INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

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MECHANICS OF POLAR CONTINUA

THEORY AND APPLICATIONS

COURSE HELD AT THE DEPARTMENT
FOR MECHANICS OF DEFORMABLE BODIES
SEPTEMBER - OCTOBER 1969

UDINE 1969

COURSES AND LECTURES - No. 2

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Foreword.

When I was invited by Professor Sobrero to give some lectures at the International Centre for Mechanical Sciences, I found it a great temptation to use the opportunity for what I had already considered for some years to be an interesting but unrealizable task: to collect the results already obtained in this branch of contemporary mechanics and to try to interpret them from a general point of view.

The number of papers on different aspects of polar continua in mechanics increases rapidly from year to year and it seems very difficult at present to treat all different models and theories on the basis of one general theory. That is not my desire, but it seems to me worth an effort to make a systematic comparative presentation of at least some of the most important treatments of the subject.

I believe that one day it will not be difficult to make a general theory so that all different theories that exist today will be only particular cases. I shall be glad if this course of 25 lectures on polar continua will help the future efforts in this direction.

I wish to express my feelings of appreciation for the possibility given to me by the International Centre for Mechanical Sciences to deliver this course of lectures here, and my thanks are particularly due to Professor Luigi Sobrero, the Secretary General of the Centre and the Director of the Institute for Mechanics of the University in Trieste, for all he did to make this Centre a reality and to make the work in it a pleasure.

1. Introduction.

There are two main approaches to the concept of generalized continua. Classical continuum mechanics considers material continua as point-continua with points having three degrees of freedom, and the response of a material to the displacements of its points is characterized by a symmetric stress tensor. Such a model is insufficient for the description of certain physical phenomena.

Already in 1843 St.Venant [381]* remarked that for the description of deformations of thin bodies a proper theory cannot be restricted to the analysis of deformations of a straight line which can be only lengthened and bent, but must also include directions which can be rotated independently of the displacements of the points.

A further generalization of this idea was to attach to each point of a three-dimensional continuum a number of directions which can be rotated independently of the displacements of the points to which they are attached. That physical bodies might be presented this way was suggested in 1893 by Duhem [68]. In the study of crystal elasticity Voigt [383,384] came to the same ideas. It is the merit of the brothers

* The numbers in square brackets refere to the list of references at the end of these lecture notes.

Eugène and François Cosserat that a theory of such oriented continua was developed, and there are three papers by them [50,51,52] published in 1907-1909, which are the basis of all later work on polar continua. However, their work remained forgotten until 1935, when Sudria [359] gave a more modern interpretation of their theory, applying the contemporary vectorial notation.

One of the essential features of polar continua is that the stress tensor is not symmetric, and the well known second law of Cauchy is to be replaced by another one from which the Cosserat equations follow.

In oriented bodies the antisymmetric part of the stress tensor, according to the Cosserat equations, is related to the divergence of a third-order tensor of couple-stresses. This tensor, through the constitutive relations, depends on the deformations of the directors, but the deformations of directors are not the only deformations responsible for the couple-stresses.

The non-symmetry of the stress tensor appears also if the higher order deformation gradients are taken into account, instead of the first-order gradients only, as it is the case in the classical continuum mechanics. According to Truesdell and Toupin [378], Hellinger [154] was the first in 1914, to obtain the general constitutive relations for stress and cou ple-stress, generalizing an analysis of E. and F. Cosserat.

In 1953 Bodaszewski [33] developed a theory of non-symmetric stress states, but without any reference to earlier works. He applied the theory to elasticity and fluid dynamics.

Since 1958, the general interest in the non-symmetric stress tensor and in the Cosserat continuum rapidly increased. In that year Ericksen and Truesdell published a paper on the exact theory of rods and shells in which they considered a generalized Cosserat continuum, i.e. a medium with deformable directors, but without any constitutive assumptions. Günther [144] gave a linear theory (statics and kinematics) of the Cosserat continuum, with a very interesting application to the continuum theory of dislocations, and Grioli [134] developed a theory of elasticity with the non-symmetric stress tensor. Ericksen's theory of liquid crystals and anisotropic fluids is also an application of the theory of oriented bodies [74] .

There are different physical and mathematical models of continua which serve as generalizations of the classical concept of a point continuum. All such models in which the stress tensor is not symmetric are regarded here as POLAR CONTINUA.

2. Physical Background.

It was already mentioned that the classical model of a material continuum is unsufficient for the description of a number of phenomena. In the case of thin bodies this can already be seen.

If we regard a very thin circular cylinder, a one-dimensional representation is the sufficient approximation for the study of its elongation, but twists are excluded from such considerations. In order to include the twist we may associate a unit vector with each point of the line, and rotations of this vector give us the needed information on the twist. Obviously, this rotation is independent of the displacements of points of the line.

For the study of a flexible string a rigid triad of unit vectors may be attached to each point of the string.

In the theory of rods, plates and shells the situation is similar. In the direct approach to the theory of rods, Green and Laws [113,115] define a rod as a curve at each point of which there are two assigned directors. The theory of plates and shells may be based on the model, consisting of a deformable surface with a single director attached to each of its points. Such a surface is called by Green and Naghdi [123] a Cosserat surface.

A crystal lattice in the continuum appro ximation is a point continuum, but the rotations of particles cannot be represented in such an approximation. In order to include the interactions of rotating particles in crystal elasticity, Voigt [383;384] was the first to generalize the classical concepts of continuum mechanics.

Ericksen [77] developed the theory of liquid crystal and anisotropic fluids assuming that a fluid is an ordinary three-dimensional point continuum with one director at each point. Particles of the fluid are assumed to be of the dumb-bell shape.

Continuum mechanics is a method for the study of mechanical properties of bodies the dimensions of which are very great in comparison with the interatomic distances. The discrete structure of matter, in fact, is to be studied if we wish to make an exact theory of the behaviour of matter. For bodies containing a large number of particles it is practically impossible. The classical point continuum is just an approximation, and some models of continua are constructed in such a way to represent a better approximation and to include some effects which cannot be interpreted from the point of view of a point continuum.

In a series of papers Stojanović, Djurić and Vujošević [343] in 1964, Green and Rivlin (for references see Rivlin [299]) have taken as the starting point the discrete structure of particles which constitute the medium. Each particle consists of a number of mass-points. The continuum representation consists of a point-continuum, the points correspond to the centres of gravity of particles and in a number of deformable vectors, the directors. The distribution of masses in such a representation is specified through some inertia coefficients. The forces acting on mass-points in the continuum representation reduce to the simple forces acting on the points of the continuum and on the director forces acting on the directors, as well as to the simple and director surface forces (stresses) and couples, measured per unit area of the deformed surface.

Kroener, Krumhansl, Kunin and other authors approach this problem of approximation from the point of view of solid state physics [186]. We shall mention here only the very impressive picture of the couple-stress given by Kroener in a dislocated crystal [188]. From the distribution of microscopical stresses, applying an averaging process, Kroener computed the macroscopic moments. The obtained couple-stress he attributed to the non-local forces, i.e. to the long-range cohesive forces.

Mindlin [219] and Eringen and Suhubi [99] introduced microstructure into the theory of elasticity and into continuum mechanics, in general. The unit cell of material with microstructure might be interpreted as a molecule of a polymer, as a crystalite of a polycrystal, or as a grain of an incoherent material. The concept of microstructure Eringen introduced also into the fluid mechanics [91].

Eringen generalized further the model and defined micromorphic materials [92]. A volume element of such a material consists of microelements which suffer micromotions and microdeformations. Micropolar materials are a subclass, in which the microelements behave as rigid bodies.

The theory of multipolar media by Green and Rivlin [128,129] represents a very fine abstract and general mathematical treatment of generalized continua, from which many theories follow as special cases.

Besides the physical models mentioned which served as a basis for different continuum-mechanical representations, there is a number of other theories and treatments inspired by the problems of solid-state physics (Teodosiu [361]), or by the structure of technical materials (Misicu [240]) or by the mathematical possibilities for generalizations of classical concepts (Grioli [134], Aero and Kuvshinskii [4]).

Granular media represent also the field in which the methods of generalized continuum mechanics are applied (Oshima [270]).

It is impossible to mention all contributors to the contemporary development of continuum mechanics and we restricted this list only to some of them whose work most inspired further research.

3. Motion and Deformation.

We shall regard material points as the fundamental entities of material bodies.

A body $\mathcal{B}$ is a three-dimensional differentiable manifold, the elements of which are called material points.*

The material points $M_1, M_2, ..$ may be regarded as a set of abstract objects M mentioned in the Appendix, section A1, so that the 1:1 correspondence of the points M_K and of the points of a three-dimensional arithmetic space establishes a general material three-dimensional space. Since bodies are available to us in Euclidean space, we shall relate the points M_K

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* This definition of a body corresponds to the definition given by Truesdell and Noll [379]. Noll [261] developed a very general approach to continuum mechanics, but we are not going to follow it since it does not include plasticity and mostly is concerned with the non-polar materials, regarding elasticity, viscoelasticity and viscosity from a unique point of view. For the general approach to this theory, because of its highest mathematical rigour and for a very complete bibliography we refere the readers to the book by Truesdell and Noll [379].

to the points of Euclidean space, establishing a 1:1 correspondence between the points M_K of a body $\mathcal{B}$ and points $\underset{\sim}{x}$ of a region $\mathcal{R}$ of this space. The numbers x^i, $i = 1, 2, 3$ represent coordinates of the material point M, and the points $\underset{\sim}{x}$ are places in the space occupied by the points M.

Any triple of real numbers $x^i, i=1,2,3$ may be regarded as an arithmetic point, which belongs to the arithmetic space A_3. A 1:1 smooth correspondence between the material points M of a body $\mathcal{B}$ and arithmetic points $\underset{\sim}{X}$, such that $X^K = X^K(M)$, $K = 1, 2, 3$ represents a system of coordinates in which individual material points are characterized by their <u>material coordinates</u> X^K, $K = 1, 2, 3$.

A 1:1 correspondence between points $\underset{\sim}{x}$ of a region $\mathcal{R}$ of Euclidean space, and points M of a body $\mathcal{B}$ is the configuration of the body.

$$x^i = x^i(M) = x^i\left(X^1, X^2, X^3\right) \tag{3.1}$$

The points x^i represent places in the space occupied by the material points M and we shall refere to the coordinates x^i as to the <u>spatial coordinates</u>. The functions $x^i = x^i(\underset{\sim}{X})$ are assumed to be continuously differentiable.

In general no assumptions are made on the geometric structure of the material manifold and it is not to be confused with one of its configurations.

It is advantageous to choose one configuration as the reference configuration and to identify material coordinates with the spatial coordinates in the reference configuration.

Thus, the material points of a body $\mathcal{B}$ in the reference configuration are referred to a system of coordinates X^K, which is an admissible system of coordinates in Euclidean space, and in the following we shall refer to X^K as to the material coordinates.

Motion of a body is a one-parameter 1:1 mapping

$$x^k = x^k(X^1, X^2, X^3, t) \equiv x^k(K, t), \qquad (3.2)$$

or shortly:

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t),$$

of the points M in the reference configuration K_0 on the points $\underset{\sim}{x}$ occupied by the material points at a moment of time t, which determines a configuration $K_t = K(t)$. The parameter t is a real parameter and it represents time. We assume that the functions $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X})$ are continuously differentiable.

We assume that

$$\det \frac{\partial x^k}{\partial X^K} \equiv \det x^k{}_{;K} \neq 0, \qquad (3.3)$$

so that there exists the inverse mapping

$$X^K = X^K(x^1, x^2, x^3, t),$$

short: (3.4)

$$\underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{x}, t)$$

The partial derivatives

$$F^k{}_{\cdot K} \equiv \partial x^k / \partial X^K \equiv x^k{}_{;K},$$

(3.5)

$$F_k{}^{\cdot K} \equiv \partial X^K / \partial x^k \equiv X^K{}_{;k},$$

are called <u>deformation gradients</u>, and the total covariant derivatives (see Appendix, section A3)

$$x^k{}_{;KL}, \ldots, x^k{}_{;K_1 \ldots K_N},$$

$$X^K{}_{;kl}, \ldots, X^K{}_{;k_1 \ldots k_N},$$

represent deformation gradients of order $2, 3, \ldots N$.

Let K_0 and K be two configurations of a body $\mathcal{B}$, K_0 referred to material coordinates X^K, and K referred to spatial coordinates x^k. The systems of reference X^K and x^k are chosen independently of one another. The <u>deformation</u> is a mapping of one configuration on the other,

$$x^l = x^l(\underset{\sim}{X}),$$

(3.6)

$$X^L = X^L(\underset{\sim}{x}).$$

If dS^2 and ds^2 are squares of the line elements in the configurations K_0 and K respectively,

$$dS^2 = G_{LM}\, dX^L\, dX^M ,$$
$$ds^2 = g_{\ell m}\, dx^\ell\, dx^m , \qquad (3.7)$$

using the mappings (3.6) we may represent the line element of the reference configuration in terms of the coordinates of the deformed configuration and conversely. From (3.6) we have

$$dX^L = X^L_{;\ell}\, dx^\ell \qquad dx^\ell = x^\ell_{;L}\, dX^L \qquad (3.8)$$

and

$$dS^2 = c_{\ell m}\, dx^\ell\, dx^m ,$$
$$ds^2 = C_{LM}\, dX^L\, dX^M . \qquad (3.9)$$

Here

$$c_{\ell m} \equiv G_{LM}\, X^L_{;\ell}\, X^M_{;m} \qquad (3.10)$$

is the <u>spatial deformation tensor</u>, and

$$C_{LM} \equiv g_{\ell m}\, x^\ell_{;L}\, x^m_{;M} \qquad (3.11)$$

is the <u>material deformation tensor</u>.

It is always possible to decompose a

non-singular matrix $\underset{\sim}{M}$ into one symmetric and one positive definit matrix,

$$M^{k}_{\cdot\ell} = R^{k}_{\cdot t}\, S^{t}_{\cdot\ell} = S^{*k}_{\cdot t}\, R^{t}_{\cdot\ell}, \tag{3.12}$$

where $\underset{\sim}{R}$, $\underset{\sim}{S}$ and $\underset{\sim}{S}^*$ are uniquely determined (cf. Ericksen [73], § 43). Applying this <u>polar decomposition theorem</u> to the matrix $\underset{\sim}{F}$ (cf. [379]) of deformation gradients, we obtain

$$\underset{\sim}{F} = \underset{\sim}{R} \cdot \underset{\sim}{U} = \underset{\sim}{V} \cdot \underset{\sim}{R} \tag{3.13}$$

where $\underset{\sim}{R}$ is orthogonal, and $\underset{\sim}{U}$ and $\underset{\sim}{V}$, determined by

$$\underset{\sim}{U}^2 = \underset{\sim}{F}^T \cdot \underset{\sim}{F} \qquad \underset{\sim}{V}^2 = \underset{\sim}{F} \cdot \underset{\sim}{F}^T \tag{3.14}$$

are the <u>right</u> and the <u>left stretch tensors,</u> respectively. The deformation tensor $\underset{\sim}{C}$ and $\underset{\sim}{B}$

$$\begin{aligned} \underset{\sim}{C} &= \underset{\sim}{U}^2 = \underset{\sim}{F}^T \underset{\sim}{F} \\ \underset{\sim}{B} &= \underset{\sim}{V}^2 = \underset{\sim}{F} \underset{\sim}{F}^T \end{aligned} \tag{3.15}$$

are accordingly called the <u>right</u> and the <u>left Caushy--Green tensors</u>.

Since $\underset{\sim}{F} = \left\{ x^{k}_{\cdot K} \right\}$, the transposed matrix $\underset{\sim}{F}^T$ is determined by

$$\underset{\sim}{F}^T = \left\{ g_{k\ell}\; x^{\ell}_{;L}\; G^{LM} \right\}$$

and for the components of the tensors $\underset{\sim}{C}$ and $\underset{\sim}{B}$ we have

$$C^K_L = g_{k\ell}\, x^k_{;L}\, x^\ell_{;M}\, G^{KM} = G^{KM} C_{LM} \tag{3.16}$$

$$B^k_\ell = G^{KL}\, x^k_{;K}\, x^m_{;L}\, g_{m\ell} = \overset{-1}{c}{}^k_\ell = g_{m\ell}\, \overset{-1}{c}{}^{km}. \tag{3.17}$$

The tensor $\overset{-1}{\underset{\sim}{c}}$, with the components

$$\overset{-1}{c}{}^{km} = G^{KM}\, x^k_{;K}\, x^m_{;M} \tag{3.18}$$

is the reciprocal of the spatial deformation tensor $\underset{\sim}{c}$,

$$c_{jk}\, \overset{-1}{c}{}^{km} = \delta^m_j$$

If a body suffers only a rigid motion, the distances between its points are preserved, there are no deformations and

$$C_{KL} = G_{KL} \qquad c_{k\ell} = g_{k\ell}\,. \tag{3.19}$$

The material and the spatial <u>strain tensors</u> are defined by the following formulae

$$E_{KL} = \frac{1}{2}\left(C_{KL} - G_{KL}\right),\ e_{k\ell} = \frac{1}{2}\left(g_{k\ell} - c_{k\ell}\right), \tag{3.20}$$

where we denote, as usualy, material tensors and material components by capital letters and capital indices, and spatial tensors and spatial components by small letters and small indices.

Velocity of a material point $\underset{\sim}{X}$ is the vector $\underset{\sim}{v}$ with the components

$$v^i = \dot{x}^i = \frac{\partial x^i(\underset{\sim}{X},t)}{\partial t} \Big/ \underset{\sim}{x} = \text{const.} \tag{3.21}$$

In general, if $\underset{\sim}{T} = \underset{\sim}{T}(\underset{\sim}{x},\underset{\sim}{X},t)$ is a time dependent double tensor field (See Appendix, section A1 and A3), the time derivatives with the material coordinates X^K kept fixed are called <u>material derivatives</u> and are denoted by a superposed dot. Sometimes it is useful to place the dot above a superposed bar, which denotes upon which quantity the operation of the material derivation is to be performed. For the tensor field $\underset{\sim}{T}$ we have

$$\dot{T}^{K\cdots}_{\cdots k\cdots} \equiv \frac{\partial T^{K\cdots}_{\cdots k\cdots}}{\partial t} + \left(\frac{\partial T^{\cdots}_{\cdots}}{\partial x^l} - \left\{ {t \atop ml} \right\} T^{K\cdots}_{\cdots t\cdots} - \cdots\right)\dot{x}^l \tag{3.22}$$

$$= \frac{\partial T^{K\cdots}_{\cdots k\cdots}}{\partial t} + T^{K\cdots}_{\cdots k\cdots,l}\,\dot{x}^l$$

Accelaration $\underset{\sim}{a}$ is a vector with the components defined by .

$$a^i = \dot{v}^i = \ddot{x}^i = \frac{dv^i}{dt} + \left\{ {i \atop j\,k} \right\} v^j v^k = \frac{\partial v^i}{\partial t} + v^i_{;j}\,\dot{x}^j \quad (3.23)$$

The rate of change of the arc element may be calculated directly from $(3.7)_2$,

$$\dot{\overline{ds^2}} = g_{ij}\left(\dot{\overline{dx^i}}\, dx^j + dx^i\, \dot{\overline{dx^j}} \right). \qquad (3.24)$$

Since

$$dx^i = x^i_{;L}\, dX^L,$$

and the material coordinates are kept fixed, we have

$$\dot{\overline{dx^i}} = \dot{\overline{x^i_{;L}}}\, dX^L = \dot{x}^i_{;L}\, dX^L = v^i_{;L}\, dX^L = v^i_{,k}\, dx^k \qquad (3.25)$$

and

$$\dot{\overline{ds^2}} = 2\, v_{j,k}\, dx^j\, dx^k = 2\, d_{jk}\, dx^j\, dx^k \qquad (3.26)$$

where

$$d_{jk} \equiv \frac{1}{2}\left(v_{j,k} + v_{k,j} \right) = v_{(j,k)} \qquad (3.27)$$

is the <u>rate of strain</u> tensor.

The gradients of velocity $v_{i,j}$ may be decomposed into the symmetric and the antisymmetric part. The antisymmetric part

$$w_{ij} \equiv v_{[i,j]} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) \qquad (3.28)$$

represents the vorticity tensor.

The tensors of the rate of strain and of the vorticity are mutually independent, but the gradients of these two tensors are related by a simple relation:

$$w_{ij,k} = \frac{1}{2}\left(v_{i,jk} - v_{j,ik}\right) \equiv \frac{1}{2}\left(v_{k,ij} + v_{i,kj} - v_{j,ki} - v_{k,ji}\right) \qquad (3.29)$$

$$= d_{ki,j} - d_{jk,i} = 2\, d_{k[i,j]} \,.$$

A motion is a rigid body motion if $ds = dS$, and the conditions for a motion to be a rigid body motion are given by (3.19). In terms of the strain tensors these conditions reduce to $\tilde{E} = 0$ and $\tilde{e} = 0$. For a rigid body motion the rate of strain vanishes and the velocity field has to satisfy the obvious equations

$$v_{(i,j)} = 0 \qquad (3.30)$$

The conditions (3.30) are necessary and sufficient for a motion to be a rigid motion. If $dx^i = u^i$ is an elementary displacement of a body, from (3.30) it follows that the necessary and sufficient conditions for displacements to determine a rigid motion are

$$u_{(i,j)} = 0 \tag{3.31}$$

These equations are called Killing equations. In Euclidean space the equations (3.30) and (3.31) are integrable and the integrals represent components of the velocity field and of the displacement field for rigid motions.

4. Compatibility Conditions.

For a given tensor field $\underset{\sim}{c}(\underset{\sim}{x})$, or $\underset{\sim}{C}(\underset{\sim}{X})$, the deformations

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}) \quad \text{or} \quad \underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{x}) \tag{4.1}$$

do not necessarily exist. The existence of the deformations depends on the integrability conditions of the equations (3.10) or (3.11), and these conditions are usually called in continuum mechanics the <u>compatibility conditions</u>.

There are six independent equations (3.10), with nine independent deformation gradients $X^K_{;k}$. In order to find the deformations we have first to find the deformation gradients, but since the number of the unknowns, regarding the equations (3.10) as a system of algebraic equations, exceeds the number of equations, we shall first differentiate partially the equations (3.10) with respect to the spatial coordinates x^ℓ, assuming that the deformations (4.1) exist. Thus we obtain a system of 18 equations with 18 unknowns $\partial_m \partial_n X^K$,

$$\partial_\ell C_{mn} = \partial_L G_{MN} X^L_{;\ell} X^M_{;m} X^N_{;n} + G_{MN}(\partial_\ell \partial_m X^M X^N_{;n} + X^M_{;m} \partial_\ell \partial_n X^N) \tag{4.2}$$

Permutating the indices ℓ, m, n we may construct the

Christoffel symbols of the first kind for the tensor $\underset{\sim}{C}$

$$[\ell m, n]_{\underset{\sim}{C}} \equiv \frac{1}{2}\left(\partial_\ell c_{mn} + \partial_m c_{n\ell} - \partial_n c_{\ell m}\right)$$

$$= [LM, N]_{\underset{\sim}{G}}\, X^L_{;\ell}\, X^M_{;m}\, X^N_{;n} + G_{MN} X^N_{;n}\, \partial_\ell \partial_m X^M, \tag{4.3}$$

where $[LM,N]_{\underset{\sim}{G}}$ are the Christoffel symbols for the fundamental tensor $\underset{\sim}{G}$. Since there are 18 equations (4.3); we easily find the derivatives $\partial_\ell \partial_m X^M$:

$$\partial_\ell \partial_m X^N = G^{NK} x^n_{;K}\, [\ell m, n]_{\underset{\sim}{C}} - \left\{ {N \atop LM} \right\}_{\underset{\sim}{G}} X^L_{;\ell}\, X^M_{;m} \tag{4.4}$$

According to (3.18) we have $G^{NK} x^n_{;K} = \overset{-1}{c}{}^{nK} X^N_{;K}$, and since

$$\overset{-1}{c}{}^{nK}\, [\ell m, n]_{\underset{\sim}{C}} = \left\{ {K \atop \ell m} \right\}_{\underset{\sim}{C}}, \tag{4.5}$$

(4.4) reduces to

$$\partial_\ell \partial_m X^N = \left\{ {n \atop \ell m} \right\}_{\underset{\sim}{C}} X^N_{;n} - \left\{ {N \atop LM} \right\}_{\underset{\sim}{G}} X^L_{;\ell}\, X^M_{;m} \equiv F^N_{\ell m} \tag{4.6}$$

The integrability conditions of (4.6) are $\partial_{[K} F_{\ell]m} = 0$.

Differentiation of (4.6) with respect to x^K and the elimination of the second-order derivatives of X's by the aid of (4.6) gives for the integrability conditions the relations

$$R_{k\ell m}{}^{n}(\underset{\sim}{c})\, X^{N}_{;n} - R_{KLM}{}^{N}(\underset{\sim}{G})\, X^{K}_{;k}\, X^{L}_{;\ell}\, X^{M}_{;m} = 0 \ ,$$

where $\underset{\sim}{R}(\underset{\sim}{c})$ and $\underset{\sim}{R}(\underset{\sim}{G})$ are the Riemann-Christoffel tensors (see Appendix, (A4.10)) for the Riemannian connections $\left\{ {k \atop \ell m} \right\}_{\underset{\sim}{c}}$ and $\left\{ {K \atop LM} \right\}_{\underset{\sim}{G}}$. However $\underset{\sim}{G}$ is the metric tensor of Euclidean space and $R(G)$ vanishes identically. Therefore the integrability conditions reduce to

$$R_{k\ell m}{}^{n}(\underset{\sim}{c}) \equiv 2\left(\partial_k \left\{ {n \atop \ell m} \right\}_{\underset{\sim}{c}} + \left\{ {n \atop kt} \right\}_{\underset{\sim}{c}} \left\{ {t \atop \ell m} \right\}_{\underset{\sim}{c}}\right)_{[k\ell]} = 0 \qquad (4.7)$$

Transvecting $R_{k\ell m}{}^{t}$ with c_{nt} we obtain the covariant Riemann-Christoffel tensor

$$R_{k\ell mn} = 2\left(\partial_k \left[\ell m, n\right]_{\underset{\sim}{c}} - \overset{-1}{c}{}^{st} \left[\ell m, s\right]_{\underset{\sim}{c}} \left[k n, t\right]_{\underset{\sim}{c}}\right)_{[k\ell]} \qquad (4.8)$$

which satisfies the following three identities (cf. Schouten [322a]):

$$\begin{aligned} R_{k\ell mn} &= -R_{\ell kmn} \ , \\ R_{k\ell mn} &= -R_{k\ell nm} \ , \\ R_{k\ell mn} &= R_{mnk\ell} \ , \end{aligned} \qquad (4.9)$$

and this reduces the number of independent components of the tensor $R_{k\ell mn}$ to six.

The Einstein curvature tensor $\underset{\sim}{A}$ with the components

$$A^{ij} \equiv R^{ij} - \frac{1}{2} R g^{ij}$$

where

$$R^{ij} \equiv g^{ki} g^{mj} R_{lkm}^{\cdot\cdot\cdot l} \qquad , \ R \equiv g_{ij} R^{ij}$$

in three-dimensional spaces may be obtained from (4.8) by

$$A^{ij} = \frac{1}{4} \epsilon^{ikl} \epsilon^{jmn} R_{klmn} , \qquad (4.10)$$

and the compatibility conditions may be expressed in terms of the Einstein tensor, which is symmetric.

The compatibility conditions are usually written in terms of the strain tensor $\underset{\sim}{e}$, and may be derived from (4.8) and (4.10), substituting $\underset{\sim}{c}$ from $(3.20)_2$,

$$\underset{\sim}{c} = \underset{\sim}{g} - 2\underset{\sim}{e}$$

and neglecting the products of the Christoffel symbols in (4.8), as small quantities of the second order. Thus,

$$\epsilon^{ikl} \epsilon^{jmn} e_{km,ln} = 0 \qquad (4.11)$$

where "," denotes covariant differentiation with respect to the fundamental tensor $\underset{\sim}{g}$.

If the compatibility conditions (4.8) for a given strain are not satisfied, we may write

$$A^{ij} = \eta^{ij}(\underset{\sim}{e}) \qquad (4.12)$$

and $\underset{\sim}{\eta}$ is the incompatibility tensor. In the linearized case we have

$$\eta^{ij} = \epsilon^{ik\ell} \epsilon^{jmn} e_{km,\ell n} . \tag{4.13}$$

When $\eta \neq 0$, a deformation of the form (4.1) does not exist and the strain tensor may be interpreted as a tensor which represents a deformation from a non- Euclidean configuration N of the body considered into one of its Euclidean configurations. This interpretation of incompatible strains is applied in the theory of dislocations and in thermoelasticity.

5. Oriented Bodies.

A body to each point of which is assigned a set of vectors $\underset{\sim}{d}_{(\alpha)}$, $\alpha = 1,2,\ldots,n$, represents an oriented body. The vectors $d_{(\alpha)}$ are directors of the body. In general, deformations of the directors are independent of the deformations of position.

Let the directors in an undeformed reference configuration K_0 be the vectors

$$\underset{\sim}{D}_{(\alpha)} = \underset{\sim}{D}_{(\alpha)}(\underset{\sim}{X}) \,, \qquad (5.1),$$

with the components $D^K_{(\alpha)}$ referred to a material system of reference X^K. A deformation of an oriented body is determined by the equations

$$\begin{aligned} \underset{\sim}{x} &= \underset{\sim}{x}(\underset{\sim}{X}) \\ \underset{\sim}{d}_{(\alpha)} &= \underset{\sim}{d}_{(\alpha)}(\underset{\sim}{D}_{(\alpha)}) = \underset{\sim}{d}_{(\alpha)}(\underset{\sim}{X}) \end{aligned} \qquad (5.2)$$

Directors are not material vectors. For material vectors $\underset{\sim}{D}(\alpha)$ the deformation is determined by the deformation of position,

$$D^k_{(\alpha)} = x^k_{;K} \, D^K_{(\alpha)} \,. \qquad (5.3)$$

In an oriented body the vectors

$$\Delta^k_{(\alpha)} = d^k_{(\alpha)} - x^k_{;K} \, D^K_{(\alpha)} \qquad . \qquad (5.4)$$

represent the differences between the deformed directors and the vectors obtained from the directors in the reference configuration by the deformation of position.

The Cosserat continuum in the strict sense is a material medium to each point of which there are assigned three directors, which represent rigid triads of unit vectors. The directors in this continuum suffer only rigid rotations, and length and angles between the directors are preserved throughout the motion, so that

$$g_{k\ell}\, d^{k}_{(\alpha)}\, d^{\ell}_{(\beta)} = G_{KL}\, D^{K}_{(\alpha)}\, D^{L}_{(\beta)} = D_{\alpha\beta} = \text{const.} \qquad (5.5)$$

A medium with deformable directors represents a <u>generalized Cosserat continuum</u>.

5. 1. Discrete Systems and Continuum Models.

The basic notion in the solid state physics is crystal lattice. A unit cell of a crystal is composed of four lattice points M_0, M_1, M_2, M_3. Let M_0 be a lattice point. Any three vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ are lattice vectors if they are position vectors of the lattice points M_1, M_2, M_3 with respect to M_0 of the unit cell. The vectors

$$\underset{\sim}{r} = l\underset{\sim}{a}_1 + m\underset{\sim}{a}_2 + n\underset{\sim}{a}_3 \quad (l, m, n \text{ - integral numbers}) \qquad (5.1.1)$$

determine the lattice points of a perfect crystal.

Motions of a crystal are determined if determined are the motions of its lattice points. However, instead of the motions of the lattice points it is possible to regard the motions of one lattice point for each cell, and the motions of the lattice vectors $\underset{\sim}{a}_\lambda$ for each individual cell. This may be considered as a four-point model which under suitable assumptions may be used for a continuum approximation of an oriented body, as was done by Stojanović, Djurić and Vujošević [343] . A more general approach to generalized Cosserat continua with an arbitrary number of directors is proposed by Rivlin [298,299] and in the following we shall consider Rivlin's n-point model.

We assume that a body consists of particles $P_1, \ldots, P_N$ and that each particle consists of n material points $M_1, \ldots, M_n$ with masses $m_1, \ldots m_n$, and with position vectors $\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_n$ with respect to a fixed origin $\underset{\sim}{0}$ in the space.

If C_P is the centre of masses of the particle P, and $\underset{\sim}{\varrho}_\nu$, $\nu = 1, \ldots, n$ position vectors of the points M_ν, from particle dynamics we obtain for the momentum, moment of momentum and kinetic energy of a particle P the following expressions:*

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* Rivlin [298,299] investigated the transition from

$$\underset{\sim}{K} = \sum_{\nu=1}^{n} m_\nu \dot{\underset{\sim}{r}}_\nu = \sum_{\nu=1}^{n} m_\nu \underset{\sim}{v}_\nu \,, \tag{5.1.2}$$

$$\underset{\sim}{\ell}^{o} = \sum_{\nu=1}^{n} m_\nu \underset{\sim}{r}_\nu \times \underset{\sim}{v}_\nu = m \underset{\sim}{r}_c \times \underset{\sim}{v}_c + \sum_{\nu=1}^{n} m_\nu \underset{\sim}{\varrho}_\nu \times \dot{\underset{\sim}{\varrho}}_\nu \tag{5.1.3}$$

$$T = \frac{1}{2}\left(m v_c^2 + \sum_{\nu=1}^{n} m_\nu \dot{\underset{\sim}{\varrho}}_\nu \cdot \dot{\underset{\sim}{\varrho}}_\nu \right) \tag{5.1.4}$$

Here we have

$$\underset{\sim}{v}_\nu = \dot{\underset{\sim}{r}}_\nu = \frac{\partial \underset{\sim}{r}_\nu}{\partial t} \,, \tag{5.1.5}$$

$$\underset{\sim}{\varrho}_\nu = \underset{\sim}{r}_\nu - \underset{\sim}{r}_c \,, \tag{5.1.6}$$

$$\sum_{\nu=1}^{n} m_\nu \underset{\sim}{\varrho}_\nu = 0 \,, \tag{5.1.7}$$

$$m = \sum_{\nu=1}^{n} m_\nu \,, \tag{5.1.8}$$

$$m \underset{\sim}{r}_c = \sum_{\nu=1}^{n} m_\nu \underset{\sim}{r}_\nu \,. \tag{5.1.9}$$

Introducing the coefficients (which are not tensors)

$$i^{\lambda\mu} = \frac{1}{m} \sum_{\nu=1}^{n} m_\nu \, \delta^{\lambda}_{\nu} \, \delta^{\mu}_{\nu} \,, \tag{5.1.10}$$

---

a discrete system to continuum, including some implications of the first and second laws of thermodynamics, without writing the expressions for momentum and moment of momentum.

the relations (5.1.3,4) may be rewritten in the form

$$\underset{\sim}{\ell}^{o} = m\left(\underset{\sim}{r}_c \times \underset{\sim}{v}_c + i^{\lambda\mu}\, \underset{\sim}{\varrho}_\lambda \times \dot{\underset{\sim}{\varrho}}_\mu\right), \qquad (5.1.11)$$

$$T = \frac{m}{2}\left(v_c^2 + i^{\lambda\mu}\, \dot{\underset{\sim}{\varrho}}_\lambda \cdot \dot{\underset{\sim}{\varrho}}_\mu\right). \qquad (5.1.12)$$

From the last two expressions we see that for the dynamical specification of the particle P we need to know the quantities: m - the mass of the particle, $i^{\lambda\mu}$ - the dimensionless coefficients which characterize the distribution of masses inside the particle, and the vectors $\underset{\sim}{\varrho}_\lambda$ which determine the configuration of the particle.

To denote that all quantities which appear in (5.1.2 - 12) correspond to the particle P we shall label them with the index P so that we write

$$\underset{\sim}{K}_P\,,\ \underset{\sim}{\ell}^{o}_P\,,\ m_P\,,\ m^P_\nu\,,\ \underset{\sim}{r}^P_\nu\,,\ \underset{\sim}{r}^P_c\,,\ \underset{\sim}{\varrho}^P_\nu\,,\ \dot{\underset{\sim}{\varrho}}^P_\nu\,,\ T_P\,,$$

and

$$m_P = \sum_{\nu=1}^{n} m^P_\nu\,, \qquad m_P\, \underset{\sim}{r}^P_c = \sum_{\nu=1}^{n} m^P_\nu\, \underset{\sim}{r}^P_\nu\,,$$

$$\underset{\sim}{K}_P = \sum_{\nu=1}^{n} m^P_\nu\, \dot{\underset{\sim}{r}}^P_\nu\,, \qquad \underset{\sim}{\ell}^{o}_P = m_P\left(\underset{\sim}{r}^P_c \times \dot{\underset{\sim}{r}}^P_c + i^{\lambda\mu}_P\, \underset{\sim}{\varrho}^P_\lambda \times \dot{\underset{\sim}{\varrho}}^P_\mu\right) \qquad (5.1.13)$$

For a body consisting of N particles we have now for the momentum

$$K = \sum_{P=1}^{N} m_P\, \dot{\underset{\sim}{r}}^P_c\,, \qquad (5.1.14)$$

for the moment of momentum

$$\underset{\sim}{\ell}^{o} = \sum_{P=1}^{N} \underset{\sim}{\ell}^{o}_{P} = \sum_{P=1}^{N} m_P \, \underset{\sim}{r}^{P}_{c} \cdot \dot{\underset{\sim}{r}}^{P}_{c} + \sum_{P=1}^{N} m_P \, i^{\lambda\mu}_{P} \underset{\sim}{\varrho}^{P}_{\lambda} \times \dot{\underset{\sim}{\varrho}}^{P}_{\mu} \, , \quad (5.1.15)$$

and for the kinetic energy

$$T = \sum_{P=1}^{N} T_P = \frac{1}{2} \sum_{P=1}^{N} m_P \left(\dot{\underset{\sim}{r}}^{P}_{c} \cdot \dot{\underset{\sim}{r}}^{P}_{c} + i^{\lambda\mu}_{P} \dot{\underset{\sim}{\varrho}}^{P}_{\lambda} \cdot \dot{\underset{\sim}{\varrho}}^{P}_{\mu} \right) \, . \quad (5.1.16)$$

To pass from this disrete system of particles to a continuum we have to replace the sums by integrals. In order to do so we assume that our system of particles occupies a domain $\mathcal{B} + \partial\mathcal{B}$, where $\partial\mathcal{B}$ is the boundary of the body $\mathcal{B}$. We assume further that the discrete vectors $\underset{\sim}{r}^{P}_{c}$, $\dot{\underset{\sim}{r}}^{P}_{c}$, $\underset{\sim}{\varrho}^{P}_{\nu}$ and $\dot{\underset{\sim}{\varrho}}^{P}_{\nu}$ may be replaced by continuous vector fields $\underset{\sim}{r}$, $\dot{\underset{\sim}{r}}$ and $\underset{\sim}{d}_{(\nu)}$ and $\dot{\underset{\sim}{d}}_{(\nu)}$, and the discrete scalars m_P and $i^{\lambda\mu}_{P}$ by continuous scalar fields ϱ and $i^{\lambda\mu}$. It must be noted that the passage from a system of particles to a continuous model can be effected only if all the quantities involved, which are connected with the particles, vary but little as we pass from one particle to its neighbours.

