove the 'largest' part in the Theorem; that is:

Claim If p is orthogonal to all NENI types then it is esn.

Proof Suppose that p is orthogonal to all NENI types. Every regular type nonorthogonal to p has the same property, hence if we prove the claim for regular types the general case will follow. So assume that A is finite and that $p\in S(A)$ is a regular type which is not esn. We show that it is nonorthogonal to a NENI type.

To simplify notation assume $A=\emptyset$.

Let $q\in S(B)$ be a type with minimal R-rank which is not orthogonal to p and has finite domain. Clearly q is regular. Also, assume that CB(q) (actually $CB_B(q)$) is minimal possible and let $\psi(\bar{x})\in q$ be a formula having the same R-rank as q and isolating q among types having CB-rank greater or equal than CB(q). Then q is aSR via $\psi(\bar{x})$. Since q is nonorthogonal to p it is not esn. Lemma 3.1.3 applies and we conclude that q is isolated, wlog by $\psi(\bar{x})$.

Let $D\supseteq B$ be finite and let $q'\in S(D)$ be a stationary nonforking extension of q. $q'\not\perp p$ implies that q' is not esn, hence there is a finite set $C\supseteq D$, $\bar{b}\models q'|C$, and an isolated type $r=tp(\bar{a}/C)$ such that $\bar{a} \not\perp_C \bar{b}$. By 3.1.2(a) we may also assume $\models \psi(\bar{a})$. Since q is aSR via $\psi(\bar{x})$, $tp(\bar{a}/C)$ is a nonforking extension of $tp(\bar{a}/B)$, hence $tp(\bar{a}/C)$ is regular. Since $tp(\bar{b}/C)$ is stationary and internally isolated, Lemma 3.1.8 applies, hence there is $\bar{b}'\in dcl(C\bar{b})$ such that $tp(\bar{b}'/C)$ is primitive. Thus $\bar{a} \not\perp_C \bar{b}'$ and by Lemma 3.1.9(b) we derive $\bar{b}'\in dcl(\bar{a}C)$. Therefore $tp(\bar{b}'/C)$ is stationary and regular (since $\bar{b}'\in dcl(\bar{b}C)$ and $tp(\bar{b}/C)$ is stationary and regular), it is isolated (since $\bar{b}'\in dcl(\bar{a}C)$ and $tp(\bar{a}/C)$ is isolated), and internally isolated (since it is $\not\perp$ to p which is not esn, it is not esn as well so Lemma 3.1.7 applies). By Lemma 3.1.12 it has to be NENI and the proof of the claim is complete.

3.2. Regular internally isolated types

In this section we study internally isolated types in more detail. The main result essentially says that 'finitely generated subspaces' have large, definable groups of automorphisms, this of course in $\mathcal{M}^{eq}$. So throughout we operate in $\mathcal{M}^{eq}$.

Let $p \in S(A)$ be a regular, stationary, internally isolated type. For $k \in \mathbb{N}$ and $\bar{a} \models p^k$ define $D_k^p(\mathcal{M}, \bar{a}) = \{x \models p \mid x \not\downarrow \bar{a}\}$. We show that $D_k^p(\mathcal{M}, \bar{y}_1) = D_k^p(\mathcal{M}, \bar{y}_2)$ is an A-definable equivalence relation on the set of realizations of p^k. Intuitively, $D_k^p(\mathcal{M}, \bar{a})$ is a 'subspace' of $p(\mathcal{M})$ generated by $\bar{a}$ and internal isolation means that the dependence (cl_p) is definable on $p(\mathcal{M})$. Formally:

Since p is stationary and internally isolated there is a formula $\phi(x, \bar{y})$ over A, so that for all $x \models p$ and $\bar{y} \models p^k$

$$\models \phi(x, \bar{y}) \quad \text{iff} \quad x \not\downarrow \bar{y}.$$

Now, let $\bar{y}_1, \bar{y}_2 \models p^k$ where $\bar{y}_i = y_i^1 y_i^2 \ldots y_i^k$ for i=1,2. Let $E_k^p(\bar{y}_1, \bar{y}_2)$ be:

$\phi(y_1^1, \bar{y}_2) \wedge \phi(y_1^2, \bar{y}_2) \wedge \ldots \wedge \phi(y_1^k, \bar{y}_2) \wedge \phi(y_2^1, \bar{y}_1) \wedge \phi(y_2^2, \bar{y}_1) \ldots \wedge \phi(y_2^k, \bar{y}_1)$.

By regularity of p, E_k^p is an A-definable equivalence relation on the set of realizations of p^k, and for $\bar{y}_1, \bar{y}_2 \models p^k$ we have

$$D_k^p(\mathcal{M}, \bar{y}_1) = D_k^p(\mathcal{M}, \bar{y}_2) \quad \text{iff} \quad E_k^p(\bar{y}_1, \bar{y}_2).$$

By compactness we can assume that E_k^p is an A-definable equivalence relation on the whole of $\mathcal{M}$.

Let $c = \bar{a}/E_k^p$ and let $p_{(k)} = tp(c/A)$. Since $p^k \in S(A)$ and E_k^p is definable over A, our $p_{(k)}$ does not depend on the particular choice of $\bar{a} \models p^k$.

<u>Lemma 3.2.1</u> Suppose that $p \in S(A)$ is a stationary, regular, internally isolated type. Let $k \in \mathbb{N}$, $c \models p_{(k)}$ and let $a_1 a_2 \ldots a_k \models p^k$ be such that $\bar{a}/E_k^p = c$.

(a) If $b_1 b_2 \ldots b_k \models p^k$ and $\bar{b} \subseteq D_k^p(\mathcal{M}, \bar{a})$ then $tp(\bar{a}/cA) = tp(\bar{b}/cA)$. In particular $\bar{b}/E_k^p = c$. More generally, if $\bar{a}' \models p^k$ and $c' = \bar{a}'/E_k^p$ then $tp(\bar{a}c/A) = tp(\bar{a}'c'/A)$.

(b) If $m \leq k$, $b_1 b_2 \ldots b_m \models p^m$ and $\bar{b} \subseteq D_m^p(\mathcal{M}, \bar{a})$ then there exists $b_{m+1} b_{m+2} \ldots b_k \subseteq D_k^p(\mathcal{M}, \bar{a})$ such that $b_1 b_2 \ldots b_k \models p^k$.

(c) If $m \leq k$ $b_1 b_2 \ldots b_m \models p^m$ and $\bar{b} \subseteq D_k^p(\mathcal{M}, \bar{a})$ then

$$tp(b_1 b_2 \ldots b_m/Ac) = tp(a_1 a_2 \ldots a_m/Ac).$$

Proof Without any loss of generality we assume that $A = \emptyset$.

(a) Let g be an automorphism of the monster model such that

$$g(a_1a_2\ldots a_k)=b_1b_2\ldots b_k.$$

By regularity of p we have: for all $x\models p$ $x \not\downarrow \bar{a}$ iff $x \not\downarrow \bar{b}$. Hence $\models E^p_k(\bar{a},\bar{b})$ and $\bar{b}/E^p_k=c$. It follows that $g(c)=c$ and $g(\bar{a}c)=\bar{b}c$. We conclude that $tp(\bar{a}c/A)=tp(\bar{b}c/A)$.

(b) Let $\{b_1,b_2,..,b_l\}\supseteq\{b_1,b_2\ldots b_m\}$ be a maximal set (under inclusion) such that $\{b_1,b_2\ldots b_l\}\subseteq D^p_k(\mathcal{M},\bar{a})$ and $b_1b_2\ldots b_l\models p^l$. By regularity of p we easily get $l=k$.

