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" DIMENZIJA F-PROSTORA "

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Želeo bih da izrazim svoju zahvalnost profesoru D. Adnadjeviću koji je pregledao rad, za stalnu podršku i korisne sugestije, Univerzitetu ALFATEH za finansijsku podršku i Mirjani Rapaić koja je prekucala ovaj rad.

S A D R Ž A J:

P R E D G O V O R

Pojam fazi topologije, koji je uopštenje pojma topologije na skupu, uveo je 1968. godine Chang u [2]. Taj pojam se bazira na Zadehovom pojmu fazi skupa koji se pojavio 1965. u [31] i koji se koristi u mnogim granama matematike. Od tada mnogi autori razvijaju različite aspekte fazi topologije što sve više privlači pažnju istraživača u raznim oblastima nauke. Mi uvodimo pojam α-kombinatorne dimenzije fazi prostora, gde je $\alpha\in[0,1)$, kao uopštenje pojma kombinatorne dimenzije topoloških prostora.

Glave II, III i IV ovog rada kao da opravdavaju uvodjenje i proučavanje α-kombinatorne dimenzije fazi prostora.

Analogno teoriji dimenzije topoloških prostora, mi razmatramo teoremu potprostora, teoremu konačne sume i teoremu proizvoda.

Svakom fazi prostoru može se pridružiti topološki prostor snabdeven tzv. modifikovanom topologijom čija je kombinatorna dimenzija jednaka α-kombinatornoj dimenziji fazi prostora. Mi koristimo ideju modifikovane topologije da bismo tvrdjenja u vezi sa α-kombinatornom dimenzijom fazi prostora sveli na tvrdjenja o kombinatornoj dimenziji topoloških prostora.

Gradja ovog rada, osim uvodnih topoloških pojmova, podeljena je na četiri glave, dok se svaka glava sastoji iz tri paragrafa u kojima se obradjuju pojedinačne teme.

U glavi I date su definicije i stavovi u vezi sa osnovnim pojmovima fazi skupova i fazi topologije. Većinu od njih ćemo kasnije koristiti.

Glava II je posvećena α-kombinatornoj dimenziji fazi prostora. Ona se definiše u prvom paragrafu a stavovi 1.1, 1.4 daju njene glavne karakterizacije. Paragraf dva bavi se teoremama podskupa i konačnom teoremom sume, dok paragraf tri obradjuje lokalne osobine α-kombinatorne dimenzije.

Prvi paragraf glave III sadrži primene prethodne glave. Paragraf dva daje relaciju izmedju α-kombinatorne dimenzije fazi prostora i kombinatorne dimenzije topološkog prostora generisanog tim fazi prostorom (stav 2.3), kao i jednakost α-kombinatorne dimenzije ultra - Tychonoffljevog fazi prostora sa - kombinatornom dimenzijom njegove ultra Stone-Čechove fazi kompaktifikacije (stav 2.4). Paragraf tri bavi se teoremom proizvoda za fazi prostore koji zadovoljavaju izvesne uslove.

Finalna glava IV posvećena je proučavanju projektivnog spektra fazi prostora i projektivnog limesa fazi prostora na sličan način kao i kod topoloških prostora. U poslednjem paragrafu ove glave proučava se α-kombinatorna dimenzija limesa fazi prostora.

Simbol □ označava kraj dokaza. On se takodje koristi na kraju nekog tvrdjenja čiji dokaz neposredno sledi iz prethodnih rezultata.

Pojedine stavke su u svakom paragrafu posebno numerisane prema njihovom imenu i položaju tj. definicija 1.1, ..., 1.2, ..., lema 1.1, 1.2,... itd. Pozivanja na svaku stavku u različitoj glavi označavamo trostruko tj. navodi se glava, paragraf i broj stavke u paragrafu. Na primer lema II.3.1 je lema 1 u paragrafu 3 glave II.

Citiranje literature stavljamo u [ ].

Glava 0:

UVODNI TOPOLOŠKI POJMOVI

Pretpostavlja se poznavanje osnova opšte topologije kao i osnovnih pojmova u vezi sa kombinatornom dimenzijom topoloških prostora. U ovom kratkom uvodu dajemo u obliku teorema sažet prikaz nekih rezultata iz ovih oblasti. Neke od tih teorema biće uopštene na slučaj fazi topoloških prostora a neke će biti od posebnog značaja u daljem toku izlaganja.

Dajemo bez dokaza kratak pregled nekih rezultata uzetih uglavnom iz [4] , [5] ,[14] , [21] ,[22] .

Znamo da je proizvod topoloških prostora kompaktan ako i samo ako je svaki činilac kompaktan. Taj rezultat je poznat kao teorema Tychonoffa dok topološki proizvod familije prebrojivo kompaktnih prostora nije u opštem slučaju prebrojivo kompaktan. Ipak važi:

<u>Teorema 1:</u> Proizvod kompaktnog prostora i prebrojivo kompaktnog prostora je prebrojivo kompaktan. □

Projektivni limes projektivnog spektra topoloških prostora može biti prazan čak i ako je svaki njegov činilac (X_s) neprazan a projekcije surjektivne. Sledeće dve teoreme daju dovoljne uslove pod kojima je projektivni limes neprazan.

<u>Teorema 2:</u> Projektivni limes projektivnog spektra nepreznih kompaktnih Hausdorffovih prostora je neprazan kompaktan Hausdorffov prostor. □

Teorema 3: Neka je $\underset{\sim}{X}$ projektivni niz nepraznih prebrojivo kompaktnih prostora. Ako su svi skupovi $f_{n,m}(X_m)$, $n \leq m$, $n \in N$, zatvoreni u X_n, onda je projektivni limes $\widetilde{X}$ neprazan i $f_n(\widetilde{X}) = \bigcap f_{n,m}(X_m)$ $n \leq m$, $n \in N$. □

Teorema 4: Projektivni limes projektivnog niza prebrojivo kompaktnih prostora i zatvorenih projekcija je prebrojivo kompaktan prostor. □

Teorema 5: Projektivni limes projektivnog niza perfektno normalnih prostora je perfektno normalan. □

Teorema 6: Ako je $\widetilde{X}$ projektivni limes projektivnog niza prostora kod kojeg su sve projekcije surjektivne, onda je i svako kanoničko preslikavanje surjektivno. □

Od prvog pojavljivanja teorije dimenzije u ranim dvadesetim godinama ovoga veka, pojam dimenzije se do danas značajno razvio. Postoje tri dimenzione funkcije za topološki prostor ali nas će interesovati samo kombinatorna dimenzija koja u suštini zavisi od reda profinjenja konačnih otvorenih pokrivača prostora.

Sledeće dve teoreme karakterišu dimenzionu funkciju dim.

Teorema 7: Neka je X topološki prostor. Tada su sledeći iskazi ekvivalentni:

(I) Prostor X zadovoljava nejednakost $\dim X \leq n$.

(II) Za svaki konačan otvoren pokrivač $\{G_i\}_{i=1}^{k}$ prostora X postoji konačan otvoren pokrivač $\{H_i\}_{i=1}^{k}$ od X čiji je red $\leq n$ i $H_i \subset G_i$ za svako $i = 1,2,\dots,k$.

(III) Za svaki otvoren pokrivač $\{G_i\}_{i=1}^{n+2}$ prostora X postoji otvoren pokrivač $\{H_i\}_{i=1}^{n+2}$ od X takav da je $H_i \subset G_i$ za svako $i = 1,2,\dots,n+2$ i $\bigcap_{i=1}^{n+2} H_i = \emptyset$. □

<u>Teorema 8:</u> Neka je X normalan prostor. Tada su sledeći iskazi ekvivalentni:

(I) Prostor X zadovoljava nejednakost dim $X \leq n$.

(II) Za svaki konačan otvoren pokrivač $\{G_i\}_{i=1}^{k}$ prostora X postoji otvoren pokrivač $\{H_i\}_{i=1}^{k}$ od X takav da je $\mathrm{cl}\, H_i \subset G_i$ za svako $i = 1,\ldots,k$ i red pokrivača $\{\mathrm{cl}\, H_i\}_{i=1}^{k} \leq n$.

(III) Za svaki konačan otvoren pokrivač $\{G_i\}_{i=1}^{k}$ prostora X postoji zatvoren pokrivač $\{F_i\}_{i=1}^{k}$ od X takav da je F_i G_i za svako $i = 1,\ldots,k$ i red pokrivača $\{F_i\}_{i=1}^{k} \leq n$.

(IV) Ako je $\{G_i\}_{i=1}^{n+2}$ otvoren pokrivač prostora X onda postoji zatvoren pokrivač $\{F_i\}_{i=1}^{k}$ od X takav da je $F_i \subset G_i$ za svako $i = 1,\ldots,n+2$ i $\bigcap_{i=1}^{n+2} F_i = \emptyset$. □

Poznato je da ako je dim X = 0 onda je X normalan prostor. Štaviše, imamo

<u>Teorema 9:</u> Ako je X kompaktan Hausdorffov prostor, onda je dim X = 0 ako i samo ako je X totalno diskoneksan. □

<u>Teorema 10:</u> Za normalan prostor X, dim X = dim βX, gde je βX Stone-Čechova kompaktifikacija prostora X. □

<u>Teorema 11:</u> Ako je M zatvoren podskup prostora X onda je dim M $\leq$ dim X. □

Predjimo na ponašanje dimenzione funkcije dim u odnosu na topološki proizvod.

<u>Teorema 12:</u> Neka je X parakompaktan Hausdorffov prostor takav da je dim X = m a Y kompaktan Hausdorffov prostor takav da je

$\dim Y = n$. Ako je bar jedan od njih neprazan onda je $\dim(X \times Y) \leq m + n$. □

Teorema 13: Za svaki par X, Y kompaktnih prostora, pri čemu je bar jedan od njih neprazan, imamo da je

$$\dim(X \times Y) \leq \dim X + \dim Y \text{ .} \quad \square$$

Teorema 14: Neka su X i Y Hausdorffovi prostori takvi da $X \times Y$ ima zvezdasto konačno svojstvo. Ako je bar jedan od njih neprazan onda je

$$\dim(X \times Y) \leq \dim X + \dim Y \text{ .} \quad \square$$

Teorema 15: Ako je X prostor a Y lokalno kompaktan, parakompaktan Hausdorffov prostor onda je

$$\dim(X \times Y) \leq \dim X + \dim Y \text{ .} \quad \square$$

Teorema 16: Neka je $\{X_s\}_{s \in S}$ familija prostora takva da je svaki prebrojivi proizvod prostora ove familije Lindelofov. Ako je $\dim X_s = 0$ za svako $s \in S$ onda je $\dim \prod X_s = 0$. □

Teorema 17: Prostor X je kompaktan Hausdorffov i $\dim X = 0$ ako i samo ako je X projektivni limes konačnih diskretnih prostora. □

Teorema 18: Neka je $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ projektivni spektar nad S kompaktnih Hausdorffovih prostora sa projektivnim limesom $\widetilde{X}$. Tada je $\dim \widetilde{X} \leq n$ ako i samo ako za svako $s \in S$ i svako otvoreno pokrivanje U_s prostora X_s postoji $t \in S$, $s \leq t$ tako da pokrivanje $f_{s,t}^{-1}(U_s)$ prostora X_t dopušta profinjenje U_t reda $\leq n+1$. □

Teorema 19: Neka je prebrojivo kompaktan prostor X projektivni limes projektivnog niza normalnih prostora i surjektivnih preslikavanja $\{X_n, f_{n,m}, N\}$ pri čemu je $\dim X_n \leq p$ za svako $n \in N$. Tada je X normalan i $\dim X \leq p$. □

Glava I:

OSNOVNI POJMOVI

Pošto mnoge definicije i pojmovi u fazi topologiji još nisu dobili svoj konačni oblik, mi posvećujemo ovu glavu osnovnim pojmovima teorije fazi skupova i fazi topologije. Dajemo brz pregled nekih od tih definicija i rezultata - bez dokaza u [1], [6], [7], [9], [10], [11], [12], [15], [16], [23], [28], [29], [30] koji se često koriste kroz ceo red.

1. TEORIJA FAZI SKUPOVA

Sa X ćemo uvek označavati neprazan skup, sa I zatvoren jedinični interval $[0,1]$ a sa I^X skup svih funkcija iz X u I.

Definicija 1.1: Fazi skup u X - u oznaci F-skup - je funkcija $f : X \to I$ koja svakoj tački $x \in X$ pridružuje vrednost funkcije $f(x) \in I$ (ili njen stepen u I).

Skup $\{x \in X : f(x) > 0\}$ naziva se nosač od f i označava sa f_0. F-skup $f \in I^X$ takav da je $f(x) = 0$ za svako $x \in X$, označavaćemo sa $\overline{0}$ i on odgovara praznom skupu $\emptyset$ a F-skup $f \in I^X$ takav da je $f(x) = 1$ za svako $x \in X$, označavaćemo sa $\overline{1}$. Ovaj F-skup odgovara skupu X.

Fazi tačka - u oznaci F-tačka - p u X je F-skup u X dat sa:

$$p(x) = \begin{cases} \alpha & \text{za } x = x_0 \quad (0 < \alpha \leq 1) \\ 0 & \text{za } x \neq x_0 \end{cases}$$

gde se x_0 naziva nosačem tačke p.

Specijalna funkcija koju ćemo koristiti u daljim razmatranjima je karakteristična funkcija podskupa, tj. ako je A podskup skupa X, onda se karakteristična funkcija $\mu(A)$ od A na X definiše sa

$$\mu(A) = \begin{cases} 1 & \text{ako je } x \in A \\ 0 & \text{ako je } x \notin A \end{cases}$$

što je jedan F-skup u X. Ako sa $P(X)$ označimo familiju svih podskupova skupa X a sa $ch(X)$ skup svih karakterističnih funkcija sa domenom X, medju njima postoji uzajamno - jednoznačna korespondencija koju ostvaruju sledeće dve funkcije:

$\Phi : P(X) \to ch(X)$ data sa $\Phi(A) = \mu(A)$

$\Psi : ch(X) \to P(X)$ data sa $\Psi(\mu) = \{x \in X : \mu(x) = 1\}$.

Pošto su F-skupovi realno vrednosne funkcije koristićemo postojeće operacije sa funkcijama $=, \leq, \vee, \wedge, \dots$ da F-skupove dovedemo u vezu sa drugim F-skupovima.

<u>Definicija 1.2:</u> Neka su f,g dva F-skupa u X. Tada:

(I) $f = g \Leftrightarrow f(x) = g(x)$ za svako $x \in X$

(II) $f \leq g \Leftrightarrow f(x) \leq g(x)$ za svako $x \in X$

(III) $p \in f \Leftrightarrow p(x) \leq f(x)$ za svako $x \in X$ gde je p F-tačka

(IV) $f \vee g = \max\{f(x), g(x)\}$ za svako $x \in X$

(V) $f \wedge g = \min\{f(x), g(x)\}$ za svako $x \in X$

Opštije, za familiju $\{f_s\}_{s\in S}$ F-skupova unija $(\bigvee_{s\in S} f_s)$ i presek $(\bigwedge_{s\in S} f_s)$ definišu se na sledeći način:

$$\Big[\bigvee_{s\in S} f_s\Big](x) = \sup_{s\in S} f_s(x), \quad x \in X$$

$$\Big[\bigwedge_{s\in S} f_s\Big](x) = \inf_{s\in S} f_s(x), \quad x \in X$$

(VI) F-skup co.f definisan sa $(co.f)(x) = 1 - f(x)$ naziva se komplement od f.

Za F-skupove f,g I^X kažemo da su disjunktni ako je $f \wedge g = \bar{0}$.

Pokazano je da ako je $f \in I^X$ onda $f \wedge co.f \neq \bar{0}$ u opštem slučaju i ako su $f_1, f_2 \in I^X$ tako da je $f_1 \wedge f_2 = \bar{0}$, onda je $f_1 \leq co.f_2$ ali da obratno u opštem slučaju ne važi što je devijacija u odnosu na obične skupove. To znači da mreža $(I^X, \leq)$ nije komplementarna, dok de Morganovi zakoni važe, tj.

$$co(\bigvee_{s\in S} f_s) = \bigwedge_{s\in S} co.f_s$$

i

$$co(\bigwedge_{s\in S} f_s) = \bigvee_{s\in S} co.f_s$$

za svaku familiju $\{f_s\}_{s\in S}$ F-skupova u X.

Iz definicije 1.2 neposredno sledi:

(I)° $f \vee g = g \vee f$ i $f \wedge g = g \wedge f$

(II)° $f \vee \bar{0} = f$, $f \wedge \bar{0} = \bar{0}$ i $f \vee \bar{1} = \bar{1}$, $f \wedge \bar{1} = f$

(III)° $co.(co.f) = f$, $co.\bar{0} = \bar{1}$ i $co.\bar{1} = \bar{0}$.

<u>Stav 1.1:</u> Neka je f F-skup u X i neka je p F-tačka u X. Tada je $f = \vee\{p : p \in f\}$. □

<u>Definicija 1.3:</u> Neka je $\psi : X \to Y$ preslikavanje skupa X u skup Y. Ako je f F-skup u X, onda je $\psi(f)$ F-skup u Y definisan sa:

$$[\psi(f)](y) = \begin{cases} \sup\limits_{x\in \psi^{-1}(y)} f(x) & \text{ako je } \psi^{-1}(y) \neq \emptyset \\ 0 & \text{ako je } \psi^{-1}(y) = \emptyset \end{cases}$$

Ako je g F-skup u Y, onda je $\psi^{-1}(g)$ F-skup u X definisan sa:

$$[\psi^{-1}(g)](x) = g(\psi(x)) .$$

<u>Stav 1.2:</u> Neka je ψ preslikavanje kao u definiciji 1.3. Tada ψ i ψ^{-1} imaju sledeće osobine:

(I) $\psi^{-1}(\bigvee_{s\in S} f_s) = \bigvee_{s\in S} \psi^{-1}(f_s)$

(II) $\Psi^{-1}(\bigwedge_{s\in S} f_s) = \bigwedge_{s\in S} \Psi^{-1}(f_s)$

(III) $\Psi(\bigvee_{s\in S} f_s) = \bigvee_{s\in S} \Psi(f_s)$

(IV) $\Psi(\bigwedge_{s\in S} f_s) \leq \bigwedge_{s\in S} \Psi(f_s)$

(V) $\Psi(\Psi^{-1}(f)) \leq f$

(VI) $\Psi^{-1}(\Psi(f)) \geq f$. □

Možemo pisati $p_{x_0}^{\alpha}$ ako je vrednost od p_{x_0} jednaka α i otuda je $p_{x_0}^{\alpha} \in f$ ako i samo ako je $f(x_0) > \alpha$ za $\alpha \in [0,1)$.

Stav 1.3: Neka je Ψ: $X \longrightarrow Y$ preslikavanje skupa X u skup Y, f F-skup u X i $p_{x_0}^{\alpha}$ F-tačka u X. Tada je:

(I) $\Psi[p_{x_0}^{\alpha}] = p_{\Psi(x_0)}^{\alpha}$

(II) $p_{x_0}^{\alpha} \in f$ povlači $\Psi[p_{x_0}^{\alpha}] \in \Psi(f)$. □

2. FAZI TOPOLOŠKI PROSTORI

Definicija 2.1: Familija T F-skupova u skupu X naziva se fazi topologija na X (F-topologija) ako su zadovoljeni sledeći uslovi:

(F.T.1) $\overline{0}, \overline{1} \in T$

(F.T.2) $f_s \in T$ povlači $\bigvee_{s\in S} f_s \in T$, $s \in S$

(F.T.3) $f_i \in T$ povlači $\bigwedge_{i=1}^{n} f_i \in T$

Članovi familije T zovu se otvoreni F-skupovi a par (X,T) zove se F-prostor. Ako je f otvoren F-skup, onda se co.f zove zatvoren F-skup.

F-prostori su prirodna generalizacija topoloških prostora jer se topologija na X može posmatrati kao familija karakterističnih funkcija sa operacijama $\leq$, $\vee$, $\wedge$ i co.

Definicija 2.2: Neka je (X,T) F-prostor. F-skup g u X zove se okolina F-skupa f u X ako je $f \leq g$ i postoji F-skup $h \in T$ takav da je $f \leq h \leq g$.

Pod unutrašnjošću F-skupa f - u oznaci $\operatorname{int}(f)$ - podrazumevaćemo F-skup.

$\operatorname{int}(f) = \bigvee_s \{ g_s : g_s \leq f, g_s \in T, \ s \in S \}$,

f je otvoren ako i samo ako je $f = \operatorname{int}(f)$.

Zatvorenje F-skupa f - u oznaci $\operatorname{cl} f$ - je F-skup dat sa:

$\operatorname{cl} f = \bigwedge_s \{ h_s : f \leq h_s, h_s \in \operatorname{co}.T, \ s \in S \}$.

f je zatvoren ako i samo ako je $f = \operatorname{cl} f$.

Stav 2.1: Svaki F-skup f u F-prostoru (X,T) zadovoljava jednakost $\operatorname{int}(f) = \operatorname{co}.(\operatorname{cl}(\operatorname{co}.f))$. □

Gornji stav daje vezu izmedju operatora unutrašnjosti i zatvorenja u F-topologiji koja je analogna dobro poznatoj vezi izmedju tih operatora u topologiji.

Definicija 2.3: Neka je (X,T) F-prostor i $\beta \subseteq T$ potfamilija od T takva da je svaki $f \in T$ unija članova iz β. Tada se β zove baza za T. $S \subseteq T$ se zove predbaza za T ako je familija svih konačnih preseka članova iz S baza za T.

Stav 2.2: Neka je (X,T) F-prostor. Tada je β baza za T ako i samo ako za bilo koje $f \in T$ i svaku F-tačku p u X, gde je $p \in f$, postoji $B \in \beta$ tako da je $p \in B \leq f$. □

Definicija 2.4: Za F-prostor (X,T) kažemo da je C_{II} ako postoji prebrojiva baza za T.

Definicija 2.5: Neka je (X,T) F-prostor i $\alpha \in [0,1)$. Kolekcija $U \subseteq T$ naziva se α-senčenje prostora X ako za svako $x \in X$ postoji $u \in U$ tako da je $u(x) > \alpha$. Potkolekcija od U koja je takodje α-senčenje od X naziva se α-podsenčenje od X.

Definicija 2.6: Neka je (X,T) F-prostor i $\alpha \in [0,1)$. Neka su U i V dva α-senčenja od X. Kažemo da je U α-profinjenje od V i pišemo $U \preccurlyeq V$ ako za svako $u \in U$ postoji $v \in V$ dako da je $u \leq v$.

Proizvoljno α-podsenčenje datog α-senčenja je α-profinjenje tog α-senčenja.

Pojam α-senčenja igra važnu ulogu u izučavanju i razvijanju teorije F-prostora analogno ulozi pojma pokrivača kod topoloških prostora. Tako se α-kompaktnost, prebrojiva α-kompaktnost itd. ... definišu i proučavaju u terminima α-senčenja kao što ćemo videti u sledećim definicijama i stavovima.

Definicija 2.7: F-prostor (X,T) je α-kompaktan (α-Lindelofov) ako svako α-senčenje prostora X ima konačno (prebrojivo) α-podsenčenje od X za $\alpha \in [0,1)$.

Definicija 2.8: F-prostor (X,T) je prebrojivo α-kompaktan, gde je $\alpha \in [0,1)$, ako svako prebrojivo α-senčenje prostora X ima konačno α-podsenčenje.

Primedba: Svaki α-kompaktan F-prostor je prebrojivo α-kompaktan a svaki α-Lindelöfov, prebrojivo α-kompaktan F-prostor je kompaktan.

Stav 2.3: Neka je (X,T) C_{II} F-prostor i $\alpha \in [0,1)$. Tada je:

(I) (X,T) je α-Lindelofov.

(II) Ako je (X,T) prebrojivo α-kompaktan, onda je (X,T) α-kompaktan. □

Definicija 2.9: F-prostor (X,T) je kompaktan ako za svaku familiju $\beta \subset T$ i svako $\alpha \in [0,1)$ takvo da je $\sup_{g \in \beta} g \geq \alpha$, i svako $\delta \in (0,\alpha]$ postoji konačna potfamilija $\beta_0 \subset \beta$ takva da je $\sup_{g \in \beta_0} g \geq \alpha - \delta$.

Definicija 2.10: Neka je $\alpha \in [0,1)$ i $B \subset I^X$. Kažemo da je familija B α-centrirana ako za svaki konačan izbor $c_1, c_2, \dots, c_n \in B$ postoji $x \in X$ tako da je $c_i(x) \geq 1 - \alpha$ za svako $i = 1,2,\dots,n$.

Stav 2.4: Neka je (X,T) F-prostor, $\alpha \in [0,1)$. Tada je (X,T) α-kompaktan (prebrojivo α-kompaktan) ako i samo ako za svaku α-centriranu familiju B ($\{c_i\}_{i=1}^{\infty}$) zatvorenih F-skupova u X postoji $x \in X$ tako da je $c(x) \geq 1 - \alpha$ za svako $c \in B$ ($c_i(x) \geq 1 - \alpha$ za svako $i = 1,2,\dots$). □

Ovaj stav pokazuje da ako je (X,T) α-kompaktan (prebrojivo α-kompaktan), $\alpha \in [0,1)$ i B ($\{c_i\}_{i=1}^{\infty}$) proizvoljna α-centrirana familija zatvorenih F-skupova, onda je $\bigwedge_{c \in B} c \neq \bar{0}$ ($\bigwedge_{i=1}^{\infty} c_i \neq \bar{0}$).

Definicija 2.11: Neka je (X,T) F-prostor i $\alpha \in [0,1)$. Kažemo da je (X,T) α-Hausdorffov (Hausdorffov) ako za $x \neq y \in X$ postoje $f,g \in T$ tako da je $f(x) > \alpha$, $g(x) > \alpha$ ($f(x)=1=g(x)$) i $f \wedge g = \bar{0}$.

Primetimo da ako je F-prostor Hausdorffov, onda je on i α-Hausdorffov za svako $\alpha \in [0,1)$.

Definicija 2.12: Neka je (X,T) F-prostor, $\alpha \in [0,1)$ i $A \subset X$. Tada:

(I) A je α-zatvoren (α^*-zatvoren) ako za svako $x \in X \setminus A$ postoji $f \in T$ tako da je $f(x) > \alpha$ ($f(x) \geq \alpha$) i $f \wedge \mu(A) = \bar{0}$.

(II) A je podesno zatvoren ako je $\mu(A)$ zatvoren F-skup u X.

Stav 2.5: Proizvoljan presek α-zatvorenih skupova je α-zatvoren i konačna unija α-zatvorenih skupova je α-zatvoren skup. □

Sledeći stav daje odnos izmedju α-zatvorenosti i podesne zatvorenosti.

Stav 2.6: Neka je (X,T) F-prostor i neka je $A \subset X$. Tada su sledeći iskazi ekvivalentni:

(I) A je podesno zatvoren skup.

(II) A je 1^*-zatvoren skup. □

Neka je (X,T) F-prostor i $Y \subset X$. Tada familija $T_Y = \{f|Y : f \in T\}$, gde je $f|Y$ restrikcija funkcije f na Y, zadovoljava sva tri uslova definicije 2.1, tj. T_Y je F-topologija na Y.

Definicija 2.13: F-topologija T_Y naziva se relativna F-topologija na Y ili F-topologija na Y indukovana F-topologijom na X, a (Y,T_Y) naziva se F-potprostor od (X,T).

Mi obično izostavljamo relativnu F-topologiju T_Y i jednostavno pišemo F-potprostor Y.

Stav 2.7: Neka je (X,T) F-prostor, Y F-potprostor od (X,T) i $f \in I^Y$. Tada:

(I) f je zatvoren u Y ako i samo ako postoji zatvoren F-skup g u X takav da je $f = g|Y$.

(II) $cl_Y f = cl_X f|Y$. □

Primetimo da ako je β baza za F-prostor (X,T), onda je

$\beta_Y = \{B/Y : B \in \beta\}$ baza za T_Y.

<u>Stav 2.8:</u> Neka je (X,T) F-prostor i $Y \subset X$. Za svako $\alpha \in [0,1)$ važi sledeće:

(i) Ako je (X,T) α-Hausdorffov, onda je (Y,T_Y) α-Hausdorffov.

(ii) Ako je (X,T) α-kompaktan (prebrojivo α-kompaktan) i Y α-zatvoren u X, onda je Y α-kompaktan (prebrojivo α-kompaktan). □

<u>Stav 2.9:</u> Neka je (X,T) α-Hausdorffov F-prostor i $Y \subset X$. Tada važi sledeće:

(i) Ako je Y α-kompaktan u X onda je Y α-zatvoren u X.

(ii) Ako je $\{Y_s\}_{s \in S}$ opadajuća familija α-kompaktnih (prebrojivo α-kompaktnih) skupova u X onda je $\bigcap_{s \in S} Y_s$ neprazan i α-kompaktan (prebrojivo α-kompaktan). □

<u>Definicija 2.14:</u> Za F-prostor (X,T) kažemo da je:

(i) FT_1-prostor ako je svaka F-tačka zatvoren F-skup.

(ii) FT_2-prostor ako za svake dve različite F-tačke p,q u X:

a) $p_0 \neq q_0$ povlači da postoje dva disjunktna otvorena F-skupa koji sadrže p i q respektivno.

b) $p_0 = q_0$, $p(x) < q(x)$ povlači da postoji $f \in T$ tako da je $p \in f$ i $q \notin \mathrm{cl}\, f$.

(iii) F-regularan prostor ako za svaki zatvoren F-skup g i svaku tačku p takvu da je $p \notin q$ postoji otvorena okolina f od g takva da $p \notin \mathrm{cl}\, f$.

<u>Definicija 2.15:</u> Za F-prostor (X,T) kažemo da je:

(i) slabo F-normalan ako za svaka dva disjunktna zatvorena F-skupa g_1 i g_2 postoje dva otvorena F-skupa h_1 i h_2

tako da je $g_1 \leq h_1$, $g_2 \leq h_2$ i $h_1 \leq co. h_2$.

(II) F-normalan ako za zatvoren F-skup g i otvoren F-skup f takav da je $g \leq f$, postoji otvoren F-skup h takav da je $g \leq h \leq cl\ h \leq f$ ili ekvivalentno: za svaka dva zatvorena F-skupa g_1 i g_2 takva da je $g_1 \leq co. g_2$, postoje dva otvorena F-skupa h_1 i h_2 takva da je $g_1 \leq h_1$, $g_2 \leq h_2$ i $h_1 \leq co. h_2$.

(III) Perfektno F-normalan ako je F-normalan i ako je svaki zatvoren F-skup prebrojiv presek otvorenih F-skupova.

Stav 2.10: Svaki F-normalan prostor je slabo F-normalan. □

Obrat ovog stava ne važi u opštem slučaju.

Stav 2.11: Neka je Y F-potprostor F-prostora (X,T). Ako je (X,T) FT_1 [FT_2] (F-regularan), onda je Y FT_1[FT_2] (F-regularan). □

F-potprostor Y F-normalnog prostora (X,T) nije F-normalan čak ni ako je Y α-zatvoren u X. U glavi III ćemo videti kada je F-potprostor F-normalnog prostora takodje F-normalan.

3. FAZI NEPREKIDNE FUNKCIJE

Definicija 3.1: Neka su (X,T) i (Y,R) F-prostori i neka je $\psi: X \to Y$ preslikavanje. Kažemo da je ψ F-neprekidno ako je $\psi^{-1}(f) \in T$ za svako $f \in R$. ψ je F-otvoreno (F-zatvoreno) ako je za svaki otvoren (zatvoren) F-skup u X, $\psi(f)$ otvoren (zatvoren) F-skup u Y. ψ je F-homeomorfizam ako je ψ F-neprekidno, bijektivno i F-otvoreno ili F-zatvoreno.

Stav 3.1: Neka su (X,T), (Y,R) F-prostori i neka je $\psi: X \to Y$.

Tada su sledeći iskazi ekvivalentni:

(I) ψ je F-neprekidno.

(II) Za svaki zatvoren F-skup g u Y, $f^{-1}(g)$ je zatvoren F-skup u X.

(III) Za svaki F-skup f u X, $\psi(\text{cl } f) \leq \text{cl}(\psi(f))$.

(IV) Za svaki F-skup g u Y, $\text{cl}(\psi^{-1}(g)) \leq \psi^{-1}(\text{cl } g)$. □

Stav 3.2: Kompozicija F-neprekidnih (F-otvorenih) [F-zatvorenih] preslikavanja je F-neprekidno (F-otvoreno) [F-zatvoreno] . □

Stav 3.3: Neka su (X,T), (Y,R) F-prostori i neka je $\psi : X \rightarrow Y$ F-neprekidno preslikavanje. Tada za $\alpha \in [0,1)$ važi sledeće:

(I) Ako je (X,T) α-kompaktan, onda je $\psi(X)$ α-kompaktan kao F-potprostor od (Y,R).

(II) Ako je (X,T) prebrojivo α-kompaktan (α-Lindelöfov), onda je $\psi(X)$ prebrojivo α-kompaktan (α-Lindelöfov).

(III) Ako je (X,T) α-kompaktan a (Y,R) α-Hausdorffov onda je za α-zatvoren skup A u X, $\psi(A)$ α-zatvoren skup u Y. □

Stav 3.4: Neka su (X,T), (Y,R) F-prostori i neka su $\phi, \psi : X \rightarrow Y$ F-neprekidna preslikavanja. Ako je (Y,R) α-Hausdorffov, $\alpha \in [0,1)$ onda je $\{x \in X : \phi(x) = \psi(x)\}$ α-zatvoren u X. □

Stav 3.5: Neka je $\psi : (X,T) \rightarrow (Y,R)$ F-neprekidno preslikavanje. Ako je A α-zatvoren u Y onda je $\psi^{-1}(A)$ α-zatvoren u X. □

Neka je (Y,T) F-prostor, X skup i $\psi : X \rightarrow Y$ funkcija. Tada je familija $\psi^{-1}(T) = \{\psi^{-1}(f) : f \in T\}$ najmanja F-topologija na X za koju je ψ F-neprekidna. Ta F-topologija naziva se inicijalna F-topologija na skupu X.

Neka je $\{(X_s,T_s)\}_{s\in S}$ familija F-prostora, $X = \prod_{s\in S} X_s$ direktan proizvod skupova X_s i neka su $p_s : X \to X_s$, $s\in S$ projekcije. Familiju $\{p_s^{-1}(f_s) : f_s\in T_s,\ s\in S\}$ F-skupova možemo uzeti za predbazu F-topologije T na X.

<u>Definicija 3.2:</u> Neka je data familija $\{(X_s,T_s)\}_{s\in S}$ F-prostora. Gore definisana F-topologija T na X zove se proizvod F-topologija a (X,T) se zove proizvod F-prostora.

Primetimo da su elementi baze proizvoda F-topologija oblika $\bigwedge_{i=1}^{n} p_i^{-1}(f_i)$, $f_i\in T_i$ za $i = 1,2,\ldots,n$.

<u>Stav 3.6:</u> Neka je (X,T) proizvod familije F-prostora $\{(X_s,T_s)\}_{s\in S}$. Tada važi:

(I) Projekcije p_s su F-neprekidne za svako $s\in S$.

(II) Proizvod F-topologija je najmanja F-topologija na X za koju su projekcije F-neprekidne. □

<u>Stav 3.7:</u> Neka je $\{(X_i,T_i)\}_{i=1}^{\infty}$ prebrojiva familija C F-prostora. Tada je proizvod (X,T) takodje C_{II}. □

<u>Stav 3.8:</u> Neka je $\{(X_s,T_s)\}_{s\in S}$ familija F-prostora. Ako je svaki (X_s,T_s) α-Hausdorffov (FT_1) [F-regularan] F-prostor onda je proizvod (X,T) α-Hausdorffov (FT_1) [F-regularan] respektivno. □

Glava II:

α-KOMBINATORNA DIMENZIJA F-PROSTORA

U ovoj glavi definišemo α-kombinatornu dimenziju F-prostora, gde je $\alpha \in [0,1)$, i dajemo neke njene karakterizacije analogne dobro poznatim karakterizacijama kombinatorne dimenzije topoloških prostora. U drugom delu glave nalazimo neke teoreme podskupa i konačne sume, dok je poslednji paragraf posvećen izučavanju lokalne α-kombinatorne dimenzije F-prostora.

1. GLAVNE KARAKTERIZACIJE

Pojam reda familije F-skupova široko ćemo koristiti u izučavanju α-kombinatorne dimenzije F-prostora.

<u>Definicija 1.1:</u> Red familije F-skupova $\{f_s\}_{s\in S}$ nekog F-prostora (X,T) je najveći ceo broj n za koji postoji $M \subset S$, sa $n+1$ elemenata, takav da je $\bigwedge_{s\in M} f_s \neq \overline{0}$. Ako takvog najvećeg broja nema onda kažemo da je red ∞.

Otuda, ako je red od $\{f_s\}_{s\in S}$ jednak n, onda za svaka $n+2$ različita elementa $s_1, s_2, \ldots, s_{n+2}$ S imamo da je $\bigwedge_{i=1}^{n+2} f_{s_i} = \overline{0}$. Specijalno, ako je red od $\{f_s\}_{s\in S} = 0$, onda se f_s $_{s\ S}$ sastoji iz uzajamno disjunktnih F-skupova različitih od $\overline{0}$.

<u>Definicija 1.2:</u> Neka je (X,T) F-prostor, n ceo broj $(n \geq 0)$ i neka je $\alpha \in [0,1)$. Mi definišemo α-kombinatornu dimenziju

F-prostora (X,T) – u oznaci F-$\dim_\alpha$ – na sledeći način:

(I) F-$\dim_\alpha X = -1$ ako je $X = \emptyset$.

(II) F-$\dim_\alpha X \leq n$ ako svako konačno α-senčenje od X ima konačno α-profinjenje reda $\leq n$.

(III) F-$\dim_\alpha X = n$ ako je F-$\dim_\alpha X \leq n$ i F-$\dim_\alpha X > n-1$.

(IV) F-$\dim_\alpha X = \infty$ ako F-$\dim_\alpha X \leq n$ nije tačno ni za jedno n.

<u>Primer 1.1:</u> Neka je $X = \{x_1, x_2, x_3, x_4\}$. Neka su f,g,h,l,m I^X definisani na sledeći način:

$$f(x) = \begin{cases} \frac{1}{6} & \text{ako je } x = x_1 \\ 0 & \text{inače} \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{4} & \text{ako je } x \in \{x_2, x_4\} \\ 0 & \text{inače} \end{cases}$$

$$h(x) = \begin{cases} \frac{1}{3} & \text{ako je } x \in \{x_3, x_4\} \\ 0 & \text{inače} \end{cases}$$

$$l(x) = \begin{cases} \frac{5}{12} & \text{ako je } x \in \{x_1, x_4\} \\ 0 & \text{inače} \end{cases}$$

$$m(x) = \begin{cases} \frac{1}{4} & \text{ako je } x = x_4 \\ 0 & \text{inače} . \end{cases}$$

Neka je $T = \{\overline{0}, \overline{1}, f, g, h, l, m, f \vee g, f \vee h, g \vee h, g \vee l\}$. Tada je (X,T) F-prostor.

Stavimo $\alpha = \frac{1}{12}$. Tada je $\{f,g,h\}$ α-senčenje od X. Jasno je da je $\{f,g,h\}$ α-profinjenje bilo kojeg α-senčenja od X i da je red od $\{f,g,h\} \leq 1$. Otuda je F-$\dim_\alpha X \leq 1$.

<u>Primer 1.2:</u> Neka je X kao u primeru 1.1. Neka su $f,g,h \in I^X$ definisani na sledeći način:

$$f(x) = \begin{cases} \frac{5}{12} & \text{ako je } x = x_1 \\ 0 & \text{inače} \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{4} & \text{ako je } x \in \{x_2, x_3\} \\ 0 & \text{inače} \end{cases}$$

$$h(x) = \begin{cases} \frac{1}{3} & \text{ako je } x = x_4 \\ 0 & \text{inače .} \end{cases}$$

Neka je $T = \{\overline{0},\overline{1},f,g,h,f\vee g,f\vee h,g\vee h\}$. Tada je (X,T) F-prostor. Ako stavimo $\alpha = \frac{1}{6}$, onda je $\{f,g,h\}$ α-senčenje od X. Lako se vidi da je $\{f,g,h\}$ α-profinjenje bilo kojeg α-senčenja od X. Kako je $f\wedge g = g\wedge h = f\wedge h = \overline{0}$ i $X \neq \emptyset$ to je $F\text{-}\dim_{\alpha} X = 0$.

Sledeći stav daje korisnu karakterizaciju α-kombinatorne dimenzije F-prostora.

<u>Stav 1.1:</u> Neka je (X,T) F-prostor, $\alpha\in[0,1)$. Tada su sledeći iskazi ekvivalentni:

(I) $F\text{-}\dim_{\alpha} X \leq n$.

(II) Za svako konačno α-senčenje $\{f_i\}_{i=1}^{k}$ od X postoji α-senčenje $\{g_i\}_{i=1}^{k}$ od X reda $\leq n$ tako da je $g_i \leq f_i$ za svako $i = 1,2,\dots,k$.

(III) Ako je $\{f_i\}_{i=1}^{n+2}$ α-senčenje od X onda postoji α-senčenje $\{g_i\}_{i=1}^{n+2}$ tako da je $g_i \leq f_i$ za svako $i = 1,2,\dots,n+2$ i $\bigwedge_{i=1}^{n+2} g_i = \overline{0}$.

<u>Dokaz.</u> (I) $\Longrightarrow$ (II) Neka je $F\text{-}\dim_{\alpha} X \leq n$ i neka je $\{f_i\}_{i=1}^{k}$ proizvoljno α-senčenje F-prostora X. Tada na osnovu definicije 1.2, $\{f_i\}_{i=1}^{k}$ ima α-profinjenje V reda $\leq n$. Ako je $v\in V$, onda je $v \leq f_i$ za neko $i\in\{1,2,\dots,k\}$. Za svako $v\in V$ izaberimo $i(v)\leq k$ tako da je $v\leq f_{i(v)}$, i neka je $g_i = \bigvee\{v\in V : i(v) = i\}$. Tada je jasno da je $\{g_i\}_{i=1}^{k}$ α-senčenje od X reda $\leq$ n i da je $g_i \leq f_i$ za svako $i = 1,2,\dots,k$.

(II) $\Longrightarrow$ (III) Neka važi (II) i neka je $\{f_i\}_{i=1}^{n+2}$ α-senčenje od X. Tada na osnovu pretpostavke postoji α-senčenje $\{g_i\}_{i=1}^{n+2}$

tako da je $g_i \leq f_i$ za $i = 1,2,\ldots,n+2$ i red od $\{g_i\}_{i=1}^{n+2} \leq n$.

Sada na osnovu definicije 1.1 dobijamo da je $\bigwedge_{i=1}^{n+2} g_i = \overline{0}$.

(III) $\Longrightarrow$ (II) Neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od X i neka važi (III). Možemo pretpostaviti da je $k > n+1$. Neka je $g_i = f_i$ za $i \leq n+1$ i $g_{n+2} = \bigvee_{i=n+2}^{k} f_i$. Tada je $\{g_i\}_{i=1}^{n+2}$ α-senčenje od X pa na osnovu pretpostavke postoji α-senčenje $\{h_i\}_{i=1}^{n+2}$ tako da je $h_i \leq g_i$ za svako $i = 1,2,\ldots,n+2$ i $\bigwedge_{i=1}^{n+2} h_i = \overline{0}$. Neka je $m_i = h_i$ za $i \leq n+1$ i $m_i = f_i \wedge h_{n+2}$ za $i > n+1$. Tada je $M = \{m_i\}_{i=1}^{k}$ α-senčenje od X, $m_i \leq f_i$ za svako i i $\bigwedge_{i=1}^{n+2} m_i = \overline{0}$. Ako neka kolekcija od n+2 člana iz M ima presek različit od $\overline{0}$ onda se članovi iz M mogu prenumerisati tako da prva n+2 člana imaju nula presek. Koristeći prethodnu konstrukciju na M, možemo dobiti α-senčenje $M^* = \{m_i^*\}_{i=1}^{k}$ tako da je $m_i^* \leq m_i$ i $\bigwedge_{i=1}^{n+2} m_i^* = \overline{0}$. Jasno je da je $m_{i_1}^* \wedge m_{i_2}^* \wedge \ldots \wedge m_{i_s}^* = \overline{0}$ uvek kada je $m_{i_1} \wedge m_{i_2} \wedge \ldots \wedge m_{i_s} = \overline{0}$ gde su $i_1, i_2, \ldots, i_s \leq k$. Ponavljajući ovaj proces konačan broj puta, dobijamo α-senčenje $\{s_i\}_{i=1}^{k}$ od X reda $\leq n$ tako da je $s_i \leq f_i$ za $i = 1,2,\ldots,k$.

(II) $\Longrightarrow$ (I) je trivijalno. □

Dok pojam normalnosti igra važnu ulogu u razvoju kombinatorne dimenzije topoloških prostora, uloga F-normalnosti je nešto manja jer činjenica $g_1 \wedge g_2 = \overline{0}$ ako i samo ako $g_1 \leq co.g_2$ ne važi u opštem slučaju kod F-skupova. To je navelo Kerrea [9] da uvede pojam slabe normalnosti za F-prostore, a nas navodi da uvedemo sledeću definiciju:

Definicija 1.3: F-prostor (X,T) zove se F_c-prostor ako za svaki par F-skupova f,g u X za koji je $\mathrm{cl}\, f \leq \mathrm{co}.g$ važi da je $\mathrm{cl}\, f \wedge g = \overline{0}$.

Primetimo da su za F_c-prostore pojmovi F-normalnosti i slabe F-normalnosti ekvivalentni. Na osnovu stava I 2.10, svaki F-normalan prostor je slabe F-normalan. Neka su sada g_1 i g_2 dva zatvorena F-skupa takva da je $g_1 \leq \mathrm{co}.\, g_2$. Otuda je $\mathrm{cl}\, g_1 \leq \mathrm{co}.g_2$ pa kako je (X,T) F_c-prostor, to na osnovu definicije 1.3 sledi da je $\mathrm{cl}\, g_1 \wedge g_2 = \overline{0}$ pa je $g_1 \wedge g_2 = \overline{0}$. Sada na osnovu definicije I.2.15 (l) postoje dva otvorena F-skupa h_1, h_2 takva da je $g_1 \leq h_1$, $g_2 \leq h_2$ i $h_1 \leq \mathrm{co}\, h_2$ a to je i trebalo dokazati. □

Definicija 1.4: Neka je $\{f_s\}_{s\ S}$ familija F-skupova u F-prostoru (X,T). Sveling ove familije je familija $\{g_s\}_{s\in S}$ F-skupova u X takva da je $f_s \leq g_s$ za svako $s \in S$ i $f_{s_1} \wedge f_{s_2} \wedge \ldots \wedge f_{s_n} = \overline{0}$ ako i samo ako je $g_{s_1} \wedge g_{s_2} \wedge \ldots \wedge g_{s_n} = \overline{0}$ za svaki izbor indeksa $s_1, s_2, \ldots, s_n \in S$.