We assume that a region V of $\mathcal{B}$, with a boundary S is sufficiently large to contain many particles. Hence we may write

$$\sum_{V} m_P = \int_V \varrho \, dV \, , \quad (5.1.17)$$

$$\sum_{V} m^{P}_{\nu} = \int_V \varrho_\nu \, dV \, , \quad (5.1.18)$$

$$\sum_V m_p \underset{\sim}{r}_c^p = \int_V \varrho \underset{\sim}{r} \, dV \, , \tag{5.1.19}$$

$$\sum_V m_p \underset{\sim}{r}_c^p \times \dot{\underset{\sim}{r}}_c^p = \int_V \varrho \underset{\sim}{r} \times \dot{\underset{\sim}{r}} \, dV \, , \tag{5.1.20}$$

$$\sum_V m_p \dot{\underset{\sim}{r}}_c^p \cdot \dot{\underset{\sim}{r}}_c^p = \int_V \varrho \dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} \, dV \, , \tag{5.1.21}$$

$$\sum_V m_p \, i_p^{\lambda\mu} \underset{\sim}{\varrho}_\lambda^p \times \dot{\underset{\sim}{\varrho}}_\mu^p = \int_V \varrho \, i^{\lambda\mu} \underset{\sim}{d}_{(\lambda)} \times \dot{\underset{\sim}{d}}_{(\mu)} \, dV \, , \tag{5.1.22}$$

$$\sum_V m_p \, i_p^{\lambda\mu} \dot{\underset{\sim}{\varrho}}_\lambda^p \cdot \dot{\underset{\sim}{\varrho}}_\mu^p = \int_V \varrho \, i^{\lambda\mu} \dot{\underset{\sim}{d}}_{(\lambda)} \cdot \dot{\underset{\sim}{d}}_{(\mu)} \, dV \, , \tag{5.1.23}$$

Thus the expressions for momentum, moment of momentum and for the kinetic energy for a part V of the body $\mathcal{B}$ obtain the form

$$\underset{\sim}{K} = \int_V \varrho \, \dot{\underset{\sim}{r}} \, dV \, , \tag{5.1.24}$$

$$\underset{\sim}{\ell}^o = \int_V \varrho \left(\underset{\sim}{r} \times \dot{\underset{\sim}{r}} + i^{\lambda\mu} \underset{\sim}{d}_{(\lambda)} \times \dot{\underset{\sim}{d}}_{(\mu)} \right) dV \, , \tag{5.1.25}$$

$$T = \frac{1}{2} \int_V \varrho \left(\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} + i^{\lambda\mu} \dot{\underset{\sim}{d}}_{(\lambda)} \; \dot{d}_{(\mu)} \right) dV \, . \tag{5.1.26}$$

The continuum representation of the originally discrete system has all the properties of a generalized Cosserat medium: to its points $\underset{\sim}{r}$ attached are the directors $\underset{\sim}{d}_{(\lambda)}$, the motions of which are independent of the motions of the points.

5. 2. Materials with Microstructure.

Let a body be composed of microelements $\Delta V'$ in which a continuous mass density P' exists, such that the microelements $\Delta V'$ represent material continua. A macro-volume element dV is composed of the micro-volume elements dV',

$$dV = \int_{dV} dV' , \tag{5.2.1.}$$

and we assume that the macro-mass dM in dV is the average of all masses in dV. Denoting by $P' dV' = dM'$ the micro-mass of the micro-volume element dV', we may write

$$\int_{dV} P' dV' = dM = P dV \tag{5.2.2}$$

With respect to a fixed Cartesian coordinate system Z^α let Z'^α be coordinates of points $\underset{\sim}{Z}'$ in a micro-volume element dV' in a reference configuration K_0. The integral over the macro-volume element

$$\int_{dV} P' Z'^\alpha dV' = P Z^\alpha dV \tag{5.2.3}$$

determines the centre of mass $\underset{\sim}{Z}$ of the macro-volume element dV. Denoting by $\underset{\sim}{R}' = Z'^\alpha \underset{\sim}{e}_\alpha$ the position vectors of the points $\underset{\sim}{Z}'$ of micro-elements, by $\underset{\sim}{R} = Z^\alpha \underset{\sim}{e}_\alpha$ the position vectors of the centres of mass of macro-volume elements dV and by $\underset{\sim}{P}' = \Xi'^\alpha \underset{\sim}{e}_\alpha$ the position vectors of the points $\underset{\sim}{R}'$ relative to the centre of gravity $\underset{\sim}{R}$,

$$\underset{\sim}{R}' = \underset{\sim}{R} + \underset{\sim}{P}' , \qquad (5.2.4)$$

all with respect to a fixed Cartesian system of reference, we have in the coordinate notation

$$Z'^{\alpha} = Z^{\alpha} + \Xi'^{\alpha} . \qquad (5.2.5)$$

In a deformed configuration $K(t)$ let the positions of points $\underset{\sim}{R}'$ be $\underset{\sim}{r}'$ and of the points $\underset{\sim}{R}$ be $\underset{\sim}{r}$. The relative position vectors of $\underset{\sim}{r}'$ with respect to the new positions of the centres of mass let be $\underset{\sim}{\varrho}'$. The equations of motion of the centres of mass of the macro-elements dV, which become dv , and of the points $\underset{\sim}{R}'$ are

$$\begin{aligned} \underset{\sim}{r} &= \underset{\sim}{r}(\underset{\sim}{R},t) \quad , \quad \underset{\sim}{R} = \underset{\sim}{R}(\underset{\sim}{r},t) \quad , \\ \underset{\sim}{r}' &= \underset{\sim}{r}(\underset{\sim}{R}',t) \quad , \quad \underset{\sim}{R}' = \underset{\sim}{R}(\underset{\sim}{r}',t) \quad , \end{aligned} \qquad (5.2.6)$$

and we assume that in the deformed configuration the positions of the points $\underset{\sim}{Z}'$ are defined by the relations

$$\underset{\sim}{r}' = \underset{\sim}{r} + \underset{\sim}{\varrho}' \qquad , \text{or} \quad z'^{\alpha} = z^{\alpha} + \xi'^{\alpha} \qquad (5.2.7)$$

The further assumption we make is that the motion (5.2.6) carries the centres of mass of dV into the centres of mass of the deformed macro-volume elements dv ,

$$\int_{dv} \varrho' \underset{\sim}{r}' \, dv' = \varrho \underset{\sim}{r} \, dv . \qquad (5.2.8)$$

From (5.2.6) we have

$$\underset{\sim}{r}' = \underset{\sim}{r}\left(\underset{\sim}{R} + \underset{\approx}{\Xi}', t\right) = \underset{\sim}{r}\left(\underset{\sim}{R}, t\right) + \underset{\sim}{\varrho}' , \tag{5.2.9}$$

where

$$\underset{\sim}{\varrho}' = \underset{\sim}{\varrho}\left(\underset{\sim}{R}, \underset{\sim}{\Xi}', t\right) . \tag{5.2.10}$$

Expanding $(5.2.9)_1$, under the assumption that $\underset{\sim}{\varrho}'$ is an analytic function of Ξ'^{α} , we obtain

$$\underset{\sim}{\varrho}' = \underset{\sim}{\varrho}\left(\underset{\sim}{R}, 0, t\right) + \frac{\partial \underset{\sim}{\varrho}}{\partial \Xi'^{\alpha}} \Xi'^{\alpha} + \ldots \tag{5.2.11}$$

Through $(5.2.9)_2$ we see that for $\underset{\sim}{P}' = 0$

$$\underset{\sim}{\varrho}\left(\underset{\sim}{R}, 0, t\right) = \underset{\sim}{0} , \tag{5.2.12}$$

and if we write

$$\frac{\partial \underset{\sim}{\varrho}}{\partial \Xi'^{\alpha}} \equiv \underset{\sim}{\chi}_{\alpha}\left(\underset{\sim}{R}, t\right) , \tag{5.2.13}$$

$$\chi^{\beta}_{\cdot\alpha} = \frac{\partial \xi^{\beta}}{\partial \Xi'^{\alpha}} ,$$

in the linear approximation we obtain the equations of motion of points $\underset{\sim}{R}'$ in the form

$$\underset{\sim}{\varrho}' = \underset{\sim}{\chi}_{\alpha} \Xi'^{\alpha} , \tag{5.2.14}$$

or

$$\xi'^{\lambda} = \chi^{\lambda}_{\cdot\alpha} \Xi'^{\alpha} . \tag{5.2.15}$$

The coefficients $\underset{\sim}{\chi}^{\alpha}$ reciprocal to $\underset{\sim}{\chi}_{\alpha}$ are defined by the relations

$$\chi_{\beta}^{\cdot\alpha} = \frac{\partial \Xi'^{\alpha}}{\partial \xi^{\cdot\beta}} , \qquad (5.2.16)$$

and

$$\chi^{\alpha}_{\cdot\beta} \; \chi^{\cdot}_{\alpha}{}^{\gamma} = \delta^{\gamma}_{\beta} , \chi^{\alpha}_{\cdot\beta} \; \chi^{\cdot\beta}_{\gamma} = \delta^{\alpha}_{\gamma} \qquad (5.2.17)$$

The velocity $\underset{\sim}{v}'$ of a point $\underset{\sim}{R}'$ is defined by

$$\underset{\sim}{v}' = \dot{\underset{\sim}{r}}' = \dot{\underset{\sim}{r}} + \dot{\underset{\sim}{\varrho}}' = \underset{\sim}{v} + \dot{\underset{\sim}{\chi}}_{\alpha} \Xi'^{\alpha} \qquad (5.2.18)$$

or, in the componental form

$$\dot{z}'^{\alpha} = \dot{z}^{\alpha} + \dot{\chi}^{\alpha}_{\cdot\beta} \; \Xi'^{\beta} . \qquad (5.2.19)$$

Eliminating Ξ'^{β} from (5.2.19) we obtain

$$v'^{\alpha} = v^{\alpha} + \dot{\chi}^{\alpha}_{\cdot\beta} \; \chi^{\cdot\beta}_{\gamma} \; \xi'^{\gamma} = v^{\alpha} + \nu^{\alpha}_{\cdot\gamma} \; \xi'^{\gamma} , \qquad (5.2.20)$$

where

$$\nu^{\alpha}_{\cdot\gamma} = \dot{\chi}^{\alpha}_{\cdot\beta} \; \chi^{\cdot\beta}_{\gamma} = \nu^{\alpha}_{\cdot\gamma} \left[\underset{\sim}{z} \left(\underset{\approx}{z} , t \right) , t \right] = \nu^{\alpha}_{\cdot\gamma} \left(\underset{\approx}{z} , t \right) \qquad (5.2.21)$$

For a macro-volume element dv the momentum is given by the relation

$$d\underset{\sim}{K} = \int_{dv} d\underset{\sim}{K}' = \int_{dv} \varrho' \underset{\sim}{v}' \, dv' = \int_{dv} \varrho' \left(\underset{\sim}{v} + \nu^{\alpha}_{\cdot\beta} \; \xi'^{\beta} \right) dv' = \varrho \underset{\sim}{v} \, dv , \qquad (5.2.22)$$

and for a portion v of a body we have

$$\underset{\sim}{K} = \int_{v} \varrho \underset{\sim}{v} \, dv \qquad (5.2.23)$$

The moment of momentum $d\underset{\sim}{\ell}^{\circ}$ for the macro-volume element dv will be

$$d\underset{\sim}{\ell}^{\circ} = \int_{dv} \varrho' \underset{\sim}{r}' \times \underset{\sim}{v}' \, dv' = \int_{dv} \varrho' (\underset{\sim}{r} + \underset{\sim}{\varrho}') \times (\underset{\sim}{v} + \dot{\underset{\sim}{\chi}}_{\alpha} \Xi'^{\alpha}) \, dv' \qquad (5.2.24)$$

Since $\underset{\sim}{\varrho}'$ are the position vectors of the points $\underset{\sim}{r}'$ relative to the centre of mass $\underset{\sim}{r}$ of the macro-volume element, we have

$$\int_{dv} \varrho' \underset{\sim}{\varrho}' \, dv' = 0 \; ,$$

and

$$d\underset{\sim}{\ell}^{\circ} = \varrho \underset{\sim}{r} \times \underset{\sim}{v} \, dv + \int_{dv} \varrho' \underset{\sim}{\varrho}' \times \dot{\underset{\sim}{\chi}}_{\alpha} \Xi'^{\alpha} \, dv' . \qquad (5.2.25)$$

In the componental form we have

$$\underset{\sim}{\varrho}' \times \dot{\underset{\sim}{\chi}}_{\alpha} \Xi'^{\alpha} = \epsilon_{\lambda\mu\nu} \, \xi'^{\lambda} \, \dot{\chi}^{\mu}_{\cdot\alpha} \, \Xi'^{\alpha} \, \underset{\sim}{e}^{\nu} . \qquad (5.2.26)$$

and using (5.2.14) this becomes

$$\begin{aligned} \underset{\sim}{\varrho}' \times \dot{\underset{\sim}{\chi}}_{\alpha} \Xi'^{\alpha} &= \epsilon_{\lambda\mu\nu} \, \chi^{\lambda}_{\cdot\beta} \, \dot{\chi}^{\mu}_{\cdot\alpha} \, \Xi'^{\alpha} \, \Xi'^{\beta} \, \underset{\sim}{e}^{\nu} \qquad (5.2.27) \\ &= \underset{\sim}{\chi}_{\beta} \, \dot{\underset{\sim}{\chi}}_{\alpha} \, \Xi'^{\alpha} \, \Xi'^{\beta} . \end{aligned}$$

Hence, for the moment of momentum $d\underset{\sim}{\ell}^{\circ}$ we may write

$$d\underset{\sim}{\ell}^{\circ} = \varrho \underset{\sim}{r} \times \underset{\sim}{v} \, dv + \underset{\sim}{\chi}_{\alpha} \times \dot{\underset{\sim}{\chi}}_{\beta} \int_{dv} \varrho' \, \Xi'^{\alpha} \, \Xi'^{\beta} \, dv' . \qquad (5.2.28)$$

Using the inverse of (5.2.15),

$$\Xi'^{\alpha} = \chi^{\cdot\alpha}_{\lambda} \, \xi'^{\lambda} \; , \qquad (5.2.29)$$

by (5.2.13) we see that

$$\int_{dv} \varrho' \, \Xi'^{\alpha} \, \Xi'^{\beta} \, dv' = \chi_{\lambda}^{\cdot\alpha} \, \chi_{\mu}^{\cdot\beta} \int_{dv} \varrho' \, \xi'^{\lambda} \, \xi'^{\mu} \, dv', \tag{5.2.30}$$

and if we introduce the "micro-inertia density" $i^{\lambda\mu}$ by the expression

$$\varrho \, i^{\lambda\mu} dv = \int_{dv} \varrho' \, \xi'^{\lambda} \, \xi'^{\mu} \, dv', \tag{5.2.31}$$

and the "macro-inertia density moments" $I^{\alpha\beta}$ by

$$I^{\alpha\beta} = \chi_{\lambda}^{\cdot\alpha} \, \chi_{\mu}^{\cdot\beta} \, i^{\lambda\mu}, \tag{5.2.32}$$

the expression (5.2.28) for the moment of momentum becomes

$$d\underset{\sim}{\ell}^{\circ} = \varrho \, \underset{\sim}{r} \times \underset{\sim}{v} \, dv + \varrho \, i^{\lambda\mu} \underset{\sim}{\chi}_{\alpha} \times \dot{\underset{\sim}{\chi}}_{\mu} \, dv \, . \tag{5.2.33}$$

For a portion v of the body we have now

$$\underset{\sim}{\ell}^{\circ} = \int_{v} d\underset{\sim}{\ell}^{\circ} = \int_{v} \varrho \left(\underset{\sim}{r} \times \underset{\sim}{v} + i^{\lambda\mu} \underset{\sim}{\chi}_{\lambda} \times \dot{\underset{\sim}{\chi}}_{\mu} \right) dv \tag{5.2.34}$$

Analoguously, we find for the kinetic energy the expression

$$T = \frac{1}{2} \int_{v} \varrho \left(\underset{\sim}{v} \cdot \underset{\sim}{v} + i^{\lambda\mu} \dot{\underset{\sim}{\chi}}_{\lambda} \cdot \dot{\underset{\sim}{\chi}}_{\mu} \right) dv \, . \tag{5.2.35}$$

Materials with micro-structure were first considered by Eringen and Suhubi in elasticity [99, 352] and in the fluid mechanics [91] . Here we diverged slightly from the original exposition of Eringen and Suhubi since we wanted to write the expressions for $\underset{\sim}{l}^{o}$ and T in a form similar to the corresponding formulae in the section 5.1, obtained from the consideration of a discrete system.

In the original papers (cf. [91]) the coefficients $I^{\alpha\beta}$ stay instead of $i^{\alpha\beta}$, and $i^{\alpha\beta}$ instead of $I^{\alpha\beta}$, and, following our notation, the coefficients

$$I^{\lambda\mu} = \chi_{\alpha}^{\cdot\lambda}\, \chi_{\beta}^{\cdot\mu} \int_{dV} \rho'\, \Xi'^{\alpha}\, \Xi'^{\beta} dV' \qquad (5.2.36)$$

are named "micro-inertia moments", and the coefficients

$$i^{\alpha\beta} = \int_{dV} \rho'\, \Xi'^{\alpha}\, \Xi'^{\beta}\, dV' \qquad (5.2.37)$$

are constant material coefficients. We prefere to use here the densities defined by (5.2.31,32)

According to Eringen [90] , materials affected by micro-motion and micro-deformation are micromorphic materials.

Micropolar media are a subclass of micromorphic materials, and they exhibit microrotational effects, i.e. the material points in a volume element can undergo only the rotational motions about the centres of mass.

The materials with microstructure of Mindlin [219,222] coincide with the model given above. Mindlin considered the infinitesimal deformations only, and his theory is restricted to the linear case. If we assume that the deformations are infinitesimal and if we make no distinction between the material and spatial coordinates Z^{α} and z^{α}, for the micro-deformation we may write

$$\xi'^{\beta} = \Xi'^{\beta} + u'^{\beta} , \qquad (5.2.38)$$

where u'^{α} are components of the micro-displacements. From (5.2.15) it follows then

$$u'^{\beta} = \left(\chi_{\lambda}^{\cdot\beta} - \delta_{\lambda}^{\beta}\right)\Xi'^{\lambda} \approx \left(\chi_{\lambda}^{\cdot\beta} - \delta_{\lambda}^{\beta}\right)\xi'^{\lambda}, \qquad (5.2.39)$$

where the quantities $\psi_{\lambda}^{\cdot\beta}$ defined by the expression

$$\psi_{\lambda}^{\cdot\beta} = \frac{\partial u'^{\beta}}{\partial \xi'^{\lambda}} = \chi_{\lambda}^{\cdot\beta} - \delta_{\lambda}^{\beta} \qquad (5.2.40)$$

are called by Mindlin the <u>micro-deformations</u>. Denoting by u^{α} the displacements of particles (which are not necessarily represented by their centres of mass),

$$u^{\alpha} = z^{\alpha} - Z^{\alpha} \qquad (5.2.41)$$

<u>the macro-strain</u> is given by

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_{\beta}}{\partial z^{\alpha}} + \frac{\partial u_{\alpha}}{\partial z^{\beta}}\right) , \qquad (5.2.42)$$

and the <u>relative deformation</u> by

$$\gamma_{\alpha\beta} = \frac{\partial u_{\beta}}{\partial z^{\alpha}} - \psi_{\alpha\beta} . \qquad (5.2.43)$$

In this theory the quantities $\psi_{\alpha\beta}$ play the role of directors, and the medium with micro-structure is a generalized Cosserat medium.

5. 3. Multipolar Theories.

In a series of papers Green and Rivlin [128, 129;131], Green [112], and Green, Naghdi and Rivlin [126] developed the theory of multipolar continua, which represents a very general, but a very formal approach. Let Z^{α} be coordinates of a particle in a reference position and z^{α} its position at time τ,

$$z^{\alpha}(\tau) = z_{\alpha}(\underset{\sim}{Z}, \tau) \quad , \quad -\infty < \tau \leq t . \tag{5.3.1}$$

It is possible to consider the position of the particle $\underset{\sim}{Z}$ at time τ also in terms of the current position at time t, so that

$$z^{\alpha}(\tau) = z^{\alpha}(z^{1}, z^{2}, z^{3}, \tau, t) . \tag{5.3.2}$$

A <u>simple 2^{γ}-pole displacement field</u> is defined in two forms,

$$z_{\alpha B_1 \ldots B_\gamma}(\tau) = z_{\alpha B_1 \ldots B_\gamma}(Z, \tau) , \tag{5.3.3}$$

and

$$z_{\alpha \beta_1 \ldots \beta_\gamma}(\tau) = z_{\alpha \beta_1 \ldots \beta_\gamma}(\underset{\sim}{z}, t, \tau) . \tag{5.3.4}$$

The examples of such multipolar displacement fields are the gradients

$$z^{\alpha}_{\cdot B_1 \ldots B_\gamma}(\tau) = \frac{\partial^{\gamma} z^{\alpha}(\tau)}{\partial Z^{B_1} \ldots \partial Z^{B_\gamma}} \tag{5.3.5}$$

and

$$z^{\alpha}_{\cdot\beta_1\cdots\beta_\gamma}(\tau) = \frac{\partial^\gamma z^\alpha(\tau)}{\partial z^{\beta_1}\dots\partial z^{\beta_\gamma}} \qquad (5.3.6)$$

The time derivatives of the multipolar displacements represent the multipolar (2^γ -pole) velocity fields.

In multipolar theories the deformation is described by the simple deformation field $\underset{\sim}{z}(\tau)$ and by ν tensor fields, say $u_{\alpha A_1\cdots A_\gamma}(\tau)$, $\gamma = 1,2,\dots,\nu$. The tensor fields $u_{\alpha A_1\dots A_\gamma}(\tau)$ are called multipolar deformation fields. In 1967 Green and Rivlin [131] showed that the multipolar theory can be considered as a special case of the director theory, with the multipolar deformation fields $u_{\alpha A_1\dots A_\gamma}$ corresponding to 3^γ directors.

The theory of multipolar media was applied by Bleustein and Green to fluids [32] .

5. 4. Strain-Gradient Theories.

The state of strain of a body at a point $\underset{\sim}{X}$ depends on the relative displacements of points in a neighbourhood $N(\underset{\sim}{X})$. If $\underset{\sim}{X} + \Delta\underset{\sim}{X}$ is a point in $N(\underset{\sim}{X})$, and the equations of motion are

$$x^i = x^i(\underset{\sim}{X}, t) \qquad (5.4.1)$$

the relative displacements of all points $\underset{\sim}{X}+\Delta\underset{\sim}{X}$ for arbitrary $\Delta\underset{\sim}{X}$ are determined by the deformation gradients

$$x^i_{;K}\,,\quad x^i_{;K_1K_2}\,,\ldots,\,x^i_{;K_1\ldots K_N}\,,\ldots \tag{5.4.2}$$

Material derivatives of these deformation gradients are the velocity gradients,

$$v^i_{;K}\,,\quad v^i_{;K_1K_2}\,,\ldots,\,v^i_{;K_1\ldots K_N}\,,\ldots \tag{5.4.3}$$

The theories which consider the influence of the higher-order deformation and velocity gradients are known as the strain-gradient theories.

According to (3.20) and (3.11), by differentiation we obtain

$$E_{KL,M} = g_{k\ell}\, x^k_{;M(L}\, x^\ell_{;K)}\,, \tag{5.4.4}$$

and we see that the first gradient of strain involves the second gradient of deformation.

The deformed directors at two points, say $\underset{\sim}{X}$ and $\underset{\sim}{X} + \Delta \underset{\sim}{X}$ in a neighbourhood $N(\underset{\sim}{X})$ will, be according to (5.2),

$$d^k_{(\alpha)} = d^k_{(\alpha)}(\underset{\sim}{X})\,,$$

$$d^k_{(\alpha)}(\underset{\sim}{X} + \Delta\underset{\sim}{X}) = d^k_{(\alpha)}(\underset{\sim}{X}) + d^k_{(\alpha);L}\,\Delta X^L + \ldots\,. \tag{5.4.5}$$

Hence, the director deformation at $\underset{\sim}{X}$ is characterized by the director gradients $d^k_{(\alpha);L}$, $d^k_{(\alpha);L_1L_2}$, ... From (5.4) it follows then that

$$d^{k}_{(\alpha);L} = \Delta^{k}_{(\alpha);L} + x^{k}_{;KL} D^{K}_{(\alpha)} + x^{k}_{;K} D^{K}_{(\alpha);L} \quad . \qquad (5.4.6)$$

If an oriented body degenerates into an ordinary body the directors will become material vectors and $\Delta^{k}_{(\alpha)}$ vanishes. In this case we may chose the directors $\underset{\sim}{D}_{(\alpha)}$ in the reference configuration to be parallel vectors fields so that $D^{k}_{(\alpha);L} = 0$. Consequently, the director gradients will be proportional to the second gradients of deformation,

$$d^{k}_{(\alpha);L} = x^{k}_{;KL} D^{K}_{(\alpha)} \ , \qquad (5.4.7)$$

and the theory of an oriented body will degenerate into a strain-gradient theory.

In Cosserat bodies the directors $\underset{\sim}{d}_{(\alpha)}$ form rigid triads, such that

$$\underset{\sim}{d}_{(\alpha)} \cdot \underset{\sim}{d}_{(\beta)} = \underset{\sim}{D}_{(\alpha)} \underset{\sim}{D}_{(\beta)} = \text{const.} \qquad (5.4.8)$$

In this case the rates of the directors will be

$$\dot{\underset{\sim}{d}}_{(\alpha)} = \underset{\sim}{\omega} \times \underset{\sim}{d}_{(\alpha)} \ , \qquad (5.4.9)$$

where $\underset{\sim}{\omega}$ is the rate of rotation of the triads of directors. In the componental form we may write

$$\dot{d}_{(\alpha)m} = \epsilon_{mij} \, \omega^{i} d^{j}_{(\alpha)} = \omega_{jm} \, d^{j}_{(\alpha)} \qquad (5.4.10)$$

If there are only three directors , $\alpha = 1,2,3$, and in the Cosserat continuum in the strict sense there are only three directors, the reciprocal triads $\underset{\sim}{d}^{(\alpha)}$ exist, and for the tensor $\underset{\sim}{\omega}$ we have

$$\omega_{nm} = d^{(\alpha)}_{n} \; \dot{d}_{(\alpha)m} \quad . \tag{5.4.11}$$

From (5.4.8) it follows that the left-hand side of (5.4.11) is an antisymmetric tensor. If the rotations of the director triads are constrained to follow the rotations of the medium determined by the displacements of the points of the medium, which are given by

$$\omega_{nm} = v_{[n,m]} \; , \tag{5.4.12}$$

where $v^i = \dot{x}^i$ is the velocity vector, for the corresponding medium it is said that it is a Cosserat continuum with constrained rotations (Toupin [372]).

6. Forces, Stresses and Couples.

In mechanics of particles it is usually proved that a system of forces, say $\underset{\sim}{f}_{(1)}, \underset{\sim}{f}_{(2)}, \ldots, \underset{\sim}{f}_{(n)}$ acting on a system of particles $M_1, \ldots, M_n$ may be reduced to the resultant force

$$\underset{\sim}{f} = \sum_{i=1}^{n} \underset{\sim}{f}_{(i)} \tag{6.1}$$

and to the resultant couple, which is defined with respect to a pole $\underset{\sim}{0}$ by the expression

$$\underset{\sim}{M}^{o} = \sum_{i=1}^{n} \underset{\sim}{r}_i \times \underset{\sim}{f}_{(i)} , \tag{6.2}$$

where $\underset{\sim}{r}_i$ are position vectors of the particles M_i with respect to $\underset{\sim}{0}$. In continuum mechanics an immediate generalization is insufficient to describe all the forces and couples which appear, even if the suitable assumptions are made for the transition from a discrete system to a continuum model.

In the following definition we partly follow Truesdell and Noll [365], but we introduce some additional definitions in order to consider more general models of continua.

Let $\mathcal{v}$ be a part of a body $\mathcal{B}$ and S the bounding surface of the $\mathcal{v}$, and let the motion of the body be given by the equations

$$x^i = x^i(\underset{\sim}{X}, t), \qquad (6.3)$$

$$d^i_{(\alpha)} = d^i_{(\alpha)}(\underset{\sim}{X}, t), \quad (\alpha = 1, 2, \ldots n)$$

and let $\varrho = \varrho(\underset{\sim}{x})$ be the density of matter.

1. At each time t there is a vector field $\underset{\sim}{f}(\underset{\sim}{x}, t)$ defined per unit mass, which we call the external body force. The vector $\underset{\sim}{F}_f(v)$ defined by the volume integral

$$\underset{\sim}{F}_f(v) = \int_v \varrho\, \underset{\sim}{f}(\underset{\sim}{x})\, dv \qquad (6.4)$$

is called the resultant external body force exerted on the part v at time t.

2. At each time t there is an antisymmetric tensor field $\ell^{ij}(\underset{\sim}{x}, t)$ defined per unit mass, which we call the external body couple. The resultant body couple is defined by the volume integral

$$M^{ij}_{\ell}(v) = \int_v \varrho\, \ell^{ij}(\underset{\sim}{x})\, dv \qquad (6.5)$$

3. At each time t, to each part v of the body $\mathcal{B}$ corresponds a vector field $\underset{\sim}{t}(x, v)$, defined for the points $\underset{\sim}{x}$ on the bounding surface s of v. It is called the stress (or the density of the contact force), acting on the part v of $\mathcal{B}$. The resultant contact force $F_t(v)$ exerted on v at time t is defined

by the surface integral

$$F_t(\mathcal{v}) = \oint_s \underset{\sim}{t} \; (\underset{\sim}{x}, \mathcal{v}) \; ds \qquad (6.6)$$

4. At each time t, to each part $\mathcal{v}$ of the body $\mathcal{B}$ corresponds an antisymmetric tensor field m^{ij} defined for the point $\underset{\sim}{x}$ on the boundary s of $\mathcal{v}$. It is called the <u>couple stress</u> (or the <u>density of the contact couple</u>), acting on the part $\mathcal{v}$ of $\mathcal{B}$. The <u>resultant contact couple</u> $M_m^{ij}(\mathcal{v})$ is defined by the surface integral

$$M_m^{ij}(\mathcal{v}) = \oint_s m^{ij}(\underset{\sim}{x}, \mathcal{v}) \; ds \qquad (6.7)$$

5. The <u>total resultant force</u> exerted on the part $\mathcal{v}$ of $\mathcal{B}$ is defined as the sum of the resultant body force and the resultant contact force,

$$\underset{\sim}{F}(\mathcal{v}) = \underset{\sim}{F}_f(\mathcal{v}) + \underset{\sim}{F}_t(\mathcal{v}) \qquad (6.8)$$

6. The <u>total resultant couple</u> exerted on the part $\mathcal{v}$ of $\mathcal{B}$ is defined as the sum of the resultant body couple and the resultant contact couple,

$$M^{ij}(\mathcal{v}) = M_\ell^{ij}(\mathcal{v}) + M_m^{ij(\mathcal{v})} . \qquad (6.9)$$

According to the <u>stress principle</u> (cf. [365]) there is a vector field $\underset{\sim}{t}(\underset{\sim}{x}, \underset{\sim}{n})$ defined for all points x in $\mathcal{B}$ and for all unit vectors $\underset{\sim}{n}$, such that the stress acting on any part $\mathcal{v}$ of $\mathcal{B}$ is given by

$$\underset{\sim}{t}(\underset{\sim}{x}, v) = \underset{\sim}{t}(\underset{\sim}{x}, \underset{\sim}{n}) \quad , \tag{6.10}$$

where $\underset{\sim}{n}$ is the exterior unit normal vector at the points $\underset{\sim}{x}$ on the boundary of S .

In elementary continuum mechanics it is proved that the <u>stress vector</u> $\underset{\sim}{t}(\underset{\sim}{x}, \underset{\sim}{n})$,

$$\underset{\sim}{t}(\underset{\sim}{x}, \underset{\sim}{n}) = t^i(\underset{\sim}{x}, \underset{\sim}{n}) \underset{\sim}{g}_i \tag{6.11}$$

may be represented in the form

$$\underset{\sim}{t}(\underset{\sim}{x}, \underset{\sim}{n}) = t^{ij}(\underset{\sim}{x})\, n_j\, g_i \quad , \tag{6.12}$$

where $t^{ij}(\underset{\sim}{x})$ are components of the <u>stress tensor</u>. From (6.6) we obtain now that the components of the resultant stress are given by the integral

$$F_t(v) = \oint_S t^{ij}(\underset{\sim}{x})\, \underset{\sim}{g}_i\, n_j\, ds \tag{6.13}$$

In analogy to the stress vector we may write for the couple stress

$$m^{ij}(\underset{\sim}{x}, v) = m^{ij}(\underset{\sim}{x}, \underset{\sim}{n}) \quad , \tag{6.14}$$

and

$$m^{ij}(\underset{\sim}{x}, \underset{\sim}{n}) = m^{ijk}(\underset{\sim}{x})\, n_k \tag{6.15}$$

where $m^{ijk} = -m^{jik}$ is the <u>couple-stress tensor</u> (cf. [364]).

7. At each time t , at each part v of the

body $\mathcal{B}$ there are vector fields $\underset{\sim}{K}^{(\alpha)}(\underset{\sim}{x},t)$ defined per unit mass, which we call the external director forces. The vectors $\underset{\sim}{F}{}^{\alpha}_{K}(v)$ defined by the integral

$$\underset{\sim}{F}{}^{\alpha}_{K}(v) = \int_{v} \rho\, \underset{\sim}{K}^{(\alpha)}(\underset{\sim}{x},t)\, dv \quad , \quad (\alpha = 1,2,\ldots,n) \tag{6.16}$$

are called the resultant director forces exerted on the part v of the body at time t .

8. At each time t , to each part v of the body $\mathcal{B}$ correspond vector fields $\underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},v)$, defined for the points $\underset{\sim}{x}$ on the boundary s of v , which we call the director stresses. We assume that there are vector fields $\underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},\underset{\sim}{n})$, defined for all points of v and for all unit vectors $\underset{\sim}{n}$, such that the director stresses acting on any part v of $\mathcal{B}$ are given by

$$\underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},v) = \underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},\underset{\sim}{n}) \quad , \quad (\alpha = 1,2,\ldots n) \tag{6.17}$$

The resultant director stresses are given by the surface integrals

$$F^{\alpha}_{h}(v) = \oint_{s} \underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},v)\, ds \quad . \quad (\alpha = 1,2,\ldots,n) \tag{6.18}$$

For the director stress vectors $\underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},\underset{\sim}{n})$ we assume that they may be represented in the form

$$\underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},\underset{\sim}{n}) = \underset{\sim}{h}^{(\alpha)i}(\underset{\sim}{x},\underset{\sim}{n})\, \underset{\sim}{g}_i \quad , \tag{6.19}$$

and that

$$\underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},v) = \underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x},\underset{\sim}{n}) = h^{(\alpha)ij}(\dot{\underset{\sim}{x}})\, \underset{\sim}{g}_i\, n_j \tag{6.20}$$

The quantities $h^{(\alpha)ij}$ we call the <u>director stress tensors</u>.