(c) Suppose that $\bar{b}\models p^m$ and $\bar{b}\subseteq D^p_k(\mathcal{M},\bar{a})$. By part (b) there exists $\bar{b}=b_1b_2\ldots b_k\models p^k$ such that $\bar{b}\subseteq D^p_k(\mathcal{M},\bar{a})$. Then by part (a) we have $tp(\bar{b}c/A)=tp(\bar{a}c/A)$. In particular

$$tp(b_1b_2\ldots b_m/cA)=tp(a_1a_2\ldots a_m/cA).$$

The following definition is justified by Lemma 3.2.1(c) for the case m=1.

<u>Definition</u> Let $p\in S(A)$ be a regular, stationary, internally isolated type. For $a_1a_2\ldots a_k\models p^k$, where $k\in\mathbb{N}$, and $c=\bar{a}/E^p_k$ we define $p_c=tp(a_1/cA)$.

Now we introduce some more notation. If $c=\bar{a}/E^p_k$ and if $c'=\bar{a}'/E^p_{k'}$ we write $c\le c'$ iff $k\le k'$ and $D^p_k(\mathcal{M},\bar{a})\subseteq D^p_{k'}(\mathcal{M},\bar{a}')$.

<u>Lemma 3.2.2</u> Suppose that A is finite and $p\in S(A)$ is a stationary, regular, internally isolated type. Further, suppose that $k'\ge k$, $c\models p_{(k)}$ and $c'\models p_{(k')}$. Then:

(a) If $c\le c'$ and $a\models p_c$ then $a \underset{c}{\downarrow} c'$.

(b) $U(p_c) \le U(p_{c'})$.

Proof To simplify notation assume $A=\emptyset$.

(a) Let $c\le c'$ and $a\models p_c$. Find $a_1a_2\ldots a_{k'}\models p^{k'}$ such that $a=a_1$, $a_1a_2\ldots a_k/E^p_k=c$ and $a_1a_2\ldots a_{k'}/E^p_{k'}=c'$; this can be done by using Lemma 3.2.1(b) twice. We show that $c'\in dcl(ca_{k+1}a_{k+2}\ldots a_{k'})$. For let f be an automorphism fixing $ca_{k+1}a_{k+2}\ldots a_{k'}$ pointwise and let $b_i=f(a_i)$, $1\le i\le k$. Then, since $f(c)=c$ and p is regular we have

$$\{x\in\mathcal{M} \mid x\vdash p \quad x \not\perp a_1a_2\ldots a_k\} = \{x\in\mathcal{M} \mid x\vdash p \quad x \not\perp b_1b_2\ldots b_k\},$$

and also

$$\{x\in\mathcal{M} \mid x\vdash p \quad x \not\perp a_1a_2\ldots a_{k'}\} = \{x\in\mathcal{M} \mid x\vdash p \quad x \not\perp b_1b_2\ldots b_ka_{k+1}\ldots a_{k'}\}.$$

Therefore $c'=f(c')$ and every automorphism fixing $ca_{k+1}a_{k+2}\cdots a_{k'}$ pointwise also fixes c'. Hence $c'\in dcl(ca_{k+1}a_{k+2}\cdots a_{k'})$.

From the independence of $\{a_1,a_k,\cdots,a_{k'}\}$ and $c\in dcl(a_1a_2\cdots a_k)$ we get $a_1 \mathop{|}_{c} a_{k+1}a_{k+2}\ldots a_{k'}$ and since $c'\in dcl(ca_{k+1}a_{k+2}\cdots a_{k'})$ we conclude $a_1 \mathop{|}_{c} c'$.

(b) Without any loss of generality we assume $c\le c'$. Let $a\vdash p_c$. Then we have:

$$U(p_c) = U(a/c) = U(a/cc') \le U(a/c') = U(p_{c'}).$$

Lemma 3.2.3 Suppose that A is finite and $p\in S(A)$ is a stationary, regular, internally isolated type of limit ordinal U-rank. Further, let E be an A-definable equivalence relation such that $q=p/E$ is primitive. Then there exists an integer $k_o=k_o(p)\ge 2$ such that for all $k\ge k_o$ and $d\vdash q_{(k)}$, q_d is nonalgebraic. Moreover, if $U(p)=\omega^{\beta+1}$ then then k_o can be chosen so that each $U(q_d)\ge\omega^{\beta}$.

Proof Clearly q is a regular internally isolated type of limit ordinal U-rank. Assume that $a\vdash q$ and let $B\supseteq A$ be such that $r=tp(a/B)$ is a nonalgebraic, forking extension of q; moreover if $U(q)=\omega^{\beta+1}$ assume in addition $U(a/B)\ge\omega^{\beta}$. Further, let $I=a_1a_2\ldots a_m$ be a Morley sequence in $stp(a/B)$ long enough so that $stp(a/B)$ is based on I. Let $k_o=k$ be the smallest integer so that $a_1a_2\ldots a_k \mathop{\not\perp}_{A} a_{k+1}$. We will show that q_d is nonalgebraic, where $d=a_1a_2\ldots a_k/E_k^p$ (by minimality of k, $a_1a_2\ldots a_k\vdash p^k$). Since I is a Morley sequence, we have

$$0 < U(a/B) \le U(a_{k+1}/a_1a_2\ldots a_kA) \le U(q_d).$$

Therefore, q_d is nonalgebraic. By Lemma 3.2.2(b), if $n\ge k$ and $c\vdash q_{(n)}$ then q_c is nonalgebraic as well.

If $U(p)=\omega^{\beta+1}$ then $U(q)=\omega^{\beta+1}$ as well and from the above inequality we would have $\omega^{\beta} \le U(q_d)$. By Lemma 3.2.2(b) the is a contradiction.

$\rightarrow$) By part $\leftarrow$) we may assume that $a_1a_2\ldots a_k \models p^k_c$. Suppose that $b_1b_2\ldots b_k \models p^k$ and $b_1b_2\ldots b_k \subseteq D^p_k(\mathcal{M},\bar{a})$. By Lemma 3.2.1(a) we get $tp(b_1b_2\ldots b_k/cA)=tp(a_1a_2\ldots a_k/cA)$ and hence $b_1b_2\ldots b_k \models p^k_c$.

<u>Lemma 3.2.5</u> Let $p\in S(A)$ be a stationary, regular, internally isolated type and $U(p)=\omega^\alpha$. Then for all $k\geq k_o(p)$ and all $c\models p_{(k)}$ $p^k_c \perp^\alpha A$.

Proof Suppose that $k\geq k_o$, $c\models p_{(k)}$ and $p^k_c \not\perp^\alpha A$. Let $\bar{a}\models p^k_c$, and let $r=stp(B/A)$ be such that $B \mathop{\smile\!\!\!\!\!\!|}_{A} c$ and $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_{Ac} B$. By Proposition 1.2.7(a), since $U(p_c)<\omega^\alpha$ we can assume that $U(r)<\omega^\alpha$. Then, since $U(p)=\omega^\alpha$ we have $p\perp r$.

Since $\bar{a}\models p^k$ and $p^k\perp r$ we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{A} B$. By Lemma 3.2.1(a) $c=\bar{a}/E^p_k$, thus $c\in dcl(\bar{a}A)$ and we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{Ac} B$. But this contradicts the above assumption and we conclude that $p^k_c \perp^\alpha A$.

<u>Lemma 3.2.6</u> Suppose that A is finite and $p\in S(A)$ is a regular, primitive, internally isolated type of limit ordinal U-rank. Let $k\geq k_o(p)$ and let $c\models p_{(k)}$. Then:

(a) If $a_1\models p_c$, $a_2\models p_c$ and $a_1\neq a_2$ then $a_1 \mathop{\smile\!\!\!\!\!\!|}_{Ac} a_2$. In particular p_c is primitive.