Stav 1.2: Neka je (X,T) F_c-normalan prostor. Tada za svaku konačnu familiju $\{f_i\}_{i=1}^{k}$ zatvorenih F-skupova u X postoji konačna familija $\{h_i\}_{i=1}^{k}$ otvorenih F-skupova u X tako da je $\{\mathrm{cl}\ h_i\}_{i=1}^{k}$ sveling familije $\{f_i\}_{i=1}^{k}$ i $f_i \leq h_i$ za svako $i = 1,2,\ldots,k$.

Dokaz. Neka je data familija $\{f_i\}_{i=1}^{k}$ i neka je g_1 unija svih preseka oblika $f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_m}$ takvih da je $f_1 \wedge f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_m} = \overline{0}$. Tada je g_1 zatvoren F-skup i $g_1 \wedge f_1 = \overline{0}$ pa je $f_1 \leq \mathrm{co}.g_1$ pri čemu je $\mathrm{co}.g_1$ otvoren F-skup. Kako je (X,T) F-normalan, postoji otvoren F-skup h_1 takav da je $f_1 \leq h_1 \leq \mathrm{cl}\ h_1 \leq \mathrm{co}.g_1$.

Familija $\{cl\ h_1, f_2, \ldots, f_k\}$ je sveling od $\{f_i\}_{i=1}^{k}$ jer ako je $f_1 \wedge f_{i_1} \wedge \ldots \wedge f_{i_m} = \bar{0}$ onda je $f_{i_1} \wedge \ldots \wedge f_{i_m} \leq g_1$ pa je $co.(f_{i_1} \wedge \ldots \wedge f_{i_m}) \geq co.g_1 \geq cl\ h_1$ tj. $cl\ h_1 \leq co(f_{i_1} \wedge \ldots \wedge f_{i_m})$. Kako je (X,T) F_c-prostor to je $cl\ h_1 \wedge (f_{i_1} \wedge \ldots \wedge f_{i_m}) = \bar{0}$. Akc primenimo ovo pravilo k puta, dobićemo familiju otvorenih F-skupova $\{h_i\}_{i=1}^{k}$ tako da je $\{cl\ h_i\}_{i=1}^{k}$ sveling od $\{f_i\}_{i=1}^{k}$ i $f_i \leq h_i$ za $i = 1, \ldots, k$. □

<u>Definicija 1.5:</u> α-senčenje $\{f_s\}_{s\in S}$ F-prostora (X,T) je stežljivo ako postoji α-senčenje $\{g_s\}_{s\in S}$ od X takvo da je $cl\ g_s \leq f_s$ za $s \in S$.

<u>Stav 1.3:</u> Neka je (X,T) F-normalan prostor. Tada je svako konačno α-senčenje stežljivo.

<u>Dokaz.</u> Neka je $\{f_i\}_{i=1}^{k}$ konačno α-senčenje od X. Stavimo
$g_1 = co.\left[\{f_2 \vee f_3 \vee \ldots \vee f_k\} \vee \bigvee_{h\in T} \{h : h \geq f_i\ \forall i=1,2,\ldots,k\}\right] \wedge \mu(f_1^*)$, gde je $f_1^* = \{x \in X : f_1(x) \geq \alpha\}$. Tada je g_1 zatvoren F-skup u X i $g_1 \leq f_1$ pa na osnovu F-normalnosti prostora (X,T) postoji otvoren F-skup m_1 takav da je $g_1 \leq m_1 \leq cl\ m_1 \leq f_1$ i $\{m_1, f_2, f_3, \ldots, f_k\}$ je α-senčenje od X. Ako primenimo ovaj metod k puta, dobićemo α-senčenje $\{m_i\}_{i=1}^{k}$ od X takvo da je $cl\ m_i \leq f_i$ za svako $i = 1,2,\ldots,k$. □

<u>Definicija 1.6:</u> Neka je (X,T) F-prostor i $\alpha \in [0,1)$. Kolekcija B zatvorenih F-skupova u X zove se α-kosenčenje od X ako za svako $x \in X$ postoji $g \in B$ tako da je $g(x) > \alpha$.

Sledeći stav daje karakterizaciju dimenzije $F\text{-}dim_\alpha$ F_c-normalnog prostora u terminima zatvorenih F-skupova.

<u>Stav 1.4:</u> Neka je (X,T) F_c-normalan prostor i $\alpha \in [0,1)$. Tada su sledeći iskazi ekvivalentni:

(I) $F\text{-dim}_\alpha X \le n$.

(II) Za svako konačno α-senčenje $\{f_i\}_{i=1}^{k}$ od X postoji α-senčenje $\{g_i\}_{i=1}^{k}$ tako da je $cl.g_i \le f_i$ i red od $\{cl.g_i\}_{i=1}^{k} \le n$.

(III) Za svako konačno α-senčenje $\{f_i\}_{i=1}^{k}$ od X postoji α-kosenčenje $\{g_i\}_{i=1}^{k}$ tako da je $g_i \le f_i$ za svako $i = 1,2,\dots,k$ i red od $\{g_i\}_{i=1}^{k} \le n$.

(IV) Ako je $\{f_i\}_{i=1}^{n+2}$ konačno α-senčenje od X onda postoji α-kosenčenje $\{g_i\}_{i=1}^{n+2}$ tako da je $\bigwedge_{i=1}^{n+2} g_i = \bar{0}$ i $g_i \le f_i$ za $i = 1,2,\dots,n+2$.

<u>Dokaz.</u> (I) $\Longrightarrow$ (II) Neka je $F\text{-dim}_\alpha X \le n$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od X. Tada na osnovu stava 1.1 postoji α-senčenje $\{h_i\}_{i=1}^{k}$ takvo da je $h_i \le f_i$ za $i = 1,2,\dots,k$ i red od $\{h_i\}_{i=1}^{k} \le n$. Kako je (X,T) F-normalan, to na osnovu stava 1.3 postoji α-senčenje $\{g_i\}_{i=1}^{k}$ takvo da je $cl\, g_i \le h_i$ za $i = 1,2,\dots,k$ i red od $\{cl\, g_i\}_{i=1}^{k} \le n$.

(II) $\Longrightarrow$ (III) i (III) $\Longrightarrow$ (IV) je jasno.

(IV) $\Longrightarrow$ (I) Neka je $\{f_i\}_{i=1}^{n+2}$ α-senčenje od X. Na osnovu pretpostavke postoji α-kosenčenje $\{g_i\}_{i=1}^{n+2}$ od X takvo da je $g_i \le f_i$ za svako $i=1,2,\dots,n+2$ i $\bigwedge_{i=1}^{n+2} g_i = \bar{0}$. Na osnovu stava 1.2 postoji familija otvorenih F-skupova $\{h_i\}_{i=1}^{n+2}$ tako da je $g_i \le h_i \le f_i$ za $i=1,2,\dots,n+2$ i $\{cl\, h_i\}_{i=1}^{n+2}$ je sveling od $\{g_i\}_{i=1}^{n+2}$.

Otuda je $\{h_i\}_{i=1}^{n+2}$ α-senčenje od X, $h_i \le f_i$ i $\bigwedge_{i=1}^{n+2} h_i = \overline{0}$ pa je na osnovu stava 1.1 $F\text{-dim}_\alpha X \le n$. □

Neka je (X,T) F-prostor i neka je $\alpha_0 \in [0,1]$ fiksirano. Definisaćemo slabiju dimenzionu funkciju - u oznaci $F_{\alpha_0}\text{-dim}_\alpha X$ - od X za $\alpha \in [0,1)$ na sledeći način: Ako je $\alpha \in [0,\alpha_0)$, onda je $F_{\alpha_0}\text{-dim}_\alpha X \le n$ ako svako konačno α-senčenje od X ima α-profinjenje čiji je red $\le n$. Ako je $\alpha \in [\alpha_0,1)$, onda je $F_{\alpha_0}\text{-dim}_\alpha X \le n$ ako svako konačno α-senčenje od X ima β-profinjenje za neko $\beta \in [0,\alpha_0)$ čiji je red $\le n$.

Primer 1.3: Neka je X = I i neka su za n = 1,2,3,4 funkcije $f_n, g_n \in I^X$ definisane na sledeći način:

$$f_n(x) = \begin{cases} \frac{2}{n+2} & \text{ako je } x \in [0,\frac{2}{3}] \\ 0 & \text{inače} \end{cases}$$

$$g_n(x) = \begin{cases} \frac{2}{2n+1} & \text{ako je } x \in [\frac{1}{3},1] \\ 0 & \text{inače} \end{cases}$$

Neka je $T = \{\overline{0},\overline{1},f_n,g_n,f_n \wedge g_n, f_n \vee g_n,\ n=1,2,3,4\}$. Tada je (X,T) F-prostor. Stavimo $\alpha_0 = \frac{5}{8}$ i neka je recimo $\alpha = \frac{1}{12}$. Tada je $\{f_4,g_4\}$ α-profinjenje svakog α-senčenja od X, gde je $\alpha \in [\frac{1}{12},\frac{1}{3})$, pri čemu je red od $\{f_4,g_4\} \le 1$. Otuda je $F_{\frac{5}{8}}\text{-dim}_\alpha X \le 1$.

Primedba: Primetimo da ako je $\alpha_0 = 1$, onda je $F_{\alpha_0}\text{-dim}\, X = F\text{-dim}_\alpha X$ za $\alpha \in [0,\alpha_0)$.

Ako je $\alpha \in [\alpha_0,1)$, onda $F_{\alpha_0}\text{-dim}_\alpha X$ i $F\text{-dim}_\alpha X$ ne moraju biti u vezi.

2. TEOREME PODSKUPA I SUME

U ovom paragrafu nalaze se neke teoreme podskupa i sume za α-kombinatornu dimenziju F-prostora. Ako je Y F-potprostor F-prostora (X,T), u opštem slučaju nije tačno da je $F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X$. Videćemo pod kojim je uslovom gornja relacija tačna.

Stav 2.1: Neka je (X,T) α-kompaktan F-prostor. Ako je Y α-zatvoren F-potprostor onda je:

$$F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X .$$

Dokaz. Neka je $F\text{-}\dim_\alpha X \leq n$, Y α-zatvoren F-potprostor od (X,T) i neka je $\mathcal{G}$ konačno α-senčenje od Y ($\alpha \in [0,1)$). Za svako $x \in X \setminus Y$, postoji $u_x \in T$ tako da je $u_x(x) > \alpha$ i $u_x \wedge \mu(Y) = \overline{0}$ na osnovu definicije α-zatvorenosti I.2.12. Tada je $\{\mathcal{G}, u_x\}_{x \in X \setminus Y}$ α-senčenje α-kompaktnog F-prostora (X,T) pa ima konačno α-podsenčenje V. Kako je $F\text{-}\dim_\alpha X \leq n$, to V ima α-profinjenje W reda $\leq n$ pa je $W|Y$ α-profinjenje od $\mathcal{G}$ reda $\leq n$. □

Stav 2.2: Neka je Y F-potprostor α-Hausdorffovog F-prostora (X,T). Tada je:

$$F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X .$$

Dokaz. Neka je $F\text{-}\dim_\alpha X \leq n$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od Y. Tada postoji familija $\{g_i\}_{i=1}^{k}$ otvorenih F-skupova takva da je $g_i/Y = f_i$ za $i = 1,2,\ldots,k$. Sada neka je $y \in Y$. Tada za svako $x \in X \setminus Y$ postoje dva otvorena F-skupa u X v_y, u_x tako da je $u_x(x) > \alpha$, $v_y(y) > \alpha$ i $v_y \wedge u_x = \overline{0}$ na osnovu definicije α-Hausdorffnosti I.2.11. Stavimo $\bigvee_{x \in X \setminus Y} u_x$. Tada je $u \in T$ i $u \wedge u_z = \overline{0}$ za $z \in Y$. Otuda je $\{g_i, u\}_{i=1}^{k}$ konačno α-senčenje od X pa na osnovu pretpostavke postoji α-profinjenje V od $\{g_i,u\}_{i=1}^{k}$ reda $\leq n$. Tada je $V|Y$ α-profinjenje od $\{f_i\}_{i=1}^{k}$ reda $\leq n$. Otuda je $F\text{-}dim_{\alpha}Y \leq n$. □

<u>Stav 2.3:</u> Neka je Y podesno zatvoren skup u F-prostoru (X,T). Tada je:

$$F\text{-}dim_{\alpha}Y \leq F\text{-}dim_{\alpha}X \ .$$

<u>Dokaz.</u> Neka je $F\text{-}dim_{\alpha}X \leq n$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od Y. Tada postoji familija $\{g_i\}_{i=1}^{k}$ otvorenih F-skupova u X takva da je $g_i|Y = f_i$. Kako je Y podesno zatvoren to je $co.\mu(Y)$ F-otvoren u X na osnovu definicije I.2.12, i otuda je $\{g_i, co.\mu(Y)\}_{i=1}^{k}$ α-senčenje od X. Iz $F\text{-}dim_{\alpha}X \leq n$ sledi na osnovu stava II.1.1 da postoji α-senčenje $\{h_i, co.\mu(Y)\}_{i=1}^{k}$ od X takvo da je $h_i \leq g_i$ za $i = 1,2,\dots,k$ i red od $\{h_i, co.\mu(Y)\}_{i=1}^{k} \leq n$. Otuda je $\{h_i|Y\}_{i=1}^{k}$ α-senčenje od Y reda $\leq n$ i $h_i|Y \leq g_i|Y = f_i$ za $i = 1,2,\dots,k$ pa je $F\text{-}dim_{\alpha}Y \leq n$. □

<u>Definicija 2.1:</u> F-potprostor Y F-prostora (X,T) je zatvoren ako je $\mu(X\setminus Y)$ otvoren F-skup u X, svaki zatvoren F-skup u Y je zatvoren F-skup u X i za svaki zatvoren F-skup f u X, $f|Y$ je zatvoren F-skup u Y.

<u>Stav 2.4:</u> Neka je (X,T) F-prostor a Y zatvoren F-potprostor od (X,T). Tada je:

$$F\text{-}dim_{\alpha}Y \leq F\text{-}dim_{\alpha}X \ .$$

<u>Dokaz.</u> Neka je $F\text{-}dim_{\alpha}X \leq n$ i neka je $\mathcal{G}$ konačno α-senčenje od Y. Tada je $U = \{u \in T : u|Y \in \mathcal{G}\}$ konačna familija otvorenih F-skupova u X i otuda je $\{U, \mu(X\setminus Y)\}$ konačno α-senčenje od X. Kako je

F-$\dim_\alpha X \leq n$ to na osnovu stava II.1.1 postoji konačno α-senčenje $\{V, \mu(X\setminus Y)\}$ od X reda $\leq n$ takvo da je $V \leq U$. Otuda je $\{V|Y\}$ konačno α-senčenje od Y reda $\leq n$ i $V|Y \leq u|Y = \mathcal{G}$ pa je F-$\dim_\alpha Y \leq n$. □

<u>Definicija 2.2:</u> Neka je Y F-potprostor F-prostora (X,T) i neka je $\mathcal{G}$ α-senčenje od Y. Tada je $\mathcal{G}$ prebrojivo ekstenzibilno (na X) ako postoji prebrojivo α-senčenje $\mathcal{H}$ od X takvo da je $\mathcal{H}|Y$ α-profinjenje od $\mathcal{G}$.

<u>Stav 2.5:</u> Neka je (X,T) α-Hausdorffov, α-Lindelöfov F-prostor. Tada je svako konačno α-senčenje ma kog F-potprostora Y od (X,T) prebrojivo ekstenzibilno.

<u>Dokaz.</u> Neka je $\mathcal{G}$ konačno α-senčenje F-potprostora Y α-Hausdorffovog, α-Lindelöfovog F-prostora (X,T). Tada postoji konačna familija $\mathcal{H} \subseteq T$ takva da je $\mathcal{H}|Y = \mathcal{G}$. Neka je $y \in Y$ proizvoljno. Na osnovu definicije α-Hausdorffnosti, za svako $x \in X\setminus Y$ postoje $u_x, v_y \in T$ tako da je $u_x(x) > \alpha$, $v_y(y) > \alpha$ i $v_y \wedge u_x = \bar{0}$. Otuda je $\{u_x\}_{x\in X\setminus Y}$ familija otvorenih F-skupova u X a $\{\mathcal{H}, u_x\}_{x\in X\setminus Y}$ je α-senčenje od X. Na osnovu definicije α-Lindelöfnosti I.2.7, postoji prebrojivo α-podsenčenje W od $\{\mathcal{H}, u_x\}_{x\in X\setminus Y}$ a $W|Y$ je α-profinjenje od $\mathcal{G}$.

<u>Stav 2.6:</u> Neka je (X,T) α-Lindelöfov F-prostor. Tada je svako konačno α-senčenje α-zatvorenog F-potprostora prebrojivo ekstenzibilno.

<u>Dokaz.</u> Neka je Y α-zatvoren F-potprostor od (X,T) i neka je $\mathcal{G}$ konačno α-senčenje od Y. Za svako $x \in X\setminus Y$ postoji otvoren F-skup u_x u X, takav da je $u_x(x) > \alpha$ i $\mu(Y) \wedge u_x = \bar{0}$ na osnovu definicije I.2.12. Otuda je $\{\mathcal{G}, u_x\}_{x\in X\setminus Y}$ α-senčenje od X pa se

dokaz kompletira na isti način kao i kod stava 2.5. □

Stav 2.7: Neka je (X,T) prebrojivo α-kompaktan F-prostor i neka je Y F-potprostor takav da je svako konačno α-senčenje od Y prebrojivo ekstenzibilno. Tada je:

$$F\text{-}\dim_\alpha Y \le F\text{-}\dim_\alpha X .$$

Dokaz. Neka je $F\text{-}\dim_\alpha X \le n$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od Y. Kako je svako konačno α-senčenje od Y prebrojivo ekstenzibilno, to postoji prebrojivo α-senčenje $\{g_i\}_{i=1}^{\infty}$ od X takvo da je $g_i|Y \le f_i$ za $i = 1,2,\dots,k$. Pošto je (X,T) prebrojivo α-kompaktan, to postoji konačno α-podsenčenje $\{g_i\}_{i=1}^{n}$ (recimo $n>k$) pa na osnovu stava II.1.1 postoji α-senčenje $\{h_i\}_{i=1}^{n}$ od X reda $\le n$ tako da je $h_i \le g_i$ za $i = 1,2,\dots,n$. Otuda je $h_i|Y \le g_i|Y \le f_i$ za $i = 1,2,\dots,k$ pa je $\{h_i|Y\}_{i=1}^{k}$ α-profinjenje od $\{f_i\}_{i=1}^{k}$ reda $\le n$ tj. $F\text{-}\dim_\alpha Y \le n$. □

Stav 2.8: Neka je (X,T) prebrojivo α-kompaktan, α-Lindelöfov F-prostor i neka je Y α-zatvoren F-potprostor od (X,T). Tada je:

$$F\text{-}\dim_\alpha Y \le F\text{-}\dim_\alpha X .$$

Dokaz. Sledi iz stavova 2.6 i 2.7 ili iz stava 2.1 i činjenice da je α-Lindelöfov i prebrojivo α-kompaktan prostor α-kompaktan. □

Teorema podskupa za $F_{\alpha_0}\text{-}\dim_\alpha$ zavisi od položaja broja α u odnosu na fiksirano α_0 .

Stav 2.9: Neka je Y zatvoren F-potprostor F-prostora (X,T). Tada je:

$$F_{\alpha_0}\text{-}\dim_\alpha Y \le F_{\alpha_0}\text{-}\dim_\alpha X \quad \text{za } \alpha \in [\alpha_0, 1).$$

Dokaz. Neka je F_{α_o}-$\dim_\alpha X \le n$, $\alpha \in [\alpha_c, 1)$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od Y. Tada postoji familija $\{g_i\}_{i=1}^{k}$ otvorenih F-skupova u X takva da je g_i Y = f_i za i = 1,2,...,k pa je $\{g_i, \mu(X\setminus Y)\}_{i=1}^{k}$ konačno α-senčenje od X. Na osnovu pretpostavke postoji β-profinjenje V od $\{g_i, \mu(X\setminus Y)\}_{i=1}^{k}$ za neko $\beta \in [0, \alpha_o)$ reda $\le n$. Otuda je $\{V|Y\}$ β-profinjenje od $\{f_i\}_{i=1}^{k}$ reda $\le n$ pa je F_{α_c}-$\dim_\alpha Y \le n$. □

Primedba: Stavovi 2.1, 2.2 i 2.3 važe za F_{α_o}-dim samo ako je $\alpha \in [\alpha_c, 1)$.

Mi želimo da determinišemo dimenziju F-prostora u terminima dimenzije nekih od njegovih F-potprostora. Sledeća dva stava daju teoremu konačne sume za F-prostor koji zadovoljava odredjene uslove.

Primetimo da ako su Y_1 i Y_2 α-zatvoreni u F-prostoru (X,T) takvi da je $Y_1 \cap Y_2 \neq \emptyset$, onda je $Y_2 \setminus Y_1$ α-zatvoren u Y_2.

Stav 2.10: Neka je (X,T) α-kompaktan F-prostor takav da je $X = \bigcup_{i=1}^{k} Y_i$ gde je za svako i = 1,2,...,k Y_i α-zatvoren F-prostor od (X,T) i F-$\dim_\alpha Y_i \le n$. Tada je F-$\dim_\alpha X \le n$.

Dokaz. Neka je $\mathcal{G}$ konačno α-senčenje od X i neka je $X = Y_1 \cup Y_2$ (i = 1,2) tako da je F-$\dim_\alpha Y_1 \le n$ i F-$\dim_\alpha Y_2 \le n$. Tada su $\{\mathcal{G}|Y_1\}$ i $\{\mathcal{G}|Y_2\}$ konačna α-senčenja od Y_1 i Y_2 respektivno. Na osnovu pretpostavke postoje α-profinjanja V_1, V_2 od $\{\mathcal{G}|Y_1\}$ i $\{\mathcal{G}|Y_2\}$ tako da je red od $V_1 \le n$ i red od $V_2 \le n$. Sada ako je $Y_1 \cap Y_2 = \emptyset$, onda je $\{V_1, V_2\}$ α-profinjenje od $\mathcal{G}$ čiji je red $\le n$ pa je F-$\dim_\alpha X \le n$.

Ako je $Y_1 \cap Y_2 \neq \emptyset$ onda je $Y_3 = Y_2 \setminus Y_1$ α-zatvoren u Y_2.

Otuda je $F\text{-}dim_{\alpha}Y_3 \leq n$, $Y_1 \cap Y_3 = \emptyset$ i $Y_1 \cup Y_3 = X$. Sada kao i u prethodnom slučaju postoji α-profinjenje $\{V_1, V_3\}$ reda $\leq n$ pa je $F\text{-}dim_{\alpha}X \leq n$. □

Stav 2.11: Neka je (X,T) α-kompaktan, α-Hausdorffov F-prostor takav da je $X = \bigcup_{i=1}^{k} Y_i$ gde je Y_i α-kompaktan F-potprostor za svako i, a $F\text{-}dim_{\alpha}Y_i \leq n$. Tada je $F\text{-}dim_{\alpha}X \leq n$.

Dokaz. Sledi neposredno iz stava 2.10 jer je svaki α-kompaktan F-potprostor α-Hausdorffovog prostora α-zatvoren na osnovu stava I.2.9 (I). □

3. LOKALNA α-KOMBINATORNA DIMENZIJA

U ovom paragrafu se definiše i proučava lokalna α-kombinatorna dimenzija F-prostora. Od sada će nam za $g \in I^X$, $\alpha \in [0,1)$ g^+ biti podskup $\{x \in X : g(x) > \alpha\}$.

Definicija 3.1: Neka je (X,T) F-prostor. Tada se lokalna α-kombinatorna dimenzija od (X,T) - u oznaci $loc\ F\text{-}dim_{\alpha}X$ - definiše na sledeći način:

(I) $loc\ F\text{-}dim_{\alpha}X = -1$ ako je $X = \emptyset$

(II) $loc\ F\text{-}dim_{\alpha}X \leq n$ $(n \geq 0)$ ako za svako $x \in X$ postoji otvoren F-skup g u X takav da je $g(x) > \alpha$ i $F\text{-}dim_{\alpha}g^+ \leq n$.

(III) $loc\ F\text{-}dim_{\alpha}X = \infty$ ako $loc\ F\text{-}dim_{\alpha}X \leq n$ nije tačno ni za jedan ceo broj n.

Stav 3.1: Neka je (X,T) α-Hausdorffov F-prostor. Tada su sledeći iskazi ekvivalentni:

(I) $loc\ F\text{-}dim_{\alpha}X \leq n$.

(II) Svako α-senčenje od X ima α-profinjenje $\mathcal{H}$ takvo da je $F\text{-}dim_{\alpha} h^{+} \leq n$ za svako $h \in \mathcal{H}$.

Dokaz. (I) $\Longrightarrow$ (II) Neka je loc $F\text{-}dim_{\alpha} X \leq n$ i neka je U α-senčenje od X. Tada na osnovu pretpostavke, za svako $x \in X$ postoji otvoren F-skup g_x u X takav da je $g_x(x) > \alpha$ i $F\text{-}dim_{\alpha} g^{+} \leq n$. Na osnovu definicije α-senčenja I.2.5, za svako $x \in X$ postoji $u_x \in U$ tako da je $u_x(x) > \alpha$. Stavimo $h_x = g_x \wedge u_x$. Tada je h_x otvoren F-skup u X, $h_x(x) > \alpha$ i $h_x \leq u_x$. Otuda je $\mathcal{H} = \{h_x : x \in X\}$ α-profinjenje od U. Kako je $h_x = g_x \wedge u_x$ to je jasno $h_x^{+} \leq g_x^{+}$. Medjutim, g_x^{+} je α-Hausdorffov F-potprostor od (X,T), $F\text{-}dim_{\alpha} g^{+} \leq n$ pa na osnovu stava 2.2 dobijamo da je $F\text{-}dim_{\alpha} h_x^{+} \leq n$ tj. $F\text{-}dim_{\alpha} h^{+} \leq n$ za $h \in \mathcal{H}$ što je i trebalo dokazati.

(II) $\Longrightarrow$ (I) Neka važi (II) i neka je $x \in X$ proizvoljna tačka. Tada za svako $y \in X \setminus \{x\}$ postoje otvoreni F-skupovi u_y, v_x takvi da je $u_y(y) > \alpha$, $v_x(x) > \alpha$ i $u_y \wedge v_x = \bar{0}$. Otuda je $\{u_y, v_x\}_{y \in X \setminus \{x\}}$ α-senčenje od X pa na osnovu pretpostavke postoji α-profinjenje $\mathcal{H}$ od $\{u_y, v_x\}_{y \in X \setminus \{x\}}$ tako da je $F\text{-}dim_{\alpha} h^{+} \leq n$ za svako $h \in \mathcal{H}$. Odatle sledi da je loc $F\text{-}dim_{\alpha} X \leq n$. □

Sledeći stavovi daju vezu izmedju α-kombinatorne dimenzije i lokalne α-kombinatorne dimenzije F-prostora.

Stav 3.2: Neka je (X,T) α-Hausdorffov, α-kompaktan F-prostor. Tada je:

$$\text{loc } F\text{-}dim_{\alpha} X \leq F\text{-}dim_{\alpha} X .$$

Dokaz. Neka je $F\text{-}dim_{\alpha} X \leq n$ i neka je $x \in X$. Tada za svako $y \in Y \setminus \{x\}$ postoje otvoreni F-skupovi u_y, v_x u X takvi da je $u_y(y) > \alpha$, $v_x(x) > \alpha$ i $u_y \wedge v_x = \bar{0}$. Otuda je familija $\{u_y, v_x\}_{y \in X \setminus \{x\}}$ α-senčenje od X pa na osnovu α-kompaktnosti F-prostora (X,T) postoji konačan broj tačaka $y_1,y_2,\dots,y_k \in X\setminus\{x\}$ tako da je $\{u_{y_i},v_x\}_{i=1}^{k}$ α-podsenčenje od $\{u_y,v_x\}_{y\in X\setminus\{x\}}$ tj. $\{u_{y_i},v_x\}_{i=1}^{k}$ je konačno α-senčenje od X pa na osnovu pretpostavke postoji α-profinjenje $\mathcal{H}$ od $\{u_{y_i},v_x\}_{i=1}^{k}$.

Sada je za svako $h\in\mathcal{H}$, h^+ F-potprostor α-Hausdorffovog F-prostora (X,T) pa je na osnovu stava 2.2 $F\text{-}dim_\alpha h^+ \leq n$ što povlači da je loc $F\text{-}dim_\alpha X \leq n$ na osnovu stava 3.1. □

<u>Stav 3.3:</u> Neka je (X,T) perfektno F-normalan, α-Hausdorffov F-prostor. Tada je:

$$\text{loc } F\text{-}dim_\alpha X \leq F\text{-}dim_\alpha X.$$

<u>Dokaz.</u> Neka je $F\text{-}dim_\alpha X \leq n$ i neka je $\mathcal{H}$ α-senčenje od X. Zbog perfektne normalnosti imamo da je za svako $h\in\mathcal{H}$, $h = \bigvee_{i=1}^{\infty} f_i$ gde je f_i zatvoren F-skup u X za $i = 1,2,\dots$. Otuda je $f_i \leq h$ za svako i pa postoji otvoren F-skup g_i takav da je $f_i \leq g_i \leq \text{cl } g_i \leq h$. Neka je $g = \bigvee_{i=1}^{\infty} g_i$. Tada je g otvoren F-skup u X i $g\leq h$. Stavimo $\mathcal{G} = \{g\in T : g\leq h \text{ za svako } h\in\mathcal{H}\}$. Tada je $\mathcal{G}$ α-profinjenje od $\mathcal{H}$ i za svako $g\in\mathcal{G}$ imamo da je $F\text{-}dim_\alpha g^+ \leq n$ na osnovu stava 2.2. Otuda je na osnovu stava 3.1 loc $F\text{-}dim_\alpha X \leq n$. □

<u>Definicija 3.2:</u> Za F-prostor (X,T) kažemo da je lokalno α-zatvoren ako za svako $x\in X$ postoji otvoren F-skup h u X takav da je $h(x)>\alpha$ i h^+ je α-zatvoren.

Sada dajemo neke uslove pod kojima se loc $F\text{-}dim_\alpha$ i $F\text{-}dim_\alpha$ podudaraju.

<u>Stav 3.4:</u> Neka je (X,T) α-kompaktan, lokalno α-zatvoren F-prostor. Tada je:

$$\text{loc } F\text{-dim}_\alpha X = F\text{-dim}_\alpha X \text{ .}$$

Dokaz. $\Longrightarrow$ Neka je $F\text{-dim}_\alpha X \leq n$. Na osnovu definicije 3.2, za svako $x \in X$ postoji otvoren F-skup h u X takav da je $h(x) > \alpha$ i h^+ je α-zatvoren. Kako je $F\text{-dim}_\alpha X \leq n$ to na osnovu stava 2.1 dobijamo da je $F\text{-dim}_\alpha h^+ \leq n$. Otuda je loc $F\text{-dim}_\alpha X \leq n$.

$\Longleftarrow$ Neka je loc $F\text{-dim}_\alpha X \leq n$. Tada za svako $x \in X$ postoji otvoren F-skup h_x u X takav da je $h_x(x) > \alpha$ i $F\text{-dim}_\alpha h_x^+ \leq n$. Familija $\{h_x\}_{x \in X}$ je α-senčenje α-kompaktnog F-prostora (X,T) pa postoji konačan broj tačaka $x_1, x_2, \ldots, x_k \in X$ tako da je $\{h_{x_i}\}_{i=1}^{k}$ konačno α-senčenje od X. Sada imamo da je $F\text{-dim}_\alpha h_{x_i}^+ \leq n$ za $i = 1,2,\ldots,k$ i $X = \bigcup_{i=1}^{k} h_i^+$. Kako je (X,T) lokalno α-zatvoren to je svaki $h_{x_i}^+$ α-zatvoren pa na osnovu stava 2.10 sledi da je $F\text{-dim}_\alpha X \leq n$. $\square$

Stav 3.5: Neka je (X,T) α-Hausdorffov F-prostor i neka je Y F-potprostor od (X,T). Tada je:

$$\text{loc } F\text{-dim}_\alpha Y \leq \text{loc } F\text{-dim}_\alpha X.$$

Dokaz. Neka je loc $F\text{-dim}_\alpha X \leq n$ i neka je $\mathcal{H}$-senčenje od Y. Tada postoji familija $\mathcal{G}$ otvorenih F-skupova u X takva da je $\mathcal{G}|Y = \mathcal{H}$. Neka je $y \in Y$. Tada za svako $x \in X \setminus Y$ postoje otvoreni F-skupovi u_x, v_y u X takvi da je $u_x(x) > \alpha$, $v_y(y) > \alpha$ i $u_x \wedge v_y = \overline{0}$. Otuda je $\{\mathcal{G}, u_x\}_{x \in X\setminus Y}$ α-senčenje pa na osnovu pretpostavke i stava 3.1, postoji α-profinjenje W od $\{\mathcal{G}, u_x\}_{x \in X \setminus Y}$ takvo da je $F\text{-dim } w^+ \leq n$ za svako $w \in W$. Tada je $W|Y$ α-profinjenje od $\mathcal{H}$. Ako stavimo $W^* = W|Y$, onda na osnovu stava 2.2 imamo da je $F\text{-dim}_\alpha w^{*+} \leq n$ za svako $w^* \in W^*$. Otuda sledi na osnovu stava 3.1 da je loc $F\text{-dim}_\alpha Y \leq n$. $\square$

Sledeći stav daje teoremu konačne sume za lokalnu α-kombinatornu dimenziju.

__Stav 3.6:__ Neka je (X,T) α-kompaktan, lokalno α-zatvoren F-prostor. Ako je $X = Y\cup Z$, pri čemu je loc $F\text{-}dim_{\alpha} Y \leq n$ i loc $F\text{-}dim_{\alpha} Z \leq n$, onda je loc $F\text{-}dim_{\alpha} X \leq n$.

__Dokaz.__ Neka je loc $F\text{-}dim_{\alpha} Y \leq n$ i loc $F\text{-}dim_{\alpha} Z \leq n$. Za svako $x\in X$, na osnovu definicije 3.2 postoji otvoren F-skup h takav da je $h(x)>\alpha$ i h^+ je α-zatvoren. Otuda je na osnovu stava I.2.8 h^+ α-kompaktan.

Sada ako je $x\in Y\setminus Z$, onda je $(h|Y\setminus Z))(x)>\alpha$ i kako je loc $F\text{-}dim_{\alpha} Y\leq n$, to je $F\text{-}dim_{\alpha}(h|(Y\setminus Z))^+\leq n$. Ako je $x\in Z\setminus Y$, onda je $(h|(Z\setminus Y))(x)>\alpha$ i na osnovu pretpostavke je $F\text{-}dim_{\alpha}(h|(Z\setminus Y))^+\leq n$. Ako je $x\in Y\cap Z$, onda je $(h|(Y\cap Z))(x)>\alpha$ i $F\text{-}dim_{\alpha}(h|(Y\cap Z))^+\leq n$. Jasno da u svakom slučaju imamo da je $(h|(Y\setminus Z))^+$, $(h|(Z\setminus Y))^+$ i $(h|(Y\cap Z))^+\subseteq h^+$ što povlači da je svaki od njih α-zatvoren u h^+.

Sada treba da dokažemo da je $F\text{-}dim_{\alpha} h^+\leq n$. Ako je h^+ jednak jednom od tri gornja podskupa, onda je jasno $F\text{-}dim_{\alpha} h^+\leq n$. Ako je $h^+ = (h|(Z\setminus Y))^+\cup(h|(Y\cap Z))^+$ ili $h^+ = (h|(Y\setminus Z))^+\cup(h|(Y\cap Z))^+$ onda u oba slučaja na osnovu stava 2.10 sledi da je $F\text{-}dim_{\alpha} h^+\leq n$ tj. za svako $x\in X$ postoji otvoren F-skup h takav da je $h(x)>\alpha$ i $F\text{-}dim_{\alpha} h^+\leq n$. Otuda je loc $F\text{-}dim_{\alpha} X\leq n$. □

Glava III:

KOMBINATORNA DIMENZIJA I MODIFIKACIJA F-PROSTORA

Veze izmedju topoloških i F-prostora proučavao je Lowen [10],[11] , koji je funkcije ω i i definisao na sledeći način:

Ako je $\mathcal{T}(X)$ skup svih topologija na X i F(X) skup svih F-topologija na X tada je:

$$a)\ i : F(X) \longrightarrow (X)$$
$$T \longrightarrow i(T)$$

i(T) je inicijalna topologija za familiju funkcija T i I sa uobičajenom topologijom

$$b)\ \omega : \mathcal{T}(X) \longrightarrow F(X)$$
$$R \longrightarrow \omega(R)$$

$\omega(R)$ je familija odozdo ograničenih poluneprekidnih funkcija iz $(X,R) \longrightarrow I$ sa uobičajenom topologijom.

Prema a) možemo pridružiti svakom F-prostoru (X,T) topološki prostor (X,i(T)) koji se naziva modifikacija od (X,T) ili modifikovana topologija. Ova ideja svodi neke osobine F-prostora na topološke osobine, tj. ako je (X,T) α-kompakt, tada je (X,i(T)) kompakt.

Na ovaj način se koriste dobro poznate osobine kombinatorne dimenzije topoloških prostora (za dokaz teoreme proizvoda za F-prostore) kao i neka druga topološka svojstva kao i Čech-Stone-ova kompaktifikacija.

Do kraja ovog rada proučavaćemo dane α-osobine i α-osobine za svako $\alpha \in [0,1)$.

1. F-dim

Definicija 1.1: Neka je (X,T) F-prostor. Tada je $F\text{-dim}_{\alpha} X \leq n$ ako je $F\text{-dim}_{\alpha} X \leq n$, za svako $\alpha \in [0,1)$.

Primer 1.1: Neka je (X,R) topološki prostor sa dim $X \leq n$.

Za u R, neka je $\mu(u)$ karakteristična funkcija za u. Neka je $T = \{\mu(u)\}_{u \in R}$. Tada je (X,T) F-prostor. Zahtevajmo da je u F-dim $X \leq n$. Vidi se da je $\{\mu(u_i)\}_{i=1}^{k}$ konačno α-senčenje od X za svako $\alpha \in [0,1)$. Tada je $\{u_i\}_{i=1}^{k}$ otvoren pokrivač $\{v_i\}_{i=1}^{k}$ od X reda $\leq n$ i $v_i \leq u_i$ za i = 1,2,...,k.

Neka je $f_i \in I^X$ definisano sa

$$f_i(x) = 1 \text{ za } x \in v_i \qquad\qquad \text{za } i = 1,2,\dots,k$$
$$\quad\;\; = 0 \text{ za } x \notin v_i$$

Tada je $\{f_i\}_{i=1}^{k}$ α-senčenje od X za svako $\alpha \in [0,1)$ takvo da je $f_i \leq \mu(u_i)$ za i = 1,2,...,k i red od $\{f_i\}_{i=1}^{k}$ je $\leq n$. Zato je F-dim $X \leq n$.

Definicija 1.2: F-prostor (X,T) je α-dopustiv ako za svako $V \subset T$, V sadrži bar dva člana, postoji $\alpha \in [0,1)$ tako da je V α-senčenje od X.

Stav 1.1: Neka je (X,T) α-dopustiv F-prostor. Ako je F-dim X=0, tada je (X,T) slabo F-normalan prostor.

Dokaz. Neka je F-dim X = 0 i neka su G_1 i G_2 disjunktni zatvoreni F-skupovi u X. Tada je $\{co.G_1,\ co.G_2\} \subset T$. Kako je (X,T) α-

dopustiv, postoji $\alpha\in[0,1)$ tako da je $\{co.G_1, co.G_2\}$ α-senčenje od X. Kako je F-dim X = 0, za svako $\alpha\in[0,1)$ je $F\text{-}dim_\alpha X = 0$, po definiciji 1.1, i zato postoji α-senčenje $\{f_1, f_2\}$ od X tako da je $f_1 \leq co.G_2$, $f_2 \leq co.G_1$ i $f_1 \wedge f_2 = 0$, tj. $f_1 \leq co.f_2$. Zato je $G_1 \leq co.f_2$, $G_1 \leq co.(co.f_1)$ i $G_1 \leq f_1$. Slično se pokazuje da je $G_2 \leq f_2$. □

Posledica 1.1: Neka je (X,T) α-dopustiv F_c-prostor. Ako je F-dim X = 0, tada je (X,T) F-normalan prostor.

Uvedimo strožiju definiciju nula dimenzije F-prostora na sledeći način:

SF-dim X = 0 ako za svako konačno α-senčenje od X postoji α-profinjenje koje se sastoji od otvorenih i zatvorenih disjunktnih F-skupova. Jasno je da, ako je $SF\text{-}dim_\alpha X = 0$, onda je $F\text{-}dim_\alpha X = 0$ za svaki F-prostor (X,T).

Definicija 1.3: Neka su G_1 i G_2 disjunktni zatvoreni F-skupovi F-prostora (X,T). Tada su G_1 i G_2 jako razdvojeni u X ako postoji otvoren i zatvoren F-skup E takav da je $G_1 \leq E$ i $E \leq co.G_2$.

Stav 1.2: Neka je (X,T) α-dopustiv F-prostor. Ako je SF-dim X=0 tada je svaki par disjunktnih zatvorenih F-skupova jako razdvojen u X.

Dokaz. Neka je SF-dim X = 0 i neka su G_1 i G_2 disjunktni zatvoreni F-skupovi. Tada je $\{co.G_1, co.G_2\}$ α-senčenje od X za neko $\alpha\in[0,1)$, jer je (X,T) α-dopustiv. Kako je SF-dim X = 0, to postoji α-senčenje $\{f_1, f_2\}$ od X takvo da su f_1 i f_2 otvoreni i zatvoreni, $f_1 \wedge f_2 = 0$ i $f_1 \leq co.G_1$, $f_2 \leq co.G_2$. Tada je, zbog $G_1 \leq co.f_1$, $G_1 \leq co.(co.f_2)$ tj. $G_1 \leq f_2 \leq co.G_2$. □

Definicija 1.4: F-prostor (X,T) je

(I) Povezan ako su otvoreni i zatvoreni F-skupovi samo $\overline{0}$ i $\overline{1}$. F-skup g je povezan ako je g_0 povezan kao F-podprostor.

(II) Nepovezan, ako nije povezan.

(III) Potpuno nepovezan, ako postoji ne povezan F-skup koji sadrži više od jedne F-tačke.

Stav 1.3: Neka je (X,T) α-dopustiv FT_1-prostor. Ako je SF-dim $X = 0$, tada je X potpuno nepovezan.

Dokaz. Neka je SF-dim $X = 0$ i neka je g povezan F-skup u X koji se sastoji od dve F-tačke p i q, $p \neq q$. Tada po stavu 1.2 postoji otvoren i zatvoren F-skup E, takav da $p \in E$ i $E \leq co\ q$. Zato je $E|g_0$ otvoren i zatvoren u g_0 i $E|g_0$ nije $\overline{0}_{g_0}$ niti $\overline{1}_{g_0}$, tj. g nije povezan, što je suprotno pretpostavci. Zato je (X,T) potpuno nepovezan. □

Vratimo se teoremi podskupa datoj u delu 2 prethodne glave, ali ovog puta posmatrajmo dimenziju dim F-prostora umesto α-dim.

Stav 1.4: Neka je Y zatvoren F-podprostor F-prostora (X,T). Tada je

$$F\text{-dim } Y \leq F\text{-dim } X .$$

Dokaz. Neka je F-dim $X \leq n$ i neka je U konačno α-senčenje od Y, za svako $\alpha \in [0,1)$. Tada postoji familija W otvorenih F-skupova u X tako da za svako $w \in W$, $w|Y \in U$ i $\{W, \mu(X \setminus Y)\}$ je konačno α-senčenje od X za svako $\alpha \in [0,1)$. Kako je F-dim $X \leq n$, tada po definiciji 1.1 postoji α-profinjenje V od $\{W, \mu(X \setminus Y)\}$ čiji je red $\leq n$ za svako $\alpha \in [0,1)$. Tako je $\{V|Y\}$ α-finiji od $W|Y = U$

i reda $\leq n$ za svako $\alpha \in [0,1)$. Zato je F-dim $Y \leq n$. □

Stav 1.5: Neka je Y F-podprostor jako α-kompaktnog Hausdorffovog F-prostora (X,T). Tada je

$$\text{F-dim } Y \leq \text{F-dim } X .$$

Dokaz. Kako je (X,T) jako -kompaktan Hausdorffov F-prostor, tada prema stavu 2.8 [6] je $\mu(X \setminus Y)$ otvoren F-skup u X i prema stavu 1.4 je F-dim $Y \leq$ F-dim X. □

Stav 1.6: Neka je Y F-podprostor jako α- Hausdorffovog F-prostora (X,T). Tada je F-dim $Y \leq$ F-dim X.

Dokaz sledi iz stava II 2.2 za svako $\alpha \in [0,1)$. □

Stav 1.7: Neka je (X,T) jako prebrojivo α-kompaktan F-prostor i Y F-podprostor od (X,T). Ako je svako konačno α-senčenje od Y $(\forall \alpha \in [0,1))$ prebrojivo ekstenzibilno tada je

$$\text{F-dim } Y \leq \text{F-dim } X .$$

Dokaz sledi iz stava II 2.6 za svako $\alpha \in [0,1)$. □

Stav 1.8: Neka je (X,T) jako α-kompaktan, takav da je $X = \bigcup_{i=1}^{k} Y_i$ gde je za svako i, Y_i jako α-zatvoren F-podprostor se F-dim $Y_i \leq n$. Tada je F-dim $X \leq n$.

Dokaz sledi primenom stava II 2.10, za svako $\alpha \in [0,1)$. □

Stav 1.9: Neka je (X,T) jako α-Hausdorffov F-prostor, $X = \bigcup_{i=1}^{k} Y_i$ i neka je svaki Y_i jako α-kompaktan F-podprostor se F-dim $Y_i \leq n$. Tada je F-dim $X \leq n$.