9. The total resultant director forces exerted on the part v of $\mathcal{B}$ are defined as the sum of the resultant director forces and the resultant director stresses,

$$\underset{\sim}{F}{}^{\alpha}_{d}(v) = F^{\alpha}_{K}(v) + F^{\alpha}_{h}(v) . \tag{6.21}$$

<u>We assume that the number of the director force vectors and of the director stress tensors is equal to the number of the directors $d_{(\alpha)}$ of the body $\mathcal{B}$.</u>

The <u>momenta</u> of forces and stresses are defined by the following expressions:

a) The moment of the external body force at a point $\underset{\sim}{x}$, with respect to the origin $\underset{\sim}{O}$:

$$\underset{\sim}{r} \times \varrho \underset{\sim}{f} , \tag{6.22}$$

and the resultant moment for the part v of $\mathcal{B}$

$$\int_{v} \varrho \underset{\sim}{r} \times \underset{\sim}{f} \, dv . \tag{6.23}$$

b) The moment of stress at $\underset{\sim}{x}$, with respect to the origin $\underset{\sim}{O}$:

$$\underset{\sim}{r} \times \underset{\sim}{t} (\underset{\sim}{x} , \underset{\sim}{n}) , \tag{6.24}$$

and the resultant moment of stress:

$$\oint_s \underset{\sim}{r} \times \underset{\sim}{t}\,(\underset{\sim}{x}\,,\underset{\sim}{n})\; ds\,. \tag{6.25}$$

c) The moment of the director forces at $\underset{\sim}{x}$:

$$\varrho\,\underset{\sim}{\Gamma} = \underset{\sim}{d}_{(\alpha)} \times \varrho\,\underset{\sim}{k}^{(\alpha)}(\underset{\sim}{x}\,,t) \tag{6.26}$$

and the resultant of the director forces for the part v of $\mathcal{B}$:

$$\int_v \varrho\,\underset{\sim}{d}_{(\alpha)} \times \underset{\sim}{k}^{(\alpha)}(\underset{\sim}{x}\,,t)\; dv\,. \tag{6.27}$$

d) The moment of the director stresses at $\underset{\sim}{x}$:

$$\underset{\sim}{d}_{(\alpha)} \times \underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x}\,,\underset{\sim}{n})\;, \tag{6.28}$$

and the resultant moment of the director stresses,

$$\oint_s \underset{\sim}{d}_{(\alpha)} \times \underset{\sim}{h}^{(\alpha)}(\underset{\sim}{x}\,,\underset{\sim}{n})\; ds\,. \tag{6.29}$$

The total resultant moment of forces acting on a part v of a body $\mathcal{B}$ at time t is the sum of the moments of body and director forces, of body and director couples, and of the moments of stress and director stresses, and of the couple stresses,

$$\begin{aligned} \underset{\sim}{L} = {} & \int_v \varrho\left(\underset{\sim}{r} \times \underset{\sim}{f} + \underset{\sim}{d}_{(\lambda)} \times \underset{\sim}{k}^{(\lambda)} + \underset{\sim}{\ell}\right) dv \; + \\ & + \oint_s \left(\underset{\sim}{r} \times \underset{\sim}{t} + \underset{\sim}{d}_{(\alpha)} \times \underset{\sim}{h}^{(\alpha)} + \underset{\sim}{m}\right) ds\,. \end{aligned} \tag{6.30}$$

This may be written in the components for as follows:

$$L^{\alpha\beta} = 2\int_{v} \varrho\left(z^{[\alpha} f^{\beta]} + d^{[\alpha}_{(\lambda)} k^{(\lambda)\beta]} + l^{\alpha\beta}\right) dv + \\ + 2 \oint_{s} \left(z^{[\alpha} t^{\beta]\gamma} + d^{[\alpha}_{(\lambda)} h^{(\lambda)\beta]\gamma} + m^{\alpha\beta\gamma}\right) n_{\gamma} ds. \tag{6.31}$$

6. 1. A Physical Interpretation.

Physical interpretations of the director forces depend on the model considered. For a medium consisting of particles which are composed of mass points, as was the medium considered in the section 5.1, we may assume (Rivlin [290,291]) that the external force $m^{(P)}_{\alpha}\, \underset{\sim}{f}^{(P)}_{\alpha}$ acts on the mass point $m^{(P)}_{\alpha}$ of the Pth particle. The resultant external force acting on the Pth particle is

$$\sum_{\alpha=1}^{n} m^{(P)}_{\alpha}\, \underset{\sim}{f}^{(P)}_{\alpha} = m^{(P)}\, \underset{\sim}{f}^{(P)} , \tag{6.1.1}$$

and if we assume that the discrete sets of vectors $\underset{\sim}{f}^{(P)}$ and $\underset{\sim}{f}^{(P)}_{\alpha}$ may be replaced by continuous vector fields $\underset{\sim}{f}$ and $\underset{\sim}{f}_{\alpha}$, defined throughout the body $\mathcal{B}$, for a part v of $\mathcal{B}$ we may write for the resultant body force

$$\underset{\sim}{F}_{f}(v) = \sum_{v} m^{(P)}\, \underset{\sim}{f}^{(P)} = \int_{v} \varrho\, \underset{\sim}{f}\, dv . \tag{6.1.2}$$

Denoting again by $\underset{\sim}{r}^{(P)}$ the position vectors of the centres of mass of the particles and by $\underset{\sim}{\varrho}^{(P)}_{\alpha}$ the position vectors of the mass points inside the particles, with respect to the corresponding centres of mass, the moment of the force $m_\alpha^{(P)} \underset{\sim}{f}_\alpha^{(P)}$ with respect to the origin $\underset{\sim}{O}$ will be

$$\left(\underset{\sim}{r}^{(P)} + \underset{\sim}{\varrho}_\alpha^{(P)} \right) \times m_\alpha^{(P)} \underset{\sim}{f}_\alpha^{(P)} . \tag{6.1.3}$$

For a particle P we have for the resultant moment of external forces the expression

$$\underset{\sim}{r}^{(P)} \times m^{(P)} \underset{\sim}{f}^{(P)} + \sum_{\alpha=1}^{n} \underset{\sim}{\varrho}_\alpha^{(P)} \times m_\alpha^{(P)} \underset{\sim}{f}_\alpha^{(P)} , \tag{6.1.4}$$

and for the part $\mathcal{v}$ of $\mathcal{B}$ under the suitable assumptions we may write

$$\sum_{\mathcal{v}} \left(\underset{\sim}{r}^{(P)} \times m^{(P)} \underset{\sim}{f}^{(P)} + \sum_{\alpha=1}^{n} \underset{\sim}{\varrho}_\alpha^{(P)} \times m_\alpha^{(P)} \underset{\sim}{f}_\alpha^{(P)} \right) = \tag{6.1.5}$$

$$= \int_{\mathcal{v}} \varrho \underset{\sim}{r} \times \underset{\sim}{f} \, d\mathcal{v} + \int_{\mathcal{v}} \varrho \underset{\sim}{d}_{(\alpha)} \times f_\alpha \, d\mathcal{v} ,$$

where according to the section 5.1 the discrete vectors $\underset{\sim}{\varrho}_\alpha^{(P)}$ are replaced by continuous vector fields $\underset{\sim}{d}_{(\alpha)}$.

According to Rivlin [291] , the field f represents the body force field, and $\underset{\sim}{f}_\alpha$ are the director force fields.

According to this model of Rivlin's, if s is the bounding surface of $\mathcal{v}$ in $\mathcal{B}$, under the assumption that <u>on the surface</u> s the discrete vectors $\underset{\sim}{f}^{(P)}$ and $\underset{\sim}{f}_\alpha^{(P)}$ may be replaced by continuous vector fields $\underset{\sim}{t}$ and $\underset{\sim}{t}_{(\alpha)}$, we may write

$$\sum_{S} m^{(P)} \underset{\sim}{f}^{(P)} = \oint_{S} \underset{\sim}{t} \cdot ds \quad , \qquad (6.1.6)$$

where $d\underset{\sim}{s} = \underset{\sim}{n} ds$ is the directed surface element and $\underset{\sim}{n}$ the unit vector, and

$$\sum_{S} m_{\alpha}^{(P)} \underset{\sim}{\varrho}_{\alpha}^{(P)} \times \underset{\sim}{f}_{\alpha}^{(P)} = \oint_{S} \underset{\sim}{d}_{(\alpha)} \times \underset{\sim}{t}_{(\alpha)} \, ds \, . \qquad (6.1.7)$$

$\underset{\sim}{t}$ represents the <u>simple surface force field</u>, or the stress, and $\underset{\sim}{t}_{(\alpha)}$ are the <u>director surface force fields</u>, or the director stresses according to the therminology introduced in the previous section.

7. Balance and Conservation Principles.

The differential equations of motion in classical continuum mechanics are usually derived from the law of conservation of mass (equation of continuity), and from the Euler's laws of balance of momentum and moment of momentum. Since we postulate here the validity of these laws, we regard them as principles.

Let v be a part of a body $\mathcal{B}$ and s the boundary of v. Let $\underset{\sim}{T}$ be the density of a quantity in balance, $\underset{\sim}{A}$ its influx (or efflux) per unit area of the bounding surface and $\underset{\sim}{B}$ its source per unit volume. The equation of balance has the general form

$$\frac{d}{dt}\int_v \underset{\sim}{T}\,dv = \oint_s \underset{\sim}{A}\cdot d\underset{\sim}{s} + \int_v \underset{\sim}{B}\,dv \qquad (7.1)$$

where $d\underset{\sim}{s}$ is the oriented surface element, $d\underset{\sim}{s} = \underset{\sim}{n}\,ds$, and $\underset{\sim}{n}$ the unit normal vector to ds. If the source vanishes, the equation of balance becomes the equation of conservation.

In classical mechanics we assume that there are neither sources nor influxes of mass. If ρ is the density of mass, so that dm,

$$\rho\,dv = dm\,, \qquad (7.2)$$

is the mass contained in the volume dv , the mass contained in the part v of the body considered will be

$$m(v) = \int_v \varrho \, dv \tag{7.3}$$

From (7.1) we may write now the law of conservation of mass,

$$\frac{dm}{dt} = \frac{d}{dt} \int_v \varrho \, dv = 0 \ , \tag{7.4}$$

which may be written in the form

$$\int_v \left(\dot{\varrho} \, dv + \varrho \, \dot{\overline{dv}} \right) = 0 \ . \tag{7.4}$$

For a body in motion the equations of motion of its points are

$$x^i = x^i \left(X^1, X^2, X^3, t \right) \quad , \left(i = 1,2,3 \right) \tag{7.5}$$

where X^K are material, and x^k spatial coordinates. If dV is the volume element of the body in an initial configuration referred to the coordinates X^K, and dv the corresponding volume element in a configuration $K(t)$ at time t , the volume elements dv and dV are related by the formula

$$dv = J \, dV \ , \tag{7.6}$$

where

$$J = \sqrt{\frac{g}{G}} \quad \det \left(x^k{}_{;K} \right) \tag{7.7}$$

From (7.6) we have now*

$$\dot{\overline{dv}} = \dot{J}\, dV \,, \tag{7.8}$$

---

* Let $d_1\underset{\sim}{r}$, $d_2\underset{\sim}{r}$ and $d_3\underset{\sim}{r}$ be three non-coplanar vectors at a point x. The element of volume defined by these vectors is

$$dv = (d_1\underset{\sim}{r} \times d_2\underset{\sim}{r}) \cdot d_3\underset{\sim}{r} = \epsilon_{k\ell m}\, d_1x^k\, d_2x^\ell\, d_3x^m .$$

Let this be the volume element at the configuration $K(t)$ of a body. Let be the corresponding volume element at the initial configuration

$$dV = \epsilon_{KLM}\, d_1X^K\, d_2X^L\, d_3X^M .$$

From (7.5) we have

$$d_m x^k = x^k_{;K}\, dX^K$$

and since

$$\epsilon_{k\ell m} = \sqrt{g}\; e_{k\ell m} \quad \text{and} \quad \epsilon_{KLM} = \sqrt{G}\; e_{KLM} \,,$$

where g and G are determinants of the fundamental tensors $\underset{\sim}{g}$ and $\underset{\sim}{G}$, respectively, we have

$$dv = \sqrt{\frac{g}{G}}\; \det\left(x^k_{;K}\right) dV .$$

and since**

$$\dot{J} = J\,\dot{x}^{k}_{;K} = J \operatorname{div} \underset{\sim}{v} , \tag{7.9}$$

from (7.4) we immediately have the global form of the law of conservation of mass,

$$\int_{v} \left(\dot{\varrho} + \varrho\, v^{k}_{;k}\right) dv = 0 \tag{7.10}$$

This has to be valid for an arbitrary part v of the body and therefore we finally obtain the local form of this law, which is often called the <u>equation of continuity</u>,

$$\dot{\varrho} + \varrho\, v^{k}_{;k} = 0 . \tag{7.11}$$

In general, the density ϱ is a function of position and time, $\varrho = \varrho(\underset{\sim}{x}, t)$ and $\dot{\varrho} = \partial\varrho/\partial t + \varrho_{,k} v^{k}$. Substituting this in (7.11) we obtain the continuity

---

**According to the rule for the differentiation of determinants, if $a = \det a^{i}_{\cdot j}$, then $\dot{a}\,\delta^{i}_{j} = \dot{a}^{i}_{j} A^{j}_{;k}$, where $A^{j}_{\cdot k}$ is the cofactor in a corresponding to the element $a^{k}_{\cdot j}$. Since $X^{K}_{;k} =$ (cofactor for $x^{k}_{;K}$) / (det $x^{k}_{;K}$), we have

$$\overline{\det x^{k}_{;K}} = \dot{x}^{k}_{;K} X^{K}_{;l} \left(\det x^{m}_{;M}\right) \delta^{l}_{k} = v^{k}_{;k} \det x^{m}_{;M}$$

where $v^{k} = \dot{x}^{k}$ is the velocity vector.

equation in another form,

$$\frac{\partial \varrho}{\partial t} + \left(\varrho v^{K}\right)_{,K} = 0 \tag{7.12}$$

The principle of balance of momentum states that the rate of the global momentum K of a part $\mathcal{v}$ of a body $\mathcal{B}$ is equal to the total resultant force exerted on the part $\mathcal{v}$ of the body. According to (6.4),(6.6),(6.8) and (6.10), for the total resultant force we have

$$\underset{\sim}{F}(\mathcal{v}) = \int_{\mathcal{v}} \varrho \underset{\sim}{f}\, d\mathcal{v} + \oint_{s} \underset{\sim}{t}\,(\underset{\sim}{x}\,.\,\underset{\sim}{n})\, ds\,. \tag{7.13}$$

We assume the momentum $\underset{\sim}{K}$ of a part $\mathcal{v}$ of a body $\mathcal{B}$ to have the form given by(5.1.24) or (5.2.23)

$$\underset{\sim}{K} = \int_{\mathcal{v}} \varrho \underset{\sim}{v}\, d\mathcal{v}\,,$$

and the balance of momentum equation reads

$$\frac{d}{dt}\int_{\mathcal{v}} \varrho \underset{\sim}{v}\, d\mathcal{v} = \int_{\mathcal{v}} \varrho \underset{\sim}{f}\, d\mathcal{v} + \oint_{s} \underset{\sim}{t}\,(\underset{\sim}{x}\,.\,\underset{\sim}{n})\, ds\,. \tag{7.14}$$

Using (6.12) and referring for the sake of simplicity all quantities to a Cartesian system of reference z^{α}, the component form of (7.14) becomes

$$\frac{d}{dt}\int_{\mathcal{v}} \varrho \dot{z}^{\alpha}\, d\mathcal{v} = \int_{\mathcal{v}} \varrho f^{\alpha}\, d\mathcal{v} + \oint_{s} t^{\alpha\beta} n_{\beta}\, ds \tag{7.15}$$

Performing the differentiation on the left-hand side and applying the divergence theorem to the surface integral on the right-hand side of (7.16), and using the continuity equation (7.11) we obtain

$$\int_{v} \varrho \dot{v}^{\alpha} \, dv = \int_{v} \left(\varrho f^{\alpha} + t^{\alpha\beta}{}_{,\beta} \right) dv , \qquad (7.17)$$

which is valid for an arbitrary part v of $\mathcal{B}$ and therefore the relation (7.17) must be valid at all points of $\mathcal{B}$, which gives the local equation for the balance of momentum;

$$\varrho \dot{v}^{\alpha} = t^{\alpha\beta}{}_{,\beta} + \varrho f^{\alpha} . \qquad (7.18)$$

This is a tensorial equation and for arbitrary curvilinear coordinates x^{i} we have

$$\varrho \dot{v}^{i} = t^{ij}{}_{;j} + \varrho f^{i} , \qquad (7.19)$$

where (see Appendix, (A3.10))

$$\dot{v}^{i} = \frac{\partial v^{i}}{\partial t} + v^{i}_{;j} \, v^{j} , \qquad (7.20)$$

and $t^{ij}{}_{;j}$ represents the covariant derivative of $\underset{\sim}{t}$ with respect to x^{j}, or the divergence of the tensor $\underset{\sim}{t}$.

In the local form (7.19), the equations of balance of momentum represent the set of three <u>differential equations of motion</u> for points of a body $\mathcal{B}$.

The principle of balance of moment of momentum states that the rate of change of the moment of momentum of a part $\mathcal{V}$ of a body is equal to the total resultant moment of forces acting on $\mathcal{V}$.

From the discussion in the section 5 we see that the expression (5.1.25) may be considered as a general form of the moment of momentum, since various physical models which lead to continuum models yield for the moment of momentum expressions of that form . Using (6.29) we may write directly the principle of balance of moment of momentum,

$$\frac{d}{dt}\int_{\mathcal{V}} \varrho\left(\underset{\sim}{r}\times\underset{\sim}{v} + i^{\lambda\mu}\,\underset{\sim}{d}_{(\lambda)}\times\dot{\underset{\sim}{d}}_{(\mu)}\right)dv =$$

$$= \int_{\mathcal{V}} \varrho\left(\underset{\sim}{r}\times\underset{\sim}{f} + \underset{\sim}{d}_{(\lambda)}\times\underset{\sim}{k}^{(\lambda)} + \underset{\sim}{l}\right)dv + \oint_{s}\left(\underset{\sim}{r}\times\underset{\sim}{t} + \underset{\sim}{d}_{(\lambda)}\times\underset{\sim}{h}^{(\lambda)} + \underset{\sim}{m}\right)ds\,. \tag{7.21}$$

For Cartesian coordinates by the application of (6.30) in the component form, the relation (7.21) reduces to

$$\frac{d}{dt}\int_{\mathcal{V}} \varrho\left(z^{[\alpha}\,\dot{z}^{\beta]} + i^{\lambda\mu}\,d^{[\alpha}_{(\lambda)}\,\dot{d}^{\beta]}_{(\mu)}\right)dv = \tag{7.22}$$

$$= \int_v \varrho \left(z^{[\alpha} f^{\beta]} + d^{[\alpha}_{(\lambda)} K^{(\lambda)\beta]} + l^{\alpha\beta} \right) dv +$$

$$+ \oint_s \left(z^{[\alpha} t^{\beta]\gamma} + d^{[\alpha}_{(\lambda)} h^{(\lambda)\beta]\gamma} + m^{\alpha\beta\gamma} \right) n_\gamma \, ds \tag{7.22}$$

Differentiating the integral on the left-hand side of (7.22), applying the divergence theorem to the surface integral, using the continuity equation and the equations of motion (7.18), and since the coefficients $i^{\lambda\mu}$ are symmetric, from (7.22) we obtain

$$\int_v \varrho \left(\dot{\overline{i^{\lambda\mu}}} \, d^{[\alpha}_{(\lambda)} \dot{d}^{\beta]}_{(\mu)} + i^{\lambda\mu} d^{[\alpha}_{(\lambda)} \ddot{d}^{\beta]}_{(\mu)} \right) dv =$$

$$= \int_v \left[\varrho \left(d^{[\alpha}_{(\lambda)} K^{(\lambda)\beta]} + l^{\alpha\beta} \right) + t^{[\alpha\beta]} + \left(d^{[\alpha}_{(\lambda)} h^{(\lambda)\beta]\gamma} + m^{\alpha\beta\gamma} \right)_{,\gamma} \right] dv. \tag{7.23}$$

However, from the analysis in the sections 5.1 and 5.2 it follows that the coefficients $i^{\lambda\mu}$ may be assumed to be independent of time, and since the relation (7.23) has to be valid for an arbitrary part v of the body, we obtain from (7.23) the local form of the principle of balance of moment of momentum,

$$\varrho \, i^{\lambda\mu} d^{[\alpha}_{(\lambda)} \ddot{d}^{\beta]}_{(\mu)} = t^{[\beta\alpha]} + \varrho \left(d^{[\alpha}_{(\lambda)} K^{(\lambda)\beta]} + l^{\alpha\beta} \right) + \left(d^{[\alpha}_{(\lambda)} h^{(\lambda)\beta]\gamma} + m^{\alpha\beta\gamma} \right)_{,\gamma} . \tag{7.24}$$

Let us introduce the notation

$$
\begin{aligned}
& i^{\lambda\mu}\, d^{[\alpha}_{(\lambda)}\, \dot{d}^{\beta]}_{(\mu)} = \sigma^{\alpha\beta} , \\
& l^{\alpha\beta} + d^{[\alpha}_{(\lambda)}\, k^{(\lambda)\beta]} = \overset{*}{l}{}^{\alpha\beta} , \\
& m^{\alpha\beta\gamma} + d^{[\alpha}_{(\lambda)}\, h^{(\lambda)\beta]\gamma} = \overset{*}{m}{}^{\alpha\beta\gamma} , \\
& d^{\alpha}_{(\lambda)}\, h^{(\lambda)\beta\gamma} = H^{\alpha\beta\gamma} , \\
& d^{\alpha}_{(\lambda)}\, k^{(\lambda)\beta} = k^{\alpha\beta} .
\end{aligned}
\tag{7.25}
$$

With this notation the relation (7.24) obtains the simple form

$$
\rho\, \dot{\sigma}^{\alpha\beta} = t^{[\beta\alpha]} + \rho\, \overset{*}{l}{}^{\alpha\beta} + \overset{*}{m}{}^{\alpha\beta\gamma}{}_{,\gamma} . \tag{7.26}
$$

The principle of moment of momentum in this form, (for elastic materials) was obtained by Toupin [372] from Hamilton's principle. He named $\sigma^{\alpha\beta}$ the spin angular momentum per unit mass, $H^{\alpha\beta\gamma}$ corresponds to Toupin's hyperstress, and $H^{[\alpha\beta]\gamma}$ he identified with the couple-stress tensor. The apparent discrepancy in the terminology and symbols is due to the fact that Toupin considered separately materials with directors, and materials which are described in terms of a strain-gradient theory. The couple stress tensor $\underset{\sim}{m}$ which we introduced independently of the hyperstress corresponds to the couple-stress tensor in Toupin's strain-gradient theory.

From (7.24) and (7.25) it is evident that it is impossible in the total effect to separate the influence of body moments from the director moments, and the influence of couple-stresses from the hyperstresses.

Assuming that there are no deformations of the directors and that there are no director forces and director stresses, the relation (7.26) reduces to

$$t^{[\alpha\beta]} = m^{\alpha\beta\gamma}{}_{,\gamma} + \varrho \ell^{\alpha\beta} \quad , \tag{7.27}$$

which substitutes Caushy's second law

$$t^{\alpha\beta} = t^{\beta\alpha} \tag{7.28}$$

valid only in the non-polar case.

In the theory of anisotropic fluids and liquid crystals, Ericksen [74a, 74 - 89] writes a separate equation of balance for the director momentum. Ericksen considers liquid crystals as packet of rod-like molecules, which corresponds to a one-director continuum model. Generalizing this idea we may introduce the <u>principle of balance of the director moments</u> (Stojanović, Djurić, Vujošević [343], Djurić [61] , Stojanović and Djurić [341]) in the form

$$\frac{d}{dt}\int_{v}\varrho\, i^{\lambda\mu}\,\dot{d}^{\alpha}_{(\mu)}\,dv = \oint_{s} h^{(\lambda)\alpha\beta}\,ds_{\beta} + \int_{v}\varrho\, k^{(\lambda)\alpha}\,dv\,, \qquad (7.29)$$

where on the right-hand side we have written in the component form the expression for the total resultant director force (6.21).

Performing the indicated differentiation and applying the divergence theorem in (7.29) we obtain

$$\varrho\, i^{\lambda\mu}\,\ddot{d}^{\alpha}_{(\mu)} = h^{(\lambda)\alpha\beta}{}_{,\beta} + \varrho\, k^{(\lambda)\alpha} \qquad (7.30)$$

as an independent set of the differential equations of motion for the directors.

Using (7.30), the equations (7.24) may be reduced to the form which does not include explicitly the inertial terms,

$$t^{[\alpha\beta]} = m^{\alpha\beta\gamma}{}_{,\gamma} + \varrho\,\ell^{\alpha\beta} + d^{[\alpha}_{(\lambda),\gamma}\; h^{(\lambda)\beta]\gamma}\,, \qquad (7.31)$$

It is obvious that the antisymmetric part of the stress tensor is affected by the director stresses if the medium considered is an oriented medium.

Since all the equations of motion (7.18), (7.26),(7.30) are tensorial equations, we shall write these equations directly in the component form valid for an arbitrary system of curvilinear coordinates x^i

$$\varrho \ddot{x}^i = t^{ij}{}_{,j} + \varrho f^i \quad , \tag{7.32}$$

$$\varrho i^{\lambda\mu} \ddot{d}^i_{(\mu)} = h^{(\lambda)ij}{}_{,j} + \varrho k^{(\lambda)i} \quad , \tag{7.33}$$

$$\varrho \dot{\sigma}^{ij} = t^{[ji]} + \varrho \overset{*}{l}{}^{ij} + \overset{*}{m}{}^{ijk}{}_{,k} \, . \tag{7.34}$$

$$\left(i, j, k = 1,2,3 \; ; \qquad \lambda, \mu = 1, 2, \ldots, n \right)$$

Eliminating from (7.34) the spin angular momentum $\underset{\sim}{\sigma}$, as it was already done in (7.31), decomposing in (7.32) the stress tensor into its symmetric and antisymmetric parts and substituting the antisymmetric part from (7.34), we obtain the set of $3n+3$ differential equations of motion,

$$\varrho \ddot{x}^i = t^{(ij)}{}_{,j} + m^{ijk}{}_{,jk} + \left(d^{[i}_{(\lambda),k} \, h^{(\lambda)j]k} \right)_{,j} + \varrho l^{ij}{}_{;j} + g f^i , \tag{7.35}$$

$$\varrho i^{\lambda\mu} \ddot{d}^i_{(\mu)} = h^{(\lambda)ij}{}_{;j} + \varrho k^{(\lambda)i} \, . \tag{7.36}$$

Obviuosly, the motion $x^i = x^i(\underset{\sim}{X}, t)$ is affected by the deformations of the directors and by the director stresses, and the motion of the directors, $d^i_{(\lambda)} = d^i_{(\lambda)}(\underset{\sim}{X}, t)$ is affected only by the director stresses and director forces.

7. 1. The Cosserat Continuum.

The Cosserat continuum is the medium in which the directors represent rigid triads of unit vectors, so that the motion is described by the motion of points and by an independent rotation of the director triads. According to (5.4.11) the rotation of the directors is determined by the field of the angular velocity tensor $\underset{\sim}{\omega}(x,t)$, so that we have

$$\dot{d}^{i}_{(\mu)} = \omega_{n}{}^{i}\, d^{n}_{(\mu)}\,, \tag{7.1.1}$$

from which follows

$$\ddot{d}^{i}_{(\mu)} = \left(\dot{\omega}_{\ell}{}^{i} + \omega_{\ell}{}^{n}\, \omega_{n}{}^{i} \right) d^{\ell}_{(\mu)}\,. \tag{7.1.2}$$

The angular velocity tensor $\underset{\sim}{\omega}$ is antisymmetric and instead of nine functions $d^{i}_{(\mu)}(\underset{\sim}{x},t)$ we have to consider only three independent components of $\underset{\sim}{\omega}$.

From (7.34) and (7.25) we easily obtain three independent equations for the determination of the angular velocity tensor,

$$\varrho \left[I^{i\ell} (\dot{\omega}_{\ell}{}^{j} + \omega_{\ell}{}^{n}\, \omega_{n}{}^{j}) \right]_{[ij]} = t^{[ij]} + \overset{*}{m}{}^{ijk}{}_{,k} + g\overset{*}{\ell}{}^{ij}\,, \tag{7.1.3}$$

where

$$I^{ij} = I^{ji} = i^{\lambda\mu}\, d^{i}_{(\lambda)}\, d^{j}_{(\mu)}\,, \tag{7.1.4}$$

which represent the density of inertia coefficients per unit mass.

According to (5.1.10), for a particle consisting of n mass points the directors $\underset{\sim}{d}_{(\lambda)}$ are position vectors of the mass points with respect to the centres of mass of corresponding particles, and therefore we have

$$I^{\alpha\beta} = \frac{1}{m} \sum_{\upsilon=1}^{n} m_\upsilon \delta_\upsilon^\pi \delta_\upsilon^\varrho \xi^\alpha_{(\pi)} \xi^\beta_{(\varrho)} = \frac{1}{m} \sum_{\upsilon=1}^{n} m_\upsilon \xi^\alpha_{(\upsilon)} \xi^\beta_{(\upsilon)} .$$

Hence, $I^{\alpha\beta}$ are components of the inertia tensor of the particle considered. Also for the media with microstructure when a curvilinear system of coordinates x^i is introduced into (5.2.32) and (5.2.36) and when the vectors $\underset{\sim}{\chi}_\alpha$ are identified with the directors, a relation of the form of (7.1.4) will be obtained .

Taking the material derivative of I^{ij} (with $i^{\lambda\mu}$ independent of time) and using (7.1.1) we find

$$\frac{\partial I^{ij}}{\partial t} + I^{ij}_{\ \ ;k} v^k - I^{kj} \omega_k^{\ \cdot i} - I^{ik} \omega_k^{\ \cdot j} = 0 . \qquad (7.1.5)$$

This relation Eringen [91] calls the conservation of micro-inertia.

The complete set of equations of motion of a Cosserat continuum consists now of the following equations:

$$1)\quad \frac{\partial \varrho}{\partial t} + \left(\varrho v^{k}\right)_{,k} = 0 \;,$$

$$2)\quad \varrho \ddot{x}^{i} = t^{\cdot ij}{}_{,j} + \varrho f^{i} \;, \tag{7.1.6}$$

$$3)\quad \dot{I}^{ij} - I^{kj}\omega_{k}^{\cdot i} - I^{ik}\omega_{k}^{\cdot j} = 0$$

$$4)\quad \varrho\left[I^{li}\left(\dot{\omega}_{l}^{\cdot j} + \omega_{l}^{\cdot n}\omega_{n}^{\cdot j}\right)\right]_{[ij]} = t^{[ji]} + \overset{*}{m}{}^{ijk}{}_{,k} + \varrho \overset{*}{l}{}^{ij} \,.$$

A very interesting field of application of the theory of Cosserat media is the dynamics of granular media. Oshima [277] considered a model of a granular medium assuming that there are no director forces and director stresses and disregarding the coefficients of inertia of the granulae. Cowin [53] assumes the same kinematical model as Oshima. A more general approach is offered by the theory of micropolar media (Eringen [90 - 94]), but this theory is not yet explicitly applied to granular materials. Satake considered first [305] a granular medium in the absence of volume and director forces and moments, but in a recent paper [306] he included these forces into the consideration. Satake approaches the problem from the point of view of a purely linear theory and, the same as Oshima, he assumes certain a priori prescribed mechanical properties of the medium (elasticity). Cowin admits the medium to be a composition of elastic and viscous phases.

A much wider field of applications is offered if the directors do not constitute rigid triedra. The micropolar theory of Eringen generalizes the idea of a Cosserat continuum admiting the directors to deform, but restricting the number of directors to three. A large number of applications is covered by the later development of the micropolar theory. (Cf. e.g. Ariman [11,12] , Ariman and Cakmak [13], Ariman, Cakmak and Hill [15], Askar and Cakmak [16], Askar , Cakmak and Ariman [17]).

A strúctural model of a micropolar continuum (Askari and Cakmak [16]), which consists of a two-dimensional network of orientable points, joined by extensible and flexible rods, yields the equations very close to those obtained by Eringen and Suhubi [99,352], Eringen [93] and Mindlin [220,223] , starting with continuum principles.

7. 2. Bodies with One Director.

The theory of liquid crystals and anisotropic fluids of Ericksen [74a,74 - 89] (cf. also Leslie [204,205]) is based on the assumption that the media such as liquid crystals and suspensions of large molecules may be described by the position vectors of the particles and by a simple director field. The differential equations of motion may be obtained from our equations (7.32-34), together with the continuity equation (7.11):

$$\dot{\varrho} + \varrho v^{k}{}_{,k} = 0 \ ,$$

$$\varrho \ddot{x}^{i} = t^{ij}{}_{,j} + \varrho f^{i} \ ,$$

$$\ddot{d}^{i} = k^{i} , \tag{7.2.1}$$

$$t^{[ij]} = k^{[i} d^{j]} \ ,$$

To obtain these equations from (7.11) and (7.25) we have to assume that there are no director stresses $\underset{\sim}{h}$, no couple-stresses $\underset{\sim}{m}$ and no volume couples $\underset{\sim}{\ell}$. Under such assumptions the equation $(7.2.1)_4$ is a direct consequence of the moment of momentum equation (7.24).

Another example of a one-director theory is the theory of <u>Cosserat surfaces</u>. (Green, Naghdi and Wainwraight [127], Green and Naghdi [121 - 125]).

A Cosserat surface is a two-dimensional material manifold s to each point of which a simple director field is assigned. This surface is embedded in a 3-dimensional Euclidean space. Let x^{α}, $\alpha = 1,2$ be coordinates defining points on the surface and $x^{3} = 0$ at all points of the surface. The position vector of a point of s at time t and the director $\underset{\sim}{d}$ are functions of position x^{α} and of time t,

$$\underset{\sim}{r} = \underset{\sim}{r}\left(x^{\alpha}, t\right) \ , \quad \underset{\sim}{d} = \underset{\sim}{d}\left(x^{\alpha}, t\right) . \tag{7.2.2}$$

The base vectors along curves x^{α} are $\underset{\sim}{g}_{\alpha}$ and we assume that $\underset{\sim}{g}_3$ is the unit normal vector to s, so that

$$\underset{\sim}{g}_{\alpha} \cdot \underset{\sim}{g}_{\beta} = g_{\alpha\beta} \, , \quad \left(\underset{\sim}{g}_{\alpha} \times \underset{\sim}{g}_{\beta}\right) \cdot \underset{\sim}{g}_3 > 0 \quad , \left(\alpha \neq \beta\right)$$

$$\underset{\sim}{g}^{\alpha} \cdot \underset{\sim}{g}_{\beta} = \delta^{\alpha}_{\beta} \, , \quad \underset{\sim}{g}^{3} \cdot \underset{\sim}{g}_{\alpha} = 0 \quad , \quad \underset{\sim}{g}_3 \cdot \underset{\sim}{g}_3 = 1 \tag{7.2.3}$$

$$\underset{\sim}{g}^{\alpha} = g^{\alpha\beta} \, \underset{\sim}{g}_{\beta}$$

From the theory of surfaces it is known that the second fundamental tensor $b_{\alpha\beta}$ of a surface is defined by

$$\underset{\sim}{g}_{\alpha/\beta} = b_{\alpha\beta} \, \underset{\sim}{g}_3 \, , \quad \frac{\partial \underset{\sim}{g}_3}{\partial x^{\beta}} = - b^{\alpha}_{\beta} \, \underset{\sim}{g}_{\alpha} \tag{7.2.4}$$

Where "|" denotes covariant differentiation with respect to the metric form on the surface s .

Let $\underset{\sim}{F}$ and $\underset{\sim}{k}$ be the assigned force and the assigned director force per unit mass,

$$\underset{\sim}{F} = F^{\alpha} \, \underset{\sim}{g}_{\alpha} + F^{3} \, \underset{\sim}{g}_3$$

$$k = k^{\alpha} \, \underset{\sim}{g}_{\alpha} + k^{3} \, \underset{\sim}{g}_3 \tag{7.2.5}$$

The stress vector $\underset{\sim}{t}^{\alpha}$ is to be regarded as a force per unit length of a curve bounding an area on

S . The same holds for the director stress $\underset{\sim}{h}^{\alpha}$, so that

$$\underset{\sim}{t}^{\alpha} = t^{\beta\alpha} \underset{\sim}{g}_{\beta} + t^{3\alpha} \underset{\sim}{g}_{3} ,$$

$$\underset{\sim}{h}^{\alpha} = h^{\beta\alpha} \underset{\sim}{g}_{\beta} + h^{3\alpha} \underset{\sim}{g}_{3} . \qquad (7.2.6)$$

To write the equation of continuity (7.11) in the appropriate form we have to calculate the divergence of the velocity vector $\underset{\sim}{v}$ considering (7.2.3). Let the velocity vector of a point on S be

$$\underset{\sim}{v} = v^{\alpha} \underset{\sim}{g}_{\alpha} + v^{3} \underset{\sim}{g}_{3} .$$

The Hamiltonian operator on the surface S is

$$\underset{\sim}{\nabla} = \underset{\sim}{g}^{\alpha} \partial_{\alpha}$$

and we have

$$v^{i}_{;i} = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \underset{\sim}{g}^{\alpha} \cdot \left(\underset{\sim}{g}_{\beta} \partial_{\alpha} v^{\beta} + v^{\beta} \partial_{\alpha} \underset{\sim}{g}_{\beta} + \underset{\sim}{g}_{3} \partial_{\alpha} v^{3} + v^{3} \partial_{\alpha} \underset{\sim}{g}_{3} \right) ,$$

which in virtue of (7.2.4) becomes

$$v^{i}_{;i} = v^{\alpha}_{|\alpha} - b^{\alpha}_{\alpha} \; v^{3}$$

Substituting this in (7.11) we obtain the continuity equation in the form

$$\dot{\varrho} + \varrho \left(v^{\alpha}{}_{|\alpha} - b^{\alpha}_{\alpha}\, v^{3} \right) = 0 \tag{7.2.7}$$

Differentiation of the stress vectors $\underset{\sim}{t}^{\alpha}$ gives

$$\underset{\sim}{t}^{\alpha}{}_{,\gamma} = t^{\beta\alpha}{}_{|\gamma}\, \underset{\sim}{g}_{\beta} + t^{\beta\alpha}\, \underset{\sim}{g}_{\beta|\gamma} + t^{3\alpha}{}_{|\gamma}\, \underset{\sim}{g}_{3} + t^{3\alpha}\, \underset{\sim}{g}_{3|\gamma} ,$$

which, because of (7.2.4), reduces to

$$\underset{\sim}{t}^{\alpha}{}_{,\gamma} = \left(t^{\beta\alpha}{}_{|\gamma} - b^{\beta}_{\gamma}\, t^{3\alpha} \right) \underset{\sim}{g}_{\beta} + \left(t^{3\alpha}{}_{|\gamma} + b_{\beta\gamma}\, t^{\beta\alpha} \right) \underset{\sim}{g}_{3} . \tag{7.2.8}$$

We obtain the similar expression for the derivatives of the director stress vectors $\underset{\sim}{h}^{\alpha}$,

$$\underset{\sim}{h}^{\alpha}{}_{,\gamma} = \left(h^{\beta\alpha}{}_{|\gamma} - b^{\beta}_{\gamma}\, h^{3\alpha} \right) \underset{\sim}{g}_{\beta} + \left(h^{3\alpha}{}_{|\gamma} + b_{\beta\gamma}\, h^{\beta\alpha} \right) \underset{\sim}{g}_{3} . \tag{7.2.9}$$

From the vectorial form of the differential equations of motion (7.19),

$$\varrho \dot{\underset{\sim}{v}} = \underset{\sim}{t}^{i}{}_{,i} + \varrho \underset{\sim}{f} ,$$

by scalar multiplication with the base vectors $\underset{\sim}{g}^{\alpha}$ and $\underset{\sim}{g}^{3}$ we obtain the following three differential equations of motion:

$$\begin{aligned} \varrho a^{\beta} &= t^{\beta\alpha}{}_{|\alpha} - b^{\beta}_{\alpha}\, t^{3\alpha} + \varrho f^{\beta} , \\ \varrho a^{3} &= t^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} t^{\beta\alpha} + \varrho f^{3} , \end{aligned} \tag{7.2.10}$$

where $\underset{\sim}{a}$ is the acceleration vector with the components (a^{α}, a^{3}).