(b) If $a\models p_c$ and $b\in acl(acA)\backslash acl(cA)$ then $a\in dcl(bcA)$.

Proof (a) Since p is primitive and $a_1\neq a_2$ we have $a_1 \mathop{\smile\!\!\!\!\!\!|}_{A} a_2$. Thus Lemma 3.2.4(b) applies and $a_1 \mathop{\smile\!\!\!\!\!\!|}_{Ac} a_2$. If p_c were not primitive we would have at least two distinct types of a pair of distinct realizations of p_c, which is not the case.

(b) If $a_1\models tp(a/bcA)$ and $a\neq a_1$ then by part (a) $a \mathop{\smile\!\!\!\!\!\!|}_{cA} a_1$, which is not possible since $b\in(acl(acA)\cap acl(a_1cA))\backslash acl(cA)$.

Corollary 3.2.7 Assume that A is finite, $p \in S(A)$ is a primitive, regular, internally nonisolated type and $U(p)=\omega^{\beta+1}$. Then for all $k \geq k_o$ and all $c \models p_{(k)}$

(i) p_c is semiregular, $U(p_c)=\omega^{\beta} \cdot n$ and

(ii) If r is an extension of p_c and $U(r) \geq \omega^{\beta}$ then $r \not\perp p_c$.

Proof Let $a \models p_c$. Then, by Theorem 1.2.4, there exists $b \in acl(acA)$ such that if $q=tp(b/cA)$ then (i) and (ii) hold with q in place of p_c, and possibly some ξ in place of β. But by the previous lemma we have $a \in dcl(bcA)$ and, since $k \geq k_o$ implies $\omega^{\beta} \leq U(a/cA) < \omega^{\beta+1}$, (i) and (ii) are true.

Lemma 3.2.8 Suppose that A is finite, $p \in S(A)$ is a regular, primitive, internally isolated type of U-rank $\omega^{\beta+1}$ and $k, k' \geq k_o$. Then for all $c \models p_{(k)}$ and $c' \models p_{(k')}$.

(a) $p_c \not\perp p_{c'}$

(b) $p_c \not\perp A$.

Proof (a) *Case 1* $c \leq c'$

Let $a \models p_c$ and let $r=tp(a/cc'A)$. Since $k \geq k_o$ we have $U(p_c) \geq \omega^{\beta}$. By Lemma 3.2.2, r and p_c are parallel, hence $U(r)=U(p_c) \geq \omega^{\beta}$. Thus 3.2.7 applies and $p_{c'} \not\perp r$. By parallellism $p_{c'} \not\perp p_c$.

Case 2 Not $(c \leq c')$

In this case let $n \geq k'+k$ and let $d \models p_{(n)}$ be such that $c \leq d$ and $c' \leq d$; to find such d, pick $\bar{a} \models p_c^k$ and $\bar{b} \models p_{c'}^{k'}$ and choose a maximal subset $\bar{b}'$ of $\bar{a}\bar{b}$ which is independent over A, let $m=|\bar{b}'|$ and let $d=\bar{b}'/E_m^p$. By Case 1 and $c \leq d$ we have $p_c \not\perp p_d$. Similarly, we have $p_d \not\perp p_{c'}$. But all these types are semiregular, hence $p_c \not\perp p_{c'}$.

(b) Let $d \models p_{(k)}$ be such that $c \underset{A}{|} d$. Then by part (a) we have $p_c \not\perp p_d$ and thus $p_c \not\perp A$.

Proposition 3.2.9 Suppose that A is finite and $p \in S(A)$ is a primitive, regular, internally isolated type of U-rank $\omega^{\beta+1}$. There exists a type $q \in S(A)$ such that $U(q) < \omega^{\beta+1}$ and for all $k \geq k_o$ and all $c \models p_{\langle k \rangle}$, p_c is q-internal; also if $G = Aut_q(p_c)$ then G is ∞-definable over cA and $\omega^{\beta} \leq U(G) < \omega^{\beta+1}$.

Proof Without loss of generality, let $A=\emptyset$. Fix $k \geq k_o(p)$, $c \models p_{\langle k \rangle}$ and $a \models p_c$. By Lemma 3.2.8(b) there exists a type $q \in S(\emptyset)$ such that $q \not\perp p_c$ and $U(q) < \omega^{\beta+1}$. By Proposition 1.2.7(b) there exists $b \in dcl(ac) \setminus acl(c)$ such that $tp(b/c)$ is q-internal. But by Lemma 3.2.6(b), $a \in dcl(bc)$ hence p_c is q-internal. Note that if $k' \geq k_o$ and $d \models p_{\langle k' \rangle}$, then by Lemma 3.2.8(a) $p_c \not\perp p_{d'}$ and by semiregularity we have $p_d \not\perp q$. Repeating the above argument we get that p_d is q-internal.

G is ∞-definable over cA by Theorem 1.3.14, so it remains to prove the inequality. So, let $Q \subseteq \mathcal{M}^{eq}$ be the set of all realizations of q and let $a_1, a_2 \models p_c$ be such that $a_1 \underset{c}{\mid} a_2$. By Lemma 3.2.5 we have $p_c \perp^a \emptyset$ and hence

$$tp(a_1/cQ) = tp(a_2/cQ) = p_c | cQ.$$

Thus there is $g \in G$ such that $g(a_1) = a_2$. Hence $a_2 \in dcl(ga_1c)$ and we have

$$U(a_2/ca_1) \leq U(g/ca_1) \leq U(g/c).$$

But from $a_2 \underset{c}{\mid} a_1$ we have $U(a_2/ca_1) = U(a_2/c) = U(p_c) \geq \omega^{\beta}$. Therefore

$$\omega^{\beta} \leq U(a_2/c) \leq U(g/c) \leq U(G).$$

On the other hand, every element of G is definable over c from a finite sequence of realizations of p_c; hence $U(G) < \omega^{\beta+1}$, completing the proof of the Proposition.

G is the definable group we promised at the start of the section.

In this section we prove Theorem B. Recall that we have a standing assumption that T is a complete, superstable, small theory in a countable language. We operate in $\mathcal{M}$.

<u>Theorem B</u> If $\sup\{U(p) \mid p \in S(T)\} \geq \omega^{\omega}$ and the generic type of any simple, ∞-definable group in T^{eq} is esn, then $I(T,\aleph_0)=2^{\aleph_0}$.

<u>Lemma 3.3.1</u> Suppose that A is finite, $p \in S(A)$ is an eventually strongly nonisolated type, $M \supseteq A$ is prime over A, and $\{a_1, a_2, \ldots\}$ is an infinite Morley sequence in p. Then for some $n \in \mathbb{N}$, $tp(a_n/M)$ is a nonforking extension of p.

Proof Let $C \supseteq A$ be a finite set such that $stp(a_1/A)|C$ is strongly nonisolated. Let M_1 be prime over C and, without loss of generality, assume that $M \subseteq M_1$. Since $wt(C/A)$ is finite and $\{a_1, a_2, \ldots\}$ is independent over A, for some $n \in \mathbb{N}$ $tp(a_n/C)$ is a nonforking extension of $stp(a_1/A)$ and thus strongly nonisolated. Then since M_1 is atomic over C, $tp(a_n/M_1)$ is a nonforking extension of p.

<u>Lemma 3.3.2</u> If $q \not\perp \emptyset$ then for all M there is a regular type $p \in S(M)$ such that $q \not\perp p$.

Proof For any M, $q \not\perp M$, so by Theorem 1.2.10 there is an sR type $p \in S(M)$ such that $p \not\perp q$. p is regular.