Dokaz se izvodi koristeći stav II 2.11, za svako $\alpha \in [0,1)$. □

Definicija 1.5: Neka je Y F-podprostor F-prostora (X,T). Tada je (X,T) konzervativan ako je za svaka dva F-skupa f i g u Y sa $f \leq g$, $f^* < g^*$ u X, gde je $f^*|Y = f$ i $g^*|Y = g$.

Stav 1.10: Neka je Y zatvoren F-podprostor konzervativnog F-normalnog prostora (X,T). Tada je Y F-normala.

Dokaz. Neka je f otvoren F-skup u Y i g zatvoren F-skup u Y takav da je $g \leq f$. Tada postoje zatvoren F-skup g^* u X i otvoren F-skup f^* u X takvi da je $g = g^*|Y$ i $f = f^*|Y$. Kako je (X,T) konzervativan, to je po definiciji $g^* < f^*$ u X. Kako je po pretpostavci (X,T) F-normalan prostor, to postoji otvoren F-skup h u X za $g^* \leq h \leq \text{cl } h \leq f^*$ i odatle je $g \leq h|Y \leq \text{cl } h|Y \leq f$. Zato je Y F-normalan prostor. □

Stav 1.11: Neka je (X,T) konzervativan F_c-normalan prostor i Y zatvoren F-podprostor od (X,T) takav da je F-dim $Y \leq n$. Ako je $\{f_i\}_{i=1}^k$ familija otvorenih F-skupova u X, gde je $\{f_i|Y\}_{i=1}^k$ α-senčenje od Y za neko $\alpha \in [0,1)$, tada postoji familija $\{g_i\}_{i=1}^k$ otvorenih F-skupova u X tako da je $\text{cl } g_i \leq f_i$ i red od $\text{cl}\{g_i\}_{i=1}^k \leq n$.

Dokaz. Neka je $\alpha \in [0,1)$ tako da je $\{f_i|Y\}_{i=1}^k$ α-senčenje od Y. Prema stavu 1.10, Y je F_c-normalan i zato, po stavu II 1.4, postoji α-kosenčenje $\{h_i\}_{i=1}^k$ od Y tako da je $h_i \leq f_i|Y$ za $i=1,2,\dots,k$ i red od $\{h_i\}_{i=1}^k \leq n$. Kako je Y zatvoren F-podprostor od (X,T), h_i je zatvoren F-skup u X za svako $i = 1,2,\dots,k$ i $h_i \leq f_i$. Zbog F-normalnosti od (X,T) postoje otvoreni F-skupovi g_i u X takvi da je $h_i \leq g_i \leq \text{cl } g_i \leq f_i$ za svako $i = 1,2,\dots,k$. Tada je $\{\text{cl } g_i\}_{i=1}^k$ nadkrivanje od $\{h_i\}_{i=1}^k$ i zato je red od $\{\text{cl } g_i\}_{i=1}^k \leq n$.

Stav 1.12: Neka je (X,T) konzervativan F_c-normalan prostor i neka je Y zatvoren F-podprostor od X takav da je F-dim $Y \leq n$. Ako je F-dim $Z \leq n$ za svaki zatvoren F-podprostor Z od (X,T) disjunktan sa Y, tada je F-dim $X \leq n$.

Dokaz. Neka je F-dim $Z \leq n$ i neka je $\{h_i\}_{i=1}^{k}$ α-senčenje od X za svako $\alpha \in [0,1)$. Tada je $\{f_i|Y\}_{i=1}^{k}$ α-senčenje od Y i prema stavu 1.11, postoji familija $\{g_i\}_{i=1}^{k}$ otvorenih F-skupova u X tako da je $g_i \leq f_i$, red od $\{g_i\}_{i=1}^{k} \leq n$ i $\{g_i|Y\}_{i=1}^{k}$ α-senčenje od Y. Neka je $Q = \text{co.}\left[\bigvee_{i=1}^{k} g_i \vee \bigvee_{t\in T}\{t : t > f_i|\forall\, i = 1,2,\ldots,k\}\right]$. Tada je Q zatvoren F-skup u X i $Q \leq \mu(X\setminus Y)$. Zbog F-normalnosti od (X,T), postoji otvoren F-skup h takav da je $Q \leq h \leq \text{cl } h \leq \mu(X\setminus Y)$. Neka je $(\text{cl } h)_o$ nosač od cl h, tada je $(\text{cl } h)_o$ zatvoren F-podprostor od X koji je disjunktan sa Y. Sada je $\{f_i|(\text{cl } h)_o\}_{i=1}^{k}$ α-senčenje od $(\text{cl } h)_o$, i kako je po pretpostavci F-dim$(\text{cl } h)_o \leq n$, to postoji α-senčenje $\{m_i\}_{i=1}^{k}$ od $(\text{cl } h)_o$ tako da je $m_i \leq f_i|(\text{cl } h)_o$ za $i = 1,2,\ldots,k$ i red od $\{m_i\}_{i=1}^{k}$ je $\leq n$.

Neka je definisano $d_i \in I^X$ za svako $i = 1,2,\ldots,k$ sa

$$d_i(x) = m_i(x) \quad \text{ako } x \in (\text{cl } h)_o$$
$$= g_i(x) \quad \text{ako } x \notin (\text{cl } h)_o.$$

Tada je $\{d_i\}_{i=1}^{k}$ α-senčenje od X takvo da je $d_i \leq f_i$ za $i = 1,2,\ldots,k$ i red od $\{d_i\}_{i=1}^{k}$ je $\leq n$. Odatle je F-dim$_\alpha X \leq n$. Primenjujući ovaj metod dokaza za svako $\alpha \in [0,1)$ dobijamo da je F-dim $X \leq n$. □

Stav 1.13: Neka je (X,T) konzervativan F_c-normalan prostor. Ako je $X = Y \cup Z$, gde su Y i Z zatvoreni F-podprostori sa F-dim $Y \leq n$ i F-dim $Z \leq n$, tada je F-dim $X \leq n$

Dokaz. Neka je $X = Y \cup Z$ i Q zatvoren F-podprostor od (X,T) disjunktan sa Y. Tada je Q zatvoren F-podprostor od Z i po stavu 1.4 je F-dim $Q \leq n$. Odatle, prema stavu 1.12, sledi da je F-dim $X \leq n$. □

U ovom delu rada proučavaćemo lokalnu kombinatornu dimenziju uvedenu u delu 3 prethodne glave.

Definicija 1.6: Neka je (X,T) F-prostor. Tada je loc F-dim $X \leq n$ ako je loc F-$\dim_\alpha X \leq n$ za svako $\alpha \in [0,1)$.

Stav 1.14: Neka je (X,T) jako α-Hausdorff-ov, jako α-kompaktan F-prostor. Tada je

$$\text{loc F-dim } X \leq \text{F-dim } X.$$

Dokaz sledi iz stava II 3.2 za svako $\alpha \in [0,1)$. □

Stav 1.15: Ako je (X,T) savršeno F-normalan jako α-Hausdorff-ov F-prostor, tada je

$$\text{loc F-dim } X \leq \text{F-dim } X.$$

Dokaz se izvodi primenom stava II 3.3 za svako $\alpha \in [0,1)$. □

Stav 1.16: Ako je (X,T) jako α-Hausdorffov F-prostor i Y F-podprostor od (X,T), tada je

$$\text{loc F-dim } Y \leq \text{loc F-dim } X.$$

Dokaz sledi iz stava II 3.5 za svako $\alpha \in [0,1)$. □

2. MODIFIKACIJA F-PROSTORA

Modifikacija F-prostora (X,T) je topološki prostor $(X,i(T))$, gde je familija $\{t^{-1}(\alpha,1] : t\in T, \alpha\in I\}$ subbaze za $i(T)$ i za svako $\alpha\in[0,1)$ je $i_\alpha(T) = \{t^{-1}(\alpha,1] : t\in T\}$. Tada je $i_\alpha(T)$ topologija na X (vidi Lowen [12]) i

$$i(T) = \vee\left\{i_\alpha(T) : \alpha\in[0,1)\right\}.$$

Poznato je da skup svih odozdo ograničenih polunepre-kidnih preslikavanja $w(\mathcal{T})$, topološkog prostora $(X,\mathcal{T})$ u I snabdeven uobičajenom topologijom, čini F-prostor $(X,w(\mathcal{T}))$ koji se naziva indukovan F-prostor.

F-prostor (X,T) se naziva topološki generisan ako postoji topologija $\mathcal{T}$ na X takva da je $T = w(\mathcal{T})$ ili ekvivalentno $T = w(i(T))$.

U ovom delu rada i nadalje $F\text{-}\dim_\alpha X_T$ znači α-kombinatornu dimenziju F-prostora (X,T) a $\dim X_{\mathcal{T}}$ znači kombinatornu dimenziju od X sa topologijom $\mathcal{T}$ na X.

U ovom delu rada dalje se utvrdjuju veze izmedju kombinatorne dimenzije F-topologije na skupu X i kombinatorne dimenzije topologije na istom skupu X.

<u>Lema 2.1:</u> Neka je (X,T) F-prostor. Tada je (X,T) α-kompakt (prebrojiv α-kompakt)[α-Lindelof] ako i samo ako je $(X,i_\alpha(T))$ kompakt (prebrojiv kompakt) [Lindelof].

<u>Dokaz.</u> Familija $u\subseteq T$ je α-senčenje od X ako i samo ako je $\bigcup_{t\in u} t^{-1}(\alpha,1] = X$. □

<u>Lema 2.2:</u> Ako je (X,T) α-Hausdorff-ov F-prostor, tada je $(X,i_\alpha(T))$ Hausdorff-ov prostor.

Dokaz. Neka je (X,T) F-prostor i neka je $X \neq Y$. Tada po definiciji α-Hausdorffovog F-prostora, postoje u i v otvoreni F-skupovi u X takvi da je $u(x)>\alpha$, $v(y)>\alpha$ i $u \wedge v = 0$. Odatle sledi da su $u^{-1}(\alpha,1]$ i $v^{-1}(\alpha,1]$ disjunktni otvoreni skupovi u $i_\alpha(T)$ takvi da je $x \in u^{-1}(\alpha,1]$ i $y \in v^{-1}(\alpha,1]$ tj. $(X,i_\alpha(T))$ je Hausdorffov prostor.

Sledeći stavovi daju odnose izmedju F-dim_α F-prostora (X,T) i dim modifikovane topologije $(X, i_\alpha(T))$.

Stav 2.1: Neka je (X,T) F-prostor i $(X,i_\alpha(T))$ modifikovana topologija na X. Tada je

$$F\text{-}dim_\alpha X_T = dim_\alpha X_{i_\alpha(T)} .$$

Dokaz. $\Longrightarrow$ Neka je $F\text{-}dim_\alpha X_T \leq n$ i neka je $\{u_i\}_{i=1}^{k}$ konačan $i_\alpha(T)$-otvoren pokrivač od X. Kako je $i_\alpha(T) = \{t^{-1}(\alpha,1] : t \in T\}$, to je za svako $i \in \{1,2,\dots,k\}$, $u_i = t_i^{-1}(\alpha,1]$ tj. $x \in u_i$ ako i samo ako iz $t_1(x)>\alpha$ sledi da je $\{t_i\}_{i=1}^{k}$ α-senčenje od X. Po pretpostavci i stavu II 1.1, postoji α-senčenje $\{S_i\}_{i=1}^{k}$ od X tako da je $S_i \leq t_i$ za $i = 1,2,\dots,k$ i red od $\{S_i\}_{i=1}^{k}$ je $\leq n$.

Neka je $v_i = \{x \in X : S_i(x)>\alpha\}$ za svako $i = 1,2,\dots,k$. Tada je v_i $i_\alpha(T)$-otvoren za svako $i = 1,2,\dots,k$,

$$\bigcup_{i=1}^{k} v_i = \bigcup_{i=1}^{k} \{S_i^{-1}(\alpha,1]\} = X, \quad v_i \subseteq u_i \quad \text{i}$$

red od $\{v_i\}_{i=1}^{k}$ je $\leq n$ tj. $\{v_i\}_{i=1}^{k}$ je konačan $i_\alpha(T)$-otvoren pokrivač od X čiji je red $\leq n$, i zato je $\dim X_{i_\alpha(T)} \leq n$.

$\Longleftarrow$ Neka je $\dim X_{i_\alpha(T)} \leq n$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od X za $\alpha \in [0,1)$. Za svako $i \in \{1,2,\dots,k\}$ neka je $u_i = f_i^{-1}(\alpha,1]$. Tada je u_i $i_\alpha(T)$-otvoren i $\bigcup_{i=1}^{k} u_i = \bigcup_{i=1}^{k}\{f_i^{-1}(\alpha,1]\} = X$ tj.

$\{u_i\}_{i=1}^{k}$ je konačan $i_\alpha(T)$-otvoren pokrivač od X. Kako je $\dim X_{i_\alpha(T)} \leq n$ to postoji konačan $i_\alpha(T)$-otvoren pokrivač $\{v_i\}_{i=1}^{k}$ od X takav da je $v_i \subseteq u_i$ za i = 1,2,...,k i red od $\{v_i\}_{i=1}^{k}$ je $\leq n$.

Neka je definisano $g_i' : X \to I$ na sledeći način:

$$g_i(x) > \alpha \quad \text{ako } x \in v_i \qquad i=1,2,\dots,k$$
$$\quad = 0 \quad \text{ako } x \notin v_i$$

Odatle je $\{g_i\}_{i=1}^{k}$ α-senčenje od X, $g_i \leq f_i$ za svako i=1,2,...,k i red od $\{g_i\}_{i=1}^{k}$ je $\leq n$. Zato je F-dim $X_T \leq n$. □

Posledica 2.1: Neka je (X,T) F-prostor i (X,i(T)) modifikovana topologija na X. Tada je

$$F\text{-}\dim X_T = \dim X_{i(T)}.$$

Dokaz se izvodi koristeći stav 2.1 za svako $\alpha \in [0,1)$. □

U sledećem stavu se dokazuje da se kombinatorna dimenzija topološkog prostora (X,T) poklapa sa kombinatornom dimenzijom indukovanog F-prostora $(X,\omega(\mathcal{T}))$.

Stav 2.2: Neka je $(X,\mathcal{T})$ topološki prostor i neka je $(X,\omega(\mathcal{T}))$ indukovan F-prostor. Tada je

$$\dim X_{\mathcal{T}} = F\text{-}\dim X_{\omega(\mathcal{T})}.$$

Dokaz. $\Rightarrow$ Neka je F-dim $X_{\omega(\mathcal{T})} \leq n$ i neka je $\{u_i\}_{i=1}^{k}$ otvoren pokrivač od X. Za svako $i \in \{1,2,\dots,k\}$ neka je $\mu(u_i)$ karakteristična funkcija od u_i tj. $\mu(u_i)(x) = 1$ ako $x \in u_i$ i $\mu(u_i)(x)=0$ ako $x \notin u_i$. Jasno je da je $\{\mu(u_i)\}_{i=1}^{k} \subset \omega(\mathcal{T})$ i da je familija $\{\mu(u_i)\}_{i=1}^{k}$ α-senčenje od X za svako $\alpha \in [0,1)$. Odatle sledi da postoji konačno α-senčenje $\{f_i\}_{i=1}^{k}$ od X za svako $\alpha \in [0,1)$ takvo da je $f_i \leq \mu(u_i)$ za svako $i = 1,2,\dots,k$ i red od $\{f_i\}_{i=1}^{k} \leq n$.

Neka je $v_i = \{x \in X : f_i(x) > \alpha, \forall \alpha \in [0,1)\}$ za svako $i = 1,2,\dots,k$. Tada je v_i otvoren podskup od X i $\{v_i\}_{i=1}^{k}$ konačan otvoren pokrivač od X čiji je red $\leq n$ i $v_i \leq u_i$ za svako $i = 1,2,\dots,k$. Zato je $\dim X_{\mathcal{F}} \leq n$.

$\Longleftarrow$ Neka je $\dim X_{\mathcal{F}} \leq n$ i neka je $\{g_i\}_{i=1}^{k}$ α-senčenje od X za svako $\alpha \in [0,1)$. Za svako $i \in \{1,2,\dots,k\}$ neka je $u_i = \{x \in X: g_i(x) > \alpha\}$. Tada je svaki u_i otvoren u X i $\{u_i\}_{i=1}^{k}$ je konačan otvoren pokrivač od X i kako je $\dim X_{\mathcal{F}} \leq n$, to postoji konačan otvoren pokrivač $\{v_i\}_{i=1}^{k}$ od X takav da je $v_i \;\; u_i$ za $i=1,2,\dots,k$ i red od $\{v_i\}_{i=1}^{k}$ je $\leq n$.

Neka je $f_i \in I^X$ za svako $i=1,2,\dots,k$ definisano na sledeći način:

$$f_i(x) > \alpha \quad \text{ako } x \in v_i$$
$$= 0 \quad \text{ako } x \notin v_i.$$

Tada je $\{f_i\}_{i=1}^{k}$ familija odozdo ograničenih poluneprekidnih preslikavanja iz $(X,\mathcal{F})$ u I tj. $\{f_i\}_{i=1}^{k}$ je α-senčenje od X, gde je $f_i \leq g_i$ za $i = 1,2,\dots,k$ i red od $\{f_i\}_{i=1}^{k}$ je $\leq n$. Odatle sledi da je $F\text{-}\dim_{\alpha} X_{\omega(\mathcal{F})} \leq n$ za svako $\alpha \in [0,1)$ i zato je $F\text{-}\dim_{\alpha} X_{\omega(\mathcal{F})} \leq n$. □

Povezujući stavove 2.1 i 2.2 dobija se sledeći rezultat koji dokazuje da se kombinatorna dimenzija topološki generisanog F-prostora poklapa sa kombinatornom dimenzijom topološkog prostora koji generiše taj F-prostor.

Stav 2.3: Ako je (X,T) topološki generisan F-prostor, tada je

$$F\text{-}\dim X_T = \dim X_{\mathcal{F}},$$

za neko $\mathcal{F}$ na X. □

Posledica 2.2: Neka je (X,T) Hausdorffov kompaktan F-prostor. Tada je

$$F\text{-dim } X_T = \dim X_{\mathcal{F}}$$

za neko $\mathcal{F}$ na X.

Dokaz. Lowen je u [13] dokazao da je svaki Hausdorffov kompaktan F-prostor topološki generisan, te je prema stavu 2.3 dokaz kompletan. □

Posledica 2.3: Neka je (X,T) jako α-kompaktan Hausdorffov F-prostor. Tada je

$$F\text{-dim } X_T = \dim X_{\mathcal{F}}$$

za neko $\mathcal{F}$ na X.

Dokaz. Kako iz jake α-kompaktnosti sledi kompaktnost u F-prostorima [12], to je jasno da iz posledice 2.2 sledi posledica 2.3. □

Posledica 2.4: Neka je U uobičajena topologija na R i neka je b uobičajena topologija na Q. Tada je

$$F\text{-dim } R_{\omega(u)} = \dim R_u = 1$$

$$F\text{-dim } Q^{*}_{\omega(b)} = \dim Q_b = 0. \quad □$$

Definicija 2.1: Neka je (X,T) F-prostor. Naziva se ultra Tychonoff F-prostor ako je (X,i(T)) Tychonoff topološki prostor.

Martin [16] je dokazao da ako je (X,T) ultra Tychonoff F-prostor, tada (X,T) ima Stone-Čech-ovu ultra F-kompaktifikaciju. Za svaki F-prostor (X,T) $(\beta X, \mathcal{F})$ označava Stone-Čech-ovu kompaktifikaciju od (X,i(T)) a $(\beta X, T_{\mathcal{F}})$ označava F-prostor koji se sastoji od odozdo ograničenih poluneprekidnih preslikavanja iz $(\beta X, \mathcal{F})$ u I (sa uobičajenom topologijom) čija restrikcija na X pripada T.

Lema 2.3: Neka je (X,T) ultra Tychonoff F-prostor. Ako je $\{f_i\}_{i=1}^{k}$ α-senčenje od X za $\alpha\in[0,1)$, tada postoji α-senčenje $\{g_i\}_{i=1}^{k}$ od βX tako da je $f_i = g_i|X$ za svako $i = 1,2,\dots,k$.

Dokaz. Neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od X. Tada je $(\beta X, \mathcal{F})$ Stone-Čech-ova kompaktifikacija od $(X,i(T))$ i $u_i = f_i^{-1}(\alpha,1]$ je $i(T)$-otvoren u X za svako $i = 1,2,\dots,k$.

Neka je $\beta u_i = \beta X\setminus cl_{\beta X}(X\setminus u_i)$, $i = 1,2,\dots,k$. Definišimo $g_i : \beta X \to I$ na sledeći način:

$$g_i(x) > \alpha \quad \text{ako } x\in\beta u_i \qquad i=1,2,\dots,k$$
$$= 0 \quad \text{u ostalim slučajevima}$$

Jasno je da je $\{\beta u_i\}_{i=1}^{k}$ familija otvorenih skupova u βX i da $g_i\in T$ za svako $i\in\{1,2,\dots,k\}$. Treba da dokažemo da je $\{g_i\}_{i=1}^{k}$ α-senčenje od βX. Ovo sledi iz činjenice da je $\{g_i\}_{i=1}^{k}$ α-senčenje od βX ako je $\bigcup_{i=1}^{k} g_i^{-1}(\alpha,1] = \beta X$. Kako imamo da je $\bigcup_{i=1}^{k} u_i = X$, onda je $\bigcup_{i=1}^{k} h_i = \beta X$ i $g_i|X = f_i$ za $i = 1,2,\dots,k$, čime je dokaz završen. □

Sledeće tvrdjenje daje ekvivalentnost α-kombinatorne dimenzije ultra Tychonoff-og F-prostora i njegove ultra Stone-Čech-ove kompaktifikacije.

Stav 2.4: Neka je (X,T) ultra Tychonoff F-prostor i neka je $(\beta X,T_{\mathcal{F}})$ njegova ultra Stone-Čech-ova F-kompaktifikacija. Tada je

$$F\text{-}dim_{\alpha}X_T = F\text{-}dim_{\alpha}\beta X_{T_{\mathcal{F}}} .$$

Dokaz. $\Rightarrow$ Neka je $F\text{-}dim_{\alpha}\beta X_{T_{\mathcal{F}}}\le n$, $\alpha\in[0,1)$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od X. Po lemi 2.3, postoji α-senčenje $\{g_i\}_{i=1}^{k}$ od βX tako da je $g_i|X = f_i$ za $i = 1,2,\dots,k$.

Kako je $F\text{-}\dim_{\alpha}\beta X_{T_{\mathcal{F}}} \leq n$, to postoji α-profinjenje V od $\{g_i\}_{i=1}^{k}$ čiji je red $\leq n$. Zato je $V|X$ α-profinjenje od $\{f_i\}_{i=1}^{k}$ reda $\leq n$ i $F\text{-}\dim_{\alpha} X_T \leq n$.

$\Longleftarrow$ Neka je $F\text{-}\dim_{\alpha} X_T \leq n$ i neka je $\{f_i\}_{i=1}^{k}$ α-senčenje od βX. Tada je $\{f_i|X\}_{i=1}^{k}$ α-senčenje od X i postoji α-profinjenje V od $\{f_i|X\}_{i=1}^{k}$ čiji je red $\leq n$. Prema lemi 2.3 βV je α-senčenje od βX tako da je $\beta V|X = V$ i βV je α-profinjenje od $\{f_i\}_{i=1}^{k}$ a red od βV je $\leq n$. Zato je $F\text{-}\dim_{\alpha}\beta X_{T_{\mathcal{F}}} \leq n$. □

3. TEOREMA PROIZVODA

U ovom delu rada utvrdjuju se uslovi pod kojima za kombinatornu dimenziju proizvoda F-prostora $(X\times Y, T\times R)$ (proizvod F-prostora (X,T) i (Y,R) važi sledeća nejednakost:

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R \text{ ,}$$

Ovaj rezultat se naziva teorema proizvoda za F-prostore. Koristeći modifikaciju F-prostora posmatranog u prethodnom delu ovog rada, možemo proširite neke dobro poznate rezultate vezane za teoremu proizvoda topoloških prostora na F-prostore.

Lema 3.1: Neka je $(X_s, T_s)_{s\in S}$ familija F-prostora i neka je $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ njihov proizvod F-prostor. Tada je $(\prod_{s\in S} X_s, i(\prod_{s\in S} T_s)) = (\prod_{s\in S} X_s, \prod_{s\in S} i_{\alpha}(T_s))$ za $\alpha\in[0,1)$.

Dokaz. Neka je $(\prod_{s\in S} X_s, \prod_{s\in S} i_{\alpha}(T_s))$ topološki proizvod modifikovanih topoloških prostora, p_s projekcije u $(\prod_{s\in S} X_s, i(\prod_{s\in S} T_s))$ modifikacija proizvoda F-prostora. Neka je u $i_{\alpha}(\prod_{s\in S} T_s)$- otvoren.

Tada je $u = f^{-1}(\alpha,1]$, $f \in \prod_{s\in S} T_s$. Po definiciji proizvoda F-topologija I.3.3., $f = \bigvee_{s\in S} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(g_{s_i})$, $g_{s_i} \in T_{s_i}$ i p_{s_i} je F-neprekidno. Za svako $s \in S$ i svako $f \in T_s$ je

$$(p_s^{-1}(f))^{-1}(\alpha,1] = p_s^{-1}(f^{-1}(\alpha,1]).$$

Tada iz $u = \bigcup_{s\in S} \bigcap_{i=1}^{n} (p_{s_i}^{-1}(g_{s_i}))^{-1}(\alpha,1] = \bigcup_{s\in S} \bigcap_{i=1}^{n} p_{s_i}^{-1}(g_{s_i}^{-1}(\alpha,1])$

Sledi da je u otvoreno u $\prod_{s\in S} i_\alpha(T_s)$.

Neka je sada u otvoren u $\prod_{s\in S} i_\alpha(T_s)$. Tako je

$u = \bigcup_{s\in S} \bigcap_{i=1}^{n} p_{s_i}^{-1}(t_{s_i}^{-1}(\alpha,1])$, $f_{s_i} \in T_{s_i}$. Tada je

$\bigcup_{s\in S} \bigcap_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}^{-1}(\alpha,1]) = \bigcup_{s} \bigcap_{i=1}^{n} (p_{s_i}^{-1}(f_{s_i}))^{-1}(\alpha,1]$ i

$\bigvee_{s} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(t_{s_i})$ $\prod_{s\in S} T_s$.

Neka je $f = \bigvee_{s} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(f_{s_i})$. Tada je $u = f^{-1}(\alpha,1]$ i zato je u otvoren u $i_\alpha(\prod_{s\in S} T_s)$. □

<u>Stav 3.1:</u> [Tychonoff-a teorema proizvoda] Neka je $(X_s,T_s)_{s\in S}$ familija F-prostora. Tada je $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ α-komaktan ako i samo ako je (X_s,T_s) α-kompaktan za svako $s \in S$.

<u>Dokaz.</u> Ovaj stav je dokazalo mnogo autora, a ovde će biti dat kratak dokaz. Prema lemi 2.1 $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ je α-kompaktan ako i samo ako je $(\prod_{s\in S} X_s, i_\alpha(T_s))$ kompaktan. Prema lemi 3.1 je

$(\prod_{s\in S} X_s, i\,(\prod_{s\in S} T_s)) = (\prod_{s\in S} X_s, \prod_{s\in S} i_\alpha(T_s))$. Poznato je da je

$(\prod_{s\in S} X_s, \prod_{s\in S} i_\alpha(T_s))$ kompaktan ako i samo ako je, za svako $s \in S$,

$(X_s, i_\alpha(T_s))$ kompaktan, odakle sledi da je za svako $s \in S$, (X_s, T_s) α-kompaktan F-prostor. □

Stav 3.2: Neka je (X,T), (Y,R) par α-kompaktnih F-prostora. Tada je

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim\, X_T + F\text{-}dim_\alpha Y_R .$$

Dokaz. Prema stavu 2.1 je

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} = dim(X\times Y)_{i_\alpha(T\times R)} .$$

Prema lemi 2.1 su $(X, i_\alpha(T))$ i $(Y, i_\alpha(R))$ kompaktni prostori, a prema stavu 3.1 proizvod topoloških prostora $(X\times Y,\ i_\alpha(T) \times i_\alpha(R))$ je takodje kompaktan. Prema lemi 3.1 je

$$(X\times Y, i_\alpha(T\times R)) = (X, i_\alpha(T)) \times (Y, i_\alpha(R)).$$

Odatle je $dim(X\times Y)_{i_\alpha(T\times R)} = dim(X_{i_\alpha(T)} \times Y_{i_\alpha(R)})$.

Dokazano je za slučaj kompaktnih topoloških prostora da je

$$dim(X_{i_\alpha(T)} \times Y_{i_\alpha(R)}) \leq dim\, X_{i_\alpha(T)} + dim\, Y_{i_\alpha(R)}.$$

Zato je $F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim_\alpha X_T + F\text{-}dim_\alpha Y_R$. □

Posledica 3.1: Ako su (X,T) i (Y,R) jako -kompaktni F-prostori, tada je

$$F\text{-}dim(X\times Y)_{(T\times R)} \leq F\text{-}dim\, X_T + F\text{-}dim\, Y_R . \quad \square$$

Stav 3.3: Neka su (X,T) i (Y,R) prebrojivi α-kompaktni C_{II} F-prostori. Tada je

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim_\alpha X_T + F\text{-}dim_\alpha Y_R .$$

Dokaz. Kako je prebrojiv α-kompaktan C_{II} F-prostor i α-kompaktan (stav 1.2) to se dokaz izvodi primenom stava 3.2. □

Posledica 3.2: Neka su (X,T) i (Y,R) prebrojivi, α-kompaktni, α-Lindelof-ovi F-prostori. Tada je

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim_\alpha X_T + F\text{-}dim_\alpha Y_R . \quad \square$$

Stav 3.4: Neka je (X,T) α-kompaktan F-prostor a (Y,R) prebrojiv α-kompaktan F-prostor. Tada je $(X\times Y,\ T\times R)$ prebrojiv -kompaktan F-prostor.

Dokaz. Kako je $i_\alpha(T)$ kompaktno a $i(R)$ prebrojivo kompaktano, kao topologije na X i Y respektivno, po lemi 2.1 je tada $(X,i_\alpha(T)) \times (Y,i_\alpha(R))$ prebrojivo kompaktan topološki prostor i prema lemi 3.1 je $(X\times Y, i_\alpha(T\times R))$ prebrojivo kompaktan. Odatle, po lemi 2.1, je $(X\times Y,\ T\times R)$ prebrojivo α-kompaktan F-prostor.

Stav 3.5: Neka je (X,T) ultra Tychonoff F-prostor a (Y,R) prebrojiv α-kompakt. Ako je svako konačno α-senčenje od $X\times Y$ prebrojivo ekstezibilno na $\beta X\times Y$, tada je

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim_\alpha X_T + F\text{-}dim_\alpha Y_R .$$

Dokaz. Neka je svako α-senčenje od $X\times Y$ prebrojivo ekstenzibilno na $\beta X\times Y$. Kako je ultra kompaktan F-prostor i α-kompaktan tj. βX je α-kompaktan, to je prema stavu 3.4 $(\beta X\times Y,\ T_{\mathcal{F}}\times R)$ prebrojivo α-kompaktan. Po pretpostavci stava II 2.7 je

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim_\alpha(\beta X\times Y)_{(T_{\mathcal{F}}\times R)} . \qquad (a)$$

Prema stavu 2.1 je

$$F\text{-}dim_\alpha(\beta X\times Y)_{(T_{\mathcal{F}}\times R)} = dim(\beta X\times Y)_{i_\alpha(T_{\mathcal{F}}\times R)} . \qquad (b)$$

Prema teoremi 0.15 Morite datoj u uvodu ovog rada je

$$dim(\beta X\times Y)_{i_\alpha(T_{\mathcal{F}}\times R)} \leq dim\ \beta X_{i_\alpha(T_{\mathcal{F}})} + dim\ Y_{i_\alpha(R)} . \qquad (c)$$

Prema stavu 2.4 je

$$F\text{-}dim_\alpha X_T = F\text{-}dim_\alpha \beta X_{T_{\mathcal{F}}} . \qquad (d)$$

U relaciji (a), (b), (c) i (d) imamo

$$F\text{-}dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}dim_\alpha X_T + F\text{-}dim_\alpha Y_R ,$$

što je trebalo dokazati. □

Posledica 3.3: Neka je (X,T) ultra Tychonoff F-prostor, a (Y,R)

jako prebrojivo α-kompaktan. Ako je svako konačno α-senčenje od XxY prebrojivo ekstenzibilno na βXxY, za svako $\alpha\in[0,1)$, tada je

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R. \quad \square$$

Stav 3.6: Neka je (X,T) jako α-Lindeloff-ov i jako prebrojivo α-kompaktan F-prostor. Ako je (Y,R) jako α-kompaktan F-prostor, tada je

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R.$$

Dokaz. Kako je α-Lindelof-ov prebrojivo α-kompaktan F-prostor i α-kompaktan, to dokaz sledi iz posledice 3.1. $\square$

Zvezdasto konačne osobine uvedene u topološkim prostorima može se koristiti u sličnom obliku i za F-prostore. Neka je (X,T) F-prostor. Kolekcija U F-prostora je zvezdasto-konačna u X, ako svako $u\in U$ seče najviše konačno mnogo drugih članova iz U.

Definicija 3.1: F-prostor (X,T) ima α-zvezdasto konačnu osobinu (α-st.f.p) ako svako α-senčenje od X ima zvezdasto-konačno α-profinjenje.

Stav 3.7: Neka je (X,T) F-prostor. Ako (X,T) ima α-zvezdasto konačnu osobinu (α-st.f.p.), tada (X,i (T)) ima zvezdasto konačnu osobinu.

Dokaz. Neka je $\mathcal{G}$ $i_\alpha(T)$-otvoren pokrivač od X. Za svako $G\in\mathcal{G}$ neka je $G = h_G^{-1}(\alpha,1]$, $h\in I^X$. Tada $f_G\in T$ i kako je $X = \bigcup_{G\in\mathcal{G}} h_G^{-1}(\alpha,1] = \bigcup_{G\in\mathcal{G}} G$, to je $\mathcal{H} = \{h_G : G\in\mathcal{G})$ α-senčenje od X. Kako (X,T) ima α-st.f.p. to postoji zvezdasto-konačno α-proširenje V od $\mathcal{H}$. Neka je $w = v^{-1}(\alpha,1]$, $v\in V$. Tada je w $i_\alpha(T)$-otvoren i familija

$\mathcal{W} = \{W : w = v^{-1}(\alpha, 1]$ za svako $v \in V\}$ je zvezdasto konačno profinjenje od $\mathcal{Y}$. □

Posledica 3.4: Ako F-prostor (X,T) ima jako α-st.f.p., onda (X, i(T)) ima st.f.p.

Dokaz sledi iz stava 3.7 za svako $\alpha \in [0,1)$. □

Stav 3.8: Neka su (X,T) i (Y,R) jako α-Hausdorff-ovi F-prostori takvi da (XxY, TxR) ima jako α-st.f.p. Tada je

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R.$$

Dokaz. Kako (XxY, TxR) ima jako α-st.f.p., tada prema posledici 3.4 (XxY, i(TxR)) ima st.f.p. . (X,i(T)) i (Y, i(R)) su Hausdorffovi topološki prostori, prema lemi 2.2. Tada koristeći teoremu 0.14 Morite za topološki slučaj, imamo

$$\dim(X\times Y)_{i(T\times R)} \leq \dim X_{i(T)} + \dim Y_{i(R)}$$

i odatle

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R.$$

Stav 3.9: Neka su (X,T) i (Y,R) α-Hausdorff-ovi F-prostori takvi da (XxY, TxR) ima α-st.f.p.. Tada je

$$F\text{-}\dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}\dim_\alpha X_T + F\text{-}\dim_\alpha Y_R.$$

Dokaz se izvodi na isti način kao za stav 3.8. □

Definicija 3.2: F-prostor (X,T) je ultra parakompaktan F-prostor ako je (X,i(T)) parakompaktan topološki prostor.

Stav 3.10: Neka je (X,T) ultra parakompaktan jako α-Hausdorff-ov F-prostor sa $F\text{-}\dim X_T = n$ i neka je (Y,R) jako α-kompaktan, jako α-Hausdorff-ov F-prostor sa $F\text{-}\dim Y_R = m$. Tada je

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq m + n.$$

Dokaz. Po pretpostavci, lemi 2.1 i stavu 2.1, $(X,i(T))$ je parakompaktan Hausdorff-ov prostor sa $\dim X_{i(T)} = n$ a $(Y,i(R))$ je kompaktan Hausdorff-ov topološki prostor (što sledi iz leme 2.1 i leme 2.2) sa $\dim Y_{i(R)} = m$, (po stavu 2.1).

Prema stavu 0.12 iz uvoda, imamo

$$\dim(X\times Y)_{i(T\times R)} = \dim(X\times Y)_{i(T)\times i(R)} \leq m + n .$$

Odatle po stavu 2.1 imamo

$$F\text{-}\dim(X\times Y)_{i(T\times R)} \leq m + n. \quad \square$$

Stav 3.11: Neka je $(X_s,T_s)_{s\in S}$ familija ultra Tychonoff-ih F-prostora takvih da je svaki prebrojiv proizvod F-prostora α-Lindelof-ov. Ako je $F\text{-}\dim_\alpha X_s = 0$ sa $s\in S$, tada je $F\text{-}\dim_\alpha \prod_{s\in S} X_s = 0$.

Dokaz. Prema stavu 2.1 je $F\text{-}\dim_\alpha X_s = 0$ ako i samo ako je $\dim X_{s\, i_\alpha(T_s)} = 0$. Kako je $(\prod_{i=1}^{\infty} X_s, \prod_{i=1}^{\infty} T_{s_i})$ α-Lindelof-ov, tada je po lemi 2.1 $(\prod_{i=1}^{\infty} X_{s_i}, \prod_{i=1}^{\infty} i(T_{s_i}))$ Lindelof-ov.

Dalje se primenjuje teorema T.0.16 iz uvoda čime je dokaz završen. $\square$

Glava IV:

PROJEKTIVNI LIMES F-PROSTORA

Pojam projektivnog sistema topoloških prostora je bio široko razmatran kako sam za sebe tako i u okviru teorije dimenzija značajno doprinoseći razvitku ove poslednje. Stoga mi se čini da je vredno pažnje uvesti ovaj pojam za F-prostore i razmatrati njegovu vezu sa kombinatornom dimenzijom F-prostora.

Posvetićemo prva dva odeljka ove glave definisanju i razmatranju pojma projektivnog sistema F-prostora i njegovog limes F-prostora i to u svetlu sličnog razmatranja u opštoj topologiji. Pokazuje se da je većinu rezultata - naročito onih koji važe za kompaktne Hausdorff-ove prostore - moguće generalizovati za F-prostore. U trećem odeljku ove glave vratićemo se α-kombinatornoj dimenziji gde ćemo dati neke karakterizacije od F-dim za limes F-prostor.

Pod α-bikompaktnim F-prostorom smatraćemo -Hausdorff-ov i α-kompaktan F-prostor.

1. PROJEKTIVNI LIMES -BIKOMPAKTNIH F-PROSTORA:

Neka je $\{(X_s,T_s) : s\in S\}$ familija F-prostora indeksirana elementima usmerenog skupa S. Za svako $s,t\in S$, gde $s\leq t$, neka je dato preslikavanje $f_{s,t} : X_t \rightarrow X_s$. Za $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$

reći ćemo da je projektivni sistem F-prostora ako je: $f_{s,t}$ je F-neprekidno za svako $s,t \in S$ $(s \leq t)$ pri čemu je $F_{s,S}$ je indentično preslikavanje za svako $s \in S$, i ako je $s \leq t \leq r$ tada je $f_{s,t} \circ f_{t,r} = f_{s,r}$.

Neka je $\widetilde{X}$ podskup Dekartovog proizvoda $\prod_{s \in S} X_s$ koji se sastoji od onih tačaka $x \in \prod_{s\in S} X_s$ za koje je $f_{s,t}(p_t(x) = p_s(x)$, za svako $s,t \in S$, $s \leq t$, gde su sa p_s, p_t označene odgovarajuće projekcije. Za svako $s \in S$, $f_s : \widetilde{X} \to X_s$ se naziva kanonskim preslikavanjem $(f_s = p_s | X)$, a F-neprekidno preslikavanje $f_{s,t}$ se naziva F-vezujuće preslikavanje od $\underset{\sim}{X}_F$.

<u>Definicija 1.1:</u> Najgrublja F-topologija $\widetilde{T}$ na $\widetilde{X}$ za koju su kanonska preslikavanja f_s $(s \notin S)$ F-neprekidna se zove projektivni limes projektivnog sistema $\underset{\sim}{X}_F$, a $(\widetilde{X},\widetilde{T})$ se naziva limes F-prostor projektivnog sistema F-prostora.

<u>Propozicija 1.1:</u> Limes F-prostor $(\widetilde{X},\widetilde{T})$ projektivnog sistema $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ je F-podprostor F-prostora $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$.

<u>Dokaz.</u> Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostora od $\underset{\sim}{X}_F$ i neka je $f_s : X \to X_s$ kanonsko preslikavanje, $s \in S$. Prema definiciji I.2.11 dovoljno je naći za svaki otvoren F-skup w u X otvoren F-skup g u $\prod_{s\in S} X_s$ tako da je $g|X = w$. Neka je w otvoren F-skup u $\widetilde{X}$. Tada je $w = \bigvee_{s} \bigwedge_{i=1}^{n} f_{s_i}(u_i)$, $u_{s_i} \in T_{s_i}$ $s \in S^* \subseteq S$. Ali $f_{s_i} = p_{s_i} | \widetilde{X}$ za svako s_i te je $p_{s_i}^{-1}(u_{s_i})$ otvoren F-skup u $\prod_{s\in S} X_s$ i ako uzmemo da je $g = \bigvee_{s} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(u_{s_i})$, tada je g otvoren F-skup u $\prod_{s\in S} X_s$ i $g|\widetilde{X} = w$. □

__Propozicija 1.2:__ Neka je $\underset{\sim}{X}_F$ projektivni sistem F-prostora (nad S) i neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor od X_F sa kanonskim preslikavanjima f_s $(s \in S)$. Tada skup svih F-skupova oblika $f_s^{-1}(u_s)$, $u_s \in T_s$, $s \in S$ čini bazu za $\widetilde{T}$.

__Dokaz.__ Neka je kolekcija svih F-skupova oblika $\{f_s^{-1}(u_s) : u_s \in T_s\}_{s\in S}$ i neka je p_s projekcija za proizvod F-prostora $(\prod_{s \in S} X_s, \prod_{s \in S} T_s)$ na (X_s,T_s) za $s \in S$ gde je $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ projektivni sistem F-prostora. Neka je w otvoren F-skup u limes F-prostor $(\widetilde{X},\widetilde{T})$. Tada po propoziciji 1.1 postoji otvoren F-skup g u $\prod_{s\in S} X_s$ tako da je $w = g|\widetilde{X}$. Po definiciji I. 3.2 proizvod F-topologije, $g = \bigvee_{s_i \in S^*} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(u_{s_i})$, $s^* \in S$, $u_{s_i} \in T_{s_i}$. Znači

$$w = \Big(\bigvee_{s_i} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(u_{s_i})\Big)|\widetilde{X} = \bigvee_{s_i} \bigwedge_{i=1}^{n} (p_{s_i}|\widetilde{X})^{-1} = \bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}),$$

tj. $w = \bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$.

Pošto je S usmeren skup, tada postoji $t \in S$ tako da je $s_i \leq t$ za svako $i = 1,2,\dots,n$.

Neka je $h_t = \bigwedge_{i=1}^{n} f_{s_i,t}^{-1}(u_{s_i})$, tada je $h_t \in T_t$ i

$$f_t^{-1}(h_t) = f_t^{-1}\Big(\bigwedge_{i=1}^{n} f_{s_i,t}^{-1}(u_{s_i})\Big) = \bigwedge_{i=1}^{n} (f_{s_i,t}^{-1} f_t^{-1})(u_{s_i}) = \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}).$$

Tada je

$$\bigvee_{t \geq s_i} f_t^{-1}(h_t) = \bigvee_{s_i \in S^*} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}) = w$$

i $f_t^{-1}(h_t) \in \beta$. Odavde je zaista baza za $\widetilde{T}$. □

Ova baza se zove standardna baza za $\widetilde{T}$ i označava se sa $\beta(X)$.

<u>Propozicija 1.3:</u> Ako je (X_s, T_s), $s \in S$ kolekcija FT_1 FT_2, α-Hausdorff ili F-regularnih F-prostora, tada je limes F-prostor $(\widetilde{X},\widetilde{T})$ projektivnog sistema $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S)\}$ FT_1, FT_2 α - Hausdorff, F- regularan F-prostor.

<u>Dokaz.</u> Iz propozicije I.3.8 i iz činjenice da su FT_1, FT_2, α-Hausdorff i F-regularan nasledna svojstva sledi dokaz naše propozicije.

Sledeća propozicija daće nam vezu izmedju indukovanog F-prostora projektivnog limesa topoloških prostora i limes F-prostora indukovanih F-prostora. Ali prvo dajmo Weiss-ovu propoziciju 3.4 [29] ako lemu bez dokaza.

<u>Lema 1.1:</u> Neka su data dva topološka prostora (X,R) $(Y,\mathcal{J})$ i preslikavanje $f : (X,R) \to (Y,\mathcal{J})$. Tada je f neprekidno ako i samo ako $f : (X,w(R)) \to (Y,w(\mathcal{J}))$ je F-neprekidno gde su $(X,w(R))$, $(Y,w(\mathcal{J}))$ indukovani F-prostori na (X,R) i $(Y,\mathcal{J})$ kao i glavi III.

<u>Lema 1.2:</u> Neka je $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ projektivni sistem topoloških prostora nad S. Tada je $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ projektivni sistem indukovanih F-prostora.

<u>Dokaz.</u> Za $s,t \in S$ sa $s \leq t$, $f_{s,t} : (X_s, R_s) \to (X_t, R_t)$ je neprekidno ako i samo ako je $f_{s,t} : (X_s, w(R_s)) \to (X_t, w(R_t))$ je F-neprekidno po lemi 1.1, što dokazuje našu lemu.