Green, Naghdi and Wainwright [127] assumed that there is an additional physical director force which they denoted by $\underset{\sim}{m}^{\alpha}$ and which acts over the curves x^{α}.

For the motion of the director $\underset{\sim}{d}(\underset{\sim}{x}^{\alpha}, t)$ we shall write also the equations (7.33) in the compact (vectorial) form to which our equations (7.33) reduce in the case of a single director field,

$$\underset{\sim}{m} = \underset{\sim}{h}^{j}{}_{,j} + \varrho\left(\underset{\sim}{k} - i\ddot{\underset{\sim}{d}}\right),$$

where $\underset{\sim}{m}$ represents the additional physical force, and ϱi is the inertia density at the points of the surface. Since the director stress depends only upon x^{α}, we may write

$$\underset{\sim}{m} = \underset{\sim}{h}^{\alpha}{}_{|\alpha} + \varrho\left(k - i\ddot{\underset{\sim}{d}}\right), \qquad (7.2.11)$$

and by scalar multiplication with $\underset{\sim}{g}^{\beta}$ and $\underset{\sim}{g}^{3}$ this equation gives the following equations in the component form:

$$m^{\beta} = h^{\beta\alpha}{}_{|\alpha} - b^{\beta}_{\alpha} h^{3\alpha} + \varrho\left(k^{\beta} - i\ddot{d}^{\beta}\right),$$

$$m^{3} = h^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} h^{\beta\alpha} + \varrho\left(k^{3} - i\ddot{d}^{3}\right). \qquad (7.2.12)$$

The equations (7.2.7),(7.2.10) and (7.2.12) represent the basic set of equations for a Cosserat surface. In the original paper of Green, Naghdi and Wainwright, as well as in the subsequent work of Green and Naghdi, the equations of motion are derived directly from the considerations of the surface, and not from a general theory of the generalized Cosserat continua.

In the applications of the theory of Cosserat surfaces to the theory of elastic plates and shells it was assumed that in the initial configuration $\underset{\sim}{D}_{(\alpha)} = 0$ and $\underset{\sim}{D}_{(3)} = \underset{\sim}{e}_3$. For further references see e.g. [121,122,124,251].*

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* Ericksen and Truesdell [72] gave a very elegant and exact theory of strain and stress in shells, assuming that three directors are assigned to each point of the surface. The work of Cohen and DeSilva [46,46a] on elastic surfaces is based also on the assumption that three directors are assigned to the points of the surface, and they based their work on the results of Ericksen and Truesdell. Their equations of equilibrium may be derived directly from our equations (7.32, 33). However, in the theory of elastic membranes [47] they consider, at the points of the membrane, a single director field. The director is taken to be normal to the surface and the only deformation it suffers is the deformation of its magnitude.

7. 3. Bodies with Two Directors. A Theory of Rods.

As an example of two-director bodies we shall consider the theory of rods by Green and Laws [113,116] , which was applied to the theory of elastic rods by Green ,Naghdi and Laws [115].

A rod is considered as a curve ℓ , imbedded in Euclidean three-dimensional space. At each point of the curve there are two assigned directors. Let Θ be a convected coordinate* defining points on the curve, and let $\tilde{r}$ be the position vector, relative to a fixed origin, of a point on the curve,

$$\tilde{r} = \tilde{r}\,(\Theta, t) \qquad (7.3.1)$$

Let $\tilde{d}_{(1)} = \tilde{g}_1$ and $\tilde{d}_{(2)} = \tilde{g}_2$ be the assigned directors and let the vector $\tilde{g}_3$, tangential to the curve,

$$\tilde{g}_3 = \frac{\partial \tilde{r}}{\partial \Theta} \,, \qquad (7.3.2)$$

*Convected coordinates, by the definition, move with the body and deform with it so that the numerical values of such coordinates for each individual point of a body remain unchanged.

be considered as the third vector of the triad, so that

$$g = (\underset{\sim}{g}_1 \times \underset{\sim}{g}_2) \cdot \underset{\sim}{g}_3 > 0 .$$

Along ℓ we may construct the reciprocal triad $\underset{\sim}{g}^i$, such that

$$\underset{\sim}{g}_i \; \underset{\sim}{g}_j = g_{ij} , \quad \underset{\sim}{g}^i \underset{\sim}{g}^j = g^{ij} , \quad g^{ij} \underset{\sim}{g}_j = \underset{\sim}{g}^i \tag{7.3.3}$$

$$\underset{\sim}{g}^i \; \underset{\sim}{g}_j = \delta^i_j \; , \quad g^{ij} g_{jk} = \delta^i_k .$$

We shall introduce the notation

$$\frac{\partial \underset{\sim}{g}_i}{\partial \Theta} \cdot \underset{\sim}{g}_j = \varkappa_{ij} , \quad g^{jk} \cdot \varkappa_{ij} \equiv \underset{\sim}{g}^k \cdot \frac{\partial \underset{\sim}{g}_i}{\partial \Theta} = \varkappa_i^{\;\,k} . \tag{7.3.4}$$

It is assumed that the stress acts along the curve ℓ . The stress vector $\underset{\sim}{t}(\Theta, \underset{\sim}{n})$ according to (6.11) is

$$\underset{\sim}{t}(\Theta, \underset{\sim}{n}) = t^i(\Theta, \underset{\sim}{n}) \; \underset{\sim}{g}_i = t^{i3} \underset{\sim}{g}_i \; n_3 \equiv t^i \underset{\sim}{g}_i . \tag{7.3.5}$$

Since $n_3 = n = 1$, the components of the stress tensor reduce to $t^{i3} = t^i$. [illegible] total resultant stress exerted on a segment (Θ_1 , Θ_2) of a rod is

$$\underset{\sim}{t}(\Theta_2) - \underset{\sim}{t}(\Theta_1) = \left[\underset{\sim}{t}(\Theta) \right]_{\Theta_1}^{\Theta_2} \tag{7.3.6}$$

For the director stress vectors $\underset{\sim}{h}^{(\alpha)}$, according to (6.19) and (6.20) we may also write

$$\underset{\sim}{h}^{(\alpha)}(\Theta, \underset{\sim}{n}) = h^{(\alpha)i}(\Theta, \underset{\sim}{n})\, \underset{\sim}{g}_i = h^{(\alpha)i3}(\Theta)\, \underset{\sim}{g}_i n_3 = h^{(\alpha)i} \underset{\sim}{g}_i, \tag{7.3.7}$$

and the moment of the director stresses, defined by (6.27), becomes

$$\underset{\sim}{\mu} \equiv \underset{\sim}{d}_{(\alpha)} \times h^{(\alpha)i} \underset{\sim}{g}_i = h^{(\alpha)i} \underset{\sim}{g}_\alpha \times \underset{\sim}{g}_i \; . \tag{7.3.8}$$

The resultant moment of the director stresses exerted on the segment (Θ_1 , Θ_2) of the rod will be according to (6.28),

$$\underset{\sim}{\mu}(\Theta_2) - \underset{\sim}{\mu}(\Theta_1) = \left[\underset{\sim}{\mu}(\Theta)\right]_{\Theta_1}^{\Theta_2} \tag{7.3.9}$$

If we assume that there are no body couples $\underset{\sim}{\ell}$ and no couple stresses $\underset{\sim}{m}$ acting on the curve ℓ , and since the mass dm of the line element ds is given by

$$dm = \varrho\, ds = \varrho \sqrt{g_{33}}\, d\Theta , \tag{7.3.10}$$

the law of conservation of mass and the principles of balance of momentum (7.14) and the moment of momentum (7.21) obtain the form

$$\frac{d}{dt} \int_{\Theta_1}^{\Theta_2} \varrho \sqrt{g_{33}}\, d\Theta = 0 , \tag{7.3.11}$$

$$\frac{d}{dt}\int_{\Theta_1}^{\Theta_2} \varrho \underset{\sim}{v} \sqrt{g_{33}}\, d\Theta = \int_{\Theta_1}^{\Theta_2} \varrho \underset{\sim}{f} \sqrt{g_{33}}\; d\Theta + \left[\underset{\sim}{t}(\Theta)\right]_{\Theta_1}^{\Theta_2} , \qquad (7.3.12)$$

$$\frac{d}{dt}\int_{\Theta_1}^{\Theta_2} \varrho \left(\underset{\sim}{r}\times\underset{\sim}{v} + i^{\lambda\mu}\underset{\sim}{d}_{(\lambda)}\times\dot{\underset{\sim}{d}}_{(\mu)}\right)\sqrt{g_{33}}\; d\Theta =$$

$$= \int_{\Theta_1}^{\Theta_2} \varrho \left(\underset{\sim}{r}\times\underset{\sim}{f} + \underset{\sim}{d}_{(\lambda)}\times\underset{\sim}{k}^{(\lambda)}\right)\sqrt{g_{33}}\; d\Theta + \left[\underset{\sim}{r}\times\underset{\sim}{t} + \underset{\sim}{\mu}\right]_{\Theta_1}^{\Theta_2} . \qquad (7.3.13)$$

Since Θ_1 and Θ_2 are convected coordinates of two points of the curve and remain unchanged under the deformations of the curve, it follows from (7.3.11) that $\varrho\sqrt{g_{33}}$ is independent of time and the law of conservation of mass may be written in the form

$$\varrho\sqrt{g_{33}} = \mathcal{Y}(\Theta) , \qquad (7.3.14)$$

where $\mathcal{Y}(\Theta)$ is an arbitrary function of position.

Using the simple relation

$$\left[f\;(\Theta)\right]_{\Theta_1}^{\Theta_2} = \int_{\Theta_1}^{\Theta_2} \frac{d\,f(\Theta)}{d\,\Theta}\; d\,\Theta ,$$

the equations (7.3.12) and (7.3.13) obtain the form+

$$\int_{\Theta_1}^{\Theta_2} \varrho \dot{\underset{\sim}{v}} \sqrt{g_{33}}\; d\Theta = \int_{\Theta_1}^{\Theta_2} \left(\varrho \underset{\sim}{f}\sqrt{g_{33}} + \frac{\partial \underset{\sim}{t}}{\partial\Theta}\right) d\Theta , \qquad (7.3.15)$$

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* We take $i^{\lambda\mu}$ to be independent of time [116].

$$\int_{\Theta_1}^{\Theta_2} \varrho\left(\underset{\sim}{r}\times\dot{\underset{\sim}{v}} + i^{\lambda\mu}\underset{\sim}{d}_{(\lambda)}\times\ddot{\underset{\sim}{d}}_{(\mu)}\right)\sqrt{g_{33}}\, d\Theta =$$

$$= \int_{\Theta_1}^{\Theta_2}\left[\varrho\left(\underset{\sim}{r}\times\underset{\sim}{f} + \underset{\sim}{d}_{(\lambda)}\times\underset{\sim}{k}^{(\lambda)}\right)\sqrt{g_{33}} + \frac{\partial}{\partial\Theta}\left(\underset{\sim}{r}\times\underset{\sim}{t}+\underset{\sim}{\mu}\right)\right] d\,\Theta\,. \tag{7.3.16}$$

These two equations must be valid for an arbitrary segment (Θ_1 , Θ_2), which yields the local form of the equations of balance, i.e. we get the equations of motion;

$$\varrho\dot{\underset{\sim}{v}} = \varrho\underset{\sim}{f} + \frac{1}{\sqrt{g_{33}}}\frac{\partial\underset{\sim}{t}}{\partial\Theta}\;; \tag{7.3.17}$$

$$\varrho\, i^{\lambda\mu}\underset{\sim}{d}_{(\lambda)}\times\ddot{\underset{\sim}{d}}_{(\mu)} = \varrho\underset{\sim}{\Gamma} + \frac{1}{\sqrt{g_{33}}}\left(\frac{\partial\underset{\sim}{\mu}}{\partial\Theta} + \underset{\sim}{g}_3\times\underset{\sim}{t}\right)\,, \tag{7.3.18}$$

$$\left(\underset{\sim}{\Gamma} \equiv \underset{\sim}{d}_{(\lambda)}\times\underset{\sim}{k}^{(\lambda)}\right).$$

where we have applied (7.3.17) to simplify the equation (7.3.18).

To write the equations of motion in the component form we have to apply the formula

$$\frac{\partial\underset{\sim}{T}}{\partial\Theta} = \left(\frac{\partial T^i}{\partial\Theta} + T^m\,\varkappa_{m}^{\cdot\, i}\right)\underset{\sim}{g}_i\;,$$

where $\underset{\sim}{T}\,(\Theta\;,t\,)$ is a tensor defined along the curve ℓ , and $\varkappa_{m}^{\cdot\, i}$ is defined by (7.3.4). Hence, the scalar products of the vectorial equations (7.3.17, 18) with the base vectors $\underset{\sim}{g}^i$ give the following six differential equations of motion:

$$\varrho \dot{\underset{\sim}{v}} \cdot \underset{\sim}{g}^i = \varrho f^i + \frac{1}{\sqrt{g_{33}}} \left(\frac{\partial t^i}{\partial \Theta} + \varkappa_m^{\cdot i} \, t^m \right) , \qquad (7.3.19)$$

$$\varrho i^{\lambda\mu} \left(\underset{\sim}{d}_{(\lambda)} \times \ddot{\underset{\sim}{d}}_{(\mu)} \right) \cdot \underset{\sim}{g}^i = \varrho \Gamma^i + \frac{1}{\sqrt{g_{33}}} \left[\frac{\partial \mu^i}{\partial \Theta} + \varkappa_m^{\cdot i} \mu^m + \left(\underset{\sim}{g}^i \times \underset{\sim}{g}_3 \right) \underset{\sim}{t} \right] \qquad (7.3.20)$$

Since we have

$$\left(\underset{\sim}{g}^i \times \underset{\sim}{g}_3 \right) \cdot \underset{\sim}{t} = g^{ij} \left(\underset{\sim}{g}_j \times \underset{\sim}{g}_3 \right) \cdot \underset{\sim}{t} = g^{ij} \, \epsilon_{j3k} \, \underset{\sim}{g}^k \, \underset{\sim}{t} = g^{ij} \, \epsilon_{j3k} \, t^k ,$$

and k must be different from 3 according to the definition of the $\underset{\sim}{\epsilon}$ -tensors, the equation (7.3.20) may be also written in the form

$$\varrho i^{\lambda\mu} \left(\underset{\sim}{d}_{(\lambda)} \times \ddot{\underset{\sim}{d}}_{(\mu)} \right) = \varrho \Gamma^i + \frac{1}{\sqrt{g_{33}}} \left(\frac{\partial \mu^i}{\partial \Theta} + \varkappa_m^{\cdot i} \mu^m + g^{ij} \, \epsilon_{j3\alpha} \, t^\alpha \right) . \qquad (7.3.21)$$

The equations (7.3.14),(7.3.19) and (7.3.21) represent the basic set of the equations of motion in the general theory of rods by Green and Laws.

Ericksen and Truesdell [72] assigned to each point of a rod three directors and discussed in detail the state of strain and stress from this point of view, without making any constitutive assumptions on the mechanical properties of the material of the rod. In their criticism of the classical description of the strain in a rod, the inadequacy of the classic al description of twist and the insufficiencies of the theories which do not assume the material to be oriented in the sense of the generalized Cosserat continuum become obvious. Cohen's theory [45] of elastic rods is based on the kinematics and statics of Ericksen and Truesdell. An independent approach to the theory of rods, but with the same form of the equations of motion as (7.3.14,19,21) is presented by Suhubi [354] .

8. Some Applications of Classical Thermodynamics.

During the last ten years a great work has been done on the development of thermodynamics of continua. Our interest here is primarly directed towards the application of thermodynamics in the derivation of the constitutive equations, and we shall restrict our considerations to the classical formulations of the first law and the second law of thermodynamics. The readers interested in the modern treatments, for the survey of the modern contributions up to 1965, may be referred to the book by Truesdell and Noll [379], and for the later work to the papers by e.g. Chen [44], Green and Laws [144], Green and Rivlin [132], Kline [180], Kline and Allen [181], Leigh [202], Truesdell [377,378], Uhlhorn [380] etc.

The experience shows that mechanical processes can not be separated from thermal phenomena. Mechanical work may make a body hotter, or, heating may produce certain mechanical effects, such as e.g. thermal dilatations and thermoelastic stresses.

To indicate how hot is a body the temperature Θ is introduced as a fundamental entity. It is assumed that there exists an absolute zero $\Theta = 0$ which is the lowest bound of Θ and for all processes $\Theta > 0$

It is postulated that the total energy of a body is the sum of the kinetic energy produced by the motion of the mass points of the body and of an internal energy E .

For the internal energy it is assumed that it is an absolutly continuous function of mass, so that for a part v of a body it may be written

$$E = \int_{v} \varepsilon \, dm = \int_{v} \varrho \varepsilon \, dv , \tag{8.1}$$

where ε is the specific internal energy,

$$\varepsilon = \varepsilon(\underset{\sim}{x}, t) . \tag{8.2}$$

The increment of the total energy per unit time depends on the rate P at which the mechanical forces do work (the mechanical working), and on the total input (output) of the nonmechanical working (heat), which we shall denote by Q .

Mechanical working is the rate at which the body forces $\underset{\sim}{f}$, the director forces $\underset{\sim}{k}^{(\lambda)}$, the body couples $\underset{\sim}{l}$, the stresses $\underset{\sim}{t}$, the director stresses $\underset{\sim}{h}^{(\lambda)}$ and couple stresses $\underset{\sim}{m}$ do work. According to the definitions of the section 6, $\underset{\sim}{t}$, $\underset{\sim}{h}^{(\lambda)}$ and $\underset{\sim}{m}$ are defined for the points on the boundary s of a part v of the body considered. Therefore the working of $\underset{\sim}{f}, \underset{\sim}{k}^{(\lambda)}$ and $\underset{\sim}{l}$ is to be summed over all points of v, and the working of the forces $\underset{\sim}{t}, \underset{\sim}{h}^{(\lambda)}$ and $\underset{\sim}{m}$ over the points on the bounding surface s.

The kinetic energy T of a part v of a body we shall assume to be in the general case represented by the expression of the form (5.1.26),

$$T = \frac{1}{2} \int_v \varrho \left(\dot{x}^i \dot{x}_i + i^{\lambda\mu} \dot{d}^i_{(\lambda)} \dot{d}_{(\mu)i} \right) dv \,, \qquad (8.3)$$

where we assume that the coefficients $i^{\lambda\mu}$ are independent of time. The rate of the kinetic energy will be now

$$\dot{T} = \int_v \varrho \left(\ddot{x}^i \dot{x}_i + i^{\lambda\mu} \ddot{d}^i_{(\lambda)} \dot{d}_{(\mu)i} \right) dv \,. \qquad (8.4)$$

Using the equations of motion (7.32,33,34),

$$\begin{aligned} \varrho \ddot{x}^i &= t^{ij}{}_{,j} + \varrho f^i \,, \\ \varrho\, i^{\lambda\mu} \ddot{d}^i_{(\mu)} &= h^{(\lambda)ij}{}_{,j} + \varrho k^{(\lambda)i} \,, \\ \varrho \dot{\sigma}^{ij} &= t^{[ji]} + \varrho \overset{*}{l}{}^{ij} + \overset{*}{m}{}^{ijk}{}_{,k} \end{aligned} \qquad (8.5)$$

where owing to the tensorial character of the quantities involved we may from (7.25) write the corresponding expressions for curvilinear coordinates,

$$\sigma^{ij} = i^{\lambda\mu} d^{[i}_{(\lambda)} \dot{d}^{j]}_{(\mu)} ,$$

$$\overset{*}{\ell}{}^{ij} = \ell^{ij} + d^{[i}_{(\lambda)} k^{(\lambda)j]} , \quad \overset{*}{m}{}^{ijk} = m^{ijk} + d^{[i}_{(\lambda)} h^{(\lambda)j]k} , \tag{8.6}$$

and for the rate of the kinetic energy we have the expression

$$\dot{T} = \oint_S \left(t^{ik} \dot{x}_i + h^{(\lambda)ik} \dot{d}_{(\lambda)i} - m^{ijk} w_{ij} \right) ds_k + \int_v \varrho \left(f^i \dot{x}_i + k^{(\lambda)i} \dot{d}_{(\lambda)i} - \ell^{ij} w_{ij} \right) dv - W \tag{8.7}$$

By W we have denoted here

$$W = \int_v w \, dv = \int_v \left(t^{(ij)} d_{ij} + h^{(\lambda)jk} d^{i}_{(\lambda),k} w_{ij} + h^{(\lambda)jk} \dot{d}_{(\lambda)j,k} - m^{ijk} w_{ij,k} \right) dv \tag{8.8}$$

The right-hand side of (8.7) represents the mechanical working P .

The non-mechanical working Q is assumed to rise from surface and volume densities,

$$Q = \oint_S q^k ds_k + \int_v h \, dm , \tag{8.9}$$

where $\underset{\sim}{q}$ is the rate at which heat flows through the surface, and h is the heat generation per unit mass (source). $\underset{\sim}{q}$ is often called the heat flux vector.

The first law of thermodynamics postulates that

$$\dot{T} + \dot{E} = P + Q \tag{8.10}$$

From (8.10), using (8.1) and (8.5-9), we obtain

$$\varrho \dot{\varepsilon} = w + q^{K}_{,K} + \varrho h \ , \tag{8.11}$$

which represents the local law of balance of energy. According to (8.1),(8.3),(8.8) and (8.9) we see that the first law of thermodynamics is also of the form of a balance law, and therefore it represents in the global form (8.10) the law of balance of the total energy.

From experience we know that at least one part of the mechanical working goes into heat, and the rest is again available for the mechanical work. Therefore we assume that W may be decomposed into a reversible part ${}_{E}W$ and into an irreversible part ${}_{D}W$ which may also be called the dissipative part of W, such that

$$W = {}_{E}W + {}_{D}W \tag{8.12}$$

The reversible part of working goes into the potential energy Σ, such that $\dot{\Sigma} = {}_E W$ and

$$\Sigma = \int_{v} {}_E w \, dv = \int_{v} \varrho \, \sigma \, dv \ , \tag{8.13}$$

where σ is the specific strain energy, or the elastic potential.

The difference between the rate of the specific internal energy and the rate of reversible work we shall denote by $\Theta \dot{\eta}$, so that

$$\varrho \dot{\varepsilon} = {}_E w + \varrho \, \Theta \dot{\eta} \ , \tag{8.14}$$

where η represents the specific entropy and is defined per unit mass and per unit temperature, and from (8.11) we obtain

$$\varrho \, \Theta \dot{\eta} = {}_D w + q^k_{\ ,k} + \varrho h \ , \tag{8.15}$$

which represents the equation for production of specific entropy.

If we assume that all stresses, director stresses and stress-couples may be decomposed into parts which do reversible work (${}_E\underset{\sim}{t}$, ${}_E\underset{\sim}{h}^{(\lambda)}$, ${}_E\underset{\sim}{m}$), and which do dissipative work (${}_D\underset{\sim}{t}$, ${}_D\underset{\sim}{h}^{(\lambda)}$, ${}_D\underset{\sim}{m}$), we may write

$$\underset{\sim}{t} = {}_E\underset{\sim}{t} + {}_D\underset{\sim}{t} \ ; \quad \underset{\sim}{h}^{(\lambda)} = {}_E\underset{\sim}{h}^{(\lambda)} + {}_D\underset{\sim}{h}^{(\lambda)} \ ; \quad \underset{\sim}{m} = {}_E\underset{\sim}{m} + {}_D\underset{\sim}{m} \ . \tag{8.16}$$

From (8.15) it follows that any portion of the stress, director stresses and couple-stresses which does recoverable work makes no contribution to the entropy (Truesdell and Toupin [375]).

On the basis of (8.11,12 and 15) we may write

$$\int_v \varrho \dot{\eta}\, dv - \int_v \left(\frac{1}{\Theta} q^k_{,k} + \frac{\varrho}{\Theta} h \right) dv = \int_v \frac{1}{\Theta}\, {}_D W \qquad (8.17)$$

The quantity H defined by

$$H \equiv \int_v \varrho \eta \, dv$$

is called the <u>total entropy</u>. Now, from (8.17) we obtain

$$\dot{H} - \oint_s \frac{q^k ds_k}{\Theta} - \int_v \frac{\varrho h}{\Theta} dv = \int_v \frac{1}{\Theta} \left({}_D W + \frac{\Theta_{,k} q^k}{\Theta} \right) dv . \qquad (8.18)$$

The <u>postulate of irreversibility</u>, also called the <u>second law of thermodynamics</u> states that

$$\dot{H} - \oint_s \frac{q^k ds_k}{\Theta} - \int_v \frac{\varrho h}{\Theta} dv \geq 0 \quad . \qquad (8.19)$$

In the form (8.19) this law is also known as the <u>Clausius-Duhem inequality</u> , or the <u>entropy inequality</u>. In the local form this law reads

$$\varrho \Theta \dot{\eta} - \varrho h - q^k_{,k} + \frac{1}{\Theta} \Theta_{,k} q^k \geq 0 \quad . \qquad (8.20)$$

Sometimes it is convenient to use the Helmholtz free energy ψ per unit mass, defined by the relation

$$\psi = \varepsilon - \Theta\eta \ . \tag{8.21}$$

On substituting this equation into (8.14) we find

$$\varrho\dot{\psi} + \varrho\eta\dot{\Theta} = {}_E w \ . \tag{8.22}$$

Using (8.11) we may rewrite (8.20) in the form which includes the mechanical working w,

$$-\varrho\dot{\varepsilon} + \varrho\Theta\dot{\eta} + w + \frac{1}{\Theta}\Theta_{,k}\, q^k \geq 0 \ , \tag{8.23}$$

and if we introduce the free energy into this inequality, it becomes

$$-\varrho\dot{\psi} - \varrho\eta\dot{\Theta} + w + \frac{1}{\Theta}\Theta_{,k}\, q^k \geq 0 \ . \tag{8.24}$$

A process in which

$\dot{\Theta} = 0$ is called isothermal

$Q = 0$ is called adiabatic,

$\dot{\eta} = 0$ is called isentropic

$\dot{\varepsilon} = 0$ is called isoenergetic

When in (8.19) we have the equality, we

have the case of equilibrium and the corresponding process is reversible.

From (8.14) and (8.22) we see that the strain energy σ is equal to the internal energy ε if the process is isentropic, and that the strain energy σ is equal to the free energy ψ if the process is isothermal.

An inspection of (8.8) shows that for the recoverable part of working we may write

$$ {}_E W = {}_E t^{(ij)} d_{ij} + {}_E h^{(\lambda)jk} d^{i}_{(\lambda),k} w_{ij} + {}_E h^{(\lambda)k}_{j} \dot{d}^{j}_{(\lambda),k} - {}_E m^{ijk} w_{ij,k} . \quad (8.25) $$

Since

$$ {}_E t^{(ij)} d_{ij} = g_{il} \, {}_E t^{(ij)} v^{l}_{,j} , $$

$$ {}_E h^{(\lambda)jk} d^{i}_{(\lambda),j} w_{ij} = g_{il} d^{[i}_{(\lambda),k} \, {}_E h^{(\lambda)j]k} v^{l}_{,j} , \quad (8.26) $$

$$ {}_E m^{ijk} w_{ij,k} = g_{il} \, {}_E m^{ijk} v^{l}_{;jk} = g_{il} \, {}_E m^{i(jk)} v^{l}_{;jk} $$

and since

$$ v^{l}_{,j} = \overline{\dot{x}^{l}_{;L}} \, X^{L}_{;j} , $$

$$ v^{l}_{;jk} = (v^{l}_{;j})_{,k} = \left(\overline{\dot{x}^{l}_{;L}} X^{L}_{;j} \right)_{,k} = \overline{\dot{x}^{l}_{;LK}} \, X^{L}_{;j} X^{K}_{;k} + \overline{\dot{x}^{l}_{;L}} \, X^{L}_{;jk} , \quad (8.27) $$

$$\dot{d}^{j}_{(\lambda),k} = \overline{d^{j}_{(\lambda);K}}^{\bullet} \, X^{K}_{;k} \quad ,$$

we see that $_{E}W$ may be expressed as a linear function in the material derivatives of the gradients of deformation and of the directors,

$$\overline{x^{\ell}_{;L}}^{\bullet} \quad , \quad \overline{x^{\ell}_{;KL}}^{\bullet} \quad , \quad \overline{d^{j}_{(\lambda);K}}^{\bullet} \quad .$$

Thus,

$$\begin{aligned} {}_{E}W = g_{i\ell}\Big[{}_{E}t^{(ij)} X^{L}_{;j} + \dot{d}^{[i}_{(\lambda),k}\, {}_{E}h^{(\lambda)j]k}\, X^{L}_{;j} - {}_{E}m^{i(jk)} X^{L}_{;jk}\Big]\, \overline{x^{\ell}_{;L}}^{\bullet} \\ + {}_{E}h^{(\lambda)\cdot k}_{j}\, X^{K}_{;k}\, \overline{d^{j}_{(\lambda);K}}^{\bullet} - g_{i\ell}\, {}_{E}m^{i(jk)} X^{L}_{;j}\, X^{K}_{;w}\, \overline{x^{\ell}_{;KL}}^{\bullet} \quad . \end{aligned} \qquad (8.28)$$

According to (8.14), we may assume that the internal energy is a function of the deformation and director gradients and of the entropy,

$$\varepsilon = \varepsilon\left(x^{\ell}_{;L}\,,\, x^{\ell}_{;KL}\,,\, d^{\ell}_{(\lambda);K}\,,\, \eta\right)$$

so that*

$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial x^{\ell}_{;L}}\, \overline{x^{\ell}_{;L}}^{\bullet} + \frac{\partial \varepsilon}{\partial x^{\ell}_{;KL}}\, \overline{x^{\ell}_{;KL}}^{\bullet} + \frac{\partial \varepsilon}{\partial d^{\ell}_{(\lambda);K}}\, \overline{d^{\ell}_{(\lambda);K}}^{\bullet} + \frac{\partial \varepsilon}{\partial \eta}\, \dot{\eta} \qquad (8.29)$$

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* We follow here the procedure applied by Stojanović and Djurić [340-342] and by Stojanović, Djurić and Vujošević [343] in the case of elasticity.

Since the relation (8.14) must be valid for any process, it must be satisfied for arbitrary rates $\overline{\dot{x}^{\ell}_{;L}}$, $\overline{\dot{x}^{\ell}_{;KL}}$, $\overline{\dot{d}^{\ell}_{(\lambda);K}}$ and $\dot{\eta}$, which yields the following relations

$$ {}_{E}m^{i(jk)} = -\varrho\, g^{i\ell} \frac{\partial \varepsilon}{\partial x^{\ell}_{;KL}} x^{j}_{;K}\, x^{k}_{;L} \,, \tag{8.30}$$

$$ {}_{E}h^{(\lambda)jk} = \varrho\, g^{j\ell} \frac{\partial \varepsilon}{\partial d^{\ell}_{(\lambda);K}} x^{k}_{;K} \,, \tag{8.31}$$

$$ {}_{E}t^{(ij)} = \varrho \left[g^{i\ell} \left(\frac{\partial \varepsilon}{\partial x^{\ell}_{;L}} x^{j}_{;L} + \frac{\partial \varepsilon}{\partial x^{\ell}_{;KL}} x^{j}_{;KL} \right) + \left(g^{i\ell} \frac{\partial \varepsilon}{\partial d^{\ell}_{(\lambda);K}} d^{j}_{(\lambda);K} \right)_{[ij]} x^{k}_{;K} \right] \tag{8.32}$$

Hence, from the first law of thermodynamics we may obtain certain relations for the reversible parts of the symmetric part of the stress tensor, of the symmetric part of the couple-stress tensor and for the director-stress tensor. The dissipative parts remain indetermined.

Regarding the dissipative parts of the stress tensor, couple-stress tensor and director--stress tensors, there is a discussion whether or no the inequalities (8.19), or (8.23,24) present any restrictions. E.g. Kline [180] demonstrated that form these inequalities without additional assumptions further conclusions can not be made, but Leigh [202] (in the nonpolar case) finds certain restrictions and applies the second law of thermodynamics to plasticity and linear viscous flow. Green and Rivlin [132] obtained the differential equations of theories of generalized continua by the systematic use of the first and second law of thermodynamics, but applied the procedure only to the reversible case (cf. also Green and Laws [114]).

I find, however, that in some cases the principle of least irreversible force by Ziegler [416] is very useful.* Ziegler applied it to a number of cases in the theory of non-polar materials.

For polar materials this principle was applied for the derivation of the constitutive relations of plasticity and viscous flow by Komljenović [184], Plavšić [286-288], Plavšić and Stojanović [290] and Djurić [63a].

Ziegler assumed that the entropy η has two parts, the irreversible part $\eta^{(i)}$ and the reversible part $\eta^{(r)}$, so that

$$\eta = \eta^{(i)} + \eta^{(r)} , \qquad (8.33)$$

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* This principle is not generally accepted and some authors have serious objections on its general validity.

and

$$\varrho \Theta \dot{\eta}^{(i)} = {}_D w \quad ,$$
$$\varrho \Theta \dot{\eta}^{(r)} = q^k_{,k} + \varrho h . \tag{8.34}$$

These relations satisfy the equation for production of entropy. Further he assumed the second law of thermodynamics (for $dt > 0$) to be of the form

$$\dot{\eta}^{(i)} \geq 0 \ . \tag{8.35}$$

From (8.20) we see that this assumption is valid only if

$$\Theta_{,k} \, q^k \leq 0 \quad , \tag{8.36}$$

which is not in contradiction with the experience, since the temperature flows from the parts of the body with higher temperature to the parts with lower temperature. It follows then from (8.34) that

$$_D w \geq 0 \quad . \tag{8.37}$$

The rate of entropy production $\dot{\eta}^{(i)}$ is independent of the heat exchange and may be a function of the rates of deformation only.

If $\dot{x}^k$, $k = 1,\ldots,n$ are variables which describe the configuration of a thermodynamical system and if $X^{(i)}_k$ are irreversible forces, we may write

$$ {}_D W = X^{(i)}_k \, dx^k \quad . \tag{8.38} $$

In an n-dimensional space of the variables x^k the <u>dissipation function</u>

$$ \Phi(\dot{\underset{\sim}{x}}) = \Theta \dot{\eta}^{(i)} \tag{8.39} $$

for each prescribed value of the velocities $\dot{x}^k$ represents a surface,

$$ \Phi(\dot{\underset{\sim}{x}}) = M \quad . \tag{8.40} $$

Assuming that a process considered is quasistatic, i.e. the change of the coördinates x^k and of the temperature Θ is sufficiently slow, the <u>principle of least irreversible force</u> states that:

If the value $M > 0$ of the dissipation function $\Phi(\dot{x}^k)$ and the direction ν_k of the irreversible force $\left(X^{(i)}_k = X\nu_k\right)$ are prescribed, then the actual quasistatic velocity $\dot{x}^k$ minimizes the magnitude X of the irreversible force $X^{(i)}_k$ subject to the condition $\Phi(\dot{x}^k) \geq 0$.

For the justification of this principle we refere to Ziegler's paper [416] .

As a consequence of this principle it

follows that the components of the irreversible force have to satisfy the equations

$$X^{(i)}_{k} = \lambda \frac{\partial \Phi}{\partial \dot{x}^{k}} , \qquad (8.41)$$

where

$$\lambda = \Phi \left(\frac{\partial \Phi}{\partial \dot{x}^{m}} \dot{x}^{m} \right)^{-1} . \qquad (8.42)$$

When we identify the components $X^{(i)}_{k}$ with the components of the irreversible parts of the stress tensor, couple stress tensor and tensor of the director stresses, and the velocities $\dot{x}^{k}$ with the corresponding rates of the deformation of position and directors, from (8.40) follow the relations for ${}_{D}\underset{\sim}{t}$, ${}_{D}\underset{\sim}{m}$ and ${}_{D}\underset{\sim}{h}^{(\lambda)}$.