<u>Lemma 3.3.3</u> Suppose that there exists a family $\{p_n \mid n \in \omega\}$ of regular, eventually strongly nonisolated types such that p_n is orthogonal to every conjugate of p_m for all $n \neq m$. Then $I(T,\aleph_0)=2^{\aleph_0}$.

Proof Let $\mathcal{A}=\{n \mid p_n \perp \emptyset\}$ and let $\mathcal{B}=\{n \mid p_n \not\perp \emptyset\}$. Then at least one of $\mathcal{A}$ and $\mathcal{B}$ is infinite and we distinguish the two cases:

Case 1 $\mathcal{A}$ is infinite.

Without loss of generality we assume that, for all $n\in\mathscr{A}$, p_n is stationary and strongly nonisolated and that $B_n=\mathrm{dom}(p_n)$ is finite. Also, assume that $\{B_n|n\in\mathscr{A}\}$ is independent over $\emptyset$; to justify this assumption note that if we replace p_n by a conjugate of itself then the conditions of the lemma are still valid. Let $B=\bigcup\{B_n|n\in\mathscr{A}\}$ and let M be a countable almost atomic model over B.

Let $X\subseteq\mathscr{A}$ be arbitrary. We will construct a countable model M_X such that

$$(*)\qquad m\in X \quad\text{iff}\quad \begin{array}{l} m\in\mathscr{A} \text{ and for all } C\subseteq M_X \text{ and } p\in S(C)\text{, if } tp(C)=tp(B_m) \\ \text{and } p \text{ is a conjugate of } p_m \text{ then } \dim(p,M_X)=\aleph_0. \end{array}$$

Inductively define a sequence of countable models $\{M_X^n|n\in\omega\}$. Let $M_X^0=M$. Further, suppose that M_X^n has already been constructed. Let $\mathscr{F}_n$ be the set of all conjugates of p_m's for all $m\in X$ whose domain is a finite subset of M_X^n. Note that $\mathscr{C}=\{\mathrm{dom}(p)|p\in\mathscr{F}_n\}$ is countable since M_X^n is countable. Since T is small for all $C\in\mathscr{C}$ there are at most countably many $p\in\mathscr{F}_n$ such that $\mathrm{dom}(p)=C$. Therefore $\mathscr{F}_n$ is countable. For $p\in\mathscr{F}_n$ choose a countable Morley sequence in $p|M_X^n$ and call it I_p. Moreover, assume that our choice is such that $I_n=\bigcup\{I_p|p\in\mathscr{F}_n\}$ is independent over M_X^n. Let M_X^{n+1} be a countable model dominated by I_n over M_X^n, and let $M_X=\bigcup\{M_X^n|n\in\omega\}$.

To prove (*) notice that, by construction, it is enough to show that $\dim(p_k,M_X)=0$ for all $k\in\mathscr{A}\setminus X$. Let $k\in\mathscr{A}\setminus X$. Since $p_k\perp\emptyset$ and B is independent over $\emptyset$ we have $p_k\vdash p_k|B$. Since M is almost atomic over B and $p_k|B$ is strongly nonisolated Lemma 2.3.3 applies, hence $p_k|B\vdash p_k|M$. Since I_n is an independent set of realizations of types which are orthogonal to p_k we derive $p_k\perp tp(I_n/M_X^n)$. Further, M_X^{n+1} is dominated by I_n over M_X^n, so $p_k\perp tp(M_X^{n+1}/M_X^n)$ and hence $p_k|M_X^n\vdash p_k|M_X^{n+1}$. We conclude that $p_k\vdash p_k|M_X$ and thus $\dim(p_k,M_X)=0$.

From $X,Y\subseteq\mathscr{A}$ and $X\neq Y$ we get by (*) $M_X\not\cong M_Y$. Since $\mathscr{A}$ is infinite $I(T,\aleph_0)=2^{\aleph_0}$.

Case 2: $\mathscr{B}$ is infinite.

Let M be a prime model. By Lemma 3.3.2, for every $n\in\mathscr{B}$ there exists a regular type $r_n\in S(M)$ such that $r_n\not\perp p_n$. For $n\in\mathscr{A}$ let $B_n\subseteq M$ be a finite set such that r_n does not fork over B_n and let

$q_n = r_n | B_n$. Note that q_n might not be stationary. Clearly, q_n is eventually strongly nonisolated, and for $n \neq m$ r_n is orthogonal to each conjugate of r_m. Let $X \subseteq \mathcal{B}$ be arbitrary. Construct M_X in the following way.

Inductively define a sequence of countable models $\{M_X^n | n \in \omega\}$. Let $M_X^0 = M$. Further, suppose that M_X^n has already been constructed. Let $\mathcal{F}_n$ be the set of all conjugates of r_m's for all $m \in X$ whose domain is a finite subset of M_X^n. As above, $\mathcal{F}_n$ is countable. For $p \in \mathcal{F}_n$ choose a countable Morley sequence in a nonforking extension of p to M_X^n. Moreover, assume that our choice is such that $I_n = \bigcup\{I_p | p \in \mathcal{F}_n\}$ is independent over M_X^n. I_n is countable, so let M_X^{n+1} be a countable model dominated by I_n over M_X^n, and let $M_X = \bigcup\{M_X^n | n \in \omega\}$. We claim that:

(**) $m \in X$ iff $m \in \mathcal{B}$ and for all $C \subseteq M_X$ and $p \in S(C)$, if $tp(C) = tp(B_m)$ and p is a conjugate of r_m, then in M_X there exists a countable Morley sequence in p.

By construction, it is enough to prove that for $m \in \mathcal{B} \setminus X$ there does not exists in M_X an infinite Morley sequence in r_m. Suppose that $\{a_1, a_2, \ldots\}$ is an infinite Morley sequence in r_m where $m \in \mathcal{B} \setminus X$. By Lemma 3.3.1, for some $k \in \mathbb{N}$ $tp(a_k / M)$ is a nonforking extension of r_m. As in the previous case we deduce that $tp(a_k / M) \vdash tp(a_k / M_X)$, so that a_k is not in M_X, completing the proof of the claim.

From $X, Y \subseteq \mathcal{B}$ and $X \neq Y$ we get by (**) $M_X \not\cong M_Y$. Since $\mathcal{B}$ is infinite we have $I(T, \aleph_0) = 2^{\aleph_0}$.

The orthogonality condition in Lemma 3.3.3 cannot be weakened in general. The example of an abnormal type from [B] XVIII.4, shows that the assumption is necessary in the Case 1 above. However, under the conditions of Case 2 it can be weakened; if A is finite and $\{p_n | n \in \omega\}$ is a family of pairwise orthogonal, regular, esn types which are nonorthogonal to A then $I(T, \aleph_0) = 2^{\aleph_0}$. Since we won't use this fact, we only outline its proof. Without any loss let $A = \emptyset$. First of all, by Lemma 3.3.3, we can assume that each p_n is a conjugate of p_1. Also, we can assume that $U(p_n) = \omega^{\alpha}$.

Then arguing as in the Proposition 1.2.7, we can find $\bar{a}_n$ and $\bar{b}_n$ such that $\bar{b}_n \in dcl(\bar{a}_n)$, $U(\bar{b}_n)<\omega^\alpha$ and $q_n=stp(\bar{a}_n/\bar{b}_n)\sqsupset(p_n)^m$ for some natural m. Hence for all n,k q_n is orthogonal to every extension of $tp(b_k)$; that is enough to control 'dimensions' of the q_n's in almost atomic models (as in the proof of Theorem C).