<u>Propozicija 1.4:</u> Neka je $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ projektivni sistem topoloških prostora. Ako je $(\widetilde{X},\widetilde{R})$ projektivni limes prostora X,

tada je $(\tilde{X}, w(\tilde{R})) = (\tilde{X}, (\widetilde{w(R)}))$.

<u>Dokaz.</u> $\Longrightarrow$ Neka je g_n $\widetilde{w(R)}$-otvoren F-skup. Tada po propoziciji 1.2 $g = \bigvee_s \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$ gde je u_{s_i} $w(R_{s_i})$ - otvoren F-skup i

$f_{s_i} : (\tilde{X}, \widetilde{w(R)}) \longrightarrow (X_{s_i}, w(R_{s_i}))$.

Ali $u_{s_i}^{-1}(\alpha, 1]$ je R_{s_i}-otvoren skup za svako $\alpha \in [0,1)$ i za svako $i = 1,2,\ldots,n$, $s \in S$. Kako je $(f_{s_i}^{-1}(u_{s_i}))^{-1}(\alpha, 1] = f_{s_i}^{-1}(u_{s_i}^{-1}(\alpha, 1])$,

to je $\bigcup_s \bigcap_{i=1}^{n} (f_{s_i}^{-1}(u_{s_i}))^{-1}(\alpha, 1]$ $\tilde{R}$-otvoren skup. Dakle,

$\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}) \in w(\tilde{R})$ tj. g je $w(\tilde{R})$-otvoren F-skup.

$\Longleftarrow$ Neka je g $w(\tilde{R})$-otvoren F-skup. Tada je $g^{-1}(\alpha, 1]$ $\tilde{R}$-otvoren skup za svako $\alpha \in [0,1)$. Znači

$g^{-1}(\alpha, 1] = \bigcup_{s_i} \bigcap_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$ gde je u_{s_i} R_{s_i}-otvoren i

$f_{s_i} : (\tilde{X}, \tilde{R}) \longrightarrow (X_{s_i}, R_{s_i})$ je kanonsko preslikavanje.

Sada za svako $i = 1,2,\ldots,n$, $\mu(u_{s_i})$ je $w(R_{s_i})$-otvoren F-skup i po lemi 1.1 $f_{s_i}^{-1}(\ (u_{s_i}))$ je otvoren F-skup u $\tilde{X}$ i otuda

$\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(\mu(u_{s_i}))$ je $\widetilde{w(R)}$-otvoren F-skup. Ali $\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}$

$(\mu(u_{s_i})) = \mu(\bigcup_{s_i} \bigcap_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}))$ tj. $\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(\mu(u_{s_i})) =$

$= (g^{-1}(\alpha, 1]) \geqslant g$.

Odavde je g $\widetilde{w(R)}$-otvoren F-skup i sledstveno $(\tilde{X}, w(\tilde{R})) = (\tilde{X}, \widetilde{w(R)})$. $\square$

Lema 1.3: [10] 3.1: Neka su (X,T), (Y,S) topološki generisana F-prostora. Tada je $f : (X,T) \to (Y,S)$ F-neprekidno ako i samo ako je $f : (X,i(T)) \to (Y,i(S))$ neprekidno gde su $(X,i(T))$ i $(Y,i(S))$ modifikovane topologije. □

Primedba. Koristeći gornju lemu mi primećujemo da ako je $X_F = \{X_s, f_{s,t}, S\}$ projektivni sistem F-prostora, tada je $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ projektivni sistem modifikovanih topologija.

Sledeća propozicija daje vezu izmedju modifikacije limes F-prostora i projektivnog limes prostora modifikovanih topologija.

Propozicija 1.5: Neka je $X_F = \{X_s, f_{s,t}, S\}$ projektivni sistem topološki generisanih F-prostora. Ako je $(\tilde{X},\tilde{T})$ limes F-prostor od X_F tada:

$$(\tilde{X}, i(\tilde{T})) = (\tilde{X}, \widetilde{i(T)}), \qquad i(T) = \prod_{s\in S} i(T_s).$$

Dokaz. $\Longrightarrow$ Neka je G $i(\tilde{T})$-otvoren skup. Tada je $G = g^{-1}(\alpha,1]$ za $g\in\tilde{T}$ i $\alpha\in[0,1)$. Po propoziciji 1.2, $g = \bigvee_{s\in S^*} f_s^{-1}(u_s)$ gde je $u_s^{-1}(\alpha,1]$ i (T_s)-otvoreno, dakle G je $\widetilde{i(T)}$-otvorensskup.

$\Longleftarrow$ Neka je G $\widetilde{i(T)}$-otvoren skup gde je $\widetilde{i(T)}$ projektivni limes prostor modifikovanih topologija $i(T_s)$, $s\in S$. Tada je $G = \bigcup_{s\, S^*} f_s^{-1}(u_s)$, u_s je $i(T_s)$-otvoren i $f_s : (X,i(T)) \to (X_s, i(T_s))$ su kanonska preslikavanja; znači $u_s = g_s^{-1}(\alpha,1]$ za $g_s\in T_s$ i $\alpha\in[0,1)$ i otuda

$$G = \bigcup_s f_s^{-1}(g_s^{-1}(\alpha,1]) = \bigcup_s (f_s^{-1}(g_s))^{-1}(\alpha,1] = \bigvee (f_s^{-1}(g_s))^{-1}(\alpha,1]$$

po lemi 1.3. tako da je $\bigvee(f_s(g_s))\in\tilde{T}$ i sledstveno

$\bigvee(f_s(g_s))(\alpha,1]\in i(T)$ tj. G je $i(\widetilde{T})$-otvoren skup.

<u>Propozicija 1.6:</u> Neka je X_F projektivni sistem α-Hausdorff F-prostora (nad S). Tada je X -zatvoren u $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ za $\alpha\in[0,1)$.

<u>Dokaz.</u> Neka je $X_F = \{X_s, f_{s,t}, S\}$ projektivni sistem α-Hausdorff F-prostora, $x\in\prod_{s\,S} X_s$ i $x\notin\widetilde{X}$. Tada postoji $s,t\in S$, $s\leq t$, tako da je $f_{s,t}\, p_t(X)\neq p_s(X)$.

Iz α-Hausdorff-nosti postoje otvoreni F-skupovi u_s, v_s u X_s tako da je $u_s(p_s(x))>\alpha$, $v_s(f_{s,t}p_t(x))>\alpha$ i $u_s\wedge v_s = 0$. Otuda $(p_s^{-1}(u_s))(x)>\alpha$, $((f_{s,t}\, p_t)^{-1}(v_s))(x)>\alpha$ u $(\prod X_s, \prod T_s)$. Neka je $w = p_s^{-1}(u_s)\wedge(f_{s,t}\, p_t)^{-1}(v_s)$. Tada je w otvoren F-skup u $\prod_{s\in S} X_s$, $w(x)>\alpha$ za $x = \prod_{s\,S} X_s\setminus X$ i za svako $y\in\widetilde{X}$, $w(y) = 0$. Znači $w\wedge\mu(\widetilde{x}) = 0$, a otuda je $\widetilde{X}$ zaista α-zatvoren. $\square$

<u>Propozicija 1.7:</u> Neka je $X_F = \{X_s, f_{s,t}, S\}$ projektivni sistem Hausdorff-ovih F-prostora. Tada je

$$\mu(\prod_{s\in S} X_s\ X) \text{ otvoren F-skup u } \prod_{s\in S} X_s\ .$$

<u>Dokaz.</u> Neka je $x = \prod_{s\in S} X_s\ X$. Tada postoji $s,t\in S$, $s\leq t$ tako da je $f_{s,t}\, p_t(x)\neq p_s(x)$, te po definiciji I.2.3 Hausdorff-ovog F-prostora postoje u_s, v_s otvoreni F-skupovi u X_s tako da je $u_s(p_s(x)) = 1 = v_s(f_{s,t}\, p_t(x))$ i $u_s\wedge v_s = 0$. Dakle, $(p_s^{-1}(u_s))(x) = 1 = ((f_{s,t}\, p_t)^{-1}(v_s))(x)$. Ako stavimo $w_x = p_s^{-1}(u_s)\wedge(f_{s,t}\, p_t)(v_s)$, tada je w_x otvoren F-skup u $\prod_{s\in S} X_s$

i $w_x(x) = 1$ za $x \in \prod_{s \in S} X_s \setminus \tilde{X}$. Znači za svako $x \in \prod_{s \in S} X \setminus \tilde{X}$ mi možemo naći otvoren F-skup w_x tako da je $w_x(x) = 1$, i sledstveno $\mu(\prod_{s \in S} X_s \setminus \tilde{X}) = \vee \{ w_x : x \in (\prod_{s \in S} X_s) \setminus \tilde{X} \}$ je otvoren F-skup u $\prod_{s \in S} X_s$.

Lema 1.4: Neka je Y pogodan zatvoren skup u F-prostoru (X,T). Tada:

(I) E je zatvoren F-skup u Y sledi E je zatvoren F-skup u X.

(II) G je zatvoren F-skup u X sledi $G|Y$ je zatvoren F-skup u Y.

Dokaz. Neka je E zatvoren F-skup u Y kao potprostoru. Tada postoji zatvoren F-skup G u X tako da je $G|Y = E$; znači $E = G \wedge \mu(Y)$ i $\bar{I}_x - E = \bar{I}_x - (G \wedge \mu(Y)) = (\bar{I}_x - G) \vee (\bar{I}_x - \mu(Y))$ otvoren F-skup u X po Demorganovom zakonu I.1.2; znači E je zatvoren F-skup u X.

(II) je jasno. □

Pošto propozicija 1.7 pokazuje da za projektivni sistem $\underset{\sim}{X}_F$ Hausdorff-ovih F-prostora, $\tilde{X}$ je 1^{*}-zatvoren, to iz propozicije I.2.6 sledi da je $\tilde{X}$ pogodan zatvoren u $\prod_{s \in S} X_s$. Ovo nas dovodi do sledeće propozicije.

Propozicija 1.8: Neka je $X_F = \{X_s, f_{s,t}, S\}$ projektivni sistem Hausdorff-ovih F-prostora. Tada je limes F-prostor $(\tilde{X}, \tilde{T})$ zatvoren F-potprostor od $(\prod_{s \in S} X_s, \prod_{s \in S} T_s)$.

Dokaz. je direktna posledica propozicije 1.7 i leme 1.4. □

Korolar 1.1: Neka je X_F projektivni sistem jako α-Hausdorff-ovih F-prostora. Tada (X,T) je zatvoren F-podprostor od $(\prod_{s \in S} X_s, \prod_{s \in S} T_s)$. □

Sledeći rezultat je generalizacija jednog od najvažnijih rezultata za inverzne sisteme topoloških prostora.

Propozicija 1.9: Limes F-prostor projektivnog sistema α-bikompaktnih F-prostora je neprazan α-bikompaktan F-prostor.

Dokaz. Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog sistema $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ α-bikompaktnih F-prostora za $\alpha \in [0,1)$. Po produk teoremi Tychonoff-a za F-prostore - propozicija III.3.2 - i propoziciji I.3.8 $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ je α-bikompaktni F-prostor. Iz propozicije 1.6 $\widetilde{X}$ je α-zatvoren u $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ te skupa sa prethodnim dobijamo da je $(\widetilde{X},\widetilde{T})$ α-bikompaktan po propoziciji I.2.7.

Moramo dokazati još da je $\widetilde{X} \neq \emptyset$. Za $t \in S$, neka je $Y_t = \{x \in \prod_{s\ S} X_s : p_s(x) = f_{s,t}\ p_t(x)$ za svako $s \in S$ takvo da je $s \leq t\}$. Tada Y_t je α-zatvoren u $\prod_{s\ S} X_s$ po propoziciji I.3.4 i $Y_t \neq \emptyset$. Naime, za $x_t \in X_t$ i za $s \in S$ neka je $Z_s \subseteq X_s$ i $Z_s = \{f_{s,t}(x_t)\}$ ako je $s \leq t$ i $Z_s = X_s$ inače. Neka je $Z = \prod_{s\in S} Z_s$. Tada je $Z \neq \neq$, Z je podskup od $\prod_{s\in S} X_s$ i $Z \subset Y_t$.

Ako su $t,s \in S$ takvi da je $s \leq t$, tada je $Y_t \subset Y_s$. Sledi da je $\{Y_t\}_{t\in S}$ opadajuća familija α-kompaktnih skupova te na osnovu propozicije I.2.8, mi imamo $\prod_{G\in S} Y_t \neq 0$.

Neka je $x \in \bigcap_{t\in} Y_t$. Ako su $t,r \in S$, $t \leq r$, tada kako je $x \in Y_r$, $p_t(x) = f_{t,r}\ p_r(x)$ imamo da je $x \in \widetilde{X}$ što se i tražilo.

Korolar 1.2: Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog sistema

$\underset{\sim}{X}_F$ jako α-bikompaktnih prostora. Tada je $(\widetilde{X},\widetilde{T})$ jako α-bikompaktan.

Korolar 1.3: Limes F-prostor projektivnog sistema $C_{\parallel}$, prebrojivo α-kompaktnih i α-Hausdorff-ovih F-prostora je neprazan.

Dokaz. Kako je $C_{\parallel}$ prebrojivo α-kompaktni prostor ujedno i α-kompaktan (po propoziciji I.2.3), tada dokaz sledi iz propozicije 1.9. □

Korolar 1.4: Limes F-prostor projektivnog sistema α-Lindelöf-ovih, prebrojivo α-kompaktnih i α-Hausdorff-ovih F-prostora je neprazan. □

Definicija 1.2: Mi ćemo reći da je $\underset{\sim}{Y}_F = \{Y_s, f_{s,t}, S\}$ F-podsistem od $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ nad istim usmerenim skupom S, ako je Y_s F-podprostor od X_s i $g_{s,t} = f_{s,t}|Y_t$ $(s,t \in S,\ s \leq t.)$ Primetimo da je $f_{s,t}(Y_t) \subset Y_s$ $(s \leq t)$.

Propozicija 1.10. Neka je $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ projektivni sistem F-prostora i neka je $(\widetilde{X},\widetilde{T})$ njegov limes F-prostor. Dalje, neka je $\underset{\sim}{Y}_F = \{Y_s, g_{s,t}, S\}$ F-podsistem od $\underset{\sim}{X}_F$. Tada je limes F-prostor $(\widetilde{Y},\widetilde{T}_Y)$ F-podprostor od $(\widetilde{X},\widetilde{T})$.

Dokaz. Pošto je za svako $s \in S$, $Y_s \subset X_s$, mi imamo da je $\widetilde{Y} = (\prod_{s\ S} Y_s) \cap \widetilde{X}$ i da je $T_{\widetilde{Y}}$ F-topologija indukovana topologijom proizvod F-prostora $(\prod_{s\in S} Y_s, \prod_{s\in S} T_{s\ Y_s})$ na $\widetilde{Y}$.

Ali $(\prod_{s\in S} Y_s, \prod_{s\in S} T_{s\ Y_s})$ je indukovan proizvod F-prostorom $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ na $\prod_{s\in S} Y_s$. Dakle $(\widetilde{Y},\widetilde{T}_Y)$ je F-podprostor od $(\widetilde{X},\widetilde{T})$.

Koloral 1.5: Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog sistema

$\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ F-prostora sa kanonskim preslikavanjima $f_s : X \longrightarrow X_s$. Neka je $Y = \{f_s(\tilde{X}), g_{s,t}, S\}$ F-podsistem od X_F. Tada je $(\tilde{X},\tilde{T})$ limes F-prostor od $\underset{\sim}{Y}_F$.

Primetimo da je svako F-vezujuće preslikavanje $g_{s,t}$ surjektivno, što takodje važi i za kanonska preslikavanja $\tilde{X} \longrightarrow f_s(X)$.

2. PROJEKTIVNI NIZOVI F-PROSTORA

Projektivni niz se posmatra kao specijalan slučaj projektivnog sistema. Primetimo da ako je $\underset{\sim}{X} = \{X_n, f_{n,m}, N\}$ projektivni niz nad N (N-prirodni brojevi sa uobičajenim poretkom) gde su svi X_n neprazni i $f_{n,m}$ surjektivno za svako $n,m \in N$, $n \leq m$, tada je projektivni limes $\tilde{X}$ od $\underset{\sim}{X}$ neprazan a da pri tome ne mora zadovoljavati nikakvo F-topološko svojstvo.

__Propozicija 2.1:__ Neka je $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ projektivni niz prebrojivo α-kompaktnih F-prostora. Ako su svi (skupovi $f_{n,m}(X_m)$ $(n \leq m)$ α-zatvoreni u X_n, tada je limes F-prostor $(\tilde{X},\tilde{T})$ od $\underset{\sim}{X}_F$ neprazan.

__Dokaz.__ Za svako $n \in N$, $\{f_{n,m}(X_m)\}_{m \geq n}$ je kolekcija α-zatvorenih skupova u X_n po pretpostavci i

$$f_{n,n+1}(X_{n+1}) \supset f_{n,n+2}(X_{n+2}) \ldots f_{n,m}(X_m) \qquad m \geq n.$$

Dakle, $\{f_{n,m}(X_m)\}_{m \geq n}$ je opadajuća kolekcija α-zatvorenih skupova u prebrojivom α-kompaktnom X_m, te je po propoziciji I.2.8 $\bigcap_{n \leq m} f_{n,m}(X_m) \neq 0$ i prebrojivo je α-kompaktan.

Stavimo $Y_n = \bigcap_{n\leq m} f_{n,m}(X_m)$. Ako je $n \leq m$, tada je $f_{n,m}(Y_m) \subset Y_n$. U stvari, $f_{n,m}(Y_m) = f_{n,m}(\bigcap_{m\leq \ell} f_{m,\ell}(X_\ell)\}) \subset \bigcap_{m\leq \ell} \{f_{n,m}(f_{m,\ell}(X_\ell))\} = \bigcap_{n\leq \ell} f_{n,\ell}(X_\ell) = Y_n$.

Sada neka je $y_n \in Y_n$, znači $y_n \in \bigcap_{n\leq \ell} f_{n,\ell}(X_\ell)$ tako da je dovoljno pokazati da je $y_m \in f_{m,\ell}(X_\ell)$ za svako $m \geq n$. Neka je m dato i izaberimo $\ell \in N$ tako da je $\ell \geq n$, $\ell \geq m$. Tada $y_n \in f_{n,\ell}(X_\ell)$. Dalje, neka je $x_\ell \in X_\ell$ takvo da je $f_{n,\ell}(X_\ell) = y_n$. Stavimo $y_m = f_{m,\ell}(X_\ell)$; tada je $y_m \in Y_m$ i $f_{n,m}(y_m) = f_{n,m}(f_{m,\ell}(X_\ell)) =$

$= f_{n,\ell}(X_\ell) = y_n$. Dakle, $y_n \in f_{n,m}(Y_m)$ i napokon $f_{n,m}(Y_m) = Y_n$ $(n \leq m)$.

Uzimajući $g_{n,m} = f_{n,m}|Y_m$, mi dobijamo projektivni niz $\underset{\sim}{Y}_F = \{Y_m, g_{n,m}, N\}$ F-prostora, gde su preslikavanja $g_{n,m}$ surjektivna. Znači limes F-prostor $(\widetilde{Y}, \widetilde{T}_{\widetilde{Y}})$ od $\underset{\sim}{Y}_F$ je neprazan i sledstveno $(\widetilde{X}, \widetilde{T})$ je neprazan F-prostor. □

<u>Definicija 2.1:</u> Neka je $\Psi : (X,T) \rightarrow (Y,R)$ F-neprekidno preslikavanje, Ψ se zove jako F-zatvoreno ako je F-zatvoreno i ako za svaki α-zatvoren skup A u X, $\Psi(A)$ je α-zatvoren u Y.

<u>Propozicija 2.2:</u> Neka je $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ projektivni niz prebrojivo α-kompaktnih α-Hausdorff-ovih F-prostora sa jako F-zatvorenim kanonskim preslikavanjima. Tada je limes F-prostor $(\widetilde{X}, \widetilde{T})$ prebrojivo α-kompaktan F-prostor.

<u>Dokaz.</u> Neka je $(\widetilde{X}, \widetilde{T})$ limes F-prostor od X_F i neka je $\{E_i\}_{i=1}^{\infty}$ α-centrirana kolekcija zatvorenih F-skupova u X. Po pretpostavci

$f_n(\widetilde{X})$ je α-zatvoren u X_n zasvako $n \in N$, gde su f_n kanonska preslikavanja, te na osnovu propozicije I.2.7, $f_n(\widetilde{X})$ je prebrojivo α-kompaktno.

Neka je $\underset{\sim}{Y}_F = \{ f_n(\widetilde{X}), g_{n,m}, N \}$ gde je $g_{n,m} = f_{n,m} / f_m(X)$ $n \leq m$. Tada je $\underset{\sim}{Y}_F$ podniz od X_F koji zadovoljava uslov propozicije 2.1, dakle, limes F-prostor $(\widetilde{Y}, \widetilde{T}_{\widetilde{Y}})$ je neprazan i $\widetilde{Y} \subseteq \widetilde{X}$.

Kako je $\{ g_m(E_i | \widetilde{Y}) \}_{i=1}^{\infty}$ α-centrirana familija zatvorenih F-skupova u Y_m i pošto je Y_m prebrojivo α-kompaktno tada po propoziciji I.2.4 postoji $y_m \in Y_m$ takvo da je $(g_m(E_i | \widetilde{Y}))\ (y_m) \geq 1-\alpha$ za svako $i = 1,2,\dots$. Odavde sledi da postoji $y \in Y$ takvo da je $g_m(y) = y_m$ i $(E_i | \widetilde{Y})\ (y) \geq 1-\alpha$ za svako $i = 1,2,\dots$.

Pošto je $\widetilde{Y} \subseteq \widetilde{X}$ to postoji $x \in X$ takvo da je $E_i(x) \geq 1-\alpha$ za svako $i = 1,2,\dots$, tj. $(\widetilde{X}, \widetilde{T})$ je prebrojivo α-kompaktno.

<u>Propozicija 2.3:</u> Limes F-prostor projektivnog niza C_{II} F-prostora je C_{II} F-prostor.

<u>Dokaz.</u> Kako je po propoziciji I.3.7 proizvod prebrojive familije C_{II} F-prostora C_{II} F-prostor, to dokaz sledi iz opšte činjenice da je svaki F-podprostor C_{II} F-prostora C_{II} F-prostor. □

Neka je (X,T) F-prostor, za $f \in I^X$ se kaže da je ℓ_∞-skup ako je $f = \bigwedge_{i=1}^{\infty} g_i$ gde su g_i otvoreni F-skupovi za svako $i = 1,2,\dots$. Dalje, za f se kaže da je H_∞-skup ako je $f = \bigvee_{i=1}^{\infty} h_i$ gde je h_i zatvoren F-skup za svako $i = 1,2,\dots$. Zapazimo da je komplement ℓ_∞-skupa H_∞-skup; tj. ako je t ℓ_∞-skup, tada je co f H_∞-skup.

<u>Definicija 2.2:</u> F-skup (X,T) se zove ℓ F-prostorom ako je svaki zatvoren F-skup u X ℓ_∞-skup.

Jasno je da je svaki F-normalan ℓ F-prostor ujedno i savršeno normalan F-prostor.

Propozicija 2.4: Ako je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog niza $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ ℓ F-prostora, tada je $(\widetilde{X},\widetilde{T})$ ℓ F-prostor.

Dokaz. Pretpostavimo da je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog niza $\underset{\sim}{X}_F$ $c\ell$ F-prostora i neka je E zatvoren F-skup u X. Tada za svako $n \in N$, $c\ell\,(f_n(E))$ je zatvoren F-prostor u X_n te iz hipoteze $c\ell\,(f_n(E)) = \bigwedge_{i=1}^{\infty} u_{n_j}$, gde je u_{n_j} otvoren F-skup u X_n za svako $j = 1,2,\ldots.$ i $f^{-1}_{n,n+1}(u_{n_j}) \geqslant u_{n+1,j}$, a $u_{n_j} \geqslant u_{n,j+1}$ za svako $n \in N$, imamo $f_n^{-1}(u_{n,n}) \geqslant E$ za $n \in N$.

Pretpostavimo da je q jedna F-tačka u $\widetilde{X}$, $q \not\leqslant E$, tako da za svako n, $q \in f_n^{-1}(u_{n,n})$. Za neko $n \in N$, $f_n(q) \notin c\ell(f_n(E))$ i sledstveno $f_n(q) \not\leqslant u_{n,n}$, za neko $m \geqslant n$. Tada je $f_m(q) \not\leqslant u_{m,n} \leqslant f(u_{n,m})$. Znači $q \not\leqslant f_m^{-1}(u_{m,m})$ i dakle $E = \bigwedge_{m=1}^{\infty} f_n^{-1}(u_{n,m})$, gde je $u_{n,m}$ otvoren F-skup u X_n. □

Definicija 2.3: Preslikavanje $\Psi : X \to Y$ ima klint svojstvo ako za neki F-skup g u Y, i za neki F-skup f u X, $\Psi^{-1}(c\ell\ g) = c\ell\,(\Psi^{-1}(g))$ i $\Psi(\mathrm{int}\ f) = \mathrm{int}(\Psi(f))$.

Propozicija 2.5: Ako je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog niza $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ savršeno normalnih F-prostora sa surjektivnim F-povezujućim preslikavanjima i kanonskim preslikavanjima koja posedujukklint svojstvo, tada je $(\widetilde{X},\widetilde{T})$ savršeno normalan F-prostor.

Dokaz. Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog niza $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ savršeno normalnih F-prostora. Po propoziciji

2.4 dovoljno je pokazati da je $(\tilde{X},\tilde{T})$ normalan F-prostor. Neka je g zatvoren F-skup u $\tilde{X}$ i h otvoren F-skup u $\tilde{X}$ takav da je $g \leq h$. Tada za svako $n \in N$, $c\ell(f_n(g))$ je zatvoren F-skup u $\tilde{X}_n$ i $int(f_n(h))$ je otvoren F-skup u X_n tako da je $c\ell(f_n(g)) \leq int(f_n(h))$ Kako je X_n normalan F-prostor, to postoji otvoren F-skup k u X_n tako da je

$$cl(f_n(g)) \leq k \leq cl\ k \leq int(f_n(h)).$$

Znači

$$f_n^{-1}(cl\ f_n(g)) \leq f_n^{-1}(k) \leq f_n^{-1}(cl\ k) \leq f_n^{-1}(int(f_n(h))).$$

Pošto svako f_n ima klint svojstvo, tada

$$f_n^{-1}(f_n(cl\ g)) \leq f_n^{-1}(k) \leq cl(f^{-1}(k)) \leq f_n^{-1}(f(int\ h)).$$

Drugim rečima

$$f_n^{-1}(f_n(g)) \leq f_n^{-1}(k) \leq cl(f^{-1}(k)) \leq f_n^{-1}(f_n(h))$$

Odatle

$$g \leq f_n^{-1}(k) \leq cl(f^{-1}(k)) \leq h.$$

Skup $u = f_n^{-1}(k)$ je otvoren F-skup u $\tilde{X}$ i $g \leq u \leq cl\ u \leq h$, što se i tražilo.

3. F-dim$_\alpha$ LIMESA F-PROSTORA

Lema 3.1: Neka je (X,T) α-kompaktan F-prostor i neka je β baza za T zatvoreno u odnosu na konačno uniranje. Ako je $\{u_i\}_{i=1}^{k}$ α-senčenje za X, tada postoji α-senčenje $\{v_i\}_{i=1}^{k}$ za X koji se sastoji od elemenata iz β tako da je $v_i \leq u_i$ za svako $i = 1,2,\dots,k$.

Dokaz. Neka je β baza za T. Za svako $x \in X$ postoji neko $i \in \{1,2,\ldots,k\}$ tako da je $u_i(x) > \alpha$, gde je $u_i = \bigvee_{b_{x} \in \beta} b_x$, što daje $b_x \leq u_i$ za neko $b_x \in \beta$; znači $\{b_x\}_{x \in X}$ čini jedno α-senčenje od X i po hipotezi postoji α-podsenčenje $\{b_1,\ldots,b_3\}$ od X. Za svako $t = 1,2,\ldots,s$ izaberimo $\varphi(t)$ tako da je $b_t \leq u_{\varphi(t)}$ i stavim $v_i = \bigvee_{\varphi(t)=i} b_t$, tada pošto je β zatvoreno u odnosu na konačno uniranje to je $v_i \leq u_i$ za $i = 1,2,\ldots,k$. Zaista, $\{v_i\}_{i=1}^{k}$ je α-senčenje od X. □

Propozicija 3.1: Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog sistema $X_F = \{X_s, f_{s,t}, S\}$ α-bikompaktnih F-prostora. Ako je $F\text{-}\dim_\alpha X \leq n$, tada svako α-senčenje od X ima konačno α-profinjenje reda $\leq n$ čiji elementi pripadaju $\beta(X)$.

Dokaz. Za $\alpha \in [0,1)$ neka je $F\text{-}\dim_\alpha \widetilde{X} \leq n$ i neka je V α-senčenje od X. Po propoziciji 1.9 $(\widetilde{X},\widetilde{T})$ je α-bikompaktan F-prostor, gde je $\widetilde{X}$ neprazan, te postoji konačno α-podsenčenje $\{u_i\}_{i=1}^{k}$ za U. Kako je $\beta(X)$ zatvoreno u odnosu na konačno uniranje dokaz sledi iz leme 3.1 i definicije za $F\text{-}\dim_\alpha$ II.1.2. □

Sledeća propozicija pokazuje da ako je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog sistema α-bikompakta koji zadovoljavaju da je $F\text{-}\dim_\alpha X_s \leq n$ tada je $F\text{-}\dim_\alpha X \leq n$.

Propozicija 3.2: Neka je $X_F = \{X_s, f_{s,t}, S\}$ projektivni sistemi α-bikompaktnih F-prostora takvih da je $F\text{-}\dim_\alpha X_s \leq n$ za $s \in S$. Tada je $F\text{-}\dim_\alpha \widetilde{X} \leq n$.

Dokaz. Neka je $U = \{u_i\}_{i=1}^{k}$ α-senčenje od X čiji elementi pripadaju $\beta(X)$, tj. u_i je oblika $u_i = f_{s_i}^{-1}(u_{s_i})$ $s_i \in S$, u_{s_i} je otvoren F-skup u X_{s_i}. Kako je S usmeren skup, to postoji $t \in S$ takvo da je $s_i \leq t$ za svako s_i, $i = 1,2,\ldots,k$.

Neka je $V_{t_i} = f_{s_i,t}^{-1}(u_{s_i})$. Tada V_{t_i} je otvoren F-skup u X_t i ako stavimo $V_t = \{V_{t_i}\}_{i=1}^{k}$ tada $f_t^{-1}(V_t) = U$.

Sada $f_t(\widetilde{X})$ je α-zatvoren u X_t po propoziciji I.2.8 i I.3.3 i V_t je α-senčenje od $f_t(\widetilde{X})$, jer neka je $x_t \in f_t(X)$. Tada je $x_t = f_t(x)$ za neko $x \in \widetilde{X}$, kako je $f_t^{-1}(V_t)$ α-senčenje za X to postoji $f_t^{-1}(v_{t_{i_o}})$ takvo da je $f_t^{-1}(v_{t_{i_o}})(x) > \alpha$; sledi $v_{t_{i_o}}(f_t(x)) > \alpha$, tj. $v_{t_{i_o}}(x_t) > \alpha$. Pošto iz propozicije II.2.1 imamo $F\text{-dim}_\alpha f_t(\widetilde{X}) \leq F\text{-dim}_\alpha X_t \leq n$, tada postoji α-profinjenje W od V_t čiji je red $\leq n$. Znači $f_t^{-1}(W)$ je α-profinjenje od $f_t^{-1}(V_t) = V$ tj. $f_t^{-1}(W)$ je α-profinjenje od V reda $\leq n$ što daje $F\text{-dim}_\alpha X \leq n$.

Korolar 3.1: Neka je $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ projektivni sistem strogo α-bikompaktnih F-prostora takvih da je $F\text{-dim}\ X_s \leq n$ za svako $s \in S$. Tada je $F\text{-dim}\ \widetilde{X} \leq n$. □

Sledeća propozicija daje karakterizaciju α-bikompaktnih F-prostora preko limes F-prostora.

Propozicija 3.3: F-prostor (X,T) je strogo α-bikompaktan i $F\text{-dim}\ X = 0$ ako i samo ako (X,T) je limes F-prostor konačno diskretnih F-prostora.

Dokaz. $\Longrightarrow$ Neka je (X,T) limes F-prostor konačno diskretnih F-prostora. Za svako $\alpha\in[0,1)$ jasno (X,T) je α-bikompaktan F-prostor po propoziciji 1.9 jer svaki konačan diskretan F-prostor je α-bikompaktan.

Da bi dokazali da je $F\text{-}\dim_\alpha X = 0$ pretpostavimo obratno, tj. neka je $F\text{-}\dim_\alpha X > 0$ za svako $\alpha\in[0,1)$. Uzmi $x_1\in X$ i $y\in X\setminus\{x_1\}$, tada po definiciji α-Hausdorff-ovog F-prostora, postoje dva otvorena F-skupa u_{x_1},u_y u X tako da je $u_{x_1}(x_1)>\alpha$, $u_y(y)>\alpha$ i $u_{x_1}\wedge u_y = 0$. Dakle, $\{u_{x_1},u_y\}_{y\in X\setminus\{x_1\}}$ je α-senčenje za X i postoji konačan skup $\{y_1,\ldots,y_y\}\subseteq X\setminus\{x_1\}$, jer je X α-kompaktan, takav da je $\{u_{x_1},u_{y_i}\}_{i=1}^{k}$ konačan α-senčenje za X. Kako je $F\text{-}\dim_\alpha X>0$, to postoji α-profinjenje V od $\{u_{x_1},u_{y_i}\}_{i=1}^{k}$ reda $\geqslant 0$, tj. postoje $v_1,v_2\in V$ tako da je $v_1\wedge v_2\neq 0$. Ali V je α-profinjenje od $\{u_{x_1},u_{y_i}\}_{i=1}^{k}$ drugačije rečeno, za u_{x_1},u_{y_2} postoje $v_1,v_2\in V$ tako da je $v_1\leq u_{x_1}$, $v_2\leq u_{y_2}$ $(v_1\wedge v_2\neq 0)$ i $v_1\wedge v_2\leq u_{x_1}\wedge u_{y_2} = 0$, što je kontradikcija. Dakle $F\text{-}\dim_\alpha X = 0$ za svako $\alpha\in[0,1)$ i sledstveno $F\text{-}\dim X = 0$.

$\Longleftarrow$ Neka je (X,T) jako α-bikompaktni F-prostor za koji važi $F\text{-}\dim X = 0$. Za svako $\alpha\in[0,1)$, neka je $\mathcal{G}_\alpha = \{G_i\}_{i\in A_\alpha}$ α-senčenje od X koji se sastoji od disjunktnih F-skupova, gde je A_α konačan skup, tada $\Omega = \{\mathcal{G}_\alpha\}_{\alpha\in[0,1)}$ je familija svih konačnih disjunktnih α-senčenja od X.

Ako je $\mathcal{G}_\alpha$ α-senčenje od $\mathcal{G}_\beta$, $\alpha\leq\beta\in I$, tada I je usmeren skup u odnosu na $\leq$. Za $\alpha\leq\beta$ mi možemo definisati $f_{\alpha,\beta} : A_\beta\to A_\alpha$ sa $f_{\alpha,\beta}(i) = j$, sledi $G_j\leq G_i$, $i\in A_\beta$, $j\in A_\alpha$.

Neka je za svako A_α data diskretna F-topologija (drugim rečima, I^{A_α} je F-topologija na A_α za svako $\alpha \in [0,1]$). Tada je jasno $f_{\alpha,\beta}$ F-neprekidno ($\alpha \leq \beta$) i $\underset{\sim}{A}_F = \{A_\alpha, t_{\alpha,\beta}, I\}$ je projektivni sistem F-prostora nad I.

Neka je $(\widetilde{A},\widetilde{T})$ limes F-prostor od $\underset{\sim}{A}_F$ i $f_\lambda = \widetilde{A} \rightarrow A_\lambda$ neka je kanonsko preslikavanje za svako $\lambda \in I$.

Definišimo $\Psi : \widetilde{A} \rightarrow X$ sa $\Psi(i) = x$ ako je $x \in \bigcap_{\alpha \in [0,1)} G_i^+$ tako da je $f_\lambda(i) = i_\lambda \in A_\lambda$.

Pokazaćemo da je Ψ F-homeomofrizam. Ψ je injektivno: Neka je $i \neq j \in \widetilde{A}$. Tada $f_\lambda(i) \neq f\lambda(j) \in A_\lambda$ tj. $i_\lambda \neq j_\lambda$ i sledstveno $G_{i_\lambda} \neq G_{j_\lambda}$ za $G_{i_\lambda}, G_{j_\lambda} \in \mathcal{G}_\lambda$. Poslednje daje $\Psi(i) = x \neq y = \Psi(j)$ ako je $x \in \bigcap_{\lambda \in [0,1)} G_{i_\lambda}^+$, $y \in \bigcap_{y \in [0,1)} G_{j_\lambda}^+$.

Ψ je surjektivno: Neka je $x \in X$. Tada postoji $\lambda \in [0,1)$ i konačno disjunktno λ-senčenje $\mathcal{G}_\lambda$ tako da je $G_{i_o}(x) > \lambda$ za neko $G_{i_{o\lambda}} \in \mathcal{G}_\lambda$ što povlači da je $x \in G_{i_{o\lambda}}^+$ i $i_{o\lambda} \in A_\lambda$.
Znači postoji $i_o \in \widetilde{A}$ takvo daa je $f_\lambda(i_o) = i_{o_\lambda}$ i sledstveno $\Psi(i_o) = x$.

Ψ je F-neprekidno: Neka je $u \neq 0$ otvoren F-skup u X. Za $\lambda \in [0,1)$ postoji $i_o \in \widetilde{A}$ i $x_{i_o} \in X$ tako da je $u(\lambda_{i_o}) > \lambda$ i $\Psi(i_o) = x_{i_o}$.
Otuda

$$\Psi^{-1}(u) = \Psi^{-1}[\vee p_{x_i} : x_i \; u^+, p_{x_i}(x_i) \geq \alpha\} = \vee\{p_{\Psi^{-1}(x_i)} : x_i \in u^+\} =$$

$$= \vee\{p_i : i \in \widetilde{A}\} = \text{otvoren F-skup u } \widetilde{A}.$$

Ψ je F-otvoreno preslikavanja: Neka je w otvoren F-skup u A. Tada $w = \vee\{p_i : p_i \in w\}$ i sledstveno $\Psi(w) = \Psi(\vee p_i) = \vee \Psi(p_i) = \vee p_x$, gde je $x \in \bigcap_{\alpha \in [0,1)} G_i^+$, $f_\lambda(i) = i_\lambda$, znači, $\Psi(w)$ je otvoren F-skup u X. Dakle $(\widetilde{A},\widetilde{T})$ je F-homeomorfan sa (X,T). □

Ako je $(\widetilde{X},\widetilde{T})$ limes F-prostor projektivnog niza F-prostora koji zadovoljavaju F-$\dim_\alpha \leq p$ gde je $p \in N$, $p \geq 0$, tada pod kojim uslovima je tačno da je F-$\dim_\alpha X \leq p$. Sledeće dve propozicije se bave tim pitanjem.

<u>Propozicija 3.4:</u> Neka je $X_F = \{X_n, t_{n,m}, N\}$ projektivni niz prebrojivo α-kompaktnih prostora koji zadovoljavaju F-$\dim_\alpha X_n \leq p$ $(p \in N,\ p \geq 0)$ i neka su svi $f_{n,m}(X_m)$ α-zatvoreni u $X_n (n \leq m)$. Ako svako konačno α-senčenje bilo kojeg F-podprostora od X_n je prebrojivo ekstenzibilno, tada je F-$\dim_\alpha \widetilde{X} \leq p$.

<u>Dokaz.</u> Neka je $U = \{u_i\}_{i=1}^{k}$ α-senčenje od $\widetilde{X}$ i neka je svako konačno α-senčenje bilo kojeg F-prostora od X_n $(n \in N)$ prebrojivo ekstenzibilno.

Neka je $u_i = f_{n_i}^{-1}(v_{n_i})$ gde je $n_i \in N$, a v_{n_i} je otvoren F-skup u X_{n_i}. Dalje, neka je $v_{m_i} = f_{n_i,m}^{-1}(V_{n_i})$, $n_i \leq m$ za svako $i = 1,2,\dots,k$. Stavimo $V_m = \{v_{m_i}\}_{i=1}^{k}$, tada je $f_m^{-1}(V_m) = U$ i $\{V_m | f_m(\widetilde{X})\}$ konačno α-senčenje od $f_m(\widetilde{X})$. Po propoziciji II.2.5 F-$\dim_\alpha f_m(\widetilde{X}) \leq p$ i sledstveno postoji konačno α-senčenje $\{G_{m_i}\}_{i=1}^{k}$ od $f_m(X)$ reda $\leq p$ i $G_{m_i} \leq V_{m_i} | f_n(\widetilde{X})$ za $i = 1,2,\dots,k$. Po pretpostavci postoji konačno α-senčenje $W = \{w_j\}_{i=1}^{\infty}$ od X_m tako da je $W | f_m(X)$ α-profinjenje od $\{G_{m_i}\}_{i=1}^{k}$ i otuda iz prebrojive α-kompaktnosti X_m-a postoji konačno α-podsenčenje $\{w_j\}_{i=1}^{L}$ $(L > k)$. Drugim rečima, $\{w_j\}_{i=1}^{k}$ je α-profinjenje od $\{G_{m_i}\}_{i=1}^{k}$ reda $\leq p$. Znači, $f_m^{-1}(\{w_j\}_{i=1}^{k})$ je α-profinjenje od $f_n^{-1}(V_m)$ što povlači da je $f_m^{-1}(\{w_j\}_{i=1}^{k})$ α-profinjenje od U reda $\leq p$, tj. $F\text{-dim}_\alpha \widetilde{X} \leq p$. □

Propozicija 3.5: Neka je $X_F = \{X_n, t_{n,m}, N\}$ projektivni niz prebrojivo α-kompaktnih α-Hausdorff-ovih F-prostora sa jako F-zatvorenim kanonskim preslikavanjima. Ako je $F\text{-dim}_\alpha X_n \leq p$ za svako $n \in N$, tada je $F\text{-dim}_\alpha \widetilde{X} \leq p$.

Dokaz. Neka je $(\widetilde{X}, T)$ limes F-prostor od $X_F = \{X_n, t_{n,m}, N\}$, $F\text{-dim}_\alpha X_n \leq p$ za svako $n \in N$ i neka je $U = \{v_i\}_{i=1}^{k}$ α-senčenje od $\widetilde{X}$. Tada kao u slučaju prethodne propozicije, mi imamo α-senčenje $f_m^{-1}(V_m) = U$ i $\{V_m | f_m(X)\}$ je konačno α-senčenje od $f_m(\widetilde{X})$, gde je $f_n(\widetilde{X})$ α-zatvoren u X_m po propoziciji 1.4 i pretpostavci i tada iz propozicije II.2.3 mi imamo $F\text{-dim}_\alpha f_m(X) \leq p$. Znači, postoji α-profinjenje W_m od $V_m | f_m(\widetilde{X})$ reda $\leq p$ i odavde $f_m^{-1}(W_m)$ je α-profinjenje od $f_m^{-1}(V_m) = V$, tj. $f_m^{-1}(W_m)$ je traženo α-profinjenje od V. Dakle, $F\text{-dim}_\alpha \widetilde{X} \leq p$. □

Propozicija 3.6: Neka je jako prebrojivo α-kompaktni F-prostor (X,T) limes F-prostor projektivnog niza slabo normalnih F-prostora sa surjektivnim F-vezujućim preslikavanjima, gde je $F\text{-dim}_\alpha X_n \leq p$ za svako $n \in N$. Tada je (X,T) slabo normalan F-prostor i $F\text{-dim}_\alpha X \leq p$.

Dokaz. Pokažimo prvo da je (X,T) slabo normalan F-prostor. Neka su H,k dva disjunktna zatvorena F-skupa u X i $f_n : X \to X_n$ kanonsko preslikavanje za svako $n \in N$. Tada

$$\bigwedge_{n\in N} f_n^{-1}(cl(f_n(k)) \wedge cl(f_n(H))) = 0$$

Kako je (X,T) strogo prebrojivo α-kompaktno po pretpostavci, to je $f_n^{-1}(cl(f_n(k)) \wedge cl(f_n(H))) = 0$ za neko $n \in N$ po propoziciji I.2.4. Kako je svako $f_{n,m}$ $(n \leq m)$ surjektivno po hipotezi, to je svako f_n surjektivno po teoremi 0.3 i sledstveno $cl(f_n(k)) \wedge cl(f_n(H)) = 0$ u X_n. Ali X_n je slabo normalan F-prostor te postoje u_1, v_2, otvoreni F-skupovi u X_n, tako da je $cl(f_n(k)) \leq u_1$, $cl(f_n(H)) \leq u_2$ i $u_1 \leq co\ v_2$.

Odavde imamo

$f_n^{-1}(cl(f_n(k)) \leq f_n^{-1}(u_1)$, $f_n^{-1}(cl(f_n(H))) \leq f_n^{-1}(u_2)$ i

$f_n^{-1}(u_1) \leq f^{-1}(co\ u_2)$, tj. $k \leq f_n^{-1}(u_1)$, $H \leq f_n^{-1}(u_2)$ i

$f_n^{-1}(u_1) \leq co\ f_n^{-1}(u_2)$, gde su $f_n^{-1}(u_1)$ i $f_n^{-1}(u_2)$ otvoreni F-skupovi u X. Dakle, (X,T) je slabo normalan F-prostor.

Neka je $U = \{u_i\}_{i=1}^{k}$ α-senčenje od X. Tada svako $u_i = f_{n_i}^{-1}(v_{n_i})$, $n_i \in N$, v_{n_i} je otvoren F-skup u X_{n_i}. Neka je $v_{m_i} = f_{n_i,m}(v_{n_i})$ $n_i \leq m$ za svako $i = 1,2,\ldots,k$. Stavimo $V_m = \{v_{m_i}\}_{i=1}^{k}$; tada je $f_m^{-1}(V_m) = U$, i kako je $f_m(X) = X_m$, to je V_m konačno α-senčenje od $X_m = f_m(X)$ za neko $\alpha \in [0,1)$. Kako je $F\text{-dim}_\alpha X_m \leq p$, to postoji α-profinjenje W od V_m reda $\leq p$ te je $f_m^{-1}(W)$ α-profinjenje od $f_m^{-1}(V_m) = U$ čiji je red $\leq p$. Dakle, $F\text{-dim}_\alpha X \leq p$. □

<u>Korolar 3.2:</u> Neka je jako prebrojivo α-kompaktni F-prostor (X,T) limes F-prostor projektivnog niza normalnih F_c-prostora sa surjektivnim F-vezama, gde je $F\text{-dim}_\alpha X_n \leq p$ za svako $n \in N$.