9. Some General Considerations on Constitutive Relations.

The relations (8.30-32) for the reversible part of the stress, director stresses and couple-stress tensors, as well as the relations for the irreversible parts which follows from (8.40), have to satisfy some additional assumptions in order to represent constitutive relations.

Constitutive relations in mechanics describe the response of a material to deformations. The response is characterized by the intrinsic properties of matter and not by the choice of coordinates, or by the choice of the way of describing deformations, rates of deformation, motions etc. Constitutive relations never describe completely mechanical properties of real materials, but only some of the dominant properties considered for some particular purposes. Therefore, a material which would completely behave according to some prescribed constitutive relations is an ideal material and does not exist in the Nature.

The first question, regarding the constitutive relations, is: which quantities are to be determined by these relations and which quantities are to be considered as variables. There are 3n+3 differential equations (7.35) and (7.36) from which the motions $\underset{\sim}{x} = \underset{\sim}{x}\left(\underset{\sim}{X},t\right)$ and $\underset{\sim}{d}_{(\alpha)} = \underset{\sim}{d}_{(\alpha)}\left(\underset{\sim}{X},t\right)$ may be determined if the forces $\underset{\sim}{f}$ and $\underset{\sim}{k}^{(\alpha)}$ and the couples $\underset{\sim}{\ell}$ are prescribed, but there are 9+9+27=45 components of the tensors $\underset{\sim}{t}$, $\underset{\sim}{h}^{(\alpha)}$ and $\underset{\sim}{m}$ which can not be determined from these equations. If we turn to the laws of thermodynamics, we obtain some relations, but then two new additional quantities are introduced, temperature Θ and entropy η . Expressing the laws of thermodynamics in terms of the internal energy ε , or in terms of the free energy ψ we may regard Θ , or η respectively, as a quantity to be determined by a constitutive relation.

There are two methods for the formulation of constitutive relations. One method is: to assume certain relations and to subject them to certain restrictions which follow from thermodynamics and from the principles which will be introduced later. The other method consists in deriving the relations from the energetic considerations based on thermodynamics; so obtained relations are then to be subject to further restrictions furnished by the additional principles.

The number of the assumed additional principles which are to be imposed on the constitutive relations varies from author to author. Since we are going to consider the constitutive relations which follow from the energetic considerations, and since we are not going to consider problems of more complex nature such as viscoelasticity and dependence of the state of stress on the history of deformation, we shall restrict the number of additional assumptions to two principles,

1° The principle of material frame indiference, and

2° The principle of local action.

The discussion of various other principles in continuum mechanics may be found e.g. in the books by Truesdell and Noll [379] and by Eringen [101a,101b].

The two mentioned principles are independent of the so called material symmetries. In order to obtain the relations for a particular class of material symmetries, we have to require, in addition, that the constitutive relations are invariant with respect to a subgroup of the group of orthogonal transformations which characterizes the class of material symmetries considered.

Let z^α and $\bar{z}^\alpha$ be two orthogonal Cartesian coordinate systems with origins at $\underset{\sim}{O}$ and $\bar{\underset{\sim}{O}}$, and let an <u>event</u> be described with respect to these two systems by $\{\underset{\sim}{z}, t\}$ and $\{\bar{\underset{\sim}{z}}, \bar{t}\}$, where t and $\bar{t}$ are times measured by two observers at $\underset{\sim}{O}$ and $\bar{\underset{\sim}{O}}$. A change of the frame of reference is expressed by the formula

$$\begin{aligned} \bar{z}^\alpha &= Q^\alpha_{\cdot\beta}(t)\, z^\beta + a^\alpha(t) \\ \bar{t} &= t - \tau \quad , \end{aligned} \tag{9.1}$$

or

$$z^{\alpha} = Q_{\beta}^{\cdot\alpha}(\bar{t})\,\bar{z}^{\beta} + b^{\alpha}(\bar{t})$$

$$t = \bar{t} + \tau \quad . \tag{9.2}$$

Here

$$Q_{\beta}^{\cdot\alpha}\ Q_{\cdot\alpha}^{\ \ \gamma} = \delta_{\beta}^{\gamma} \quad , \quad Q_{\beta}^{\cdot\alpha}\ Q_{\cdot\gamma}^{\ \ \beta} = \delta_{\gamma}^{\alpha} \quad , \tag{9.3}$$

and we assume that $\underset{\sim}{Q}$ is an orthogonal matrix, $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^{T}$.

If $\underset{\sim}{T}$ is a tensor field with components $T^{\cdots}_{\cdots}$ and $\bar{T}^{\cdots}_{\cdots}$ with respect to the coordinate systems $\underset{\sim}{O}\underset{\sim}{z}$ and $\underset{\sim}{\bar{O}}\underset{\sim}{\bar{z}}$ respectively, and if the components transform according to the transformation law for tensors when both, the dependent and independent variables, are transformed according to (9.1,2), the tensor field $\underset{\sim}{T}$ is said to be <u>frame-indifferent</u>, or <u>objective</u>.

The components of the position vector $\underset{\sim}{r} = z^{\alpha}\underset{\sim}{e}_{\alpha}$ are obviously not objective quantities since they transform according to (9.1,2).

The components of the velocity vector $\underset{\sim}{v}$ are defined with respect to the two considered reference frames by

$$v^{\alpha} = \dot{z}^{\alpha} \quad , \quad \bar{v}^{\alpha} = \dot{\bar{z}}^{\alpha} . \tag{9.4}$$

From (9.1) we have

$$\bar{v}^{\alpha} = \dot{Q}^{\alpha}_{\cdot\beta}\, z^{\beta} + Q^{\alpha}_{\cdot\beta}\, v^{\beta} + \dot{a}^{\alpha} \tag{9.5}$$

and obviously the velocity vector is not an objective vector. Writing (9.5) in the form

$$\bar{v}_{\alpha} = \dot{Q}_{\alpha\lambda}\, z^{\lambda} + Q_{\dot{\alpha}}^{\ \lambda}\, v_{\lambda} + \dot{a}_{\alpha} \tag{9.6}$$

we obtain for the velocity gradients the following transformation law,

$$\begin{aligned} \frac{\partial \bar{v}_{\alpha}}{\partial \bar{z}^{\beta}} &= \dot{Q}_{\alpha\lambda} \frac{\partial z^{\lambda}}{\partial \bar{z}^{\beta}} + Q_{\dot{\alpha}}^{\ \lambda} \frac{\partial v_{\lambda}}{\partial z^{\mu}} \frac{\partial z^{\mu}}{\partial \bar{z}^{\beta}} \\ &= \dot{Q}_{\alpha\lambda}\, Q_{\dot{\beta}}^{\ \lambda} + Q_{\dot{\alpha}}^{\ \lambda} \frac{\partial v_{\lambda}}{\partial z^{\mu}}\, Q_{\dot{\beta}}^{\ \mu} \ . \end{aligned} \tag{9.7}$$

Hence, the velocity gradients are not objective quantities. However, the rate of strain tensor is an objective tensor. From (9.7) we have

$$\bar{d}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \bar{v}_{\alpha}}{\partial \bar{z}^{\beta}} + \frac{\partial \bar{v}_{\beta}}{\partial \bar{z}^{\alpha}} \right) = Q_{(\dot{\beta}}^{\ \lambda}\, \dot{Q}_{\alpha)\lambda} + Q_{\dot{\alpha}}^{\ \lambda}\, Q_{\dot{\beta}}^{\ \mu}\, v_{(\lambda,\mu)} \ ,$$

but in view of (9.3)

$$Q_{\dot{\beta}}^{\ \lambda}\, \dot{Q}_{\alpha\lambda} + Q_{\dot{\alpha}}^{\ \lambda}\, \dot{Q}_{\beta\lambda} = \frac{d}{dt}\left(Q_{\dot{\beta}}^{\ \lambda}\, Q_{\alpha\lambda} \right) = 0 \ ,$$

and obviously

$$\bar{d}_{\alpha\beta} = Q_{\dot{\alpha}}{}^{\lambda} Q_{\dot{\beta}}{}^{\mu} d_{\lambda\mu} \quad .$$

From (9.7) it may be seen that the vorticity tensor $w_{\alpha\beta} = v_{[\alpha,\beta]}$ is not an objective tensor, but the gradients of this tensor are objective quantities. We have

$$w_{\alpha\beta} = Q_{[\dot{\beta}}{}^{\lambda} \dot{Q}_{\alpha]\lambda} + Q_{\dot{\alpha}}{}^{\lambda} Q_{\dot{\beta}}{}^{\mu} w_{\lambda\mu}$$

and

$$\bar{w}_{\alpha\beta,\gamma} = Q_{\dot{\alpha}}{}^{\lambda} Q_{\dot{\beta}}{}^{\mu} Q_{\dot{\gamma}}{}^{\nu} w_{\lambda\mu,\nu} \quad . \qquad (9.8)$$

If points of a body are referred to a system of material Cartesian coordinates Z^{λ} and if z^{α} and $\bar{z}^{\alpha}$ are two spatial reference frames, we see from (9.1) that

$$\frac{\partial \bar{z}^{\alpha}}{\partial Z^{\lambda}} = Q^{\alpha}_{\cdot\beta} \frac{\partial z^{\beta}}{\partial Z^{\lambda}} \quad , \qquad (9.9)$$

and the deformation gradients are objective. The same holds for the higher order deformation gradients

$$\frac{\partial^2 \bar{z}^{\alpha}}{\partial Z^{\lambda} \partial Z^{\mu}} = Q^{\alpha}_{\cdot\mu} \frac{\partial^2 z^{\mu}}{\partial Z^{\lambda} \partial Z^{\mu}} \quad , \text{ etc,} \qquad (9.10)$$

The principle of material frame indifference requires that: Constitutive equations must be invariant with respect to rigid motions of the spatial frame of reference.

A function $F\left(V^{\alpha}_{(1)}, V^{\alpha}_{(2)}, \ldots, z^{\alpha}\right)$ of vectors $\underset{\sim}{V}_{(\lambda)}$ is objective or frame-indifferent if it remains invariant under rigid motions of the spatial frame.
If only translations are regarded, $\bar{z}^{\alpha} = z^{\alpha} + a^{\alpha}$, it follows that

$$\bar{V}^{\alpha}_{(\lambda)} = V^{\alpha}_{(\lambda)}$$

and the condition of objectivity for the function F reduces to

$$F\left(\underset{\sim}{V}_{(1)}, \ldots, z^{\alpha} + a^{\alpha}\right) = F\left(\underset{\sim}{V}_{(1)}, \ldots, z^{\alpha}\right).$$

If the translations a^{α} are small quantities, from the Taylor series expansion of the function F we obtain that it will be objective only if

$$\frac{\partial F}{\partial z^{\alpha}} = 0$$

i.e. if it does not depend explicity on spatial coordinates of position.

Let us see now which restrictions are imposed on the function F by arbitrary rigid rotations of the spatial frame, if F is an objective function.

Let Q be the matrix

$$\underset{\sim}{Q} = \left(\delta^{\alpha}_{\beta} + \omega^{\alpha}_{\cdot\beta} \right) ,$$

where $\omega_{\alpha\beta} = - \omega_{\beta\alpha}$ is an arbitrary infinitesimal rotation, and

$$\bar{z}^{\alpha} = Q^{\alpha}_{\cdot\beta}\, z^{\beta} = \left(\delta^{\alpha}_{\beta} + \omega^{\alpha}_{\cdot\beta} \right) z^{\beta} . \qquad (9.11)$$

If F is an objective function of vectors $V^{\alpha}_{(\nu)}$, $\nu = 1,2,\ldots,n$, it will satisfy the relation

$$F\left(V^{\alpha}_{(1)} , \ldots , V^{\alpha}_{(n)} \right) = \bar{F}\left(\bar{V}^{\alpha}_{(1)} , \ldots , \bar{V}^{\alpha}_{(n)} \right) . \qquad (9.12)$$

From (9.11) we have

$$\bar{V}^{\alpha}_{(\nu)} = V^{\beta}_{(\nu)} \frac{\partial \bar{z}^{\alpha}}{\partial z^{\beta}} = V^{\alpha}_{(\nu)} + V^{\beta}_{(\nu)}\, \omega^{\alpha}_{\cdot\beta} \qquad (9.13)$$

and the invariance requirement reduces to the relation

$$F\left(V^{\alpha}_{(1)} , \ldots \right) = F\left(V^{\alpha}_{(1)} + V^{\beta}_{(1)}\, \omega^{\alpha}_{\cdot\beta} , \ldots \right) . \qquad (9.14)$$

For sufficiently small $\omega^{\alpha}_{\cdot\beta}$ we may expand F into the Taylor series,

$$F\left(V^{\alpha}_{(1)} + V^{\beta}_{(1)}\, \omega^{\alpha}_{\cdot\beta} , \ldots \right) = F\left(V^{\alpha}_{(1)} , \ldots \right) + \sum_{\nu=1}^{n} \frac{\partial F}{\partial V^{\alpha}_{(\nu)}}\, V^{\beta}_{(\nu)}\, \omega^{\alpha}_{\cdot\beta} + \ldots ,$$

Hence, if F is an objective function, for infinitesimal rotations $\underset{\sim}{\omega}$ we obtain that F has to satisfy the condition

$$\sum_{\nu=1}^{n} \frac{\partial F}{\partial V^{\alpha}_{(\nu)}} V^{\beta}_{(\nu)} \, \omega^{\alpha}{}_{\cdot\beta} = 0 \ . \qquad (9.15)$$

But ω is an arbitrary antisymmetric tensor and (9.15) reduces to the system of three differential equations (Toupin [370])

$$\left(\sum_{\nu=1}^{n} \frac{\partial F}{\partial V^{\alpha}_{(\nu)}} V_{(\nu)\beta} \right)_{[\alpha\beta]} = 0 \ . \qquad (9.16)$$

The equations (9.16) are tensorial equations. If the variables are objective quantities, we may write (9.16) in the form appropriate for arbitrary curvilinear coordinates x^{k},

$$\left(\sum_{\nu=1}^{n} g^{il} \frac{\partial F}{\partial V^{l}_{(\alpha)}} V^{j}_{(\alpha)} \right)_{[ij]} = 0 \ . \qquad (9.17)$$

The <u>principle of local action</u> states that: the state of stress at a point $\underset{\sim}{Z}$ of a medium is determined by the motion inside an arbitrary neighbourhood $N(\underset{\sim}{Z})$ of the point $\underset{\sim}{Z}$, and the motion outside this neighbourhood may be disregarded.

Under the "state of stress" we understand the values of all the quantities which describe the stress field ($\underset{\sim}{t}$, $\underset{\sim}{h}^{(\lambda)}$, $\underset{\sim}{m}$ etc.). If $\varphi(z(Z))$ is a function which describes the state of stress at Z at time t , according to this principle, at a configuration $K(t)$ the state of stress at Z is determined by the instantaneous configuration of the neighbourhood $N(\underset{\sim}{Z})$. Let $\underset{\sim}{Z}'$ be a point in $N(\underset{\sim}{Z})$. At the configuration $K(t)$ the relative position of $\underset{\sim}{Z}'$ with respect to $\underset{\sim}{Z}$ is given by

$$\Delta \underset{\sim}{z} = \underset{\sim}{z}\left(\underset{\sim}{Z}',t\right) - \underset{\sim}{z}\left(\underset{\sim}{Z},t\right) .$$

If $\quad Z'^{\alpha} - Z^{\alpha} = \Delta Z^{\alpha}$, we may write

$$\Delta z^{\alpha} = \frac{\partial z^{\alpha}}{\partial Z^{\lambda}} \Delta Z^{\lambda} + \frac{1}{2} \frac{\partial^2 z^{\alpha}}{\partial Z^{\lambda}\partial Z^{\mu}} \Delta Z^{\lambda} \Delta Z^{\mu} \quad . \tag{9.18}$$

Since the state of stress at $\underset{\sim}{Z}$ is determined by the local configuration of an arbitrary neighbourhood $N(\underset{\sim}{Z})$, it follows that φ must be a function of the deformation gradients,

$$\varphi = \varphi\left(z^{\alpha}_{;\lambda_1}, z^{\alpha}_{;\lambda_1\lambda_2}, \ldots z^{\alpha}_{;\lambda_1\ldots\lambda_N}, \ldots, Z, t\right) . \tag{9.19}$$

If φ is the internal energy function ε and if N is the highest order of the deformation gradients which appears in the expression for the energy, according to Toupin [372] , the corresponding material is said to be of <u>order N</u> .

Stojanović and Djurić [340,341] generalized this notion to directed elastic bodies, considering the strain energy as a function of the deformation gradients of an order N, and of the director gradients of an order M, such that ε is a function of the form*

$$\varepsilon = \varepsilon\left(x^{k}_{;K}, x^{k}_{;K_1K_2}, \ldots, x^{k}_{;K_1\ldots K_N};\ d^{k}_{(\lambda);K}, d^{k}_{(\lambda);K_1K_2}, d^{k}_{(\lambda)K_1K_2\ldots K_M}; \eta, \underset{\sim}{X}\right). \tag{9.20}$$

In the following we restrict our considerations to the materials of the order $N=2$ and $M=1$, i.e. the constitutive variables, which are to be considered as independent variables, in the expression for the internal energy density are first and second order deformation gradients and the director gradients, so that

$$\varepsilon = \varepsilon\left(x^{k}_{;K}\, x^{k}_{;KL}, d^{k}_{(\lambda);K}, \eta, \underset{\sim}{X}\right). \tag{9.21}$$

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* A number of authors considered the strain energy as a function of the components $d^{k}_{(\lambda)}$ of directors, and not only as a function of the gradients of the directors (mostly in linear theories). From our considerations in the section 8 (see eq. (8.28)) it does not follow that the components of the directors appear explicitly as constitutive variables and therefore we omit them here.

Generalizations to higher order materials are in principle simple, but require more involved notation which makes the expressions less clear. The higher order gradients of deformation and directors may be identified with the multipolar theory then might be directly applied.

The materials for which the constitutive relations do not depend explicitly on $\underset{\sim}{X}$ are called homogeneous and we shall consider only such materials.

9. 1. The Internal Energy Function.

The internal energy function ε in the form (9.21) has, according to the principle of material frame indifference, to satisfy the conditions of the form (9.17). When the constitutive variables are identified with the components of the vectors $V(\alpha)$ according to the table

$$V^{\ell}_{(1)}\ ,V^{\ell}_{(2)}\ ,V^{\ell}_{(3)} \longrightarrow x^{\ell}_{;1}\ ,x^{\ell}_{;2}\ ,x^{\ell}_{;3}$$

$$V^{\ell}_{(4)}\ ,\ldots ,V^{\ell}_{(9)} \longrightarrow x^{\ell}_{;11}\ \ldots\ x^{\ell}_{;33}$$

$$V^{\ell}_{(10)}\ ,\ldots ,V^{\ell}_{(3n+9)} \longrightarrow d^{\ell}_{(1);1}\ ,\ldots ,d^{\ell}_{(n);3}\ ,$$

the equations (9.17) obtain the form

$$\left[g^{il}\left(\frac{\partial \varepsilon}{\partial x^{l}_{;K}}\, x^{j}_{;K} + \frac{\partial \varepsilon}{\partial x^{l}_{;KL}}\, x^{j}_{;KL} + \frac{\partial \varepsilon}{\partial d^{l}_{(\lambda);K}}\, d^{j}_{(\lambda);K}\right)\right]_{[ij]} = 0 \quad (9.1.1)$$

This represents a system of 3 linear partial differential equations with 3 x (3n+9) variables $V^{l}_{(\nu)}$, $l = 1,2,3$; $\nu = 1,2,\ldots,3n+9$. The internal energy ε is an arbitrary function of $3 \times (3n+9) - 3 = 9n+24$ independent integrals of the system (9.1.1).

It is matter of a direct calculation to verify that integrals of the system (9.1.1) are the material tensors

$$C_{AB} \equiv g_{ab}\, x^{a}_{;A}\, x^{b}_{;B}, \qquad (9.1.2)$$

$$G_{CAB} = g_{ab}\, x^{a}_{;CA}\, x^{b}_{;B}, \qquad (9.1.3)$$

$$F_{\alpha AB} = g_{ab}\, x^{a}_{;A}\, d^{b}_{(\alpha);B}. \qquad (9.1.4)$$

These tensors are invariants under the transformations of spatial coordinates. Since

$$C_{AB} = C_{BA} \qquad G_{CAB} = G_{ACB} \qquad (9.1.5)$$

there are $6 + 18 + 9n$ independent integrals $\underset{\sim}{C}, \underset{\sim}{G}, \underset{\sim}{F}_{(\alpha)}$ and the internal energy is an arbitrary function of these quantities,

$$\mathcal{E} = \mathcal{E} \left(C_{AB}, G_{CAB}, F_{\alpha AB}, X^K \right) . \qquad (9.1.6)$$

9. 2. Irreversible Processes.

The dissipation function Φ in (8.39) is a function of certain generalized velocities. According to the principle of material frame indifference Φ has to be a function of objective variables. Such variables are the components of the rate of strain tensor $d_{ij} = v_{(i,j)}$, the gradients of vorticity $w_{ij,k}$, as well as the second gradients $v_{i,jk}$ of the velocity vector.

For oriented media the rates of directors $\dot{d}^i_{(\alpha)}$ and the gradients $\dot{d}^i_{(\alpha);k}$ of these rates are objective tensors. With respect to rigid motions (9.1.3) of Cartesian frames, it follows that the directors are objective vectors,

$$\bar{d}^{\lambda}_{(\alpha)} = d^{\mu}_{(\alpha)} \, Q^{\lambda}_{\cdot\mu}$$

but the rates

$$\dot{\bar{d}}^{\lambda}_{(\alpha)} = \dot{d}^{\mu}_{(\alpha)} \, Q^{\lambda}_{\cdot\mu} + d^{\mu}_{(\alpha)} \dot{Q}^{\lambda}_{\cdot\mu}$$

and the gradients of the rates

$$\dot{\bar{d}}^{\lambda}_{(\alpha),\beta} = \dot{d}^{\mu}_{(\alpha),\nu} \, Q^{\lambda}_{\cdot\mu} \, Q_{\beta}^{\cdot\nu} + d^{\mu}_{(\alpha),\nu} \dot{Q}^{\lambda}_{\cdot\mu} \, Q_{\beta}^{\cdot\nu}$$

are obviously not objective quantities.

From (8.8) we have for the dissipative part ${}_D W$ of the mechanical power the expression

$$ {}_D W = {}_D t^{(ij)} d_{ij} + {}_D h^{(\lambda)jk} \left(\dot{d}_{(\lambda)j,k} - w_j^{\;i} d_{(\lambda)i,k} \right) - {}_D m^{ijk} w_{ij,k} . \qquad (9.2.1) $$

However, we may write

$$ \dot{d}_{(\lambda)j,k} - w_j^{\;i} d_{(\lambda)i,k} = \left(\dot{d}_{(\lambda)j} - w_j^{\;i} d_{(\lambda)i} \right)_{,k} + w_{j\;,k}^{\;i} d_{(\lambda)i} , \qquad (9.2.2) $$

where

$$ \hat{d}_{(\lambda)j} \equiv \dot{d}_{(\lambda)j} - w_j^{\;i} d_{(\lambda)i} \qquad (9.2.3) $$

is the <u>co-rotational time flux</u> (cf. [375]) of the vector $\underset{\sim}{d}_{(\lambda)}$. It may be directly verified that $\hat{d}_{(\lambda)j}$ is an objective vector. Hence, we may rewrite now (9.2.1) in the form

$$ {}_D W = {}_D t^{(ij)} d_{ij} + {}_D h^{(\lambda)jk} \hat{d}_{(\lambda)j,k} - \left(m^{ijk} + d_{(\lambda)}^{\;i} h^{(\lambda)jk} \right) w_{ij,k} . \qquad (9.2.4) $$

Hence, all rates which appear here,

$$ d_{ij} \quad , \quad w_{ij,k} \quad , \quad \hat{d}_{(\lambda)j,k} \qquad (9.2.5) $$

are objective. It would be natural to assume now that the dissipation function Φ depends on the objective

rates (9.2.5). But, according to the definition, Φ is a function of velocities, and therefore it might be regarded as a function of $\dot{x}^{k}_{;K}$, $\dot{x}^{k}_{;KL}$, $\dot{d}^{k}_{(\lambda);K}$ via the objective variables (9.2.5).
For the derivation of the constitutive relations for irreversible processes we may turn now to Ziegler's principle, or to consider the Clausius-Duhem inequality. Ziegler's principle of least irreversible force is so far applied only to the case of non-oriented polar media, where it was assumed (for refernces see section 8) that

$$\Phi = \Phi\left(d_{ij} , w_{ij,k} \right) . \qquad (9.2.6)$$

Formal difficulties for the application of the Clausius-Duhem inequality are evident, since the internal energy function ε , or the free energy ψ , have to be regarded as functions of $x^{k}_{;K}$, $x^{k}_{;KL}$, $d^{k}_{(\lambda);K}$, and not of the rates (9.2.5). Therefore we may only quote Rivlin [298] , who said that "The application of the Clausius-Duhem inequality to inelastic materials is.... questionable. It should, however, be realized that the results obtained from much applications are, in the main, not very strong".

The only possibility which remains is to introduce the constitutive ralations by assumption, and in the form which will not violate the laws of motion and the laws of thermodynamics. The form of the assumed relations depends on the mechanical properties which are to be considered. Often in the applications of this method is used the principle of equipresence: A quantity present as an independent variable in one constitutive equation should be also present in all, unless its presence contradicts the laws of physics, or the rules of invariance (cf. [379]). It should be noted that this principle is not generally accepted.

In the theory of inelastic properties of non-polar media, owing to the recent developments of the thermodynamics of continua, some progress is made by Leigh 202 , and Dillon 59 a .

In the following sections we shall discuss the constitutive relations of some particular media, when the constitutive relations are expressed in the form of functions. More general theories, based on functionals, are not yet much developed*.

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* For some aspects of viscoelasticity we refere the readers to the papers by DeSilva and Kline [59] and by Eringen [90,98] .

10. Elasticity.

In some modern treatments the difference is made between <u>elastic</u> and <u>hyperelastic</u> materials. Hyperelastic materials are those for which an elastic potential exists and the stresses may be derived from this potential. For elastic materials the existence of such a potential is not necessary. Hyperelastic materials are elastic, but elastic materials are not necessarily hyperelastic. We restrict our considerations, according to this division, to hyperelastic materials.

In the sense of thermodynamics the mechanical work done by a deformation of an elastic material is reversible and it is accumulated in the elastic potential energy σ , so that from (8.12,13) we have

$$w = {}_E w \quad , \quad \sigma = \Sigma \quad , \tag{10.1}$$

The local law of balance of energy (8.11) may be written in one of the forms corresponding to (8.14) or (8.22),

$$\varrho \dot{\varepsilon} = {}_E w + \varrho \Theta \dot{\eta} \quad , \tag{10.2}$$

or

$$\varrho\dot{\psi} = {}_{E}w - \varrho\eta\dot{\Theta} \ . \tag{10.3}$$

Since the dissipative part of working vanishes we shall drop the subscript "E".

According to the section 8, we assume the specific internal energy to be a function of the form

$$\varepsilon = \varepsilon\left(x^{\ell}_{;L}, x^{\ell}_{;LK}, d^{\ell}_{(\lambda);K}, \eta\right) \tag{10.4}$$

and the specific free energy to be a function of the form

$$\psi = \psi\left(x^{\ell}_{;L}, x^{\ell}_{;LK}, d^{\ell}_{(\lambda);K}, \Theta\right) . \tag{10.5}$$

If we take the energy balance equation in the form (10.2), from (8.29-32) we obtain the following expressions for the temperature, stress, director stress and couple-stress:

$$\Theta = \frac{\partial\varepsilon}{\partial\eta} \ , \tag{10.6}$$

$$t^{(ij)} = \varrho\left[g^{i\ell}\left(\frac{\partial\varepsilon}{\partial x^{\ell}_{;L}}x^{j}_{;L} + \frac{\partial\varepsilon}{\partial x^{\ell}_{;KL}}x^{j}_{;KL}\right) + \left(g^{i\ell}\frac{\partial\varepsilon}{\partial d^{\ell}_{(\lambda);K}}d^{j}_{(\lambda);k}\right)_{[ij]}x^{k}_{;K}\right] , \tag{10.7}$$

$$m^{i(jk)} = -\varrho g^{i\ell}\frac{\partial\varepsilon}{\partial x^{\ell}_{;KL}}x^{j}_{;K}\,x^{k}_{;L} \ , \tag{10.8}$$

$$h^{(\lambda)ik} = \varrho g^{il} \frac{\partial \varepsilon}{\partial d^{l}_{(\lambda);K}} x^{k}_{;K} . \qquad (10.9)$$

The similar set of equations follows if the free energy function ψ is used instead of ε, but since in ψ the temperature Θ is regarded as one of the constitutive variables, the corresponding constitutive equation for entropy will be

$$\eta = - \frac{\partial \Psi}{\partial \Theta} . \qquad (10.10)$$

The relations (10.7 - 9) can not be regarded yet as constitutive relations. First, the internal energy must be an objective function, and second, the symmetry properties of the left and right hand-sides of the relations (10.7,8) have to be the same, i.e. the necessary and sufficient conditions for the tensorial equations (10.7 - 9) to be satisfied are that the irreducible parts of the left and right-hand sides of each of the equations are equal (Toupin [371]).

According to this requirement the relations (10.6) and (10.9) present no restrictions on the function ε, since the requirements are identically fulfilled, but the relations (10.6) and (10.7) present considerable restrictions.

On the left-hand side of (10.7) we have the symmetric part of the stress tensor, and hence the antisymmetric part of the right-hand side must vanish. This yields the set of three euqations

$$\left[g^{il}\left(\frac{\partial \varepsilon}{\partial x^{l}_{;L}} x^{j}_{;L} + \frac{\partial \varepsilon}{\partial x^{l}_{;KL}} x^{j}_{;KL} + \frac{\partial \varepsilon}{\partial d^{l}_{(\lambda);K}} d^{j}_{(\lambda);K} \right) \right]_{[ij]} = 0 \qquad (10.11)$$

If we compare this with (9.1.1), which followed from the principle of material frame indifference, we see that (10.11) is identical with (9.1.1). Accordingly, the internal energy must be a function of the form

$$\varepsilon = \varepsilon \left(C_{AB}\ , G_{CAB}\ , \dot{F}_{\alpha AB}\ , \eta\ , X^{K} \right) . \qquad (10.12)$$

To investigate the restrictions imposed by the symmetries of (10.8) we have first to find the irreducible parts of the tensor $m^{i(jk)} \equiv M^{ijk}$, knowing that $m^{ijk} = - m^{jik}$. According to the Appendix, (A2.26-29), the irreducible parts of the tensor M^{ijk} are

$$_{S}M^{ijk} = 0$$

$$_{A}M^{ijk} = 0$$

$$_{P}M^{ijk} = \frac{1}{6}\left(2\, m^{ijk} - m^{jki} - m^{kij} \right),$$

$$\qquad (10.13)$$

$$_{\bar{P}}M^{ijk} = \frac{1}{6}\left(m^{ijk} + m^{jki} - 2\, m^{kij} \right) = -\,_{P}M^{kij} .$$

Hence, the right-hand side of (10.8) has to satisfy 10 conditions $(10.13)_1$,

$$\left(g^{il} \frac{\partial \varepsilon}{\partial x^{l}_{;KL}} x^{j}_{;K} \; x^{k}_{;L} \right)_{(ijk)} = 0 \quad , \tag{10.14}$$

and one condition $(10.13)_2$,

$$\left(g^{il} \frac{\partial \varepsilon}{\partial x^{l}_{;KL}} x^{j}_{;K} \; x^{k}_{;L} \right)_{[ijk]} = 0 \quad . \tag{10.15}$$

and the tensor $m^{i(jk)}$ has only 8 independent components.

Owing to the symmetry of the gradients $x^{l}_{;KL} = x^{l}_{;LK}$, (10.15) is identically satisfied.

Relations (10.14) represent an additional system of 10 partial differential equations which must be satisfied simultaneously with the system (10.11). According to the definitions of the tensors $\underset{\sim}{C}$, $\underset{\sim}{G}$ and $\underset{\sim}{F}_{(\alpha)}$, (9.1.2-4), it is obvious that (10.14) will yield restrictions only on the tensor $\underset{\sim}{G}$. It may be directly verified that the system (10.14) is satisfied by the material tensor

$$D_{ABC} \equiv G_{C[BA]} = C_{C[A,B]} \quad . \tag{10.16}$$

Hence, the specific internal energy ε is an arbitrary function of the tensors $\underset{\sim}{C}$, $\underset{\sim}{D}$, $\underset{\sim}{F}_{(\alpha)}$ and of Θ and X^K . For homogeneous materials ε does not depend on X^K,

$$\varepsilon = \varepsilon \left(C_{AB} \, , \, D_{ABC} \, , F_{\alpha AB}, \eta \right) . \quad (10.17)$$

To write the mechanical constitutive equations (10.7-9) we have to perform the differentiations of the internal energy function considering it as an arbitrary function of the form (10.17), which gives for the derivatives the following expressions:

$$\frac{\partial \varepsilon}{\partial x^{\ell}_{;L}} = \frac{\partial \varepsilon}{\partial C_{AB}} \frac{\partial C_{AB}}{\partial x^{\ell}_{;L}} + \frac{\partial \varepsilon}{\partial D_{ABC}} \frac{\partial D_{ABC}}{\partial x^{\ell}_{;L}} - \frac{\partial \varepsilon}{\partial F_{\alpha AB}} \frac{\partial F_{\alpha AB}}{\partial x^{\ell}_{;L}} \, ,$$

$$\frac{\partial \varepsilon}{\partial x^{\ell}_{;KL}} = \frac{\partial \varepsilon}{\partial D_{ABC}} \frac{\partial D_{ABC}}{\partial x^{\ell}_{;KL}} \, , \quad (10.18)$$

$$\frac{\partial \varepsilon}{\partial d^{\ell}_{(\lambda);K}} = \frac{\partial \varepsilon}{\partial F_{\alpha AB}} \frac{\partial F_{\alpha AB}}{\partial d^{\ell}_{(\lambda);K}} \, .$$

According to (10.11), the equation for the symmetric part of the stress tensor becomes now

$$t^{(ij)} = \varrho \left[g^{i\ell} \left(\frac{\partial \varepsilon}{\partial x^{\ell}_{;L}} \, x^{j}_{;L} + \frac{\partial \varepsilon}{\partial x^{\ell}_{;KL}} \, x^{j}_{;KL} \right) \right]_{(ij)} \, , \quad (10.19)$$

and the complete set of the mechanical constitutive relations is

$$t^{(ij)} = \varrho \left(2 \, \frac{\partial \varepsilon}{\partial C_{KL}} x^{i}_{;K} x^{j}_{;L} + \frac{\partial \varepsilon}{\partial D_{KLM}} x^{(i}_{;K} x^{j)}_{;LM} + \frac{\partial \varepsilon}{\partial F_{\alpha KL}} x^{i}_{;K} d^{j}_{(\alpha);L} \right) , \quad (10.20)$$

$$m^{i(jK)} = - \varrho \, \frac{\partial \varepsilon}{\partial D_{KLM}} \, x^{i}_{;K} \cdot x^{(j}_{;L} \, x^{K)}_{;M} \, , \quad (10.21)$$

$$h^{(\lambda)ij} = \varrho \frac{\partial \varepsilon}{\partial F_{\lambda KL}} x^i_{;K} x^j_{;L} . \tag{10.22}$$

For applications it is advantageous to substitute the deformation tensor $\underset{\sim}{C}$ by the strain tensor $\underset{\sim}{E}$ (3.10). It is also possible to represent the tensor $\underset{\sim}{D}$ in terms of the strain gradients,

$$D_{ABC} = 2 E_{C[A,B]} . \tag{10.23}$$

From the constitutive relations (10.20-22) we see that the symmetric part of the stress tensor is affected by the strain of position, by the strain gradients and by the deformations of the directors, but couple-stresses depend (explicitly) only on the strain gradients, and the director stresses depend explicitly only on the deformations of the directors.

It is also to be explicitly mentioned that in the thermodynamical approach to the constitutive relations the couple stress tensor remains indetermined. Out of its nine components only eight appear in the equation of energy balance and only eight are determined by the constitutive relations.

So far, except in the theory of dislocations (Kroener and Hehl [152], Stojanović [335,337], Stojanović and Djurić [340]) the general relations (10.20-22) were not used in the applications. The applications are mostly concerned with more special classes of materials, i.e. with materials of grade two (the strain gradient theory), and with different kinds of oriented (directed) materials. For materials of grade two the internal energy is assumed to be of the form

$$\varepsilon = \varepsilon\left(\underset{\sim}{C}, \underset{\sim}{D}, \eta\right), \tag{10.24}$$

and for oriented materials of the form

$$\varepsilon = \varepsilon\left(\underset{\sim}{C}, \underset{\sim}{F}_{\alpha}, \eta\right). \tag{10.25}$$

10. 1. A Principle of Virtual Work and Boundary Conditions.

To derive the boundary conditions for elastic polar materials we shall generalize the principle of virtual work used by Toupin [371] for static equilibrium in the theory of elastic materials of grade two. In a slightly more general form this principle was also applied to generalized Cosserat continua by Stojanović and Djurić [341].

We assume the principle of virtual work in the form

$$\delta T + \delta E = \delta w, \tag{10.1.1}$$

where δT is the virtual work of inertial forces, δE

is the first variation of the internal energy and δw is the virtual work of all body and contact forces acting on a part $\mathfrak{v}$ of a body. At the points of the boundary s of $\mathfrak{v}$ the normal derivatives $D\delta x^i$ and $D\delta d^i_{(\alpha)}$ of (by assumption) independent variations δx^i and $\delta d^i_{(\alpha)}$ are to be considered also as independent.