Lemma 3.3.4 Suppose that A is finite, $\psi(x)$ and q are over A, $\psi(x)$ is q-internal, and q is a NENI type with $U(q)=\omega^\alpha$. Then for all finite $B\supseteq A$, $\{x \mid \models \psi(x) \text{ and } U(x/B)<\omega^\alpha\}$ is definable over B.

Proof Without any loss of generality, assume that $dom(q)=A=B=\emptyset$. Let $D=\{x \mid \models \psi(x) \text{ and } U(x)<\omega^\alpha\}$. We show that D is both closed and open in the topology induced by formulas over $acl^{eq}(\emptyset)$. Since D is closed under automorphisms of $\mathcal{M}$ it will follow that it is $\emptyset$-definable.

Let $\phi(y)\in q$ be a formula which isolates q. Suppose that $a\in D$. Since ψ is ϕ-internal, there are $\bar{c}$ and $\bar{b}=b_1\ldots b_n$ such that $a \mid \bar{c}$, $\bar{b}\subseteq\phi(\mathcal{M})$ and $a\in dcl(\bar{b}\bar{c})$. Moreover, we can assume that $U(\bar{b}/\bar{c})<\omega^\alpha$; this because $U(a/\bar{c})<\omega^\alpha$. Let $\varphi(y,\bar{c})$ be a formula which isolates $q|\bar{c}$ and let $x=f(\bar{y},\bar{z})$ witness that $a\in dcl(\bar{b}\bar{c})$. Let $r=stp(\bar{c})$and consider the following formula

$$\models d_r\bar{z}\exists y_1y_2\ldots y_n(\bigwedge_i(\phi(y_i)\wedge\neg\varphi(y_i,\bar{z}))\wedge x=f(\bar{y},\bar{z})).$$

Denote it by $\theta(x)$. Note that $\models\theta(a)$. Suppose that $\models\theta(a')$ and pick $\bar{b}'=b_1'\ldots b_n'$ and $\bar{c}'\models r|\bar{a}'$ such that

$$\models\bigwedge_i(\phi(b_i')\wedge\neg\varphi(b_i',\bar{c}'))\wedge a'=f(\bar{b}',\bar{c}').$$

Then $tp(b_i'/\bar{c}')$ is a forking extension of q so $U(b_i'/\bar{c}')<\omega^\alpha$ for all $i\leq n$, so that $U(a'/\bar{c}')<\omega^\alpha$ and, because $a' \mid \bar{c}'$, $U(a')<\omega^\alpha$. Hence D is open.

On the other hand, if $U(a)\geq\omega^\alpha$ and $\models\psi(a)$ then $stp(a)\not\perp q$, hence there are $\bar{c}$ and $b\models q|\bar{c}$ such that $a \mid \bar{c}$ and $a \not\mid_{\bar{c}} b$. Let $\varphi(y,\bar{c})$ be a formula which isolates $q|\bar{c}$, and let ρ be such that $\models\rho(b,a,\bar{c})$ and $\rho(y,a,\bar{c})$ forks over $\bar{c}$. Let $r=stp(\bar{c})$ and consider the following formula

$$d_r\bar{z}\exists y(\rho(y,x,\bar{z})\wedge\varphi(y,\bar{z})).$$

Denote it by $\theta(x)$ and note that $\models \theta(a)$. Suppose that $\models \theta(a')$ and pick b' and $\bar{c}'$ such that $\bar{c}' \models r|a'$ and $\models \theta(b',a',\bar{c}') \wedge \varphi(b',\bar{c}')$. Thus $b' \models q|\bar{c}'$ and $a' \not\downarrow_{\bar{c}'} b'$ so that $stp(a'/\bar{c}') \not\perp q$ and therefore $U(a'/\bar{c}') \geq \omega^{\alpha}$. Hence $U(a') \geq \omega^{\alpha}$ and D is closed.

<u>Lemma 3.3.5</u> (in $\mathcal{M}^{eq}$) Suppose that A is finite, G is an A-definable abelian group, $p \in S(A)$ is a regular, stationary, internally isolated type of an element of G and E is an A-definable equivalence relation such that:

(a) $U(p) = \omega^{\alpha+1}$,

(b) $q = p/E$ is a primitive NENI type, and

(c) G is q-internal.

Then one of the following two conditions holds:

(i) There exists a definable field F such that

$$\omega^{\alpha} \leq U(F) < \omega^{\alpha+1}.$$

(ii) There exists an ∞-definable simple group S such that

$$\omega^{\alpha} \leq U(S) < \omega^{\alpha+1}.$$

Proof To simplify notation assume $A=\emptyset$.

By $\mathcal{P}$ and $\mathcal{Q}$ we denote the sets of all realizations of p and q respectively. For $c \models p_{(k)}$ and $\bar{a} \models p_C^k$ let $c_E \models q_{(k)}$ denote the name for the set $\{x \models q \mid x \not\perp a_1/E\, a_2/E \ldots a_k/E\}$. Note that c_E and c are interdefinable; every automorphism fixing c also fixes c_E and vice versa.

Claim 1 There exists an integer k_1 such that for all $k \geq k_1$ and all $c \models p_{(k)}$, if $d = c_E$ then p_C is q_d-internal.

Proof: Since p is q-internal, by Proposition 1.2.6 there exists a $\emptyset$-definable function $f(\bar{y},\bar{z})$ such that $dom(f) \supseteq \mathcal{P}^s \times \mathcal{Q}^t$, $ran(f) \supseteq \mathcal{P}$ and if $a\bar{b} \models p^{s+1}$ then there exists $\bar{u} \in \mathcal{Q}^t$ such that $a = f(\bar{b},\bar{u})$ (here $\mathcal{X}^n$ stands for the cartesian power of $\mathcal{X}$). Let $k_1 = s+t$. We show that k_1 satisfies the claim, i.e. for $k \geq k_1$, $c \models p_{(k)}$ and $d = c_E$ we show that p_C is q_d-internal.

Let $a \models p_C$, let $ab_1b_2\ldots b_s \models p^{s+1}$ and let $e_1e_2\ldots e_t \in \mathcal{P}^t$ be such that $a=f(\bar{b},\bar{e}/E)$, where $\bar{e}/E$ denotes $e_1/E\ e_2/E\ldots e_t/E$. From

$$\dim(a\bar{b}\bar{e}) = \dim(\bar{b}\bar{e}) \le s+t = k_1 \le k,$$

there exists $\bar{b}'$ and $\bar{e}'$ such that $b'_i \models p_c$ and $e'_j \models p_c$ for $1\le i\le s$ and $1\le j\le t$ such that $tp(a\bar{b}'\bar{e}')=tp(a\bar{b}\bar{e})$. To see this choose $c'\models p_{(k)}$ so that $a,b_1,b_2,\ldots b_s,e_1,e_2,\ldots,e_t$ all realize $p_{c'}$. By Lemma 3.2.1 $tp(ac)=tp(ac')$, hence there exists an automorphism of the monster model, say θ, which maps c to c' and fixes a; let $\bar{b}'=\theta(\bar{b})$ and let $\bar{e}'=\theta(\bar{e})$). Hence $a\bar{b}'\models p^{s+1}$ and $a=f(\bar{b}',\bar{e}'/E)$.

We show that a, $\bar{b}'$ and $\bar{e}'$ witness that p_c is q_d-internal. From $a\bar{b}'\models p^{s+1}$ and $s+1\le k$, by Lemma 3.2.4(b), we have $a\bar{b}'\models p_c^{s+1}$. For $1\le j\le t$ we have $e'_j/E\models q_d$. Also $a=f(\bar{b}',\bar{e}'/E)$ hence p_c is q_d-internal, as we claimed.