Tada je (X,T) normalan F-prostor i $F\text{-}dim_\alpha X \leq p$. □

Mi ćemo završiti ovaj odeljak dajući generalizaciju Delinić-Mardešićeve teoreme [19], drugim rečima daćemo dovoljan i potreban uslov za $F\text{-}dim_\alpha X \leq n$. Koristićemo sličnu tehniku koja je korišćena prilikom dokaza te teoreme u topološkom slučaju.

<u>Definicija 3.1.</u> Projektivni sistem $X_F = \{X_s, f_{s,t}, S\}$ se zove α-reverzibilnim ako za svaki otvoren F-skup u_s u X_s, $f_{s,t}^{-1}(X_s \setminus \text{supp. } u_s)$ je α-zatvoren u X_t, za $\alpha \in [0,1)$ i $s \leq t$.

<u>Lema 3.2:</u> Neka je $(\widetilde{X},\widetilde{T})$ limes F-prostor α-reverzibilnog sistema $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ α-bikompaktnih F-prostora. Ako je $s \in S$ i u_s je otvoren F-skup u X_s takav da je supp. $u_s \supset f_s(\widetilde{X})$, tada postoji $t \in S$, $s \leq t$ takvo da je $f_{s,t}(X_t) \subset$ supp. u_s.

<u>Dokaz.</u> Pretpostavimo suprotno, tj. za svako $t \in S$, $s \leq t$ mi imamo $X_t^* = f_t^{-1}(X_s \setminus \text{supp } u_s) \neq 0$. Po hipotezi X_t^* je α-zatvoreno u X_t te otuda ono je α-kompaktno. Dalje, $X_F^* = \{X_t^*, f_{t,r}^*, S\}$, $t,r \geq S$ i $f_{t,r}^* = f_{t,r}|X_t^*$ je projektivni sistem α-bikompaktnih F-prostora te mora imati neprazan limes F-prostor $(\widetilde{X^*},\widetilde{T^*}) \subset (\widetilde{X},\widetilde{T})$ iz propozicije 1.9. Jasno $f_s(\widetilde{X^*}) \subseteq X_s \setminus \text{sup } u_s$ što je suprotno pretpostavci $f_s(\widetilde{X}) \subset \text{supp } u_s$. □

<u>Lema 3.3:</u> Neka je $(\widetilde{X},\widetilde{T})$ dat kao u prošloj lemi, $U = \{u_i\}_{i=1}^k$ α-senčenje od X čiji članovi pripadaju $\beta(\widetilde{X})$ i neka je $s \in S$. Tada postoji $t \in S$, $s \leq t$ i konačno α-senčenje $V_t = \{v_{t_i}\}_{i=1}^k$ od X_t tako da je $f_t^{-1}(v_{t_i}) \leq u_i$ i red od $V_t \leq$ Od reda V (ord $(V_t) \leq$ ord (v)).

<u>Dokaz.</u> Za svako $u_i \in V$, $u_i = f_{s_i}^{-1}(u_{s_i})$, $s_i \in S$, u_{s_i} je otvoren F-skup u X_{s_i}. Izaberi $s_o \in S$, $s_o \geq S$, $s_1, \dots, s_k$. Neka je $U_{s_o i} = f_{s_i s_o}^{-1}(u_{s_i})$, $i = 1,2,\dots,k$. Tada je $V_{s_o} = \{u_{s_o i}\}_{i=1}^{k}$ i $f_{s_o}^{-1}(u_{s_o i}) = f_{s_o}^{-1} f_{s_i s_o}^{-1}(u_{s_i}) = f_{s_i}^{-1}(u_{s_i}) = u_i$, a odavde $f_{s_o}^{-1}(U_{s_o}) = V$, tj. $f_{s_o}(u_{s_o})$ je α-senčenje od X. Znači V_{s_o} je α-senčenje od $f_{s_o}(\widetilde{X})$ koje je α-kompaktno i sledi $f_{s_o}(\widetilde{X})$ sup u_{s_o} gde je $u_{s_o} = \bigvee_{i=1}^{k} u_{s_o i}$. Po lemi 3.2 postoji $t \in S$ takvo da je $t \geq s_o \geq s$ i $f_{s_o t}(X_t) \subset \operatorname{supp} u_{s_o}$. Stavimo $v_{t_i} = f_{s_o,t}^{-1}(u_{s_o i})$, tada je $V_t = \{v_{t_i}\}_{i=1}^{k}$ α-senčenje od X_t i $f_t^{-1}(v_{t_i})$ u_i za $i = 1,2,\dots,k$. Sada ako je $v_{t\, i_1} \wedge \dots \wedge v_{t\, i_k} \neq 0$, to je $f_{s_o,t}^{-1}(u_{s_o\, i_1}) \wedge f_{s_o t}^{-1}(u_{s_o\, i_2}) \wedge \dots \wedge f_{s_o t}^{-1}(u_{s_o\, i_k}) \neq 0$ te je takođe i

$$f_t^{-1} f_{s_o,t}^{-1}(u_{s_o, i_1}) \wedge \dots \wedge f_t^{-1} f_{s_o,t}^{-1}(u_{s_o\, i_k}) = f_{s_o}^{-1}(u_{s_o\, i_1}) \wedge \dots \wedge f_{s_o}^{-1}(u_{s_o\, i_k}) \neq 0.$$

Sledi $u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_k} \neq 0$ te na kraju dakle dobijamo da je $\operatorname{ord}(v_t) \leq \operatorname{ord}(V)$. $\square$

<u>Lema 3.4:</u> Neka je $(\widetilde{X},\widetilde{T})$ dato kao u lemi 3.2. Ako je u_s otvoren F-skup u X_s i v_t otvoren F-skup u X_t, s,t S, tako da je $f_t^{-1}(v_t) \leq f_s^{-1}(u_s)$, tada postoji $r \in S$, $r \geq s,t$ tako da je $f_{t,r}^{-1}(v_t) \leq f_{s,r}^{-1}(u_s)$.

Dokaz. Pretpostavimo da tvrdnja nije tačna, drugim rečima za svako $r \in S$, $r \geq s,t$ mi imamo $X_r^* = f_{s,r}^{-1}(X_s \setminus \text{supp } u_s) \setminus f_{t,r}^{-1}(X_t$ supp $v_t) \neq \emptyset$. Tada je X_r^* α-zatvoren u X_r i kao u lemi 3.2, mi bi smo imali inverzan podsistem α-bikompaktnih F-prostora koji imaju neprazan limes F-prostor. Ovo bi vodilo ka:

$$\text{Supp. } f_t^{-1}(v_t) \setminus \text{Supp. } f_s(u_s) \neq \emptyset ,$$

što je suprotno našoj hipotezi da je $f_t^{-1}(v_t) \leq f_s^{-1}(u_s)$. □

Ova lema sada odmah daje sledeći rezultat.

Primedba. U_s je α-senčenje od X_s i V_t je konačno α-senčenje od X_t takvo da $f_t^{-1}(V_t)$ profinjuje $f_s^{-1}(U_s)$; tada postoji $r \in S$, $r \geq s,t$, takvo da $f_{t,r}^{-1}(V_t)$ profinjuje $f_{s,r}^{-1}(U_s)$.

Propozicija 3.7: Neka je $\underset{\sim}{X}_F = \{X_s, t_{s,t}, S\}$ α-reverzibilan projektivni sistem α-bikompaktnih F-prostora sa limes F-prostorom $(\tilde{X},\tilde{T})$. Tada je $F\text{-dim}_\alpha X \leq n$ ako i samo ako za svako $s \in S$ i svako α-senčenje U_s od X_s postoji $t \in S$, $s \leq t$, takvo da α-senčenje $f_{s,t}^{-1}(U_s)$ od X_t poseduje α-profinjenje reda $\leq n$.

Dokaz. Neka je $F\text{-dim}_\alpha \tilde{X} \leq n$ i neka je U_s α-senčenje od X_s. Tada je $f_s^{-1}(U_s)$ α-senčenje od $\tilde{X}$. Po propoziciji 3.1 $f_s^{-1}(U_s)$ ima α-profinjenje $U = \{u_1,\dots,u_k\}$ čiji članovi pripadaju standardnoj bazi $\beta(\tilde{X})$ i red od V $\leq$ od n. Po lemi 3.3, postoji $t \in S$, $s \leq t$, i konačno α-senčenje od $V_t = \{v_{t_i}\}_{i=1}^{k}$ tako da je

$$f_s^{-1}(v_{t_i}) \leq u_i, \quad \text{za } i = 1,2,\dots,k \text{ i } \text{ord}(V_t) \leq \text{ord}(U) \leq n.$$

Na osnovu primedbe iza leme 3.4 gde za svako $i = 1,2, \ldots, k$, mi imamo $f_t^{-1}(v_{t_i}) \leq u_i$, te $f_t^{-1}(v_t)$ profinjuje $f_s^{-1}(u_s)$ sledi da postoji $r \in S$, $r \geq s,t$ tako da $f_{t,r}^{-1}(V_t)$ profinjuje $f_{s,r}^{-1}(U_s)$. Znači ako je $W_r = f_{t,r}^{-1}(V_t)$, onda je W_r α-senčenje od X_r jer V_t je α-senčenje od X_t i jasno $\mathrm{ord}(W_r) \leq \mathrm{ord}(V_t) \leq n$. Dakle, r i W_r zadovoljavaju traženo svojstvo.

Neka je U konačno α-senčenje od X sa članovima za koje možemo pretpostaviti da su iz standardne baze $\mathcal{B}(X)$. Tada po lemi 3.3 postoji $s \in S$ i konačno α-senčenje U_s od X_s takvo da $f_s^{-1}(U_s)$ profinjuje V. Po pretpostavci postoji $t \in S$, $s \leq t$, i konačno α-senčenje V_t od X_t koje profinjuje $f_{s,t}^{-1}(U_s)$ reda $\leq n$. Sledi da je $f_t^{-1}(V_t)$ konačno α-senčenje od X koje profinjuje $f_s^{-1}(U_s)$, te $f_t^{-1}(V_t)$ profinjuje U. Jasno, $\mathrm{ord}\,(f_t^{-1}(V_t)) \leq \mathrm{ord}\,(V_t) \leq n$. Dakle, $F\text{-}\dim_\alpha \widetilde{X} \leq n$ što je i trebalo dokazati.

□

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" DIMENSION OF F-SPACES "

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A C K N O W L E D G M E N T S:

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C O N T E N T S:

INTRODUCTION

The notion of fuzzy topology which is a generalization of the notion of topology on a set was introduced by chang in [2] 1968. This notion based on Zadeh's concept of fuzzy sets appeared in [31] 1965, which has been applied to several branches of mathematics. Since then many authors have developed various aspects of fuzzy topology, but still attracting the attention of researchers in a number of different fields. We introduce the concept of α-covering dimension of fuzzy spaces, where $\alpha \in [0,1)$, as a generalization of the concept of the covering dimension of topological spaces.

Chapters II, III and IV of this work may justify the introduction and the study of covering dimension of fuzzy spaces.

Analogous to the theory of the covering dimension of topological spaces, we consider the subset theorem, the finite sum theorem and the product theorem.

With each fuzzy space, a topological space can be associated called the modificated topology with covering dimension equal to the α-covering dimension of the fuzzy space. We use the idea of the modificated topology to reduce statements concering α-covering dimension of fuzzy spaces to statements about the covering dimension of topological spaces.

The material of this work, apart from the topological preliminary, is organized into four main chapters, each chapt being made up of three sections which treat individual topics

The definitions and propositions given in chapter I a basic concepts of fuzzy sets and fuzzy topology where most of them will be needed in the subsequent.

Chapter II is devoted to the α-covering dimension of fuzzy spaces.

This is defined in section one and propositions 1.1, give its main characterization. Section two deals with the su theorems and the finite sum theorem, while section three trea the local properties of the α-covering dimension.

The first section of chapter III contains application of the previous chapter. Section two gives the relation betwe the covering dimension of a fuzzy space and the covering dime sion of a topologic al space generating that fuzzy space prop sition 2.3, as well as the equality of the α-covering dimens of an ultra-Tychonoff fuzzy space with the α-covering dimens of its ultra Stone-Čech fuzzy compactification proposition 2. Section three deals with the product theorem for fuzzy spaces satisfying certain conditions.

Finally chapter IV is devoted to the study of the inv se system of fuzzy spaces and the limit fuzzy space in a sim lar way to the case of the inverse limit space of topological spaces, and of the covering dimension of the limit fuzzy spac

The symbol □ is used to denote the end of a proof. I is also used at the end of a statement whose proof follows ea ly from previous results.

Items in each section are numbered separately accordi to their names and positions i.e. definition 1.1,...,1.2,..., lemma 1.1, 1.2,... and so on. References to each item in a different chapter are by triples indicating, respectively, the chapter, the section and the number of the item in the section i.e. lemma II.3.1 is lemma 1 in section 3 of chapter II.

Bibliographical references are put in [ ].

Chapter 0:

TOPOLOGICAL PRELIMINARY

The knowldge of the basic general topology and the elementary concepts of the covering dimension of topological spaces is assumed. In this short introduction a brief summary of some results in these areas which will be classified under theorems is considered. Those listed theorems either to be generalized to fuzzy topological spaces or because they are of particular importance in the subsequent.

We shall give a quick review without proofs of some results selected mainly from [4] [5] [14] [21] [22]

We know that the product of topological spaces is compact if and only if each factor is compact. This result is known by Tychonoff theorem, while the topological product of a family of countably compact is not in general countably compact. But we have:

<u>Theorem 1:</u> The product of a compact space with a countably compact space is countably compact. □

The inverse limit of an inverse system of topological spaces may be empty even if each factor space (X_s) is non-empty and the bonding maps are surjective. Useful conditions under which the inverse limit is non-empty are given by the following two theorems.

<u>Theorem 2:</u> The inverse limit space of an inverse system of non-empty compact Hausdorff spaces is a non-empty compact

Hausdorff space. □

Theorem 3: Let $\underset{\sim}{X}$ be an inverse sequence of non-empty countably compact spaces. If all sets $f_{n,m}(X_m), n \leq m, n \in N$, are closed in X_n, then the limit space $\widetilde{X}$ is non-empty and $f_n(\widetilde{X}) = \bigcap f_{n,m}(X_m) \quad n \leq m, n \in N$. □

Theorem 4: The inverse limit space of an inverse sequence of countably compact spaces and closed bonding mappings is countably compact space. □

Theorem 5: The inverse limit of an inverse sequence of perfectly normal spaces is perfectly normal. □

Theorem 6: If $\widetilde{X}$ is the inverse limit space of an inverse sequence of spaces with all bonding mappings surjective, then each canonical mapping is surjective. □

Since the first appearance of the notion of dimension theory in the early twenties there have been remarkable developments in this notion. There are three dimension functions for a topological space but we are interested in the covering dimension (dim) which depends essensially on the order of open refinements of finite open coverings of the space.

The next two theorems characterize the dimension function dim.

Theorem 7: Let X be a topological space. Then the following are equivalent:

(I) The space X satisfies the inequality dim $X \leq n$

(II) For every finite open cover $\{G_i\}_{i=1}^{k}$ of X there is a finite open cover $\{H_i\}_{i=1}^{k}$ of X whose order $\leq n$ and

$H_i \subset G_i$ for each $i = 1,2,\ldots,k$.

(III) For every open cover $\{G_i\}_{i=1}^{n+2}$ of the space X there is an open cover $\{H_i\}_{i=1}^{n+2}$ of X with $H_i \subset G_i$ for each $i = 1,2,\ldots n+2$ and $\bigcap_{i=1}^{n+2} H_i = \emptyset$. □

Theorem 8: Let X be a normal space. Thenthe following statements are equivalent:

(I) The space X satisfies $\dim X \leq n$

(II) For every finite open cover $\{G_i\}_{i=1}^{k}$ of X there is an open cover $\{H_i\}_{i=1}^{k}$ of X such that each $\text{cl } H_i \subset G_i$ and the order of $\{\text{cl } H_i\}_{i=1}^{k} \leq n$.

(III) For every finite open cover $\{G_i\}_{i=1}^{k}$ of X there is a closed cover $\{F_i\}_{i=1}^{k}$ of X such that each $F_i \subset G_i$ and order of $\{F_i\}_{i=1}^{k} \leq n$.

(IV) If $\{G_i\}_{i=1}^{n+2}$ is an open cover of X there is a closed cover $\{F_i\}_{i=1}^{n+2}$ of X such that each $F_i \subset G_i$ and $\bigcap_{i=1}^{n+2} F_i = \emptyset$. □

It is observed that if $\dim X = 0$, then X is normal space. More over we have

Theorem 9: If X is compact Hausdorff space then $\dim X = 0$ if and only if X is totally disconnected. □

Theorem 10: For a normal space X, $\dim X = \dim \beta X$, where βX is the Stone-Čech compactification of X. □

Theorem 11: If M is a closed subset of a space X, then $\dim M \leq \dim X$. □

We turn to the behaviour of the dimension function dim under the cartesian multiplication.

Theorem 12: Let X be a paracompact Hausdorff space with $\dim X = m$ and Y a compact Hausdorff space with $\dim Y = n$. If either X or Y is not empty, then:

$$\dim(X\times Y) \leq m + n. \qquad \square$$

Theorem 13: For every pair X,Y of compact spaces of which at least one is non-empty we have

$$\dim(X\times Y) \leq \dim X + \dim Y. \quad \square$$

Theorem 14: Let X and Y be Hausdorff spaces such that $X\times Y$ has the star finite property. If X or Y is non-empty, then

$$\dim(X\times Y) \leq \dim X + \dim Y. \quad \square$$

Theorem 15: If X is a space and Y is a locally compact, paracompact Hausdorff space, then

$$\dim(X\times Y) \leq \dim X + \dim Y. \quad \square$$

Theorem 16: Let $\{X_s\}_{s\in S}$ be a family of spaces such that any countable product of the spaces in this family is Lindelof. If $\dim X_s = 0$ for each $s \in S$, then $\dim \prod X_s = 0$. $\square$

Theorem 17: A space X is a compact Hausdorff space with $\dim X = 0$ if and only if X is an inverse limit of finite discrete spaces. $\square$

Theorem 18: Let $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ be an inverse system over S of compact Hausdorff spaces with the inverse limit $\widetilde{X}$. Then $\dim \widetilde{X} \leq n$ if and only if for every $s \in S$ and every open covering U of X_s there exists $t \in S$, $s \leq t$ such that the covering $f^{-1}_{s,t}(U_s)$ of X_t admits a refinement U_t of order $\leq n+1$. $\square$

Theorem 19: Let a countably compact space X be the inverse limit of an inverse sequence of normal spaces and surjective maps $\{X_n, f_{n,m}, N\}$ with $\dim X_n \leq p$ for each $n \in N$. Then X is normal and $\dim X \leq p$. □

Chapter I:

FUNDAMENTAL CONCEPTS

Since many definitions and concepts in fuzzy topology have not yet taken their final forms, we devote this chapter to the basic concepts of the theory of fuzzy sets and fuzzy topology. We give a quick review for some of those definitions and results - without proof - given in [1] [6] [7] [9] [10] [11] [12] [15] [16] [23] [28] [29] [30] which will be used frequintly throughout this work.

1. FUZZY SET THEORY:

X stands always for a non-empty set, I for the closed unit interval $[0,1]$ and I^X for the set of all functions from X to I.

Definition 1.1: A fuzzy set in X-denoted by F-set-is a function $f : X \longrightarrow I$ which associates with each point $x \in X$ its value $f(x) \in I$ (or its grade in I).

The set $\{x \in X : f(x) > 0\}$ is called the support of f and is denoted by f_o. The F-set $f \in I^X$ such that $f(x) = 0$ for all $x \in X$, will be denoted by $\overline{0}$ which corresponds to the empty set $\emptyset$ and the F-set $f \in I^X$ such that $f(x) = 1$ for all $x \in X$, will be denoted by $\bar{1}$. This F-set corresponds to the set X.

A fuzzy point - denoted by F-point - p in X is an F-set in X given by:

$$p(x) = \alpha \qquad \text{for } x = x_0 \quad (0 < \alpha \leq 1)$$
$$= 0 \qquad \text{for } x \neq x_0$$

where x_0 is called the support of p.

We note that any two F-points p and q in X are distinct if and only if their supportes are distinct.

A special function which we shall find useful in the subsequent discussion is the characteristic function of a subset i.e. if A is a subset of a set X, then the characteristic function $\mu(A)$ of A on X is defined by:

$$\mu(A) = 1 \qquad \text{if } x \in A$$
$$= 0 \qquad \text{if } x \notin A$$

which is an F-set in X. There is a one-one correspondance between the family of all subsets of a set X, P(X) and the set of all characteristic functions which have domain X, ch(X) i.e.

There are two functions:

$$\phi : P(X) \longrightarrow ch(X) \text{ given by } \phi(A) = \mu(A)$$
$$\Psi : ch(X) \longrightarrow P(X) \text{ given by } \Psi(\mu) = \{x \in X \mid (x) = 1\}$$

Since F-sets are real valued functions, we will use the existing function operations of $=, \leq, \vee, \wedge, \ldots$. to relate F-sets to other F-sets.

<u>Definition 1.2:</u> Let f,g be two F-sets in X. Then:

(I) $f = g \iff f(x) = g(x)$ for all $x \in X$

(II) $f \leq g \iff f(x) \leq g(x)$ for all $x \in X$

(III) $p \in f \iff p(x) \leq f(x)$ for all $x \in X$, where p is an F-point

(IV) $f \vee g = \max\{f(x), g(x)\}$ for all $x \in X$

(V) $f \wedge g = \min\{f(x), g(x)\}$ for all $x \in X$.

More generally, for a family $\{f_s\}_{s \in S}$ of F-sets the join $(\bigvee_{s \in S} f_s)$ and the meet $(\bigwedge_{s \in S} f_s)$ are defined by:

$$\left[\bigvee_{s \in S} f_s\right](x) = \sup_{s \in S} f_s(x) \qquad x \in X$$

$$\left[\bigwedge_{s \in S} f_s\right](x) = \inf_{s \in S} f_s(x) \qquad x \in X$$

(VI) The F-set cof defined by $(cof)(x) = 1 - f(x)$ is called the complement of f.

The F-sets $f, g \in I^X$ are said to be disjoint if $f \wedge g = 0$.

It is observed that if $f \in I^X$, then $f \wedge cof \neq 0$ in general and if $f_1, f_2 \in I^X$ such that $f_1 \wedge f_2 = 0$, then $f_1 \leq cof_2$ but this does not imply that $f_1 \wedge f_2 = 0$ generally, which is a deviation from ordinary sets; that is the lattice $(I^X, \leq)$ is not a complemented lattice, while Demorgan's law holds i.e.

$$co(\bigvee_{s \in S} f_s) = \bigwedge_{s \in S} co\, f_s$$

and

$$co(\bigwedge_{s \in S} f_s) = \bigvee_{s \in S} co\, f_s$$

for any family $\{f_s\}_{s \in S}$ of F-sets in a set X.

It follows from definition 1.2 that:

(I)° $f \vee g = g \vee f$ and $f \wedge g = g \wedge f$

(II)° $f \vee \bar{0} = f$, $f \wedge \bar{0} = 0$ and $f \vee \bar{1} = \bar{1}$, $f \wedge \bar{1} = f$

(III)° $co(cof) = f$, $co\,\bar{0} = \bar{1}$ and $co\,\bar{1} = \bar{0}$.

<u>Proposition 1.1:</u> Let f be an F-set in X and let p be an F-point in X. Then $f = \bigvee\{p : p \in f\}$. □

Definition 1.3: Let $\Psi: X \longrightarrow Y$ be a mapping of a set X into a set Y. If f is an F-set in X, then $\Psi(f)$ is an F-set in Y defined by:

$$[\Psi(f)](x) = \begin{cases} \sup\limits_{x \in \Psi^{-1}(y)} f(x) & \text{if } \Psi^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Psi^{-1}(y) = \emptyset \end{cases}$$

If g is an F-set in Y, then $\Psi^{-1}(g)$ is an F-set in X defined by:

$$[\Psi^{-1}(g)](x) = g(\Psi(x)).$$

Proposition 1.2: Let Ψ be a mapping as in definition 1.3. Then Ψ and Ψ^{-1} have the following properties:

(I) $\Psi^{-1}(\bigvee_{s \in S} f_s) = \bigvee_{s \in S} \Psi^{-1}(f_s)$

(II) $\Psi^{-1}(\bigwedge_{s \in S} f_s) = \bigwedge_{s \in S} \Psi^{-1}(f_s)$

(III) $\Psi(\bigvee_{s \in S} f_s) = \bigvee_{s \in S} \Psi(f_s)$

(IV) $\Psi(\bigwedge_{s \in S} f_s) \leq \bigwedge_{s \in S} \Psi(f_s)$

(V) $\Psi(\Psi^{-1}(f)) \leq f$

(VI) $\Psi^{-1}(\Psi(f)) \geq f$. □

We may write $p_{x_0}^{\alpha}$ if the value of p_{x_0} is α and hence $p_{x_0}^{\alpha} \in f$ if and only if $f(x_0) > \alpha$ for $\alpha \in [0,1)$.

Proposition 1.3: Let $\Psi : X \longrightarrow Y$ be a mapping from a set X to a set Y, f be an F-set in X and $p_{x_0}^{\alpha}$ be an F-point in X. Then:

(I) $\Psi[p_{x_0}^{\alpha}] = p_{\Psi(x_0)}^{\alpha}$

(ii) $P_{x_0}^{\alpha} \in f$ implies $\Psi\left[P_{x_0}^{\alpha}\right] \in \Psi(f)$. □

2. FUZZY TOPOLOGICAL SPACES

Definition 2.1: A family T of F-sets in a set X is called fuzzy topology on X (F-topology)if it satisfies the following conditions:

(F.T.1) $\bar{0}, \bar{I} \in T$

(F.T.2) $f_s \in T$ implies $\bigvee_{s \in S} f_s \in T$ $s \in S$

(F.T.3) $f_i \in T$ implies $\bigwedge_{i=1}^{n} f_i \in T$.

The members of T are called open F-sets and the pair (X,T) is called F-space. If f is an open F-set, then co.f is called closed F-set.

F-spaces are a very natural generalization of topological spaces, as a mater of fact a topology on X can be regarded as a family of characteristic functions with the function operations of $\leq$, $\vee$, $\wedge$ and co.

Definition 2.2: Let (X,T) be an F-space. An F-set g in X is called a neighbourhood of an F-set f in X if $f \leq g$ and there is an F-set $h \in T$ such that $f \leq h \leq g$.

By the interior of an F-set - denoted by int(f) - we mean the F-set:

$$\text{int}(f) = \bigvee_{s} \left\{ g_s : g_s \leq f,\ g_s \in T ,\quad s \in S \right\}.$$

f is open if and only if $f = \text{int}(f)$.

The closure of an F-set f-denoted by cl f is the F-set given by:

$$cl\ f = \bigwedge_s \left\{ h_s : f \leq h_s \quad h_s \in co\ T,\ s \in S \right\}$$

f is closed if and only if $f = cl\ f$.

<u>Proposition 2.1:</u> Every F-set f in an F-space (X,T) satisfies $int(f) = co(cl(cof))$. □

The above proposition gives the connection between the interior and closure operator in F-topology which is similar to the well konown connection between them in topology.

<u>Definition 2.3:</u> Let (X,T) be an F-space and $\beta \subseteq T$ be a subfamily of T such that each $f \in T$ is the join of members of β . Then β is called a basis for T. Also $S \subseteq T$ is called a subbasis for T if the family of all finite meets of members of S is a base for T.

<u>Proposition 2.2:</u> Let (X,T) be an F-space. Then β is a base for T if and only if, for any $f \in T$ and for every F-point p in X with $p \in f$, there exists $B \in \beta$ such that $p \in B \leq f$. □

<u>Definition 2.4:</u> An F-space (X,T) is said to be C_{II} if there exists a countable base for T.

<u>Definition 2.5:</u> Let (X,T) be an F-space and $\alpha \in [0,1)$. A collection $U \leq T$ is called and α-shading of X if for each $x \in X$ there exists $u \in U$ such that $u(x) > \alpha$. A subcollection of U which is also an α-shading of X is called an α-subshading of X.

<u>Definition 2.6:</u> Let (X,T) be an F-space and $\alpha \in [0,1)$. Let U and V be two α-shadings of X. Then U is said to be an α-refinement of V , written $U \leq_\alpha V$, if for each $u \in U$ there is $v \in V$ such $u \leq v$.

Any α-subshading of a given α-shading is an α-refinement of that α-shading.

The notion of α-shading plays an important role in the study and development of F-spaces analogous to the role of the notion of covering in topological spaces, and so α-compactness, countably α-compactness and so on... are defined and studied in terms of α-shadings as we shall see in the following definitions and propositions.

Definition 2.7: An F-space (X,T) is said to be α-compact (α-Lindelof) if each α-shading of X has a finite (countable) α-subshading of X for $\alpha\in[0,1)$.

Definition 2.8: An F-space (X,T) is called countably α-compact, where $\alpha\in[0,1)$, if every countable α-shading of X has a finite α-subshading.

Remark: Every α-compact F-space is countably α-compact, and every α-Lindelof countably α-compact is α-compact.

Proposition 2.3: Let (X,T) be a C_{II} F-space and $\alpha\in[0,1)$. Then the following hold:

(I) (X,T) is α-Lindelof.

(II) If (X,T) is countably α-compact, then (X,T) is α-compact. □

Definition 2.9: An F-space (X,T) is compact if for each family $\beta\subset T$ and for each $\alpha\in[0,1)$ such that $\sup_{g\in\beta} g\geq\alpha$ and for each $\delta\in(0,\alpha]$ there exists a finite subfamily $\beta_0\subset\beta$ such that $\sup_{g\in\beta_0} g\geq\alpha-\delta$.

Definition 2.10: Let $\alpha \in [0,1)$ and $B \subset I^X$. Then b is called α-centred if for all $c_1, c_2, \dots, c_n \in B$, there exists $x \in X$ with $C_i(x) \geq 1 - \alpha$ for all $i = 1, 2, \dots, n$.

Proposition 2.4: Let (X,T) be an F-space, $\alpha \in [0,1)$. Then (X,T) is α-compact (countably α-compact) if and only if, for every α-centred $B(\{c_i\}_{i=1}^{\infty})$ of closed F-sets in X there exists $x \in X$ such that $c(x) \geq 1 - \alpha$ for all $c \in B (c_i(x) \geq 1 - \alpha$ for all $i = 1, 2, \dots)$. □

This proposition shows that if (X,T) is α-compact (countably α-compact), $\alpha \in [0,1)$, and for every α-centered $B(\{c_i\}_{i=1}^{\infty})$ of closed F-sets, then $\bigwedge_{c \in B} c \neq 0$ $(\bigwedge_{i=1}^{\infty} c_i \neq 0)$.

Definition 2.11: Let (X,T) be an F-space and $\alpha \in [0,1)$. Then (X,T) is α-Hausdorff (Hausdorff) if for $x \neq y \in X$, there exist $f, g \in T$ such that $f(x) > \alpha$, $g(x) > \alpha$ ($f(x) = 1 = g(x)$) and $f \wedge g = 0$.

Note that Hausdorff F-space implies α-Hausdorff F-space for any $\alpha \in [0,1)$.

Definition 2.12: Let (X,T) be an F-space, $\alpha \in [0,1)$, and $A \subset X$. Then:

(I) A is α-closed (α^*-closed) if for each $x \in X \setminus A$ there is $f \in T$ such that $f(x) > \alpha$ $(f(x) \geq \alpha)$ and $f \wedge \mu(A) = 0$

(II) A is suitable closed if $\mu(A)$ is closed F-set in X.

Proposition 2.5: Arbitrary intersections of α-closed sets are α-closed, and finite unions of α-closed sets are α-closed. □

The following proposition relates the α-closedness with suitable closedness.

Proposition 2.6: Let (X,T) be an F-space and let $A \subset X$. Then the following are equivalent:

(I) A is suitable closed set.

(II) A is 1^*- closed set. □

Let (X,T) be an F-space and $Y \subset X$. Then the family $T_y = \{ f|y : f \in T \}$, where $f | y$ is the restriction of f to y satisfies the three conditions of definition 2.1, that is T_y is an F-topology on Y.

Definition 2.13: The F-topology T_y is called the relative F-topology on Y or the F-topology on Y induced by the F-topology T on X, and (Y,T_y) is called the F-subspace of (X,T).

We usually omit the relative F-topology T_y and simply write the F-subspace Y.

Proposition 2.7: Let (X,T) be an F-space, Y be F-subspace of (X,T) and $f \in I^Y$. Then:

(I) f is closed in Y if and only if there exists a closed F-set in X such that $f = g | Y$.

(II) $cl_y f = cl_x f | Y$. □

We note that if β is a base for an F-space (X,T), then $\beta_y = \{B | y : B \in \beta\}$ is a base for T_y.

Proposition 2.8: Let (X,T) be an F-space and $Y \subset X$. For $\alpha \in [0,1)$ the following hold:

(I) (X,T) is α-Hausdorff, then (Y,T_y) is α-Hausdorff.

(II) (X,T) is α-compact (countably α-compact) and Y is α-closed in X, then Y is α-compact (countably α-compact). □

Proposition 2.9: Let (X,T) be an α-Hausdorff F-space and $Y \subset X$. Then the following hold:

(I) Y is α-compact in X, then Y is α-closed in X.

(II) $\{Y_s\}_{s\in S}$ is a decreasing family of α-compact (countably α-compact) in X, then $\bigcap_{s\in S} Y_s \neq \phi$ and is α-compact (countably α-compact). □

Definition 2.14: An F-space (X,T) is called:

(I) FT_1-space if each F-point is closed F-set.

(II) FT_2-space if for each two distinct F-points p,q in X:

a) $p_0 \neq q_0$ implies that there exist two disjoint open F-sets containing p and q respect.

b) $p_0 = q_0$, $p(x) < q(x)$ implies that there is an $f \in T$ such that $p \in f$ and $q \notin cl\ f$.

(III) F-regular space if for each closed F-set g and each F-point p such that $p \notin q$ there exists an open nbhd. f of g such that $p \notin cl(f)$.

Definition 2.15: An F-space (X,T) is called:

(I) weakly F-normal if for every two disjoint closed F-sets g_1 and g_2 there exist two open F-sets h_1 and h_2 such that $g_1 \leq h_1$, $g_2 \leq h_2$ and $h_1 \leq co.\ h_2$.

(II) F-normal if for a closed F-set g and an open F-set with $g \leq f$, there exists an open F-set h such that $g \leq h \leq cl\ h \leq f$ or equivalently: for every closed F-sets g_1 and g_2 such that $g_1 \leq co\ g_2$, there exist two open F-sets h_1 and h_2 such that $g_1 \leq h_1$, $g_2 \leq h_2$ and $h_1 \leq co\ h_2$.

(III) Perfectly F-normal if it is F-normal and every closed F-set is a countable meet of open F-sets.

Proposition 2.10: Every F-normal space is weakly F-normal. □

The converse of this proposition is not true in general.

Proposition 2.11: Let Y be an F-subspace of an F-space (X,T). If (X,T) is FT_1, [FT_2] (F-regular), then Y is FT_1 [FT_2] (F-regular). □

The F-subspace Y of an F-normal space (X,T) is not F-normal even if Y is α-closed in X. We shall see in chapter III when an F-subspace of an F-normal is F-normal.

3. FUZZY CONTINUOUS FUNCTIONS:

Definition 3.1: Let (X,T) and (Y,R) be F-spaces, and let $\Psi: X \longrightarrow Y$ be a map. Ψ is called F-continuous if $\Psi^{-1}(f) \in T$ whenever $f \in R$ and is called F-open (F-closed) if every open (closed) F-set f in X, $\Psi(f)$ is open (closed) F-set in Y. Ψ is F-homeomorphism if Ψ is F-continuous bijective and F-open or F-closed.

Proposition 3.1: Let (X,T), (Y,R) be F-spaces and let $\Psi: X \longrightarrow Y$. Then the following are equivalent:

(I) Ψ is F-continuous.

(II) For every closed F-set g in Y, $\Psi^{-1}(g)$ is closed F-set in X.

(III) For every F-set f in X, $\Psi(\text{cl } f) \leq \text{cl}(\Psi(f))$.

(IV) For every F-set g in Y, $\text{cl}(\Psi^{-1}(g)) \leq \Psi^{-1}(\text{cl } g)$. □

Proposition 3.2: The composite of F-continuous (F-open) [F-closed] is F-continuous (F-open) [F-closed]. □

Proposition 3.3: Let (X,T), (Y,R) be F-spaces and $\Psi: X \longrightarrow Y$

be an F-continuous map. Then for $\alpha\in[0,1)$, the following hold:

(I) (X,T) is α-compact, then $\Psi(X)$ is α-compact as an F-subspace of (Y,R).

(II) (X,T) is countably α-compacet (α-Lindelof), then $\Psi(X)$ is countably α-compact (α-Lindelof).

(III) (X,T) is α-compact and (Y,R) is α-Hausdorff, then for α-closed set A in X, $\Psi(A)$ is α-closed set in Y. □

Proposition 3.4: Let (X,T), (Y,R) be F-spaces and let Φ, $\Psi : X \longrightarrow Y$ be F-continuous. If (Y,R) is α-Hausdorff, $\alpha\in[0,1)$, then:

$$\{x \in X : \Phi(x) = \Psi(x)\} \text{ is } \alpha\text{-closed in } X. \quad \square$$

Proposition 3.5: Let $\Psi : (X,T) \longrightarrow (Y,R)$ be an F-continuous map. If A is α-closed in Y, then $\Psi^{-1}(A)$ is α-closed in X. □

Let (Y,T) be an F-space, X is a set and $\Psi : X \longrightarrow Y$ is a function. Then the family $\Psi^{-1}(T) = \{\Psi^{-1}(f) : f \in T\}$ is the smallest F-topology on X making Ψ F-continuous. This F-topology is called the initial F-topology on the set X.

Let $\{(X_s,T_s)\}_{s\in S}$ be a family of F-spaces, $X = \prod_{s\in S} X_s$ be the cartisian product of X_s (as sets) and let $P_s : X \longrightarrow X_s$, $s \in S$ be the projections. The family $\{P_s^{-1}(f_s) : f_s \in T_s,\ s \in S\}$ of F-sets can be taken as a subbase for an F-topology T on X.

Definition 3.2: Given a family $\{(X_s,T_s)\}_{s\in S}$ of F-spaces. The F-topology T on X defined as above is called the product F-topology and (X,T) is called the product F-space.

It will be observed that the members of a base for the product F-topology T are of the form: $\bigwedge_{i=1}^{n} P_i^{-1}(f_i)$, $f_i \in T_i$ for $i = 1,2,\ldots,n$.

Proposition 3.6: Let (X,T) be the product F-space of the family $\{(X_s, T_s)\}_{s\in S}$ of F-spaces. Then the following are equivalent:

(I) The projection P_s is F-continuous for every $s \in S$.

(II) The product F-topology is the smallest F-topology on X for which the projections are F-continuous. □

Proposition 3.7: Let $\{(X_i,T_i)\}_{i=1}^{\infty}$ be a countable family of C_{II} F-spaces. Then the product F-space (X,T) is also C_{II}. □

Proposition 3.8: Let (X_s,T_s) be a family of F-spaces. If each (X_s,T_s) is α-Hausdorff (F,T) [F-regular] F-space. Then the product F-space (X,T) is α-Hausdorff (FT_1) [F-regular] respectively. □

Chapter II:

α-COVERNIG DIMENSION OF F-SPACES

In this chapter we define the α-covering dimension of F-spaces up to α-level where $\alpha \in [0,1)$ and give some of its characterization similar to the well known characterization of the covering dimension of topological spaces. Also we find some subset and finite sum theorems in the second part of this chapter, while the last section is devoted to the study of the local α-covering dimension of F-spaces.

1. MAIN CHARACTERIZATIONS:

The notion of the order of a family of F-sets will be widly used in our study of the covering dimension of F-spaces.

Definition 1.1: The order of a family of F-sets $\{f_s\}_{s \in S}$ of an F-space (X,T) is the largest integer n for which there exists $M \subset S$ with n+1 members such that $\bigwedge_{s \in M} f_s \neq 0$. Or ∞ if there is no such largest integer.

Thus, if the order of $\{f_s\}_{s \in S} = n$, then for each n+2 distinct $s_1, s_2, \ldots, s_{n+2} \subset S$ we have $\bigwedge_{i=1}^{n+2} f_{s_i} = 0$, and if the order of $\{f_s\}_{s \in S} = 0$, then $\{f_s\}_{s \in S}$ consists of pairwise disjoint non-zero F-sets.

Definition 1.2: Let (X,T) be an F-space, n be an integer $(n \geq 0)$, and let $\alpha \in [0,1)$. We define the α-covering dimension of (X,T) - denoted by $F\text{-}\dim_\alpha X$ - as follows:

(I) $F\text{-}\dim_\alpha X = -1$ if $X = \emptyset$.

(II) $F\text{-}\dim_\alpha X \leq n$ if every finite α-shading of X has an α-refinement of order $\leq n$.

(III) $F\text{-}\dim_\alpha X = n$ if $F\text{-}\dim_\alpha X \leq n$ and $F\text{-}\dim_\alpha X > n-1$.

(IV) $F\text{-}\dim_\alpha X = \infty$ if $F\text{-}\dim_\alpha X \leq n$ is false for every n.

Example 1.1: Let $X = \{x_1,x_2,x_3,x_4\}$. Let f, g, h, ℓ, $m \in I^X$ be defined as follows:

$$f(x) = \tfrac{1}{6} \text{ if } x = x_1$$
$$= 0 \text{ otherwise}$$
$$g(x) = \tfrac{1}{4} \text{ if } x \in \{x_2,x_4\}$$
$$= 0 \text{ otherwise}$$
$$h(x) = \tfrac{1}{3} \text{ if } x \in \{x_3,x_4\}$$
$$= 0 \text{ otherwise}$$
$$\ell(x) = \tfrac{5}{12} \text{ if } x \in \{x_1,x_4\}$$
$$= 0 \text{ otherwise}$$
$$m(x) = \tfrac{1}{4} \text{ if } x = x_4$$
$$= 0 \text{ otherwise.}$$

Let $T = \{\bar{0},\bar{1},f,g,h,\ell,m,f \vee g,f \vee h,g \vee h,g \vee \ell\}$. Then (X,T) is an F-space.

Put $\alpha = \frac{1}{12}$, then $\{f,g,h\}$ is an α-shading of X, which is clearly an α-refinement of any α-shading of X and the order of $\{f,g,h\} \leq 1$. Hence $F\text{-}\dim_\alpha X \leq 1$.

Example 1.2: Let X be as in example 1.1. Let $f,g,h \in I^X$ be defined as follows:

$$f(x) = \frac{5}{12} \text{ if } x = x_1$$
$$= 0 \text{ otherwise}$$
$$g(x) = \frac{1}{4} \text{ if } x \in \{x_2, x_3\}$$
$$= 0 \text{ otherwise}$$
$$h(x) = \frac{1}{3} \text{ if } x = x_4$$
$$= 0 \text{ otherwise.}$$

Let $T = \{\overline{0}, \overline{1}, f, g, h, f \vee g, f \vee h, g \vee h\}$. Then (X,T) is an F-space. If we put $\alpha = \frac{1}{6}$, then $\{f,g,h\}$ is an α-shading of X, which is an α-refinement of any α-shading of X.

Since $f \wedge g = g \wedge h = f \wedge h = 0$, and $X \neq \emptyset$, so $F\text{-}\dim_\alpha X = 0$.

The next proposition gives a useful characterization of the α-covering dimension of F-spaces.

<u>Proposition 1.1:</u> Let (X,T) be an F-space, $\alpha \in [0,1)$. Then the following are equivalent:

(I) $F\text{-}\dim_\alpha X \leq n$.

(II) For any finite α-shading $\{f_i\}_{i=1}^{k}$ of X, there is an α-shading $\{g_i\}_{i=1}^{k}$ of X of order $\leq n$, and $g_i \leq f_i$ for each $i = 1,2,\ldots,k$.

(III) If $\{f_i\}_{i=1}^{n+2}$ is an α-shading of X, then there is an α-shading $\{g_i\}_{i=1}^{n+2}$ such that $g_i \leq f_i$ for $i = 1,2,\ldots,n+2$, and $\bigwedge_{i=1}^{n+2} g_i = 0$.

<u>Proof</u>. (I) $\Longrightarrow$ (II) Let $F\text{-}\dim_\alpha X \leq n$, and let $\{f_i\}_{i=1}^{k}$ be an α-shading of X. Then by definition 1.2, $\{f_i\}_{i=1}^{k}$ has an α-refinement V of order $\leq n$. If $v \in V$, then $v \leq f_i$ for some $i \in \{1,2,\ldots,k\}$. For each $v \in V$ choose $i(v) \leq k$ such that $v \leq f_{i(v)}$ and let $g_i = \bigvee\{v \in V : i(v) = i\}$. Then clearly

$\{g_i\}_{i=1}^{k}$ is an α-shading of X whose order $\leq n$, and for each $i = 1,2,\ldots,k$, $g_i \leq f_i$.

(II) $\Longrightarrow$ (III) Let (II) be hold and let $\{f_i\}_{i=1}^{n+2}$ be an α-shading of X. Then by assumption there is an α-shading $\{g_i\}_{i=1}^{n+2}$ such that $g_i \leq f_i$ for $i = 1,\ldots,n+2$, and order of $\{g_i\}_{i=1}^{n+2} \leq n$. So from definition 1.1 we get $\bigwedge_{i=1}^{n+2} g_i = 0$.