In general, it may be assumed that the boundary s consists of a finite number of surfaces $\mathcal{S}$ bounded by curves $\mathcal{C}$. The boundary curves represent edges.

The gradients $\varphi_{,k}$ of a function φ, defined in the interior and on the boundary of $\mathfrak{v}$, may be decomposed on the boundary of $\mathfrak{v}$ into the surface gradient $D_k \varphi$ and the normal gradient $D\varphi$,

$$\varphi_{,k} = D_k \varphi + n_k D\varphi , \qquad (10.1.2)$$

where $\underset{\sim}{n}$ is the unit normal to the boundary surface S. Toupin introduced a three-dimensional extension of the second fundamental tensor $\underset{\sim}{b}$ of a surface by*

$$b_{ij} = -D_i n_j = -D_j n_i . \qquad (10.1.3)$$

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* Let u^α, $\alpha = 1,2$ be coordinates on S, and the equations of the surface are $x^i = x^i(u^\alpha)$. From (10.1.3) it follows that

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For any smooth tensor field $f\ldots$ defined at points of a smooth surface $\mathcal{S}$ Toupin introduced the integral identity

$$\int_{\mathcal{S}} D_i f\ldots n_j \, ds = \int_{\mathcal{S}} (b^k_k n_i n_j - b_{ij}) f\ldots ds + \oint_{\mathcal{C}} m_i n_j f\ldots dl, \quad (10.1.4)$$

where $\underset{\sim}{m} = \underset{\sim}{\tau} \times \underset{\sim}{n}$ and $\underset{\sim}{\tau}$ is the unit tangent to $\mathcal{C}$, and dl is the scalar line element of $\mathcal{C}$.

If the integral transformation (10.1.4) is applied to all surfaces $\mathcal{S}$, i.e, to the whole boundary s of v, one gets

$$\int_{s} D_i f\ldots n_j \, ds = \oint_{s} (b^k_k n_i n_j - b_{ij}) f\ldots ds + \int_{c} [m_i n_j f\ldots] \, dl, \quad (10.1.5)$$

where [ ] represents the jumps of the enclosed quantity when an edge is approached from either side. We assume that the boundary s of v has no edges and that $f\ldots$ is smooth throughout s, so that the

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$$b_{ij} x^i_{;\alpha} x^j_{;\beta} = + n_j x^j_{;\alpha\beta} \equiv b_{\alpha\beta},$$

where $b_{\alpha\beta}$ is the second fundamental tensor and $x^j_{;\alpha\beta}$ are covariant derivatives of $x^j_{;\alpha}$ with respect to the surface metric. It is to be noted that for the points on the surface $n_j x^j_{;\alpha} = 0$.

line integral in (10.1.5) vanishes.

For the virtual work of inertia forces we assume the expression

$$\delta T = \int_v \varrho \left(\ddot{x}^i \, \delta x_i + i^{\lambda\mu} \ddot{d}^i_{(\lambda)} \, \delta d_{(\mu)i} \right) dv , \tag{10.1.6}$$

and for the variation of the internal energy we may write

$$\delta E = \int_v \varrho \left(\frac{\partial \varepsilon}{\partial x^k_{;K}} \delta x^k_{;K} - \frac{\partial \varepsilon}{\partial x^k_{;KL}} \delta x^k_{;KL} + \frac{\partial \varepsilon}{\partial d^k_{(\lambda);K}} \delta d^k_{(\lambda);K} \right) dv . \tag{10.1.7}$$

Since the spatial coordinates only are subject to variations we shall use the following relations:

$$\begin{aligned} \delta x^k_{;K} &= \left(\delta x^k\right)_{,m} x^m_{;K} \\ \delta x^k_{;KL} &= \left[\left(\delta x^k\right)_{,m} x^m_{;K}\right]_{;L} = \left(\delta x^k\right)_{;m\ell} x^m_{;K} x^\ell_{;L} + \left(\delta x^k\right)_{,m} x^m_{;KL} \\ \delta d^k_{(\lambda);K} &= \left(\delta d^k_{(\lambda)}\right)_{,m} x^m_{;K} , \end{aligned} \tag{10.1.8}$$

and (10.1.7) may be rewritten in the form

$$\delta E = \int_v \varrho \left[\left(\frac{\partial \varepsilon}{\partial x^k_{;K}} x^m_{;K} + \frac{\partial \varepsilon}{\partial x^k_{;KL}} x^m_{;KL} \right) \left(\delta x^k\right)_{,m} \right. \tag{10.1.9}$$

$$+ \frac{\partial \varepsilon}{\partial x^{\kappa}_{;KL}} x^{m}_{;K} x^{\ell}_{;L} \left(\delta x^{\kappa}\right)_{,m\ell} + \frac{\partial \varepsilon}{\partial d^{\kappa}_{(\lambda);K}} x^{m}_{;K} \left(\delta d^{\kappa}_{(\lambda)}\right)_{,m} \Big] dv .$$

For the sake of brevity in writing let us introduce the notation

$$A_{\kappa}^{\cdot m} \equiv \varrho \left(\frac{\partial \varepsilon}{\partial x^{\kappa}_{;K}} x^{m}_{;K} + \frac{\partial \varepsilon}{\partial x^{\kappa}_{;KL}} x^{m}_{;KL} \right) ,$$

$$B_{\kappa}^{\cdot m\ell} \equiv \varrho \frac{\partial \varepsilon}{\partial x^{\kappa}_{;KL}} x^{m}_{;K} x^{\ell}_{;L} , \tag{10.1.10}$$

$$P^{(\lambda)\cdot m}_{\cdot \kappa} \equiv \varrho \frac{\partial \varepsilon}{\partial d^{\kappa}_{(\lambda);K}} x^{m}_{;K} , .$$

and

$$J_1 \equiv \int_v A_{\kappa}^{\cdot m} \left(\delta x^{\kappa}\right)_{,m} dv ,$$

$$J_2 \equiv \int_v B_{\kappa}^{\cdot m\ell} \left(\delta x^{\kappa}\right)_{,m\ell} dv , \tag{10.1.11}$$

$$J_3 \equiv \int_v P^{(\lambda)\cdot m}_{\cdot \kappa} \left(\delta d^{\kappa}_{(\lambda)}\right)_{,m} dv .$$

For J_1 we have

$$J_1 = \int_v \left[\left(A_{\kappa}^{\cdot m} \, \delta x^{\kappa} \right)_{,m} - A_{\kappa,m}^{\cdot m} \, \delta x^{\kappa} \right] dv$$

$$= \oint_s A_{\kappa}^{\cdot m} \, \delta x^{\kappa} n_m \, ds - \int_v A_{\kappa}^{\cdot m}{}_{,m} \, \delta x^{\kappa} dv . \tag{10.1.12}$$

J_2 may be written in the form

$$J_2 = \int_v \left[\left(B_{\kappa}^{\cdot m\ell} \delta x^{\kappa} \right)_{,m\ell} + B_{\kappa}^{\cdot m\ell}{}_{,m\ell} \, \delta x^{\kappa} \right] dv - 2 \oint_s B_{\kappa}^{\cdot (m\ell)}{}_{,m} \, \delta x^{\kappa} n_{\ell} \, ds . \tag{10.1.13}$$

Since we may write

$$J_2' \equiv \int_v \left(B_{k}^{\cdot m\ell}\delta x^{k}\right)_{,m\ell} dv = \oint_s \left(B_{k}^{\cdot m\ell}\delta x^{k}\right)_{,m} n_\ell \, ds ,$$

applying the integral identity (10.1.5) this becomes

$$J_2' = \oint_s \left\{\left[D B_{k}^{\cdot m\ell} n_m n_n + \left(b_t^t \, n_m n_\ell - b_{m\ell}\right) B_{k}^{\cdot m\ell}\right] \delta x^{k} + \right.$$

$$\left. + B_{k}^{\cdot m\ell} n_m n_\ell \left(D \delta x^{k}\right)\right\} ds ,$$

and for J_2 we definitively have

$$\begin{aligned} J_2 = &\int_v B_{k\;\;,m\ell}^{\cdot m\ell} \delta x^{k} dv + \\ &\pm \oint_s \left\{\left[D B_{k}^{\cdot m\ell} n_m n_\ell + \left(b_t^t \, n_m n_\ell - b_{m\ell}\right) B_{k}^{\cdot m\ell} - 2 B_{k\;\;\;,m}^{\cdot (m\ell)} n_\ell\right] \delta x^{k}\right. \\ &\left. + B_{k}^{\cdot m\ell} n_m n_\ell \left(D \delta x^{k}\right)\right\} ds . \end{aligned} \tag{10.1.14}$$

For J_3 we obtain similarly

$$J_3 = \oint_s P^{(\alpha)\cdot m}_{\cdot\;\; k} \, \delta d_{(\alpha)}^{k} n_m ds - \int_v P^{(\alpha)\cdot m}_{\cdot\;\; k\;\;,m} \, \delta d_{(\alpha)}^{k} dv . \tag{10.1.15}$$

Collecting the results we obtain for δE the expression

$$\delta E = \int_v \left[\left(-A_{k\;\;,m}^{\cdot m} + B_{k\;\;,m\ell}^{\cdot m\ell}\right) \delta x^{k} - P^{(\alpha)\cdot m}_{\cdot\;\; k\;\;,m} \, \delta d_{(\alpha)}^{k}\right] dv +$$

$$+\oint_s \Big\{\Big[A_k^{\cdot m} n_m + \left(DB_k^{\cdot m\ell}\right) n_m n_\ell + \left(b_t^t n_m n_\ell - b_{m\ell}\right) B_k^{\cdot m\ell} - 2B_{k\ ,m}^{\cdot m\ell} n_\ell\Big] dx^k$$

$$+ P^{(\alpha)\cdot m}_{\cdot k} n_m \delta d^k_{(\alpha)} + B_k^{\cdot m\ell} n_m n_\ell \left(D\delta x^k\right)\Big\} ds . \qquad (10.1.16)$$

According to the form of (10.1.16) it is natural to assume for the virtual work δw the expression

$$\delta w = \int_v \left(L_k \delta x^k + S^{(\alpha)}_k \delta d^k_{(\alpha)}\right) dv +$$

$$+ \oint_S \left[M_k \delta x^k + N_k \left(D\delta x^k\right) + T^{(\alpha)}_k \delta d^k_{(\alpha)}\right] ds . \qquad (10.1.17)$$

where $\underset{\sim}{L}$, $\underset{\sim}{M}$, $\underset{\sim}{N}$, $\underset{\sim}{S}^{(\alpha)}$ and $\underset{\sim}{T}^{(\alpha)}$ are some generalized forces.

Introducing now δT , δE and δw from (10.1.6,16,17) into (10.1.1) and assuming that the variations δx^k , $D\delta x^k$ and $\delta d^k_{(\alpha)}$ in v and on S are independent, we obtain the following relations:

$$\varrho \ddot{x}^\ell - A^{\ell m}_{\ ,m} + B^{\ell mn}_{\ ,mn} = L^\ell , \qquad (10.1.18)$$

in v :

$$\varrho i^{\alpha\mu} \ddot{d}^\ell_{(\alpha)} - P^{(\alpha)\ell m}_{\ ,m} = S^{(\alpha)\ell} , \qquad (10.1.19)$$

$$A^{\ell m} n_m + \left(DB^{\ell mn}\right) n_m n_n + \left(b_t^t n_m n_n - b_{mn}\right) B^{\ell mn} - 2B^{\ell mn}_{\ ,m} n_n = M^\ell , \qquad (10.1.20)$$

on S :

$$p^{(\alpha)\ell m} n_m = T^{(\alpha)\ell}, \tag{10.1.21}$$

$$B^{\ell mn} n_m n_n = N^{\ell} . \tag{10.1.22}$$

From (10.8),(10.9),(10.19) and (10.1.10) we see that

$$\begin{aligned} A^{(\ell j n)} &= t^{(\ell j n)} , \\ B^{\ell mn} &= -m^{\ell(mn)} , \\ p^{(\alpha)\ell m} &= h^{(\alpha)\ell m} . \end{aligned} \tag{10.1.23}$$

According to (10.11) we also have*

$$A^{[\ell m]} - d^{[\ell}_{(\alpha),p} h^{(\alpha)m]p} = 0 , \tag{10.1.24}$$

which substituted in (10.1.18) yields

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* The equation (10.1.24) follows also from the requirement that δE is invariant under virtual rigid displacements. Let x^i be Cartesian coordinates. The virtual rigid displacements are $\delta x^k = a^k + \epsilon^{kij} K_i x_j$ and $\delta d^k_{(\alpha)} = \epsilon^{kij} K_i d_{(\alpha)j}$, where a^k and K_i are arbitrary constants. Introducing this into (10.1.14) and requireing that the energy of every part of the body is separately invariant under all rigid variations we obtain (10.1.24)

$$\varrho \ddot{x}^{\ell} = t^{(\ell m)}{}_{,m} + m^{\ell(mn)}{}_{,mn} + d^{[\ell}_{(\alpha),p} h^{(\alpha)m]p} + L^{\ell} ,$$

This, together with (10.1.19),

$$\varrho i^{\alpha\mu} \ddot{d}^{\ell}_{(\alpha)} = h^{(\mu)\ell m}_{(\alpha)}{}_{,m} + s^{(m)\ell} ,$$

represents the equations of motion. Here we may identify L^{ℓ} with $\varrho\left(F^{\ell} + \ell^{\ell m}{}_{,m}\right)$, and $s^{\alpha(\ell)}$ with $\varrho \,k^{(\alpha)\ell}$. The boundary conditions follow from (10.1.20-22),

$$t^{(\ell m)} n_m + d^{[\ell}_{(\alpha),p} h^{(\alpha)m]p} n_m + D m^{\ell(mn)} n_m n_n - \left(b^t_t n_m n_n - b_{mn}\right) m^{\ell(mn)} + 2 m^{\ell(mn)}{}_{,m} n_n = M^{\ell},$$

$$h^{(\alpha)\ell m} n_m = T^{(\alpha)\ell} , \qquad (10.1.25)$$

$$-m^{\ell(mn)} n_m n_n = N^{\ell} .$$

The generalized forces $\underset{\sim}{M}$, $\underset{\sim}{T}^{(\alpha)}$ and $\underset{\sim}{N}$ are certain surface tractions which are to be prescribed on the boundary of the body.

10. 2. Elastic Materials of Grade Two.

When the internal enrgy is a function of deformation gradients $x^{k}_{;K}$ and $x^{k}_{;KL}$ and of X^{K} and η only, the mechanical constitutive relations (10.20, 21) obtain the form

$$t^{(ij)} = \varrho \left(2 \frac{\partial \varepsilon}{\partial C_{KL}} x^i_{;K} x^j_{;L} + \frac{\partial \varepsilon}{\partial D_{KLM}} x^{(i}_{;K} x^{j)}_{;LM} \right) , \tag{10.2.1}$$

$$m^{i(jk)} = - \varrho \frac{\partial \varepsilon}{\partial D_{KLM}} x^i_{;K} x^{(j}_{;L} x^{k)}_{;M} . \tag{10.2.2}$$

According to the Appendix (A1.32), the couple-stress tensor m^{ijk} may be represented by the second order tensor $m_l^{\cdot k}$, and this tensor may be decomposed into its deviatoric and spherical part, where the deviatoric part is

$$\mu_l^{\cdot k} \equiv m_l^{\cdot k} - m_p^{\cdot p} \delta_l^k = \frac{1}{2} \epsilon_{lij} m^{ijk} - \frac{1}{2} \epsilon_{pqr} m^{pqr} \delta_l^k \tag{10.2.3}$$

or

$$m^{ijk} = \mu^{ijk} + m_p^{\cdot p} \epsilon^{ijk} , \tag{10.2.4}$$

where

$$\mu^{ijk} = \epsilon^{ijl} \mu_l^{\cdot k} . \tag{10.2.5}$$

In the constitutive relations (10.2.2) only the symmetric part $m^{i(jk)}$ of the couple-stress tensor appears, and from (10.2.4) we see that

$$m^{i(jk)} = \mu^{i(jk)} . \tag{10.2.6}$$

Since there are only eight independent components of the tensor $m^{i(jk)}$ (cf.(10.13)), and since the deviator has only eight components (cf. App. (A2.4)), we may represent the deviator μ^{ijk} in terms of the tensor $m^{i(jk)}$,

$$\mu^{ijk} = \frac{2}{3}\left(2\, m^{i(jk)} + m^{k(ij)}\right) . \qquad (10.2.7)$$

The invariant $\epsilon_{ijk}\, m^{ijk} = m^{k}_{\cdot k}$ of the couple-stress tensor remains undetermined since there are only eight constitutive equations (10.2.2), and also in the boundary conditions (10.1.25) only the symmetric part of the couple-stress tensor appears. According to Koiter [183], without any loss in generality we may assume that $m_{k}^{\cdot k}$ is equal to zero.

The tensor D_{KLM} is antisymmetric in K and L and if we introduce the second-order material tensor

$$D^{N}_{\cdot M} \equiv \frac{1}{2}\,\epsilon^{NKL}\, D_{KLM} \ , \qquad (10.2.8)$$

the constitutive equations (10.2.1) obtain the form

$$t^{(ij)} = \varrho\left(\frac{\partial \varepsilon}{\partial E_{KL}}\, x^{i}_{;K}\, x^{j}_{;L} + \frac{1}{2}\frac{\partial \varepsilon}{\partial D^{N}_{\cdot M}}\,\epsilon^{NKL}\, x^{(i}_{;K}\, x^{j)}_{;LM}\right) , \qquad (10.2.9)$$

where we have used (3.10), and for the deviator $\mu_{l}^{\cdot k}$ we get from (10.2.3,7,8) the relation

$$\mu_{l}^{\cdot k} = -\frac{1}{3}\,\varrho\,\frac{\partial \varepsilon}{\partial D^{N}_{\cdot M}}\,\epsilon^{NKL}\,\epsilon_{ijl}\, x^{i}_{;K}\, x^{(j}_{;L}\, x^{k)}_{;M} \ . \qquad (10.2.10)$$

For isotropic materials the internal energy must be a function of isotropic invariants (see App. section A2) of the tensor $\underset{\sim}{E}$ and $\underset{\sim}{D}$,

$$\varepsilon = \varepsilon \left(\mathrm{I}_E, \mathrm{II}_E, \mathrm{III}_E, {}^1\mathrm{II}_D, {}^2\mathrm{II}_D, {}^2\mathrm{III}_{ED}, {}^3\mathrm{III}_{ED}, {}^4\mathrm{III}_{ED}, \cdots \right). \quad (10.2.11)$$

Teodosiu [361-366] applied the general theory of elastic materials of grade two to media with internal and initial stresses and particularly to the determination of internal stresses produced by dislocations. He also considered a more general theory in which the couple-stress tensor is not indetermined. A proposal for such a generalization was already given by Toupin [371] on the basis of the analysis of the boundary conditions $(10.1.25)_3$. From the antisymmetry of the couple-stress tensor it follows that the traction $\underset{\sim}{N}$ has to be orthogonal to the boundary surface, $\underset{\sim}{N} \cdot \underset{\sim}{n} = 0$, but this requirement for the traction $\underset{\sim}{N}$ is without a physical motivation. For that reason Toupin proposed a more general theory in which the complete couple-stress tensor would be determined.

For <u>infinitesimal deformations</u> we may assume that the coordinates X^K and x^k coincide in the reference configuration, such that

$$x^k = X^K \delta^k_K + u^k, \quad (10.2.12)$$

$$x^{k}_{;L} = \delta^{k}_{L} + u^{k}_{,l}\,\delta^{l}_{L} \quad ,$$

$$x^{k}_{;LM} = u^{k}_{,lm}\,\delta^{l}_{L}\,\delta^{m}_{M} \quad , \tag{10.2.12}$$

where $\underset{\sim}{u}$ is an infinitesimal displacement. The deformation tensors in the linear approximation are

$$E_{KL} \approx e_{kl}\,\delta^{k}_{K}\,\delta^{l}_{L} = u_{(k,l)}\,\delta^{k}_{K}\,\delta^{l}_{L} \quad ,$$

$$D_{KLM} \approx D_{klm}\,\delta^{k}_{K}\,\delta^{l}_{L}\,\delta^{m}_{M} \quad , \tag{10.2.13}$$

where

$$D_{klm} = 2\,e_{m[k,l]} = 2\,\dot{w}_{kl,m} \quad ,$$

$$\omega_{kl} = u_{[k,l]} \;. \tag{10.2.14}$$

It is accustomed, however, to represent the third-order tensors $\underset{\sim}{\mu}$ and $\underset{\sim}{D}$ by their second-order duals. Since the rotation tensor w_{kl} may be represented by the vector $w^{i} = \frac{1}{2}\epsilon^{ikl}w_{kl}$, we may put

$$k_{ij} \equiv w_{i,j} \quad , \tag{10.2.16}$$

and the linear constitutive relations may be written in the form

$$t^{(ij)} = C_{1}{}^{ijkl}\,e_{kl} + C_{2}{}^{ijkl}\,k_{kl} \quad ,$$

$$\mu^{ij} = M_{1}{}^{ijkl}\,e_{kl} + M_{2}{}^{ijkl}\,k_{kl} \;. \tag{10.2.17}$$

For isotropic materials the fourth-order tensors $\underset{\sim}{C}$ and $\underset{\sim}{M}$ are linear combinations of the fundamental tensors g^{ij} , such that

$$C_\nu^{ijk\ell} = \alpha_\nu g^{ij} g^{k\ell} + \beta_\nu g^{ik} g^{j\ell} + \gamma_\nu g^{i\ell} g^{jk} , \tag{10.2.18}$$

$$M_\nu^{ijk\ell} = a_\nu g^{ij} g^{k\ell} + b_\nu g^{ik} g^{j\ell} + c_\nu g^{i\ell} g^{jk} . \qquad (\nu = 1,2)$$

Since the constitutive relations (10.2.17) for isotropic materials have to be invariant under the full orthogonal group of trasformations, we shall obtain them substituting the elasticity tensors from (10.2.18) into (10.2.17).

In the linear theory we may assume that the density ϱ is approximatively equal to the density in the reference configuration, $\varrho \approx \varrho_0$. For isotropic materials in this approximation the internal energy function may be approximated by a quadratic polynominal in the isotropic invariants I_e , II_e and ${}^1\mathrm{II}_D$, ${}^2\mathrm{II}_D$ of the tensors $\underset{\sim}{e}$ and $\underset{\sim}{D}$, and it may be written in the form (Koiter [183])

$$\varrho_0 \varepsilon = G \left[\frac{\nu}{1-2\nu} \mathrm{I}_e^2 + e^i_j e^j_i + 2\ell^2 \left(k^{\cdot}_i{}_j k_i{}^{\cdot j} + \eta k^{\cdot i}{}_j k^j{}_i \right) \right] , \tag{10.2.19}$$

where G is the shear modulus, ν is the Poisson ratio and $2G\ell^2$ and $2\eta G\ell^2$ are two additional new elastic constants. The constant ℓ has the dimension of length and is called the <u>characteristic length</u> of the material. η is a non-dimensional number.

The constitutive relations (10.2.9,10) may be written now in the form

$$t^{(ij)} = \rho_0 \frac{\partial \varepsilon}{\partial e_{ij}} = 2G\left(e^{ij} + \frac{\nu}{1-2\nu}\, I_e\, g^{ij}\right),$$
$$\mu^{ij} = \rho_0 \frac{\partial \varepsilon}{\partial k_{ji}} = 4Gl^2\left(k^{ij} + \eta k^{ji}\right). \qquad (10.2.20)$$

These relations were obtained by Aero and Kuvshinskii [4] in 1960. Grioli [134a] studied the non-linear theory and in the linearization he obtained the similar expressions, but he neglected the terms involving η . Mindlin and Teirsten [217] considered the linear constitutive equations as a result of linearization of the relations derived by Toupin, and they applied the linear theory to a number of problems in vibrations and stress concentration (cf. also Mindlin [221]). One of the most interesting effects of couple-stresses is its influence on the stress concentration factor which appears to be a function of the characteristic length l and to be less than what is usually assumed in the non-polar theories to be its value. For detailed study of the influence of couple--stresses in linear elasticity we refere the reader, among others, to the papers by Mindlin and Tiersten [217] , Mindlin [218], [221] , Mindlin and Eshel [225] Koiter [183] , Neuber [256] , and, for the problems of stress concentration, to the book by Savin [308] which appeared in 1968 and where detailed references may be found.

Within the theory of materials of grade two (or, within the strain-gradient theory) a generalization of Rivlin's method for the construction of general solutions in non-linear elasticity was presented by Stojanović and Blagojević [339] and by Blagojević [28,28a]. It is found that owing to the influence of couple-stresses the Poynting effect, which is in the non-linear theory of elasticity attributed to the second-order terms, appears as an effect of the first order in hemitropic materials.

A very fine and general synthesis of work of Grioli, Aero and Kuvshinskii, Bressan [35a] and other authors is presented by Galletto [107].

10. 3. The Elastic Cosserat Continuum.

When the influence of the strain gradients in the internal energy function is neglected, according to (10.20-22) the couple-stress tensor $\underset{\sim}{m}$ will vanish and the constitutive relations obtain the form

$$t^{(ij)} = \varrho \left(\frac{\partial \varepsilon}{\partial E_{KL}} x^i_{;K} x^j_{;L} + \frac{\partial \varepsilon}{\partial F_{\alpha KL}} x^i_{;K} d^j_{(\alpha);L} \right) , \qquad (10.3.1)$$

$$h^{(\lambda)ij} = \varrho \frac{\partial \varepsilon}{\partial F_{(\lambda)KL}} x^i_{;K}\, x^j_{;L} \quad . \tag{10.3.2}$$

The directors in a Cosserat medium represent rigid triads and therefore we may assume that in the initial (reference) configuration the directors $D^K_{(\alpha)}$ coincide with the base vectors of a Cartesian system of refernce X^K , i.e.

$$\underset{\sim}{D}_{(\alpha)} = D^K_{(\alpha)} \underset{\sim}{e}_K \quad , \quad D^K_{(\alpha)} = \delta^K_\alpha \quad . \tag{10.3.3}$$

For infinitesimal deformation we may write

$$x^k = X^K \delta^k_K + u^k \quad ,$$
$$\underset{\sim}{d}_{(\alpha)} = \underset{\sim}{D}_{(\alpha)} + \underset{\sim}{\Omega} \times \underset{\sim}{D}_{(\alpha)} \tag{10.3.4}$$

or

$$d_{(\alpha)}{}^k = \delta_\alpha{}^k + \Omega\, \dot{\alpha}^k \tag{10.3.5}$$

where $\underset{\sim}{u}$ is an infinitesimal displacement vector, and $\underset{\sim}{\Omega}$ is an independent rotation of the director triads However,

$$x^k_{;K} = \delta^k_K + u^k_{,l}\, \delta^l_K \quad ,$$
$$d^k_{(\alpha);L} = \Omega\, \dot{\alpha}^k{}_{,l}\, \delta^l_L \quad , \tag{10.3.6}$$

and the deformation tensors are

$$E_{kl} \approx u_{(k,l)} \delta^k_K \delta^l_L$$

$$F_{\alpha KL} \approx \Omega_{\alpha k,l} \delta^k_K \delta^l_L \equiv \varkappa_{\alpha kl} \delta^k_K \delta^l_L \tag{10.3.7}$$

Thus, we may consider as the constitutive variables the strain tensor e_{kl} and the gradients of rotation $\varkappa_{\alpha kl} = - \varkappa_{k\alpha l}$ or

$$\varkappa^m_{\cdot l} \equiv \frac{1}{2} e^{\alpha k m} \Omega_{\alpha k,l} \, . \tag{10.3.8}$$

The constitutive relations (10.3.1,2) for $\varrho \approx \varrho_0$ become now

$$t^{(ij)} = \varrho \left(\frac{\partial \varepsilon}{\partial e_{ij}} + \frac{\partial \varepsilon}{\partial \varkappa_{\lambda i l}} \varkappa_{\lambda \cdot l}^{\cdot j} \right) ,$$

$$h^{(\lambda)ij} = \varrho_0 \frac{\partial \varepsilon}{\partial \varkappa_{\lambda ij}} \, . \tag{10.3.9}$$

However, $\varkappa_{\lambda ij}$ is an antisymmetric tensor and the index λ is of the tensorial character. Applying (10.3.8) we may now write

$$t^{(ij)} = \varrho_0 \left(\frac{\partial \varepsilon}{\partial e_{ij}} + \frac{\partial \varepsilon}{\partial \varkappa^m_{\cdot n}} \varkappa^m_{\cdot n} \delta^{ij} - \frac{\partial \varepsilon}{\partial \varkappa^m_{\cdot n}} \varkappa^i_{\cdot n} \delta^{jm} \right)$$

$$h_{\lambda}^{\cdot j} = \varrho_0 \frac{\partial \varepsilon}{\partial \varkappa^i_{\cdot j}} \, , \tag{10.3.10}$$

where

$$h_{i}^{\cdot j} \equiv \frac{1}{2} \epsilon_{i\lambda n} h^{\lambda n j} \tag{10.3.11}$$

The internal energy ε may be approximated now by a quadratic polynomial,

$$\varrho_0 \varepsilon = G \left[\frac{\nu}{1-2\nu} I_e^2 + e_{j}^{i} e_{i}^{j} + 2 \overset{*}{\ell}{}^2 \left(\varkappa_{\cdot j}^{i} \varkappa_{i}^{\cdot j} + \overset{*}{\eta} \varkappa_{\cdot j}^{i} \varkappa_{\cdot i}^{j} \right) \right] \tag{10.3.12}$$

and the linear constitutive relations have the form completely analoguous to (10.2.20),

$$\begin{aligned} t^{(ij)} &= 2G \left(e^{ij} + \frac{\nu}{1-2\nu} I_e \overset{\bullet}{g}{}^{ij} \right) , \\ h_{j}^{\cdot i} &= 4G \overset{*}{\ell}{}^2 \left(\varkappa_{j}^{\cdot i} + \overset{*}{\eta} \varkappa_{\cdot j}^{i} \right) . \end{aligned} \tag{10.3.13}$$

Here again we have, a "characteristic length" $\overset{*}{\ell}$ of the material, and a nondimensional constant $\overset{*}{\eta}$.

The linear theory of elasticity of Cosserat materials is studied extensively by Schaefer [310-316] , who also elaborated a method for solving the equilibrium problems in terms of the stress-functions [314,315] , and applied the theory to the theory of dislocations* [317-319] .

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* I mostly appreciate Prof. Schaefer's kindness to put at my disposal his yet unpublished results on the dislocation theory in the Cosserat continuum.

The theory of nonsymmetric elasticity developed since 1960 by Aero, Bul'gin and Kuvshinskii [3-6,36,199] is based on the assumption that particles of a medium may suffer rotations independent of the displacements, which makes their theory to be, in fact, a theory of Cosserat media.

The equations of motion $(7.1.6)_{2,4}$ in the linearized theory of Cosserat continua obtain the form

$$
\begin{aligned}
&\varrho \ddot{x}^i = t^{ij}{}_{,j} + \varrho f^i \\
&\varrho I^{l[i} \dot{\omega}_l{}^{j]} = t^{[ji]} + H^{[ij]k}{}_{,k} + \varrho d^{[i}_{(\alpha)} k^{(\alpha)j]} \quad ,
\end{aligned}
\tag{10.3.14}
$$

where the hyperstress tensor H^{ijk} defined by $(7.25)_4$, appears only as an antisymmetric tensor.
The moments of director forces appear here in the form of body couples. The effect of hyperstresses in the linear theory of an elastic Cosserat continuum is obviously the same as the effect of couple-stresses in the strain-gradient theory. For that reason many authors consider both kinds of "materials" as Cosserat materials, or simply as materials with couple-stresses without making any distinction between the two kinds of materials.

10. 4. Elastic Materials with Microstructure.

a) Micromorphic and micropolar materials.- The basic theory is developed by Eringen and Suhubi [90-101,352]. It is assumed that for the microelements are valid the Caushy laws of motion,

$$
\begin{aligned}
&\varrho' a'^{i} = t'^{ij}{}_{,j} + \varrho f'^{i} , \\
&t'^{ij} = t'^{ji}
\end{aligned}
\qquad (10.4.1)
$$

where primes denote that the quantities are related to microelements. For macromaterial the corresponding quantities are obtained through the averaging, e.g.

$$
\int_{ds} t'^{ij} ds'_j \equiv t^{ij} ds_j , \quad \int_{dv} \varrho' f'^{i} dv' = \varrho f^{i} dv , \text{ etc.} \qquad (10.4.2)
$$

The stress and volume moments are defined by the relations

$$
\begin{aligned}
&\int_{ds} t'^{ij} \xi'^{k} ds'_j \equiv \lambda^{ijk} ds_j . \\
&\int_{dv} \varrho' f'^{i} \xi'^{j} dv' \equiv \varrho \ell^{ij} dv ,
\end{aligned}
\qquad (10.4.3)
$$

and λ^{ijk} represents the "first stress moment", which is not the same as the couple-stress. Further, in the relation

$$
\int_{dv} \varrho' a'^{i} \xi'^{j} dv' \equiv \varrho \dot{\sigma}^{ij} dv \qquad (10.4.4)
$$

the quantity σ^{ij} is defined as the "inertial spin", and the symmetric tensor s^{ij}, defined by

$$\int_{dv} t'^{ij}\, dv' \equiv s^{ij}\, dv \tag{10.4.5}$$

represents the "microstress average".

The constitutive relations, according to our notation (cf. section 5.2) read

$$t^{k}_{\cdot l} = \varrho \frac{\partial \varepsilon}{\partial x^{l}_{;K}} x^{k}_{;K} \quad ,$$

$$s^{k}_{\cdot l} = \varrho \left(\frac{\partial \varepsilon}{\partial x^{l}_{;K}} x^{k}_{;K} + \frac{\partial \varepsilon}{\partial d^{l}_{(\lambda)}} d^{k}_{(\lambda)} + \frac{\partial \varepsilon}{\partial d^{l}_{(\lambda);K}} d^{k}_{(\lambda);K} \right) \quad , \tag{10.4.6}$$

$$\lambda^{k \cdot m}_{\cdot l} = \varrho \frac{\partial \varepsilon}{\partial d^{l}_{(\lambda);L}} x^{k}_{;L}\, d^{m}_{(\lambda)} \; . \qquad \left(\lambda = 1,2,3 \right)$$

where it is assumed that the internal energy ε is a function of the mechanical constitutive variables

$$x^{k}_{;K} \quad , \quad d^{l}_{(\lambda)} \quad , \quad d^{l}_{(\lambda);K} \; . \tag{10.4.7}$$

The stress moment λ coincides with our hyperstress, and this theory may be regarded also as a theory of generalized Cosserat continua.

The difference between the general theory outlined in the section 10.3 and the theory of micromorphic continua is in the assumption that the internal energy depends explicitly on the components of the directors , and, also, in the assumed existence of two independent stresses - the macro-stress $\underset{\sim}{t}$ and the micro - stress average $\underset{\sim}{s}$.

In micropolar bodies the micro-elements are rigid. The directors in this case represent rigid triads and the theory reduces to the theory of elastic Cosserat media (cf. section 10.3).

b) Microstructure.- The linear theory of elastic bodies with microstructure was developed by Mindlin [218,219]. The continuum is composed of unit cells which have some properties of crystal lattices. The theory represents, in the mechanical sense, the linearized version of the theory of generalized Cosserat continua with deformable directors (section 5.2). The directors represent microdeformations, and since there are only three directors in this theory we may put $d_{(\alpha)j} = \psi_{\alpha j}$, where ψ_{ij} are displacement-gradients in the micro-medium,

$$\psi_{ij} = \partial u'^{j} / \partial x'^{i}.$$

Denoting by x^i and u^i Cartesian coordinates and components of the macro-displacements, resp. , the relative deformation is given by .

$$\gamma_{ij} = \frac{\partial u_j}{\partial x^i} - \psi_{ij} \ ,$$

and the macro-strain by

$$\varepsilon_{ij} = \partial_{(i} u_{j)} \ .$$

Macro-deformation gradients are determined by the tensor

$$x_{ijk} = \partial_i \psi_{jk}$$

which represents the tensor of director-gradients.

The state of stress is described by the ordinary (Caushy) stress t^{ij}, by the relative stress σ^{ij} and by the double stress μ^{ijk}, such that (for $\varrho \approx \varrho_0 = 1$)

$$t^{ij} = \frac{\partial \varepsilon}{\partial \varepsilon_{ij}} \ , \quad \sigma^{ij} = \frac{\partial \varepsilon}{\partial \gamma_{ij}} \ , \quad \mu_{ijk} \equiv \frac{\partial \varepsilon}{\partial x_{ijk}} \ , \qquad (10.4.8)$$

and the equations of motion are

$$\begin{aligned} &\left(t^{ij} + \sigma^{ij}\right)_{,j} + \varrho f^i = \varrho \ddot{u}^i \\ &\mu^{ijk}{}_{,i} + \sigma^{jk} + \Phi^{jk} = i^{jl} \ddot{\psi}_l{}^k \ . \end{aligned} \qquad (10.4.9)$$

This theory contains the linearized equations of Cosserat continua as a special case, and the linear version of the strain-gradient theory as a special case, too. Eringen [98] showed, however, that this theory coincides with the theory of micromorphic materials. The theory of Mindlin, however, is elaborated only in the linear version and it is difficult to say from the coincidence of two theories in their linear forms if they agree in general, or they represent two different theories.

10. 5. Incompatible Deformations.