Now we apply Proposition 3.2.9 to q, which is a primitive, NENI type of U-rank $\omega^{\alpha+1}$.

Let $k_o=k_o(q)$ and let $r\in S(\emptyset)$ be such that:

(i) $\omega^{\alpha} \le U(r) < \omega^{\alpha+1}$, and

(ii) for all $k\ge k_o$ and $d\models q_{(k)}$, q_d is r-internal.

By Corollary 3.2.7

(iii) q_d is semiregular, $U(q_d)=\omega^{\alpha}\cdot n_k$, and

(iv) whenever q' is an extension of q_d and $U(q')\ge\omega^{\alpha}$ then $q' \not\perp q_d$; semiregularity of q_d then implies $q' \not\perp r$.

Claim 2 For all $k\ge k_o$ and $c\models p_{(k)}$, $U(p_c)=\omega^{\alpha}\cdot m_k$ for some $m_k\in\mathbb{N}$.

Proof: Suppose that the claim is not true. Let $c\models p_{(k)}$ where $k\ge k_o$ contradicts the claim. Then, since $U(p_c)<\omega^{\alpha+1}$, we must have $U(p_c)=\omega^{\alpha}\cdot m_k+\xi$ where $0<\xi<\omega^{\alpha}$. Let $a_1\models p_c$. By Theorem 1.2.3, p_c is nonorthogonal to a type of U-rank ω^{η}, where $0\le\eta<\alpha$. By Proposition 1.2.7(a) there is $e\in dcl(a_1c)$ such that $0 < U(e/c) < \omega^{\alpha}$. Because c and d are interdefinable we have $U(e/d) < \omega^{\alpha}$.

Let $a_2\models tp(a_1/ec)$ be such that $a_1 \underset{ec}{\mid} a_2$. Clearly, $a_1 \underset{c}{\not\perp} a_2$. Further, let $d=c_E$ and let $b_1=a_1/E$ and $b_2=a_2/E$. Since $b_1,b_2\models q_d$, and $U(q_d)=\omega^{\alpha}\cdot n_k$ and $U(e/d)<\omega^{\alpha}$ we have $b_1 \underset{d}{|} e$, hence $b_1 \underset{c}{|} e$. From $a_1 \underset{ec}{|} a_2$ and $b_1 \in dcl(a_1)$ we get $b_1 \underset{ec}{|} a_2$, and by transitivity $b_1 \underset{c}{|} ea_2$. Therefore $b_1 \underset{c}{|} a_2$ and thus $b_1 \underset{c}{|} b_2$. By Lemma 3.2.4(b) applied to q we get $b_1 \mid b_2$ so $a_1 \mid a_2$. By Lemma 3.2.4(b) again applied to p we have $a_1 \underset{c}{|} a_2$, and that contradicts the above observation.

Continuing the proof of the Lemma we fix $k \geq \max\{k_0, k_1, 2\}$, $c \models p_{(k)}$ and $d=c_E \models q_{(k)}$. Let $\mathcal{P}_c \subseteq G$ be the set of all realizations of p_c and let $\mathcal{R} \subseteq \mathcal{M}^{eq}$ be the set of all realizations of r. p_c is stationary by Lemma 3.2.4(a) and $U(p_c)=\omega^{\alpha}\cdot n$ by Claim 2; the conditions of Proposition 1.3.10 are satisfied, hence $\mathcal{P}_c$ is α-indecomposable.

Let $G_0=\{x \in G \mid U(x/c)<\omega^{\alpha+1}\}$. If $a,b \in G_0$ then $a\cdot b^{-1} \in dcl(a^\frown b^\frown c)$ so $U(a\cdot b^{-1}/c) \leq U(a^\frown b/c) \leq U(a/c)\oplus U(b/c) < \omega^{\alpha+1}$ and $a\cdot b^{-1} \in G_0$, which implies that G_0 is a subgroup of G. But G is q-internal and q is NENI so Lemma 3.3.4 applies and G_0 is definable over c. It follows that $U(G_0)<\omega^{\alpha+1}$. Also, $\mathcal{P}_c \subseteq G_0$ and

$$\omega^{\alpha} \leq U(\mathcal{P}_c) \leq U(G_0) < \omega^{\alpha+1}.$$

Now we can apply Theorem on Indecomposables to $\mathcal{P}_c \subseteq G_0$ so that $H=Gp(\mathcal{P}_c^{-1}\cdot\mathcal{P}_c)$ is ∞-definable over c and $H=(\mathcal{P}_c^{-1}\cdot\mathcal{P}_c)^n$ for some $n \in \mathbb{N}$.

Fix $a \models p_c$.

p_c is q_d-internal by Claim 1 and q_d is r-internal, hence p_c is r-internal and $K=Aut_{ac\mathcal{R}}(\mathcal{P}_c)$ is ∞-definable over ac, by Theorem 1.3.14. It is also α-connected by the same theorem.

We extend the action of K to H in the following way. For all $g \in K$ and $e \in H$ if $e=a_1\cdot b_1^{-1}\cdot\ldots\cdot a_l\cdot b_l^{-1}$ where $a_1,b_1,\ldots,a_l,b_l \in \mathcal{P}_c$ define

$$g(e) = g(a_1)\cdot g(b_1)^{-1}\ldots\cdot g(a_l)\cdot g(b_l)^{-1}.$$

Claim 4 *K is a group of (all) $ac\mathcal{R}$-automorphisms of H.*

Proof There are a few points to be checked in this claim:

(1) For each $g \in K$ the action of g on H is well defined.

Suppose that $a_1, b_1, \ldots a_l, b_l \in \mathcal{P}_C$ and $a'_1, b'_1, \ldots a'_l, b'_l \in \mathcal{P}_C$ are such that $a_1 \cdot b_1^{-1} \cdot \ldots \cdot a_l \cdot b_l^{-1} = a'_1 \cdot b_1'^{-1} \ldots \cdot a'_l \cdot b_l'^{-1}$. Then since g is an automorphism of $\mathcal{P}_C$ we must have

$$g(a_1) \cdot g(b_1)^{-1} \ldots g(a_l) \cdot g(b_l)^{-1} = g(a'_1) \cdot g(b'_1)^{-1} \ldots g(a'_l) \cdot g(b'_l)^{-1},$$

so the action of g on H is well defined.

(2) Each $g \in K$ is an $ac\mathcal{R}$-automorphism of H.

Suppose that $\models \phi(e_1, e_2, \ldots, e_m)$ holds, where $\bar{e} \subseteq H$ and ϕ is a formula over $ac\mathcal{R}$. For each $i \leq m$ and $j \leq n$, let $a_i^j, b_i^j \in \mathcal{P}_C$ be such that $e_j = \prod_i a_i^j \cdot (b_i^j)^{-1}$. We have:

$$\models \phi(\prod_i a_i^1 \cdot (b_i^1)^{-1}, \prod_i a_i^2 \cdot (b_i^2)^{-1}, \ldots, \prod_i a_i^n \cdot (b_i^n)^{-1})$$

and since g is a $ac\mathcal{R}$-automorphism of $\mathcal{P}_C$ we must have:

$$\models \phi(\prod_i g(a_i^1) \cdot g(b_i^1)^{-1}, \prod_i g(a_i^2) \cdot g(b_i^2)^{-1}, \ldots, \prod_i g(a_i^n) \cdot g(b_i^n)^{-1})$$

Therefore $\models \phi(g(e_1), g(e_2), \ldots, g(e_m))$ and the conclusion follows.

(3) If $g, h \in K$ and $e \in H$ then $g(h(e)) = (g \circ h)(e)$ and $g(e^{-1}) = g(e)^{-1}$.