(III) $\Longrightarrow$ (II) Let $\{f_i\}_{i=1}^{k}$ be an α-shading of X, and assume that (III) is true. We may assume that $k > n+1$. Let $g_i = f_i$ for $i \leq n+1$ and $g_{n+2} = \bigvee_{i=n+2}^{k} f_i$. Then $\{g_i\}_{i=1}^{n+2}$ is an α-shading of X and so by assumption there is an α-shading $\{h_i\}_{i=1}^{n+2}$ such that $h_i \leq g_i$ for each $i = 1,2,\ldots,n+2$, and $\bigwedge_{i=1}^{n+2} h_i = 0$. Let $m_i = h_i$ for $i \leq n+1$, and $m_i = g_i \wedge h_{n+2}$ for $i > n+1$. Then $M = \{m_i\}_{i=1}^{k}$ is an α-shading of X, each $m_i \leq f_i$, and $\bigwedge_{i=1}^{n+2} m_i = 0$. If some collection of n+2 - members of M has a non-zero meet, then they can be renumbered so that the first n+2 - members have a zero meet. By applying the above construction to M, we can get an α-shading $M^* = \{m_i^*\}_{i=1}^{k}$ such that $m_i^* \leq m_i$ and $\bigwedge_{i=1}^{n+2} m_i^* = 0$. Clearly $m_{i_1}^* \wedge m_{i_2}^* \wedge \cdots m_{i_s}^* = 0$ whenever $m_{i_1} \wedge m_{i_2} \cdots \wedge m_{i_s} = 0$ where $i_1, i_2, \ldots, i_s$ is less than or equal to k. So by a finite number of repetitions of this process we obtain an α-shading $\{s_i\}_{i=1}^{k}$ of X of order $\leq n$ and $s_i \leq f_i$ for $i = 1,2,\ldots,k$.

(II) $\Longrightarrow$ (I) is obvious. □

The notion of normality plays an important role in the development of the covering dimension of topological spaces, but the role of F-normality is slightly weaker because of the fact that: $g_1 \wedge g_2 = 0$ if and only if $g_1 \leq co\, g_2$ does not hold in general in case of F-sets. This led Kerre[ ] to introduce the notion of weak normality for F-spaces, and it leads us to the following definition.

<u>Definition 1.3:</u> An F-space (X,T) is called F_c-space if for each pair of F-sets f,g in X with cl $f \leq co\, g$ then cl $f \wedge g = 0$.

We observe that for F_c-spaces, F-normality and weak F-normality are equivalent. For : by proposition I.2.4(g), every F-normal space is weakly F-normal. Now let g_1, g_2 be two closed F-sets in an F_c-normal space (X,T), $g_1 \leq co\, g_2$. Then by definition I.2.15 (ii), there exist two open F-sets h_1, h_2 such that $g_1 \leq h_1$, $g_2 \leq h_2$ and $h_1 \leq co\, h_2$. Since (X,T) is F_c-normal so by definition 1.3, cl $g_1 \leq co\, g_2$ implies cl $g_1 \wedge g_2 = 0$ and hence $g_1 \wedge g_2 = 0$ as required. □

<u>Definition 1.4:</u> Let $\{f_s\}_{s \in S}$ be a family of F-sets in an F-space (X,T). A swelling of this family is a family $\{g_s\}_{s \in S}$ of F-sets in X such that $f_s \leq g_s$ for every $s \in S$ and:

$f_{s_1} \wedge f_{s_2} \ldots f_{s_n} = 0$ if and only if $g_{s_1} \wedge g_{s_2} \wedge \ldots \wedge s_n = 0$ for every choice of indices $s_1 \ldots s_n \in S$.

<u>Proposition 1.2:</u> Let (X,T) be an F_c-normal space. Then for every finite family $\{f_i\}_{i=1}^{k}$ of closed F-sets in X, there is a finite family $\{h_i\}_{i=1}^{k}$ of open F-sets in X such that $\{cl\, h_i\}_{i=1}^{k}$ is a swelling of $\{f_i\}_{i=1}^{k}$ and $f_i \leq h_i$ for each $i = 1,2,\ldots,k$.

Proof. Let $\{f_i\}_{i=1}^{k}$ be given, and let g_1 be the join of all meets of the form $f_{i_1} \wedge f_{i_2} \wedge \dots \wedge f_{i_m}$ such that $f_1 \wedge f_{i_1} \wedge \dots \wedge f_{i_m} = 0$. Then g_1 is closed F-set and $g_1 \wedge f_1 = 0$, so $f_1 \leq \mathrm{co}\ g_1$, where $\mathrm{co}\ g_1$ is open F-set. Since (X,T) is F-normal, there exists an open F-set h_1 such that $f_1 \leq h_1 \leq \mathrm{cl}\ h_1 \leq \mathrm{co}\ g_1$.

The family $\{\mathrm{cl}\ h_1, f_2 \dots f_k\}$ is a swelling of $\{f_i\}_{i=1}^{k}$ because: If $f_1 \wedge f_{i_1} \wedge f_{i_2} \dots \wedge f_{i_m} = 0$ then $f_{i_1} \wedge \dots \wedge f_{i_m} \leq g_1$ and so $\mathrm{co}(f_{i_1} \wedge \dots \wedge f_{i_m}) \geq \mathrm{co}\ g_1 \geq \mathrm{cl}\ h_1$ i.e. $\mathrm{cl}\ h_1 \leq \mathrm{co}(f_{i_1} \wedge \dots \wedge f_{i_m})$. Since (X,T) is F_c-space then $\mathrm{cl}\ h_1 \wedge (f_{i_1} \wedge \dots \wedge f_{i_m}) = 0$. Thus if we apply thus principle K-times, we obtain a family of open F-sets $\{h_i\}_{i=1}^{k}$ such that $\{\mathrm{cl}\ h_i\}_{i=1}^{k}$ is a swelling of $\{f_i\}_{i=1}^{k}$ and $f_i \leq h_i$ for $i = 1,2,\dots,k$. □

Definition 1.5: An α-shading $\{f_s\}_{s\in S}$ of an F-space (X,T) is said to be shrinkable if there is an α-shading $\{g_s\}_{s\in S}$ of X such that $\mathrm{cl}\ g_s \leq f_s$ for $s \in S$.

Proposition 1.3: Let (X,T) be an F-normal space. Then each finite α-shading of X is shrinkable.

Proof. Let $\{f_i\}_{i=1}^{k}$ be a finite α-shading of X. Let us put $g_1 = \mathrm{co}\left[\{f_2 \vee f_3 \dots \vee f_k\} \vee \bigvee_{h\in T} \{h : h \geq f_i\ \forall\ i=1,2,\dots,k\}\right] \wedge \mu(f_1^*)$, where $f_1^* = \{x \in X \mid f_1(x) \geq \alpha\}$. Then g_1 is closed F-set in X and $g_1 \leq f_1$, so by the F-normality of (X,T), there exists an open F-set m, such that $g_1 \leq m_1 \leq \mathrm{cl}\ m_1 \leq f_1$, and $\{m_1, f_2, f_3, \dots, f_k\}$ is α-shading of X. If we apply this method K-times, we obtain an α-shading $\{m_i\}_{i=1}^{k}$ of X such that $\mathrm{cl}\ m_i \leq f_i$ for each $i = 1,2,\dots,k$. □

Definition 1.6: Let (X,T) be an F-space. A collection B of closed F-sets in X is called an α-coshading of X if for each $x \in X$, there exists $g \in B$, such that $g(x) > \alpha$. $\alpha \in [0,1)$.

The next proposition gives a characterization for F-$\dim_{\alpha}$ of F_c-normal space in terms of closed F-sets.

Proposition 1.4: Let (X,T) be an F_c-normal space, $\alpha \in [0,1)$. Then the following are equivalent:

(I) F-$\dim_{\alpha} X \leq n$.

(II) For every finite α-shading $\{f_i\}_{i=1}^{k}$ of X, there exists an α-shading $\{g_i\}_{i=1}^{k}$ of X such that $\mathrm{cl}\ g_i \leq f_i$ and order of $\{\mathrm{cl}\ g_i\}_{i=1}^{k} \leq n$.

(III) For every finite α-shading $\{f_i\}_{i=1}^{k}$ of X, there exists an α-coshading $\{g_i\}_{i=1}^{k}$ such that $g_i \leq f_i$ for each $i = 1,2,\ldots, k$, and order of $\{g_i\}_{i=1}^{k} \leq n$.

(IV) If $\{f_i\}_{i=1}^{n+2}$ is a finite α-shading of X, then there exists an α-coshading $\{g_i\}_{i=1}^{n+2}$ such that $\bigwedge_{i=1}^{n+2} g_i = 0$ and $g_i \leq f_i$ for $i = 1,2,\ldots,n+2$.

Proof. (I) $\Longrightarrow$ (II) Let F-$\dim_{\alpha} X \leq n$, and let $\{f_i\}_{i=1}^{k}$ be an α-shading of X. Then by proposition 1.1 there exists an α-shading $\{h_i\}_{i=1}^{k}$ such that $h_i \leq f_i$ for $i = 1,2,\ldots,k$ and order of $\{h_i\}_{i=1}^{k} \leq n$. But (X,T) is F-normal so by proposition 1.3 there is an α-shading $\{g_i\}_{i=1}^{k}$ such that $\mathrm{cl}\ g_i \leq h_i$ for $i = 1,2,\ldots,k$, and order of $\{\mathrm{cl}\ g_i\}_{i=1}^{k} \leq n$.

(II) $\Longrightarrow$ (III) and (III) $\Longrightarrow$ (IV) are clear.

(IV) $\Rightarrow$ (I) Let $\{f_i\}_{i=1}^{n+2}$ be an α-shading of X, by hypothesis there exists an α-coshading $\{g_i\}_{i=1}^{n+2}$ of X such that $g_i \leq f_i$ for each $i = 1,2,\dots,n+2$, and $\bigwedge_{i=1}^{n+2} g_i = 0$. By proposition 1.2 there exists a family of open F-sets $\{h_i\}_{i=1}^{n+2}$ such that $g_i \leq h_i \leq f_i$ for $i = 1,2,\dots,n+2$, and $\{cl\ h_i\}_{i=1}^{n+2}$ is a swelling of $\{g_i\}_{i=1}^{n+2}$. Thus $\{h_i\}_{i=1}^{n+2}$ is α-shading of X, $h_i \leq f_i$ and $\bigwedge_{i=1}^{n+2} h_i = 0$.

Hence by proposition 1.1: $F\text{-}\dim_\alpha X \leq n$. □

Let (X,T) be an F-space, and let $\alpha_0 \in [0,1]$ be fixed. We define a weaker dimension function - denoted by $F_{\alpha_0}\text{-}\dim_\alpha X$ - of X at $\alpha \in [0,1)$ as follows: If $\alpha \in [0,\alpha_0)$, then $F_{\alpha_0}\text{-}\dim_\alpha X \leq n$ if every finite α-shading of X has an α-refinement whose order $\leq n$. If $\alpha \in [\alpha_0,1)$, then $F_{\alpha_0}\text{-}\dim_\alpha X \leq n$ if every finite α-shading of X has a β-refinement for some $\beta \in [0,\alpha_0)$ whose order $\leq n$.

<u>Example 1.3:</u> Let X = I, for n = 1,2,3,4 and let $f_n, g_n \in I^X$ be defined as follows:

$$f_n(x) = \frac{2}{n+2} \quad \text{if } x \in \left[0,\tfrac{2}{3}\right]$$
$$= 0 \quad \text{otherwise}$$
$$g_n(x) = \frac{2}{2n+1} \quad \text{if } x \in \left[\tfrac{1}{3},1\right]$$
$$= 0 \quad \text{otherwise}$$

Let $T = \{\bar{0},\bar{1},f_n,g_n,f_n \wedge g_n,f_n \vee g_n, n=1,2,3,4\}$.

Then (X,T) is an F-space. Put $\alpha_0 = \frac{5}{8}$ and let $\alpha = \frac{1}{12}$ (say). Then $\{f_4,g_4\}$ is α-refinement of every α-shading of X, where $\alpha \in [\frac{1}{12},\frac{1}{3})$ whose order ≤ 1. So $F_{\frac{5}{8}}\text{-}\dim_\alpha X \leq 1$.

Remark: We notice that if $\alpha_0 = 1$, then F_{α_0}-$\dim_\alpha X$ = F-$\dim_\alpha X$ for $\alpha \in [0, \alpha_0)$.

If $\alpha \in [\alpha_0, 1)$, then F_{α_0}-$\dim_\alpha X$ and F-$\dim_\alpha X$ may not related.

2. THE SUBSET AND SUM THEOREMS:

In this section we find some subset and sum theorems for α-covering dimension of F-spaces. It is not true in general that if (X,T) is an F-space and Y is an F-subspace of (X,T) then F-$\dim_\alpha Y \leq$ F-$\dim_\alpha X$. We will see under what condition the above relation is true.

Proposition 2.1: Let (X,T) be an α-compact F-space. If Y is an α-closed F-subspace, then:

$$\text{F-}\dim_\alpha Y \leq \text{F-}\dim_\alpha X.$$

Proof: Let F-$\dim_\alpha X \leq n$, Y be an α-closed F-subspace of (X,T), and $\mathcal{G}$ be a finite α-shading of Y ($\alpha \in [0,1)$). For each $x \in X \setminus Y$, there exists $u_x \in T$ with $u_x(x) > \alpha$ and $u_x \wedge \mu(Y) = 0$ by definition of α-closedness I.2.10. Then $\{\mathcal{G}, u_x\}_{x \in X \setminus Y}$ is α-shading of X, and hence has finite α-subshading V(say) from the α-compactness of (X,T). Since F-$\dim_\alpha X \leq n$, V has an α-refinement W whose order $\leq n$, and so $W|Y$ is an α-refinement of $\mathcal{G}$ whose order $\leq n$. □

Proposition 2.2: Let Y be an F-subspace of an α-Hausdorff F-space (X,T). Then:

$$\text{F-}\dim_\alpha Y \leq \text{F-}\dim_\alpha X.$$

Proof. Let F-$\dim_\alpha X$, and let $\{f_i\}_{i=1}^{k}$ be an α-shading of Y. Then

$\{g_i\}_{i=1}^{k}$ is a family of open F-sets in X such that $g_i|Y = f_i$ for $i = 1,2,\ldots,k$. Now let $y \in Y$, then for each $x \in X\setminus Y$ there exist two open F-sets in X, V_y, u_x such that $u_x(x) > \alpha$, $V_y(y) > \alpha$ and $V_y \wedge u_x = 0$ by definition of α-Hausdorff 1.2.11 Put $u = \bigvee_{x \in X\setminus Y} u_x$, then $U \in T$ and $u \wedge u_z = 0$ for $z \in Y$. Hence $\{g_i, u\}_{i=1}^{k}$ is finite α-shading of X, and by assumption there is an α-refinement V of $\{g_i, u\}_{i=1}^{k}$ whose order $\leq n$. Then $V|Y$ is an α-refinement of $\{f_i\}_{i=1}^{k}$ of order $\leq n$.

Therefore $F\text{-}\dim_\alpha Y \leq n$. □

<u>Proposition 2.3:</u> Let Y be a suitable closed set in an F-space (X,T). Then:

$$F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X.$$

<u>Proof.</u> Let $F\text{-}\dim_\alpha X \leq n$, and let $\{f_i\}_{i=1}^{k}$ be an α-shading of Y. Then $\{g_i\}_{i=1}^{k}$ is a family of open F-sets in X such that $g_i|Y = f_i$. Since Y is suitable closed then $co\,\mu(Y)$ is F-open in X by definition 1.2.10, and hence $\{g_i, co\,\mu(Y)\}_{i=1}^{k}$ is an α-shading of X. But $F\text{-}\dim_\alpha X \leq n$ so by proposition II.1.1 there is an α-shading $\{h_i, co\,\mu(Y)\}_{i=1}^{k}$ of X such that $h_i \leq g_i$ for $i = 1,2,\ldots,k$ and order of $\{h_i, co\,\mu(Y)\}_{i=1}^{k} \leq n$. Therefore $\{h_i|Y\}_{i=1}^{k}$ is an α-shading of Y of order $\leq n$ and $h_i|Y \leq g_i|Y = f_i$ for $i = 1,2,\ldots,k$. So $F\text{-}\dim_\alpha Y \leq n$. □

<u>Definition 2.1:</u> An F-subspace Y of an F-space (X,T) is closed if $\mu(X\setminus Y)$ is open F-set in X, each closed F-set in Y is closed F-set in X, and for each closed F-set f in X, $f|Y$ is closed F-set in Y.

<u>Proposition 2.4:</u> Let (X,T) be an F-space, and Y be a closed F-subspace of (X,T). Then:

$$F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X.$$

<u>Proof.</u> Let $F\text{-}\dim_\alpha X \leq n$ and let $\mathcal{G}$ be a finite α-shading of Y. Then $U = \{u \in T : u|Y \in \}$ is a finite family of open F-sets in X and hence $\{U, \mu(X\setminus Y)\}$ is a finite α-shading of X. Since $F\text{-}\dim_\alpha X \leq n$, there is a finite α-shading $\{V, \mu(X\setminus Y)\}$ of X order $\leq n$ by proposition II.1.1 and $V \leqslant U$. But $\{V|Y\}$ is a finite α-shading of Y of order $\leq n$ and $V|Y \leqslant U|Y = \mathcal{G}$. So $F\text{-}\dim_\alpha Y \leq n$. □

<u>Definition 2.2:</u> Let Y be an F-subspace of an F-space (X,T) and let $\mathcal{G}$ be an α-shading of Y. Then $\mathcal{G}$ is called countably extendable (to X) if there exists a countable α-shading $\mathcal{H}$ of X such that $\mathcal{H}|Y$ is a precise α-refinement of $\mathcal{G}$.

<u>Proposition 2.5:</u> Let (X,T) be an α-Hausdorff α-Lindelof F-space. Then every finite α-shading of any F-subspace Y of (X,T) is countably extendable.

<u>Proof.</u> Let $\mathcal{G}$ be a finite α-shading of an F-subspace Y of an α-Hausdorff α-Lindeloff F-space (X,T). Then there is a finite $\mathcal{H} \subseteq T$ such that $\mathcal{H}|Y = \mathcal{G}$. Let $y \in Y$ by definition of α-Hausdorff, for each $x \in X\setminus Y$ there are $u_x, v_y \in T$ such that $u_x(x) > \alpha$, $v_y(y) > \alpha$ and $v_y \wedge u_x = 0$. So $\{u_x\}_{x\in X\setminus Y}$ is a family of open F-sets in X and $\{\mathcal{H}, U_x\}_{x\in X\setminus Y}$ is an α-shading of X. By definition of α-Lindelof I.2.7, there exists a countable α-subshading W of $\{\mathcal{H}, U_x\}_{x\in X\setminus Y}$ and $W|Y$ is a precise α-refinement of $\mathcal{G}$. □

<u>Proposition 2.6:</u> Let (X,T) be an α-Lindelof F-space. Then every finite α-shading of an α-closed F-subspace is countably extendable.

Proof. Let Y be an α-closed F-subspace of (X,T) and let $\mathcal{G}$ be a finite α-shading of Y. For each $x \in X\backslash Y$, there is an open F-set u_x in X, such that $u_x(x)>\alpha$ and $\mu(Y) \wedge u_x = 0$ from definition I.2.12. So $\{\mathcal{G}, u_x\}_{x \in X\backslash Y}$ is an α-shading of X and hence the proof. is completed by the same way of proposition 2.5. □

Proposition 2.7: Let (X,T) be a countably α-compact F-space and let Y be an F-subspace such that every finite α-shading of Y is countably extendable. Then:

$$F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X.$$

Proof. Let $F\text{-}\dim_\alpha X \leq n$, and $\{f_i\}_{i=1}^{k}$ be an α-shading of Y. Suppose that every finite α-shading of Y is countably extendable. Then there is a countable α-shading $\{g_i\}_{i=1}$ of X such that $g_i|Y \leq f_i$ for $i = 1,2,\ldots,k$. Since (X,T) is countably α-compact, there is a finite α-subshading $\{g_i\}_{i=1}^{n}$ of X ($n > k$ say) and so there is an α-shading $\{h_i\}_{i=1}^{n}$ of X whose order $\leq n$ and $h_i \leq g_i$ for $i = 1,2,\ldots,n$, from proposition II.1.1. Hence $h_i|Y \leq g_i|Y \leq f_i$ for $i = 1,2,\ldots,k$ and so $\{h_i\}_{i=1}^{k}$ is α-refinement of $\{f_i\}_{i=1}^{k}$ of order $\leq n$. i.e. $F\text{-}\dim_\alpha Y \leq n$. □

Proposition 2.8: Let (X,T) be countably α-compact α-Lindelof F-space and let Y be an α-closed F-subspace of (X,T), Then:

$$F\text{-}\dim_\alpha Y \leq F\text{-}\dim_\alpha X.$$

Proof. Follows from propositions 2.6 and 2.7 or from the observation that α-Lindelof and countably α-compact is α-compact and proposition 2.1. □

The subset theorem for $F_{\alpha_0}\text{-}\dim_\alpha$ depends on the position of α with respect to the fixed α_0.

Proposition 2.9: Let Y be a closed F-subspace of an F-space (X,T). Then:

$$F_{\alpha_0}\text{-dim}_{\alpha} Y \leq F_{\alpha_0}\text{-dim}_{\alpha} X.$$

For $\alpha \in [\alpha_0, 1)$.

Proof. Let $F_{\alpha_0}\text{-dim}_{\alpha} X \leq n$, $\alpha \in [\alpha_0, 1)$ and let $\{f_i\}_{i=1}^{k}$ be an α-shading of Y. Then there is a family $\{g_i\}_{i=1}^{k}$ of open F-sets in X such that $g_i | Y = f_i$ for $i = 1,2,\ldots,k$, and so $\{g_i, \mu(X\setminus Y)\}_{i=1}^{k}$ is a finite α-shading of X. By assumption there is an β-refinement V of $\{g_i, \mu(X\setminus Y)\}_{i=1}^{k}$ for some $\beta \in [0, \alpha_0)$ whose order $\leq n$. Hence $\{V|Y\}$ is β-refinement of $\{f_i\}_{i=1}^{k}$ of order $\leq n$. So $F_{\alpha_0}\text{-dim}\ Y \leq n$. □

Remark: Propositions 2.1, 2.2 and 2.3 hold for $F_{\alpha_0}\text{-dim}_{\alpha}$ only if $\alpha \in [\alpha_0, 1)$.

We wish to determine the dimension of an F-space in terms of the dimension of some of its F-subspaces. The following two propositions give the sum theorem for an F-space satistying some conditions in terms of a finite F-subspaces.

We observe that if Y_1 and Y_2 are α-closed in an F-space (X,T) such that $Y_1 \cap Y_2 \neq \emptyset$, then $Y_2 \setminus Y_1$ is α-closed in Y_2.

Proposition 2.10: Let (X,T) be an α-compact F-space such that $X = \bigcup_{i=1}^{k} Y_i$ where for each $i = 1,2,\ldots,k$, Y_i is α-closed F-subspace of (X,T) with F-dim $Y_i \leq n$. Then $F\text{-dim}_{\alpha} X \leq n$.

Proof. Let $\mathcal{G}$ be a finite α-shading of X and let $X = Y_1 \cup Y_2$ $(i = 1,2)$ such that $F\text{-dim}_{\alpha} Y_1 \leq n$ and $F\text{-dim}_{\alpha} Y_2 \leq n$. Then $\{\mathcal{G}|Y_1\}$ $\{\mathcal{G}|Y_2\}$ are finite α-shadings of Y_1 and Y_2 respectively. By assumption there are α-refinements V_1, V_2 of $\{\mathcal{G}|Y_1\}$ and $\{\mathcal{G}|Y_2\}$ with order of $V_1 \leq n$, and order of $V_2 \leq n$. Now if $Y_1 \cap Y_2 = \emptyset$, then $\{V_1, V_2\}$ is an α-refinement of $\mathcal{G}$ whose order $\leq n$ and so F-$\dim_\alpha X \leq$.

If $Y_1 \cap Y_2 \neq \emptyset$, then $Y_2 \setminus Y_1$ is α-closed in Y_2, let us call it Y_3, so F-$\dim_\alpha Y_3 \leq n$, $Y_1 \cap Y_3 = \emptyset$ and $Y_1 \cup Y_3 = X$. Hence by the same argument there is an α-refinement $\{V_1, V_3\}$ of whose order $\leq n$ and so F-$\dim_\alpha X \leq n$. □

<u>Proposition 2.11:</u> Let (X,T) be an α-compact α-Hausdorff F-space such that $X = \bigcup_{i=1}^{k} Y_i$ where Y_i is α-compact F-subspace for each i, with F-$\dim_\alpha Y_i \leq n$. Then F-$\dim_\alpha X \leq n$.

<u>Proof:</u> Since each α-compact F-subspace of an α-Hausdorff is α-closed proposition I.2.9. Then the proof. follows at once from proposition 2.10. □

3. LOCAL α-COVERING DIMENSION

The local α-covering dimension of F-spaces is defined and studied in this section. From now on, by g^+ for $g \in I^X$, $\alpha \in [0,1)$ we mean the subset $\{x \in X : g(x) > \alpha\}$.

<u>Definition 3.1:</u> Let (X,T) be an F-space. Then the local - covering dimension of (X,T)- denoted by loc F-$\dim_\alpha X$ - is defined as follows:

(I) loc F-$\dim_\alpha X = -1$ if $X = \emptyset$

(II) loc F-$\dim_\alpha X \leq n$ $(n \geq 0)$ if for every $x \in X$, there exists an open F-set g in X, with $g(x) > \alpha$ such that F-$\dim_\alpha g^+ \leq n$.

(III) loc $F\text{-}\dim_\alpha X = \infty$ if loc $F\text{-}\dim_\alpha X \leq n$ is false for every integer n.

Proposition 3.1: Let (X,T) be an α-Hausdorff F-space. Then the following statements are equivalent:

(I) loc $F\text{-}\dim_\alpha X \leq n$

(II) Every α-shading of X has an α-refinement $\mathcal{H}$ with $F\text{-}\dim_\alpha h^+ \leq n$ for every $h \in \mathcal{H}$.

Proof. (I)$\Longrightarrow$(II) Let loc $F\text{-}\dim_\alpha X \leq n$, and U be an α-shading of X. Then by assumption for each $x \in X$, there exists an open F-set g_x in X such that $g_x(x) > \alpha$ and $F\text{-}\dim_\alpha g^+ \leq n$. By definition of α-shading of X I.2.3, for each $x \in X$ there exists $u_x \in U$ such that $U_x(x) > \alpha$. Put $h_x = g_x \wedge u_x$, then h_x is open F-set in X, $h_x(x) > \alpha$ and $h_x \leq u_x$. If $\mathcal{H} = \{h_x : x \in X\}$, then $\mathcal{H}$ is an α-refinement of U. Since $u_x = g_x \wedge u_x$ then clearly $h_x^+ \leq g_x^+$, but g_x^+ is α-Hausdorff F-subspace of (X,T) with $F\text{-}\dim_\alpha g_x^+ \leq n$, so by proposition 2.2 we get $F\text{-}\dim_\alpha h_x^+ \leq n$; that is $F\text{-}\dim_\alpha h^+ \leq n$ for $h \in \mathcal{H}$ as required.

(II)$\Longrightarrow$(I) Assume (II) and let x be any point of X. Then for every $y \in X \setminus \{x\}$ there are open F-sets u_y, v_x such that $u_y(y) > \alpha$, $v_x(x) > \alpha$ and $u_y \wedge v_x = 0$ from definition I.2. So $\{u_y, v_x\}_{y \in X \setminus \{x\}}$ is an α-shading of X and then by assumption there is an α-refinement $\mathcal{H}$ of $\{u_y, v_x\}_{y \in X \setminus \{x\}}$ such that $F\text{-}\dim_\alpha h^+ \leq n$ for every $h \in \mathcal{H}$; that is loc $F\text{-}\dim_\alpha X \leq n$. □

The following propositions give the relation betwen the α-covering dimension and the local α-covering dimension of F-spaces.

Proposition 3.2: Let (X,T) be α-Hausdorff α-compact F-space.

Then:

$$\text{loc } F\text{-dim}_\alpha X \leq F\text{-dim}_\alpha X.$$

Proof. Let $F\text{-dim}_\alpha X \leq n$. Let $x \in X$. Then for each $y \in X\setminus\{x\}$ there are open F-sets u_y, v_x in X such that $u_x(y) > \alpha$, $v_x(x) > \alpha$ and $u_y \wedge v_x = 0$. Hence the family $\{u_y, v_x\}_{y\in X\setminus\{x\}}$ is an α-shading of X and by α-compactness of (X,T) there is a finite numbers of points $y_1, y_2, \ldots, y_k \in X\setminus\{x\}$ (say) such that $\{u_{y_i}, v_i\}_{i=1}^{k}$ is α-subshading of $\{u_y, v_x\}_{y\in X\setminus\{x\}}$ i.e. $\{u_{y_i}, v_x\}_{i=1}^{k}$ is a finite α-shading of X and then by assumption there is an α-refinement $\mathcal{H}$ of $\{u_{y_i}, v_x\}_{i=1}^{k}$.

Now for each $h \in \mathcal{H}$, h^+ is an F-subspace of an α-Hausdorff F-space (X,T) so by proposition 2.2 $F\text{-dim } h^+ \leq n$ which implies that loc F-dim $X \leq n$ from proposition 3.1. □

Proposition 3.3: Let (X,T) be a perfectly F-normal α-Hausdorff F-space. Then:

$$\text{loc } F\text{-dim}_\alpha X \leq F\text{-dim}_\alpha X.$$

Proof. Let $F\text{-dim}_\alpha X \leq n$ and let $\mathcal{H}$ be an α-shading of X. By perfectly F-normality for each $h \in \mathcal{H}$, $h = \bigvee_{i=1}^{\infty} f_i$ where f_i is closed F-set in X for each $i = 1,2,\ldots$. So $f_i \leq h$ for each i and hence there is an open F-set g_i such that $f_i \leq g_i \leq \text{cl } g_i \leq h$. Let $g = \bigvee_{i=1}^{\infty} g_i$. Then g is open F-set in X and $g \leq h$. Put $\mathcal{G} = \{g \in T : g \leq h \text{ for each } h \in \mathcal{H}\}$, then $\mathcal{G}$ is an α-refinement of $\mathcal{H}$ and for each $g \in \mathcal{G}$ we have $F\text{-dim}_\alpha g^+ \leq n$ from proposition 2.2.

Hence by proposition 3.1 loc $F\text{-dim}_\alpha X \leq n$.

<u>Definition 3.2:</u> An F-space (X,T) is called locally α-closed if for each $x \in X$ there exists an open F-set h in X such that $h(x) > \alpha$ and h^+ is α-closed.

Next we give some conditions under which loc F-dim$_\alpha$ and F-dim$_\alpha$ coincide.

<u>Proposition 3.4:</u> Let (X,T) be an α-compact locally α-closed F-space. Then:

$$\text{loc F-dim}_\alpha X = \text{F-dim}_\alpha X.$$

<u>Proof.</u> $\Longrightarrow$ Let F-dim$_\alpha X \leq n$. By definition 3.2 for each $x \in X$, there exists an open F-set h in X such that $h(x) > \alpha$ and h^+ is α-closed. Since F-dim$_\alpha X \leq n$ then by proposition 2.1 we get F-dim$_\alpha h^+ \leq n$. Hence loc F-dim$_\alpha X \leq n$.

$\Longleftarrow$ Let loc F-dim$_\alpha X \leq n$. Then for each $x \in X$ there is an open F-set h_x in X such that $h_x(x) > \alpha$ and F-dim$_\alpha h_x^+ \leq n$. But the family $\{h_x\}_{x\in X}$ is an α-shading of X which is α-compact so there is a finite $x_1, x_2, \ldots, x_k \in X$ such that $\{h_{x_i}\}_{i=1}^{k}$ is a finite α-shading of X with F-dim$_\alpha h_{x_i}^+ \leq n$ for $i = 1,2,\ldots,k$, and clearly $X = \bigcup_{i=1}^{k} h_{x_i}^+$. Since (X,T) is locally α-closed then each $h_{x_i}^+$ is α-closed, so by finite sum theorem proposition 2.10 F-dim$_\alpha X \leq n$. $\square$

<u>Proposition 3.5:</u> Let (X,T) be an α-Hausdorff F-space and let Y be an F-subspace of (X,T). Then:

$$\text{loc F-dim}_\alpha Y \leq \text{loc F-dim}_\alpha X.$$

<u>Proof.</u> Let loc F-dim$_\alpha X \leq n$ and let $\mathcal{H}$ be an α-shading of Y. Then there is a family $\mathcal{G}$ of open F-sets in X such that $\mathcal{G}|Y = \mathcal{H}$. Let $y \in Y$. Then for any $x \in X\setminus Y$, there are open F-sets u_x, v_y in

X such that $u_x(x)>\alpha$, $v_y(y)>\alpha$ and $u_x \wedge v_y = 0$. Hence $\{\mathcal{G}, u_x\}_{x\in X\setminus Y}$ is an α-shading and hence by assumption and proposition 3.1 there is an α-refinement $\mathcal{W}$ of $\{\mathcal{G}, u_x\}_{x\in X\setminus Y}$ such that $F\text{-}\dim_\alpha w^+ \leq n$ for any $w \in \mathcal{W}$. Then $\mathcal{W}|Y$ is an α-refinement $\mathcal{H}$. If we put $\mathcal{W}^* = \mathcal{W}|Y$, then by proposition 2.2 we have $F\text{-}\dim_\alpha w^{*+} \leq n$ for any $w^* \in \mathcal{W}^*$. Hence by proposition 3.1, loc $F\text{-}\dim_\alpha Y \leq n$. □

The next proposition gives the finite sum theorem of local α-covering dimension.

<u>Proposition 3.6:</u> Let (X,T) be an α-compact locally α-closed F-space. If $X = Y \cup Z$ with loc $F\text{-}\dim_\alpha Y \leq n$ and loc $F\text{-}\dim_\alpha Z \leq n$, then loc $F\text{-}\dim_\alpha X \leq n$.

<u>Proof.</u> Let loc $F\text{-}\dim_\alpha Y \leq n$ and loc $F\text{-}\dim_\alpha Z \leq n$. For each $x \in X$ by definition 3.2 there exists an open F-set h such that $h(x)>\alpha$ and h^+ is α-closed. Hence by proposition I.2.8 h^+ is α-compact.

Now if $x \in Y\setminus Z$, then $(h|(Y\setminus Z))(x)>\alpha$ and since loc $F\text{-}\dim_\alpha Y \leq n$, so $F\text{-}\dim\,(h|(Y\setminus Z))^+ \leq n$. If $x \in Z\setminus Y$, then $(h|(Z\setminus Y))(x) >\alpha$ and from the assumption $F\text{-}\dim_\alpha (h|(Z\setminus Y))^+ \leq n$. If $x \in Y \cap Z$, then $(h|(Y\cap Z))(x)>\alpha$ and $F\text{-}\dim_\alpha (h|(Y\cap Z))^+ \leq n$. Clearly in either case we have $(h|(Y\setminus Z))^+$, $(h|(Z\setminus Y))^+$ and $(h|(Y\cap Z))^+ \subseteq h^+$, which implies that each of them is α-closed in h^+.

Now we have to show that $F\text{-}\dim_\alpha h^+ \leq n$. If h^+ equals one of the above three subsets then clearly $F\text{-}\dim_\alpha h^+ \leq n$. If $h^+ = (h|(Z\setminus Y))^+ \cup (h|(Y\cap Z))^+$ or $h^+ = (h|(Y\setminus Z))^+ \cup (h|(Y\cap Z))^+$, then in both cases by the (finite sum theorem) proposition 2.10 we have $F\text{-}\dim_\alpha h^+ \leq n$. i.e. for each $x \in X$ there is an open F-set h such that $h(x)>\alpha$ and $F\text{-}\dim_\alpha h^+ \leq n$. Therefore loc $F\text{-}\dim_\alpha X \leq n$. □

Chapter III:

COVERING DIMENSION AND MODIFICATION OF F-SPACES

The relation between topological spaces and F-spaces was studied by Lowen [10] [11] where the two functions w and i were used as follows:

If $\mathcal{F}(X)$ is the set of all topologies on X, and F(X) is the set of all F-topologies on X then:

a) $i : F(X) \longrightarrow \mathcal{F}(X)$

$T \longrightarrow i(T)$

i(T) is the initial topology for the family of functions T and I with the usual topology.

b) $w : \mathcal{F}(X) \longrightarrow F(X)$

$R \longrightarrow w(R)$

w(R) is the family of lower semicontinuous functions from (X,R) to I with the usual topology.

From a) we can associate with each F-space (X,T) a topological space (X, i(T)) called the modification of (X,T) or simply a modificated topology. This concept reduces some properties of F-spaces to a topological properties i.e. if (X,T) is α-compact, then (X, i(T)) is compact.

We use this concept to benifit from the well known results of the covering dimension of topological spaces (in particular to get a product theorem for F-spaces), as well as from some other topological properties like Čechstone compactification.

Throughout the rest of this work we mean by a strong α-property an α-property for all $\alpha \in [0,1)$.

1. F-dim

<u>Definition 1.1:</u> Let (X,T) be an F-space. Then F-dim $X \leq n$ if F-$\dim_{\alpha} X \leq n$ for all $\alpha \in [0,1)$.

<u>Example 1.1:</u> Let (X,R) be a topological space with dim $X \leq n$.

For $u \in R$, let $\mu(u)$ be the characteristic function of u. Put $T = \{\mu(u)\}_{u \in R}$. Then (X,T) is an F-space. We claim that F-dim $X \leq n$. To see that let $\{\mu(u_i)\}_{i=1}^{k}$ be a finite α-shading of X for any $\alpha \in [0,1)$. Then $\{u_i\}_{i=1}^{k}$ is an open cover of X, and so there is a finite open cover $\{v_i\}_{i=1}^{k}$ of X of order $\leq n$ and $v_i \leq u_i$ for $i = 1,2,\ldots,k$.

Let $f_i \in I^X$ be defined by

$$f_i(x) = 1 \quad \text{if} \quad x \in v_i \qquad \text{for each } i = 1,2,\ldots,k$$
$$= 0 \quad \text{if} \quad x \notin v_i$$

Then $\{f_i\}_{i=1}^{k}$ is an α-shading of X for all $\alpha \in [0,1)$ such that $f_i \leq \mu(u_i)$ for $i = 1,2,\ldots,k$, and order of $\{f_i\}_{i=1}^{k}$ n. Hence F-dim $X \leq n$.

<u>Definition 1.2:</u> An F-space (X,T) is called α-permissible if for each $V \subset T$, V contains at least two members, there is $\alpha \in [0,1)$ such that V is an α-shading of X.

<u>Proposition 1.1:</u> Let (X,T) be an α-permissible F-space. If F-dim $X = 0$, then (X,T) is weakly F-normal space.

<u>Proof.</u> Let F-dim $X = 0$ and let G_1, G_2 be two disjoint closed

F-sets in X. Then $\{co\ G_1,\ co\ G_2\} \subset T$. Since (X,T) is α-permissible, so there exists $\alpha \in [0,1)$ such that $\{co\ G_1,\ co\ G_2\}$ is α-shading of X. But F-dim X = 0 implies for all $\alpha \in [0,1)$, F-dim$_\alpha$X = 0 by definition 1.1 and hence there is an α-shading $\{f_1, f_2\}$ of X such that $f_1 \leq co\ G_2$, $f_2 \leq co\ G_1$ and $f_1 \wedge f_2 = 0$ i.e. $f_1 \leq co\ f_2$. Therefore $G_1 \leq co\ f_2$ so $G_1 \leq co(co\ f_1)$ and then $G_1 \leq f_1$. Similarly $G_2 \leq f_2$. □

<u>Corollary 1.1:</u> Let (X,T) be an α-permissible F_c-space. If F-dim X = 0, then (X,T) is F-normal space. □

We introduce a stronger definition of zero dimension of F-spaces as follows:

SF-dim$_\alpha$X = 0 if for every finite α-shading of X there is an α-refinement consisting of open and closed disjoint F-sets. Clearly if SF-dim$_\alpha$X = 0 then F-dim$_\alpha$X = 0 for any F-space (X,T).

<u>Definition 1.3:</u> Let G_1, G_2 be two disjoint closed F-sets of an F-space (X,T). Then G_1, G_2 are called strongly separated in X if there is an open and closed F-set E such that $G_1 \leq E$ and $E \leq co\ G_2$.

<u>Proposition 1.2:</u> Let (X,T) be an α-permissible F-space. If SF-dim X = 0, then each pair of disjoint closed F-sets is strongly separated in X.

<u>Proof.</u> Let SF-dim X = 0 and suppose that G_1, G_2 are disjoint closed F-sets. Then $\{co\ G_1,\ co\ G_2\}$ is an α-shading of X for some $\alpha \in [0,1)$ because (X,T) is α-permissible. Since SF-dim X = 0, so there is an α-shading $\{f_1, f_2\}$ of X such that f_1, f_2 are open and closed, $f_1 \wedge f_2 = 0$ and $f_1 \leq co\ G_1$, $f_2 \leq co\ G_2$. So

$G_1 \leq co\ f_1$ implies $G_1 \leq co(co\ f_2)$ i.e. $G_1 \leq f_2 \leq co\ G_2$. □

Definition 1.4: An F-space (X,T) is said to be:

(I) Connected if and only if the only open and closed F-sets are $\bar{0},\bar{1}$. An F-set g is connected if g_0 is connected as an F-subspace.

(II) Disconnected if it is not connected.

(III) Totally disconnected if there is no connected F-set containing more that one F-point.

Proposition 1.3: Let (X,T) be an α-permissible FT_1-space. If SF-dim X = 0, then X is totally disconnected.

Proof. Let SF-dim X = 0 and let g be a connected F-set of X consisting of two F-points p,q, $p \neq q$. Then by proposition 1.2 there is an open and closed F-set E such that $p \in E$ and $E \leq co\ q$. Hence $E|g_0$ is open and closed in g_0 and $E|g_0$ is not $\bar{0}_{g_0}$ and not $\bar{1}_{g_0}$. i.e. g is not connected contradicting our assumption that g is connected. Therefore (X,T) is totally disconnected. □

We return to the subset theorem given in section 2 of the previous chapter, but this time we consider the covering dimension of F-spaces instead of the α-covering dimension.

Proposition 1.4: Let Y be a closed F-subspace of an F-space (X,T). Then:

$$\text{F-dim } Y \leq \text{F-dim } X.$$

Proof. Let F-dim $X \leq n$, and let U be a finite α-shading of Y for all $\alpha \in [0,1)$. There is a family W of open F-sets in X such that for every $w \in W$, $w|Y \in U$, and $\{W, \mu(X \setminus Y)\}$ is a finite α-shading of X for all $\alpha \in [0,1)$. Since F-dim $X \leq n$, then by

definition 1.1. there is an α-refinement V of $\{W, \mu(X\ Y)\}$ whose order $\leq n$ for all $\alpha \in [0,1)$. So $\{V|Y\}$ is α-refinement of $W|Y = U$ whose order $\leq n$ for all $\alpha \in [0,1)$.
Hence F-dim Y $\leq n$. □

Proposition 1.5: Let Y be an F-subspace of a strong α-compact, Hausdorff F-space (X,T). Then:

$$\text{F-dim } Y \leq \text{F-dim } X$$

Proof. Since (X,T) is strong α-compact Hausdorff, then by proposition 2.8 [6] $\mu(X \setminus Y)$ is open F-set in X and it follows from proposition 1.4 that F-dim Y $\leq$ F-dim X.

Proposition 1.6: Let Y be an F-subspace of a strong α-Hausdorff F-space (X,T). Then

$$\text{F-dim } Y \leq \text{F-dim } X$$

Proof. Follows from proposition II.2.2 for all $\alpha \in [0,1)$. □

Proposition 1.7: Let (X,T) be a strong countably α-compact F-space and Y be an F-subspace of (X,T). If each finite α-shading of Y ($\forall\ \alpha \in [0,1)$) is countably extendable, then:

$$\text{F-dim } Y \leq \text{F-dim } X$$

Proof. Follows from proposition II.2.6 for all $\alpha \in [0,1)$. □

Proposition 1.8: Let (X,T) be a strong α-compact such that $X = \bigcup_{i=1}^{k} Y_i$ where for each i, Y_i is strong α-closed F-subspace with F-dim $Y_i \leq n$. Then F-dim X $\leq n$.

Proof. Apply proposition II.2.10 for each $\alpha \in [0,1)$. □

Proposition 1.9: Let (X,T) be a strong α-Hausdorff F-space that $X = \bigcup_{i=1}^{k} Y$, let each Y_i be strong α-compact F-subspace with

F-dim $Y_i \leq n$. Then F-dim $X \leq n$. □

Proof. Apply proposition II.2.11 for each $\alpha \in [0,1)$. □

Definition 1.5: Let Y be an F-subspace of an F-space (X,T). Then (X,T) is called conservative if for any two F-sets f,g in Y with $f \leq g$, then $f^* \leq g^*$ in X, where $f^*|Y = f$, $g^*|Y = g$.

Proposition 1.10: Let Y be a closed F-subspace of a conservative F-normal space (X,T). Then Y is F-normal.

Proof. Let f be an open F-set in Y and g be a closed F-set in Y such that $g \leq f$. So there are a closed F-set g^* in X, and an open F-set f^* in X such that $g = g^*|Y$ and $f = f^*|Y$. Since (X,T) is conservative then by the above definition $g^* \leq f^*$ in X, but by assumption (X,T) is F-normal, so there is an open F-set h in X with $g^* \leq h \leq \mathrm{cl}\ h \leq f^*$ and hence $g \leq h|Y \leq \mathrm{cl}\ h|Y \leq f$. Thus Y is F-normal. □

Proposition 1.11: Let (X,T) be a conservative F_c-normal space and let Y be a closed F-subspace of (X,T) such that F-dim $Y \leq n$. If $\{f_i\}_{i=1}^k$ is a family of open F-sets in X with $\{f_i|Y\}_{i=1}^k$ is an α-shading of Y for some $\alpha \in [0,1)$. Then there exists a family $\{g_i\}_{i=1}^k$ of open F-sets in X such that cl $g_i \leq f_i$ and order of cl $\{g_i\}_{i=1}^k \leq n$.

Proof. Let $\alpha \in [0,1)$ such that $\{f_i|Y\}_{i=1}^k$ is an α-shading of Y. By proposition 1.10, Y is F_c-normal and hence by proposition II.1.4 there exists an α-coshading $\{h_i\}_{i=1}^k$ of Y such that $h_i \leq f_i|Y$ for $i = 1,2,\ldots,k$ and order of $\{h_i\}_{i=1}^k \leq n$. Since Y is closed F-subspace of (X,T) so h_i is closed F-set in X for each

$i = 1,2,\ldots,k$, and $h_i \leq f_i$. By F-normality of (X,T) there are open F-sets g_i in X such that $h_i \leq g_i \leq \mathrm{cl}\ g_i \leq f_i$ for each $i = 1,2,\ldots,k$. In fact $\{\mathrm{cl}\ g_i\}_{i=1}^{k}$ is a swelling of $\{h_i\}_{i=1}^{k}$ and hence the order of $\{\mathrm{cl}\ g_i\}_{i=1}^{k} \leq n$. □

<u>Proposition 1.12:</u> Let (X,T) be a conservative F_c-normal space and let Y be a closed F-subspace of X such that F-dim $Y \leq n$. If F-dim $Z \leq n$ for any closed F-subspace Z of (X,T) disjoint from Y, then F-dim $X \leq n$.