Under certain circumstances a field of stresses can not be associated to a field of deformations which satisfies the compatibility conditions (see App. sections A4 and section 4). Such situations appear in thermoelasticity and in the theory of dislocations. In the classical linear thermoelasticity, in the Duhamel-Neumann law, it is assumed that the total strain $\underset{\sim}{e}$, which satisfies the compatibility conditions, is composed of two strains which do not satisfy these conditions, of an elastic strain $\underset{\sim}{e}^{\varepsilon}$ which produces thermal stresses, and of a strain $\underset{\sim}{e}^{T}$ which depends on the distribution of temperature in a body. This idea was used in the linear theory of dislocations for the determination of internal stresses produced by dislocations (cf. Kroener [195a]) and later it was generalized first in the theory of dislocations by Kroener and Seeger [193a] . Guenther [143] established a very important and interesting relation between the incompatibilities of the Cosserat continuum and the structural curvature of a dislocated crystal.

Stojanović, Djurić and Vujoshević [335-337, 344,347-349,385-388] developed a general theory of elastic incompatible deformations, which was applied to thermoelasticity and dislocations [335,337].

The theory is based on the assumption that the deformation gradients corresponding to a deformation $x^k = x^k(\underset{\sim}{X})$ of a body from an initial (and unstressed) configuration K_0 into a deformed (and stressed) configuration K may be decomposed in two deformations, such that

$$x^k_{;K} = \Phi^k_{(\lambda)} \, \Theta^{(\lambda)}_K \, , \quad X^K_{;k} = \Theta^K_{(\lambda)} \, \Phi^{(\lambda)}_k \, , \qquad (10.5.1)$$

where $\underset{\sim}{\Theta}_{(\lambda)}$ and $\underset{\sim}{\Theta}^{(\lambda)}$ represent reciprocal triads of vectors, as well as $\underset{\sim}{\Phi}_{(\lambda)}$ and $\underset{\sim}{\Phi}^{(\lambda)}$.

The linear differential forms

$$du^\lambda = \Phi^{(\lambda)}_k dx^k \quad \text{and} \quad du^\lambda = \Theta^{(\lambda)}_K dX^K \qquad (10.5.2)$$

are in general non integrable. The vectors $\Phi^{(\lambda)}$ represent <u>elastic distorsions</u>, and $\underset{\sim}{\Theta}^{(\lambda)}$ are <u>plastic</u> or thermal distorsions (the terminology depends on the applications; in the theory of dislocations these distorsions are plastic). The coordinates u^λ owing to the non-integrability of (10.5.2) may be interpreted as coordinates of points of a non-Euclidean, linearly connected space with the coefficients of connection (with respect to the systems of reference x^k and X^K)

$$\Gamma^{k}_{\ell m} = \bar{\Phi}^{k}_{(\lambda)} \partial_\ell \Phi^{(\lambda)}_{m} \quad , \qquad \Gamma^{K}_{LM} = \Theta^{k}_{(\lambda)} \partial_L \Theta^{(\lambda)}_{M} \quad . \tag{10.5.3}$$

Under the assumption that the state of stress is described by a nonsymmetric stress tensor t^{ij} and by the antisymmetric tensor m^{ijk} of couple-stresses, and that the internal energy σ is a function of elastic distorsions $\bar{\Phi}^{(\lambda)}_{\ell}$ and their gradients $\Phi^{(\lambda)}_{\ell,m}$, obtained are the constitutive relations in the form

$$t^{(ij)} = 2\varrho \left(\frac{\partial \sigma}{\partial c_{\lambda\mu}} \Phi^{i}_{(\lambda)} \Phi^{j}_{(\mu)} + \frac{\partial \sigma}{\partial D_{\lambda\mu\nu}} \Phi^{(i}_{(\lambda)} \Phi^{j)}_{(\mu),m} \Phi^{m}_{(\nu)} \right) ,$$

$$m^{ijk} = -\varrho \frac{\partial \sigma}{\partial D_{\lambda\mu\nu}} \bar{\Phi}^{i}_{(\lambda)} \Phi^{j}_{(\mu)} \Phi^{k}_{(\nu)} \quad , \tag{10.5.4}$$

where

$$c_{\lambda\mu} \equiv g_{\ell m} \Phi^{\ell}_{(\lambda)} \Phi^{m}_{(\mu)} \quad , \quad D_{\lambda\mu\nu} \equiv \left(g_{\ell m} \Phi^{\ell}_{(\lambda)} \Phi^{m}_{(\mu),n} \Phi^{n}_{(\nu)} \right)_{[\lambda\mu]} . \tag{10.5.5}$$

These constitutive relations reduce to the constitutive relations of the strain-gradient theory (elastic materials of grade two) (10.2.1,2) if we put

$$\Phi^{\ell}_{(\lambda)} = x^{\ell}_{;\lambda} \quad \text{and} \quad \Theta^{(\lambda)}_{L} = \delta^{\lambda}_{L} \quad . \tag{10.5.6}$$

The most important consequence of this approach to the problem of incompatible deformations in elasticity is that in the constitutive relations

(10.5.4) the couple-stress tensor is completely determined, while in the theory of materials of grade two, to which this theory reduces in the limiting case, there are only eight constitutive relations for the tensor $m^{i(jk)}$.

For applications to thermoelasticity $\Theta^{(\lambda)}_{L}$ are interpreted as tensors of thermal distorsions (e.g. for isotropic materials with small thermal dilatations α we have $\Theta^{(\lambda)}_{L} = (1+\alpha\Theta)\,\delta^{\lambda}_{L}$).

In the applications to the theory of dislocations the generalized Cosserat continuum with three directors is considered. The directors may be related to lattice vectors, but in the absence of dislocations the directors are then material vectors and the continuum reduces to an ordinary continuum. In a dislocated crystal with a given distribution of dislocation in terms of the dislocation density tensor $\alpha_{ij}{}^{k} = -\alpha_{ji}{}^{k}$ there is the fundamental connection between directors and dislocation-density tensor (Stojanović [335,337])

$$\alpha_{ij}^{\cdot\cdot k} = d^{k}_{(\lambda)}\, d^{(\lambda)}_{[j,i]} \quad , \tag{10.5.7}$$

such that

$$F_{\alpha[AB]} = \alpha_{ijk}\, x^{i}_{;A}\, x^{j}_{;B}\, d^{k}_{(\alpha)} \tag{10.5.8}$$

where $F_{\alpha AB}$ is the tensor of the director-deformation (9.1.4).

The linear theory of moving dislocations in the Cosserat continuum is developed recently by Schaefer [317-319] .

The problem of couple-stresses in thermo elasticity, without a reference to incompatibilities of the thermal strains, but both for Cosserat continua and for strain-gradient materials is studied by Nowacki [262-269]. Thermoelasticity of materials with micro structure, also without entering into the problems of incompatibilities, is presented in a number of papers by Wozniak [404-410] .

11. Viscous and Plastic Flow.

In this section we shall give only a short review of some of the results in the theory of polar fluids and in the theory of plasticity of polar materials.

a) Polar fluids.- In comparison with elasticity, the theory of polar fluids (or of fluids with non-symmetric stress tensor) is much less developed, although there are certain effects which might be experimentally observed and which might serve for the verification of the theory.

Liquid crystals were already mentioned as an example of a Cosserat continuum with one director (section 7.2). Ericksen's theory was further developed by Leslie [204,205], who applied some of the modern concepts of thermodynamics and obtained Ericksens's equations of statics as the limiting case. Coleman [48] and Wang [90,91] considered liquid crystals from the point of view of the more general theories of materials with memory.

Aero, Bul'gin and Kuvshinskii [3] considered (1964) a fluid with rotating particles, which is in fact a Cosserat fluid. Dahler and Scriven [56] and Condif and Dahler [49] (1964) gave an interpretation of the antisymmetric stress in fluids, based on the considerations of the spin of molecular substructure.

Stokes [350] in 1966 attributed the couple-stresses to the influence of the gradient of vorticity. Such fluids, according to the terminology of the multipolar continuum mechanics (section 5.3) may be named dipolar fluids. Plavšić [285-288] considered such fluids from the thermodynamical point of view, applying Ziegler's principle of least irreversible force for the derivation of the general (non linear) constitutive relations. The linear theory of dipolar fluids is given by Bleustein and Green [32] and Hills [157]. The theory of generalized Cosserat continua Allen and DeSilva [9,10] applied to fluid mechanics and obtained from the continuum approach the results of Dahler and Scriven. Alblas [8] also considered a fluid with rotating particles.

Theory of micropolar fluids represents an application of the general concept of micromorphic continua. The basis is presented in the papers by Eringen in 1964 [90,91]. Later it was applied to a study of some particular cases of flow (Eringen [94], Ariman [12] Ariman and Cakmak [13-15], Willson [400], Hudimoto and Tokuoka [164], Kirwan and Newman [179] etc.).

Independently of the model of a fluid considered by each of the authors, there is one impressive characteristic result mutual to all models in the linear theory. In the linear theory of Newtonian fluids there are two viscosity coefficients (volume viscosity and shear viscosity). In the theory of linear polar fluids there appears a third coefficient which might be named "rotational viscosity". Owing to this coefficient the distribution of velocities is different from what follows from the theory of Newtonian fluids e.g. in the theory of channel flow the velocities in the middle of the channel are smaller than it is predicted for Newtonian fluids. Similar results are also obtained for other flows and the results may be applied for experimental viscometric measurments and for the determination of the coefficient of rotational viscosity. Plavšić [286,287] studied the whole series of the so-called viscometric flows of dipolar fluids and obtained the results which might be directly applied for experimental investigations.

b) Plasticity.- So far little is done in the domain of the theory of plastic flow of polar materials. Komljenović [184] applied Ziegler's principle of least irreversible force to materials of the strain-gradient type and obtained the non-linear constitutive relations for elastic-plastic materials. The yield condition in his work is $\Phi = \kappa^2 \geq 0$, where Φ is the dissipation function. In the linearized case this reduces for isotropic materials to the Hencky-Mises yield condition.

For ideally plastic materials Sawczuk [309] and Lippmann [207] formulated the theories of plastic flow of Cosserat continua. The yield conditions follow from the requirement that the constitutive relations are not explicitly depending on time. Lippmann [207]* applied the theory to numerous problems of practical interest. Randenković and Plavšić [289] formulated a theory of visco-plasticity for the strain-gradient type of materials*, and applied it to the study of flow of a visco-plastic material between two parallel plates. The obtained solution reduces in the non-polar case to the well-known solution of Prager.

* * * * * * * * *

Time has not allowed me to pay more attention to application although in the applications and not in the general theory the effects of couple stresses may be realized to the full extent. The role of more general concepts of material continua is to be detected completely only when the effects are predicted and presentedby the theories in such a way that the experimental verification might be realized.

These lecture notes were written during the time the cours itself was held and I am aware of many errors not only of the technical and linguistical character, but also of the conceptual nature. I am aware that most of the points of view adopted here may

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* The author is mostly indebted to Prof. Lippmann and to Dr. Plavšić for giving him the manuscripts of their yet unpublished papers.

be subjected to serious criticism. To defend myself I can only say that many different approaches, many different mathematical and physical models which serve as basis for various theories of continua with non--symmetric stress tensor make it difficult (at least to me) to find a unique and sufficiently general way of presentation of the subject of my lectures. Neglecting all the weaknesses of these lectures and of these lecture notes, if this course of Mechanics of Polar Continua contributed to increase the interest in this field of mechanics to numerous listeners I had, the main purpose of this course is fulfilled.

At the end I wish to thank to all listeners for their interest and cooperation throughtout the whole series of 30 lectures.

The autorities of CISM and particularly the Secretary General, Prof. L. Sobrero and the Rector, Prof. W. Olszak have done everything to make the conditions for my work in Udine perfect. The responsibility for the failure to make this course as it has to be according to the excellent conditions for work - absolutely is mine.

My sincere thanks are also due to the members of the technical staff of CISM at Udine for all their unselfish help and cooperation which made it possible to prepare this text in such a short time. Their cooperation was far beyond what might be considered as the official cooperation.

Appendix.

For theoretical considerations it seems to me that the most suitable in the nonlinearized expositions is the notation of the double tensor field theory (cf.Ericksen's Appendix "Tensor Fields"in [73]) Assuming that the readers are familiar with the tensor analysis, the aim of this Appendix is to present only a survey of notation and some basic properties of ordinary and double tensor fields which are used in the lectures.

A. 1. Coordinates. Tensors.

An ordered set of numbers $\underset{\sim}{x}=\{x^1, x^2, \ldots, x^n\}$(we consider only real numbers) represents an arithmetic point. The numbers x^k are coordinates of the point $\underset{\sim}{x}$. The set of all possible arithmetic points, obtained when the coordinates take all possible values, represents an n-dimensional arithmetic space A_n .

If M is a set of objects m , such that there is a 1:1 correspondence between the objects of the set M and the points $\underset{\sim}{x}$ of a region A of A_n we may say that the numbers x^i are coordinates of the objects m , and that the objects m are pictures of the arithmetic points $\underset{\sim}{x}$.

If there is a 1:1 mapping of points $\underset{\sim}{x}$ of a region A in A_n upon points $\underset{\sim}{\bar{x}}$ of a region $\bar{A}$ in the same A_n,

$$x^k = x^k(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n),$$
$$\bar{x}^k = \bar{x}^k(x^1, x^2, \ldots, x^n), \qquad \text{(A1.1)}$$

we say that the $\bar{x}^k$ represent another coordinate system with respect to which the objects m are determined. The set M of objects m, together with the coordinate system x^k, and a group of transformations (A1.1) which introduces all admissible systems, represents an n-dimensional geometric space X_n. The objects m are now points of the space X_n.

The coordinate transformations are transformations of the numbers characterizing the same point m.

If R is a region in X_n with points $A \in R$ referred to a coordinate system x^k, and if $\bar{R}$ is another region in X_n with points B referred to a system of coordinates X^K, the 1:1 mappings of the points of R upon the points of $\bar{R}$,

$$x^k_A = x^k(X^1_B, \ldots, X^n_B),$$
$$X^K_B = X^K(x^1_A, \ldots, x^n_A), \qquad \text{(A1.2)}$$

represents a point transformation.

In the following, if x^k are coordinates of a point in X_n, we say it is the point $\underset{\sim}{x}$.

A geometric quantity in X_n at a point $\underset{\sim}{x}$ is defined by a set of numbers, say N, and by a transformation law which enables us to determine these numbers when a coordinate transformation is performed. If x_P^k and $\bar{x}_P^k$ are coordinates of a point P in X_n given with respect to two coordinate systems, and F_Ω, $\Omega = 1,2,\ldots,N$ are the components of a geometric object $\underset{\sim}{F}$, the general transformation law has the form

$$F_\Omega\{\bar{\underset{\sim}{x}}_P\} = \Phi_\Omega\left(F_1\{\underset{\sim}{x}_P\},\ldots,F_N\{\underset{\sim}{x}_P\}, \underset{\sim}{x}_P, \bar{\underset{\sim}{x}}_P, \frac{\partial \bar{x}^k}{\partial x^m},\ldots,\frac{\partial^q \bar{x}^k}{\partial x^{m_1}\ldots\partial x^{m_q}},\ldots\right).$$

If the transformation law does not depend explicitly on the coordinates of the point P, and on the partial derivatives of order higher then the first, the geometric object is a geometric quantity.

A scalar is a geometric quantity with one component, and with the transformation law

$$\varphi(x^1,\ldots,x^n) = \varphi(\bar{x}^1,\ldots,\bar{x}^n) \; . \tag{A1.3}$$

Covariant vectors are quantities with the number of components equal to the number of the dimensions of the space, $n = N$. If v_k and $\bar{v}_l$ are components of a covariant vector $\underset{\sim}{v}$ at a point $\underset{\sim}{x}$, the trans formation law for covariant vectors reads

$$\bar{v}_\ell = v_k \frac{\partial x^k}{\partial \bar{x}^\ell}, \qquad (k, \ell = 1, 2, \ldots, n) \quad (A1.4)$$
$$v_k = \bar{v}_\ell \frac{\partial \bar{x}^\ell}{\partial x^k}.$$

Here and in the following we apply the usual summation convention for repeated indices.

For a <u>contravariant vector</u> $\underset{\sim}{w}$ with components w^k and $\bar{w}^\ell$ the transformation law reads

$$\bar{w}^\ell = w^k \frac{\partial \bar{x}^\ell}{\partial x^k}, \qquad (A1.5)$$
$$w^k = \bar{w}^\ell \frac{\partial x^k}{\partial \bar{x}^\ell},$$

A <u>tensor</u> $\underset{\sim}{T}$ of covariant order p and contravariant order q is a quantity with n^{p+q} components $T_{i_1 \ldots i_p}^{\cdots\cdots j_1 \ldots j_q}$ and with the transformation law

$$\bar{T}_{i_1 \ldots i_p}^{\cdots\cdots j_1 \ldots j_q} = T_{k_1 \ldots k_p}^{\cdots\cdots \ell_1 \ldots \ell_q} \frac{\partial x^{k_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{k_p}}{\partial \bar{x}^{i_p}} \frac{\partial \bar{x}^{j_1}}{\partial x^{\ell_1}} \cdots \frac{\partial \bar{x}^{j_q}}{\partial x^{\ell_q}} \quad (A1.6)$$

The <u>order</u> of this tensor is $p+q$.

A tensor all of whose indices are superscripts (subscripts) is said to be a contravariant (covariant) tensor.

If the components of a tensor remain unchanged when two of its co- or contravariant indices interchange their places, we say that the tensor is symmetric with respect to these two indices, e.g.

$$T_{ijkl} = T_{ikjl} \quad , \quad T^{pqr} = T^{qpr} .$$

If components of a tensor change sign when two of its co- or contravariant indices interchange their positions, the tensor is antisymmetric, e.g.

$$T_{ijkl} = -T_{ikjl} \, , \quad T^{pqr} = -T^{qpr} .$$

A second-order tensor may always be decomposed into its symmetric part,

$$\begin{aligned} T_{(ij)} &\equiv \tfrac{1}{2}\left(T_{ij} + T_{ji}\right) , \\ T^{(ij)} &\equiv \tfrac{1}{2}\left(T^{ij} + T^{ji}\right) , \end{aligned} \qquad \text{(A1.7)}$$

and into its antisymmetric part,

$$\begin{aligned} T^{[ij]} &\equiv \tfrac{1}{2}\left(T^{ij} - T^{ji}\right) , \\ T_{[ij]} &\equiv \tfrac{1}{2}\left(T_{ij} - T_{ji}\right) , \end{aligned} \qquad \text{(A1.8)}$$

such that

$$\begin{aligned} T^{ij} &= T^{(ij)} + T^{[ij]} , \\ T_{ij} &= T_{(ij)} + T_{[ij]} . \end{aligned} \qquad \text{(A1.9)}$$

There are tensors defined simultaneously with respect to two points of the space, and these two points are, in general, referred to two different coordinate systems, say x^k and X^K. Such tensors represent the double tensor fields. Let $t^k_{\cdot K}(\underset{\sim}{x}, X)$ be such a tensor. With respect to coordinate transformations at $\underset{\sim}{x}$ it transforms like a contravariant vector, and with respect to coordinate transformations at $\underset{\sim}{X}$ it transforms like a covariant vector,

$$\bar{t}^{l}_{\cdot L} = t^k_{\cdot K} \frac{\partial \bar{x}^l}{\partial x^k} \cdot \frac{\partial X^K}{\partial \bar{X}^L} . \qquad \text{(A1.10)}$$

Further examples of the double tensor fields are partial derivatives of the point transformations (A1.2)

$$F^k_{\cdot K} \equiv \frac{\partial x^k}{\partial X^K} \equiv x^k_{;K} \; ; \quad F^{\cdot K}_{k} \equiv \frac{\partial X^K}{\partial x^k} \equiv X^K_{;k} \cdot \text{(A1.11)}$$

In Euclidean spaces E_n there exist rectilinear orthogonal (Cartesian) coordinate systems z^α, $\alpha = 1, 2, \ldots, n$, and if such a coordinate system is admissible in an X_n, besides some other properties which will be mentioned later, we say that it is Euclidean space. The unit vectors in the directions of the coordinate lines z^α we shall denote by $\underset{\sim}{e}_\alpha = \underset{\sim}{e}^\alpha$. The position of a point $\underset{\sim}{z}$ in E_n is determined by the position vector $\underset{\sim}{r}$,

$$\underset{\sim}{r} = z^{\alpha}\,\underset{\sim}{e}_{\alpha}\ , \tag{A1.12}$$

where $r^{\alpha} = z^{\alpha}$ are the components of $\underset{\sim}{r}$. If x^{i} is an admissible coordinate system in Euclidean space, i.e. if there exist the coordinate transformations

$$\begin{aligned} x^{i} &= x^{i}\left(z^{1}, \ldots, z^{n}\right)\ , \\ z^{\alpha} &= z^{\alpha}\left(x^{1}, \ldots, x^{n}\right)\ , \end{aligned} \tag{A1.13}$$

which are analytic functions in the neighbourhood of the point $\underset{\sim}{z}$, the components of the position vector $\underset{\sim}{r}$ with respect to the system x^{i} are given by

$$\begin{aligned} r^{i} &= z^{\alpha}\,\frac{\partial x^{i}}{\partial z^{\alpha}}\ , \\ z^{\alpha} &= r^{i}\,\frac{\partial z^{\alpha}}{\partial x^{i}}\ . \end{aligned} \tag{A1.14}$$

Denoting by $\underset{\sim}{g}_{i}$ the base vectors of the coordinate system x^{i}, the position vector $\underset{\sim}{r}$ may be expressed now in the form

$$\underset{\sim}{r} = r^{i}\,\underset{\sim}{g}_{i}\ , \tag{A1. 15}$$

where

$$\underset{\sim}{g}_{i} = \underset{\sim}{e}_{\alpha}\,\frac{\partial z^{\alpha}}{\partial x_{i}} = \frac{\partial \underset{\sim}{r}}{\partial x^{i}}\ . \tag{A1.16}$$

The reciprocal base vectors $\underset{\sim}{g}^{i}$, defined by the relations

$$\underset{\sim}{g}^{i} = \underset{\sim}{e}^{\alpha} \frac{\partial x^{i}}{\partial z^{\alpha}} \tag{A1.17}$$

represent the reciprocal vectorial base. For Cartesian coordinates, the scalar products of the base vectors are

$$\underset{\sim}{e}^{\alpha} \cdot \underset{\sim}{e}_{\beta} = \delta^{\alpha}_{\beta} \ , \ \underset{\sim}{e}_{\alpha} \cdot \underset{\sim}{e}_{\beta} = \delta_{\alpha\beta} \ , \ \underset{\sim}{e}^{\alpha} \cdot e^{\beta} = \delta^{\alpha\beta}, \tag{A1.18}$$

where $\delta^{\alpha}_{\beta} = \delta_{\alpha\beta} = \delta^{\alpha\beta} = \begin{cases} 1, \alpha=\beta \\ 0, \alpha\neq\beta \end{cases}$ are the Kronecker symbols. Hence,

$$\underset{\sim}{g}^{i} \cdot \underset{\sim}{g}_{j} = \underset{\sim}{e}^{\alpha} \cdot \underset{\sim}{e}_{\beta} \frac{\partial x^{i}}{\partial z^{\alpha}} \frac{\partial z^{\beta}}{\partial x^{j}} = \delta^{\alpha}_{\beta} \frac{\partial x^{i}}{\partial z^{\alpha}} \frac{\partial z^{\beta}}{\partial x^{j}} = \delta^{i}_{j} \ .$$

We shall use the symbol $\underset{\sim}{1}$ for the matrix $\left\{\delta^{\alpha}_{\beta}\right\}$.

The scalar products of the base vectors $\underset{\sim}{g}_{i}$ and $\underset{\sim}{g}^{i}$ give the components of the fundamental tensor (g_{ij} and g^{ij}) for the systems of coordinates x^{i} , which is a symmetric tensor,

$$g_{ij} \equiv \underset{\sim}{g}_{i} \cdot \underset{\sim}{g}_{j} = g_{ji} = \delta_{\alpha\beta} \frac{\partial z^{\alpha}}{\partial x^{i}} \frac{\partial z^{\beta}}{\partial x^{j}} \ , \tag{A1.19}$$

and also

$$g^{ij} = \underset{\sim}{g}^{i} \cdot \underset{\sim}{g}^{j} = g^{ji} = \delta^{\alpha\beta} \frac{\partial x^{i}}{\partial z^{\alpha}} \frac{\partial x^{j}}{\partial z^{\beta}} . \qquad (A1.20)$$

Transvection of co- and contravariant components of the fundamental tensor gives the components of the unit tensor,

$$g^{ij} g_{jk} = \delta^{i}_{k} . \qquad (A1.21)$$

Denoting by G^{ij} the cofactor in the determinant $g = \det g_{ij}$, corresponding to the element g_{ji} such that

$$g \delta^{j}_{k} = G^{ji} g_{ik} , \qquad (A1.22)$$

from (A1.22) we have

$$g^{ji} = \frac{G^{ji}}{g} , \quad g_{ji} = g G_{ji} , \qquad (A1.23)$$

where G_{ji} is the cofactor in $\det g^{ij}$ corresponding to the element g^{ij}, and

$$\det g^{ij} = \left(\det g_{ij}\right)^{-1} . \qquad (A1.24)$$

In 3-dimensional Euclidean space the vectorial product of two base vectors $\underset{\sim}{e}_{\alpha}$ and $\underset{\sim}{e}_{\beta}$, $\alpha \neq \beta$ is the vector $\pm\, \underset{\sim}{e}_\gamma$, α , β , γ all different. If $\alpha\beta\gamma$ is an even permutation of the numbers 123 , we have

$$\underset{\sim}{e}_\alpha \times \underset{\sim}{e}_\beta = \underset{\sim}{e}_\gamma \qquad (\alpha, \beta, \gamma \neq) \tag{A1.25}$$

and if it is an odd permutation,

$$\underset{\sim}{e}_\alpha \times \underset{\sim}{e}_\beta = -\underset{\sim}{e}_\gamma \qquad (\alpha, \beta, \gamma \neq) \tag{A1.26}$$

Hence, we may define completely antisymmetric unit tensors $e_{\alpha\beta\gamma}$ and $e^{\alpha\beta\gamma}$ by the scalar products

$$\begin{aligned} \left(\underset{\sim}{e}_\alpha \times \underset{\sim}{e}_\beta\right) \cdot \underset{\sim}{e}_\gamma &= e_{\alpha\beta\gamma} \\ \left(\underset{\sim}{e}^\alpha \times \underset{\sim}{e}^\beta\right) \cdot \underset{\sim}{e}^\gamma &= e^{\alpha\beta\gamma} \end{aligned} \tag{A1.27}$$

Under arbitrary coordinate transformations the unit tensors $\underset{\sim}{e}$ do not behave as tensors. The transformation law involves the Jacobian of the coordinate transformation and such tensors are named <u>relative tensors</u>. However, if we make the scalar products analoguous to (A1.27), we obtain using the relations (A1. 16,17)

$$(g_i \times g_j)\, \underset{\sim}{g}_k = e_{\alpha\beta\gamma} \frac{\partial z^\alpha}{\partial x^i} \frac{\partial z^\beta}{\partial x^j} \frac{\partial z^\gamma}{\partial x^k} = \left(\det \frac{\partial z^\lambda}{\partial x^l}\right) e_{ijk} , \tag{A1.28}$$

where e_{ijk} are now numerical symbols with the same meaning the unit tensors for Cartesian coordinates have. From (A1.19) we have now

$$g = \left(\det \delta_{\alpha\beta}\right)\left(\det \frac{\partial z^\alpha}{\partial x^i}\right)^2 = \left(\det \frac{\partial z^\alpha}{\partial x^i}\right)^2 ,$$

and therefore for (A1.28) we may write

$$\epsilon_{ijk} \equiv \left(\underset{\sim}{g}_i \times \underset{\sim}{g}_j\right) \cdot \underset{\sim}{g}_k \ . \tag{A1.29}$$

Simmilarly we have

$$\epsilon^{ijk} \equiv \left(g^i \times g^j\right) \cdot g^k = \frac{1}{\sqrt{g}}\ e^{ijk} . \tag{A1.30}$$

The quantities ϵ_{ijk} and ϵ^{ijk} are true tensors under arbitrary coordinate transformations and often they are referred to as the <u>Ricci tensors</u>.

Using Ricci tensors an antisymmetric tensor may be represented by a vector. For instance, if $M^{ij} = -M^{ji}$ the tensor M has three independent nonvanishing components in E_3 and we may represent it by a covariant vector

$$M_i = \frac{1}{2}\, \epsilon_{ijk} M^{jk} ,$$

$$\left(M^{jk} = \epsilon^{ijk} M_i\right). \tag{A1.31}$$

Analoguously, if $m^{ijk} = -m^{jik}$ is an antisymmetric third order tensor, we may represent it as a second order mixed tensor,

$$m_{\ell}^{\cdot k} = \frac{1}{2}\,\epsilon_{\ell ij}\, m^{ijk} \,,$$

$$\left(m^{ijk} = \epsilon^{\ell ij}\; m_{\ell}^{\;\cdot k}\right) \,. \qquad \text{(A1.32)}$$

Using the components of the fundamental tensor the operation of raising and lowering of indices may be defined, such that

$$g_{ij}\, T^{\cdots i \cdots}_{\cdots \bullet \cdots} = T^{\cdots \bullet \cdots}_{\cdots j \cdots} \,, \qquad \text{(A1.33)}$$

and

$$g^{ij}\, T^{\cdots \bullet \cdots}_{\cdots j \cdots} = T^{\cdots i \cdots}_{\cdots \bullet \cdots} \,. \qquad \text{(A1.34)}$$

Thus,

$$t^{ij} = g^{jk}\, t^{i}_{\cdot k} = g^{jk}\, g^{i\ell}\, t_{\ell k} = g^{i\ell}\, t_{\ell}^{\cdot j} \,,$$

and for the scalar product of two vectors, say $\underset{\sim}{u}$ and $\underset{\sim}{v}$, we may write

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u^{i} v_{i} = g_{ij}\, u^{i} v^{j} = g^{ij}\, u_{i} v_{j} = u_{i} v^{i} \,. \qquad \text{(A1.35)}$$

The vectorial product of two vectors, say $\underset{\sim}{a}$ and $\underset{\sim}{b}$, is a second-order antisymmetric tensor,

$$\underset{\sim}{a} \times \underset{\sim}{b} = \left\{ a^i b^j - a^j b^i \right\} = \left\{ c^{ij} \right\} , \qquad \text{(A1.36)}$$
$$c^{ij} = - c^{ji} ,$$

and using the Ricci tensor we may represent it as a vector $\underset{\sim}{c}$,

$$c_k = \frac{1}{2} \epsilon_{ijk} c^{ij} = \epsilon_{ijk} a^i b^j . \qquad \text{(A1.37)}$$

Tensors, as geometrical quantities, are defined at points of the space, and the operations of addition may be performed only if the tensors considered are brought to the same point of the space. If we have to add two tensors, or to compare them, and they are not defined at the same point, one of the tensors must be shifted parallelly to the point in which the other tensor is defined. In Cartesian coordinates the components of a vector which represents a field of parrallel vectors at all points of the space are equal, but with respect to curvilinear coordinates this is not true and we have to define the operation of parallel shifting which will enable us to compare components of tensors which are not given at the same point.

Let $\underset{\sim}{v}$ be a field of parallel vectors in E_3 and let v^k be its components at a point $\underset{\sim}{x}$, and V^K its components at a point $\underset{\sim}{X}$. The two points may, in general, be determined with respect to two different coordinate systems, x^k and X^K. Let $\underset{\sim}{z}$ and $\underset{\sim}{Z}$ be the coordinates of the two points considered with respect to an absolute Cartesian system of reference and v^λ and V^Λ the components of the vector field $\underset{\sim}{v}$ with respect to this Cartesian system. Since by assumption $\underset{\sim}{v}$ is a field of parallel vectors, we have

$$v^\lambda = \delta^\lambda_\Lambda V^\Lambda \quad \text{, or} \quad V^\Lambda = \delta^\Lambda_\lambda v^\lambda . \tag{A1.38}$$

According to the transformation law for vectors we have

$$v^\lambda = v^k \frac{\partial z^\lambda}{\partial x^k} , \quad V^\Lambda = V^K \frac{\partial Z^\Lambda}{\partial X_K} , \tag{A1.39}$$

and the relations (A1.38)may be written in the form

$$v^k = \delta^\lambda_\Lambda \frac{\partial x^k}{\partial z^\lambda} \frac{\partial Z^\Lambda}{\partial X_K} V^K , \quad V^K = \delta^\Lambda_\lambda \frac{\partial X^K}{\partial Z^\Lambda} \frac{\partial z^\lambda}{\partial x^k} v^k \tag{A1.40}$$

The quantities

$$g^k_{\cdot K} \equiv \delta^\lambda_\Lambda \frac{\partial x^k}{\partial z^\lambda} \frac{\partial Z^\Lambda}{\partial X^K} , \quad g_k^{\cdot K} \equiv \delta^\Lambda_\lambda \frac{\partial X^K}{\partial Z^\Lambda} \frac{\partial z^\lambda}{\partial x^k} , \tag{A1.41}$$

$$\left(\text{with} \quad g^k_{\cdot K} g_l^{\cdot K} = \delta^k_l \quad , \quad g^k_{\cdot K} g_k^{\cdot L} = \delta^L_K \right) ,$$

are the <u>Euclidean shifters</u> (Doyle and Ericksen [67] , Toupin [370]). Using the shifters we may perform the shifting of an arbitrary tensor from one point of the space to another.

As an example let us consider a vector

field $\underset{\sim}{v}$ at a point (R,Φ) given with respect to a system of polar coordinates in the Euclidean plane, and let us shift it to a point (r,φ) given with respect to the same system of coordinates. Since $Z^1 = X$, $Z^2 = Y$; $z^1 = x$, $z^2 = y$; $X^1 = R$, $X^2 = \Phi$; $x^1 = r$, $x^2 = \varphi$ and since the coordinate transformations at the two considered points are

$$X = R\cos\Phi \quad , \quad Y = R\sin\Phi$$

$$x = r\cos\varphi \quad , \quad y = r\sin\varphi$$

from (A1.41) we obtain the following expressions for the components of the shifter:

$$g^{1}_{\cdot 1} = \cos(\varphi - \Phi) \quad , \quad g^{1}_{\cdot 2} = R\sin(\varphi - \Phi)$$

$$g^{2}_{\cdot 1} = \frac{1}{r}\sin(\Phi - \varphi) \quad , \quad g^{2}_{\cdot 2} = \frac{R}{r}\cos(\Phi - \varphi)$$

Using now $(A1.40)_1$ we easily obtain the components v^k of the vector $\underset{\sim}{v}$ when shifted from the point (R,Φ) to the point (r,φ):

$$v^1 = V^1\cos(\varphi - \Phi) + RV^2\sin(\varphi - \Phi) \ ,$$

$$v^2 = \frac{1}{r}V^1\sin(\Phi - \varphi) + \frac{R}{r}V^2\cos(\Phi - \varphi) \ .$$

The shifters $g^{k}_{\cdot K}$ represent another example of double tensors, and applying them to an arbitrary tensor by parallel shifting we perform the conversion of indices, e.g. .

$$g^{l}_{\cdot K}\, T^{K}_{\cdot PQ} = T^{l}_{\cdot PQ} \; .$$

If g_{mn} and G_{MN} are components of the fundamental tensors corresponding to the coordinate systems x^k and X^K at the points $\underset{\sim}{x}$ and $\underset{\sim}{X}$ of the space, from (A1.19) and (A1. 41) we obtain

$$g^{k}_{\cdot K}\, g^{l}_{\cdot L}\, g_{kl} = \delta_{AB} \frac{\partial Z^A}{\partial X^K} \frac{\partial Z^B}{\partial X^L} \equiv G_{KL}$$

Let $\underset{\sim}{g}_k$, $\underset{\sim}{g}^k$, $\underset{\sim}{G}_K$ and $\underset{\sim}{G}^K$ be base vectors for curvilinear coordinate systems x^k and X^K respectively. According to (A1.16,17) we have

$$\underset{\sim}{g}_k = \frac{\partial \underset{\sim}{r}}{\partial x^k} = \frac{\partial z^\alpha}{\partial x^k}\, \underset{\sim}{e}_\alpha \quad , \quad \underset{\sim}{g}^k = \frac{\partial x^k}{\partial z^\alpha}\, \underset{\sim}{e}^\alpha \quad ,$$

$$\underset{\sim}{G}_K = \frac{\partial \underset{\sim}{R}}{\partial X^K} = \frac{\partial Z^\alpha}{\partial X^K}\, \underset{\sim}{e}_\alpha \quad , \quad G^K = \frac{\partial X^K}{\partial Z^\alpha}\, \underset{\sim}{e}^\alpha \quad .$$

The Euclidean shifters may be defined as scalar products of the base vectors considered at two different points of the space,

$$\underset{\sim}{g}_k\, \underset{\sim}{G}^K = g_k^{\cdot K} \quad , \quad \underset{\sim}{g}^k\, \underset{\sim}{G}_K = g^{k}_{\cdot K} \quad , \qquad \text{(A1.42)}$$

and we may write the following formulae:

$$G_{KL}\, g_{k}^{\cdot K} = g_{kL} = \underset{\sim}{g}_{k}\, \underset{\sim}{G}_{L}\,,$$
$$g_{Kl}\, g^{l}_{\cdot K} = g_{kK} = \underset{\sim}{g}_{k}\, \underset{\sim}{G}_{K}\,. \tag{A1.43}$$

The infinitesimal displacements $d\underset{\sim}{r}$ at a point $\underset{\sim}{x}$ are vectors of the form

$$d\underset{\sim}{r} = dx^{i}\, \underset{\sim}{g}_{i} = \frac{\partial z^{\alpha}}{\partial x^{i}}\, dx^{i}\, \underset{\sim}{e}_{\alpha}\,, \tag{A1.44}$$

and the square of the displacement $d\underset{\sim}{r}$ represents the fundamental (metric) form for the space and for the considered system of coordinates,

$$ds^{2} = d\underset{\sim}{r}\, d\underset{\sim}{r} = \underset{\sim}{g}_{i}\, \underset{\sim}{g}_{j}\, dx^{i}\, dx^{j} = g_{ij}\, dx^{i}\, dx^{j}\,. \tag{A1.45}$$

Hence, the fundamental tensor in the Euclidean space is the metric tensor.