Let $e = \prod_i a_i \cdot b_i^{-1}$ where $a_i, b_i \in \mathcal{P}_C$. Then, since g, h are $ac\mathcal{R}$-auts of $\mathcal{P}_C$ we have:

$$g(h(e)) = g(h(\prod_i a_i \cdot b_i^{-1})) = g(\prod_i h(a_i) \cdot h(b_i)^{-1}) =$$

$$= \prod_i (g \circ h)(a_i) \cdot (g \circ h)(b_i)^{-1} = (g \circ h)(\prod_i a_i \cdot b_i^{-1}) = (g \circ h)(e).$$

The proof of $g(e^{-1}) = g(e)^{-1}$ is similar.

(4) $K \cong Aut_{ac\mathcal{R}}(H)$.

Suppose that $g(x) = x$ for all $x \in H$. In particular, for all $y \in \mathcal{P}_C$, $y \cdot a^{-1} \in H$, hence $g(y \cdot a^{-1}) = y \cdot a^{-1}$. Thus $g(y) \cdot g(a)^{-1} = y \cdot a^{-1}$ and since $g(a) = a$ we have $g(y) = y$. g is the identity map on $\mathcal{P}_C$ so $g = 1$. Hence K acts faithfully on H.

If $h \in Aut_{ac\mathcal{R}}(H)$ then by $\hat{h}(y) = h(y \cdot a^{-1}) \cdot a^{-1}$ we define an $ac\mathcal{R}$-automorphism of $\mathcal{P}_C$. Also $\hat{h}$ is identity if and only if h is identity, and the conclusion follows.

Claim 5 K acts transitively on $a^{-1}\mathcal{P}_C \setminus cl_r(a)$ and

$$\omega^{\alpha} \le U(K) < \omega^{\alpha+1}.$$

Proof: Since every element of K is definable over $a\frown c$ from a finite sequence of realizations of p_C we have $U(K) < \omega^{\alpha+1}$.

Let $b_1\frown b_2\frown a \models p_C^3$. Since stp(c) is p-semiregular and $p \perp stp(\mathcal{R})$ we have $c \mid \mathcal{R}$. But by Lemma 3.2.5 we have $p_C^2 \mid^{\alpha} \emptyset$ (recall that $k \ge 2$) and hence $b_1\frown a \models p_C^2 | c\mathcal{R}$ and $b_2\frown a \models p_C^2 | c\mathcal{R}$. Therefore

$$tp(b_2 \cdot a^{-1}/a\frown c\frown \mathcal{R}) = tp(b_1 \cdot a^{-1}/a\frown c\frown \mathcal{R}).$$

Because $b_2 \cdot a^{-1}$ and $b_1 \cdot a^{-1}$ both belong to H, there exists $g \in K$ such that $g(b_2 \cdot a^{-1}) = b_1 \cdot a^{-1}$. We have

$$\omega^{\alpha} \le U(p_C) = U(b_1/c\frown a) = U(b_1/b_2\frown a\frown c) = U(b_1 \cdot a^{-1}/b_2\frown a\frown c)$$

The second equality holds since $b_1 \mathop{|}_{C} b_2\frown a$; and the third one since $b_1 \cdot a^{-1}$ and b_1 are interdefinable over $b_2\frown a\frown c$. Further, since $b_1 \cdot a^{-1} \in dcl(g\frown b_2\frown a\frown c)$ we have $U(b_1 \cdot a^{-1}/b_2\frown a\frown c) \le U(g/b_2\frown a\frown c)$.

We conclude that $\omega^{\alpha} \le U(g/b_2\frown a\frown c) \le U(g/c\frown a) \le U(K)$ and the proof of the claim is complete.

Let H' be a K-invariant, ∞-definable, proper subgroup of H of maximal r-weight. By Proposition 1.3.4 it is an intersection of $ac\mathcal{R}$-definable supergroups, so we can assume that $H' = D \cap H$, where D is an $ac\mathcal{R}$-definable group. The action of K on H/H' is defined by: for $g \in K$ and $b \in H$ $g(b_{H'}) = g(b)_{H'}$. Now, we replace H' by

$$H'' = \{x \mid x_{H'} \text{ is fixed by all } g \in K\},$$

which is maximal, K-invariant, proper subgroup of H (by Claim 5 K acts transitively on $a^{-1}\mathcal{P}_C \setminus cl_r(a)/H'$, so H'' is a proper subgroup of H). Hence H' is K- minimal. Further, define:

$$K' = \{g \in K \mid g(x) = x \text{ for all } x \in H/H'\}.$$

Claim 6 K/K' is nontrivial and α-connected.

$a^{-1}\mathcal{P}_C$ is not included in H' so, since H is α-connected, there are $b, b' \in a^{-1}\mathcal{P}_C \setminus cl_r(a)$ which are distinct modulo H'. Further, by Claim 5 there is $g \in K$ such that $g(b) = b'$. We conclude that $g \notin K'$ and K/K' is nontrivial. Since K is α-connected, K/K' is α-connected as well.

What we have up to this moment is that K/K' is a group of automorphisms of H/H', H/H' is abelian and K/K'-minimal, and both H/H' and K/K' are nontrivial and α-connected. Suppose that the condition (ii) from the conclusion of the Lemma does not hold. Then every α-connected group is solvable and in particular K/K' is solvable. Thus all the conditions of Theorem 1.3.13 are satisfied, and (ii) holds.

Lemma 3.3.6 If $F \subseteq \mathcal{M}^{eq}$ is a definable field and $\omega^{\alpha+1} \leq U(F) < \omega^{\alpha+2}$, then the generic of F is esn.

Proof Suppose on the contrary that the generic of F is not esn. Then it is nonorthogonal to an NENI type p such that $U(p)=\omega^{\alpha+1}$. By Corollary 1.3.7, F is p-internal. Let $a \in F$, $U(a) \geq \omega^{\alpha+1}$ and let B be finite such that $\omega^{\alpha} \leq U(a/B) < \omega^{\alpha+1}$. Let a' and B' be such that $tp(a'B')=tp(aB)$ and $a'B' | aB$. Thus $\omega^{\alpha} \leq U(a'/BB') < \omega^{\alpha+1}$ and $\omega^{\alpha+1} \leq U(a/B')$. Define

$$E=\{x \in F \mid U(x/B)<\omega^{\alpha+1}\} \quad \text{and} \quad E'=\{x \in F \mid U(x/BB')<\omega^{\alpha+1}\}.$$

By Lemma 3.3.4 E is a B-definable subfield of F and E' is a BB'-definable subfield of F and $U(E), U(E')<\omega^{\alpha+1}$. Also $E \subseteq E'$, and because $a' \in E' \setminus E$ we have $E \neq E'$.

By Theorem 1.3.5 both E and E' are algebraically closed, hence E' is an infinite-dimensional vector space over E. Since every element of an n-dimensional vector space is interdefinable with an element of E^n over a generic basis, $U(E) \cdot n \leq U(E')$ holds for all integers n. Therefore $\omega^{\alpha+1} \leq U(E')$, contradicting the above.

Proof of Theorem B Suppose that $U(T) \geq \omega^{\omega}$ and that there is no simple ∞-definable group in T^{eq} whose generic type is nonorthogonal to an NENI type. We'll show that $I(\aleph_0, T)=2^{\aleph_0}$. We operate in $\mathcal{M}^{eq}$. Let I be the set of all $n \in \omega$ such that there exists a regular, eventually strongly nonisolated type p_n such that $U(p_n)=\omega^n$.

Claim I is infinite.

Proof: Suppose, on the contrary, that I is finite and let $n=\max(I)+1$. We show that there is a finite set A, an A-definable group G and $p\in S(A)$ such that the conditions of Lemma 3.3.5 are satisfied.