<u>Proof.</u> Let F-dim $Z \leq n$ and let $\{f_i\}_{i=1}^{k}$ be an α-shading of X for any $\alpha \in [0,1)$, then $\{f_i|Y\}_{i=1}^{k}$ is an α-shading of Y and so by proposition 1.11 there is a family $\{g_i\}_{i=1}^{k}$ of open F-sets in X such that $g_i \leq f_i$, order $\{g_i\}_{i=1}^{k} \leq n$ and $\{g_i|Y\}_{i=1}^{k}$ is an α-shading of Y. Put $Q = \mathrm{co}\left[\bigvee_{i=1}^{k} g_i \vee \bigvee_{t \in T} \{t : t > f_i \ \forall \ i = 1,2,\ldots,k\}\right]$

then Q is closed F-set in X and $Q \leq \mu(X\setminus Y)$. By F-normality of (X,T) there is an open F-set h such that $Q \leq h \leq \mathrm{cl}\ h \leq \mu(X\setminus Y)$. Let $(\mathrm{cl}\ h)_0$ be the support of cl h, then $(\mathrm{cl}\ h)_0$ is a closed F-subspace of X which is disjoint from Y. Now $\{f_i|(\mathrm{cl}\ h)_0\}_{i=1}^{k}$ is an α-shading of $(\mathrm{cl}\ h)_0$, where from hypothesis F-dim $(\mathrm{cl}\ h_0) \leq n$, so there is an α-shading $\{m_i\}_{i=1}^{k}$ of $(\mathrm{cl}\ h)_0$ such that $m_i \leq f_i|(\mathrm{cl}\ h)_0$ for $i = 1,2,\ldots,k$ and order of $\{m_i\}_{i=1}^{k} \leq n$.

Let us define $d_i \in I^X$ for each $i = 1,2,\ldots,k$ by:

$$d_i(x) = m_i(x) \text{ if } x \in (\mathrm{cl}\ h)_0$$

$$= g_i(x) \text{ if } x \notin (\mathrm{cl}\ h)_0$$

So $\{d_i\}_{i=1}^{k}$ is an α-shading of X such that $d_i \leq f_i$ for $i = 1,2,\ldots,k$ and order of $\{d_i\}_{i=1}^{k} \leq n$.

Hence F-$\dim_\alpha X \leq n$. Apply this method for every $\alpha \in [0,1)$, we get F-dim $X \leq n$. □

Proposition 1.13: Let (X,T) be a conservative F_c-normal. If $X = Y \cup Z$ where Y, Z are closed F-subspaces with F-dim $Y \leq n$, and F-dim $Z \leq n$, then F-dim $X \leq n$.

Proof. Let $X = Y \cup Z$ and let Q be a closed F-subspace of (X,T) disjoint from Y. Then Q is closed F-subspace of Z and by proposition 1.4 F-dim $Q \leq n$. Hence by proposition 1.12 it follows that F-dim $X \leq n$. □

We end this section by returing to the local covering dimension introduced in section 3 of the previous chapter.

Definition 1.6: Let (X,T) be an F-space. Then loc F-dim $X \leq n$ if loc F-$\dim_\alpha X \leq n$ for all $\alpha \in [0,1)$.

Proposition 1.14: Let (X,T) be strong α-Hausdorff strong α-compact F-space. Then:

$$\text{loc F-dim } X \leq \text{F-dim } X$$

Proof. Follows from proposition II.3.2 for each $\alpha \in [0,1)$. □

Proposition 1.15: If (X,T) is perfectly F-normal strongly α-Hausdorff F-space, then:

$$\text{loc F-dim } X \leq \text{F-dim } X$$

Proof. Apply proposition II.3.3 for each $\alpha \in [0,1)$. □

Proposition 1.16: If (X,T) is strongly α-Hausdorff F-space and Y is an F-subspace of (X,T), then:

$$\text{loc F-dim } Y \leq \text{loc F-dim } X$$

Proof. Apply proposition II.3.5 for each $\alpha \in [0,1)$. □

2. MODIFICATION OF F-SPACES:

The modification of an F-space (X,T) is a topological space $(X,i(T))$, where the family $\{t^{-1}(\alpha,1] : t \in T, \alpha \in I\}$ is a subbase for $i(T)$, and if for all $\alpha \in [0,1)$ we put $i_\alpha(T) = \{t^{-1}(\alpha,1] : t \in T\}$. Then $i_\alpha(T)$ is a topology on X (see Lowen [12]) and

$$i(T) = \vee\{i_\alpha(T) : \alpha \in [0,1)\}.$$

It is known that the set of all lower semicontinuous maps $w(\mathcal{F})$ from a topological space $(X,\mathcal{F})$ into I equipped with the usual topology forms an F-space $(X,w(\mathcal{F}))$ called the induced F-space.

An F-space (X,T) is called topologically generated if there is a topology $\mathcal{F}$ on X such that $T = w(\mathcal{F})$ or equivalently if $T = w(i(T))$.

Throughout this section and the rest of our work, $F\text{-dim}_\alpha X_T$ means the α-covering dimension of the F-space (X,T), and $\dim X_{\mathcal{F}}$ means the covering dimension of X with the topology $\mathcal{F}$ on X.

Our main purpose in this section is to find a relation between the covering dimension of an F-topology on a set X and the covering dimension of a topology on the same set X and vice versa.

Lemma 2.1: Let (X,T) be an F-space. Then (X,T) is α-compact (countably α-compact) [α-Lindelof] if and only if $(X,i_\alpha(T))$ is compact (countably compact) [Lindelof].

Proof. The family $u \subseteq T$ is an α-shading of X if and only if $\bigcup_{t \in u} t^{-1}(\alpha,1] = X$ completes the proof. □

Lemma 2.2: If (X,T) is an α-Hausdorff F-space, then $(X, i_\alpha(T))$ is Hausdorff space.

Proof. Let (X,T) be an F-space and let $x \neq y$. Then from the definition of α-Hausdorff, there are u,v open F-sets in X such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \wedge v = 0$. Hence $u^{-1}(\alpha,1]$, $v^{-1}(\alpha,1]$ are disjoint open sets in $i_\alpha(T)$ such that $x \in u^{-1}(\alpha,1], y \in V^{-1}(\alpha,1]$ i.e. $(X, i_\alpha(T))$ is Hausdorff. □

The following proposition gives the relation between $F\text{-dim}_\alpha$ of an F-space (X,T) and dim of the modificated topology $(X, i_\alpha(T))$.

Proposition 2.1: Let (X,T) be an F-space and $(X, i_\alpha(T))$ be the modificated topology on X. Then:

$$F\text{-dim}_\alpha X_T = \dim X_{i_\alpha(T)}$$

Proof. $\Rightarrow$ Let $F\text{-dim}_\alpha X_T \leq n$, and let $\{u_i\}_{i=1}^{k}$ be a finite $i_\alpha(T)$-open cover of X. Since $i_\alpha(T) = \{t^{-1}(\alpha,1] : t \in T\}$ so for each $i \in \{1,2,\ldots,k\}$, $u_i = t_i^{-1}(\alpha,1]$ i.e. $x \in u_i$ if and only if $t_i(x) > \alpha$ which implies that $\{t_i\}_{i=1}^{k}$ is an α-shading of X. By assumption and proposition II.1.1, there is an α-shading $\{S_i\}_{i=1}^{k}$ of X such that $S_i \leq t_i$ for $i = 1,2,\ldots,k$ and order of $\{S_i\}_{i=1}^{k} \leq n$.

Let $v_i = \{x \in X : s_i(x) > \alpha\}$ for each $i = 1,2,\ldots,k$. Then v_i is $i_\alpha(T)$-open for each $i = 1,2,\ldots,k$,

$$\bigcup_{i=1}^{k} v_i = \bigcup_{i=1}^{k} \{S_i^{-1}(\alpha,1] = X, \quad v_i \subseteq u_i \quad \text{and}$$

order of $\{v_i\}_{i=1}^{k} \leq n$. i.e. $\{v_i\}_{i=1}^{k}$ is a finite $i_\alpha(T)$-open cover of X whose order $\leq n$, so $\dim X_{i_\alpha(T)} \leq n$

$\Leftarrow$ Let $\dim_{i_\alpha} X_{(T)} \leq n$, and let $\{f_i\}_{i=1}^{k}$ be an α-shading of X for $\alpha \in [0,1)$. For each $i \in \{1,2,\ldots,k\}$, let $u_i = f_i^{-1}(\alpha,1]$, so u_i is $i_\alpha(T)$-open and $\bigcup_{i=1}^{k} u_i = \bigcup_{i=1}^{k} \{f_i^{-1}\ f_i^{-1}(\alpha,1]\} = X$. i.e. $\{u_i\}_{i=1}^{k}$ is a finite $i_\alpha(T)$-open cover of X. Since $\dim X_{i_\alpha(T)} \leq n$, there exists a finite $i_\alpha(T)$-open cover $\{v_i\}_{i=1}^{k}$ of X such that $v_i \subseteq u_i$ for $i = 1,2,\ldots,k$ and order of $\{v_i\}_{i=1}^{k} \leq n$.

Let us define $g_i : X \longrightarrow I$ as follows:

$$g_i(x) \begin{cases} > \alpha & \text{if } x \in v_i \\ = 0 & \text{if } x \notin v_i \end{cases} \qquad \text{for } i=1,2,\ldots,k$$

Hence $\{g_i\}_{i=1}^{k}$ is α-shading of X, $g_i \leq f_i$ for each $i=1,2,\ldots,k$ and clearly order of $\{g_i\}_{i=1}^{k} \leq n$. So $F\text{-}\dim_\alpha X_T \leq n$. □

<u>Corollary 2.1:</u> Let (X,T) be an F-space and (X, i(T)) be the modificated topology on X. Then:

$$F\text{-}\dim X_T = \dim X_{i(T)}$$

<u>Proof.</u> Apply proposition 2.1 for each $\alpha \in [0,1)$. □

The next proposition shows that the covering dimension of a topological space $(X, \mathcal{F})$ coincides with the covering dimension of the induced F-space $(X, w(\mathcal{F}))$.

<u>Proposition 2.2:</u> Let $(X, \mathcal{F})$ be a topological space and let $(X, w(\mathcal{F}))$ be the induced F-space. Then:

$$\dim X_{\mathcal{F}} = F\text{-}\dim X_{w(\mathcal{F})}$$

<u>Proof.</u> Let $F\text{-}\dim X_{w(\mathcal{F})} \leq n$ and assume that $\{u_i\}_{i=1}^{k}$ is an open

cover of X. For each $i \in \{1,2,\ldots,k\}$, let $\mu(u_i)$ be the characteristic function on u_i. i.e. $\mu(u_i)(x) = 1$ if $x \in u_i$ and $\mu(u_i)(x) = 0$ if $x \notin u_i$ clearly $\{\mu(u_i)\}_{i=1}^{k} \subset w(\mathcal{F})$ and the family $\{\mu(u_i)\}_{i=1}^{k}$ is α-shading of X for all $\alpha \in [0,1)$.

Hence there exists a finite α-shading $\{f_i\}_{i=1}^{k}$ of X for all $\alpha \in [0,1)$ such that $f_i \leq \mu(u_i)$ for each $i = 1,2,\ldots,k$ and order $\{f_i\}_{i=1}^{k} \leq n$.

Let $v_i = \{x \in X : f_i(x) > \alpha \ \forall\ \alpha \in [0,1)\}$ for each $i = 1, 2,\ldots,k$. Then each v_i is open subset of X and $\{v_i\}_{i=1}^{k}$ is clearly a finite open cover of X whose order $\leq n$ and $v_i \leq u_i$ for each $i = 1,2,\ldots,k$. So $\dim X_{\mathcal{F}} \leq n$.

$\Longleftarrow$ Let $\dim X_{\mathcal{F}} \leq n$ and let $\{g_i\}_{i=1}^{k}$ be an α-shading of X for any $\alpha \in [0,1)$. For each $i \in \{1,2,\ldots,k\}$ let $u_i = \{x \in X : g_i(x) > \alpha\}$. So each u_i is open in X and $\{u_i\}_{i=1}^{k}$ is a finite open cover of X and since $\dim X_{\mathcal{F}} \leq n$, there is a finite open cover $\{v_i\}_{i=1}^{k}$ of X such that $v_i \leq u_i$ for $i=1,2,\ldots,k$ and order of $\{v_i\}_{i=1}^{k} \leq n$.

Let us define $f_i \in I^X$ for each $i = 1,2,\ldots,k$ as follows

$$f_i(x) > \alpha \quad \text{if } x \in v_i$$
$$= 0 \quad \text{if } x \notin v_i$$

Then $\{f_i\}_{i=1}^{k}$ is a family of lower semicontinuous maps from $(X, \mathcal{F})$ into I. i.e. $\{f_i\}_{i=1}^{k}$ is α-shading of X, where $f_i \leq g_i$ for $i = 1,2,\ldots,k$ and order $\{f_i\}_{i=1}^{k} \leq n$. So $F\text{-}\dim_{\alpha} X_{w(\mathcal{F})} < n$ for any $\alpha \in [0,1)$, which implies that

$$\text{F-dim } X_{w(\mathcal{F})} \leq n. \quad \square$$

Combining propositions 2.1 and 2.2 together we get a main result shows that the covering dimension of a topologically generated F-space coincides with the covering dimension of the topological space generating that F-space.

Proposition 2.3: If (X,T) is topologically generated F-space, then:

$$\text{F-dim } X_T = \dim X_{\mathcal{F}}$$

for some $\mathcal{F}$ on X. $\square$

Corollary 2.2: Let (X,T) be a Hausdorff compact F-space. Then

$$\text{F-dim } X_T = \dim X_{\mathcal{F}}$$

from some $\mathcal{F}$ on X.

Proof. Lowen in [13] proved that every Hausdorff compact F-space is topologically generated, then by proposition 2.3 the proof is completed. $\square$

Corollary 2.3: Let (X,T) be strong α-compact Hausdorff F-space. Then:

$$\text{F-dim } X_T = \dim X_{\mathcal{F}}$$

for some $\mathcal{F}$ on X.

Proof. Since strong α-compactness implies compactness in F-spaces [12]. Then it is clear that corollary 2.2 implies corollary 2.3. $\square$

Corollary 2.4: Let U be the usual topologyon the real line R, and let b be the usual topology on the rationals Q. Then:

$$\text{F-dim } R_{w(u)} = \dim R_u = 1$$

$$\text{F-dim } Q_{w(b)} = \dim Q_b = 0. \quad \square$$

Definition 2.1: Let (X,T) be an F-space. Then it is called an ultra Tychonoff F-space if $(X, i(T))$ is a Tychonoff topological space.

Martin [16] proved that if (X,T) is an ultra Tychonoff F-space, then (X,T) has a Stone-Čech ultra F-compactification. For an F-space (X,T), $(\beta X,\mathcal{F})$ denotes the Stone-Cech compactification of $(X, i(T))$ and $(\beta X,T_{\mathcal{F}})$ is an F-space consisting of lower semicontinuous maps from $(\beta X,\mathcal{F})$ into I (with the usual topology) whose restriction to X belongs to T.

Lemma 2.3: Let (X,T) be an ultra Tychonoff F-space. If $\{f_i\}_{i=1}^{k}$ is an α-shading of X for $\alpha \in [0,1)$, then there exists an α-shading $\{g_i\}_{i=1}^{k}$ of βX such that $f_i = g_i | X$ for each $i = 1,2,\ldots,k$.

Proof. Let $\{f_i\}_{i=1}^{k}$ be an α-shading of X. Then $(\beta X,\mathcal{F})$ is the Stone-Cech compactification of $(X, i(T))$ and $u_i = f_i^{-1}(\alpha,1]$ is $i(T)$-open in X for each $i = 1,2,\ldots,k$.

Let $\beta u_i = \beta X \setminus cl_{\beta X}(X \setminus u_i) \quad i = 1,2,\ldots,k.$

Define $g_i : \beta X \longrightarrow I$ as follows:

$$g_i(x) > \alpha \qquad \text{if } x \in \beta u_i \qquad i=1,2,\ldots,k$$
$$= 0 \quad \text{otherwise}$$

Clearly $\{\beta u_i\}_{i=1}^{k}$ is a family ofopen sets in βX, and so $g_i \in T_{\mathcal{F}}$ for each $i \in \{1,2,\ldots,k\}$. We need to show that $\{g_i\}_{i=1}^{k}$ is α-shading of βX. This follows from the fact that $\{g_i\}_{i=1}^{k}$ is α-shading of βX if $\bigcup_{i=1}^{k} g_i^{-1}(\alpha,1] = \beta X$. Since we have $\bigcup_{i=1}^{k} u_i = X$, then $\bigcup_{i=1}^{k} u_i = \beta X$ and $g_i | X = f_i$ for $i = 1,2,\ldots,k$, completes the proof. □

The following proposition gives the equality of the α-covering dimension of an ultra Tychonoff F-space and its ultra Stone-Čech F-compactification.

Proposition 2.4: Let (X,T) be an ultra Tychonoff F-space and let $(\beta X;T_{\mathcal{F}})$ be its ultra Stone-Čech F-compactification. Then:

$$F\text{-}\dim_{\alpha} X_T = F\text{-}\dim_{\alpha} \beta X_{T_{\mathcal{F}}}$$

Proof. $\Longrightarrow$ Let $F\text{-}\dim_{\alpha} \beta X_{T_{\mathcal{F}}} \leq n$, $\alpha \in [0,1)$ and let $\{f_i\}_{i=1}^{k}$ be an α-shading of X. By the above lemma 2.3, there is an α-shading $\{g_i\}_{i=1}^{k}$ of βX such that $g_i|X = f_i$ for $i = 1,2,\ldots,k$. Since $F\text{-}\dim_{\alpha} \beta X_{T_{\mathcal{F}}} \leq n$, so there exists an α-refinement V of $\{g_i\}_{i=1}^{k}$ whose order $\leq n$. Hence $V|X$ is an α-refinement of $\{f_i\}_{i=1}^{k}$ of order $\leq n$. i.e. $F\text{-}\dim_{\alpha} X_T \leq n$.

$\Longleftarrow$ Let $F\text{-}\dim_{\alpha} X_T \leq n$, and let $\{f_i\}_{i=1}^{k}$ be an α-shading of βX. Then $\{f_i|X\}_{i=1}^{k}$ is α-shading of X and so there is an α-refinement V of $\{f_i|X\}_{i=1}^{k}$ whose order $\leq n$. By lemma 2.3 βV is α-shading of βX such that $\beta V|X = V$ so βV is α-refinement of $\{f_i\}_{i=1}^{k}$ and order of $\beta V \leq n$.

Hence $F\text{-}\dim_{\alpha} \beta X_{T_{\mathcal{F}}} \leq n$. □

3. THE PRODUCT THEOREM:

In this section we shall find conditions under which the covering dimension of the product F-space $(X \times Y, T \times R)$ of

F-spaces (X,T) and (Y,R) such that the inequality:

$$F\text{-dim}(X\times Y)_{(T\times R)} \leq F\text{-dim } X_T + F\text{-dim } Y_R$$

holds. We call such results product theorems for F-spaces. Using the modification of F-spaces considered in the previous section, we can extend some well known results of the product theorems of topological spaces to F-spaces.

<u>Lemma 3.1:</u> Let $(X_s, T_s)_{s\in S}$ be a family of F-spaces and let $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ be its product F-space. Then $(\prod_{s\in S} X_s, i_\alpha(\prod_{s\in S} T_s)) = (\prod_{s\in S} X_s, \prod_{s\in S} i_\alpha(T_s))$ for $\alpha\in[0,1)$.

<u>Proof.</u> Let $(\prod_{s\in S} X_s, \prod_{s\in S} i_\alpha(T_s))$ be the topological product of the modificated topological spaces, P_s be the projections and $(\prod_{s\in S} X_s, i_\alpha(\prod_{s\in S} T_s))$ be the modification of the product F-space. Let U be $i_\alpha(\prod_{s\in S} T_s)$-open. Then $u = f^{-1}(\alpha,1]$, $f \in \prod_{s\in S} T_s$. By the definition of product F-topology I.3.3, $f = \bigvee_{s\in S} \bigwedge_{i=1}^{n} P_{s_i}^{-1}(g_{s_i})$, $g_{s_i} \subseteq T_{s_i}$, P_{s_i} is F-continuous. For any $s \in S$, and any $f \in T_s$ we have $(P_s^{-1}(f))^{-1}(\alpha,1] = P_s^{-1}(f^{-1}(\alpha,1])$

So

$$u = \bigcup_s \bigcap_{i=1}^{n} (P_{s_i}^{-1}(g_{s_i}))^{-1}(\alpha,1] = \bigcup_s \bigcap_{i=1}^{n} P_{s_i}^{-1}(g_{s_i}^{-1}(\alpha,1])$$ implies that

u is open in $\prod_{s\in S} i_\alpha(T_s)$.

Now let u be open in $\prod_{s\in S} i_\alpha(T_s)$. So

$$u = \bigcup_s \bigcap_{i=1}^{n} P_{s_i}^{-1}(f_{s_i}^{-1}(\alpha,1]) \qquad f_{s_i} \in T_{s_i}$$

Then

$$\bigcup_s \bigcap_{i=1}^{n} P_{s_i}^{-1}(f_{s_i}^{-1}(\alpha,1]) = \bigcup_s \bigcap_{i=1}^{n} (P_{s_i}^{-1}(f_{s_i}))^{-1}(\alpha,1]$$

So

$\bigvee_{s} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(f_{s_i}) \in \prod_{s\in S} T_s$

Let $f = \bigvee_{s} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(f_{s_i})$. Then $u = f^{-1}(\alpha,1]$ and hence u is open in $i_\alpha(\prod_{s\in S} T_s)$.

<u>Proposition 3.1:</u> [Tychonoff product theorem] Let $(X_s,T_s)_{s\in S}$ be a family of F-spaces. Then $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ is α-compact if and only if (X_s,T_s) is α-compact for all $s \in S$.

<u>Proof.</u> In fact this proposition has been proved by many authers and here we give the short proof. By lemma 2.1 $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ is α-compact if and only if $(\prod_{s\in S} X_s, i_\alpha(T_s))$ is compact. By lemma 3.1 $(\prod_{s\in S} X_s, i_\alpha(\prod_{s\in S} T_s) = (\prod_{s\in S} X_s, \prod_{s\in S} i_\alpha(T_s))$. We know that $(\prod_{s\in S} X_s, \prod_{s\in S} i_\alpha(T_s))$ is compact if and only if for each $s \in S$, $(X_s, i_\alpha(T_s))$ is compact which gives that for each $s \in S$, (X_s,T_s) is α-compact F-space. □

<u>Proposition 3.2:</u> Let (X,T), (Y,R) be a pair of α-compact F-spaces. Then:

$$\text{F-dim}_\alpha(X\times Y)_{(T\times R)} \leq \text{F-dim}_\alpha X_T + \text{F-dim}_\alpha Y_R .$$

<u>Proof.</u> From proposition 2.1 we have

$$\text{F-dim}_\alpha(X\times Y)_{(T\times R)} = \dim (X\times Y)_{i_\alpha(T\times R)} .$$

By lemma 2.1 $(X, i_\alpha(T))$ and $(Y, i_\alpha(R))$ are compact spaces and by proposition 3.1 the product topological space $(X\times Y, i_\alpha(T)\times i_\alpha(R))$ is compact also. But by lemma 3.1

$$(X\times Y, i_\alpha(T\times R)) = (X, i_\alpha(T)) \times (Y, i_\alpha(R))$$

Hence $\dim(X\times Y)_{i_\alpha(T\times R)} = \dim(X_{i_\alpha(T)} \times Y_{i_\alpha(R)})$.

We know in case of compact topological spaces that

$$\dim(X_{i_\alpha(T)} \times Y_{i_\alpha(R)}) \leq \dim X_{i_\alpha(T)} + \dim Y_{i_\alpha(R)}.$$

Therefore

$$F\text{-}\dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}\dim_\alpha X_T + \dim_\alpha Y_R. \quad \square$$

Corollary 3.1: If (X,T), (Y,R) are strongly α-compact F-spaces, then:

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R. \quad \square$$

Proposition 3.3: Let (X,T), (Y,R) be countably α-compact C_{II} F-spaces. Then:

$$F\text{-}\dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}\dim_\alpha X_T + F\text{-}\dim_\alpha Y_R.$$

Proof. Since countably α-compact C_{II} F-space is α-compact proposition I.2.3. Then the proof is completed by proposition 3.2. $\square$

Corollary 3.2: Let (X,T), (Y,R) be countably α-compact, α-Lindelof F-spaces. Then:

$$F\text{-}\dim_\alpha(X\times Y)_{(T\times R)} \leq F\text{-}\dim_\alpha X_T + F\text{-}\dim_\alpha Y_R. \square$$

Proposition 3.4: Let (X,T) be an α-compact F-space, and (Y,R) be countably α-compact F-space. Then $(X\times Y, T\times R)$ is countably α-compact F-space.

Proof. Since $i_\alpha(T)$ is compact, and $i\ (R)$ is countably compact as topologies on X and Y respectively by lemma 2.1. Then from the topological $(X, i_\alpha(T)) \times (Y, i_\alpha(R))$ is countably compact topological space and lemma 3.1 implies that $(X\times Y, i_\alpha(T\times R))$ is countably compact. Hence by lemma 2.1 $(X\times Y, T\times R)$ is countably α-compact F-space. $\square$

Proposition 3.5: Let (X,T) be an ultra Tychonoff F-space, and (Y,R) be a countably α-compact. If each finite α-shading of

XxY is countably extendable to $\beta X\times Y$, then:

$$F\text{-}\dim_{\alpha}(X\times Y)_{(T\times R)} \leq F\text{-}\dim_{\alpha} X_T + F\text{-}\dim_{\alpha} Y_R \ .$$

Proof. Let each finite α-shading of XxY is countably extendable to $\beta X\times Y$. Since ultra compact F-space is α-compact i.e. βX is α-compact, so by proposition 3.4 ($\beta X\times Y$, $T_{\mathcal{F}}\times R$) is countably α-compact. By assumption and proposition II.2.7, we have

$$F\text{-}\dim_{\alpha}(X\times Y)_{(T\times R)} \leq F\text{-}\dim_{\alpha}(\beta X\times Y)_{(T_{\mathcal{F}}\times R)} \ . \qquad (a)$$

By proposition 2.1, we have

$$F\text{-}\dim_{\alpha}(\beta X\times Y)_{(T_{\mathcal{F}}\times R)} = \dim(\beta X\times Y)_{i_{\alpha}(T_{\mathcal{F}}\times R)} . \qquad (b)$$

We appeal to Morita's theorem 0.15 given in the topological preliminary we get

$$\dim(\beta X\times Y)_{i_{\alpha}(T_{\mathcal{F}}\times R)} \leq \dim \beta X_{i_{\alpha}(T_{\mathcal{F}})} + \dim Y_{i_{\alpha}(R)} \qquad (c)$$

But from proposition 2.4 we have

$$F\text{-}\dim_{\alpha} X_T = F\text{-}\dim_{\alpha} \beta X_{T_{\mathcal{F}}} \qquad (d)$$

From the relations (a), (b), (c) and (d) we have

$$F\text{-}\dim_{\alpha}(X\times Y)_{(T\times R)} \leq F\text{-}\dim_{\alpha} X_T + F\text{-}\dim_{\alpha} Y_R$$

which completes the proof. □

Corollary 3.3: Let (X,T) be an ultra Tychonoff F-space, and (Y,R) be a strong countably α-compact. If each finite α-shading of XxY is countably extendable to $\beta X\times Y$ for all $\alpha\in[0,1)$, then

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R . \qquad \square$$

Proposition 3.6: Let (X,T) be strong α-Lindelof and strong countably α-compact F-space. If (Y,R) is strong α-compact F-space, then

$$F\text{-}\dim(X\times Y)_{(T\times S)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R$$

Proof. Since α-Lindelof countably α-compact F-space is α-compact. Then corollary 3.1 finishes the proof.

The star finite property introduced in topological spaces can be used in a similar way for F-spaces. Let (X,T) be an F-space. A collection U of F-sets in X is said to be star - finite if every $u \in U$ meets at most a finite number of other members of U.

Definition 3.1: An F-space (X,T) is said to have α-star - finite property (α-st.f.p) if every α-shading of X has a star-finite α-refinement.

Proposition 3.7: Let (X,T) be an F-space. If (X,T) has an α-star finite propery, then $(X, i_\alpha(T))$ has a star finite property.

Proof. Let $\mathcal{G}$ be an $i_\alpha(T)$-open cover of X. For each $G \in$, let $G = h_G^{-1}(\alpha,1]$, $h \in I^X$. Then $f_G \in T$ and since $X = \bigcup_{G\in\mathcal{G}} h_G^{-1}(\alpha,1] = \bigcup_{G\in\mathcal{G}} G$ so $\mathcal{H} = \{h_G : G \in \mathcal{G}\}$ is an α-shading of X. By the α-st.f.p of (X,T), there is a star finite α-refinement V of $\mathcal{H}$. Let $w = v^{-1}(\alpha,1]$, $v \in V$. Then w is an $i_\alpha(T)$-open and the family $W = \{w : w = v^{-1}(\alpha,1] \text{ for all } v \in V\}$ is clearly a star finite refinement of $\mathcal{G}$. □

Corollary 3.4: If an F-space (X,T) has a strong α-st.f.p., then (X, i(T)) has a st.f.p.

Proof. Follows from proposition 3.7 for each $\alpha \in [0,1)$. □

Proposition 3.8: Let (X,T), (Y,R) be strong α-Hausdorff F-spaces such that (XxY, TxR) has the strong α-st.f.p. Then:

$$\text{F-dim}(X\times Y)_{(T\times R)} \leq \text{F-dim } X_T + \text{F-dim } Y_R .$$

Proof. Since (XxY, TxR) has the strong α-st.f.p. then by corollary 3.4, (XxY, i(TxS)) has the st.f.p. (X i(T)) and (Y,i(R)) are Hausdorff topological spaces from lemma 2.2. Then we apply Morita's theorem [14] of the topological part, we have

$$\dim(X\times Y)_{i(T\times R)} \leq \dim X_{i(T)} + \dim Y_{i(R)}$$

and hence

$$F\text{-}\dim(X\times Y)_{(T\times R)} \leq F\text{-}\dim X_T + F\text{-}\dim Y_R \quad \square$$

Proposition 3.9: Let (X,T), (Y,R) be α-Hausdorff F-spaces such that (XxY, TxR) has the α-st.f.p. Then

$$F\text{-}\dim_\alpha(X\times Y)_{(T\times S)} \leq F\text{-}\dim_\alpha X_T + F\text{-}\dim_\alpha Y_R$$

Proof. Follows by the same way of proof of proposition 3.8. $\square$

Definition 3.2: An F-space (X,T) is called ultra paracompact F-space if (X, i(T)) is paracompact topological space.

Proposition 3.10: Let (X,T) be an ultra paracompact strongly α-Hausdorff F-space with F-dim $X_T = n$ and (Y,R) a strong α-compact strong α-Hausdorff F-space with F-dim $Y_R = m$. Then:

$$F\text{-}\dim\ (X\times Y)_{(T\times R)} \leq m + n$$

Proof. By assumption and lemma 2.1 and proposition 2.1, (X, i(T)) is paracompact Hausdorff with dim $X_{i(T)} = n$, and (X,i(R)) is compact Hausdorff topological space from lemmas 2.1 and 2.2 with dim $Y_{i(R)} = m$ from proposition 2.1. From the topological preliminary 0.12, we have

$$\dim(X\times Y)_{i(T\times R)} = \dim(X\times Y)_{i(T)\times i(R)} \leq m + n.$$

Hence by proposition 2.1, we get

$$F\text{-}\dim(X\times Y)_{i(T\times R)} \leq m + n . \quad \square$$

Proposition 3.11: Let $(X_s, T_s)_{s\in S}$ be a family of ultra Tychonoff F-spaces such that any countable product F-space is α-Lindelof. If $F\text{-}\dim_\alpha X_s = 0$ for $s \in S$, then $F\text{-}\dim_\alpha \prod_{s\in S} X_s = 0$.

Proof. From proposition 2.1, $F\text{-}\dim_\alpha X_s = 0$ if and only if $\dim X_s\ i_\alpha(T_s) = 0$. Since $(\prod_{i=1}^{\infty} X_{s_i}, \prod_{i=1}^{\infty} T_{s_i})$ is α-Lindelof, then by lemma 2.1, we have $(\prod_{i=1}^{\infty} X_{s_i}, \prod_{i=1}^{\infty} i_\alpha(T_{s_i}))$ is Lindelof. The proof is completed by appling Theorem 0.1 6 of the topological preliminary □

Chapter IV:

INVERSE LIMIT OF F-SPACES

The notion of the inverse limit of topological spaces has been widly considered and its connection with the dimension of the spaces has a great contribution to the development of the dimension theory of topological spaces. It is worth while to introduce this notion for F-spaces and look at its relation with the covering dimension of F-spaces.

We devote the first two sections of this chapter to the introduction and the study of the notion of inverse system of F-spaces and its limit F-space in the light of the study of this notion in case of general topology, where most of the results - specially for compact Hausdorff spaces - can be generalized analogously to F-spaces. We return in the third section to the α-covering dimension, where we give some characterization of F-$\dim_\alpha$ for the limit F-space.

By α-bicompact F-space we mean an α-Hausdorff and α-compact F-space.

1. INVERSE LIMIT OF α-BICOMPACT F-SPACES:

Let $\{(X_s,T_s) : s \in S\}$ be a family of F-spaces indexed by a directed set S. For each $s,t \in S$ with $s \leq t$, let $f_{s,t} : X_t \longrightarrow X_s$ be a mapping. We call $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ an inverse system of

F-spaces over S if: $f_{s,t}$ is F-continuous for each $s,t \in S$ $(s \leq t)$ such that $f_{s,S}$ is the identity (for every S), and if $s \leq t \leq r$, then $f_{s,t} \circ f_{t,r} = f_{s,r}$.

Let $\widetilde{X}$ be a subset of the cartesian product $\prod_{s\in S} X_s$, consisting of the points $x \in \prod_{s\in S} X_s$ such that $f_{s,t}(p_t(x)) = p_s(x)$ for each pair $s,t \in S$ with $s \leq t$, where p_s, p_t denote the projections. For each $s \in S$, $f_s : \widetilde{X} \longrightarrow X_s$ is called the canonical mapping ($f_s = p_s|\widetilde{X}$), and the F-continous mappings $f_{s,t}$ are called the F-bonding mappings of $\underset{\sim}{X}_F$.

<u>Definition 1.1:</u> The coarsest F-topology $\widetilde{T}$ on $\widetilde{X}$ for which the canonical mappings $f_s (s \in S)$ are F-continuous is called the inverse limit of the inverse system $\underset{\sim}{X}_F$, and $(\widetilde{X},\widetilde{T})$ is called the limit F-space of the inverse system of F-spaces.

<u>Proposition 1.1:</u> The limit F-space $(\widetilde{X},\widetilde{T})$ of an inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t} : S\}$ is the relative F-subspace of the product F-space $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ on $\widetilde{X}$.

<u>Proof.</u> Let $(\widetilde{X},\widetilde{T})$ be the limit F-space of $\underset{\sim}{X}_F$ and let $f_s : \widetilde{X} \longrightarrow X_s$ be the canonical projections for $s \in S$. According to definition I.2.11 it is enough to find for any open F-set w in $\widetilde{X}$ an open F-set g in $\prod_{s\in S} X_s$ such that $g|\widetilde{X} = w$. Let w be an open F-set in $\widetilde{X}$. Then $w = \bigvee_s \bigwedge_{i=1}^{n} f_{s_i}(u_i)$, $u_{s_i} \in T_{s_i}$ $s \in S^* \subseteq S$. But $f_{s_i} = p_{s_i}|\widetilde{X}$ for each s_i so $p_{s_i}^{-1}(u_{s_i})$ is open F-set in $\prod_{s\in S} X_s$ and if we put $g = \bigvee_s \bigwedge_{i=1}^{n} p_{s_i}^{-1}(u_{s_i})$, then g is open F-set in $\prod_{s\in S} X_s$ and clearly $g|\widetilde{X} = w$. □

<u>Proposition 1.2:</u> Let $\underset{\sim}{X}_F$ be an inverse system of F-spaces (over S) and let $(\widetilde{X},\widetilde{T})$ be the limit F-space of $\underset{\sim}{X}_F$ with canonical mappings $f_s (s \in S)$. Then the set of all F-sets of the form $f_s^{-1}(u_s)$, $u_s \in T_s$ $s \in S$ forms a basis for $\widetilde{T}$.

<u>Proof.</u> Let β be the set of all F-sets $\{f_s^{-1}(u_s) : u_s \in T_s\}_{s\in S}$ and let p_s be the projection from the product F-space $(\prod_{s\in S} X_s$, $\prod_{s\in S} T_s)$ onto (X_s,T_s) for $s \in S$ where $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ is an inverse system of F-spaces. Let w be an open F-set in the limit F-space $(\widetilde{X},\widetilde{T})$. Then by proposition 1.1 there exists an open F-set g in $\prod_{s\in S} X_s$ such that $w = g|\widetilde{X}$. By definition I.3.2 of the product F-topology, $g = \bigvee_{s_i\in S^*} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(u_{s_i})$, $S^* \in S$, $u_{s_i} \in T_{s_i}$. So

$$w = (\bigvee_{s_i} \bigwedge_{i=1}^{n} p_{s_i}^{-1}(u_{s_i}))|\widetilde{X} = \bigvee_{s_i} \bigwedge_{i=1}^{n} (p_{s_i}|\widetilde{X})^{-1}(u_{s_i}) = \bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$$

That is $w = \bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$.

Since S is directed set, then there is $t \in S$ such that $s_i \leq t$ for each $i = 1,2,\ldots,n$.

Put $h_t = \bigwedge_{i=1}^{n} f_{s_i,t}^{-1}(u_{s_i})$, then $h_t \in T_t$ and

$$f_t^{-1}(h_t)=f_t^{-1}(\bigwedge_{i=1}^{n} f_{s_i,t}^{-1}(u_{s_i})) = \bigwedge_{i=1}^{n} (f_{s_i,t}^{-1} f_t^{-1})(u_{s_i}) = \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}).$$

Then

$$\bigvee_{t\geq s_i} f_t^{-1}(h_t) = \bigvee_{s_i\in S^*} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}) = w$$

and $f_t^{-1}(h_t) \in \beta$.

Hence β is a base for $\widetilde{T}$ as required. □

This base is called the standard basis for $\widetilde{T}$ and denoted by $\beta(\widetilde{X})$.

Proposition 1.3: If (X_s,T_s) $s \in S$ are FT_1, FT_2, α-Hausdorff or F-regular F-spaces, then the limit F-space $(\tilde{X},\tilde{T})$ of the inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ is FT_1, FT_2, α-Hausdorff or F-regular F-space.

Proof. By proposition I.3.8 each of FT_1, FT_2, α-Hausdorff and F-regular is productive and clearly each of them is heredatory which completes the proof. □

The next proposition gives the connection between the induced F-space of the inverse limit of topological spaces and the limit F-space of the induced F-spaces. But first we restate weiss's proposition 3.4 [29] as a lemma without proof.

Lemma 1.1: Let (X,R), $(Y,\mathcal{J})$ be two topological spaces and $f : (X,R) \longrightarrow (Y,\mathcal{J})$ be a mapping. Then f is continuous if and only if $f : (X,w(R)) \longrightarrow (Y,w(\mathcal{J}))$ is F-continuous where $(X,w(R))$, $(Y,w(\mathcal{J}))$ are the induced F-spaces on (X,R) and $(Y,\mathcal{J})$ as in chapter III.

Lemma 1.2: Let $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ be an inverse system of topological spaces over S. Then $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ is an inverse system of induced F-spaces.

Proof. For $s,t \in S$ with $s \le t$, $f_{s,t} : (X_s,R_s) \longrightarrow (X_t,R_t)$ is continuous if and only if $f_{s,t} : (X_s,w(R_s)) \longrightarrow (X_t,w(R_t))$ is F-continuous by lemma 1.1 finishes the proof. □

Proposition 1.4: Let $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ be an inverse system of topological spaces. If $(\tilde{X},\tilde{R})$ is the inverse limit space of $\underset{\sim}{X}$, then

$$(\tilde{X},w(\tilde{R})) = (\tilde{X},(\widetilde{w(R)})).$$

Proof. $\Longrightarrow$ Let g be $\widetilde{w(R)}$-open F-set. Then by proposition 1.2 $g = \bigvee_s \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$ where u_{s_i} is $w(R_{s_i})$-open F-set and $f_{s_i} : (\widetilde{X}, \widetilde{w(R)}) \longrightarrow (X_{s_i}, w(R_{s_i}))$. But $u_{s_i}^{-1}(\alpha,1]$ is R_{s_i}-open set for any $\alpha \in [0,1)$ for each $i = 1,2,\dots,n$, $s \in S$. Since $(f_{s_i}^{-1}(u_{s_i}))^{-1}(\alpha,1]$ $= f_{s_i}^{-1}(u_{s_i}^{-1}(\alpha,1])$, then $\bigcup_s \bigcap_{i=1}^{n} (f_{s_i}^{-1}(u_{s_i}))^{-1}(\alpha,1]$ is $\widetilde{R}$-open set.
So

$\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}) \in w(\widetilde{R})$ i.e. g is $w(\widetilde{R})$-open F-set.

$\Longleftarrow$ Let g be $w(\widetilde{R})$-open F-set. Then $g^{-1}(\alpha,1]$ is $\widetilde{R}$-open set for any $\alpha \in [0,1)$.
So

$g^{-1}(\alpha,1] = \bigcup_{s_i} \bigcap_{i=1}^{n} f_{s_i}^{-1}(u_{s_i})$ where u_{s_i} is R_{s_i}-open and

$f_{s_i} : (\widetilde{X}, \widetilde{R}) \longrightarrow (X_{s_i}, R_{s_i})$ is the canonical. Now for each $i = 1,2,\dots,n$, $\mu(u_{s_i})$ is $w(R_{s_i})$-open F-set and by lemma 1.1 $f_{s_i}^{-1}(\mu(u_{s_i}))$ is open F-set in $\widetilde{X}$ and hence $\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(\mu(u_{s_i}))$ is $\widetilde{w(R)}$-open F-set. But $\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(\mu(u_{s_i})) = \mu(\bigcup_{s_i} \bigcap_{i=1}^{n} f_{s_i}^{-1}(u_{s_i}))$
i.e. $\bigvee_{s_i} \bigwedge_{i=1}^{n} f_{s_i}^{-1}(\mu(u_{s_i})) = \mu(g^{-1}(\alpha,1]) \geq g$.

So g is $\widetilde{w(R)}$-open F-set. Therefore $(\widetilde{X}, w(\widetilde{R})) = (\widetilde{X}, \widetilde{w(R)})$. □

Lemma 1.3: [10] 3.1: Let (X,T), (Y,S) be two topologically generated F-spaces. Then $f : (X,T) \longrightarrow (Y,S)$ is F-continuous if and only if $f : (X,i(T)) \longrightarrow (Y,i(S))$ is continuous where (X,i(T)) and (Y,i(S)) are the modificated topologies. □

Remark: Using the above lemma we observe that if $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ is an inverse system of F-spaces, then $\underset{\sim}{X} = \{X_s, f_{s,t}, S\}$ is an inverse system of modificated topologies.

The following proposition shows the relation between the modification of the limit F-space and the inverse limit space of the modificated topologies.

Proposition 1.5: Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of topologically generated F-spaces. If $(\tilde{X}, \tilde{T})$ is the limit F-space of $\underset{\sim}{X}_F$ then:

$$(\tilde{X}, i(\tilde{T})) = (\tilde{X}, \widetilde{i(T)}) \qquad i(T) = \prod_{s\in S} i(T_s) .$$

Proof. $\Longrightarrow$ Let G be an $i(\tilde{T})$-open set. Then $G = g^{-1}(\alpha,1]$ for $g \in \tilde{T}$ and $\alpha \in [0,1)$. By proposition 1.2, $g = \bigvee_{s\in S^*} f_s^{-1}(u_s)$ where $f_s : \tilde{X} \longrightarrow X_s$ and $u_s \in T_s$. But

$G = (\bigvee_s f_s^{-1}(u_s))^{-1}(\alpha,1] = \bigcup_s f_s^{-1}(u_s^{-1}(\alpha,1])$ where $u_s^{-1}(\alpha,1]$ is $i(T_s)$-open, so G is $\widetilde{i(T)}$-open set.

$\Longleftarrow$ Let G be an $\widetilde{i(T)}$-open set where $\widetilde{i(T)}$ is the inverse limit space of the modificated topologies $i(T_s)$ $s \in S$. Then $G = \bigcup_{s\in S^*} f_s^{-1}(u_s)$, u_s is $i(T_s)$-open and $f_s : (X, i(T)) \longrightarrow (X_s, i(T_s))$ is the canonical mappings, so $u_s = g_s^{-1}(\alpha,1]$ for $g_s \in T_s$ and $\alpha \in [0,1)$ and hence

$$G = \bigcup_s f_s^{-1}(g_s^{-1}(\alpha,1]) = \bigcup_s (f_s^{-1}(g_s))^{-1}(\alpha,1] = \bigvee (f_s^{-1}(g_s))^{-1}(\alpha,1]$$

by lemma 1.3 so $\bigvee(f_s(g_s)) \in \tilde{T}$ and so $\bigvee(f_s(g_s))(\alpha,1] \in i(T)$ i.e. G is $i(\tilde{T})$-open set. □

Proposition 1.6. Let $\underset{\sim}{X}_F$ be an inverse system of α-Hausdorff

F-spaces (over S). Then $\widetilde{X}$ is α-closed in $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ for $\alpha \in [0,1)$.