Physical components of vectors and tensors are defined only for orthogonal systems of coordinates $\left(g_{ij} = 0 \quad \text{for } i \neq j\right)$. If we write for the base vectors $\underset{\sim}{g}_{i} = h_{i}\, \underset{\sim}{g}_{0i}$, with $h_{i} = |\underset{\sim}{g}_{i}|$, where $\underset{\sim}{g}_{0i}$ are unit vectors colinear with the base vectors, evidently we have

$$h_{i} = \sqrt{g_{ii}} \tag{A1.46}$$

and

$$\underset{\sim}{g}_{oi} = \frac{\underset{\sim}{g}_i}{\sqrt{g_{ii}}} \quad \text{(not summed)} . \tag{A1.47}$$

We may also write $\underset{\sim}{g}^i = h^i \underset{\sim}{g}_0^i$ with

$$h^i = \sqrt{g^{ii}} \quad , \quad \underset{\sim}{g}_o^i = \frac{1}{\sqrt{g^{ii}}} \underset{\sim}{g}^i \quad \text{(not summed)}, \tag{A1.48}$$

and from (A1.23) we see that for orthogonal coordinate systems

$$g^{ii} = \frac{1}{g^{ii}} \quad . \tag{A1.49}$$

The physical components of a vector are scalar products of the vector and of unit vectors co-linear with the base vectors. Thus, for the physical components of a vector $\underset{\sim}{V}$, which will be denoted by $V(i)$ since the indices are neither co- nor contra-variant we have

$$V(i) = \underset{\sim}{V}\, \underset{\sim}{g}_{oi} = \frac{1}{\sqrt{g_{ii}}} V^k \underset{\sim}{g}_k \underset{\sim}{g}_i = V_i / \sqrt{g_{ii}} \tag{A1.50}$$

$$= \underset{\sim}{V} \cdot \underset{\sim}{g}_o^i = V^i / \sqrt{g^{ii}} .$$

Physical components of tensors are defined in analogy to the definition just introduced for vectors, e. g. for a second-order tensor we have

$$t(ij) = \frac{t^{ij}}{\sqrt{g^{ii} g^{jj}}} = \frac{t_{ij}}{\sqrt{g_{ii} g_{jj}}} = \frac{t^{i}_{\cdot j}}{\sqrt{g^{ii} g_{jj}}} . \tag{A1.51}$$

Besides the decomposition of a second-order tensor into its symmetric and antisymmetric parts, for mixed tensors also may be introduced a decomposition into its deviatoric and spherical parts. The deviator of a tensor $\underset{\sim}{T}$ is defined by the expression

$$ {}^{D}T^{i}_{\cdot j} \equiv T^{i}_{\cdot j} - \frac{1}{3} T^{k}_{\cdot k} \delta^{i}_{j} , \tag{A1.52} $$

and its spherical tensor will be

$$ {}^{S}T^{i}_{\cdot j} \equiv \frac{1}{3} T^{k}_{\cdot k} \delta^{i}_{j} , $$

such that for the considered tensor we have

$$ T_{\cdot j} = {}^{D}T^{i}_{\cdot j} + {}^{S}T^{i}_{\cdot j} \quad . $$

A. 2. Invariants.

Let $\underset{\sim}{T}_{(1)}, \ldots , \underset{\sim}{T}_{(k)}$ be tensor variables. Any scalar function of these variables,

$$ f\left(\underset{\sim}{T}_{(1)}, \ldots , \underset{\sim}{T}_{(k)} \right) , \tag{A2.1} $$

which remains invariant with respect to arbitrary coordinate transformations is an absolute invariant of the tensor $\underset{\sim}{T}_{(1)}, \ldots , \underset{\sim}{T}_{(k)}$. However, there are invariants only with respect to some particular groups of transformations. We are mostly interested in orthogonal transformations.

For a linear transformation of Cartesian co ordinates

$$\bar{z}^{\lambda} = Q^{\lambda}_{\cdot\mu}\, z^{\mu} + a^{\lambda}, \quad z^{\lambda} = Q_{\mu}^{\cdot\lambda}\, \bar{z}^{\mu} + b^{\lambda}, \qquad \text{(A2.2)}$$

we say that it is orthogonal if

$$Q \equiv \det Q^{\lambda}_{\cdot\mu} = \pm 1 \ , \qquad \text{(A2.3)}$$

and the matrix of the coefficients of this transformation has the properties $\underset{\sim}{Q}^{T} = \underset{\sim}{Q}^{-1}$, where T denotes the transposition of a matrix, If $Q = \pm 1$, the transformation (A2.2) belongs to the group of full orthogonal transformations, and if $Q = +1$, we have the group of proper transformations.

Functions (A2.1) invariant with respect to the full orthogonal group are called isotropic invariants, and if they are invariant only with respect to a subgroup of the full orthogonal group, then it is said that they are relative invariants with respect to that subgroup. If a function is invariant only under the transformations of the group of proper orthogonal transformations, such invariants are called hemitropic invariants.

If $\underset{\sim}{T}$ is a symmetric tensor of the second order, the principal invariants of $\underset{\sim}{T}$ are:

$$I_{T} = \frac{1}{1!}\, \delta^{i}_{\ell}\, T^{\ell}_{i} \ , \qquad \text{(A2.4)}$$

$$\mathrm{II}_T = \frac{1}{2!}\,\delta^{ij}_{lm}\,T^l_i\,T^m_j \,,$$

$$\mathrm{III}_T = \frac{1}{3!}\,\delta^{ijk}_{lmn}\,T^l_i\,T^m_j\,T^n_k \quad ,$$

and all three invariants are isotropic.
Here we have used the symbols

$$\delta^{ijk}_{lmn} \equiv \epsilon^{ijk}\epsilon_{lmn} \;,$$

$$\delta^{ij}_{lm} \equiv \delta^{ijn}_{lmn} = \delta^i_l\,\delta^j_m - \delta^i_m\,\delta^j_l \;.$$

The principal directions of a second-order symmetric tensor are the directions determined by the unit vectors $\underset{\sim}{n}$, such that $T^\alpha_\beta n^\beta = T n^\alpha$, or

$$\left(T^\alpha_\beta - T\delta^\alpha_\beta\right) n^\beta = 0 \quad , \qquad \text{(A2.5)}$$

and there are three such directions. Since the equations (A2.5) are homogeneous, the nontrivial solutions for $\underset{\sim}{n}$ exist if

$$\det\left(T^\alpha_\beta - T\delta^\alpha_\beta\right) = 0 \;, \qquad \text{(A2.6)}$$

which represents a third-order equation in T ,

$$-T^3 + \mathrm{I}_T T^2 - \mathrm{II}_T T + \mathrm{III}_T = 0 \quad , \qquad \text{(A2.7)}$$

and the solutions $T_{(\lambda)}$ are the <u>principal values</u> (eigenvalues, proper values) of the tensor $\underset{\sim}{T}$.

If we denote by $\underset{\sim}{n}^{(\alpha)}$ the vectors of a triad reciprocal to the triad of the vectors $\underset{\sim}{n}_{(\alpha)}$ obtained for $\lambda = 1, 2, 3$ from (A2.5), it is possible to introduce a coordinate transformation so that the new Cartesian coordinates $\bar{z}^{\alpha}$ are colinear with the principal directions,

$$\bar{z}^{\lambda} = n^{(\lambda)}_{\alpha} z^{\alpha} \quad , \qquad\qquad \text{(A2.8)}$$

$$z^{\alpha} = n^{\alpha}_{(\lambda)} \bar{z}^{\lambda} \quad ,$$

where

$$n^{(\lambda)}_{\alpha} n^{\alpha}_{(\mu)} = \delta^{\lambda}_{\mu} \quad , \quad n^{(\lambda)}_{\alpha} n^{\beta}_{(\lambda)} = \delta^{\beta}_{\alpha} \ . \qquad\qquad \text{(A2.9)}$$

The components $\bar{T}^{\lambda}_{\mu}$ of $\underset{\sim}{T}$ with respect to the new coordinates $\bar{z}^{\alpha}$ are

$$\bar{T}^{\lambda}_{\mu} = T^{\alpha}_{\beta} n^{\beta}_{(\mu)} n^{(\lambda)}_{\alpha} \quad ,$$

and according to (A2.5) and (A2.9) we have

$$\bar{T}^{\lambda}_{\mu} = T_{(\mu)} n^{\alpha}_{(\mu)} n^{(\lambda)}_{\alpha} = T_{(\mu)} \delta^{\lambda}_{\mu} \ . \qquad\qquad \text{(A2.10)}$$

Hence, the principal values of a tensor $\underset{\sim}{T}$ are its components with respect to a Cartesian coordinate system with coordinate axes colinear with the principal directions. With respect to this system of coordinates the matrix of the tensor $\underset{\sim}{T}$ has only diagonal elements.

The powers of a tensor $\underset{\sim}{T}$ are defined by the expressions

$$\overset{2}{T}{}^{\lambda}_{\cdot\mu} = T^{\lambda}_{\cdot\alpha}\, T^{\alpha}_{\cdot\mu} \ ,$$

$$\overset{3}{T}{}^{\lambda}_{\cdot\mu} = T^{\lambda}_{\cdot\alpha}\, T^{\alpha}_{\cdot\beta}\, T^{\beta}_{\cdot\mu} , \tag{A2.11}$$

$$\cdots\cdots$$

and from (A2.10) it follows that

$$\overset{2}{T}{}^{\lambda}_{\mu} = T^{2}_{(\mu)}\, \delta^{\lambda}_{\mu} \ , \quad \overset{3}{T}{}^{\lambda}_{\mu} = T^{3}_{(\mu)}\, \delta^{\lambda}_{\mu} \ , \ \ldots \ .$$

Since $T_{(\mu)}$ are the solutions of (A2.7) we have obviously

$$T^{3}_{(\mu)}\delta^{\lambda}_{\mu} = \mathrm{I}_T T_{(\mu)}\delta^{\lambda}_{\mu} - \mathrm{II}_T T_{(\mu)}\delta^{\lambda}_{\mu} + \mathrm{III}_T\, \delta^{\lambda}_{\mu} \tag{A2.12}$$

or

$$\overset{3}{\underset{\sim}{T}} = \mathrm{I}_T \overset{2}{\underset{\sim}{T}} - \mathrm{II}_T \underset{\sim}{T} + \mathrm{III}_T \ \underset{\sim}{1} \ , \tag{A2.13}$$

which represents the Cayley-Hamilton theorem.

For an antisymmetric tensor $M^{\alpha\beta\gamma} = -M^{\beta\alpha\gamma}$ of the third order, the corresponding second order tensor, according to (A1.32) is given by

$$M_{\lambda}^{\cdot\gamma} = \frac{1}{2}\, e_{\lambda\alpha\beta}\, M^{\alpha\beta\gamma} \ . \tag{A2.14}$$

Because of the nonsymmetry of $\underset{\sim}{M}$ for the construction of the invariants we have to regard besides its components $M_{\lambda}^{\cdot\gamma}$ also the components

$M^{\mu}_{\cdot\nu} = g^{\lambda\mu} g_{\gamma\nu} M_{\lambda}^{\cdot\gamma}$, which makes the list of invariants larger than the list of invariants of a symmetric second-order tensors. There is one linear invariant,

$$I_M = \delta^{k}_{l}\; M_{k}^{\cdot l} \;, \tag{A2.15}$$

but there are two independent quadratic invariants,

$$\begin{aligned} {}^1\mathrm{II}_M &= \frac{1}{2!}\,\delta^{ij}_{lm}\, M_i^{\cdot l} M_j^{\cdot m} \;, \\ {}^2\mathrm{II}_M &= \frac{1}{2!}\,\delta^{ij}_{lm}\, M_i^{\cdot l}\; M^{m}_{\cdot\cdot j} \;, \end{aligned} \tag{A2.16}$$

and there are eight independent cubic invariants, etc.

If we write for I_M the expression

$$I_M = \frac{1}{2}\, e_{\alpha\beta\gamma}\, M^{\alpha\beta\gamma} \;, \tag{A2.17}$$

and apply the orthogonal transformation (A2.2) to the components of $\underset{\sim}{M}$, we obtain

$$I_M = \frac{1}{2}\, e_{\alpha\beta\gamma}\, Q_{\lambda}^{\cdot\alpha} Q_{\mu}^{\cdot\beta} Q_{\nu}^{\cdot\gamma}\, \bar{M}^{\lambda\mu\nu} \,.$$

Since

$$e_{\alpha\beta\gamma}\, Q_{\lambda}^{\cdot\alpha} Q_{\mu}^{\cdot\beta} Q_{\nu}^{\cdot\gamma} = \left(\det\; Q_{\lambda}^{\cdot\alpha}\right) e_{\lambda\mu\nu} = \pm 1\, e_{\lambda\mu\nu} \;,$$

and it follows that I_M is a hemitropic invariant.

The invariants ${}^{1}\underline{\mathrm{II}}_{M}$ and ${}^{2}\underline{\mathrm{II}}_{M}$ are isotropic.

The joint invariants of a symmetric tensor $\underset{\sim}{T}$ and of a non-symmetric tensor $\underset{\sim}{M}$ are:

quadratic $\quad \underline{\mathrm{II}}_{TM} = T^{i}_{\ell} M^{\ell}_{\cdot i} = T^{i}_{\ell} M_{i}^{\cdot \ell}$; (A2.18)

cubic
$$
\begin{aligned}
{}^{1}\underline{\mathrm{III}}_{TM} &= T^{i}_{\ell}\, T^{\ell}_{m}\, M_{i}^{\cdot m} ,\\
{}^{2}\underline{\mathrm{III}}_{TM} &= T^{i}_{\ell}\, M_{i}^{\cdot m} M_{m}^{\cdot \ell} ,\\
{}^{3}\underline{\mathrm{III}}_{TM} &= T^{i}_{\ell}\, M^{m}_{\cdot i} M_{m}^{\cdot \ell} ,\\
{}^{4}\underline{\mathrm{III}}_{TM} &= T^{i}_{\ell}\, M^{m}_{\cdot i} M^{\ell}_{\cdot m} ,
\end{aligned} \tag{A2.19}
$$

Possible are also other combinations of one symmetric and one non-symmetric second-order tensor, which are not listed in (A2. 18,19), but it may easily be varified that the listed invariants (cubic and quadratic) are the only independent invariants. For higher order invariants I have not tried to establish the list of the independent invariants.

Among the listed joint invariants, $\underline{\mathrm{II}}_{TM}$ and ${}^{1}\underline{\mathrm{III}}_{TM}$ are hemitropic, and the remaining invariants are isotropic.

The principal invariants of a symmetric tensor $\underset{\sim}{T}$ may be expressed also in terms of the principal values of $T_{(\lambda)}$,

$$\begin{aligned} \mathrm{I}_T &= T_{(1)} + T_{(2)} + T_{(3)} \,, \\ \mathrm{II}_T &= T_{(2)}T_{(3)} + T_{(3)}T_{(1)} + T_{(1)}T_{(2)} \,, \\ \mathrm{III}_T &= T_{(1)}T_{(2)}T_{(3)} \,. \end{aligned} \tag{A2.20}$$

Sometimes it is useful to consider the <u>moments</u> $\overline{\mathrm{II}}_T, \overline{\mathrm{III}}_T$, instead of the principal invariants. The moments are related to the principal invariants by the formulae

$$\begin{aligned} \overline{\mathrm{II}}_T &= T^{i}_{\cdot j} T^{j}_{\cdot i} = \mathrm{I}_T^2 - 2\,\mathrm{II}_T = \sum_{\alpha=1}^{3} T_{(\alpha)}^2 \,, \\ \overline{\mathrm{III}}_T &= T^{i}_{\cdot j} T^{j}_{\cdot k} T^{k}_{\cdot i} = \mathrm{I}_T^3 - 3\,\mathrm{I}_T \mathrm{II}_T + 3\,\mathrm{III}_T = \sum_{\alpha=1}^{3} T_{(\alpha)}^3 \,. \end{aligned} \tag{A2.21}$$

In the theory of plasticity often is used the so-called <u>octahedral invariant</u> Δ_T;

$$3\,\Delta_T = \left[2\,\mathrm{I}_T^2 - 6\,\mathrm{II}_T\right]^{1/2} = \sum_{\alpha>\beta}\left[\left(T_{(\alpha)} - T_{(\beta)}\right)^2\right]^{1/2} . \tag{A2.22}$$

If a tensor is decomposed into its spherical and deviatoric parts,

$$T^{i}_{\cdot j} = \frac{1}{3}\,\mathrm{I}_T\,\delta^i_j + \left(T^{i}_{\cdot j} - \frac{1}{3}\,\mathrm{I}_T\,\delta^i_j\right) = {}^{s}T^{i}_{j} + \tau^{i}_{\cdot j} \,, \tag{A2.23}$$

the principal invariants of the spherical part are

$${}^{s}\mathrm{I}_T = \mathrm{I}_T \,;\, {}^{s}\mathrm{II}_T = \frac{1}{3}\,\mathrm{I}_T^2 \,;\, {}^{s}\mathrm{III}_T = \frac{1}{27}\,\mathrm{I}_T^3 \,, \tag{A2.24}$$

and the first invariant of the deviatoric part vanishes identically,

$$\underline{I}_{\tau} \equiv {}^{D}\underline{I}_{T} = 0 . \tag{A2.25}$$

Since (A2.25) represents a relation between nine components of a tensor, it follows that a deviator has only eight independent components.

A second-order tensor can be uniquely decomposed into its symmetric and antisymmetric parts. For a third-order tensor such a decomposition is more involved because we are searching for its irreducible parts. Toupin [371] introduced the following decomposition.

Let M^{ijk} be an arbitrary tensor of the third order. Its irreducible parts are:
the symmetric part

$${}_{S}M^{ijk} = M^{(ijk)} = \frac{1}{3!}\left(M^{ijk} + M^{jki} + M^{kij} + M^{ikj} + M^{jik} + M^{kji}\right), \tag{A2.26}$$

the antisymmetric part

$${}_{A}M^{ijk} = M^{[ijk]} = \frac{1}{3!}\left(M^{ijk} + M^{jki} + M^{kij} - M^{ikj} - M^{jik} - M^{kji}\right), \tag{A2.27}$$

the principal parts

$$\begin{aligned} {}_{P}M^{ijk} &= \frac{1}{3}\left(M^{ijk} + M^{kji} - M^{jik} - M^{kij}\right), \\ {}_{\bar{P}}M^{ijk} &= \frac{1}{3}\left(M^{ijk} + M^{jik} - M^{kji} - M^{jki}\right) . \end{aligned} \tag{A2.28}$$

The symmetric part ${}_{S}\underset{\sim}{M}$ has 10 independent components, the antisymmetric part has 1, and the principal parts ${}_{P}\underset{\sim}{M}$ and ${}_{\bar{P}}\underset{\sim}{M}$ have 8 independent components each, so that the tensor $\underset{\sim}{M}$ is determined by 27 independent components of its irreducible parts, and

$$\underset{\sim}{M} = {}_{S}\underset{\sim}{M} + {}_{A}\underset{\sim}{M} + {}_{P}\underset{\sim}{M} + {}_{\bar{P}}\underset{\sim}{M} \ . \tag{A2.30}$$

A. 3. Differentiation.

If $\underset{\sim}{V}$ is a vector field in E_3 with components V^k and V_ℓ with respect to a coordinate system x^i, the partial derivatives of the vector $\underset{\sim}{V}$ are given by the expressions

$$\frac{\partial \underset{\sim}{V}}{\partial x^m} = \frac{\partial V^k}{\partial x^m}\underset{\sim}{g}_k + V^k \frac{\partial \underset{\sim}{g}_k}{\partial x^m} = \left(\frac{\partial V^k}{\partial x^m} + \begin{Bmatrix} k \\ m\ell \end{Bmatrix} V^\ell\right)\underset{\sim}{g}_k \ , \tag{A3.1}$$

or

$$\frac{\partial \underset{\sim}{V}}{\partial x^m} = \frac{\partial V_\ell}{\partial x^m}\underset{\sim}{g}^\ell + V_\ell \frac{\partial \underset{\sim}{g}^\ell}{\partial x^m} = \left(\frac{\partial V_\ell}{\partial x^m} - \begin{Bmatrix} k \\ \ell m \end{Bmatrix} V_k\right)\underset{\sim}{g}^\ell \ , \tag{A3.2}$$

where

$$V^k_{,m} \equiv \frac{\partial V^k}{\partial x^m} + \begin{Bmatrix} k \\ m\ell \end{Bmatrix} V^\ell \ , \tag{A3.3}$$

$$V_{\ell,m} \equiv \frac{\partial V_\ell}{\partial x^m} - \left\{ {k \atop m\ell} \right\} V_k \quad , \qquad \text{(A3.4)}$$

represent the <u>covariant derivatives</u> of co- and contravariant components of the vector field $\underset{\sim}{V}$.

The quantities

$$[\ell m,n] \equiv \frac{\partial \underset{\sim}{g}_m}{\partial x^\ell} \cdot \underset{\sim}{g}_n = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial x^\ell} + \frac{\partial g_{n\ell}}{\partial x^m} - \frac{\partial g_{\ell m}}{\partial x^n} \right) \qquad \text{(A3.5)}$$

are the <u>Christoffel symbols</u> of the first kind, and

$$\left\{ {k \atop m\ell} \right\} \equiv g^{kn} [\ell m,n] = \frac{\partial \underset{\sim}{g}_\ell}{\partial x^m} \cdot \underset{\sim}{g}^k \qquad \text{(A3.6)}$$

are the Christoffel symbols of the second kind.

In general, if $\underset{\sim}{T}$ is a tensor of contravariant order p and covariant order q , the covariant derivatives of its components are tensors of contravariant order p and covariant order $q+1$,

$$\begin{aligned} T^{i_1 \dots i_p}{}_{j_1 \dots j_q , k} = {} & \frac{\partial}{\partial x^k} T^{i_1 \dots i_p}{}_{j_1 \dots j_q} + \\ & + \sum_{\alpha=1}^{p} \left\{ {i_\alpha \atop k\ell} \right\} T^{i_1 \dots i_{\alpha-1}\, \ell\, i_{\alpha+1} \dots i_p}{}_{j_1 \dots j_q} \\ & - \sum_{\beta=1}^{q} \left\{ {\ell \atop k j_\beta} \right\} T^{i_1 \dots i_p}{}_{j_1 \dots j_{\beta-1}\, \ell\, j_{\beta+1} \dots j_q} \ . \end{aligned} \qquad \text{(A3.7)}$$

For the sake of brevity we write sometimes for partial derivatives

$$\frac{\partial}{\partial x^m} = \partial_m \; . \tag{A3.8}$$

The covariant differential of a tensor $\underset{\sim}{T}$ is a tensor of the same order, defined by the expression

$$\delta T^{\cdots}_{\cdots} = T^{\cdots}_{\cdots ,k} \, dx^k \; . \tag{A3.9}$$

Let $\underset{\sim}{T}$ be a time-dependent tensor field. The absolute time derivatives of the components of $\underset{\sim}{T} = \underset{\sim}{T}(\underset{\sim}{x}, t)$ are defined by the formula

$$\frac{DT^{\cdots}_{\cdots}}{dt} = \frac{\partial T^{\cdots}_{\cdots}}{\partial t} + T^{\cdots}_{\cdots ,k} \frac{dx^k}{dt} \equiv \dot{T}^{\cdots}_{\cdots} \; . \tag{A3.10}$$

For double tensor fields we define partial and total covariant derivatives. If $T^{k}_{\cdot K}(\underset{\sim}{x}, \underset{\sim}{X})$ is such a tensor, the partial covariant derivatives are defined by

$$T^{k}_{\cdot K,\ell} = \frac{\partial T^{k}_{\cdot K}}{\partial x^{\ell}} + \left\{ {k \atop \ell m} \right\} T^{m}_{\cdot K} \; , \tag{A3.11}$$

$$T^{k}_{\cdot K,L} = \frac{\partial T^{k}_{\cdot K}}{\partial X^{L}} - \left\{ {M \atop LK} \right\} T^{k}_{\cdot M} \; . \tag{A3.12}$$

If there is a mapping $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X})$, the <u>total covariant</u>

However, when the vector field $\underset{\sim}{V}$ is refered to an arbitrary system of curvilinear coordinates x^i, (A4.1) will obtain the form

$$\overset{*}{d} V^k = -\left\{ {k \atop lm} \right\} V^m \, dx^l . \qquad \text{(A4.2)}$$

The vector field V^k at a point $\underset{\sim}{x} + d\underset{\sim}{x}$ has the components

$$V^k \left(\underset{\sim}{x} + d\underset{\sim}{x} \right) = V^k \left(\underset{\sim}{x} \right) + \partial_l V^k dx^l + \ldots \, . \qquad \text{(A4.3)}$$

The difference between the field value of the vector $\underset{\sim}{V}$ at $\underset{\sim}{x} + d\underset{\sim}{x}$ and $V^k + dV^k$ is the covariant differential,

$$\delta V^k = V^k \left(\underset{\sim}{x} + d\underset{\sim}{x} \right) - \overset{*}{V}{}^k = \left(\partial_l V^k + \left\{ {k \atop lm} \right\} V^m \right) dx^l . \qquad \text{(A4.4)}$$

According to (A4.2) parallelism in Euclidean space is defined (in the sense of differential geometry) as a linear connection of the increment $\overset{*}{d} V^k$ of the components of the vector V^k and the components dx^l of the displacement.

The law (A4.2) may be generalized writing

$$dV^k = - \Gamma^k_{lm} V^m \, dx^l , \qquad \text{(A4.5)}$$

where Γ^k_{lm} are arbitrary functions of position and are called <u>coefficients of connection</u> of a <u>linearly connected space</u> L_3.

derivative with respect to x^{ℓ} and X^{L} are defined as a generalization of the classical rule

$$T^{k}_{\cdot K;\ell} = T^{k}_{\cdot K,\ell} + T^{k}_{\cdot K,L}\, X^{L}_{;\ell} \quad , \tag{A3.13}$$

$$T^{k}_{\cdot K;L} = T^{k}_{\cdot K,L} + T^{k}_{\cdot K,\ell}\, x^{\ell}_{;L} \quad , \tag{A3.14}$$

where $X^{L}_{;\ell}$ and $x^{\ell}_{;L}$ are the gradients of the mapping $\underset{\sim}{x} \longleftrightarrow \underset{\sim}{X}$. The chain rule of ordinary differential calculus also holds for total covariant differentiation,

$$\begin{aligned} T^{\cdots}_{\cdots;K} &= T^{\cdots}_{\cdots;k}\, x^{k}_{;K} \\ T^{\cdots}_{\cdots;k} &= T^{\cdots}_{\cdots;K}\, X^{K}_{;k} \quad . \end{aligned} \tag{A3.15}$$

A. 4. Linearly Connected Spaces.

Let V^{α} be components of a vector field in E_3 , refered to a system of Cartesian coordinates and let us perform a parallel displacement of the vec tor $\underset{\sim}{V}$ from a point $\underset{\sim}{z}$ to a neighbouring point $\underset{\sim}{z} + d\underset{\sim}{z}$. The components of the vector $\underset{\sim}{V}$ will remain unchanged. Denoting by $\overset{*}{d} V^{\alpha}$ the change of the components at a parallel displacement along $d\underset{\sim}{z}$ we may write

$$\overset{*}{d} V^{\alpha} = 0 \ . \tag{A4. 1}$$

In general, the coefficients $\Gamma^{k}_{\ell m}$ are not symmetric, and the antisymmetric part $S_{\ell m}^{\cdot\cdot k} \equiv \Gamma^{k}_{[\ell m]}$ is the <u>torsion tensor</u> of the space L_3.

Generalizing the rules for covariant differentiation to linearly connected spaces we may write for the covariant derivatives of a contravariant vector

$$V^{k}_{;\ell} = \partial_\ell V^k + \Gamma^{k}_{\ell m} V^m , \qquad \text{(A4.6)}$$

and from the requirements that $V^{k}_{;\ell}$ transformes like a mixed second-order tensor we obtain the transformation law for the coefficients of connection:

$$\bar{\Gamma}^{i}_{jk} = \Gamma^{\ell}_{mn} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^\ell} + \frac{\partial \bar{x}^i}{\partial x^\ell} \frac{\partial^2 x^\ell}{\partial \bar{x}^j \partial \bar{x}^k} = \qquad \text{(A4.7)}$$

$$= \Gamma^{\ell}_{mn} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^\ell} - \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} .$$

From (A4.7) it follows that $S_{mn}^{\cdot\cdot\ell}$ is a tensor indeed.

Parallelism in an L_n is, according to (A4.5), defined only for infinitesimal displacements. If ABC is a curve in L_3, the total increment ΔV^k of the components V^k of a vector transported parallelly from A to C along the curve will be

$$\Delta V^k = \int_{ABC} \overset{*}{d} V^k = - \int_{ABC} \Gamma^{k}_{\ell m} V^m dx^\ell .$$

If $AB'C$ is another curve connecting the points A and C, the increment of the components of the vector V^k along this curve will be

$$\Delta'' V^k = \int_{AB'C} \overset{*}{d} V^k \, ,$$

and the increments $\Delta' V^k$ and $\Delta'' V^k$ are, in general, not equal, i.e. the integral along the closed contour $ABCB'A$ is not vanishing,

$$\Delta V^k = \oint_{ABCB'A} \overset{*}{d} V^k = - \oint \Gamma^k_{\ell m} V^m dx^\ell = \Delta' V^k - \Delta'' V^k.$$

Denoting $-\Gamma^k_{\ell m} V^m$ by f^k_ℓ and applying the Stokes theorem,

$$\oint f^k_\ell \, dx^\ell = \iint_F f^k_{[\ell, m]} \, dF^{m\ell}$$

where F is the surface enclosed by the contour $ABCB'A$ and $dF^{m\ell}$ are components of the surface element, $\Delta F^{m\ell} = -\Delta F^{\ell m}$, we have

$$\Delta V^k = \iint_F R_{nm\ell}^{\cdot\cdot\cdot k} \, V^\ell \, dF^{mn} \, , \tag{A4.9}$$

where

$$R_{nm\ell}^{\cdot\cdot\cdot k} = \partial_n \Gamma^k_{m\ell} - \partial_m \Gamma^k_{n\ell} + \Gamma^k_{nt} \, \Gamma^t_{m\ell} - \Gamma^k_{mt} \, \Gamma^t_{n\ell} \tag{A4.10}$$

is the <u>Riemann-Christoffel curvature tensor</u>.

If $R_{nm\ell}^{\cdot\cdot\cdot k}$ vanishes at all points of the space, we say that this space is with absolute parallelism (or with teleparallelism).

In Euclidean space the fundamental tensor g_{ij} is covariant constant, i.e. its covariant derivatives are identically equal to zero. If an L_3 admits a symmetric covariant constant vector field g_{ij}, we say thet the space L_3 is metric. Let us assume that an L_3 with the coefficients of connection Γ^k_{lm} is metric and that its fundamental metric tensor is g_{ij}, then we have

$$g_{ij,k} = \partial_k g_{ij} - \Gamma^l_{ki}\, g_{lj} - \Gamma^l_{kj}\, g_{il} = 0. \qquad \text{(A4.11)}$$

The integrability conditions of (A4.11) are

$$\left(\partial_l \partial_k - \partial_k \partial_l\right) g_{ij} = 0 ,$$

and after some calculations they reduce to

$$R^{\cdot\,\cdot\,\cdot\,\cdot}_{\;nm(lk)} = 0 . \qquad \text{(A4.12)}$$

Hence, if the Riemann-Christoffel tensor for a linear connection Γ^k_{lm} is symmetric with respect to the second pair of indices, the connection is metric.

The linearly connected space is Euclidean if: 1° the coefficients of connection are symmetric, 2° it is a metric space, 3° the fundamental form of the space.

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{A4.13}$$

is positive definit, and 4° if the Riemann-Christoffel tensor vanishes everywhere in the space. If all these conditions are satisfied, it is possible to find a coordinate transformation

$$\begin{aligned} x^i &= x^i\left(z^1, z^2, z^3\right) \\ z^\alpha &= z^\alpha\left(x^1, x^2, x^3\right), \end{aligned} \tag{A4.14}$$

such that the fundamental tensor with respect to the new coordinate system z^α obtains the form

$$g_{\alpha\beta} = \frac{\partial x^i}{\partial z^\alpha} \frac{\partial x^j}{\partial z^\beta}\, g_{ij} = \delta_{\alpha\beta} \;. \tag{A4.15}$$

In some problems we have to deal with the correspondence of a set of points of Euclidean space with a set of points of a linearly connected space L_3 . If x^α is a system of coordinates in Euclidean space, and u^α a system of coordinates in L_3 there do not exist 1:1 finite mappings of the form

$$\begin{aligned} x^i &= x^i\left(u^1, u^2, u^3\right) \\ u^\alpha &= u^\alpha\left(x^1, x^2, x^3\right) \end{aligned} \tag{A4.16}$$

but only the local mappings of infinitesimal elements dx^i and du^α,

$$dx^i = \Phi^i_{(\alpha)}\, du^\alpha \;. \tag{A4.17}$$

We assume that the relations (A4.17) are linearly independent,

$$\det \Phi^{i}_{(\alpha)} \neq 0 \qquad (A4;18)$$

so that there exist the inverse relations

$$du^{\alpha} = \Phi^{(\alpha)}_{i}\, dx^{i} \qquad (A4.19)$$

The integrability conditions of (A4.17) read

$$2\, S^{(\alpha)}_{ij} \equiv \partial_j \Phi^{(\alpha)}_{i} - \partial_i \Phi^{(\alpha)}_{j} = 0 \qquad (A4.20)$$

and those conditions are, in general, not satisfied.

The vectors $\underset{\sim}{\Phi}_{(\alpha)}$ constitute in E_3 three vector fields and at each point there are lines the tangents of which are colinear with the vectors $\underset{\sim}{\Phi}_{(\alpha)}$. The differential equations of these lines are

$$\frac{dx^1}{\Phi^1_{(\alpha)}} = \frac{dx^2}{\Phi^2_{(\alpha)}} = \frac{dx^3}{\Phi^3_{(\alpha)}} \quad . \qquad (A4.21)$$

Let us assume that there is a linearly connected space with the coefficients of connection Γ^{k}_{ij} such that the vector fields $\Phi^{i}_{(\alpha)}$ constitute fields of absolutely parallel vectors, i.e. with respect to the connection considered the vectors $\underset{\sim}{\Phi}_{(\alpha)}$ are covariant constant everywhere in the space,

$$\partial_m \Phi^k_{(\alpha)} + \Gamma^{k}_{m\ell} \Phi^\ell_{(\alpha)} = 0 . \qquad (A4.22)$$

Transvection of this with $\Phi^{(\alpha)}_n$ and using the relations

$$\Phi^{(\alpha)}_n \Phi^k_{(\alpha)} = \delta^k_n \quad , \quad \Phi^{(\alpha)}_\ell \Phi^\ell_{(\beta)} = \delta^\alpha_\beta \qquad (A4.23)$$

we obtain

$$\Gamma^{k}_{mn} = - \Phi^{(\alpha)}_n \partial_m \Phi^k_{(\alpha)} = \Phi^k_{(\alpha)} \partial_m \Phi^{(\alpha)}_n . \qquad (A4.24)$$

It may be easily verified that substituting Γ^{k}_{mn} from (A4.24) into the expression (A4.10) for the components of $R_{nm\ell}^{\cdot\cdot\cdot k}$ will identically vanish. According to (A4.12) it follows that the conditions for the space considered to be metric are identically fulfilled.

From the preceding it follows that it is always possible to associate a linearly connected metric space to a non-integrable mapping, and the torsion of this space does not necessarily vanish.

The torsion tensor of the connection (A4.23) is given by

$$S_{mn}^{\cdot\cdot k} = \Phi^k_{(\alpha)} \partial_{[m} \Phi^{(\alpha)}_{n]} = \Phi^k_{(\alpha)} S^{(\alpha)}_{mn} \quad , \qquad (A4.25)$$

and it is obvious that the space associated to a non--integrable mapping will be Euclidean only if the torsion vanishes i.e. if the mapping is integrable (this is a necessary, but not a sufficient condition).

The quantities obtained by transvecting vectors, tensors etc. of Euclidean space with the components of the vectors $\underset{\sim}{\Phi}_{(\alpha)}$, $\underset{\sim}{\Phi}^{(\alpha)}$, e.g.

$$V^{(\alpha)} = V^i \Phi_i^{(\alpha)} \quad , \quad T^{(\alpha)}_{\cdot(\beta)} = T^i_{\cdot j} \Phi_i^{(\alpha)} \Phi^j_{(\beta)} \qquad \text{(A4.26)}$$

are often called <u>non-holonomic components</u> of those quantities.

References.

This list of references includes besides the quoted papers also a number of other papers dealing with polar continua, or containing material which might be of interest for mechanics of polar continua. This is not a complete list of all papers dealing with polar continua and includes only the papers which were available to the author, or the reference to which the author has found in other literature. For some journals are used the usual abbreviations, e.g. PMM = Prikladnaja Matematika i Mekhanika, ZAMM = Zeitschrift fuer Angewandte Mathematik und Mechanik, ZAMP = Zeitschrift fuer Angewandte Mathematik und Physik. Other abbreviations are also generally accepted and in wide use.

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