By our assumption on n, every regular type of U-rank ω^{n+2} is not esn and hence by Theorem A there exists an NENI type of U-rank ω^{n+2}. By Lemma 3.1.8 there exists a primitive NENI type of U-rank ω^{n+2}. Proposition 3.2.9 applies, hence there exists a finite set C and an ∞-definable group H over C such that $\omega^{n+1}\le U(H)<\omega^{n+2}$. By Proposition 1.3.1 we may replace H by a C-definable supergroup of H. Further, by Theorem 1.3.8 and 1.3.9, after possibly passing to a subgroup of H and adding a few parameters to C, we assume that H is abelian and $U(H)=\omega^{n+1}\cdot m$. Also, assume that m is minimal possible; then from the U-rank inequalities for groups we derive that H has no proper definable subgroup of infinite index which has U-rank bigger than ω^{n+1}.

Further, the generic of H is nonorthogonal to a regular type of U-rank ω^{n+1} which is not eventually strongly nonisolated and so is, by Theorem A, nonorthogonal to a NENI type r of U-rank ω^{n+1}. After possibly slightly enlarging C we can assume that $r\in S(C)$. By Proposition 1.3.6 there exists a C-definable subgroup $H_1\subseteq H$ such that H/H_1 is r-internal and infinite. Since $U(H)=\omega^{n+1}\cdot m$, from the U-rank inequalities for groups and the minimality of m, we derive $U(H/H_1)=\omega^{n+1}\cdot m$. Therefore if we replace H by H/H_1, we have the following situation:

> C is a finite set, $r\in S(C)$ is NENI, $U(r)=\omega^{n+1}$ and H is a C-definable abelian group which is r-internal, $U(H)=\omega^{n+1}\cdot m$ and H has no proper, definable subgroup F such that $\omega^{n+1}\le U(F)<U(H)$.

Now we proceed as in the proof of Theorem A. Let p_1 be a type of an element of H whose domain is finite and contains C, which is nonorthogonal to r and has minimal possible R-rank and minimal possible CB-rank. As in the proof of Theorem A we conclude that p_1 is regular, isolated by $\psi(x)$ say, and internally isolated. Moreover p_1 is not esn; this is because it is nonorthogonal to r which is NENI.

We show that $U(p_1)=\omega^{n+1}$. If $U(p_1)>\omega^{n+1}$ then there would exist a forking extension of p_1 whose rank is at least ω^{n+1}. Since H is r-internal and $U(r)=\omega^{n+1}$, this extension is nonorthogonal to r, and that contradicts the minimality assumption on $R(p_1)$. Hence $U(p_1)=\omega^{n+1}$.

Let $d\models p_1$ and $B=dom(p_1)$. Then $stp(d/B)|Bd$ is not esn, so pick witnesses for this; according to 3.1.2(a), there are A, a and b such that $A\supseteq Bd$ is finite, $p_2=stp(d/B)|A$, $a\models p_2$, $\models \psi(b)$, $tp(b/A)$ is isolated and $a \not\downarrow_A b$.

p_2 is stationary and internally isolated, so let E_1 be an A-definable equivalence relation, given by 3.1.8, such that p_2/E_1 is primitive. As in the proof of Theorem A, we show that p_2/E_1 is NENI. Let $a'=a/E_1$. Note that $p_2/E_1=tp(a'/A)$ is stationary and internally isolated, because $p_2=tp(a/A)$ is. By Lemma 3.1.9(b) $a'\in dcl(bA)$ and since $tp(b/A)$ is isolated, p_2/E_1 is isolated, too. Hence p_2/E_1 is stationary, internally isolated and isolated, so by Lemma 3.1.12 it is NENI.

p_2/E_1 is nonorthogonal to r, hence it is nonorthogonal to the generic type of H; by Proposition 1.3.6 there exists an A-definable subgroup $K\leq H$ such that $G=H/K$ is p_2/E_1-internal and infinite.

Let $p=tp(a_K/A)$ (recall that $a_K\in G$ is the name of the K-coset which contains a). Since K is definable over A and $p_2\in S(A)$, p does not depend on the particular choice of $a\models p_2$. Let E be an A-definable equivalence relation such that $q=p/E$ is primitive. We show that our G, p and E satisfy all the conditions of Lemma 3.3.5, i.e. that:

(i) p is a stationary, regular, internally isolated type nonorthogonal to the generic of G and $U(p)=\omega^{n+1}$,

(ii) q is a primitive NENI type, and

(iii) G is q-internal.

By the minimality assumption on U(H) we must have $U(K)<\omega^{n+1}$. From $\omega^{n+1}= U(a/A) \leq U(K) \oplus U(a_K/A)$ we derive $U(a_K/A)=\omega^{n+1}$. Since $p_2=tp(a/A)$ is stationary and $a_K\in dcl(aA)$, $p=tp(a_K/A)$ is stationary too, and since $p_2 \not\perp r$ we must have that $p \not\perp r$, and p is nonorthogonal to the generic of G. Therefore (i) is satisfied. (ii) holds by definition, so it remains to show that G is q-internal. Actually, we show that each realization of q is interdefinable over A with a realization of p_2/E_1; since G is p_2/E_1-internal it will follow that G is q-internal, as well.

From $a \underset{A}{\not\perp} a'$ and $a \underset{A}{\not\perp} a_K/E$ we get $a_K/E \underset{A}{\not\perp} a'$ and since both $p_2/E_1=tp(a')$ and $q=tp(a_K/E)$ are primitive, regular types we conclude by Lemma 3.2.6(b) that a' and a_K/E are interdefinable over A. Since $a \models p_2$ was arbitrary, the conclusion follows.

The conditions of Lemma 3.3.5 are fulfilled and we conclude that there exists a definable field $F \subseteq \mathcal{M}^{eq}$ such that $\omega^n \leq U(F) < \omega^{n+1}$, or there is an ∞-definable simple group S such that $\omega^n \leq U(S) < \omega^{n+1}$. In the first case, Lemma 3.3.6 implies that the generic of F is eventually strongly nonisolated, hence if s is regular, $U(s)=\omega^n$ and s is nonorthogonal to the generic of F we have that s is esn and thus $n \in I$. In the second case, it follows from our assumption on T that the generic of S is esn and, similarly to the first case, that $n \in I$. The proof of the Claim is complete.

Continuing the proof of the theorem let $\{p_n \mid n \in I\}$ be an infinite family of regular, eventually strongly nonisolated types such that $U(p_n)=\omega^n$ for all $n \in I$. Clearly for $m,n \in I$ and $m \neq n$, p_n is orthogonal to every conjugate of p_m. Thus the family $\{p_n \mid n \in I\}$ satisfies the conditions of Lemma 3.3.3 and $I(\aleph_0,T)=2^{\aleph_0}$.

<u>Example</u> Let $L=\{+,\cdot,0,1,V_i \mid 0 \leq i \leq n\}$ and let $\mathcal{F}=(F,+,\cdot,0,1,F_i)_{0 \leq i \leq n}$ be an L-structure such that $F \supseteq F_n \supseteq \ldots \supseteq F_1 \supseteq F_0$ is a sequence of algebraically closed fields of characteristic 0, such that each of them has infinite transcendence degree over the previous one. Let $T=Th(\mathcal{F})$. Then T is superstable and if p_i is the generic of F_i then $U(p_i)=\omega^i$. $U(p)=\omega^{n+1}$ where p is the generic of F. Every model of T is determined, up to isomorphism, by $(\dim(p),\dim(p_i))_{0 \leq i \leq n}$. Hence T has only $\aleph_0$ nonisomorphic countable models.

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