<u>Proof.</u> Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of α-Hausdorff F-spaces, $x \in \prod_{s\in S} X_s$ and $x \notin \widetilde{X}$. Then there exist $s,t \in S$, $s \leq t$, such that $f_{s,t}p_t(x) \neq p_s(x)$. By α-Hausdorffness there are open F-sets u_s, v_s in X_s such that $u_s(p_s(x)) > \alpha$,

$v_s(f_{s,t}\, p_t(x)) > \alpha$ and $u_s \wedge v_s = 0$. Hence $(p_s^{-1}(u_s))(x) > \alpha$,

$((f_{s,t}\, p_t)^{-1}(v_s))(x) > \alpha$ in $(\prod X_s, \prod T_s)$.

Let $w = p_s^{-1}(u_s) \wedge (f_{s,t}\, p_t)^{-1}(v_s)$. Then w is open F-set in $\prod_{s\in S} X_s$, $w(x) > \alpha$ for $x \in \prod_{s\in S} X_s \setminus \widetilde{X}$ and for any $y \in \widetilde{X}$, $w(y) = 0$. Hence $w \wedge \mu(\widetilde{X}) = 0$, that is $\widetilde{X}$ is α-closed as required. □

<u>Proposition 1.7:</u> Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of Hausdorff F-spaces. Then

$$\mu(\prod_{s\in S} X_s \setminus \widetilde{X}) \text{ is open F-set in } \prod_{s\in S} X_s.$$

<u>Proof.</u> Let $x \in \prod_{s\in S} X_s \setminus \widetilde{X}$. Then there are $s,t \in S$, $s \leq t$ such that $f_{s,t}p_t(x) \neq p_s(x)$, so by definition I.2.3 of Hausdorff F-space there exist u_s, v_s open F-sets in X_s such that $u_s(p_s(x)) = 1 = v_s(f_{s,t}p_t(x))$ and $u_s \wedge v_s = 0$. So $(p_s^{-1}(u_s))(x) = 1 = ((f_{s,t}p_t)^{-1}(v_s))(x)$. If we put $w_x = p_s^{-1}(u_s) \wedge (f_{s,t}p_t)(v_s)$, then w_x is open F-set in $\prod_{s\in S} X_s$ and $w_x(x) = 1$ for $x \in \prod_{s\in S} X_s \setminus \widetilde{X}$. So for each $x \in \prod_{s\in S} X_s \setminus \widetilde{X}$ we can find an open F-set w_x such that $w_x(x) = 1$, and hence $\mu(\prod_{s\in S} X_s \setminus \widetilde{X}) = \bigvee \{w_x : x \in (\prod_{s\in S} X_s) \setminus \widetilde{X}\}$ which is open F-set in $\prod_{s\in S} X_s$. □

Lemma 1.4: Let Y be a suitable closed set in an F-space (X,T). Then:

(I) E is closed F-set in Y implies E is closed F-set in X.

(II) G is closed F-set in X implies G|Y is closed F-set in Y.

Proof. Let E be closed F-set in Y as subspace. Then there exists a closed F-set G in X such that $G|Y = E$; that is $E = G \wedge \mu(Y)$ and so $\bar{I}_X - E = \bar{I}_X - (G \wedge \mu(Y)) = (\bar{I}_X - G) \vee (\bar{I}_X - \mu(Y)) =$ open F-set in X by Demorgan's law I.1.2; that is E is closed F-set in X.

(II) is clear. □

Since proposition 1.7 shows that for an inverse system $\underset{\sim}{X}_F$ of Hausdorff F-spaces $\widetilde{X}$ is I^*-closed, then it follows from proposition I.2.6 that $\widetilde{X}$ is suitable closed in $\prod_{s \in S} X_s$. This leads to the following proposition.

Proposition 1.8: Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of Hausdorff F-spaces. Then the limit F-space $(\widetilde{X}, \widetilde{T})$ is a closed F-subspace of $(\prod_{s \in S} X_s, \prod_{s \in S} T_s)$.

Proof: is an immediate consequence of proposition 1.7 and lemma 1.4. □

Corollary 1.1: Let $\underset{\sim}{X}_F$ be an inverse system of strong α-Hausdorff F-spaces. Then $(\widetilde{X}, \widetilde{T})$ is a closed F-subspace of $(\prod_{s \in S} X_s, \prod_{s \in S} T_s)$ □

The following result is a generalization of one of the most important result in the inverse limit of topological spaces 0.2.

Proposition 1.9: The limit F-space of an inverse system of α-

bicompact F-spaces is non-empty α-bicompact F-space.

Proof. Let $(\widetilde{X},\widetilde{T})$ be the limit F-space of an inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ of α-bicompact F-spaces for $\alpha \in [0,1)$. Then by Tychonoff product theorem for F-spaces - proposition III.3.2 - and proposition I.3.8 $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ is α-bicompact F-space, form proposition 1.6 $\widetilde{X}$ is α-closed in $(\prod_{s S} X_s, \prod_{s S} T_s)$ which implies that $(\widetilde{X},\widetilde{T})$ is α-bicompact by proposition I.2.7.

Now we have to show that $\widetilde{X} \neq \emptyset$. For $t \in S$, let $Y_t = \{x \in \prod_{s\in S} X_s : p_s(x) = f_{s,t}\, p_t(x)$ for each $s \in S$ such that $s \leq t\}$. Then Y_t is α-closed in $\prod_{s\in S} X_s$ from proposition I.3.4 and $Y_t \neq \emptyset$, for let $x_t \in X_t$ and for $s \in S$ let $Z_s \subseteq X_s$ with just one point $f_{s,t}(x_t)$ if $s \leq t$ and $Z_s = X_s$ otherwise. Let $Z = \prod_{s\in S} Z_s$. Then $Z \neq \emptyset$, Z is a subset of $\prod_{s\in S} X_s$ and $Z \subset Y_t$.

If $t,s \in S$ such that $s \leq t$, then $Y_t \subset Y_s$. It follows that $\{Y_t\}_{t\in S}$ is a decreasing family of α-compact and hence by proposition I.2.8, we get $\bigcap_{G\in S} Y_t \neq \emptyset$.

Let $x \in \bigcap_{t\in S} Y_t$. If $t,r \in S$ with $t \leq r$, then since $x \in Y_r$, $p_t(x) = f_{t,r}\, p_r(x)$ which implies that $x \in \widetilde{X}$ as required. □

Corollary 1.2: Let $(\widetilde{X},\widetilde{T})$ be the limit F-space of an inverse system $\underset{\sim}{X}_F$ of strong α-bicompact. Then $(\widetilde{X},\widetilde{T})$ is strong α-bicompact. □

Corollary 1.3: The limit F-space of an inverse system of C_{II}, countably α-compact and α-Hausdorff F-spaces is non-empty.

Proof. Since C_{II} countably α-compact is α-compact by proposition I.2.3, then the proof is completed by proposition 1.9. □

Corollary 1.4: The limit F-space of an inverse system of α-Lindelof, countably α-compact and α-Hausdorff F-spaces is non-empty. □

Definition 1.2: We shall say that $\underset{\sim}{Y}_F = \{Y_s, g_{s,t}, S\}$ is an F-subsystem of $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ over the same directed set S, if Y_s is F-subspace of X_s and $g_{s,t} = f_{s,t}|Y_t$ $(s,t \in S, s \leq t)$. We note that $f_{s,t}(Y_t) \subset Y_s$ $(s \leq t)$.

Proposition 1.10: Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of F-spaces and let $(\widetilde{X},\widetilde{T})$ be its limit F-space. Let $\underset{\sim}{Y}_F = \{Y_s, g_{s,t}, S\}$ be an F-subsystem of $\underset{\sim}{X}_F$. Then the limit F-space $(\widetilde{Y},\widetilde{T}_{\widetilde{Y}})$ is an F-subspace of $(\widetilde{X},\widetilde{T})$.

Proof. Since for each $Y_s \subset X_s$ so we have $\widetilde{Y} = (\prod_{s\in S} Y_s) \cap \widetilde{X}$ and $\widetilde{T}_{\widetilde{Y}}$ is the F-topology induced by the product F-space $(\prod_{s\in S} Y_s, \prod_{s\in S} T_{s\ Y_s})$ on $\widetilde{Y}$.

But $(\prod_{s\in S} Y_s, \prod_{s\in S} T_{s\ Y_s})$ is induced by the product F-space $(\prod_{s\in S} X_s, \prod_{s\in S} T_s)$ on $\prod_{s\in S} Y_s$. So $(\widetilde{Y},\widetilde{T}_{\widetilde{Y}})$ is an F-subspace of $(\widetilde{X},\widetilde{T})$. □

Corollary 1.5: Let $(\widetilde{X},\widetilde{T})$ be a limit F-space of an inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ of F-spaces with canonical mappings $f_s : \widetilde{X} \longrightarrow X_s$. Let $Y = \{f_s(\widetilde{X}), g_{s,t}, S\}$ be an F-subsystem of $\underset{\sim}{X}_F$. Then $(\widetilde{X},\widetilde{T})$ is the limit F-space of $\underset{\sim}{Y}_F$. □

We note that each F-bonding map $g_{s,t}$ is surjective and the canonical maps $\widetilde{X} \longrightarrow f_s(\widetilde{X})$ are also surjectives.

2. INVERSE SEQUENCES OF F-SPACES:

The inverse sequence is considered as a special case of an inverse system. We notice that if $\underset{\sim}{X} = \{X_n, f_{n,m}, N\}$ is an inverse sequence over N (N is the set of positive integers with its usual ordering) with all the sets X_n are non-empty and all bonding maps $f_{n,m}$ ($n \leq m$) are surjectives, then the inverse limit $\widetilde{X}$ of $\underset{\sim}{X}$ is non-empty without imposing any F-topological conditions.

Proposition 2.1: Let $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ be an inverse sequence of countably α-compact F-spaces. If all $f_{n,m}(X_m)$ ($n \leq m$) are α-closed in X_n, then the limit F-space $(\widetilde{X}, \widetilde{T})$ of $\underset{\sim}{X}_F$ is non-empty.

Proof. For any $n \in N$, $\{f_{n,m}(X_m)\}_{m \geq n}$ is a family of α-closed sets by assumption in X_n and

$$f_{n,n+1}(X_{n+1}) \supset f_{n,n+2}(X_{n+2}) \cdots f_{n,m}(X_m) \qquad m \geq n.$$

So $\{f_{n,m}(X_m)\}_{m \geq n}$ is a decreasing family of α-closed sets in a countably α-compact X_m, and then by proposition I.2.8 $\bigcap_{n \leq m} f_{n,m}(X_m) \neq \emptyset$ and is countably α-compact.

Let us put $Y_n = \bigcap_{n \leq m} f_{n,m}(X_m)$.

If $n \leq m$, then $f_{n,m}(Y_m) \subset Y_n$. In fact

$$f_{n,m}(Y_m) = f_{n,m}(\bigcap_{m \leq \ell} \{f_{m,\ell}(X_\ell)\}) \subset \bigcap_{m \leq \ell} \{f_{n,m}(f_{m,\ell}(X_\ell))\}$$

$$= \bigcap_{n \leq \ell} f_{n,\ell}(X_\ell) = Y_n.$$

Now let $y_n \in Y_n$, that is $y_n \in \bigcap_{n \leq \ell} f_{n,\ell}(X_\ell)$, so it is enough to show that $y_m \in f_{m,\ell}(X_\ell)$ for any $m \geq n$. Let m be given and choose $\ell \in N$ such that $\ell \geq n$, $\ell \geq m$. Then $y_n \in f_{n,\ell}(X_\ell)$, let $x_\ell \in X_\ell$ such that $f_{n,\ell}(x_\ell) = y_n$. Define $y_m = f_{m,\ell}(x_\ell)$ so we have $y_m \in Y_m$ and $f_{n,m}(y_m) = f_{n,m}(f_{m,\ell}(x_\ell)) = f_{n,\ell}(x_\ell) = y_n$. Hence $y_n \in f_{n,m}(Y_m)$ and then $f_{n,m}(Y_m) = Y_n$ $(n \leq m)$.

By letting $g_{n,m} = f_{n,m}|Y_m$ to be the restriction of $f_{n,m}$ to Y_m, we obtain an inverse sequence $\underset{\sim}{Y}_F = \{Y_m, g_{n,m}, N\}$ of F-spaces, where the F-bonding mappings $g_{n,m}$ are surjectives. Hence the limit F-space $(\tilde{Y}, \tilde{T}_{\tilde{Y}})$ of $\underset{\sim}{Y}_F$ is non-empty and consequently $(\tilde{X}, \tilde{T})$ is non-empty F-space. □

<u>Definition 2.1:</u> Let $\psi: (X,T) \longrightarrow (Y,R)$ be an F-continuous map, ψ is called strongly F-closed if it is F-closed and for any α-closed set A in X, $\psi(A)$ is α-closed in Y.

In the following proposition we give the condition under which the limit F-space of an inverse sequence of countably α-compact α-Hausdorff F-spaces is countably α-compact.

<u>Proposition 2.2:</u> Let $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ be an inverse sequence of countably α-compact α-Hausdorff F-spaces with strongly F-closed canonical mappings. Then the limit F-space $(\tilde{X}, \tilde{T})$ is countably α-compact F-space.

<u>Proof.</u> Let $(\tilde{X}, \tilde{T})$ be the limit F-space of $\underset{\sim}{X}_F$ and let $\{E_i\}_{i=1}^{\infty}$ be an α-centered family of closed F-sets in $\tilde{X}$. By assumption $f_n(\tilde{X})$ is α-closed in X_n for each $n \in N$, where f_n is the canonical mapping and hence by proposition I.2.7, $f_n(\tilde{X})$ is countably α-compact.

Let $\underset{\sim}{Y}_F = \{f_n(\tilde{X}), g_{n,m}, N\}$ where $g_{n,m} = f_{n,m}|f_m(\tilde{X})$ $n \leq m$. Then $\underset{\sim}{Y}_F$ is a subsequence of $\underset{\sim}{X}_F$ satisfying condition of proposition 2.1, therefore the limit F-space $(\tilde{Y},\tilde{T}_{\tilde{Y}})$ is non-empty and $\tilde{Y} \leq \tilde{X}$.

Now $\{g_m(E_i|\tilde{Y})\}_{i=1}^{\infty}$ is α-centered family of closed F-sets in Y_m and since Y_m is countably α-compact then by proposition I.2.4 there is $y_m \in Y_m$ such that $(g_m(E_i|\tilde{Y}))(y_m) \geq 1-\alpha$ for all $i = 1,2,\ldots$. So there is $y \in \tilde{Y}$ such that $g_m(y) = y_m$ and $(E_i|\tilde{Y})(y) \geq 1-\alpha$ for all $i = 1,2,\ldots$.

Since $\tilde{Y} \leq \tilde{X}$ so there is $x \in \tilde{X}$ such that $E_i(x) \geq 1-\alpha$ for all $i = 1,2,\ldots$. i.e. $(\tilde{X},\tilde{T})$ is countably α-compact. □

<u>Proposition 2.3:</u> The limit F-space of an inverse sequence of C_{II} F-spaces is C_{II} F-space.

<u>Proof.</u> Since the product of a countable family of C_{II} F-spaces is C_{II} F-space by proposition I.3.7, then the proof follows from the clear observation that any F-subspace of a C_{II} F-space is C_{II} F-space. □

Let (X,T) be an F-space, $f \in I^X$ is called ℓ_∞-set if $f = \bigwedge_{i=1}^{\infty} g_i$ where g_i is open F-set for $i = 1,2,\ldots$, and f is called H_∞- set if $f = \bigvee_{i=1}^{\infty} h_i$ where h_i is closed F-set for $i = 1,2,\ldots$. We note that the complement of an ℓ_∞-set is H_∞-set; that is if f is an ℓ_∞-set, then co $f = H_\infty$-set.

<u>Definition 2.2:</u> An F-space (X,T) is called ℓF-space if each closed F-set in X is an ℓ_∞-set.

It is clear that every F-normal ℓF-space is perfectly normal F-space.

<u>Proposition 2.4:</u> If $(\tilde{X},\tilde{T})$ is a limit F-space of an inverse sequence $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ of ℓF-spaces, then $(\tilde{X},\tilde{T})$ is ℓF-space.

Proof. Suppose that $(\widetilde{X},\widetilde{T})$ is the limit F-space of an inverse sequence $\underset{\sim}{X}_F$ of cl F-spaces and let E be a closed F-set in $\widetilde{X}$. Then for each $n \in N$ $cl(f_n(E))$ is a closed F-set in X_n and hence by hypothesis $cl(f_n(E)) = \bigwedge_{i=1}^{\infty} u_{nj}$, where u_{nj} is open F-set in X_n for $j = 1,2,\ldots$ and for each $n \in N$, $f_{n,n+1}^{-1}(u_{nj}) \geq u_{n+1,j}$, $u_{nj} \geq u_{n,j+1}$ clearly for $n \in N$, $f_n^{-1}(u_{n,n}) \geq E$.

Suppose that q is an F-point in $\widetilde{X}$, $q \notin E$ such that for each n, $q \in f_n^{-1}(u_{n,n})$. For some $n \in N$, $f_n(q) \notin cl(f_n(E))$ and hence for some $m \geq n$, $f_n(q) \notin u_{n,m}$. Then $f_m(q) \notin u_{m,m} \leq f(u_{n,m})$. So $q \notin f_m^{-1}(u_{m,m})$ and therefore $E = \bigwedge_{m=1}^{\infty} f_n^{-1}(u_{n,m})$, where $u_{n,m}$ is open F-set in X_n. □

Definition 2.3: A mapping $\psi : X \longrightarrow Y$ has a clint property if for an F-set g in Y, and an F-set $f \in X$, $\psi^{-1}(cl\ g) = cl(\psi^{-1}(g))$ and $\psi(int\ f) = int\ (\psi(f))$.

Proposition 2.5: If $(\widetilde{X},\widetilde{T})$ is a limit F-space of an inverse sequence $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ of perfectly normal F-space with surjective F-bondings and canonical mappings have a clint property, then $(\widetilde{X},\widetilde{T})$ is perfectly normal F-space.

Proof. Let $(\widetilde{X},\widetilde{T})$ be a limit F-space of an inverse sequence $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ of perfectly normal F-spaces. By proposition 2.4, it is enough to show that $(\widetilde{X},\widetilde{T})$ is normal F-space. Let g be a closed F-set in $\widetilde{X}$ and h be an open F-set in $\widetilde{X}$ with $g \leq h$. Then for each $n \in N$, $cl(f_n(g))$ is closed F-set in X_n and $int(f_n(h))$ is open F-set in X_n with $cl(f_n(g)) \leq int(f_n(h))$ from the assumption. Since X_n is normal F-space, then there is an open F-set k in X_n such that

$$cl(f_n(g)) \le k \le cl\ k \le int(f_n(h)).$$

So

$$f_n^{-1}(cl\ f_n(g)) \le f_n^{-1}(k) \le f_n^{-1}(cl\ k) \le f_n^{-1}(int(f_n(h)))$$

Since each f_n has a clint property, then

$$f_n^{-1}(f_n(cl\ g)) \le f_n^{-1}(k) \le cl(f^{-1}(k)) \le f_n^{-1}(f(int\ h))$$

That is

$$f_n^{-1}(f_n(g)) \le f_n^{-1}(k) \le cl(f^{-1}(k)) \le f_n^{-1}(f_n(h))$$

Hence

$$g \le f_n^{-1}(k) \le cl(f^{-1}(k)) \le h\ .$$

Let $u = f_n^{-1}(k)$ the u is open F-set in $\widetilde{X}$ and $g \le u \le cl\ u \le h$ finishes the proof. □

3. F-dim$_\alpha$ OF LIMIT F-SPACE:

Lemma 3.1: Let (X,T) be an α-compact F-space and let β be a base for T closed under finite joins. If $\{u_i\}_{i=1}^{k}$ is an α-shading of X, then there is an α-shading $\{v_i\}_{i=1}^{k}$ of X by member of β such that $v_i \le u_i$ for each $i = 1,2,\ldots,k$.

Proof. Let β be a base for T. For each $x \in X$ there is some $i \in \{1,2,\ldots,k\}$ such that $u_i(x) > \alpha$ so $u_i = \bigvee_{b_x \in \beta} b_x$ which gives that $b_x \le u_i$ for some $b_x \in \beta$; that is $\{b_x\}_{x \in X}$ forms an α-shading of X and hence by assumption there is a finite α-subshading $\{b_1 \ldots b_s\}$ (say) of X. For each $t = 1,2,\ldots,s$ choose $\mathcal{G}(t)$ such that $b_t \le u_{(t)}$ and set $v_i = \bigvee_{\mathcal{G}(t)=i} b_t$, then since β is closed under finite joins, so $v_i \in \beta$ and $v_i \leq u_i$ for $i = 1,2, \ldots,k$. Clearly $\{v_i\}_{i=1}^{k}$ is an α-shading of X. □

<u>Proposition 3.1:</u> Let $(\tilde{X},\tilde{T})$ be a limit F-space of an inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ of α-bicompact F-spaces. If F-$\dim_\alpha X \leq n$, then every α-shading of $\tilde{X}$ has a finite α-refinement with order $\leq n$ and whose members belong to $\beta(\tilde{X})$.

<u>Proof.</u> For $\alpha \in [0,1)$ let F-$\dim_\alpha \tilde{X} \leq n$ and let U be an α-shading of $\tilde{X}$. By proposition 1.9 $(\tilde{X},\tilde{T})$ is α-bicompact F-space where $\tilde{X}$ is non-empty and hence there is a finite α-subshading $\{u_i\}_{i=1}^{k}$ of U. Since $\beta(X)$ is closed under finite joins, the proof follows from lemma 3.1 and the definition of F-$\dim_\alpha$ II.1.2. □

The next proposition shows that if $(\tilde{X},\tilde{T})$ is the limit F-space of an inverse system of α-bicompact each with F-$\dim_\alpha X_s \leq n$ then F-$\dim_\alpha \tilde{X} \leq n$.

<u>Proposition 3.2:</u> Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of α-bicompact F-spaces such that F-$\dim_\alpha X_s \leq n$ for $s \in S$. Then F-$\dim_\alpha \tilde{X} \leq n$.

<u>Proof.</u> Let $U = \{u_i\}_{i=1}^{k}$ be an α-shading of X whose members belong to $\beta(\tilde{X})$; that is u_i is of the form $u_i = f_{s_i}^{-1}(u_{s_i})$ $s_i \in S$, u_{s_i} open F-set in X_{s_i}. Since S is directed, so there is $t \in S$ such that $s_i \leq t$ for all s_i, $i = 1,2,\ldots,k$.

Let $v_{t_i} = f_{s_i,t}^{-1}(u_{s_i})$. Then v_{t_i} is open F-set in X_t and if we put $V_t = \{v_{t_i}\}_{i=1}^{k}$, then $f_t^{-1}(V_t) = U$.

Now $f_t(\tilde{X})$ is α-closed in X_t by proposition I.2.8 and

I.3.3 and V_t is an α-shading of $f_t(\widetilde{X})$ for: let $x_t \in f_t(\widetilde{X})$. Then $x_t = f_t(x)$ for some $x \in \widetilde{X}$, since $f_t^{-1}(V_t)$ is α-shading of $\widetilde{X}$ so there is $f_t^{-1}(v_{t_{i_0}})$ such that $f_t^{-1}(v_{t_{i_0}})(x) > \alpha$, then $v_{t_{i_0}}(f_t(x)) > \alpha$ i.e. $v_{t_{i_0}}(x_t) > \alpha$. Since F-dim$_\alpha f_t(\widetilde{X}) \leq$ F-dim$_\alpha$ $X_t \leq n$ by proposition II.2.1, then there exists an α-refinement W of V_t whose order $\leq n$. Hence $f_t^{-1}(W)$ is α-refinement of $f_t^{-1}(V_t) = U$ i.e. $f_t^{-1}(W)$ is α-refinement of U of order $\leq n$ which implies that F-dim$_\alpha X \leq n$. □

<u>Corollary 3.1:</u> Let $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ be an inverse system of strong α-bicompact F-spaces such that F-dim $X_s \leq$ for every $s \in S$. Then F-dim $\widetilde{X} \leq n$. □

The next proposition gives the characterization of the α-bicompact F-space in term of the limit F-space.

<u>Proposition 3.3:</u> An F-space (X,T) is strong α-bicompact with F-dim $X = 0$ if and only if (X,T) is a limit F-space of finite discrete F-spaces.

<u>Proof.</u> $\Longrightarrow$ Let (X,T) be the limit F-space of finite discrete F-spaces. For each $\alpha \in [0,1)$ clearly (X,T) is α-bicompact F-space by proposition 1.9 since each finite discrete F-space is α-bicompact.

To show that F-dim$_\alpha X = 0$ assume the contrary i.e. let F-dim$_\alpha X > 0$ for each $\alpha \in [0,1)$. Take $x_1 \in X$ and $y \in X \setminus \{x_1\}$, then by the definition of α-Hausdorff F-space, there are two open F-sets u_{x_1}, u_y in X such that $u_{x_1}(x_1) > \alpha$, $u_y(y) > \alpha$ and $u_{x_1} \wedge u_y = 0$. So $\{u_{x_1}, u_y\}_{y \in X \setminus \{x_1\}}$ is an α-shading of X and then

there is a finite set $\{y_1,\ldots,y_y\} \subseteq X\setminus\{x_1\}$ from the α-compactness of X, such that $\{u_{x_1},u_{y_i}\}_{i=1}^{k}$ is a finite α-shading of X. Since $F\text{-}\dim_\alpha X > 0$, so there is an α-refinement V of $\{u_{x_1},u_{y_i}\}_{i=1}^{k}$ of order ≥ 0; that is there are at least $v_1,v_2 \in V$ such that $v_1 \wedge v_2 \neq 0$. But V is α-refinement of $\{u_{x_1},u_{y_i}\}_{i=1}^{k}$ i.e. for u_{x_1}, u_{y_2} (say) there are v_1, $v_2 \in V$ such that $v_1 \leq u_{x_1}$, $v_2 \leq u_{y_2}$ ($v_1 \wedge v_2 \neq 0$) and $v_1 \wedge v_2 \leq u_{x_1} \wedge u_{y_2} = 0$ contradiction. Hence $F\text{-}\dim_\alpha X = 0$ for each $\alpha\in[0,1)$.
Therefore $F\text{-}\dim X = 0$.

$\Longleftarrow$ Let (X,T) be a strong α-bicompact F-space with $F\text{-}\dim X = 0$. For each $\alpha\in[0,1)$, let $\mathcal{G}_\alpha = \{G_i\}_{i\in A_\alpha}$ be an α-shading of X by disjoint F-sets where A_α is a finite set, then $\Omega = \{\mathcal{G}_\alpha\}_{\alpha\in[0,1)}$ is the family of all finite disjoint α-shadings of X.

If $\mathcal{G}_\alpha$ is an α-refinement of $\mathcal{G}_\beta$, we say $\alpha \leq \beta \in I$, then I is a directed set by $\leq$. For $\alpha\leq\beta$ we define $f_{\alpha,\beta}: A_\beta \longrightarrow A_\alpha$ by $f_{\alpha,\beta}(i) = j$ implies $G_j \leq G_i$, $i \in A_\beta$, $j\in A_\alpha$. Let each A_α be given the discrete F-topology (i.e. I^{A_α} is the F-topology on A_α for each $\alpha\in[0,1]$). Then clearly $f_{\alpha,\beta}$ is F-continuous ($\alpha \leq \beta$) and so $\underset{\sim}{A}_F = \{A_\alpha, f_{\alpha,\beta}, I\}$ is an inverse system of F-spaces over I.

Let $(\widetilde{A},\widetilde{T})$ be the limit F-space of $\underset{\sim}{A}_F$ and $f_\lambda = \widetilde{A} \longrightarrow A_\lambda$ be the canonical mapping for each $\lambda \in I$.

Define $\Psi: \widetilde{A} \longrightarrow X$ by $\Psi(i) = x$ if $x\in \bigcap_{\lambda\in[0,1)} G_i^+$ such that

$$f_\lambda(i) = i_\lambda \in A_\lambda.$$

We show that Ψ is F-homeomorphism. Ψ is injective: Let

$i \neq j \in \widetilde{A}$. Then $f_\lambda(i) \neq f_\lambda(j) \in A_\lambda$ i.e. $i_\lambda \neq j_\lambda$ and hence $G_{i_\lambda} \neq G_{j_\lambda}$ for $G_{i_\lambda}, G_{j_\lambda} \in \mathcal{G}_\lambda$ this implies that $\Psi(i) = x \neq y = \Psi(j)$ if $x \in \bigcap_{\lambda \in [0,1)} G^+_{i_\lambda}$, $y \in \bigcap_{\lambda \in [0,1)} G^+_{j_\lambda}$.

Ψ is surjective: Let $x \in X$. Then there is an $\lambda \in [0,1)$ and a finite disjoint λ-shading $\mathcal{G}_\lambda$ such that $G_{i_{o\lambda}}(x) > \lambda$ for some $G_{i_{o\lambda}} \in \mathcal{G}_\lambda$ which implies that $x \in G^+_{i_{o\lambda}}$ and $i_{o\lambda} \in A_\lambda$. So there is $i_o \in \widetilde{A}$ such that $f_\lambda(i_o) = i_{o\lambda}$ and hence $\Psi(i_o) = x$.

Ψ is F-continuous: Let $u \neq 0$ be an open F-set in X. So for some $\lambda \in [0,1)$ there exist $i_o \in \widetilde{A}$ and $x_{i_o} \in X$ such that $u(x_{i_o}) > \lambda$ and $\Psi(i_o) = x_{i_o}$. Hence

$$\begin{aligned}\Psi^{-1}(u) &= \Psi^{-1}\left[V\left\{p_{x_i} : x_i \in u^+,\ p_{x_i}(x_i) > \lambda\right\}\right] = V\left\{p_{\Psi^{-1}(x_i)} : x_i \in u^+\right\} \\ &= V\left\{p_i : i \in \widetilde{A}\right\} \\ &= \text{open F-set in } \widetilde{A}.\end{aligned}$$

is F-open map: Let w be open F-set in $\widetilde{A}$. Then $w = V\{p_i : p_i \in w\}$ and hence $\Psi(w) = \Psi(V\ p_i) = V\ \Psi(p_i) = V\ p_{\Psi(i)} = V\ p_x$ where $x \in \bigcap_{\lambda \in [0,1)} G^+_{i_\lambda}$, $f_\lambda(i) = i_\lambda$. so $\Psi(w)$ is open F-set in X. Therefore $(\widetilde{A},\widetilde{T})$ is F-homeomorphic to (X,T). □

If $(\widetilde{X},\widetilde{T})$ is the limit F-space of an inverse sequence of F-spaces with $\text{F-dim}_\alpha \leq p$ where $p \in N$, $p \geq 0$. Under what conditions is it true that $\text{F-dim}_\alpha \widetilde{X} \leq p$. The following two propositions deal with this question.

<u>Proposition 3.4:</u> Let $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ be an inverse sequence of countably α-compact with $\text{F-dim}_\alpha \underset{\sim}{X} \leq p$ $(p \in N,\ p \geq 0)$ and all $f_{n,m}(X_m)$ be α-closed in X_n $(n \leq m)$. If each finite α-shading of any F-subspace of X_n is countably extendable, then $\text{F-dim}_\alpha \widetilde{X} \leq p$.

<u>Proof.</u> Let $U = \{u_i\}_{i=1}^{k}$ be an α-shading of $\widetilde{X}$ and let each finite α-shading of any F-subspace of X_n $(n \in N)$ is countably extendable.

Each $u_i = f_{n_i}^{-1}(v_{n_i})$ where $n_i \in N$, v_{n_i} is open F-set in X_{n_i}. Let $v_{m_i} = f_{n_i,m}^{-1}(v_{n_i})$, $n_i \leq m$ for all $i = 1,2,\ldots,k$.

Put $V_m = \{v_{m_i}\}_{i=1}^{k}$. then $f_m^{-1}(V_m) = U$ and $\{V_m | f_m(\widetilde{X})\}$ is a finite α-shading of $f_m(\widetilde{X})$. By proposition II.2.5 F-dim$_\alpha$ $f_m(\widetilde{X}) \leq p$ and hence there is a finite α-shading $\{G_{m_i}\}_{i=1}^{k}$ of $f_m(\widetilde{X})$ whose order $\leq p$ and $G_{m_i} \leq v_{m_i} | f_n(\widetilde{X})$ for $i = 1,2,\ldots,k$. By assumption there is a countable α-shading $W = \{w_j\}_{i=1}^{\infty}$ of X_m such that $W | f_m(\widetilde{X})$ is a precise α-refinement of $\{G_{m_i}\}_{i=1}^{k}$ and hence from countably α-compactness of X_m there is a finite α-subshading $\{w_j\}_{i=1}^{L}$ $(L > k$ say) of X_m. i.e. $\{w_j\}_{i=1}^{k}$ is α-refinement of $\{G_{m_i}\}_{i=1}^{k}$ whose order $\leq p$. So $f_m^{-1}(\{w_j\}_{i=1}^{k})$ is α-refinement of $f_m^{-1}(V_m)$ which implies that $f_m^{-1}(\{w_j\}_{i=1}^{L})$ is an α-refinement of U of order $\leq p$. i.e. F-dim$_\alpha \widetilde{X} \leq p$. □

<u>Proposition 3.5:</u> Let $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$ be an inverse sequence of countably α-compact α-Hausdorff F-spaces with strongly F-closed canonical mappings. If F-dim$_\alpha X_n \leq p$ for each $n \in N$, then F-dim$_\alpha \widetilde{X} \leq p$.

<u>Proof.</u> Let $(\widetilde{X},\widetilde{T})$ be the limit F-space of $\underset{\sim}{X}_F = \{X_n, f_{n,m}, N\}$, F-dim$_\alpha X_n \leq p$ for each $n \in N$ and let $U = \{u_i\}_{i=1}^{k}$ be an α-shading

of $\widetilde{X}$. Then as in the case of the previous proposition, we get a finite α-shading $f_m^{-1}(V_m) = U$ and $\{V_m | f_m(X)\}$ is a finite α-shading of $f_m(\widetilde{X})$ where $f_m(\widetilde{X})$ is α-closed in X_m by proposition 1.4 and the hypothesis and then from proposition II.2.3 we have $F\text{-}\dim_\alpha f_m(\widetilde{X}) \leq p$. So there is an α-refinement W_m of $V_m | f_m(\widetilde{X})$ whose order $\leq p$ and hence $f_m^{-1}(W_m)$ is α-refinement of $f_m^{-1}(V_m) = U$ i.e. $f_m^{-1}(W_m)$ is the required α-refinement of U. Therefore $F\text{-}\dim_\alpha \widetilde{X} \leq p$. □

<u>Proposition 3.6:</u> Let a strong countably α-compact F-space (X,T) be the limit F-space of an inverse sequence of weakly normal F-spaces and surjective F-bondings with $F\text{-}\dim_\alpha X_n \leq p$ for each $n \in N$. Then (X,T) is weakly normal F-space and $F\text{-}\dim_\alpha X \leq p$.

<u>Proof.</u> Let us show first that (X,T) is weakly normal F-space. Let H,k be two disjoint closed F-sets in X and $f_n : X \longrightarrow X_n$ be the canonical mapping for each $n \in N$. Then

$$\bigwedge_{n \in N} f_n^{-1}(cl(f_n(k)) \wedge cl(f_n(H))) = 0$$

Since (X,T) is strong countably α-compact by assumption, then $f_n^{-1}(cl(f_n(k)) \wedge cl(f_n(H))) = 0$ for some $n \in N$ from proposition I. 2.4. Now each $f_{n,m}$ $(n \leq m)$ is surjective by hypothesis, so each f_n is surjective by theorem 0.3 and hence $cl(f_n(k)) \wedge cl(f_n(H)) = 0$ in X_n. But X_n is weakly normal F-space so there are u_1, u_2 open F-sets in X_n such that $cl(f_n(k)) \leq u_1$, $cl(f_n(H)) \leq u_2$ and $u_1 \leq co\ u_2$.

Hence

$f_n^{-1}(cl(f_n(k)) \leq f_n^{-1}(u_1)$, $f_n^{-1}(cl(f_n(H))) \leq f_n^{-1}(u_2)$ and

$f_n^{-1}(u_1) \leq f^{-1}(co\ u_2)$.

That is

$k \leq f_n^{-1}(u_1)$, $H \leq f_n^{-1}(u_2)$ and $f_n^{-1}(u_1) \leq co\ f_n^{-1}(u_2)$

where $f_n^{-1}(u_1)$ and $f_n^{-1}(u_2)$ are open F-sets in X.

Therefore (X,T) is weakly normal F-space.

Let $U = \{u_i\}_{i=1}^{k}$ be an α-shading of X. Then each $u_i = f_{n_i}^{-1}(v_{n_i})$, $n_i \in N$, v_{n_i} is open F-set in X_{n_i}. Let $v_{m_i} = f_{n_i,m}(v_{n_i})$ $n_i \leq m$ for all i = 1,2,...,k. Put $V_m = \{v_{m_i}\}_{i=1}^{k}$ then $f_m^{-1}(V_m) = U$ and since $f_m(X) = X_m$, then V_m is a finite α-shading of $X_m = f_m(X)$ for some $\alpha \in [0,1)$. Since $F\text{-}dim_\alpha X_m \leq p$, then there exists an α-refinement W of V_m of order $\leq p$ and so $f_m^{-1}(W)$ is an α-refinement of $f_m^{-1}(V_m) = U$ whose order $\leq p$. Hence $F\text{-}dim_\alpha X \leq p$. □

<u>Corollary 3.2:</u> Let a strong countably α-compact F-space (X,T) be the limit F-space of an inverse sequence of normal F_c-spaces and surjective F-bondings with $F\text{-}dim_\alpha X_n \leq p$ for each $n \in N$. Then (X,T) is normal F-space and $F\text{-}dim_\alpha X \leq p$. □

We end this section by giving a generalization to Delinić and Mardešić's theorem [19] i.e. to give a necessary and sufficient condition for $F\text{-}dim_\alpha X \leq n$. Following a similar technique we start with some lemmas.

<u>Definition 3.1.</u> An inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ is called α-reversable if for an open F-set u_s in X_s, $f_{s,t}^{-1}(X_s \setminus supp. u_s)$ is α-closed in X_t for $\alpha \in [0,1)$ and $s \leq t$.

<u>Lemma 3.2:</u> Let $(\tilde{X},\tilde{T})$ be the limit F-space an α-reversable inverse system $\underset{\sim}{X}_F = \{X_s, f_{s,t}, S\}$ of α-bicompact F-spaces.

If $s \in S$ and u_s is an open F-set in X_s such that supp. $u_s \supset f_s(\tilde{X})$, then there exists a $t \in S$, $s \leq t$ such that $f_{s,t}(X_t) \subset$ supp. u_s.

<u>Proof.</u> Assume the contrary: that is for every $t \in S$, $s \leq t$ we have $X_t^* = f_{s,t}^{-1}(X_s \setminus \text{supp } u_s) \neq \emptyset$. By hypothesis X_t^* is α-closed in X_t and hence it is α-compact. Then $\underset{\sim}{X}_F^* = \{X_t^*, f_{t,r}^*, S\}$, $t,r \geq S$ and $f_{t,r}^* = f_{t,r}|X_t^*$ is an inverse system of α-bicompact F-spaces and would have a non-empty limit F-space $(\tilde{X}^*, \tilde{T}^*) \subset (\tilde{X}, \tilde{T})$ from proposition 1.9. Clearly $f_s(\tilde{X}^*) \leq X_s \setminus$ supp. u_s which contracts the assumption $f_s(\tilde{X}) \subset$ supp. u_s. □

<u>Lemma 3.3:</u> Let $(\tilde{X}, \tilde{T})$ be as in the prvious lemma $U = \{u_i\}_{i=1}^k$ be an α-shading of X whose members belong to $\beta(\tilde{X})$ and let $s \in S$. Then there exists $t \in S$, $s \leq t$ and finite α-shading $V_t = \{v_{t_i}\}_{i=1}^k$ of X_t such that $f_t^{-1}(v_{t_i}) \leq u_i$ and order $V_t \leq$ order U.

<u>Proof.</u> For each $u_i \in U$, $u_i = f_{s_i}^{-1}(u_{s_i})$, $s_i \in S$, u_{s_i} open F-set in X_{s_i}. Choose $s_o \in S$, $s_o \geq s, s_1, \ldots, s_k$. Let $u_{s_o i} = f_{s_i s_o}^{-1}(u_{s_i})$ $i = 1,2,\ldots,k$. Then $U_{s_o} = \{u_{s_o i}\}_{i=1}^k$ and $f_{s_o}^{-1}(u_{s_o i}) =$

$f_{s_o}^{-1} f_{s_i\, s_o}^{-1}(u_{s_i}) = f_{s_i}^{-1}(u_{s_i}) = u_i$. Then $f_{s_o}^{-1}(U_{s_o}) = U$. i.e. $f_{s_o}(u_{s_o})$ is α-shading of $\tilde{X}$. Hence $\underset{\sim}{U}_{s_o}$ is an α-shading of $f_{s_o}(\tilde{X})$ which is α-compact and then $f_{s_o}(\tilde{X}) \subset$ supp. u_{s_o} where $u_{s_o} = \bigvee_{i=1}^k u_{s_o i}$.

By lemma 3.2 there exists $t \in S$ such that $t \geq s_o \geq s$ and $f_{s_o t}(X_t) \subset$ supp. u_{s_o} put $v_{t_i} = f_{s_o,t}^{-1}(u_{s_o i})$, then $V_t = \{v_{t_i}\}_{i=1}^k$

is an α-shading of X_t and $f_t^{-1}(v_{t_i}) \leq u_i$ for $i = 1,2,\ldots,k$.

Now if $v_{t\ i_1} \wedge \ldots \wedge v_{t\ i_k} \neq 0$, then $f^{-1}_{s_o,t}(u_{s_o\ i_1}) \wedge f^{-1}_{s_o\ t}(u_{s_o\ i_2}) \wedge \ldots \wedge f^{-1}_{s_o\ t}(u_{s_o\ i_k}) \neq 0$ and hence

$$f_t^{-1}\ f^{-1}_{s_o,t}(u_{s_o\ i_1}) \wedge \ldots \wedge f_t^{-1}\ f^{-1}_{s_o,t}(u_{s_o\ i_k}) = f^{-1}_{s_o}(u_{s_o\ i_1}) \wedge \ldots \wedge f^{-1}_{s_o}(u_{s_o\ i_k}) \neq 0.$$

Consequently

$$u_{i_1} \wedge u_{i_2} \wedge \ldots \wedge u_{i_k} \neq 0.$$

Therefore order $V_t \leq$ order U. □

<u>Lemma 3.4:</u> Let $(\widetilde{X},\widetilde{T})$ be as in lemma 3.2. If u_s is an open F-set in X_s and v_t is an open F-set in X_t, $s,t \in S$ such that $f_t^{-1}(v_t) \leq f_s^{-1}(u_s)$, then there exists $r \in S$, $r \geq s,t$ such that $f^{-1}_{t,r}(v_t) \leq f^{-1}_{s,r}(u_s)$.

<u>Proof.</u> Suppose that the assertion is false i.e for every $r \in S$, $r \geq s,t$ we have $X_r^* = f^{-1}_{s,r}(X_s \setminus \text{supp. } u_s) \setminus f^{-1}_{t,r}(X_t\ \text{supp. } v_t) \neq \emptyset$. Then X_r^* is α-closed in X_r and as in case of lemma 3.2, we would have an inverse subsystem of α-bicompact F-space which has a non-empty limit F-space. This would lead to

$$\text{Supp. } f_t^{-1}(v_t) \setminus \text{Supp. } f_s(u_s) \neq \emptyset$$

Conctracts our assumption that

$$f_t^{-1}(v_t) \leq f_s^{-1}(u_s). \quad □$$

This lemma gives at once the following result .

Remark: U_s is an α-shading of X_s and V_t is a finite α-shading of X_t such that $f_t^{-1}(V_t)$ refines $f_s^{-1}(U_s)$ then there is $r \in S$, $r \geq s,t$ such that $f_{t,r}^{-1}(V_t)$ refines $f_{s,r}^{-1}(U_s)$. □

Proposition 3.7: Let $X_F = \{X_s, t_{s,t}, S\}$ be an α-reversable inverse system of α-bicompact F-spaces with the limit F-space $(\tilde{X},\tilde{T})$. Then F-$\dim_\alpha X \leq n$ if and only if for each $s \in S$ and each α-shading U_s of X_s there is $t \in S$, $s \leq t$ such that the α-shading $f_{s,t}^{-1}(U_s)$ of X_t has an α-refinement V_t of order $\leq n$.

Proof. Let F-$\dim_\alpha \tilde{X} \leq n$ and let U_s be an α-shading of X_s. Then $f_s^{-1}(U_s)$ is an α-shading of $\tilde{X}$. By proposition 3.1 $f_s^{-1}(U_s)$ has an α-refinement $U = \{u_1,\dots,u_k\}$ whose members belong to the standard basis $\beta(\tilde{X})$ and order of $U \leq n$. By lemma 3.3, there exists $t \in S$ with $s \leq t$ and a finite α-shading of $V_t = \{v_{t_i}\}_{i=1}^{k}$ of X_t such that $f_s^{-1}(v_{t_i}) \leq u_i$ for $i = 1,2,\dots,k$ and order $V_t \leq$ order $U \leq n$.

By the remark ofter lemma 3.4 where for each $i = 1,2,\dots,k$, we have $f_t^{-1}(v_{t_i}) \leq u_i$, then $f_t^{-1}(V_t)$ refines $f_s^{-1}(U_s)$ and hence there exists $r \in S$, $r \geq s,t$ such that $f_{t,r}^{-1}(V_t)$ refines $f_{s,r}^{-1}(U_s)$. So if $W_r = f_{t,r}^{-1}(V_t)$, then W_r is α-shading of X_r because V_t is α-shading of X_t and clearly order $W_r \leq$ order $V_t \leq n$. Therefore r and W_r satisfy the required property.

Let U be a finite α-shading of $\tilde{X}$ where members of U can be assumed in the standard basis $\beta(X)$. Then by lemma 3.3 there is $s \in S$ and a finite α-shading U_s of X_s such that $f_s^{-1}(U_s)$ refines U. By assumption there exists $t \in S$, $s \leq t$, and a finite α-shading V_t of X_t which refines $f^{-1}_{s,t}(U_s)$ and order of $V_t \leq n$. Hence $f^{-1}_t(V_t)$ is a finite α-shading of X which refines $f^{-1}_s(U_s)$ and then $f^{-1}_t(V_t)$ refines U. Clearly order $f^{-1}_t(V_t)$ $\leq$ order $V_t \leq n$. Therefore $F\text{-}\dim_\alpha \widetilde{X} \leq n$ as required. □

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LIST OF SYMBOLS